Kragujevac Journal of Mathematics
Volume 49(2) (2025), Pages 267–285.

ESSENTIAL APPROXIMATE PSEUDOSPECTRA OF MULTIVALUED LINEAR RELATIONS

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Abstract. One of the fundamental ideas investigated in A. Ammar, A. Jeribi and K. Mahfoudhi in [5] is that of providing conditions under which the essential approximate pseudospectrum of closed, densely defined linear operators have a relationship with Fredholm theory and perturbation theory. In this paper the approximate pseudospectrum and the essential approximate pseudospectrum of closed, densely defined multivalued linear relations are introduced and studied, and work done in the aforementioned papers are extended to general multivalued linear relations

1. Introduction

A vast number of the problems that have been investigated in the Banach algebra setting originated in the context of bounded linear operators or multivalued linear relations on a Banach space. Let X denote a linear vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. T multivalued linear operator on X is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T, into the collection of non empty subsets of X such that

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2),$$

for all non zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator, which is equivalent to $T(0) = \{0\}$.

Key words and phrases. Pseudospectrum, approximate pseudospectra, essential approximate pseudospectra, multivalued linear relations.

2020 *Mathematics Subject Classification.* Primary: 47A53. Secondary: 47A55, 54B35, 47A13.

DOI 10.46793/KgJMat2502.267J

Received: February 03, 2022.

Accepted: April 04, 2022.

We denote by $\mathrm{L}\mathcal{R}(X)$ the class of linear relations everywhere defined. $T \in \mathrm{L}\mathcal{R}(X)$ is uniquely determined by its graph $\mathcal{G}(T)$, which is defined by

$$\mathcal{G}(T) := \{(x, y) \in X \times X : x \in \mathcal{D}(T),\ y \in Tx\},$$

so that we can identify T with $\mathcal{G}(T)$. The closure and completion of T, denoted by $\overline{T}$ and $\widetilde{T}$, respectively, is the linear relation defined by

$$\mathcal{G}(\overline{T}) := \overline{\mathcal{G}(T)},$$
$$\mathcal{G}(\widetilde{T}) := \widetilde{\mathcal{G}(T)}.$$

We denote by $\mathrm{C}\mathcal{R}(X)$ the class of all closed linear relations from X into X. The inverse of T is a linear relation T^{-1} given by

$$\mathcal{G}(T^{-1}) := \{(y, x) : (x, y) \in \mathcal{G}(T)\}.$$

If $\mathcal{G}(T)$ is closed, then T is said to be closed,

$$\mathcal{N}(T) := \{x \in \mathcal{D}(T) : (x, 0) \in \mathcal{G}(T)\} \quad \text{and} \quad \mathcal{R}(T) := T(\mathcal{D}(T))$$

denote kernel structure and the range of the relation T, respectively. The linear relation $T + S$ is defined by

$$\mathcal{G}(T + S) := \{(x, y) \in X \times X : y = u + v \text{ with } (x, u) \in \mathcal{G}(T), (x, v) \in \mathcal{G}(S)\}.$$

Let $T \in \mathrm{L}\mathcal{R}(X)$ and $S \in \mathrm{L}\mathcal{R}(X)$ where $\mathcal{R}(T) \cap \mathcal{D}(S) \neq \emptyset$. The product of ST is defined by

$$\mathcal{G}(ST) := \{(x, z) \in X \times X : (x, u) \in \mathcal{G}(T) \text{ and } (u, z) \in \mathcal{G}(S) \text{ for some } u \in X\}.$$

Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$. We shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly, $Q_T T$ is a single valued operator and the norm of T is defined by

$$\|T\| := \|Q_T T\|.$$

We say that T is continuous if for each neighborhood V in $\mathcal{R}(T)$, $T^{-1}(V)$ is a neighborhood in $\mathcal{D}(T)$ ($\|T\| < \infty$), bounded if it is continuous with $\mathcal{D}(T) = X$, open if T^{-1} is continuous, equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) := \sup\Big\{\lambda \geq 0 : \lambda d(x, \mathcal{N}(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T)\Big\},$$

where $d(x, \mathcal{N}(T))$ is the distance between x and $\mathcal{N}(T)$. If $\mathcal{D}(T)$ and if $\|T\| < \infty$, then we shall say that T is bounded.

The class of such relations is denoted by $\mathrm{L}\mathcal{R}(X)$ and we denote by $\mathcal{L}(X)$ the set of all bounded linear operators from X. For $T \in \mathrm{L}\mathcal{R}(X)$, we write

$$\alpha(T) := \dim \mathcal{N}(T), \quad \beta(T) := \dim X/\mathcal{R}(T), \quad \overline{\beta}(T) := \dim X/\overline{\mathcal{R}(T)},$$

and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite. We say T is upper semi-Fredholm, if there exists a finite codimensional subspace M of $\mathcal{D}(T)$ for which $T_{|M}$ is injective and open. If M and N are subspaces of X and of the dual space $X^{'}$ respectively, then

$$M^{\perp} := \left\{x^{'} \in X^{'} : x^{'}(x) = 0 \text{ for all } x \in M\right\}$$

and

$$N^{\top} := \left\{x \in X : x^{'}(x) = 0 \text{ for all } x^{'} \in N\right\}.$$

The conjugate of $T \in \mathrm{L}\mathcal{R}(X)$ is the linear relation $T^{'}$ defined by

$$\mathcal{G}(T^{'}) := \mathcal{G}(-T^{-1})^{\perp} \subset Y^{'} \times X^{'},$$

so that $(y^{'}, x^{'}) \in \mathcal{G}(T^{'})$ if, and only if, $y^{'}(y) = x^{'}(x)$ for all $(x, y) \in \mathcal{G}(T)$.

A closed linear relation T acts from X into X.

Definition 1.1. Let $T \in \mathrm{L}\mathcal{R}(X)$.

(i) T is said to be upper semi-Fredholm, if there exists a closed, finite, codimensional subspace M of X, such that the restriction $T|_M$ has a single valued continuous inverse.

(ii) T is said to be lower semi-Fredholm linear relation if its conjugate $T^{'}$ is uppersemi-Fredholm linear relation.

We denote by $\mathcal{F}_+(X)$, the set of upper semi-Fredholm linear relations and by $\mathcal{F}_-(X)$ the set of lower semi-Fredholm linear relations.

In the case when X is Banach space, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators, and note that the definitions of the classes $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are consistent, respectively, with

$$\Phi_+(X) = \left\{T \in \mathrm{C}\mathcal{R}(X) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } X\right\},$$

$$\Phi_-(X) = \left\{T \in \mathrm{C}\mathcal{R}(X) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } X\right\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm relations from X and $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ denotes the set of semi-Fredholm relations from X.

We say that T is strictly singular, if there is no infinite dimensional subspace M of $\mathcal{D}(T)$ for which the restriction $T|_M$ has a single valued continuous inverse.

The families of all compact and strictly singular linear relations will be denoted by $\mathrm{K}\mathcal{R}(X)$ and $SS\mathcal{R}(X)$, respectively.

Let $T \in \mathrm{L}\mathcal{R}(X)$, the set

$$\rho(T) := \left\{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open with dense range on X}\right\}.$$

Referring back to the closed theorem of linear relations (see [16,17]), when T is closed and X is a Banach space, this coincides with the set

$$\left\{\lambda \in \mathbb{C} : (\lambda - \widetilde{T})^{-1} \text{ is everywhere defined and single valued}\right\}.$$

Therefore, our definition of a resolvent set coincides with the standard definition for bounded or closed operators in Banach spaces.

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$. The set $\rho(T)$ is open, whereas the spectrum $\sigma(T)$ of a closed linear relation T is closed. The approximate point spectrum of T is the set defined by

$$\sigma_{ap}(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below}\Big\}.$$

The defect spectrum of T is the set defined by

$$\sigma_{\delta}(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\Big\}.$$

Let $T \in \mathrm{L}\mathcal{R}(X)$ and $\varepsilon > 0$. We define the pseudospectra of a linear relation T by

$$\sigma_{\varepsilon}(T) := \sigma(T) \cup \Big\{\lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\Big\}.$$

The approximate pseudospectrum of a linear relation T by the set

$$\sigma_{ap,\varepsilon}(T) := \sigma_{ap}(T) \cup \Big\{\lambda \in \mathbb{C} : \inf_{\substack{x\in\mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| \leq \varepsilon\Big\},$$

and the defect pseudospectrum of a linear relation T by

$$\sigma_{\delta,\varepsilon}(T) = \sigma_{ap,\varepsilon}(T^{'}).$$

Our concern in this paper is mainly the following essential pseudospectra

$$\begin{aligned}
\sigma_{e1,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{+}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e2,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{-}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e3,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi_{\pm}(X) \text{ for all } S \in \mathcal{I}_T(X)\},\\
\sigma_{e4,\varepsilon}(T) =& \mathbb{C} \setminus \{\lambda \in \mathbb{C} : \lambda - T + S \in \Phi(X) \text{ for all } S \in \mathcal{I}_T(X)\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_T(X) :=& \{S \in \mathrm{L}\mathcal{R}(X) \text{ is continuous} : \|S\| < \varepsilon, \mathcal{D}(S) \supset \mathcal{D}(T) \text{ and } S(0) \subset T(0)\},\\
\sigma_{w,\varepsilon}(T) =& \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon}(T + K),\\
\sigma_{eap,\varepsilon}(T) =& \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T + K),\\
\sigma_{e\delta,\varepsilon}(T) =& \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\delta,\varepsilon}(T + K)
\end{aligned}$$

and

$$\mathcal{K}_T(X) := \{K \in \mathrm{K}\mathcal{R}(X) : \mathcal{D}(K) \supset \mathcal{D}(T) \text{ and } K(0) \subset T(0)\}.$$

We turn our attention to the following inclusions

$$\begin{aligned}
\sigma_{e1,\varepsilon}(T) \cap \sigma_{e2,\varepsilon}(T) =& \sigma_{e3,\varepsilon}(T) \subset \sigma_{e4,\varepsilon}(T) \subset \sigma_{w,\varepsilon}(T) \subset \sigma_{\varepsilon}(T),\\
\sigma_{e1,\varepsilon}(T) \subset& \sigma_{eap,\varepsilon}(T) \text{ and } \sigma_{e2,\varepsilon}(T) \subset \sigma_{e\delta,\varepsilon}(T),\\
\sigma_{w,\varepsilon}(T) =& \sigma_{eap,\varepsilon}(T) \cup \sigma_{e\delta,\varepsilon}(T).
\end{aligned}$$

If ε tends to 0, we recover the usual definition of the essential spectra of a closed operator T. The subsets $\sigma_{e1}(.)$ and $\sigma_{e2}(\cdot)$ are the Gustafson and Weidmann essential spectra, $\sigma_{e3}(\cdot)$ is the Kato essential spectrum, $\sigma_{e4}(\cdot)$ is the Wolf essential spectrum, $\sigma_w(\cdot)$ is the Schechter essential spectrum, $\sigma_{eap}(\cdot)$ is the essential approximate point spectrum and $\sigma_{e\delta}(\cdot)$ is the essential defect spectrum.

Remark 1.1. Let $T \in L\mathcal{R}(T)$.
(*i*) If $\varepsilon_1 < \varepsilon_2$, then $\sigma_{j,\varepsilon_1}(T) \subset \sigma_{j,\varepsilon_2}(T)$ with $j = 1, 2, 3, 4, w, eap, \delta$.
(*ii*) It is clear that $\sigma_{j,\varepsilon}(T)$, with $j = w, eap, \delta$ has a remarkable stability, any compact perturbation $K \in \mathcal{K}_T(X)$ leaves the essential pseudospectrum invariant, then we have $\sigma_{j,\varepsilon}(T + K) = \sigma_{j,\varepsilon}(T)$, with $j = w, eap, \delta$.

This paper is a continuation of the research which was undertaken by A. Ammar and A. Jeribi in works [3,5,6,13] and was devoted to special subsets of the pseudospectrum and the essential pseudospectrum of closed, densely defined multivalued linear relations

$$\sigma_{w,\varepsilon}(T) := \bigcap_{K\in\mathcal{K}_T(X)} \sigma_\varepsilon(T + K) := \bigcup_{\|D\|<\varepsilon} \sigma_w(T + D),$$

where

$$\sigma_w(T) := \bigcap_{K\in\mathcal{K}_T(X)} \sigma(T + K).$$

Also, for the benefit of the reader we review an important result about pseudospectrum from [7–10] and [11,12].

After compressing or depressing them, certain parts of pseudospectrum of an linear relations acting between Banach space may be distinguished. Among these parts, we are interested in two: one is the approximate pseudospectrum and the other is the essential approximate pseudospectrum. Motivated by the approximate pseudospectrum versions introduced by M. P. H. Wolf [22] in the case of linear operator, it becomes possible to extend this definition to the case of multivalued linear relations of closed, densely defined multivalued linear relations. Recently, J. M. Varah [21], has introduced the first idea of pseudospectra. L. N. Trefethen [18,19], not only initiated the study of pseudospectrum for matrices and operators, but he also talked of approximate eigenvalues and pseudospectrum and used this notion to study interesting problems in mathematical physics. In the same vein, several authors have worked on this field. For example, we may refer to E. B. Davies [15].

The main aims of this work are the following: we introduce and study the approximate pseudospectrum and the essential approximate pseudospectrum of closed, densely defined multivalued linear relations. We begin by the definition and we investigate the characterization, the stability and some properties of these pseudospectrum.

We organize our paper in the following way. In Section 2 contains preliminary and auxiliary properties that will be necessary in order to prove the main results of the other sections. Some results concerning approximate pseudospectrum and essential approximate pseudospectrum are established in Sections 3 and 4. The main focus of this section are Theorems 3.4, 3.5 and 4.1. Subsequently, we apply the obtained results to study the invariance and the characterization of the essential approximate pseudospectrum of a closed multivalued linear operator.

2. Preliminary Results

In this section we collect some results of the theory of multivalued linear operators which will be needed in the following sections.

Definition 2.1. Let $S \in \mathrm{L}\mathcal{R}(X)$ be continuous where, X is normed spaces.

(i) S is called a Fredholm perturbation if $T + S \in \Phi(X)$, whenever $T \in \Phi(X)$.

(ii) S is called an upper semi-Fredholm perturbation if $T + S \in \Phi_+(X)$, whenever $T \in \Phi_+(X)$.

(iii) S is called a lower semi-Fredholm perturbation if $T + S \in \Phi_-(X)$, whenever $T \in \Phi_-(X)$.

The sets of Fredholm, upper and lower semi-Fredholm perturbations are denoted by $\mathcal{P}(\Phi(X))$, $\mathcal{P}(\Phi_+(X))$, and $\mathcal{P}(\Phi_-(X))$, respectively.

We denote also the set

$$\mathcal{P}_T(\Phi(X)) := \{S \in \mathcal{P}(\Phi(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\},$$

$$\mathcal{P}_T(\Phi_+(X)) := \{S \in \mathcal{P}(\Phi_+(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\}$$

and

$$\mathcal{P}_T(\Phi_-(X)) := \{S \in \mathcal{P}(\Phi_-(X)) : S(0) \subset T(0) \text{ and } \mathcal{D}(S) \supset \mathcal{D}(T)\}.$$

In general by [2] we have

$$\mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_+(X)) \subset \mathcal{P}_T(\Phi(X)) \quad \text{and} \quad \mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_-(X)) \subset \mathcal{P}_T(\Phi(X)).$$

Lemma 2.1 ([2])**.** *Let $T \in C\mathcal{R}(X)$, where X is Banach spaces. Then the following hold.*

(i) *If $T \in \Phi_+(X)$ and $S \in \mathcal{P}_T(\Phi_+(X))$, then $T + S \in \Phi_+(X)$ and $i(T + S) = i(T)$.*

(ii) *If $T \in \Phi_-(X, Y)$ and $S \in \mathcal{P}_T(\Phi_-(X))$, then $T+S \in \Phi_-(X)$ and $i(T+S) = i(T)$.*

Lemma 2.2 ([16])**.** *Let $T \in L\mathcal{R}(X)$. Then for $x \in \mathcal{D}(T)$, we have the following equivalence:*

(i) $y \in Tx \Leftrightarrow Tx = y + T(0)$.

In particular,

(ii) $0 \in Tx \Leftrightarrow Tx = T(0)$.

Lemma 2.3 ([16, Corollary I.2.11])**.** *Let $T \in L\mathcal{R}(X)$. Then*

(i) $T^{-1}Tx = x + T^{-1}(0)$ *for all* $x \in \mathcal{D}(T)$*;*

(ii) $TT^{-1}y = y + T(0)$ *for all* $y \in R(T)$.

Lemma 2.4 ([16, Proposition II.1.4 and II.1.6])**.** *Let X is normed spaces and $T \in L\mathcal{R}(X)$. Then*

(i) $\|Tx\| = d(y, T(0))$ for any $y \in Tx$*;*

(ii) $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$ $(x \in \mathcal{D}(T))$*;*

(iii) $\|T\| = \sup_{x \in B_X} \|Tx\|$ with $B_X := \{x \in X : \|x\| \leq 1\}$.

Lemma 2.5 ([12, Proposition 3.1]). *Let S, $T \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$. If $T \in C\mathcal{R}(X)$ and S is continuous, then $S + T \in C\mathcal{R}(X)$.*

Theorem 2.1 ([16, Theorem III.4.2]). *Let $T \in C\mathcal{R}(X)$, then*
(*i*) *T is continuous if and only if $\mathcal{D}(T)$ is closed;*
(*ii*) *T is open if and only if $R(T)$ is closed.*

Lemma 2.6 ([16, Proposition II.3.20]). *Let T be open and injective and let $S \in L\mathcal{R}(X)$ be a linear operator satisfying $\|S\| < \gamma(T)$. Then $T + S$ is open and injective.*

Proposition 2.1 ([16, Proposition I.4.2], [14, Lemma 2.4]). *Let $R, S, T \in L\mathcal{R}(X)$. Then*
(*i*) *$(R + S)T \subset RT + ST$ with equality if T is single valued.*
(*ii*) *Let $T \in L\mathcal{R}(X)$ and $S, R \in L\mathcal{R}(Y, Z)$. If $T(0) \subset \mathcal{N}(S)$ or $T(0) \subset \mathcal{N}(R)$, then $(R + S)T = RT + ST$.*

Theorem 2.2 ([16, Theorem III.5.3]). *Let X, Y be Banach spaces and let $T \in C\mathcal{R}(X)$. Then T is open if and only if $R(T)$ is closed.*

Theorem 2.3 ([12, Theorem 2.2]). *Let $S, T \in L\mathcal{R}(X)$ be closed. We have the following.*
(*i*) *If $S, T \in \Phi_+(X)$, then $ST \in \Phi_+(X)$ and $TS \in \Phi_+(X)$.*
(*ii*) *If $S, T \in \Phi_-(X)$, with TS (resp. ST) closed, then $TS \in \Phi_-(X)$ (resp. $ST \in \Phi_-(X)$).*
(*iii*) *If $S, T \in \Phi(X)$, then $TS \in \Phi(X)$ and $i(TS) = i(T) + i(S) + \dim X\big/(R(S) + D(T)) - \dim[S(0) \cap N(T)]$.*
(*iv*) *If S and T are everywhere defined and $TS \in \Phi_+(X)$, then $S \in \Phi_+(X)$.*
(*v*) *If S and T are everywhere defined such that $TS \in \Phi(X)$ and $ST \in \Phi(X)$, then $S \in \Phi(X)$ and $T \in \Phi(X)$.*

3. The Approximate Pseudospectrum of Linear Relations

The goal of this section is to study the approximate of pseudospectrum of closed, densely defined multivalued linear relations.

Proposition 3.1. *Let $T \in L\mathcal{R}(X)$ where X is a normed space. Then*

$$\sigma_{ap,\varepsilon}(T) \subset \sigma_\varepsilon(T).$$

Proof. Let $\lambda \notin \sigma_\varepsilon(T)$, then $\|(\lambda - T)^{-1}\| \leq \frac{1}{\varepsilon}$. Moreover,

$$\frac{1}{\inf\limits_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\|} = \sup_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \frac{\|x\|}{\|(\lambda - \tilde{T})x\|}$$

$$= \sup_{0\neq x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)}} \frac{\|x\|}{\|(\lambda - \tilde{T})x\|}. \tag{3.1}$$

Putting $x := (\lambda - T)^{-1}y$ we have

$$\begin{aligned}(\lambda - T)x &= (\lambda - T)(\lambda - T)^{-1}y \quad \text{(by Lemma 2.3)}\\ &= y + (\lambda - T)(0).\end{aligned}$$

Using Lemma 2.2, we obtain that $y \in (\lambda - T)x$. On the other hand, $(\lambda - T)(0) = \lambda(0) - T(0) = 0 - T(0) = T(0)$. Also by Lemma 2.2, $0 \in T(0)$ and from Lemma 2.4 we have

$$\|(\lambda - T)x\| = d(y, (\lambda - T)(0)) = d(y, T(0)) \leq d(y, 0) = \|y\|. \tag{3.2}$$

Combining (3.1) and (3.2) that

$$\sup_{y \in X\setminus\{0\}} \frac{\|(\lambda - \tilde{T})^{-1}y\|}{\|y\|} = \|(\lambda - \tilde{T})^{-1}\| \leq \frac{1}{\varepsilon}.$$

Consequently,

$$\inf_{\substack{x \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| > \varepsilon.$$

Hence,

$$\lambda \notin \sigma_{ap,\varepsilon}(T).$$

□

Example 3.1. Let $X = \mathbb{C}^n$ and let T be the single-valued linear operator on $\mathcal{L}(\mathbb{C}^n)$ given for all $n \geq 2$ with the infinity norm by

$$\begin{cases} T : \mathbb{C}^n \to \mathbb{C}^n,\\ e_i \mapsto Te_i, \quad \text{where } Te_i = e_{(n+1)-i}.\end{cases}$$

It is easily checked that

$$\begin{cases} T = T^{-1},\\ \|T\| = 1,\\ \sigma(T) \cup \{\infty\} = \{-1, 1\}.\end{cases}$$

Therefore, T is everywhere defined closed linear relation. We will check that if

$$\|(\lambda - T)e_i\| = \lambda e_i - e_{(n+1)-i},$$

then

$$\inf_{\substack{e_i \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|e_i\|=1}} \|(\lambda - T)e_i\| = |\lambda| + 1,$$

and if

$$\|(\lambda - T)^{-1}e_i\| = \frac{\lambda e_i - e_{(n+1)-i}}{\lambda^2 - 1},$$

then

$$\|(\lambda - T)^{-1}\| = \frac{|\lambda| + 1}{|\lambda^2 - 1|}.$$

Moreover, for $\varepsilon > 1$ we obtain

$$\sigma_{ap,\varepsilon}(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon - 1\},$$
$$\sigma_{\varepsilon}(T) = \left\{\lambda \in \mathbb{C} : \frac{|\lambda| + 1}{|\lambda^2 - 1|} > \frac{1}{\varepsilon}\right\}.$$

It is easy to verify that, for all λ with $0 \leq |\lambda| \leq 1$ we have

$$\sigma_{\varepsilon}(T) \neq \sigma_{ap,\varepsilon}(T).$$

Proposition 3.2. *Let $T \in L\mathcal{R}(X)$, where X is a normed space. Then*

$$\bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T).$$

Proof. It is clear that $\sigma_{ap}(T) \subset \sigma_{ap,\varepsilon}(T)$ for all $\varepsilon > 0$, then

$$\sigma_{ap}(T) \subset \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T).$$

Conversely, let $\lambda \notin \sigma_{ap}(T)$. Then $\lambda - T$ is bounded below, hence $\lambda - T$ is injective, open with dense range on X and $(\lambda - \tilde{T})^{-1}$ is a bounded linear operator, so there exists $\varepsilon > 0$ such that

$$\|(\lambda - \tilde{T})^{-1}\| \leq \frac{1}{\varepsilon}.$$

Therefore,

$$\lambda \notin \sigma_{\varepsilon}(T),$$

and we conclude from Proposition 3.1 that $\lambda \notin \sigma_{ap,\varepsilon}(T)$. So, $\lambda \notin \bigcap_{\varepsilon>0} \sigma_{ap,\varepsilon}(T)$. □

Theorem 3.1. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, for any α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ we have the following.*

(i) If $\alpha \in \mathbb{C}$ and $\varepsilon > 0$, then $\sigma_{ap,\varepsilon}(T + \alpha I) = \alpha + \sigma_{ap,\varepsilon}(T)$.

(ii) If $\beta \in \mathbb{C}\backslash\{0\}$ and $\varepsilon > 0$, then $\sigma_{ap,|\beta|\varepsilon}(\beta T) = \beta\sigma_{ap,\varepsilon}(T)$.

Proof. (i) Let $\lambda \in \sigma_{ap,\varepsilon}(T + \alpha I)$, then

$$\lambda \in \sigma_{ap}(T + \alpha I) \quad \text{or} \quad \inf_{\substack{x \in \mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - \alpha I - T)x\| \leq \varepsilon.$$

Hence, $(\lambda - \alpha)I - T$ is not bounded below (not injective) or

$$\inf_{\substack{x \in \mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|((\lambda - \alpha)I - T)x\| \leq \varepsilon.$$

This yields to $\lambda \in \alpha + \sigma_{ap,\varepsilon}(T)$. For the second inclusion it is the same reasoning.

(ii) Let $\lambda \in \sigma_{ap,|\alpha|\varepsilon}(\alpha T)$, then $\lambda \in \sigma_{ap}(\alpha T)$ or

$$\inf_{\substack{x \in \mathcal{D}(T)\backslash\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - \alpha T)x\| \leq |\alpha|\varepsilon.$$

It follows that $\lambda - \alpha T$ is not bounded below (not injective) or

$$\inf_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\| \alpha\left(\frac{\lambda}{\alpha} - T\right)x \right\| \leq |\alpha|\varepsilon.$$

Hence, $\alpha(\frac{\lambda}{\alpha} - T)$ is not bounded below (not injective) or

$$\inf_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\| \alpha\left(\frac{\lambda}{\alpha} - T\right)x \right\| = |\alpha| \inf_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\| \left(\frac{\lambda}{\alpha} - T\right)x \right\| \leq |\alpha|\varepsilon.$$

Thus, $\frac{\lambda}{\alpha} \in \sigma_{ap}(T)$ or

$$\inf_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \left\| \left(\frac{\lambda}{\alpha} - T\right)x \right\| \leq \varepsilon.$$

So, $\sigma_{ap,|\alpha|\varepsilon}(\alpha T) \subseteq \alpha\sigma_{ap,\varepsilon}(T)$. However, the reverse inclusion is similar. □

Corollary 3.1. *Let $T \in L\mathcal{R}(X)$ and $\varepsilon > 0$. Then, for any α, $\beta \in \mathbb{C}$ with $\beta \neq 0$ we have*

$$\sigma_{ap,\varepsilon}(\alpha I + \beta T) = \alpha + \beta\sigma_{ap,\varepsilon|\beta|}(T).$$

Definition 3.1. Given a polynomial $P(z) = \sum_{k=0}^{n} \alpha_k z^k$ with coefficients $\alpha_k \in \mathbb{C}$, we define the polynomial in T by $P(T) = \sum_{k=0}^{n} \alpha_k T^k$.

Theorem 3.2. *Let T be closed relation and assume that V is closed single valued bounded relation such that $0 \in \rho(V)$. Let $S = VTV^{-1}$. Then, for all polynomial $P(T)$ of degree n we have $P(S)$ is closed and*

$$P(S) = VP(T)V^{-1}.$$

Proof. We need to show that $P(S)$ is closed. Let T is closed relation, then from [1, Lemma 2.7] we obtain that $P(T)$ is closed. On the other hand, V is closed single valued bounded relation such that $0 \in \rho(V)$, then V has a closed range ($R(V) = X$). By the fact that V injective and open we have

$$\alpha(V) = 0 < \infty \quad \text{and} \quad \gamma(V) > 0.$$

By using [16, Proposition II.5.17], we deduce that $VP(T)$ is closed. Moreover, since V^{-1} is single valued and bounded, then from [16, Exercise II.5.18] we obtain $VP(T)V^{-1}$ is closed. Hence, $P(S)$ is closed. Now, let

$$P(S) = \sum_{k=0}^{n} \alpha_k S^k = \sum_{k=0}^{n} \alpha_k (VTV^{-1})^k.$$

Since, V is single valued injective we infer that

$$P(S) = \sum_{k=0}^{n} \alpha_k (VTV^{-1})^k = V\left(\sum_{k=0}^{n} \alpha_k T^k\right)V^{-1} = VP(T))V^{-1}.$$ □

Theorem 3.3. *Let $T \in C\mathcal{R}(X)$ where X is a complete space and assume that V as in Theorem 3.2. Let $k = \|V\|\|V^{-1}\|$ and $P(S) = VP(T)V^{-1}$. Then*

$$\sigma_{ap}(P(S)) = \sigma_{ap}(P(T)).$$

Proof. We have

$$\begin{aligned}\lambda - P(S) =& \lambda - VP(T)V^{-1},\\ (\lambda - P(S))V =& (\lambda - VP(T)V^{-1})V\\ =& (\lambda V - VP(T)V^{-1}V) \quad \text{(using [16, Proposition I.4.2])}\\ =& (\lambda V - VP(T)) \quad \text{(as V is injective)}\end{aligned}$$

and

$$\begin{aligned}V^{-1}(\lambda - P(S))V =& V^{-1}(\lambda V - VP(T)),\\ V^{-1}(\lambda - P(S))V =& (\lambda V^{-1}V - V^{-1}VP(T)) \quad \text{(using [16, Proposition I.4.2])}\\ V^{-1}(\lambda - P(S))V =& (\lambda - P(T)) \quad \text{(as V is injective)}\\ (\lambda - P(S)) =& V(\lambda - P(T))V^{-1} \quad \text{(as V is single valued).}\end{aligned}$$

Now, if $\lambda \notin \sigma_{ap}(P(T))$ then the closed relation $\lambda - P(T)$ is bounded below (injective, open). By [16, Proposition VI.5.2])

$$V(\lambda - P(T))V^{-1} = \lambda - P(S)$$

is closed, bounded below (injective, open). Hence, $\lambda \in\notin \sigma_{ap}(P(S))$.

Conversely, if $\lambda \notin \sigma_{ap}(P(S))$ then the closed relation $\lambda - P(S)$ is bounded below (injective, open). By [16, Proposition VI.5.2])

$$V^{-1}(\lambda - P(S))V = \lambda - P(T)$$

is also closed, bounded below (injective, open). Hence, $\lambda \notin \sigma_{ap}(P(T))$, which implies the result. ☐

Now, we are ready to give our first main result of this section.

Theorem 3.4. *Let $T \in C\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If $\lambda \in \sigma_{ap,\varepsilon}(T)$, then there is $S \in L\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T + S)$.*

Proof. Let $\lambda \in \sigma_{ap,\varepsilon}(T)$. We will discuss these two cases.

1. case. If $\lambda \in \sigma_{ap}(T)$, we may put $S = 0$.
2. case. If $\lambda \notin \sigma_{ap}(T)$, then there exists $x_0 \in X$, $\|x_0\| = 1$ such that

$$\|(\lambda - T)x_0\| < \varepsilon,$$

and by the Hahn Banach Theorem (see [20]), there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = \|x_0\|$. We define the relation $S : X \to X$ by

$$S(x) := x'(x)(\lambda - T)x_0.$$

It is clear that S is everywhere defined and single valued (as $S(0) = 0$).

$$\|Sx\| = \|x'(x)(\lambda - T)x_0)\| \leq \|x'\|\|x\|\|(\lambda - T)x_0\|,$$

for $x \neq 0$, we have

$$\frac{\|Sx\|}{\|x\|} \leq \|(\lambda - T)x_0\|,$$

so,

$$\|S\| \leq \|(\lambda - T)x_0\| < \varepsilon.$$

We can rewrite

$$\begin{aligned}
\inf_{\substack{x \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T - S)x\| \leq & \|(\lambda - T - S)x_0\| \\
\leq & \|(\lambda - T)x_0 - Sx_0\| \\
\leq & \|(\lambda - T)x_0 - x'(x_0)(\lambda - T)x_0\| \\
\leq & \|(\lambda - T)(0)\| \\
\leq & \|\lambda(0) - T(0)\| \\
\leq & \|T(0)\| = d(T(0), T(0)) = 0.
\end{aligned}$$

Then, $\lambda \in \sigma_{ap}(T + S)$. ☐

Theorem 3.5. *Let $T \in C\mathcal{R}(X)$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If there is $S \in L\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon$ such that $\lambda \in \sigma_{ap}(T + S)$. Then $\lambda \in \sigma_{ap,\varepsilon}(T)$.*

Proof. Suppose that there exists a continuous linear relation $D \in \mathrm{L}\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$ such that

$$\lambda \in \sigma_{ap}(T + S),$$

which means that

$$\inf_{\substack{x \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T - S)x\| = 0.$$

In order to prove that

$$\inf_{\substack{x \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| < \varepsilon,$$

we can write

$$\begin{aligned}
\|(\lambda - T)x_0\| = \|(\lambda - T - S + S)x_0\| \leq & \|(\lambda - T - S)x_0\| + \|Sx_0\| \\
\leq & \|T(0)\| + \varepsilon \\
\leq & \varepsilon.
\end{aligned}$$

Then

$$\inf_{\substack{x \in \mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T)x\| < \varepsilon.$$ ☐

Corollary 3.2. *In summary, at the present moment we have shown that from Theorems* 3.5 *and* 3.4*, that for* $T \in C\mathcal{R}(X)$ *and* $\varepsilon > 0$

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\mathcal{J}_T(X)} \sigma_{ap}(T+S),$$

where

$$\mathcal{J}_T(X) := \Big\{ S \in L\mathcal{R}(X) \text{ is continuous} : \|S\| < \varepsilon, \mathcal{D}(T) \subset \mathcal{D}(S) \text{ and } S(0) \subset T(0) \Big\}.$$

Theorem 3.6. *Let* $T \in C\mathcal{R}(X)$ *where* X *is a complete space, then for any* $\varepsilon > 0$ *and* $E \in L\mathcal{R}(X)$ *such that* $E(0) \subset T(0)$ *and* $\mathcal{D}(E) \supset \mathcal{D}(T)$

$$\sigma_{ap,\varepsilon-\|E\|}(T) \subseteq \sigma_{ap,\varepsilon}(T+E) \subseteq \sigma_{ap,\varepsilon+\|E\|}(T).$$

Proof. Let $\lambda \in \sigma_{ap,\varepsilon-\|E\|}(T)$. Then, by Theorem 3.4 there is $S \in \mathrm{L}\mathcal{R}(X)$ satisfying $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$, $\|S\| < \varepsilon - \|E\|$ such that

$$\lambda \in \sigma_{ap}(T+S) = \sigma_{ap}\Big((T+E) + (S-E)\Big).$$

Using [16, Proposition II.1.7] we get

$$\begin{aligned} \|S - E\| \leq & \|S\| + \|-E\| \\ = & \|S\| + \|E\| < \varepsilon \quad \text{(using [16, Proposition II.1.7])}. \end{aligned}$$

Then, from Theorem 3.5, we deduce that $\lambda \in \sigma_{ap,\varepsilon}(T+E)$. Using a similar reasoning to the first inclusion, we deduce that $\lambda \in \sigma_{ap,\varepsilon+\|E\|}(T)$. ☐

4. Essential Approximate Pseudospectra of Linear Relations

We begin this section by showing that the essential approximate pseudospectra of linear relations are closed, and then illustrate some characteristic properties.

Theorem 4.1. *Let* $T \in C\mathcal{R}(X)$ *and* $\varepsilon > 0$*. Then the following statements are equivalent:*

(i) $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(ii) *For all continuous linear relations* $S \in L\mathcal{R}(X)$ *such that* $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ *and* $\|S\| < \varepsilon$*, we have*

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

(iii) *For all continuous single valued relations* $D \in L\mathcal{R}(X)$ *such that* $\mathcal{D}(T) \subset \mathcal{D}(D)$ *and* $\|D\| < \varepsilon$*, we have*

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

Proof. $(i) \Rightarrow (ii)$ Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$. Then there exists $K \in \mathcal{K}_T(X)$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+K).$$

Using Theorems 3.5 and 3.4, for all continuous linear relations $S \in L\mathcal{R}(X)$ such that

$$\begin{aligned}\mathcal{D}(T+K) =& \mathcal{D}(T)\cap\mathcal{D}(K)\\ =& \mathcal{D}(T)\subset\mathcal{D}(S) \quad (\text{as } \mathcal{D}(\ T)\subset\mathcal{D}(K))\\ (T+K)(0) =& T(0)\supset S(0) \quad (\text{as } K(0)\subset T(0)),\end{aligned}$$

and $\|S\| < \varepsilon$, we have $\lambda \notin \sigma_{ap}(T+S+K)$. Then, $\lambda - T - S - K$ is open, injective with dense range. On the other hand, T is closed and K is compact then K is continuous hence $\lambda - S - K$ is continuous, furthermore $(\lambda - S - K)(0) \subset T(0)$, then using Lemma 2.5, we obtain that $\lambda - T - S - K$ is closed. Hence, from Theorem 2.1, $R(\lambda - T - S - K)$ is closed. We conclude that, $R(\lambda - T - S - K) = X$. Therefore

$$\lambda - T - S - K \in \Phi_+(X) \ \text{ and } \ i(\lambda - T - S - K) \leq 0,$$

for all continuous linear relations $S \in L\mathcal{R}(X)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$. It is obvious from [1, Lemma 2.3] that for all continuous linear relations $S \in L\mathcal{R}(X)$ such that $\mathcal{D}(T) \subset \mathcal{D}(S)$, $S(0) \subset T(0)$ and $\|S\| < \varepsilon$ we have

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

$(ii) \Rightarrow (iii)$ Is trivial.

$(iii) \Rightarrow (i)$ We assume that for all $D \in L\mathcal{R}(X)$ a continuous single valued relations such that $\mathcal{D}(T) \subset \mathcal{D}(D)$ and $\|D\| < \varepsilon$, then we have

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

By virtue of [2, Theorem 3.5 (i)], $\lambda - T - D$ can be expressed in the form

$$\lambda - T - D = S + K,$$

where $K \in K_{\lambda-T-D}(X) = \mathcal{K}_T(X)$ since

$$\begin{aligned}K(0) \subset& T(0) = (\lambda - T - D)(0),\\ \mathcal{D}(\lambda - T - D) =& \mathcal{D}(T) \subset \mathcal{D}(K),\end{aligned}$$

and S is a linear relation with closed range and S is injective linear relation (i.e., $\alpha(S) = 0$). So,

$$\lambda - T - D - K = S \quad \text{and} \quad \alpha(\lambda - T - D - K) = 0.$$

Since $\lambda - T - D - K$ is injective linear relation (bounded below), then there exists a constant $M > 0$ such that

$$\|(\lambda - T - D - K)x\| \geq M\|x\|, \quad \text{for all } x \in \mathcal{D}(T).$$

This proves that

$$\inf_{\substack{x\in\mathcal{D}(T)\setminus\overline{\mathcal{N}(T)},\\ \|x\|=1}} \|(\lambda - T - D - K)x\| \geq M > 0.$$

This is equivalent to say that

$$\lambda \notin \sigma_{ap}(T + D + K),$$

and therefore, $\lambda \notin \sigma_{eap,\varepsilon}(T)$. ☐

Remark 4.1. In summary, we have shown that from Theorem 4.1, that for $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\substack{\|D\|<\varepsilon \\ \mathcal{D}(T)\subset\mathcal{D}(D)}} \sigma_{eap}(T+D) = \bigcup_{\substack{\|S\|<\varepsilon \\ S(0)\subset T(0) \\ \mathcal{D}(S)\supset\mathcal{D}(T)}} \sigma_{eap}(T+S).$$

Theorem 4.2. *Let $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$. Then $\sigma_{eap,\varepsilon}(T)$ is a closed set.*

Proof. Let $\lambda \notin \sigma_{eap,\varepsilon}(T)$ and D be a single valued continuous linear relation such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$. Hence, by Theorem 4.1, we have

$$\lambda - T - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - D) \leq 0.$$

So, $R(\lambda - T - D)$ is closed and from Lemma 2.5, we have $\lambda - T - D$ is closed. Then by using Theorem 2.1, $\lambda - T - D$ is open and hence $\gamma(\lambda - T - D) > 0$. Let $r > 0$ such that $r < \gamma(\lambda - T - D)$, let $\mu \in B_f(\lambda, r)$ then $|\mu - \lambda| \leq r < \gamma(\lambda - T - D)$. According to Lemma 2.6, it is clear that

$$\mu - T - D = \lambda - T - D + \mu - \lambda$$

is open and injective. Since $\mu - T - D$ is closed and open, then from Theorem 2.1 we deduce $R(\mu - T - D)$ is closed. Then, $\mu - T - D \in \Phi_+(X)$. On the other hand, using [16, Corollary V.15.7], we have

$$i(\mu - T - D) = i(\lambda - T - D) \leq 0.$$

Consequently, $\mu \notin \sigma_{eap,\varepsilon}(T)$ and we infer that $\sigma_{eap,\varepsilon}(T)$ is a closed. ☐

Observe that as a direct consequence of Theorem 4.1, we infer the following result.

Proposition 4.1. *Let $T \in C\mathcal{R}(X)$.*

(*i*) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma_{eap}(T) \subset \sigma_{eap,\varepsilon_1}(T) \subset \sigma_{eap,\varepsilon_2}(T)$.*

(*ii*) *If $\varepsilon > 0$, then $\sigma_{eap,\varepsilon}(T) \subset \sigma_{ap,\varepsilon}(T)$.*

(*iii*) $\bigcap\limits_{\varepsilon>0} \sigma_{eap,\varepsilon}(T) = \sigma_{eap}(T)$.

Theorem 4.3. *Let $T \in L\mathcal{R}(X)$ where X is a complete space, then the following hold.*

(*i*) *For any $\varepsilon > 0$ and $S \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \varepsilon$ we have*

$$\sigma_{eap,\varepsilon-\|S\|}(T) \subseteq \sigma_{eap,\varepsilon}(T+S) \subseteq \sigma_{eap,\varepsilon+\|S\|}(T).$$

(*ii*) *For every $\alpha, \beta \in \mathbb{C}$, with $\beta \neq 0$*

$$\sigma_{eap,\varepsilon}(\alpha I + \beta T) = \alpha + \beta\sigma_{eap,\varepsilon|\beta|}(T).$$

Proof. The proof of this theorem is inspired from the proof of Corollary 3.1 and Theorem 3.6 and [4, Propositions 4.2 and 4.4]. ☐

Theorem 4.4. *Let $T \in C\mathcal{R}(X)$ and $\varepsilon > 0$. Then*

(*i*) $\sigma_{eap,\varepsilon}(T) = \bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P)$;

(*ii*) $\sigma_{eap,\varepsilon}(T) = \bigcap_{S \in SS\mathcal{R}(X)} \sigma_{ap,\varepsilon}(T+S)$.

Proof. (*i*) Because, $\mathcal{K}_T(X) \subset \mathcal{P}_T(\Phi_+(X))$, we have that

$$\bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P) \subset \sigma_{eap,\varepsilon}(T).$$

Conversely, let $\lambda \notin \bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P)$ then there exists $P \in \mathcal{P}_T(\Phi_+(X))$ such that

$$\lambda \notin \sigma_{ap,\varepsilon}(T+P).$$

Since, P is continuous and the use of Lemma 2.5 we infer that $T+P$ is closed. Now, by using of Corollary 3.2 we see that $\lambda \notin \sigma_{ap}(T+S+P)$ for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $\|S\| < \varepsilon$ and

$$\begin{aligned} \mathcal{D}(T+P) =& \mathcal{D}(T) \cap \mathcal{D}(P) = \mathcal{D}(T) \subset \mathcal{D}(S), \\ (T+P)(0) =& T(0) \quad (\text{as } P(0) \subset T(0)) \\ \supset& S(0). \end{aligned}$$

On the other hand, $(\lambda - S - P)(0) \subset T(0)$, $\mathcal{D}(\lambda - S - P) = \mathcal{D}(S) \cap \mathcal{D}(P) \supset \mathcal{D}(T)$ and T is closed, by using of Lemma 2.5, $\lambda - T - S - P$ is closed, and $\lambda - T - S - P$ is open as $\lambda \notin \sigma_{ap}(T+P+S)$, then from Theorem 2.1, $R(\lambda - T - S - P)$ is closed. Hence $\lambda - T - S - P$ is injective and open. Therefore

$$\lambda - T - S - P \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S - P) \leq 0.$$

Since

$$\begin{aligned} P \in \mathcal{P}_T(\Phi_+(X)), \quad P(0) \subset& (\lambda - T - S - P)(0), \\ \mathcal{D}(P) \supset \mathcal{D}(\lambda - T - S - P) =& \mathcal{D}(T) \cap \mathcal{D}(S) \cap \mathcal{D}(P), \quad P \in \mathcal{P}_{\lambda - T - D - P}(\Phi_+(X)) \end{aligned}$$

Using Lemma 2.1, we obtain that for all continuous linear relations $S \in \mathrm{L}\mathcal{R}(X)$ such that $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, $\|S\| < \varepsilon$

$$\lambda - T - S \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T - S) \leq 0.$$

Finally, it follows from Theorem 4.1 that $\lambda \notin \sigma_{eap,\varepsilon}(T)$.

(*ii*) From [2, Theorem 3.3], we have the inclusion $\mathcal{K}_T(X) \subset SS\mathcal{R}(X) \subset \mathcal{P}_T(\Phi_+(X))$. Then

$$\begin{aligned} \sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{ap,\varepsilon}(T+K) \subset& \bigcap_{S \in SS\mathcal{R}(X)} \sigma_{ap,\varepsilon}(T+S) \\ \subset& \bigcap_{P \in \mathcal{P}_T(\Phi_+(X))} \sigma_{ap,\varepsilon}(T+P) = \sigma_{eap,\varepsilon}(T). \qquad \square \end{aligned}$$

We finally close this paper with the following theorem.

Theorem 4.5. *Let $T, S \in L\mathcal{R}(X)$ such that $S(0) \subset T(0)$ and $\varepsilon > 0$. If $D \in L\mathcal{R}(X)$ such that $D(0) \subset S(0) \subset \mathcal{N}(T)$, $\|D\| < \varepsilon$ and $T(S+D) \in \mathcal{F}_+(X)$, then*

$$\sigma_{eap,\varepsilon}(T+S) \subset \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T).$$

If, further, $(S+D)T \in \mathcal{F}_+(X)$ and $T(0) \subset \mathcal{N}(S)$ or $T(0) \subset \mathcal{N}(D)$ we have

$$\sigma_{eap,\varepsilon}(T+S) = \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T).$$

Proof. Since $\mathcal{D}(T) = X$, then using Proposition 2.1, we obtain that

$$(\lambda - T)(\lambda - S - D) = \lambda(\lambda - S - D) - T(\lambda - S - D).$$

Also, by Proposition 2.1, we have

$$\begin{aligned}(\lambda - T)(\lambda - S - D) =& \lambda^2 - \lambda S - \lambda D - \lambda T + TS + TD \\ =& \lambda(\lambda - S - D - T) + T(S + D)\end{aligned}$$

Let $\lambda \notin \sigma_{eap,\varepsilon}(S) \cup \sigma_{eap}(T)$, then $\lambda \notin \sigma_{eap,\varepsilon}(S)$ and $\lambda \notin \sigma_{eap}(T)$. Using [12, Corollary 4.1] we have

$$(\lambda - T) \in \Phi_+(X) \quad \text{and} \quad i(\lambda - T) \leq 0.$$

According to Theorem 4.1 we obtain

$$\lambda - S - D \in \Phi_+(X) \quad \text{and} \quad i(\lambda - S - D) \leq 0,$$

for all continuous linear relations $D \in \mathrm{L}\mathcal{R}(X)$ such that $D(0) \subset S(0)$ and $\|D\| < \varepsilon$. We infer that

$$(\lambda - T)(\lambda - S - D) \in \Phi_+(X).$$

Since

$$T(S + D)(0) = TS(0) \subset TT^{-1}(0) = T(0) = (\lambda - S - D - T)(0) = T(0),$$

and $T(S + D) \in \mathcal{F}_+(X)$, then $\lambda - S - D - T \in \Phi_+(X)$ for all $D \in \mathrm{L}\mathcal{R}(X)$ such that $D(0) \subset S(0)$ and $\|D\| < \varepsilon$

$$\begin{aligned}i(\lambda - S - D - T) =& i(\lambda - T) + i(\lambda - S - D) - \dim\Big(T(0) \cap \mathcal{N}(\lambda - S - D)\Big) \\ =& i(\lambda - T) + i(\lambda - S - D) \leq 0.\end{aligned}$$

Hence, from Theorem 4.1, we have $\lambda \notin \sigma_{eap,\varepsilon}(T+S)$. The second inclusion is analogous to the previous one. □

Acknowledgements. The authors are very grateful for the comments and corrections of the referee, whose detailed attention allowed us to make very worthwhile improvements.

References

[1] F. Abdmouleh, T. Àlvarez, A. Ammar and A. Jeribi, *Spectral mapping theorem for Rakocević and Schmoeger essential spectra of a multivalued linear operator*, Mediterr. J. Math. **12**(3) (2015), 1019–1031. https://doi.org/10.1007/s00009-014-0437-7

[2] A. Ammar, *A characterization of some subsets of essential spectra of a multivalued linear operator*, Complex Anal. Oper. Theory **11**(1) (2017), 175–196. https://doi.org/10.1007/s11785-016-0591-y

[3] A. Ammar, H. Daoud and A. Jeribi, *Pseudospectra and essential pseudospectra of multivalued linear relations*, Mediterr. J. Math. **12**(4) (2015), 1377–1395. https://doi.org/10.1007/s00009-014-0469-z

[4] A. Ammar, H. Daoud and A. Jeribi, *Stability of pseudospectra and essential pseudospectra of linear relations*, J. Pseudo-Differ. Oper. Appl. **7**(4) (2015), 473–491. https://doi.org/10.1007/s11868-016-0150-3

[5] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the essential approximation pseudospectrum on a Banach space*, Filomat **31** (11) (2017), 3599–3610. https://doi.org/10.2298/FIL1711599A

[6] A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the condition pseudospectrum on Banach space*, Funct. Anal. Approx. Comput. **10**(2) (2018), 13–21. http://www.pmf.ni.ac.rs/faac

[7] A. Ammar, A. Jeribi and K. Mahfoudhi, *The condition pseudospectrum of a operator pencil*, Asian-Eur. J. Math. **14**(4) (2021), 12 pages. https://doi.org/10.1142/S1793557121500571

[8] A. Ammar, A. Jeribi and K. Mahfoudhi, *Some description of essential structured approximate and defect pseudospectrum*, Korean J. Math. **28**(4) (2020), 673–697. http://dx.doi.org/10.11568/kjm.2020.28.4.673

[9] A. Ammar, A. Jeribi and K. Mahfoudhi, *The condition pseudospectrum subset and related results*, J. Pseudo-Differ. Oper. Appl. **11**(1) (2020), 491–504. https://doi.org/10.1007/s11868-018-0265-9

[10] A. Ammar, A. Jeribi and K. Mahfoudhi, *Browder essential approximate pseudospectrum and defect pseudospectrum on a Banach space*, Extracta Math. **34**(1) (2019), 29–40. https://doi.org/10.17398/2605-5686.34.1.29

[11] A. Ammar, A. Jeribi and K. Mahfoudhi, *Measure of noncompactness, essential approximation and defect pseudospectrum*, Methods Funct. Anal. Topology **25**(1) (2019), 1–11.

[12] T. Àlvarez, A. Ammar and A. Jeribi, *On the essential spectra of some matrix of lineair relations*, Math. Methods Appl. Sci. **37** (2014), 620–644. https://doi.org/10.1002/mma.2818

[13] T. Àlvarez, A. Ammar and A. Jeribi, *A characterization of some subsets of $\mathcal{S}$-essential spectra of a multivalued linear operator*, Colloq. Math. **135**(2) (2014), 171–186. https://doi.org/10.4064/cm135-2-2

[14] E. Chafai and M. Mnif, *Perturbation of normally solvable linear relations in paracomplete spaces*, Linear Algebra Appl. **439**(7) (2013), 1875–1885. https://doi.org/10.1016/j.laa.2013.05.019

[15] E. B. Davies, *Linear Operators and their Spectra,* Cambridge Studies in Advanced Mathematics **106**, Camb. Univ. Press, Cambridge, 2007.

[16] R. W. Cross, *Multivalued Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics **213**, Marcel Dekker, Inc., New York, 1998.

[17] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer-Verlag, New York, 2015.

[18] L. N. Trefethen,*Pseudospectra of Matrices*, reprinted from: D. F. Griffiths and G. A. Watson (Eds.), *Numerical Analysis*, Longman Science and Technology, Harlow, 1992, 234–266.

[19] L. N. Trefethen, *Pseudospectra of linear operators*, SAIM Review **39**(3) (1997), 383–406. https://doi.org/10.1137/s0036144595295284

[20] M. Schechter, *Principles of Functional Analysis*, Second Edition, Graduate Studies in Mathematics **36**, American Mathematical Society, Providence, RI, 2002.

[21] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator*, Ph.D. Thesis, Stanford University ProQuest LLC, (1967).

[22] M. P. H. Wolff, *Discrete approximation of unbounded operators and approximation of their spectra*, J. Approx. Theory **113** (2001), 229–244. https://doi.org10.1006/jath.2001.3588

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