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MULTIPLE SOLUTIONS FOR A NONLOCAL KIRCHHOFF PROBLEM IN FRACTIONAL ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, using the three critical points theorem we obtain the existence of three weak solutions for a Kirchhoff type problem driven by a nonlocal operator of the elliptic type in a fractional Orlicz-Sobolev space, with homogeneous Dirichlet boundary conditions.

1. Introduction

In the last decade, great attention has been devoted to the study of nonlinear problems involving non-local operators. These types of operator come up in a quite natural way in several applications such as phase transition phenomena, crystal dislocation, soft thin films, minimal surfaces and finance; see for instance [2, 18] and references therein. We also refer the interested reader to [33], where a more extensive bibliography and an introduction to the subject are given.

In this paper, we are concerned with a class of nonlocal problems in fractional Orlicz-Sobolev spaces of the form

$$(P_a)\begin{cases} M\left(\displaystyle\int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right)(-\Delta)^s_{a(\cdot)}u & \\ =\lambda f(x,u)+\beta g(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

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where Ω is an open bounded subset in $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary $\partial\Omega$, $0 < s < 1$, A is an N-function, $M : [0, \infty) \to (0, \infty)$ is a nondecreasing continuous function, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, λ and β are two real parameters and $(-\Delta)^s_{a(\cdot)}$ is a nonlocal integro-differential operator of elliptic type defined as follows

$$(-\Delta)^s_{a(\cdot)} u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u(x) - u(y)}{|x - y|^s} \cdot \frac{dy}{|x - y|^{N+s}},$$

for all $x \in \mathbb{R}^N$, where $a : \mathbb{R} \to \mathbb{R}$ which will be specified later.

This problem (P_a) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = h(u, x), \tag{1.1}$$

presented by Kirchhoff [29] in 1883 which is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.1), L is the length of string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [23]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [35] for some related works. In [7], the authors obtained the existence of three weak solutions for a Kirchhoff type elliptic system involving nonlocal fractional p-Laplacian by using the three point critique theorem. In [10], by means of mountain pass theorem of Ambrosetti and Rabinowitz, direct variational approach and Ekeland's variational principle, the authors showed the existence of nontrivial weak solutions to a class of $p(x)$-Kirchhoff type problem. For the problems involving fractional Kirchhoff type, we refer the reader to the works [11, 13]. They use different methods to establish the existence of solutions.

Problems of this type have been intensively studied in the last few years, due to numerous and relevant applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids), calculus of variations, nonlinear potential theory, the theory of quasiconformalmappings, differential geometry, geometric function theory, probability theory and image processing (see, for instance [22]).

The problem (P_a) involves the fractional $a(\cdot)$-Laplacian operator, the most appropriate functional framework for dealing with this problem is the fractional Orlicz Sobolev space [8, 16], namely a fractional Sobolev space constructed from an Orlicz space at the place of $L^p(\Omega)$. As we know, the Orlicz spaces represent a generalization of classical Lebesgue spaces in which the role usually played by the convex function t^p is assumed by a more general convex function $A(t)$; they have been extensively studied in the monograph of Krasnoselśkii and Rutickii [28] as well as in Luxemburg's doctoral thesis [31]. If the role played by $L^p(\Omega)$ in the definition of fractional Sobolev spaces $W^{s,p}(\Omega)$ is assigned to an Orlicz $L_A(\Omega)$ space, the resulting space $W^sL_A(\Omega)$ is exactly a fractional Orlicz-Sobolev space . Many properties of fractional Sobolev spaces have been extended to fractional Orlicz-Sobolev spaces (see [4, 5, 8, 9, 12, 16, 17]). For this, many researchers have studied the existence of solutions for the eigenvalue problems involving nonhomogeneous operators in the divergence form through Orlicz-Sobolev spaces by using variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory (see for instance [14, 15, 20, 32]).

The problem (P_a) is motivated by the class of problems on the form

$$(P)\begin{cases} Au = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \partial\Omega,\end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$, $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions and λ, β are two real parameters. For $Au = -\Delta_p = -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$, the problem (P) has been studied in many papers, we refer to [35, 36], in which the authors have used different methods to get the existence of solutions for (P). In the case when $Au = -\Delta_{p(\cdot)} = -\mathrm{div}\left(|\nabla u|^{p(\cdot)-2}\nabla u\right)$, where $p(\cdot)$ is a continuous function, problem (P) has also been studied by many authors, see for examples [19, 24, 25]. On the other hand, Chung in [26], studied the problem (P) with $Au = -M\left(\int_\Omega \phi(|\nabla u|)dx\right)\mathrm{div}(a(|\nabla u|\nabla u))$. That is, the following problem in Orlicz-Soblev spaces:

$$(P_\phi)\quad \begin{cases} -M\left(\int_\Omega \phi(|\nabla u|)dx\right)\mathrm{div}(a(|\nabla u|)\nabla u) = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \partial\Omega,\end{cases}$$

where ϕ is an N-function, defend as

$$\phi(t) = \int_0^t a(\tau)\tau d\tau,$$

and $M : [0,\infty) \to (0,\infty)$ is a nondecreasing continuous Kirchhoff function. Under some suitable conditions, the author obtained the existence of three weak solutions of (P_Φ), by using the three critical point theorem. For $M \equiv 1$ in the problem (P_Φ), Cammaroto and Vilasti in [20], by the same theorem, they showed the existence of three weak solutions.

In the fractional case, i.e., when we take $Au = M\left([u]^p_{s,p}\right)(-\Delta)^s_p u$. That is, we consider the following problem

$$(P_s)\begin{cases} M\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}}dxdy\right)(-\Delta)^s_p u = \lambda f(x,u) + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where Ω is an open bounded subset in $\mathbb{R}^N$ and $(-\Delta)^s_p$ is the fractional p-Laplace operator. In [6], by using the three critical point theorem, the authors obtained the existence of three weak solutions of (P_s).

To our knowledge, this is the first contribution to studying of non-local problems in this class of functional spaces. More precisely, using the ideas first presented in articles [6, 20, 26]. Our result in this article generalizes special cases, in which we will consider the problem (P_a) with $M(t) = 1$ or $M(t) \neq 1$ and $A(t) = \frac{1}{p}t^p$ (the problem (P_s)).

This paper is organized as follows. In the second section, we recall some properties of fractional Sobolev spaces. In the third section, using the three critical points theorem which introduced by Ricceri [34], we obtain the existence of a three weak solutions of problem (P_a). Finally, the fourth section is devoted to giving an example which illustrates the mains abstracts results.

2. Some Preliminaries Results

To deal with this situation we introduce the fractional Orlicz-Sobolev space to investigate problem (P_a). Let us recall the definitions and some elementary properties of this spaces. We refer the reader to [1, 3, 8, 16, 33] for further reference and for some of the proofs of the results in this section.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. We assume that $a : \mathbb{R} \to \mathbb{R}$ in (P_a) is such that : $\varphi : \mathbb{R} \to \mathbb{R}$ defined by:

$$\varphi(t) = \begin{cases} a(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$A(t) = \int_0^t \varphi(\tau) d\tau.$$

Then, A, is N-function, see [1], i.e., $A : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, convex, increasing function, with $\frac{A(t)}{t} \to 0$ as $t \to 0$ and $\frac{A(t)}{t} \to \infty$ as $t \to \infty$.

For the function A introduced above we define the Orlicz space:

$$L_A(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ mesurable } \int_\Omega A(\lambda |u(x)|) dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_A = \inf \left\{ \lambda > 0 : \int_\Omega A\left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

The conjugate N-function of A is defined by $\overline{A}(t) = \int_0^t \overline{\varphi}(\tau) d\tau$, where $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ is given by $\overline{\varphi}(t) = \sup \{ s : \varphi(s) \leq t \}$. Furthermore, it is possible to prove a Hölder type inequality, that is

$$\left| \int_\Omega uv dx \right| \leq 2 \|u\|_A \|v\|_{\overline{A}}, \quad \text{for all } u \in L_A(\Omega) \text{ and } v \in L_{\overline{A}}(\Omega).$$

Throughout this paper, we assume that

$$1 < p^- := \inf_{t\geq 0} \frac{t\varphi(t)}{A(t)} \leq p^+ := \sup_{t\geq 0} \frac{t\varphi(t)}{A(t)} < +\infty. \tag{2.1}$$

The above relation implies that $A \in \Delta_2$, i.e., A satisfies the global Δ_2-condition (see [32]):

$$A(2t) \leq KA(t), \quad \text{for all } t \geq 0,$$

where K is a positive constant.

Furthermore, we assume that A satisfies the following condition

$$\text{the function } [0,\infty) \ni t \mapsto A(\sqrt{t}) \text{ is convex.} \tag{2.2}$$

The above relation assures that $L_A(\Omega)$ is an uniformly convex space (see [32]).

Lemma 2.1 ([16])**.** *Assume that $A \in \Delta_2$. Then we have*

$$\overline{A}(\varphi(t)) \leq cA(t), \quad \text{for all } t \geq 0,$$

where $c > 0$.

Now, we defined the fractional Orlicz-Sobolev space $W^sL_A(\Omega)$ as follows

$$W^sL_A(\Omega) = \left\{ u \in L_A(\Omega) : \int_\Omega \int_\Omega A\left(\frac{\lambda |u(x)-u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N} < \infty \text{ for some } \lambda > 0 \right\}.$$

This space is equipped with the norm

$$\|u\|_{s,A} = \|u\|_A + [u]_{s,A},$$

where $[\cdot]_{s,A}$ is the Gagliardo seminorm, defined by

$$[u]_{s,A} = \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dxdy}{|x-y|^N} \leq 1 \right\}.$$

We work in the closed linear subspace

$$W_0^sL_A(\Omega) = \left\{ u \in W^sL_A(\mathbb{R}^N) : u = 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega \right\},$$

which can be equivalently renormed by setting $\|\cdot\| := [\cdot]_{s,A}$. By [16], $W^sL_A(\Omega)$ and is Banach space, also separable (resp. reflexive) space if and only if $A \in \Delta_2$ (resp. $A \in \Delta_2$ and $\overline{A} \in \Delta_2$). Furthermore, if $A \in \Delta_2$ and $A(\sqrt{t})$ is convex, then the space $W^sL_A(\Omega)$ is uniformly convex.

To simplify the notation, we set

$$\varPhi(u) = \int_\Omega \int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N}, \quad D^su = \frac{u(x)-u(y)}{|x-y|^s}, \quad d\mu = \frac{dxdy}{|x-y|^s},$$

and the dual space of $(W^sL_A(\Omega), \|\cdot\|)$ is denoted by $((W^sL_A(\Omega))^*, \|\cdot\|_*)$. Note that $d\mu$ is a regular Borel measure on the set $\Omega \times \Omega$.

Theorem 2.1 ([8])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then*

$$C_0^2(\Omega) \subset W^sL_A(\Omega).$$

Remark 2.1. A trivial consequence of Theorem 2.1, $C_0^\infty(\Omega) \subset W^s L_A(\Omega)$ and $W^s L_A(\Omega)$ is non-empty.

Proposition 2.1 ([8]). *Let Ω be an open subset of $\mathbb{R}^N$ and let A be an N-function. Assume condition* (2.1) *is satisfied, then the following relations hold true*

$$[u]_{s,A}^{p^-} \le \Phi(u) \le [u]_{s,A}^{p^+}, \quad \textit{for all } u \in W^s L_A(\Omega), \textit{with } [u]_{s,A} > 1,$$

$$[u]_{s,A}^{p^+} \le \Phi(u) \le [u]_{s,A}^{p^-}, \quad \textit{for all } u \in W^s L_A(\Omega), \textit{with } [u]_{s,A} < 1.$$

Theorem 2.2 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$, with $C^{0,1}$-regularity and bounded boundary, let $0 < s' < s < 1$. Let A be an N-function, assume condition* (2.1) *is satisfied and we define*

$$p_{s'}^* = \begin{cases} \frac{Np^-}{N - s'p^-}, & \textit{if } N > s'p^-, \\ \infty, & \textit{if } N \le s'p^-. \end{cases}$$

- *If $s'p^- < N$, then $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, p_{s'}^*]$ and the embedding $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, is compact for all $q \in [1, p_{s'}^*)$.*
- *If $s'p^- = N$, then $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in [1, \infty]$ and the embedding $W^s L_A(\Omega) \hookrightarrow L^q(\Omega)$, is compact for all $q \in [1, \infty)$.*
- *If $sp^- > N$, then the embedding $W^s L_A(\Omega) \hookrightarrow L^\infty(\Omega)$, is compact.*

Definition 2.1. Let X be a real Banach space. We denote by $\mathcal{W}_A$ the class of all functionals $A : X \to \mathbb{R}$ possessing the following propositionerty: if $\{u_n\}$ is a sequence in X weakly converging to $u \in X$ and $\liminf_{n\to\infty} A(u_n) \le A(u)$, then $\{u_n\}$ has a subsequence strongly converging to u.

Definition 2.2. Let $0 < s' < s < 1$, if $N > s'p^-$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1 + |t|^{q-1}} < \infty,$$

where $q \in [1, p_{s'}^*)$.

While when $N < s'p^-$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for each $C > 0$, the function $x \mapsto \sup_{|t|\le C} |f(x,t)|$ belongs to $L^1(\Omega)$.

Theorem 2.3 ([34]). *Let X be a separable and reflexive real Banach space with norm $\|\cdot\|$, let $\Psi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_A$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and let $J : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that Ψ has a strict local minimum x_0, with $\Psi(x_0) = J(x_0) = 0$. Finally, assume that*

$$\max\left\{\limsup_{\|x\|\to+\infty} \frac{J(x)}{\Psi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Psi(x)}\right\} \le 0$$

and that

$$\sup_{x\in X}\min\{\Psi(x),J(x)\}>0.$$

Let

$$\theta^*:=\inf\left\{\frac{\Psi(x)}{J(x)}:x\in X,\min\{\Psi(x),J(x)\}>0\right\}.$$

Then, for each compact interval $\Lambda\subset(\theta^*,+\infty)$*, there exists a number* $\delta>0$ *with the following propositionerty: for every* $\lambda\in\Lambda$ *and every* C^1 *functional* $\Gamma:X\to\mathbb{R}$ *with compact derivative, there exists* $\beta^*>0$ *such that for each* $\beta\in[0,\beta^*]$*, the equation*

$$\Psi'(x)=\lambda J'(x)+\beta\Gamma'(x),$$

has at least three solutions whose norms are less than δ*.*

3. Mains Results

In this section, we prove the existence of three weak solutions in fractional Orlicz-Sobolev spaces applying Theorem 2.3. For this, we suppose that the Kirchhoff function $M:[0,\infty)\to(0,\infty)$ is a continuous and nondecreasing function satisfying the following condition:

(M_0) there exists $m_0>0$ such that $M(t)\geq m_0$, for all $t\geq 0$.

For $f\in\mathcal{A}$, we assume that

$$(F_1)\qquad \sup_{u\in W_0^sL_A(\Omega)}\int_\Omega F(x,u)dx>0,$$

$$(F_2)\qquad \limsup_{t\to 0}\frac{\sup_{x\in\Omega}F(x,t)}{|t|^{p^+}}\leq 0,$$

$$(F_3)\qquad \limsup_{|t|\to\infty}\frac{\sup_{x\in\Omega}F(x,t)}{|t|^{p^-}}\leq 0,$$

where $F(x,t)=\int_0^t f(x,\tau)d\tau$.

Under such hypothesis, we set

$$\theta^*=\inf\left\{\frac{\widehat{M}(\Phi(u))}{\displaystyle\int_\Omega F(x,u)dx}:u\in W_0^sL_A(\Omega),\int_\Omega F(x,u)dx>0\right\}.$$

Definition 3.1. We say that $u\in W_0^sL_A(\Omega)$ is a weak solution of problem (P_a) if

$$M(\Phi(u))\int_{\mathbb{R}^N\times\mathbb{R}^N}a(|D^su|)D^suD^svd\mu=\lambda\int_\Omega f(x,u)vdx+\beta\int_\Omega g(x,u)vdx,$$

for all $v\in W_0^sL_A(\Omega)$.

Theorem 3.1. *Let* A *be an* N*-function. Suppose that* M *satisfy* (M_1) *and for* $f\in\mathcal{A}$*, we suppose that* (F_1)*,* (F_2) *and* (F_3) *hold true. If* $p^+<p^*_{s'}$*, then for each compact interval* $\Lambda\subset(\theta^*,\infty)$*, there exists a number* $\delta>0$ *with the following propositionerty:*

for every $\lambda \in \Lambda$ and every $g \in \mathcal{A}$ there exists $\beta^* > 0$ such that, for each $\beta \in [0, \beta^*]$, problem (P_a) has at least three weak solutions whose norms are less than δ.

We first prove the following useful result, which helps us to apply Theorem 2.3. For this, we define the functionals $\Psi, J : W_0^s L_A(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega F(x,u)dx, \quad \Psi(u) = \widehat{M}\left(\int_\Omega \int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}\right),$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

Lemma 3.1. *Let $f \in \mathcal{A}$. Then the functional $J \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ with derivative given by*

$$\langle J'(u), v\rangle = \int_\Omega f(x,u)vdx,$$

for all $u, v \in W_0^s L_A(\Omega)$. Moreover $J' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is compact.

By using Theorem 2.2, the proof of this Lemma is seminary to Lemma 3.3 in [6].

Lemma 3.2. *Let (M_1) and (2.1) hold true. Then $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and*

$$\langle \Psi'(u), v\rangle = M(\mathit{\Phi}(u))\int_{\Omega\times\Omega} a(|D^s u|)D^s u D^s v d\mu,$$

for all $u, v \in W_0^s L_A(\Omega)$. Moreover, for each $u \in W_0^s L_A(\Omega)$, $\Psi'(u) \in (W_0^s L_A(\Omega))^*$.

Proof. First, it is easy to see that

$$\langle \Psi'(u), v\rangle = M(\mathit{\Phi}(u))\int_{\Omega\times\Omega} a(|D^s u|)D^s u D^s v d\mu, \tag{3.1}$$

for all $u, v \in W_0^s L_A(\Omega)$. It follows from (3.1) that $\Psi'(u) \in (W_0^s L_A(\Omega))^*$ for each $u \in W_0^s L_A(\Omega)$.

Next, we prove that $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$. Let $\{u_n\} \subset W_0^s L_A(\Omega)$ with $u_n \to u$ strongly in $W_0^s L_A(\Omega)$, then $D^s u_n \to D^s u$ in $L_A(\Omega\times\Omega, d\mu)$. So by dominated convergence theorem, there exist a subsequence $\{D^s u_{n_k}\}$ and a function h in $L_A(\Omega\times\Omega, d\mu)$ such that

$$a(|D^s u_{n_k}|)D^s u_{n_k} \to a(|D^s u|)D^s u$$

and

$$|a(|D^s u_{n_k}|)D^s u_{n_k}| \leq |a(|h|)h|,$$

for almost every (x,y) in $\Omega\times\Omega$, by Lemma 2.1, we have $|a(|h|)h| \in L_{\overline{A}}(\Omega\times\Omega, d\mu)$. So, for $v \in W_0^s L_A(\Omega)$, $D^s v \in L_A(\Omega\times\Omega, d\mu)$ and by Hölder's inequality

$$\begin{aligned}
&\left|\int_{\Omega\times\Omega}\left[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right] D^s v d\mu\right| \\
\leq& 2\left\|a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right\|_{L_{\overline{A}}} \|D^s v\|_{L_A} \\
\leq& 2\left\|a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right\|_{L_{\overline{A}}} \|v\|.
\end{aligned}$$

Then by dominated convergence theorem we obtain that

$$\sup_{\|v\|\le 1}\left|\int_{\Omega\times\Omega}\left[a(|D^s u_{n_k}|)D^s u_{n_k} - a(|D^s u|)D^s u\right] D^s v d\mu\right| \to 0. \tag{3.2}$$

On the other hand, the continuity of M and Proposition 2.1, we have

$$M\left(\Phi(u_n)\right) \to M\left(\Phi(u)\right). \tag{3.3}$$

Combining (3.2)–(3.3) with the Hölder inequality, we have

$$\|\Psi'(u_n) - \Psi'(u)\|_* = \sup_{v\in W_0^s L_A(\Omega), \|v\|\le 1} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| \to 0. \qquad \square$$

Lemma 3.3. *The following properties hold true:*
(i) *the functional* Ψ *is sequentially weakly lower semi continuous;*
(ii) *the functional* Ψ *belongs to the class* $\mathcal{W}_{W_0^s L_A(\Omega)}$.

Proof. (i) First, note that the map

$$u \mapsto \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N},$$

is lower semi-continuous in the weak topology of $W_0^s L_A(\Omega)$. Indeed, similar to Lemma 3.1, we obtain $\Phi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_\Omega\int_\Omega a(|D^s u|)D^s u D^s v d\mu,$$

for all $u, v \in W_0^s L_A(\Omega)$. On the other hand, since A is a convex function so Φ is also convex.

Now, let $\{u_n\} \subset W_0^s L_A(\Omega)$ with $u_n \rightharpoonup u$ weakly in $W_0^s L_A(\Omega)$, then by convexity of Φ we have

$$\Phi(u_n) - \Phi(u) \ge \langle \Phi'(u), u_n - u\rangle,$$

and hence, we obtain $\Phi(u) \le \liminf \Phi(u_n)$, that is, the map

$$u \mapsto \int_\Omega\int_\Omega A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^N}$$

is lower semi-continuous. On the other hand by the continuity and monotonicity of the function $t \mapsto \widehat{M}(t)$, we get

$$\liminf_{n\to\infty}\Psi(u_n) = \liminf_{n\to\infty}\widehat{M}(\Phi(u_n)) \ge \widehat{M}(\liminf_{n\to\infty}\Phi(u_n)) \ge \widehat{M}(\Phi(u)).$$

Thus, the functional Ψ is sequentially weakly lower semicontinuous.

(ii) Since $\widehat{M}$ is continuous and strictly increasing, it suffices to show that $\Phi \in \mathcal{W}_{W_0^s L_A(\Omega)}$. Then, let $\{u_n\}$ be a sequence weakly converging to in $W_0^s L_A(\Omega)$ and let $\liminf_{n\to\infty}\Phi(u_n) \le \Phi(u)$. Since the functional Φ is sequentially weakly lower semicontinuous, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$\lim_{n\to\infty}\Phi(u_n) = \Phi(u).$$

On the other hand, since $\left\{\frac{u_n+u}{2}\right\}$ converges weakly to u in $W_0^sL_A(\Omega)$, from (i), we have

$$\liminf_{n\to\infty} \Phi\left(\frac{u_n+u}{2}\right) \geq \Phi(u). \tag{3.4}$$

We assume by contradiction that $\{u_n\}$ does not converge to u in $W_0^sL_A(\Omega)$. Hence, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and there exits $\varepsilon_0 > 0$ such that

$$\left\|\frac{u_n-u}{2}\right\| \geq \frac{\varepsilon_0}{2},$$

by Proposition 2.1, we have

$$\Phi\left(\frac{u_n-u}{2}\right) \geq \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\}.$$

On the other hand, by the conditions (2.1) and (2.2), we can apply [30, Lemma 2.1] in order to obtain

$$\frac{1}{2}\Phi(u_n) + \frac{1}{2}\Phi(u) - \Phi\left(\frac{u_n+u}{2}\right) \geq \Phi\left(\frac{u_n-u}{2}\right) \geq \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\}. \tag{3.5}$$

It follows from (3.5) that

$$\Phi(u) - \max\left\{\varepsilon_0^{p^-}, \varepsilon_0^{p^+}\right\} \geq \limsup_{n\to\infty} \Phi\left(\frac{u_n+u}{2}\right), \tag{3.6}$$

from (3.4) and (3.6) we obtain a contradiction. This shows that $\{u_n\}$ converges strongly to u and the functional Ψ belongs to the class $\mathcal{W}_{W_0^sL_A(\Omega)}$. $\square$

Lemma 3.4. *Assume that the sequence $\{u_n\}$ converges weakly to u in $W_0^sL_A(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega\int_\Omega a(|D^su_n|)D^su_n\,(D^su_n - D^su)\,d\mu \leq 0. \tag{3.7}$$

Then the sequence $\{u_n\}$ converges strongly to u in $W_0^sL_A(\Omega)$.

Proof. Since u_n converges weakly to u in $W_0^sL_A(\Omega)$, then $\{\|u_n\|\}$ is a bounded sequence of real numbers, that fact and Proposition 2.1, implies that the $\{\Phi(u_n)\}$ is bounded, then for a subsequence, we deduce that $\Phi(u_n) \to c$. Or since Φ is weak lower semi continuous, we get $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n) = c$. On the other hand, by the convexity of Φ, we have

$$\Phi(u) \geq \Phi(u_n) + \langle \Phi'(u_n), u_n - u\rangle.$$

Next, by the hypothesis (3.7), we conclude that $\Phi(u) = c$. Since $\left\{\frac{u_n+u}{2}\right\}$ converges weakly to u in $W_0^sL_A(\Omega)$, so since Φ is sequentially weakly lower semicontinuous:

$$c = \Phi(u) \leq \liminf_{n\to\infty} \Phi\left(\frac{u_n+u}{2}\right).$$

Seminary to proof of Lemma 3.3, we assume by contradiction that u_n converges strongly to u in $W_0^sL_A(\Omega)$. $\square$

Lemma 3.5. *Let (M_1) hold, then the operator $\Psi' : W_0^sL_A(\Omega) \to (W_0^sL_A(\Omega))^*$ is invertible and Ψ'^{-1} is continuous.*

Proof. First, we assume that the operator $\Psi' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is invertible on $W_0^s L_A(\Omega)$. By the Minty-Browder theorem (see [37]), it suffices to prove that Ψ' is strictly monotone, hemicontinuous and coercive in the sense of monotone operators.

So, let $u, v \in W_0^s L_A(\Omega)$, with $u \neq v$ and let $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$. Since $a(|t|)t$ is increasing, then

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle = \int_\Omega \int_\Omega (a(|D^s u|)D^s u - a(|D^s u|)D^s v)\,(D^s u - D^s v)\,d\mu > 0.$$

So, $\Psi' : W_0^s L_A(\Omega) \to (W_0^s L_A(\Omega))^*$ is strictly monotone, so by [37, Proposition 25.10], Φ is strictly convex. Moreover, since M is nondecreasing the function $\widehat{M}$ is convex in $\mathbb{R}^+$. Thus,

$$\widehat{M}(\Phi(\lambda u + \mu v)) < \widehat{M}(\lambda\Phi(u) + \mu\Phi(v)) \leq \lambda\widehat{M}(\Phi(u)) + \mu\widehat{M}(\Phi(v)).$$

This shows that Ψ is strictly convex and already said, that Ψ' is strictly monotone.

Let $u \in W_0^s L_A(\Omega)$, with $\|u\| > 1$, by (M_1) and Proposition 2.1, we have

$$\frac{\langle \Psi'(u), u\rangle}{\|u\|} = \frac{M(\Phi(u))\,\langle \Phi'(u), u\rangle}{\|u\|} \geq \frac{m_0 p^- \Phi(u)}{\|u\|} \geq m_0 p^- \|u\|^{p^- -1}.$$

Thus,

$$\lim_{\|u\|\to\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|} = \infty,$$

that is, Ψ' is coercive.

Now, by Lemma 3.1, we have $\Psi \in C^1(W_0^s L_A(\Omega), \mathbb{R})$, then Ψ is hemicontinuous. Thus, in view of the Minty-Browder theorem, there exists $\Psi'^{-1} : (W_0^s L_A(\Omega))^* \to W_0^s L_A(\Omega)$ and it is bounded.

Let us prove that Ψ'^{-1} is continuous by showing that its is sequentially continuous. Let $\{u_n\} \subset (W_0^s L_A(\Omega))^*$ be a sequence strongly is converging to $u \in (W_0^s L_A(\Omega))^*$ and let $v_n = \Psi'^{-1}(u_n)$ and $v = \Psi'^{-1}(u)$. Then, $\{v_n\}$ bounded in $W_0^s L_A(\Omega)$, then, we can assume that it converges weakly to a certain $v_0 \in W_0^s L_A(\Omega)$. Since u_n converges strongly to u, we have

$$\lim_{n\to\infty} \langle \Psi'(v_n), v_n - v_0\rangle = \lim_{n\to\infty} \langle u_n, v_n - v_0\rangle = 0,$$

i.e.,

$$\lim_{n\to\infty} M(\Phi(v_n)) \int_\Omega \int_\Omega a(|D^s v_n|) D^s v_n\,(D^s v_n - D^s v_0)\,d\mu = 0. \tag{3.8}$$

Since $\{v_n\}$ is bounded in $W_0^s L_A(\Omega)$, then by Proposition 2.1, $\Phi(v_n)$ is also bounded, then

$$\Phi(v_n) \to t_0 \geq 0, \quad \text{as } n \to \infty.$$

If $t_0 = 0$, then using Proposition 2.1, we get $\{v_n\}$ that strongly converges to v_0 in $W_0^s L_A(\Omega)$, by the continuity and injectivity of Ψ'^{-1} we obtain the desired result.

If $t_0 > 0$, it follows from the continuity of the function M that

$$M(\Phi(v_n)) \to M(t_0), \quad \text{as } n \to \infty.$$

Thus, by (M_1), for sufficiently large n, we get

$$M(\Phi(v_n)) \geq C_0 > 0. \tag{3.9}$$

By (3.8) and (3.9), we have

$$\int_\Omega \int_\Omega a(|D^s v_n|) D^s v_n \, (D^s v_n - D^s v_0) \, d\mu = 0. \tag{3.10}$$

From (3.10) and since v_n converges weakly to v_0 in $W_0^s L_A(\Omega)$, we can apply Lemma 3.4, in order to deduce that v_n converge strongly to v_0 in $W_0^s L_A(\Omega)$. □

Proof of Theorem 3.1. We wish to apply Theorem 2.3 taking $X = W_0^s L_A(\Omega)$, Ψ and J are as before, by Lemma 3.1 J is C^1-functional with compact derivative. Moreover by Lemma 3.3, Ψ is a sequentially weakly lower continuous and C^1-functional belongs to the class $\mathcal{W}_{W_0^s L_A(\Omega)}$, also by Lemma 3.5, the operator Ψ' admits a continuous inverse on $(W_0^s L_A(\Omega))^*$.

On the other hand, we show that Φ is coercive. In fact, if $\|u\| > 1$, by (M_1) and Proposition 2.1, we have

$$\Psi(u) = \widehat{M}(\Phi(u)) \geq m_0 \Phi(u) \geq m_0 \|u\|^{p^-},$$

from which we have the coercivety of Ψ.

It is evident that $u_0 = 0$ is the global minimum of Ψ and that $\Psi(u_0) = J(u_0) = 0$. Moreover, Ψ is bounded on each bounded subset of $W_0^s L_A(\Omega)$. Indeed, if $\|u\| \leq C$, then

$$\Psi(u) = \widehat{M}(\Phi(u)) \leq \begin{cases} \widehat{M}(C^{p^-}), & \text{if } \|u\| > 1, \\ \widehat{M}(1), & \text{if } \|u\| \leq 1. \end{cases}$$

So, $\Psi(u) \leq \max\left\{\widehat{M}(1), \widehat{M}(C^{p^-})\right\}$.

Now, by the assumption (F_2) for all $\varepsilon > 0$, there exits $\eta_1 > 0$ such that

$$|F(x,t)| \leq \varepsilon |t|^{p^+},$$

for each $x \in \Omega$ and $|t| \leq \eta_1$. Since $p^+ < p^*_{s'}$, so by Theorem 2.2, the embedding $W_0^s L_A(\Omega)$ in $L^{p^+}(\Omega)$ is compact. Then for some positive constant C_2, one has for all $u \in W_0^s L_A(\Omega)$ with $|u| \leq \eta_1$ and $\|u\| < 1$

$$J(u) \leq \varepsilon \|u\|^{p^+}_{L^{p^+}} \leq \varepsilon C_2 \|u\|^{p^+} \leq \varepsilon C_2 \Phi(u).$$

Or by (M_1), we have $\Phi(u) \leq \frac{1}{m_0}\Psi(u)$, then

$$J(u) \leq \varepsilon C_2 \frac{1}{m_0} \Psi(u).$$

Consequently, we have

$$\limsup_{u \to 0} \frac{J(u)}{\Psi(u)} \leq \varepsilon C_2 \frac{1}{m_0}. \tag{3.11}$$

By (F_3), for all $\varepsilon > 0$, there exists $\eta_2 > 0$ such that

$$|F(x,t)| \leq \varepsilon |t|^{p^-}, \tag{3.12}$$

for all $x \in \Omega$ and $|t| > \eta_2$.

For $\|u\| > 1$ large enough, from (3.12), Proposition 2.1 and Theorem 2.2, we have

$$\begin{aligned}\frac{J(u)}{\Psi(u)} &= \frac{J(u)}{\widehat{M}(\Phi(u))}\\ &\leq \frac{\displaystyle\int_{\{x\in\Omega:|u|\leq\eta_2\}} F(x,u)dx}{m_0\|u\|^{p^-}} + \frac{\displaystyle\int_{\{x\in\Omega:|u|>\eta_2\}} F(x,u)dx}{m_0\|u\|^{p^-}},\\ &\leq \frac{|\Omega| \displaystyle\sup_{\Omega\times[-\eta_2,\eta_2]} F}{m_0\|u\|^{p^-}} + \frac{\varepsilon\|u\|^{p^-}_{L^{p^-}(\Omega)}}{m_0\|u\|^{p^-}},\\ &\leq \frac{|\Omega| \displaystyle\sup_{\Omega\times[-\eta_2,\eta_2]} F}{m_0\|u\|^{p^-}} + C_3\varepsilon.\end{aligned}$$

So,

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\Psi(u)} \leq \varepsilon C_3. \tag{3.13}$$

Since $\varepsilon > 0$ is arbitrary, relations (3.11) and (3.13) imply that

$$\max\left\{\limsup_{\|x\|\to+\infty} \frac{J(x)}{\Psi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Psi(x)}\right\} \leq 0.$$

Hence, all assumptions of Theorem 2.3 are satisfied. So, for each compact interval $\Lambda \subset (\theta^*, +\infty)$, there exists a number $\delta > 0$ with the propositionerty described in the conclusion of Theorem 2.3. Fix $\lambda \in \Lambda$ and $g \in \mathcal{A}$. Put

$$\Gamma(u) = \int_\Omega G(x,u)dx \text{ and } G(x,t) = \int_0^t g(x,s)ds,$$

for all $u \in W_0^s L_A(\Omega)$. Then Γ is a C^1 functional on $W_0^s L_A(\Omega)$ with compact derivative. So, there exists $\beta^* > 0$ such that, for each $\beta \in [0, \beta^*]$, the equation

$$\Psi'(x) = \lambda J'(x) + \beta\Gamma'(x),$$

has at least three solutions whose norms are less than δ. But the solutions in $W_0^s L_A(\Omega)$ of the above equation are exactly the weak solutions of problem (P_a) and thus, the proof of Theorem 3.1 is completed. □

4. Example

We present in this section an example of functions that satisfies the conditions of Theorem 3.1. Let

$$\varphi(t) = \log(1 + |t|)|t|^{p-2}t, \tag{4.1}$$

where $p \in [2, N)$. Let $b > \max\{2, p^+\}$, $a > 0$, $b \geq 0$ and $\alpha \geq 1$ we consider

$$f(t) = b\cos(t)\sin(t)|\sin(t)|^{b-2}, \quad \text{for all } t \in \mathbb{R}, \tag{4.2}$$

$$M(t) = a + bt^{\alpha-1}, \quad \text{for all } t \geq 0. \tag{4.3}$$

So, from (4.1), (4.2) and (4.3), we have

$$A(t) = \frac{1}{p}\log(1+|t|)|t|^p - \frac{1}{p}\int_0^{|t|}\frac{t^p}{1+t}dt, \quad \widehat{M}(t) = at + \frac{b}{\alpha}t^\alpha, \tag{4.4}$$

$$F(x,t) = F(t) = |\sin(t)|^b. \tag{4.5}$$

We will next show that all the hypotheses of Theorem 3.1 are satisfied.

By Example 2 in [21, page 243], it follows that

$$p^+ = p + 1 \quad \text{and} \quad p^- = p.$$

On the other hand, we point out that trivial computations imply that

$$\frac{d^2A(\sqrt{t})}{dt^2} = \frac{1}{4}\left[\frac{1}{1+|\sqrt{t}|} + (p-2)\log(1+|\sqrt{t}|)\right] \geq 0,$$

for all $t \in \mathbb{R}$ and thus, relations (2.1)–(2.2) are satisfied.

- For each $t \in \mathbb{R}$, we claim that $f \in \mathcal{A}$. Actually, the inequality

$$\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{q-1}} < b < \infty,$$

holds for any $1 < q < p_s^*$ and on the other hand, we have

$$\lim_{|t|\to 0}\frac{|\sin(t)|^b}{|t|^{p^+}} = 0 \quad \text{and} \quad \lim_{|t|\to\infty}\frac{|\sin(t)|^b}{|t|^{p^-}} = 0.$$

Select a compact set $V \subset \Omega$ of positive measure and $v \in W_0^sL_A(\Omega)$ such that $v(x) = \frac{\pi}{2}$ in V and $0 \leq v(x) \leq \frac{\pi}{2}$ in $\Omega \setminus V$. We obtain

$$\int_\Omega |\sin(v(x)|^b dx = |V| + \int_{\Omega\setminus V}|\sin(v(x)|^b dx > 0,$$

which means that (F_1), (F_2) and (F_3) are verified. Also, for $m_0 = a$ the condition (M_1) is satisfied, we set

$$\theta^* = \inf\left\{\frac{a\Phi(u) + \frac{b}{\alpha}(\Phi(u))^\alpha}{\int_\Omega|\sin(u(x))|^b dx} : u \in W_0^sL_A(\Omega), \int_\Omega|\sin(u(x))|^b dx > 0\right\}.$$

Then, for a bounded domain Ω in $\mathbb{R}^N$ of class $C^{0,1}$, it follows from Theorem 3.1, that for each compact interval $\Lambda \subset (\theta^*, +\infty)$, there exist a number $\delta > 0$ and $\beta^* > 0$ such that, for every $\lambda \in \Lambda$ such that for all $\beta \in [0, \beta^*]$, and all $g \in \mathcal{A}$ the following problem

$$\begin{cases} (a + b(\Phi(u))^{\alpha-1})\,(-\Delta)^s_{\log}u = \lambda b\cos(u)\sin(u)|\sin(u)|^{b-2} + \beta g(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N\setminus\Omega,\end{cases}$$

where

$$(-\Delta)^s_{\log} u = 2 \text{ p.v} \int_{\mathbb{R}^N} \log(1 + |D^s u|)|D^s u|^{p-2} D^s u d\mu$$

has at least three weak solutions whose norms are less than δ.

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