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POSITIVITY AND PERIODICITY IN NONLINEAR NEUTRAL MIXED TYPE LEVIN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this work, we give sufficient conditions for the existence of periodic and positive periodic solutions for a nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delays by using Krasnoselskii's fixed point theorem. Also, we obtain the existence of a unique periodic solution of the posed equation by means of the contraction mapping principle. As an application, we give an example to illustrate our results. Previous results are extended and generalized.

1. Introduction

Differential and integro-differential equations with delays have received great attention and have become an active area of research. This is due to the fact that several phenomena in life sciences, engineering, chemistry and physics can be described by means of delay equations. Indeed, problems concerning the positivity, periodicity and stability of solutions for differential and integro-differential equations with delays have received the considerable attention of many authors, see [1]–[24], [26,27] and the references therein.

Key words and phrases. Fixed points, positivity, periodicity, Levin-Nohel integro-differential equations.

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In this paper, we consider the following nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delays

$$\begin{aligned} \frac{d}{dt}x(t) = &-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,x(s)\,ds - \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,x(s)\,ds \\ &+ \frac{d}{dt}g\left(t, x\left(t-\tau_1(t)\right),\ldots,x\left(t-\tau_m(t)\right)\right), \end{aligned} \tag{1.1}$$

where a_j, b_j, τ_j, σ_j and g are continuous functions with $\tau_j(t) > 0$, $\sigma_j(t) > 0$, $j = 1,\ldots,m$. In this work, we use the idea of integrating factor to convert the equation (1.1) into an integral equation. Then, we employ Krasnoselskii's fixed point theorem to show the existence of periodic and positive periodic solutions of (1.1). Also, we obtain the existence of a unique periodic solution by using the contraction mapping principle. An example is given to illustrate our main results.

In [9], we investigated the asymptotic stability of the zero solution for (1.1) by using the contraction mapping theorem. Also, in the special case $a_j(t,s) = 0$, $j = 2,\ldots,m$, $b_j(t,s) = 0$, $j = 1,\ldots,m$ and $g(t,x_1,x_2,\ldots,x_m) = g_1(t,x_1)$, in [10], we proved the existence and uniqueness of periodic solutions and the existence of positive solutions for (1.1) by appealing Krasnoselskii's fixed point theorem and the contraction mapping theorem. Then, the results presented in this paper extend and generalize the main results in [10].

2. EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS

For $T > 0$ let P_T be the set of all continuous scalar functions x periodic in t of period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}}|x(t)| = \sup_{t\in[0,T]}|x(t)|.$$

Since we are searching for the existence of periodic solutions for (1.1), it is natural to suppose that

$$\begin{aligned} a_j(t+T,s+T) =&a(t,s), \quad b_j(t+T,s+T) = b_j(t,s), \\ \tau_j(t+T) =&\tau_j(t), \quad \sigma_j(t+T) = \sigma_j(t), \end{aligned} \tag{2.1}$$

with τ_j and σ_j being scalar continuous functions, $\tau_j(t) \geq \tau_j^* > 0$ and $\sigma_j(t) \geq \sigma_j^* > 0$, $j = 1,\ldots,m$. Also, we suppose

$$\int_0^T A(z)\,dz > 0, \quad A(t) = \sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t} a_j(t,s)\,ds + \sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)} b_j(t,s)\,ds. \tag{2.2}$$

The function $g(t,x_1,x_2,\ldots,x_m)$ is periodic in t of period T, it is also globally Lipschitz continuous in x_j, $j = 1,\ldots,m$. That is

$$g(t+T,x_1,x_2,\ldots,x_m) = g(t,x_1,x_2,\ldots,x_m), \tag{2.3}$$

and there are positive constants E_j, $j = 1, \dots, m$, such that

$$|g(t, x_1, x_2, \dots, x_m) - g(t, y_1, y_2, \dots, y_m)| \le \sum_{j=1}^{m} E_j |x_j - y_j| . \tag{2.4}$$

The next lemma is crucial to our results.

Lemma 2.1. *Suppose that* (2.1)–(2.3) *hold. Then,* $x \in P_T$ *is a solution of the equation* (1.1) *if and only if* x *satisfies*

$$\begin{aligned} x(t) =& G_x(t) - \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \\ & \times \int_{t-T}^{t} \left[L_x(s) + N_x(s) + A(s) G_x(s)\right] e^{-\int_{s}^{t} A(z)dz} ds, \end{aligned} \tag{2.5}$$

where

$$G_x(t) = g(t, x(t - \tau_1(t)), \dots, x(t - \tau_m(t))), \tag{2.6}$$

and

$$\begin{aligned} L_x(t) =& \sum_{j=1}^{m} \int_{t-\tau_j(t)}^{t} a_j(t,s) \left(\int_s^t \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} N_x(t) =& \sum_{j=1}^{m} \int_{t}^{t+\sigma_j(t)} b_j(t,s) \left(\int_s^t \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} a_k(u,\nu) x(\nu) d\nu \right.\right. \\ & \left.\left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} b_k(u,\nu) x(\nu) d\nu \right) du + G_x(s) - G_x(t) \right) ds. \end{aligned} \tag{2.8}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du.$$

Inserting this relation into (1.1), we obtain

$$\begin{aligned} & \frac{d}{dt} x(t) + \sum_{j=1}^{m} \int_{t-\tau_j(t)}^{t} a_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds \\ & + \sum_{j=1}^{m} \int_{t}^{t+\sigma_j(t)} b_j(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du \right) ds - \frac{d}{dt} G_x(t) \ = 0. \end{aligned}$$

So,

$$\begin{aligned}&\frac{d}{dt}x\left(t\right)+x\left(t\right)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j\left(t,s\right)ds+\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j\left(t,s\right)ds\right)\\&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j\left(t,s\right)\left(\int_s^t\frac{\partial}{\partial u}x\left(u\right)du\right)ds\\&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j\left(t,s\right)\left(\int_s^t\frac{\partial}{\partial u}x\left(u\right)du\right)ds-\frac{d}{dt}G_x\left(t\right)\;=\;0.\end{aligned}$$

Substituting $\frac{\partial x}{\partial u}$ from (1.1), we get

$$\begin{aligned}&\frac{d}{dt}x\left(t\right)+x\left(t\right)\left(\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j\left(t,s\right)ds+\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j\left(t,s\right)ds\right)\\&-\sum_{j=1}^{m}\int_{t-\tau_j(t)}^{t}a_j\left(t,s\right)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k\left(u,\nu\right)x\left(\nu\right)d\nu\right.\right.\\&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k\left(u,\nu\right)x\left(\nu\right)d\nu+\frac{\partial}{\partial u}G_x\left(u\right)\right)du\right]ds\\&-\sum_{j=1}^{m}\int_{t}^{t+\sigma_j(t)}b_j\left(t,s\right)\left[\int_s^t\left(-\sum_{k=1}^{m}\int_{u-\tau_k(u)}^{u}a_k\left(u,\nu\right)x\left(\nu\right)d\nu\right.\right.\\&\left.\left.-\sum_{k=1}^{m}\int_{u}^{u+\sigma_k(u)}b_k\left(u,\nu\right)x\left(\nu\right)d\nu+\frac{\partial}{\partial u}G_x\left(u\right)\right)du\right]ds-\frac{d}{dt}G_x\left(t\right)\;=\;0.\end{aligned}\tag{2.9}$$

By performing the integration, we obtain

$$\int_s^t\frac{\partial}{\partial u}G_x\left(u\right)du=G_x\left(t\right)-G_x\left(s\right).\tag{2.10}$$

Substituting (2.10) into (2.9), we get

$$\frac{d}{dt}x\left(t\right)+A\left(t\right)x\left(t\right)+L_x\left(t\right)+N_x\left(t\right)-\frac{d}{dt}G_x\left(t\right)=0,$$

where A and L_x and N_x are given by (2.2), (2.7) and (2.8), respectively. We rewrite this equation as

$$\frac{d}{dt}\left\{x\left(t\right)-G_x\left(t\right)\right\}=-A\left(t\right)\left(x\left(t\right)-G_x\left(t\right)\right)-A\left(t\right)G_x\left(t\right)-L_x\left(t\right)-N_x\left(t\right).\tag{2.11}$$

Multiply both sides of (2.11) with $e^{\int_0^t A(z)dz}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t}\frac{\partial}{\partial s}\left[x\left(s\right)-G_x\left(s\right)\right]e^{\int_0^s A(z)dz}ds\\&=-\int_{t-T}^{t}\left[L_x\left(s\right)+N_x\left(s\right)+A\left(s\right)G_x\left(s\right)\right]e^{\int_0^s A(z)dz}ds.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&\left(x\left(t\right)-G_{x}\left(t\right)\right)e^{\int_{0}^{t}A(z)dz}-\left(x\left(t-T\right)-G_{x}\left(t-T\right)\right)e^{\int_{0}^{t-T}A(z)dz}\\ =&-\int_{t-T}^{t}\left[L_{x}\left(s\right)+N_{x}\left(s\right)+A\left(s\right)G_{x}\left(s\right)\right]e^{\int_{0}^{s}A(z)dz}ds.\end{aligned}$$

Dividing both sides of the above equation by $e^{\int_{0}^{t}A(z)dz}$ and using the fact that $x\left(t\right)=x\left(t-T\right)$, we obtain

$$\begin{aligned}&x\left(t\right)-G_{x}\left(t\right)\\ =&-\left(1-e^{-\int_{t-T}^{t}A(z)dz}\right)^{-1}\int_{t-T}^{t}\left[L_{x}\left(s\right)+N_{x}\left(s\right)+A\left(s\right)G_{x}\left(s\right)\right]e^{-\int_{s}^{t}A(z)dz}ds.\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. ☐

Define the mapping H by

$$\begin{aligned}\left(H\varphi\right)\left(t\right)=&G_{\varphi}\left(t\right)-\left(1-e^{-\int_{t-T}^{t}A(z)dz}\right)^{-1}\\ &\times\int_{t-T}^{t}\left[L_{\varphi}\left(s\right)+N_{\varphi}\left(s\right)+A\left(s\right)G_{\varphi}\left(s\right)\right]e^{-\int_{s}^{t}A(z)dz}ds.\end{aligned}\tag{2.12}$$

It is clear form (2.12) that $H:P_{T}\rightarrow P_{T}$ by the way it was constructed in Lemma 2.1.

Next, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of periodic and positive periodic solutions. For the proof of Krasnoselskii's fixed point theorem we refer the reader to [25].

Theorem 2.1 (Krasnoselskii). *Let M be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B},\|\cdot\|)$. Suppose that C and B map M into $\mathbb{B}$ such that*
(i) $x,y\in M$ implies $Cx+By\in M$;
(ii) C is continuous and CM is contained in a compact set;
(iii) B is a contraction mapping.
Then there exists $z\in M$, with $z=Cz+Bz$.

We note that to apply the above theorem we need to construct two mappings; one is contraction and the other is continuous and compact. Therefore, we express (2.12) as

$$\left(H\varphi\right)\left(t\right)=\left(B\varphi\right)\left(t\right)+\left(C\varphi\right)\left(t\right),$$

where $C,B:P_{T}\rightarrow P_{T}$ are given by

$$\left(B\varphi\right)\left(t\right)=G_{\varphi}\left(t\right),\tag{2.13}$$

and

$$\left(C\varphi\right)\left(t\right)=-\left(1-e^{-\int_{t-T}^{t}A(z)dz}\right)^{-1}\int_{t-T}^{t}\left[L_{\varphi}\left(s\right)+N_{\varphi}\left(s\right)+A\left(s\right)G_{\varphi}\left(s\right)\right]e^{-\int_{s}^{t}A(z)dz}ds.\tag{2.14}$$

To simplify notations, we introduce the following constants

$$\eta = \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1}, \quad \rho = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} \sum_{j=1}^{m} \left(\int_{s-\tau_j(s)}^{s} |a_j(s,w)|\, dw \right) \right),$$

$$\varrho = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} \sum_{j=1}^{m} \left(\int_{s}^{s+\sigma_j(s)} |b_j(s,w)|\, dw \right) \right), \quad \gamma = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} e^{-\int_s^t A(z)dz} \right),$$

$$\delta = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} \left(\sup_{w\in[t-T,t]} \int_w^s \left(\sum_{k=1}^{m} \int_{u-\tau_k(u)}^{u} |a_k(u,\nu)|\, d\nu \right.\right.\right.$$

$$\left.\left.\left. + \sum_{k=1}^{m} \int_{u}^{u+\sigma_k(u)} |b_k(u,\nu)|\, d\nu \right) du \right) \right), \quad \alpha = \sup_{t\in[0,T]} |G_0(t)|. \tag{2.15}$$

Lemma 2.2. *Let C be given in* (2.14). *Suppose that* (2.1)–(2.4) *hold. Then $C : P_T \to P_T$ is continuous and the image of C is contained in a compact set.*

Proof. To see that C is continuous, let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\beta = \frac{\epsilon}{N}$ with $N = \eta\gamma T\left(\rho + \varrho\right)\left(\delta + 3\sum_{j=1}^{m} E_j\right)$ where E_j, $j = 1, \ldots, m$, are given by (2.4). Now, for $\|\varphi - \psi\| < \beta$, we get

$$\begin{aligned} &\|C\varphi - C\psi\| \\ \leq &\eta\gamma \int_{t-T}^{t} \left[\rho \left(\delta + 2\sum_{j=1}^{m} E_j \right) \|\varphi - \psi\| + \varrho \left(\delta + 2\sum_{j=1}^{m} E_j \right) \|\varphi - \psi\| \right. \\ &\left. + (\rho + \varrho) \left(\sum_{j=1}^{m} E_j \right) \|\varphi - \psi\| \right] ds \\ \leq &\eta\gamma \int_{t-T}^{t} (\rho + \varrho) \left(\delta + 3\sum_{j=1}^{m} E_j \right) \|\varphi - \psi\|\, ds \\ \leq &N \|\varphi - \psi\| < \epsilon. \end{aligned}$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \leq R\}$ where R is a fixed positive constant. Let $\varphi \in D$. Observe that in view of (2.4) we have

$$|G_\varphi(t)| = |G_\varphi(t) - G_0(t) + G_0(t)| \leq |G_\varphi(t) - G_0(t)| + |G_0(t)| \leq \sum_{j=1}^{m} E_j \|\varphi\| + \alpha.$$

Consequently,

$$\begin{aligned}\|C\varphi\| \leq& \eta\gamma \int_{t-T}^{t} \left[\rho \left(\delta R + 2\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right)\right.\\ &\left.+\varrho\left(\delta R + 2\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right) + (\rho+\varrho)\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right] ds\\ \leq& \eta\gamma T(\rho+\varrho)\left(\delta R + 3\left(R\sum_{j=1}^{m} E_j + \alpha\right)\right) = L.\end{aligned}$$

So, $C(D)$ is uniformly bounded. Next, we calculate $(C\varphi)'(t)$ and prove that $C(D)$ is equicontinuous. By making use of (2.1)–(2.3) we get by taking the derivative in (2.14) that

$$(C\varphi)'(t) = -A(t)(C\varphi)(t) - L_\varphi(t) - N_\varphi(t) - A(t)G_\varphi(t).$$

Thus, the above expression yields $\left\|(C\varphi)'\right\| \leq F$, for some positive constant F. So, $C(D)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $C(D)$ is relatively compact. Then, $C(D)$ is contained in a compact set. □

Lemma 2.3. *Suppose that* (2.1), (2.3) *and* (2.4) *hold, and*

$$\sum_{j=1}^{m} E_j < 1, \tag{2.16}$$

where E_j, $j = 1,\ldots,m$, *are given by* (2.4). *If* B *is given by* (2.13), *then* B *is a contraction mapping.*

Proof. Let B be defined by (2.13). Then for $\varphi, \psi \in P_T$ we obtain

$$\begin{aligned}\|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)|\\ &\leq \sum_{j=1}^{m} E_j \sup_{t\in[0,T]} |\varphi(t-\tau_j(t)) - \psi(t-\tau_j(t))|\\ &\leq \left(\sum_{j=1}^{m} E_j\right)\|\varphi - \psi\|.\end{aligned}$$

Hence, B defines a contraction mapping. □

Theorem 2.2. *Assume that* (2.1)–(2.4) *and* (2.16) *hold. Let* J *be a positive constant satisfying the inequality*

$$J\sum_{j=1}^{m} E_j + \alpha + \eta\gamma T(\varrho+\rho)\left(\delta J + 3\left(J\sum_{j=1}^{m} E_j + \alpha\right)\right) \leq J. \tag{2.17}$$

Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. *Then the equation* (1.1) *has a solution in* M.

Proof. By Lemma 2.2, $C : M \to P_T$ is continuous and $C(M)$ is contained in a compact set. Also, by Lemma 2.3, the mapping B is a contraction and it is clear that $B : M \to P_T$. Next, we prove that if $\varphi, \psi \in M$, we have $\|C\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in M$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned}
&\|C\varphi + B\psi\| \\
&\leq \left(\sum_{j=1}^{m} E_j\right) \|\psi\| + \alpha + \eta\gamma \int_{t-T}^{t} \left[\rho \left(\delta \|\varphi\| + 2 \left(\sum_{j=1}^{m} E_j \|\varphi\| + \alpha\right)\right)\right. \\
&\qquad \left. + \varrho \left(\delta \|\varphi\| + 2 \left(\sum_{j=1}^{m} E_j \|\varphi\| + \alpha\right)\right) + (\varrho + \rho) \left(\left(\sum_{j=1}^{m} E_j\right) \|\varphi\| + \alpha\right)\right] ds \\
&\leq J \sum_{j=1}^{m} E_j + \alpha + \eta\gamma T (\varrho + \rho) \left(\delta J + 3 \left(J \sum_{j=1}^{m} E_j + \alpha\right)\right) \\
&\leq J.
\end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in M such that $z = Cz + Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence, (1.1) has a T-periodic solution. ☐

Theorem 2.3. *Suppose that* (2.1)–(2.4) *hold. If*

$$\sum_{j=1}^{m} E_j + \eta\gamma T (\varrho + \rho) \left(\delta + 3 \sum_{j=1}^{m} E_j\right) < 1, \tag{2.18}$$

then the equation (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (2.12). For $\varphi, \psi \in P_T$, in view of (2.12), we obtain

$$\|H\varphi - H\psi\| \leq \left(\sum_{j=1}^{m} E_j + \eta\gamma T (\varrho + \rho) \left(\delta + 3 \sum_{j=1}^{m} E_j\right)\right) \|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. ☐

Corollary 2.1. *Suppose that* (2.1)–(2.3) *hold. Let J be a positive constant and define $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Suppose there are positive constants E_j^*, $j = 1, \ldots, m$, so that for $x, y \in M$ we have*

$$\begin{aligned}
&|g(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) - g(t, y(t - \tau_1(t)), \ldots, y(t - \tau_m(t)))| \\
&\leq \sum_{j=1}^{m} E_j^* |x(t - \tau_j(t)) - y(t - \tau_j(t))|.
\end{aligned}$$

If $\sum_{j=1}^{m} E_j^* < 1$ *and* $\|H\varphi\| \leq J$ *for* $\varphi \in M$, *then* (1.1) *has a* T*-periodic solution in* M. *Moreover, if*

$$\sum_{j=1}^{m} E_j^* + \eta\gamma T\left(\varrho + \rho\right)\left(\delta + 3\sum_{j=1}^{m} E_j^*\right) < 1,$$

then (1.1) *has a unique* T*-periodic solution in* M.

Proof. Let the mapping H be given by (2.12). Then, the results follow immediately from Theorem 2.2 and Theorem 2.3. □

Example 2.1. For small positive ϵ_1, ϵ_2 and ϵ_3, we consider the nonlinear neutral mixed type Levin-Nohel integro-differential equation with variable delay

$$\begin{aligned} &\frac{d}{dt}x\left(t\right) + \epsilon_1 \int_{t-\frac{2\pi}{\omega}}^{t} \left(1 + \sin\omega\left(t-s\right)\right) x\left(s\right) ds \\ &+ \epsilon_2 \int_{t}^{t+\frac{\pi}{\omega}} \left(2 + \cos\omega\left(s-t\right)\right) x\left(s\right) ds - \epsilon_3 \frac{d}{dt}\left(\sin\left(\omega t\right) x^2\left(t - \frac{2\pi}{\omega}\right)\right) = 0, \end{aligned} \tag{2.19}$$

where ω is a positive constant. So, we have

$$\begin{aligned} a_1\left(t,s\right) =& \epsilon_1\left(1+\sin\omega\left(t-s\right)\right), \quad b_1\left(t,s\right) = \epsilon_2\left(2+\cos\omega\left(s-t\right)\right), \\ a_j\left(t,s\right) =& b_j\left(t,s\right) = \tau_j\left(t\right) = \sigma_j\left(t\right) = 0, \quad j = 2,\ldots,m, \\ \tau_1\left(t\right) =& \frac{2\pi}{\omega}, \quad \sigma_1\left(t\right) = \frac{\pi}{\omega}, \end{aligned}$$

and

$$g\left(t, x\left(t-\tau_1\left(t\right)\right), \ldots, x\left(t-\tau_m\left(t\right)\right)\right) = \epsilon_3 \sin\left(\omega t\right) x^2\left(t - \frac{2\pi}{\omega}\right).$$

Proof. Define $M = \left\{\varphi \in P_{\frac{2\pi}{\omega}} : \|\varphi\| \leq J\right\}$, where J is a positive constant. For $\varphi \in M$, we have

$$\|H\varphi\| \leq \epsilon_3 J^2 + \left(1 - e^{-\left(\epsilon_1+\epsilon_2\right)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} \left(8\epsilon_1 + 6\epsilon_2\right) \frac{\pi^2}{\omega^2}\left[8\epsilon_1 \frac{\pi^2}{\omega^2} J + 6\epsilon_2 \frac{\pi^2}{\omega^2} J + 3\epsilon_3 J^2\right].$$

Thus, the inequality

$$\epsilon_3 J^2 + \left(1 - e^{-\left(\epsilon_1+\epsilon_2\right)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} \left(8\epsilon_1 + 6\epsilon_2\right) \frac{\pi^2}{\omega^2}\left[8\epsilon_1 \frac{\pi^2}{\omega^2} J + 6\epsilon_2 \frac{\pi^2}{\omega^2} J + 3\epsilon_3 J^2\right] \leq J, \tag{2.20}$$

which is satisfied for small ϵ_1, ϵ_2 and ϵ_3, implies $\|H\varphi\| \leq J$. Hence, (2.19) has a $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.1.

For the uniqueness of the periodic solution, we let $\varphi, \psi \in M$. From (2.19) we see that

$$\eta = \left(1 - e^{-\left(\varepsilon_1+\varepsilon_2\right)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1}, \quad \rho = \frac{2\pi}{\omega}\varepsilon_1, \quad \varrho = \frac{2\pi}{\omega}\varepsilon_2, \quad \gamma \leq 1.$$

Also $\alpha = 0$, $E = 2\varepsilon_3 J^2$, where J is given by (2.20). If

$$2\varepsilon_3 J + \left(1 - e^{-\left(\varepsilon_1+\varepsilon_2\right)\left(\frac{2\pi}{\omega}\right)^2}\right)^{-1} \left(8\varepsilon_1 + 6\varepsilon_2\right) \frac{\pi^2}{\omega^2}\left[8\varepsilon_1 \frac{\pi^2}{\omega^2} + 6\varepsilon_2 \frac{\pi^2}{\omega^2} + 6\varepsilon_3 J\right] < 1,$$

is satisfied for small ε_1, ε_2 and ε_3, then (2.19) has a unique $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.1. □

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

For a non-negative constant L and a positive constant K, we define the set

$$\mathbb{M} = \left\{\varphi \in P_T : L \leq \varphi \leq K\right\},$$

which is a closed convex and bounded subset of the Banach space P_T. To simplify notation, we let

$$\theta = \max_{t\in[0,T]}\left(\max_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}\right), \quad \lambda = \min_{t\in[0,T]}\left(\min_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}\right).$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $G_x(t) \geq 0$ and $G_x(t) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{M}$.

In the case $G_x(t) \geq 0$, we assume that there exist non-negative constants k_{1j} and positive constants k_{2j}, $j = 1, \ldots, m$, such that

$$\sum_{j=1}^{m} k_{1j} x(t-\tau_j(t)) \leq G_x(t) \leq \sum_{j=1}^{m} k_{2j} x(t-\tau_j(t)), \tag{3.1}$$

$$\sum_{j=1}^{m} k_{2j} < 1, \tag{3.2}$$

and for all $t \in [0,T]$, $x \in \mathbb{M}$

$$\frac{L\left(1-\sum_{j=1}^{m} k_{1j}\right)}{\eta\lambda T} \leq F_x(t) \leq \frac{K\left(1-\sum_{j=1}^{m} k_{2j}\right)}{\eta\theta T}, \tag{3.3}$$

where $F_x(t) = -L_x(t) - N_x(t) - A(t)G_x(t)$.

Theorem 3.1. *Assume that* (2.1)–(2.4), (2.16) *and* (3.1)–(3.3) *hold. Then the equation* (1.1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$.

Proof. By Lemma 2.1 x is a solution of (1.1) if $x = Cx + Bx$, where C and B are given by (2.14) and (2.13), respectively. By Lemma 2.2, C is continuous and compact. Moreover, by Lemma 2.3, B is a contraction. We just need to prove that condition (i) of Theorem 2.1 is satisfied. Toward this, let $\varphi, \psi \in \mathbb{M}$, then

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t)\\
=&G_\psi(t) - \eta\int_{t-T}^{t}\left[L_\varphi(s) + N_\varphi(s) + A(s)G_\varphi(s)\right]e^{-\int_s^t A(z)dz}ds\\
\leq&K\sum_{j=1}^{m} k_{2j} + \eta\theta T\frac{K\left(1-\sum_{j=1}^{m} k_{2j}\right)}{\eta\theta T} = K.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
=&G_\psi(t) - \eta \int_{t-T}^{t} \left[L_\varphi(s) + N_\varphi(s) + A(s) G_\varphi(s)\right] e^{-\int_s^t A(z)dz} ds \\
\geq& L \sum_{j=1}^{m} k_{1j} + \eta\lambda T \frac{L\left(1 - \sum_{j=1}^{m} k_{1j}\right)}{\eta\lambda T} = L.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{M}$ such that $x = Bx + Cx$. By Lemma 2.1 this fixed point is a solution of (1.1) and the proof is complete. □

In the case $G_x(t) \leq 0$, we substitute conditions (3.1)–(3.3) with the following conditions respectively. We suppose that there exist negative constants k_{3j} and non-positive constants k_{4j}, $j = 1, \ldots, m$, such that

$$\sum_{j=1}^{m} k_{3j} x(t - \tau_j(t)) \leq G_x(t) \leq \sum_{j=1}^{m} k_{4j} x(t - \tau_j(t)), \tag{3.4}$$

$$-\sum_{j=1}^{m} k_{3j} < 1, \tag{3.5}$$

and for all $t \in [0, T]$, $x \in \mathbb{M}$

$$\frac{L - K \sum_{j=1}^{m} k_{3j}}{\eta\lambda T} \leq F_x(t) \leq \frac{K - L \sum_{j=1}^{m} k_{4j}}{\eta\theta T}. \tag{3.6}$$

Theorem 3.2. *Suppose that* (2.1)–(2.4), (2.16) *and* (3.4)–(3.6) *hold. Then the equation* (1.1) *has a positive T-periodic solution x in the subset* $\mathbb{M}$.

The proof follows along the lines of Theorem 3.1, and hence we omit it.

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