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LOCAL EXISTENCE AND BLOW UP FOR A NONLINEAR VISCOELASTIC KIRCHHOFF-TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. The aim of this paper is to consider the initial boundary value problem of nonlinear viscoelastic Kirchhoff-type equation with logarithmic source term. Firstly, we prove the local existence of weak solution by applying Banach fixed theorem. Later, we derive the blow-up results by the combination of the perturbation energy method, concavity method and differential-integral inequality technique.

1. Introduction

In this article, we study the following viscolelastic Kirchhoff type problem

$$(1.1)\quad \begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int\limits_0^t g\left(t-s\right) \triangle u\left(s\right) ds = u \ln\left|u\right|, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u\left(x,0\right) = u_0\left(x\right), \quad u_t\left(x,0\right) = u_1\left(x\right), & x \in \Omega, \\ u\left(x,t\right) = 0, \quad x \in \partial\Omega \times \mathbb{R}^+, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $M(s) = \beta_1 + \beta_2 s^\gamma$, $\gamma, s \geq 0$. Specially, we take $\beta_1 = \beta_2 = 1$. We impose some conditions to be specified on the kernel function $g(t)$.

The equation with the logarithmic source term is related with many branches of physics. Cause of this is interest in it occures naturally in inflation cosmology, nuclear physics, supersymmetric field theories and quantum mechanics (see [3, 5, 10]). Later,

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by the motivation of this work, some authors gave necessary and sufficient conditions for the hyperbolic equation with logarithmic source term (see [6, 12, 15, 16]).

The Kirchhoff-type problem without the viscoelastic term has been extensively studied and many results for the existence, blow up and asymptotic behaviour of solutions have been established. For example, the following equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + |u_t|^{p-1} u_t - \Delta u_t = u^{k-1} \ln |u|,$$

has been considered by Yang et al. [19], where $M(s) = \alpha + \beta s^{\gamma}$, $\gamma > 0$, $\alpha \geq 1$, $\beta > 0$. They studied the local existence, asymptotic behavior and finite time blow up of solutions in cases subcritical energy and critical energy. And also, they proved the finite time blow up solutions in case arbitrary high energy.

In 2019, Pişkin and Irkıl [9] considered the global existence for the following equation

$$u_{tt} + M\left(\|\Delta u\|^2\right) \triangle^2 u + g(u_t)\, u_t = |u|^{p-1} \ln |u|^k.$$

In recent years, when by $g \neq 0$ and M is a constant function, problem have been offered by many authors. Al-Gharabli et al. [2] considered the following equation

$$|u_t|^{\rho} u_{tt} + \Delta^2 u_{tt} + \Delta^2 u - \int_0^t g(t-s)\, \Delta^2 u ds + u = u \ln |u|^k. \tag{1.2}$$

They investigated the local existence, global existence and stability for the problem (1.2). Later, they [11] proved the existence and decay results of problem (1.2) for $\rho = 0$ and absence $\Delta^2 u_{tt}$ term. Pişkin and Irkıl [18] studied the exponential growth of solutions of problem (1.2) for $\rho = 0$ and higher order viscoelastic term. In [17], the same authors studied the following equation

$$u_{tt} + [P u_{tt} + P u_t] + P u + u - \int_0^t g(t-s)\, P u ds + u_t = u \ln |u|^k,$$

where $P = (-\triangle)^m$, $m \geq 1$, and $m \in \mathbb{N}$. They obtained local existence by using Faedo-Galerkin method and a logaritmic Sobolev inequality. Later, they proved general decay results of solutions.

In [13], Peyravi considered

$$u_{tt} - \Delta u + u + \int_0^t g(t-s)\, \Delta u \; ds + h(u_t)\, u_t + |u|^2 u = u \ln |u|^k, \tag{1.3}$$

in $\Omega \subset \mathbb{R}^3$ with $h(s) = k_0 + k_1 |s|^{m-1}$. He studied the decay estimate and exponential growth of solutions for the problem (1.3).

In [20], Ye studied the logarithmic viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s) \triangle u(s)\, ds = u \ln |u|,$$

in three-dimensional space. The local and global existence for this problem are proved and the blow up of solutions is obtained.

In 2019, Boulaaras et al. [4] studied viscoleastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity. They obtained an arbitrary rate of decay, which is not necessarily of polynomial or exponential decay.

In wiev of the articles mentioned above, much less effort has been devoted to initial boundary value problem for viscoelastic Kirchhoff type equation with logarithmic nonlinearity to our knowledge. Our purposes of this paper are to prove the local existence and blow up result by combining of Banach fixed point theorem, potential well theory and Logarithmic Sobolev inequality.

The structure of the work is as follows. To facilitate the description, firstly we give some definitions, notations, energy functional and some lemmas which will be used in our proof in Section 1. In Section 2 and in Section 3, respectively, we pove the local existence and blow up results for the solution of problem (1.1).

2. Preliminaries

In this part, we will present some notations and lemmas which will be used throughout this paper. We will write $\|\cdot\|_2$ and $\|\cdot\|_p$ for the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. We will use the Standart Lebesque Space $L^2(\Omega)$ with the inner product and the norm. The inner product can take as

$$\langle u, v\rangle = \int u(x)v(x)dx,$$

and the norm is defined as

$$\|u\|_2 = \langle u, u\rangle^{\frac{1}{2}}.$$

Let us begin with defining the following total energy functional

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{4}\|u\|^2 \\ &+ \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}\int_\Omega u^2\ln|u|\,dx. \end{aligned} \tag{2.1}$$

The potential energy functional

$$\begin{aligned} J(u) =& \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \frac{1}{4}\|u\|^2 \\ &+ \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}\int_\Omega u^2\ln|u|\,dx, \end{aligned}$$

and the Nehari functional

$$I(u) = \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + (g\circ\nabla u)(t) - \int_\Omega u^2\ln|u|\,dx, \tag{2.2}$$

for $u \in H_0^1(\Omega)$, where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \|\nabla u(s) - \nabla u(t)\|^2 ds.$$

Then, it is easy to show that for $u \in H_0^1(\Omega)$,

$$J(u) = \frac{1}{2} I(u) + \frac{1}{4} \|u\|^2 - \frac{\gamma}{\gamma+1} \|\nabla u\|^{2(\gamma+1)}, \tag{2.3}$$

$$E(t) = \frac{1}{2} \|u_t\|^2 + J(u). \tag{2.4}$$

The potential well depth is defined as

$$W = \left\{ u \in H_0^1(\Omega) \mid J(u) < d, I(u) > 0 \right\} \cup \{0\},$$

and the outer space of the potential well

$$V = \left\{ u \in H_0^1(\Omega) \mid J(u) < d, I(u) < 0 \right\}.$$

The depth of potential well is defined as

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{2.5}$$

Now, we present following assumptions and some useful lemmas.

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) \geq 0, 1 - \int_0^\infty g(s) ds = l_0 > 0,$$

where

$$\int_0^\infty g(s) ds > \frac{\|\nabla u\|^2 + (g \circ \nabla u)(t) - \int_\Omega u^2 \ln |u| dx}{\|\nabla u\|^2}.$$

$(A2)$ There exists positive constant ϑ such that

$$g'(t) \leq \vartheta g(t), \quad t \geq 0.$$

Lemma 2.1 ([7,8] Logarithmic Sobolev Inequality). *Let u be any function $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain and $a > 0$ be any number. Then*

$$\int_\Omega \ln |u| u^2 dx < \frac{\alpha^2}{2\pi} \|\nabla u\|^2 + \ln \|u\| \|u\|^2 - \frac{3}{2}(1 + \ln \alpha) \|u\|_2^2.$$

Lemma 2.2 ([1, 14]). *Let $n = 3$. Then $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and there exists a constant c_p, the smallest positive number, satisfying*

$$\|u\|_6 \leq c_p \|\nabla u\|_2, \quad \textit{for all } u \in H_0^1(\Omega).$$

Lemma 2.3. *Suppose that $(A1)$ and $(A2)$ hold. Then the energy functional $E(t)$ is decresing with respect to t and*

$$E'(t) = \frac{1}{2}\left[(g' \circ \nabla u)(t) - g(t)\|\nabla u(t)\|^2\right] \leq 0,$$

where

$$(g' \circ \nabla u)(t) = \int_0^t g'(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2\, dxdt. \tag{2.6}$$

Proof. Multiplyingboth sides of (1.1) by u_t and then integrating from 0 to t, we have

$$E(t) = \int_0^t \frac{1}{2}\left[(g' \circ \nabla u)(t) - g(t)\|\nabla u(t)\|^2\right] + E(0),$$

which yields (2.6) by a simple calculation. □

Lemma 2.4. *For any $u \in H_0^1(\Omega)$, $\|u\| \neq 0$, we have*

i) $\lim\limits_{\lambda\to 0} J(\lambda u) = 0$, $\lim\limits_{\lambda\to\infty} J(\lambda u) = -\infty$*;*

ii) for $0 < \lambda < \infty$ there exists a unique λ_1 such that

$$\frac{d}{d\lambda} J(\lambda u)\,|_{\lambda=\lambda_1} = 0,$$

where λ_1 is the unique root of equation

$$l_0\|\nabla u\|^2 + (g \circ \nabla u)(t) - \int_\Omega u^2 \ln|u|\, dx = \ln\lambda \int_\Omega u^2 dx - \lambda^{2\gamma}\|\nabla u\|^{2\gamma+2};$$

iii) $J(\lambda u)$ is strictly decreasing on $\lambda_1 < \lambda < \infty$, strictly increasing on $0 < \lambda < \lambda_1$ and attains the maximum at $\lambda = \lambda_1$;

iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda_1$, $I(\lambda u) > 0$ for $\lambda_1 < \lambda < \infty$, and $I(\lambda_1 u) = 0$

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 \leq \lambda \leq \lambda_1, \\ = 0, & \lambda = \lambda_1, \\ < 0, & \lambda_1 \leq \lambda. \end{cases}$$

Proof. i) By the definition of $J(u)$, we get

$$\begin{aligned} J(\lambda u) =& \frac{\lambda^2}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u\|^2 + \frac{\lambda^2}{2}(g \circ \nabla u)(t) \\ &+ \frac{\lambda^{2\gamma+2}}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{\lambda^2}{4}\int_\Omega u^2 dx \\ &- \frac{\lambda^2}{2}\int_\Omega u^2 \ln|u|\, dx - \frac{\lambda^2 \ln\lambda}{2}\int_\Omega u^2 dx. \end{aligned} \tag{2.7}$$

Considering $\|u\| \neq 0$, so $\lim\limits_{\lambda\to 0} J(\lambda u) = 0$ and $\lim\limits_{\lambda\to\infty} J(\lambda u) = -\infty$ hold.

ii) Taking derivative of $J(\lambda u)$ with respect to λ, (2.7) yields

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda u) =&\lambda\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\lambda(g\circ\nabla u)(t)\\&+\lambda^{2\gamma+1}\|\nabla u\|^{2(\gamma+1)}-\lambda\int_\Omega u^2\ln|u|\,dx-\lambda\ln\lambda\int_\Omega u^2dx\\=&\lambda\Bigg(l_0\|\nabla u\|^2+(g\circ\nabla u)(t)+\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega u^2\ln|u|\,dx\\&-\ln\lambda\int_\Omega u^2dx\Bigg),\end{aligned}$$

which means that there is a unique λ_1 such that $\frac{d}{d\lambda}J(\lambda u)\,|_{\lambda=\lambda_1}=0$, where λ_1 is the unique root of equation

$$l_0\|\nabla u\|^2+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx=\ln\lambda\int_\Omega u^2dx-\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)},$$

where $l_0\|\nabla u\|^2+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx<0$.

iii) A simple corollary of the *ii*) we get

$$\frac{d}{d\lambda}J(\lambda u)>0,\qquad \text{for } 0<\lambda<\lambda_1,$$

and

$$\frac{d}{d\lambda}J(\lambda u)<0,\qquad \text{for } \lambda_1<\lambda<\infty.$$

iv) From (2.2), we get

$$\begin{aligned}I(\lambda u)=&\lambda^2\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\lambda^2(g\circ\nabla u)(t)\\&-\int_\Omega(\lambda u)^2\ln|\lambda u|\,dx\\=&\lambda^2\left(l_0\|\nabla u\|^2+(g\circ\nabla u)(t)+\lambda^{2\gamma}\|\nabla u\|^{2(\gamma+1)}-\int_\Omega u^2\ln|u|\,dx-\ln\lambda\int_\Omega u^2dx\right)\\=&\lambda^2\frac{d}{d\lambda}J(\lambda u),\end{aligned}$$

which implies $I(\lambda_1 u)=0$, then $I(\lambda u)>0$ for $0<\lambda<\lambda_1$, $I(\lambda u)>0$ for $\lambda_1<\lambda<\infty$. □

Lemma 2.5. *Assume that $u\in H_0^1(\Omega)$. Then $d=\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3$.*

Proof. Combining Logarithmic Sobolev inequality and $(A1)$ yields that

$$
\begin{aligned}
I(u) &= \left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)-\int_\Omega u^2\ln|u|\,dx \\
&\geq \left(l_0-\frac{\alpha^2}{2\pi}\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+(g\circ\nabla u)(t)+\left[\frac{3}{2}(1+\ln\alpha)-\ln\|u\|\right]\|u\|^2, \qquad (2.8)
\end{aligned}
$$

for any $\alpha>0$. Taking $\alpha=\sqrt{2\pi l_0}$, by (2.8) and $(A1)$, we arrive that

$$I(u)>\left[\frac{3}{2}(1+\ln\alpha)-\ln\|u\|\right]\|u\|^2. \qquad (2.9)$$

From Lemma 2.4 and (2.3), we conclude that

$$
\begin{aligned}
\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda_1 u) &= \frac{1}{2}I(\lambda_1 u)+\frac{1}{4}\|\lambda_1 u\|^2-\frac{\gamma}{\gamma+1}\|\lambda_1\nabla u\|^{2(\gamma+1)} \\
&\geq \frac{1}{2}I(\lambda_1 u)+\frac{1}{4}\|\lambda_1 u\|^2. \qquad (2.10)
\end{aligned}
$$

It follows from (2.9) and Lemma 2.4 that

$$0=I(\lambda_1 u)\geq\left[\frac{3}{2}(1+\ln\alpha)-\ln\|\lambda_1 u\|\right]\|\lambda_1 u\|^2,$$

which implies that

$$\|\lambda_1 u\|^2\geq(2\pi l_0)^{\frac{3}{2}}e^3. \qquad (2.11)$$

We gain from (2.10) and (2.11) that

$$\sup_{\lambda\geq 0} J(\lambda u)\geq\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3. \qquad (2.12)$$

By (2.5) and (2.12), $d=\frac{1}{4}(2\pi l_0)^{\frac{3}{2}}e^3>0$. □

3. Local Existence

In this part, we state and prove the local existence result for the problem (1.1). Firstly, we consider linear problem

$$
\begin{cases}
u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+\int_0^t g(t-s)\triangle u(s)\;ds+u=v\ln|v|, & (x,t)\in\Omega\times(0,T), \\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in\Omega, \\
u(x,t)=0, \quad x\in\partial\Omega\times\mathbb{R}^+,
\end{cases} \qquad (3.1)
$$

in which $T>0$.

Lemma 3.1. *Assume that* $(A1)$ *and* $(A2)$ *hold. Then for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $v \in C([0,T]; H_0^1(\Omega))$, *problem* (3.1) *has a unique local solution for some* $T > 0$

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right).$$

Proof. Suppose that $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the Laplace operator with the Dirichlet boundary condition

$$-\Delta w_j = \lambda_j w_j, \quad w_j |_{\partial\Omega} = 0.$$

Then, we choose an orthogonal basis $\{w_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let V_m be the subspace of $H_0^1(\Omega)$ generated by $\{w_1, w_2, \ldots, w_m\}$, $m \in \mathbb{N}$. We search for an approximate solution

$$u^m(x,t) = \sum_{j=1}^{m} h_j^m(t)\, w_j(x),$$

which satisfies the following Cauchy problem in V_m

$$\begin{cases} \left(u_{tt}^m(t), w_j\right) - M\left(\|\nabla^m u\|^2\right)\left(\Delta^m u(t), w_j\right) + \int_0^t g(t-s)\left(\triangle^m u(s), w_j\right) ds \\ = \left(v \ln|v|, w_j\right), \quad j = 1,2,\ldots,m \in V_m, \\ u^m(0) = u_0^m = \sum_{j=1}^{m} \left(u_{0,} w_j\right) w_j, \quad \text{in } H_0^1(\Omega), m \to \infty, \\ u_t^m(0) = u_1^m = \sum_{j=1}^{m} \left(u_{1,} w_j\right) w_j, \quad \text{in } L^2(\Omega), m \to \infty. \end{cases} \tag{3.2}$$

This leads to the initial value problem for a system second-order differantial equations for unknown functions $h_j^m(t)$

$$\begin{cases} h_{jtt}^m(t) + M\left(\|\nabla^m u\|^2\right)\lambda_j h_j^m(t) = G_j\left(h_j^m(t)\right), \quad j = 1,2,\ldots,m, \\ h_j^m(0) = \int_\Omega u_0 w_j dx, \quad h_{jt}^m(0) = \int_\Omega u_1 w_j dx, \quad j = 1,2,\ldots,m, \end{cases} \tag{3.3}$$

where

$$G_j\left(h_j^m(t)\right) = \int_0^t g(t-s)\lambda_j h_j^m(s)\; ds + \int_\Omega v \ln|v|\, w_j, \quad j = 1,2,\ldots,m.$$

Multiplying (3.3) by $h_{jt}^m(t)$ and sum over j from 1 to m, and later integrating over $[0,t]$, we obtain

$$\begin{aligned} &\|u_t^m(t)\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)} + (g \circ \nabla u^m)(t) \\ = &\|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} \\ &+ 2\int_0^t \int_\Omega v(s)\ln|v(s)|\, u_t^m(s)\,dxds + \int_0^t \left[(g' \circ \nabla u)(s) - g(s)\|\nabla u(s)\|^2\right] ds \end{aligned}$$

$$\leq \|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} + 2\int_0^t\int_\Omega v(s)\ln|v(s)|\,u_t^m(s)\,dxds. \tag{3.4}$$

We estimate the last term in the right-hand side as follows. By Hölder's and Young's inequalities, we have

$$\begin{aligned} 2\int_0^t\int_\Omega v(s)\ln|v(s)|\,u_t^m(s)\,dxds \leq & 2\int_0^t\int_\Omega |v(s)\ln|v(s)||^2\,dxds \int_0^t\int_\Omega |u_t^m(s)|^2\,dxds \\ \leq & \int_0^t\int_\Omega |v(s)\ln|v(s)||^2\,dxds + \int_0^t \||u_t^m(s)|\|^2\,ds. \end{aligned} \tag{3.5}$$

For $v \in H_0^1(\Omega)$, by direct calculation and using of Lemma 2.2, we obtain

$$\begin{aligned} \int_\Omega |v\ln|v||^2\,dx = & \int_{\{x\in\Omega;|v(x)|\leq 1\}} v^2(\ln|v|)^2\,dx + \int_{\{x\in\Omega;|v(x)|>1\}} v^2(\ln|v|)^2\,dx \\ \leq & e^{-2}|\Omega| + \frac{1}{4}\int_{\{x\in\Omega;|v(x)|>1\}} |v|^6\,dx \leq e^{-2}|\Omega| + \frac{1}{4}\|v\|_6^6 \\ \leq & e^{-2}|\Omega| + \frac{1}{4}c_p\|\nabla v\|^6 = C, \end{aligned} \tag{3.6}$$

since

$$\begin{cases} \ln|u| < \frac{u^2}{2}, & |u(x)| > 1, \\ u\ln|u| < e^{-1}, & |u(x)| \leq 1. \end{cases}$$

It follows from $(A1)$, (3.4), (3.5) and (3.6) that

$$\begin{aligned} & \|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} \\ & \leq \|u_1^m(t)\|^2 + \|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2(\gamma+1)} + CT + \int_0^t \||u_t^m(s)|\|^2\,ds \\ & \leq C_* + \int_0^t \left[\|u_t^m(s)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)}\right] ds, \end{aligned} \tag{3.7}$$

where $C_* = \|u_1^m(t)\|^2 + l_0\|\nabla u_0^m\|^2 + \frac{1}{\gamma+1}\|\nabla u_0^m\|^{2\gamma+2} + CT$. By using of Gronwall inequality and (3.7), we get

$$\|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)} \leq C_2 e^T. \tag{3.8}$$

We obtain from (3.8) that

$$\begin{cases} u^m \text{ is a bounded sequence in } L^\infty([0,T];H_0^1(\Omega)), \\ u_t^m \text{ is a bounded sequence in } L^\infty([0,T];L^2(\Omega)). \end{cases}$$

Hence, there exists a subsequence of $\{u^m\}$, still denoted by $\{u^m\}$, such that

$$\begin{cases} u_m \to u, \text{ weakly star in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ u_{mt} \to u_t, \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega)\right), \\ u_{mtt} \to u_{tt}, \text{ weakly in } L^2\left(0,T;H_0^{-1}(\Omega)\right). \end{cases} \tag{3.9}$$

Setting up $m \to \infty$ and passing to the limit in (3.2), and combining by (3.9), we obtain

$$(u_{tt}(t), w_j) - M\left(\|\nabla u\|^2\right)(\Delta u(t), w_j) + \int_0^t g(t-s)(\triangle u(s), w_j)\,ds = (v\ln|v|, w_j),$$

for $j = 1, 2, \ldots$ Since $\{w_j\}_{j=1}^\infty$ is a base in the corresponding space, we deduce that u satisfies the equation in (3.1). We finished this section by proving a local existence result of the problem (1.1). □

Theorem 3.1. *Suppose that (A1) holds. Assume further that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then problem (1.1) has a unique local solution*

$$u \in C\left([0,T]; H_0^1(\Omega)\right), \quad u_t \in C\left([0,T]; L^2(\Omega)\right).$$

Proof. We define the following set

$$X_{r_0,T} = \left\{u \in \Pi \mid \|u(t)\|_\Pi \le r_0^2, t \in [0,T]\right\},$$

here the space

$$\Pi = \left\{u \mid u \in C\left([0,T]; H_0^1(\Omega)\right), u_t \in C\left([0,T]; L^2(\Omega)\right)\right\},$$

equipped with the norm

$$\|u(t)\|_\Pi = \sup_{0\le t\le T}\left(\|u_t^m(t)\|^2 + l_0\|\nabla u^m\|^2 + \frac{1}{\gamma+1}\|\nabla u^m\|^{2(\gamma+1)}\right).$$

Then $X_{r_0,T}$ is a complete metric space with the distance

$$d(u_1,u_2) = \|u_1 - u_2\|_\Pi.$$

By Lemma 3.1, we define the nonlinear mapping $\Psi : v \to u = \Psi v$ in the following way. For $v \in X_{r_0,T}$, $u = \Psi v$ is the unique solution of problem (3.1). We claim that Ψ is a contraction mapping from $X_{r_0,T}$ into itself for $r_0 > 0$ and $T > 0$.

Let $v \in X_{r_0,T}$, for $t \in [0,T]$, we get from $(A1)$ and (3.4) that

$$\begin{aligned} &\|u_t\|^2 + l_0\|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} \\ &\le \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + 2\int_0^t\int_\Omega v(s)\ln|v(s)|\,u_t(s)\,dxds \end{aligned}$$

(3.10)

$$\leq \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + \int_0^t \|v(s)\ln|v(s)|\|^2\,ds + \int_0^t \||u_t(s)|\|^2\,ds.$$

Next we estimate the $\int_0^t \|v(s)\ln|v(s)|\|^2\,ds$ term in (3.10), by using of Hölder ineqality, Lemma 2.2, the definition of $\|u(t)\|_\Pi$ and the inequality $\ln x < x$ as $x > 1$ such that we obtain

$$\begin{aligned}\|v(s)\ln|v(s)|\|^2 &= \int_{\{x\in\Omega;|v(x)|\leq 1\}} v^2(\ln|v|)^2\,dx + \int_{\{x\in\Omega;|v(x)|>1\}} v^2(\ln|v|)^2\,dx\\ &\leq \int_{\{x\in\Omega;|v(x)|>1\}} |v|^4\,dx\\ &\leq \sqrt[3]{\Omega}\,\|v\|_6^4 \leq \sqrt[3]{\Omega}c_p^4\|\nabla v\|^4 \leq \frac{\sqrt[3]{\Omega}c_p^4 r_0^4}{l_0^2}.\end{aligned} \tag{3.11}$$

By combining of (3.10) and (3.11) and using of the definition of $\|u(t)\|_\Pi$, we have

$$\begin{aligned}\|u_t\|^2 + l_0\|\nabla u\|^2 + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)} &\leq \Xi(u_0,u_1,r_0,T) + \int_0^t \||u_t(s)|\|^2\,ds\\ &\leq \Xi(u_0,u_1,r_0,T) + \int_0^t \|u(s)\|_\Pi\,ds,\end{aligned} \tag{3.12}$$

where $\Xi(u_0,u_1,r_0,T) = \|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)} + \frac{\sqrt[3]{\Omega}c_p^4 r_0^4}{l_0^2}T$.

We get from (3.12) and Gronwall's inequality that

$$\|u\|_\Pi \leq \Xi(u_0,u_1,r_0,T)\,e^T. \tag{3.13}$$

Choosing

$$r_0 > \sqrt{\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}}$$

and

$$T < \left[\frac{r_0^2 - \left(\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right) l_0^2}{\sqrt[3]{\Omega}c_p^4 r_0^4}\right],$$

such that $\Xi(u_0,u_1,r_0,T) \leq r_0^2$, we see that $u \in X_{r_0,T}$ by (3.13). This shows that Ψ maps $X_{r_0,T}$ into itself.

Next, we shall show that Ψ is a contraction mapping. Let $v_1, v_2 \in X_{r_0,T}$ and $u_1 = \Psi v_1,\ u_2 = \Psi v_2,$ be the corresponding solution for problem (3.1). Taking $U = u_1 - u_2$,

$V = v_1 - v_2$, then U satisfies the following problem

$$\begin{cases} U_{tt} - M\left(\|\nabla U\|^2\right)\Delta U + \int\limits_0^t g(t-s) \triangle U(s)\,ds \\ = v_1 \ln|v_1| - v_2 \ln|v_2|, \quad (x,t) \in \Omega \times (0,T), \\ U(x,0) = U_t(x,0) = 0, \quad x \in \Omega, \\ \frac{\partial^j U(x,t)}{\partial v^j} = 0, \quad j = 0,1,2,\dots,m-1, (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{3.14}$$

Multiplying (3.14) by U_t and then integrate it over $\Omega \times (0,T)$, we obtain

$$\|U_t\|^2 + \left(1 - \int\limits_0^t g(s)\,ds\right)\|\nabla U(t)\|^2 + \frac{1}{\gamma+1}\|\nabla U(t)\|^{2(\gamma+1)}$$

$$+ (g \circ \nabla U)(t) - \int\limits_0^t \left[(g' \circ \nabla U)(s) - g(s)\|\nabla U(s)\|^2\right] ds$$

$$=2\int\limits_0^t \int\limits_\Omega (v_1 \ln|v_1| - v_2 \ln|v_2|)\, U_t(x,s)\,dxds. \tag{3.15}$$

Thanks to Lagrange mean value Theorem, we get $v_1 \ln|v_1| - v_2 \ln|v_2| = V(1 + \ln|\beta|)$, where $|\beta| = |v_1 + \theta(v_2 - v_1)| = |(1-\theta)v_1 + \theta v_2|$, $0 < \theta < 1$. Thus, by applying the same process as (3.11), we estimate the last term in (3.15) as follows

$$\int\limits_0^t \int\limits_\Omega (v_1 \ln|v_1| - v_2 \ln|v_2|)\, U_t(x,s)\,dxds$$

$$\le \int\limits_0^t \int\limits_\Omega V U_t(x,s)\,dxds + \int\limits_0^t \int\limits_\Omega V(|v_1| + |v_2|)\, U_t(x,s)\,dxds$$

$$\le \int\limits_0^t \|V\|\,\|U_t\|\,ds + \int\limits_0^t \|V\|_6\,\||v_1| + |v_2|\|_3\,\|U_t\|\,ds$$

$$\le c_p \int\limits_0^t \|\nabla V\|\,\|U_t\|\,ds + c_p^2 \int\limits_0^t \|\nabla V\|\,(|\nabla v_1| + |\nabla v_2|)\,\|U_t\|\,ds$$

$$\le \int\limits_0^t c_p\left(1 + 2l_0^{-\frac{1}{2}} c_p r_0\right)\|\nabla V\|\,\|U_t\|\,ds$$

$$\le \frac{1}{2}\left[c_p\left(1 + 2l_0^{-\frac{1}{2}} c_p r_0\right)\right]^2 \int\limits_0^t \|\nabla V\|^2 + \frac{1}{2}\int\limits_0^t \|U_t(s)\|^2\,ds. \tag{3.16}$$

We have from $(A1)$, (3.15) and (3.16) that

$$\|U_t\|^2 + l_0\|\nabla U(t)\|^2 + \frac{1}{\gamma+1}\|\nabla U(t)\|^{2(\gamma+1)}$$

$$\leq \left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^2 \int_0^t \|\nabla V\|^2 + \int_0^t \|U_t(s)\|^2\, ds,$$

which implies that

$$\|U\|_\Pi \leq l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^2 T\|V\|_\Pi + \int_0^t \|U\|_\Pi\, ds. \tag{3.17}$$

By the Gronwall inequality and (3.17), we have

$$\|U\|_\Pi \leq l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^2 T\|V\|_\Pi e^T.$$

By choosing

$$T < l_0\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^{-2} e^{-T},$$

such that

$$l_0^{-1}\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^2 T\|V\|_\Pi e^T < 1,$$

then Ψ is a contraction mapping.

In summary, when we choose

$$r_0 > \sqrt{\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}},$$

and

$$T < \min\left\{\frac{r_0^2 - \left(\|u_1\|^2 + \|\nabla u_0\|^2 + \frac{1}{\gamma+1}\|\nabla u_0\|^{2(\gamma+1)}\right) l_0^2}{\sqrt[3]{\Omega}c_p^4 r_0^4},\right.$$
$$\left. l_0\left[c_p\left(1+2l_0^{-\frac{1}{2}}c_p r_0\right)\right]^{-2} e^{-T}\right\},$$

Ψ is a contraction mapping from $X_{r_0,T}$ to itself. According to Banach fixed point theorem, we have the local existence result. The proof is completed. □

4. Blow Up

In this part, we prove the blow up result of solution for the problem (1.1). We give some lemmas which will e used in our proof.

Lemma 4.1. *If a solution u of the problem* (1.1) *meets $u \in V$, then*

$$I(u(t)) < 2(J(u) - d).$$

Proof. By $u \in V$ and Lemma 2.4, there exists a λ_1 such that $0 < \lambda_1 < 1$ and $I(\lambda_1 u) = 0$. By taking of $I(\lambda_1 u) = 0$, definition of d in (2.5) and (2.3), we get

$$d < J(\lambda_1 u) = \frac{1}{2}I(\lambda_1 u) + \frac{1}{4}\|\lambda_1 u\|^2 - \frac{\gamma}{\gamma+1}\|\lambda_1 \nabla u\|^{2(\gamma+1)}$$

$$< \lambda_1^2 \left(\frac{1}{4} \|u\|^2 - \frac{\gamma}{\gamma+1} \|\nabla u\|^{2(\gamma+1)} \right)$$

$$< \frac{1}{4} \|u\|^2 - \frac{\gamma}{\gamma+1} \|\nabla u\|^{2(\gamma+1)} . \tag{4.1}$$

Combining (4.1) and (2.3) yields that

$$d < J(u) - \frac{1}{2} I(u),$$

which implies that

$$I(u) < 2 \left(J(u) - d \right) . \tag{4.2}$$

□

Lemma 4.2. *Assume that $u(t)$ is a solution of the problem* (1.1)*. If $u_0 \in V$ and $E(0) < d$, then $E(t) < d$ for all $t \geq 0$.*

Proof. By Lemma 2.3 and (2.1), we get

$$J(u) \leq E(t) \leq E(0) < d, \quad \text{for all } t \geq 0.$$

Suppose that there exists $t^* \in [0, \infty)$ such that $u(t^*) \notin V$, then by continuity of $I(u(t))$, we obtain $I(u(t^*)) = 0$. This means that $u(t^*) \in \mathcal{N}$. Thus, from definition of d, we get that $J(u(t^*)) \geq d$, which is a contradiction with (4.2). Consequently, Lemma 4.1 is valid. □

Theorem 4.1. *Assume that $u_0 \in V$, $u_1 \in L^2(\Omega)$, $\int_\Omega u_0 u_1 dx > 0$ and $E(0) < d$. Then the solution $u(t)$ in Theorem* 3.1 *of the problem* (1.1) *blows up as time t goes to infinity.*

Proof. We set

$$G(t) = \int_\Omega u^2 dx, \tag{4.3}$$

for all $t \in [0, \infty)$. It is obvious that $G(t) > 0$. Moreover, by using of (4.3) and (1.1), we get

$$G'(t) = 2 \int_\Omega u_t u dx \tag{4.4}$$

and

$$\begin{aligned} G''(t) =& 2 \|u_t\|^2 + 2 \int_\Omega u_{tt} u dx \\ =& 2 \|u_t\|^2 - 2 \int_\Omega M\left(\|\nabla u\|^2 \right) \|\nabla u\|^2 dx \\ & + 2 \int_0^t g(t-s) \nabla u(s) \nabla u(t) ds dx + 2 \int_\Omega u^2 \ln |u| \end{aligned}$$

$$=2\|u_t\|^2-2\|\nabla u\|^2-2\|\nabla u\|^{2(\gamma+1)}+2\int_0^t g(t-s)\,ds\,\|\nabla u\|^2$$

$$+2\int_0^t g(t-s)\int_\Omega \nabla u(t)\left(\nabla u(s)-\nabla u(t)\right)dxds+2\int_\Omega u^2\ln|u|\,. \tag{4.5}$$

By using Young inequality, we have

$$\int_0^t g(t-s)\int_\Omega |\nabla u(t)|\,|\nabla u(s)-\nabla u(t)|\,dxds\le\int_0^t g(s)\,ds\,\|\nabla u\|^2+\frac{1}{4}(g\circ\nabla u)(t)\,. \tag{4.6}$$

Combining (4.5) and (4.6) yields that

$$\begin{aligned}G''(t)\ge&2\|u_t\|^2-2\|\nabla u\|^2-2\|\nabla u\|^{2(\gamma+1)}\\&-2\int_0^t g(s)\,ds\,\|\nabla u\|^2+2\int_\Omega u^2\ln|u|-\frac{1}{2}(g\circ\nabla u)(t)\\\ge&2\|u_t\|^2-2I(u).\end{aligned} \tag{4.7}$$

From (4.4) and (4.3) and using of the Cauchy inequality, we have

$$|G'(t)|^2\le 4\int_\Omega |u_t|^2dx\int_\Omega |u|^2dx=4G(t)\|u_t\|^2\,. \tag{4.8}$$

Combining (4.7), (4.8) and (2.4), we arrive at

$$\begin{aligned}G''(t)G(t)-(G'(t))^2&\ge G(t)\left(2\|u_t\|^2-2I(u)\right)-4G(t)\|u_t\|^2\\&=-2G(t)\left(\|u_t\|^2+I(u(t))\right)\\&\ge-2G(t)\left(2E(t)-2J(u(t))+I(u(t))\right).\end{aligned} \tag{4.9}$$

Combining $u_0\in V$, $E(0)<d$ with Lemma 4.2 obtain $u\in V$, $E(t)<d$. By Lemma 4.1, we have

$$2E(t)-2J(u(t))+I(u)\le 2d-2J(u(t))+2\left(J(u(t))-d\right)=0. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$G''(t)G(t)-(G'(t))^2>0.$$

By directly calculation, we have

$$(\ln|G(t)|)'=\frac{G'(t)}{G(t)}$$

and

$$(\ln|G(t)|)''=\frac{G''(t)G(t)-(G'(t))^2}{(G(t))^2}>0. \tag{4.11}$$

By (4.11), we know that $(\ln|G(t)|)'$ is increasing with respect to t. Integrating both sides of (4.11) over $[0,t]$, we get

$$\ln|G(t)| - \ln|G(0)| = \int_0^t (\ln|G(\tau)|)' \, d\tau = \int_0^t \frac{G'(\tau)}{G(\tau)} d\tau \geq \frac{G'(0)}{G(0)} t,$$

which implies that

$$G(t) \geq G(0) \exp\left(\frac{G'(0)}{G(0)} t\right).$$

$G(t)$ tends to infinity as time goes to infinity. This completed our proof. □

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PERMUTING TRI-DERIVATIONS ON POSETS

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ABSTRACT. Let P be a partially ordered set (poset). The main objective of the present paper is to introduce and study the idea of permuting tri-derivations of posets. Several characterization theorems involving permuting tri-derivations are given. In particular, we prove that if d_1 and d_2 are two permuting tri-derivations of P with traces ϕ_1 and ϕ_2, then $\phi_1 \leq \phi_2$ if and only if $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$.

1. INTRODUCTION

Motivated by the ideas of derivations and related maps in rings and algebras (see [1, 2, 7, 9] and references therein), the notions of derivation on lattices were introduced and studied in [10] and [11], respectively. Recently, several authors have studied and verified a lot of meaningful conclusions by applying derivations and its generalized forms to lattices (see [3] for more details). In see of over mentioned development, it is very common to exchange the idea of derivations to partially ordered sets. In this direction some progress have already been made (see [14]). In the year 2009, Öztürk et al. [8] brought about the idea of permuting tri-derivations to lattices and investigated some related properties (for more information see also [4] and [13]).

In the present paper, the notion of permuting tri-derivation of a partially ordered sets is introduced and some related properties are investigated. Precisely, in Section 2, the notion of permuting tri-derivations of partially ordered sets is presented and concentrate their essential properties. Further, the fixed sets (for more information about fixed sets see [12]) are examined in light of the permuting tri-derivations. Finally,

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Section 3 is devoted to the study of the properties of ideals and the operations related with the permuting tri-derivations.

Throughout this paper, $(P, \leq)$ always denotes a partially ordered set (poset). We additionally utilize the shorthand P to indicate a poset. According to [14], for $z \in P$, we write, $\downarrow z = \{p \in P : p \leq z\}$ and $\uparrow z = \{p \in P : z \leq p\}$. For $W \subseteq P$, we denote $l(W) = \{p \in P : p \leq w, \text{ for all } w \in W\}$ the lower cone of W and $u(W) = \{p \in P : w \leq p, \text{ for all } w \in W\}$ the upper cone of W dually. It is quickly clear that both are antitone and their compositions $l(u(\cdot))$ and $u(l(\cdot))$ are monotone. Also, we have $l(u(l(\cdot))) = l(\cdot)$, $u(l(u(\cdot))) = u(\cdot)$ from [5]. If $W = \{w_1, w_2, \ldots, w_n\}$ is a finite subset, then we write $l(W) = l(w_1, w_2, \ldots, w_n)$ and $u(W) = u(w_1, w_2, \ldots, w_n)$ simply. Moreover, for $W_1 \subseteq P$ and $W_2 \subseteq P$, we will denote $l(W_1, W_2)$ for $l(W_1 \cup W_2)$ and $u(W_1, W_2)$ for $u(W_1 \cup W_2)$. For $A \subseteq P$, we write $\downarrow A = \{p \in P : p \leq a \text{ for some } a \in A\}$. From [6], we find that if $A = \downarrow A$, then A is said to be a lower set. A is directed if it is nonempty and every finite subset of A has an upper bound in A. From nonemptiness, it is ample to expect each combine of components in A has an upper bound in A. A subset J of P is called an ideal if it is a directed lower set.

2. Permuting Tri-Derivations on Posets

The following notions are essential in our discussions.

Definition 2.1. ([14, Definition 2.1]) Let $(P, \leq)$ be a poset and $d : P \to P$ be a function. We call d a derivation on P if it satisfies the following conditions:

(i) $d(l(x, y)) = l(u(l(d(x), y), l(x, d(y))))$ for all $x, y \in P$;
(ii) $l(d(u(x, y))) = l(u(d(x), d(y)))$ for all $x, y \in P$.

Let $(P, \leq)$ be a poset. A mapping $f : P \times P \times P \to P$ is called permuting if $f(x, y, z) = f(x, z, y) = f(y, x, z) = f(y, z, x) = f(z, x, y) = f(z, y, x)$ for all $x, y, z \in P$. A mapping $d : P \to P$ defined by $d(x) = f(x, x, x)$ for all $x \in P$, is called the trace of f where f is a permuting mapping.

Inspired by the notion permuting tri-derivations on rings [2, 7] and lattices [8, 13] the following notion on posets is introduced.

Definition 2.2. Let $(P, \leq)$ be a poset and $d : P \times P \times P \to P$ be a permuting mapping. Nextly, d is called a permuting tri-derivation on P if for all $x, y, z, w \in P$ the following conditions hold:

(i) $d(l((x, w), y, z) = l(u(l(d(x, y, z), w), l(x, d(w, y, z))))$ for all $x, y, z, w \in P$;
(ii) $l(d(u(x, w), y, z))) = l(u(d(x, y, z), d(w, y, z))$ for all $x, y, z, w \in P$.

Remark 2.1. Note that, a permuting tri-derivation on P satisfies the following conditions:

(i) $d(x, l(y, w), z) = l(u(l(d(x, y, z), w), l(y, d(x, w, z))))$ for all $x, y, z, w \in P$;
(ii) $l(d(x, u(y, w), z))) = l(u(d(x, y, z)), d(x, w, z)))$ for all $x, y, z, w \in P$;
(iii) $d(x, y, l(z, w)) = l(u(l(d(x, y, z), w), l(z, d(x, y, w))))$ for all $x, y, z, w \in P$;

(iv) $l(d(x, y, u(z, w)))) = l(u(d(x, y, z)), d(x, y, w)))$ for all $x, y, z, w \in P$.

Example 2.1. Let $(P, \leq) = (\mathbb{N}, \leq)$. Define the function $d : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $d(x, y, z) = \min\{x, y, z\}$ for all $x, y, z \in P$. It is straightforward to check that d is a permuting tri-derivation on P.

Proposition 2.1. *Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. Then the followings hold:*

(1) *$d(x, y, z) \leq x$, $d(x, y, z) \leq y$ and $d(x, y, z) \leq z$ for all $x, y, z \in P$;*
(2) *$d(x, y, z) \in l(x, y, z)$, for all $x, y, z \in P$;*
(3) *if $x_1 \leq x_2$ and $y, z \in P$, then $d(x_1, y, z) \leq d(x_2, y, z)$;*
(4) *if $y_1 \leq y_2$ and $x, z \in P$, then $d(x, y_1, z) \leq d(x, y_2, z)$;*
(5) *if $z_1 \leq z_2$ and $x, y \in P$, then $d(x, y, z_1) \leq d(x, y, z_2)$;*
(6) *$\phi(x) \leq x$, for all $x \in P$;*
(7) *$\phi(l(x)) \subseteq l(\phi(x))$, for all $x \in P$;*
(8) *if $x \leq y$, then $\phi(x) \leq \phi(y)$;*
(9) *$\phi^2(x) = \phi(x)$, for all $x \in P$.*

Proof. (1) Let d be a permuting tri-derivation on P. Then

$$\begin{aligned} d(l(x, x), y, z) &= l(u(l(d(x, y, z), x), l(x, d(x, y, z)))) \\ &= l(u(l(x, d(x, y, z)))) \\ &= l(x, d(x, y, z)), \end{aligned}$$

for all $x, y, z \in P$. Since $d(x, y, z) \in d(l(x, x), y, z)$, the above relation gives $d(x, y, z) \in l(x, d(x, y, z))$ for all $x, y, z \in P$. In this way, we conclude that $d(x, y, z) \leq x$ for all $x, y, z \in P$. Similarly, we can prove $d(x, y, z) \leq y$ and $d(x, y, z) \leq z$. Hence, $d(x, y, z) \leq x$, $d(x, y, z) \leq y$ and $d(x, y, z) \leq z$ for all $x, y, z \in P$.

(2) It is obvious from (1).

(3) Let $x_1 \leq x_2$ and $y, z \in P$. Then

$$l(d(u(x_1, x_2)), y, z) = l(d(u(x_2), y, z)) = l(u(d(x_1, y, z), d(x_2, y, z))),$$

for all $x_1, x_2, y, z \in P$. Since $d(x_1, y, z) \in l(u(d(x_1, y, z), d(x_2, y, z)))$, we find that $d(x_1, y, z) \in l(d(u(x_2), y, z))$ for all $x_1, x_2, y, z \in P$. Hence, $d(x_1, y, z) \leq d(x_2, y, z)$ for all $x_1, x_2, y, z \in P$.

(4), (5) Proofs run on comparable lines as in (3).

(6) By the definition,

$$\begin{aligned} d(l(x, x), x, x) &= l(u(l(d(x, x, x), x), l(x, d(x, x, x)))) \\ &= l(u(l(x, d(x, x, x)))) \\ &= l(x, d(x, x, x)), \end{aligned}$$

for all $x \in P$. Since $d(x, x, x) \in d(x, l(x, x), x))$, the last relation gives

$$\phi(x) = d(x, x, x) \in l(x, d(x, x, x)), \quad \text{for all } x \in P.$$

Consequently, we get $\phi(x) = d(x, x, x) \leq x$ for all $x \in P$.

(7) Let $x \in P$. Then

$$\begin{aligned}\phi(l(x)) &= \{d(y,y,y) : y \in P \text{ and } y \leq x\},\\ &\subseteq d(l(y,y),y,y)\\ &= l(u(l(d(y,y,y),y),l(y,d(y,y,y))))\\ &= l(u(l(d(y,y,y),y)))\\ &= l(u(l(d(y,y,y))))\\ &= l(d(y,y,y))\\ &= l(\phi(y)), \quad \text{for all } y \in P \text{ and } y \leq x.\end{aligned}$$

This implies that $\phi(l(x)) \subseteq l(\phi((x)))$ for all $x \in P$.

(8) Let $x, y \in P$ such that $x \leq y$. Then, applications of part (7) we get $\phi(l(y)) \subseteq l(\phi(y))$. Since $\phi(x) \in \phi(l(y))$, we find that $\phi(x) \in l(\phi(y))$ for all $x, y \in P$. Hence, we conclude that $\phi(x) \leq \phi(y)$ for all $x, y \in P$.

(9) In view of part (5), we get $\phi^2(x) = \phi(\phi(x)) \leq \phi(x) \leq x$ for all $x \in P$. Then for all $x \in P$

$$\begin{aligned}\phi(l(x)) &\subseteq l(\phi(x)),\\ &\subseteq d(l(x),y,y)\\ &= d(l(x,x),y,y)\\ &= l(u(l(d(x,y,y),x),l(d(x,y,y)),x))\\ &= l(u(l(d(x,y,y)),l(d(x,y,y))))\\ &= l(u(l(d(x,y,y)))\\ &= l(d(x,y,y))\\ &\subseteq l(x,y)\\ &= l(y), \quad \text{for all } y \in P \text{ and } y \leq x.\end{aligned}$$

Then for all $x \in P$ we have $\phi(l(x)) \subseteq l(y)$ for all $y \in P$ such that $y \leq x$. Since $\phi^2(x) \leq x$ for all $x \in P$, we observe that $\phi(l(x)) \subseteq l(\phi^2(x))$ for all $x \in P$. Since $\phi(x) \in \phi(l(x))$ for all $x \in P$, so $\phi(x) \in l(\phi^2(x))$ for all $x \in P$. This implies that $\phi(x) \leq \phi^2(x)$ for all $x \in P$. Hence, finally, $\phi^2(x) = \phi(x)$ for all $x \in P$. ☐

Example 2.2. Let $(P, \leq) = (\mathbb{N}, \leq)$. Define the function $d : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $d(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in P$. Then, d is not a permuting tri-derivation on P.

Corollary 2.1. *Let P be a poset with the least element 0 and let d be a permuting tri-derivation on P. Then $d(0,y,z) = 0$ for all $y, z \in P$.*

Lemma 2.1. *Let P be a poset and I be an ideal of P. Next, let d be a permuting tri-derivation on P. Then $d(x,y,z) \in I$ for all $x,y,z \in I$.*

Proof. Let $x, y, z \in I$. Then in view of Proposition 2.1 (1), we get $d(x, y, z) \leq x$ for all $x, y, z \in I$. The last expression yields $d(x, y, z) \in I$, since $x \in I$. Hence, the result holds. ☐

Lemma 2.2. *Let d be a permuting tri-derivation on P with trace ϕ. Then the following statements hold:*

(1) *If $d(l(x), x, x) = l(y)$, then $\phi(x) = y$ for all $x, y \in P$;*
(2) *If $d(u(x), x, x) = u(y)$, then $\phi(x) = y$ for all $x, y \in P$.*

Proof. (1) Let $x, y \in P$ such that $d(l(x), x, x) = l(y)$. Then, by the definition of $l(\cdot)$, we get $y \in l(y)$ for all $y \in P$. This gives $y \in d(l(x), x, x)$. Hence, there exists $z \in l(x)$ such that $d(z, x, x) = y$. Application of Proposition 2.1(3) yields $y = d(z, x, x) \leq d(x, x, x) = \phi(x)$ for $x \in P$. Therefore, the above relation forces that $y \leq \phi(x)$ for all $x, y \in P$. On the other hand if $\phi(x) \in d(l(x), x, x) = l(y)$, then we obtain $\phi(x) \leq y$. Hence $\phi(x) = y$ for all $x, y \in P$.

(2) By using comparable approach with fundamental variety, we can prove (2). ☐

Theorem 2.1. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. Then $\phi(1) = 1$ if and only if $d(x, 1, 1) = x$ for all $x \in P$.*

Proof. By the assumption, $\phi(1) = d(1, 1, 1) = 1$. In view of Proposition 2.1(1), it is easy to see that $d(x, 1, 1) \leq x$ for all $x \in P$. Secondly, to prove that $x \leq d(x, 1, 1)$ for all $x \in P$. Let $x \in P$. Then, we have

$$\begin{aligned} d(l(x), 1, 1) &= d(l(x, 1), 1, 1) \\ &= l(u(l(d(x, 1, 1), 1), l(x, d(1, 1, 1))) \\ &= l(u(l(d(x, 1, 1), 1), l(x, 1)) \\ &= l(u(l(d(x, 1, 1)), l(x)) \\ &= l(u(l(x))) \quad (\text{since } d(x, 1, 1) \leq x) \\ &= l(x). \end{aligned}$$

By another way, observe that

$$\begin{aligned} d(l(x), 1, 1) &= d(l(x, x), 1, 1) \\ &= l(u(l(d(x, 1, 1), x), l(x, d(x, 1, 1))) \\ &= l(u(l(d(x, 1, 1))), l(d(x, 1, 1))) \\ &= l(u(l(d(x, 1, 1)))) \\ &= l(d(x, 1, 1). \end{aligned}$$

On comparing the above two expressions, we get $l(x) = l(d(x, 1, 1))$ for all $x \in P$. Hence $d(x, 1, 1) = x$ for all $x \in P$. The converse part is clear. ☐

Theorem 2.2. *Let P be a poset with a least element 0 and a greatest element 1. Next, let d be a permuting tri-derivation on P. Then $d(1, 0, 0) = 0$ if and only if $d(x, 0, 0) = 0$ for all $x \in P$.*

Proof. Suppose that $d(1,0,0)=0$ and $x\in P$. Then

$$\begin{aligned} d(l(1),0,0) &= d(l(1,1),0,0) \\ &= l(u(l(d(1,0,0),1),l(1,d(1,0,0)))) \\ &= l(u(l(0,1),l(1,0)) \\ &= l(u(l(0),l(0)) = l(u(l(0))) \\ &= l(0)=\{0\}. \end{aligned}$$

But $l(1)=P$ and $x\in P$, the above relation gives $d(x,0,0)\in d(l(1),0)=l(0)=\{0\}$. Hence, $d(x,0,0)=0$ for all $x\in P$. For the converse part, proof is obvious. □

Theorem 2.3. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. If $x\le\phi(1)$, then $d(x,1,1)=x$ for all $x\in P$.*

Proof. Let $x\le\phi(1)=d(1,1,1)$ for all $x\in P$. Then for all $x\in P$, we have

$$\begin{aligned} d(l(x),1,1) &= d(l(x,1),1,1) \\ &= l(u(l(d(x,1,1),1),l(x,d(1,1,1)))) \\ &= l(u(l(d(x,1,1)),l(x))) \\ &= l(u(l(x))) \quad (\text{since } d(x,1,1)\le x) \\ &= l(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(l(x),1,1) &= d(l(x,x),1,1) \\ &= l(u(l(d(x,1,1),x),l(x,d(x,1,1)))) \\ &= l(u(l(d(x,1,1)),l(d(x,1,1)))) \\ &= l(u(l(d(x,1,1)))) \\ &= l(d(x,1,1)). \end{aligned}$$

By comparing the above two expressions, we infer that $l(d(x,1,1))=l(x)$. Hence, $d(x,1,1)=x$ for all $x\in P$. This proves the theorem completely. □

Corollary 2.2. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. Then $\phi(1)=1$ if and only if $\phi=id_P$ (identity map on P).*

Proof. Assume that $\phi(1)=d(1,1,1)=1$. Now we prove that $x=\phi(x)=d(x,x,x)$ for all $x\in P$. Let $x\in P$. Then, we have

$$\begin{aligned} d(l(x),x,x) &= d(l(x,1),x,x) \\ &= l(u(l(d(x,x,x),1),l(x,d(1,x,x))) \\ &= l(u(l(d(x,x,x)),l(d(1,x,x)) \\ &= l(u(l(d(1,x,x))) \quad (\text{since } d(x,x,x)\le d(1,x,x)) \\ &= l(d(1,x,x)). \end{aligned}$$

By another way, observe that

$$\begin{aligned} d(l(x),x,x) &= d(l(x,x),x,x) \\ &= l(u(l(d(x,x,x),x),l(x,d(x,x,x))) \\ &= l(u(l(d(x,x,x))),l(d(x,x,x))) \\ &= l(u(l(d(x,x,x)))) \\ &= l(d(x,x,x). \end{aligned}$$

On comparing the above two expressions, we get $l(d(x,x,x)) = l(d(1,x,x))$ for all $x \in P$. Hence $d(x,x,x) = d(1,x,x)$ for all $x \in P$. Again

$$\begin{aligned} d(l(x),x,1) &= d(l(x,1),x,1) \\ &= l(u(l(d(x,x,1),1),l(x,d(1,x,1))) \\ &= l(u(l(d(x,x,1)),l(d(1,x,1)) \\ &= l(u(l(d(1,x,1))) \quad (\text{since } d(x,x,1) \leq d(1,x,1)) \\ &= l(d(1,x,1)). \end{aligned}$$

Similarly we observe that

$$\begin{aligned} d(l(x),x,1) &= d(l(x,x),x,1) \\ &= l(u(l(d(x,x,1),x),l(x,d(x,x,1))) \\ &= l(u(l(d(x,x,1))),l(d(x,x,1))) \\ &= l(u(l(d(x,x,1)))) \\ &= l(d(x,x,1). \end{aligned}$$

From the above two expressions, we get $l(d(1,x,1)) = l(d(x,x,1))$ for all $x \in P$. So $d(1,x,1) = d(x,x,1)$ for all $x \in P$. Since d is permuting map then $d(1,x,1) = d(x,1,1) = d(1,x,x) = d(x,x,1)$ for all $x \in P$. Hence, $\phi(x) = d(x,x,x) = d(x,1,1)$ for all $x \in P$. Applications of Theorem 2.3 gives $\phi(x) = x$ for all $x \in P$, i.e., $\phi = id_P$. The converse part is obvious. □

Theorem 2.4. *Let P be a poset and $d : P \times P \times P \to P$ be a permuting map. Then, d is a permuting tri-derivation on P if and only if*

(1) *$d(l(x,y),z,w)) = l(d(x,z,w),y)) = l(x,d(y,z,w))$ for all $x,y,z,w \in P$;*
(2) *$l(d(u(x,y),z,w))) = l(u(d(x,z,w),d(y,z,w)))$ for all $x,y,z,w \in P$.*

Proof. Essentially ought to appear that the condition (1) in Definition 2.1 is identical to the one (1) in this hypothesis. First, we suppose that the condition in this hypothesis holds. Then

$$\begin{aligned} d(l(x,y),z,w) &= l(d(x,z,w),y) \\ &= l(u(l(d(x,z,w),y))) \\ &= l(u(l(d(x,z,w),y),l(x,d(y,z,w)))), \end{aligned}$$

for all $x, y, z, w \in P$. Secondly, suppose that d is a permuting tri-derivation on P. Then

$$\begin{aligned} l(d(x,y,z),w) &= l(u(l(d(x,y,z),w))) \\ &\subseteq l(u(l(d(x,y,z),w),l(x,d(y,z,w)))) \\ &= d(l(x,w),y,z)), \end{aligned}$$

for all $x, y, z, w \in P$. On the other hand, suppose that $v \in d(l(x,y),z,w))$, then there exists $t \in l(x,y)$ satisfying the relation $d(t,z,w) = v$. By using Proposition 2.1 (1) and (3), it is easy to see that $d(t,z,w) \leq d(x,z,w)$, $d(t,z,w) \leq d(y,z,w) \leq y$. This shows that $v = d(t,z,w) \in l(d(x,z,w),y)$. Thus $d(l(x,y),z,w)) \subseteq l(d(x,z,w),y)$. Hence,

$$d(l(x,y),z,w)) = l(d(x,z,w),y), \quad \text{for all } x,y,z,w \in P.$$

Similarly, the case $d(l(x,y),z,w)) = l(x,d(y,z,w))$ for all $x, y, z, w \in P$. This proves the theorem. □

Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. Put $Fix_\phi(P) = \{x \in P : \phi(x) = x\}$. If P has a least element 0, then $0 \in Fix_\phi(P)$. In view of Proposition 2.1, it is easy to get $Fix_\phi(P) \neq \emptyset$.

Proposition 2.2. *Let d, t be two permuting tri-derivations on P with traces ϕ_1, ϕ_2, respectively. Then $\phi_1 = \phi_2$ if and only if $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$.*

Proof. It is clear that if $\phi_1 = \phi_2$, then $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$. Conversely, assume that $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$, and $x \in P$. Then by Proposition 2.1 (9), obtain $\phi_1(x) \in Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$. This implies that $\phi_2(\phi_1(x)) = \phi_1(x)$. By a similar way we get $\phi_1(\phi_2(x)) = \phi_2(x)$ for all $x \in P$. Application of Proposition 2.1 (6), (8) yields that $\phi_1(x) \leq \phi_2(x)$ and $\phi_2(x) \leq \phi_1(x)$ for all $x \in P$. Consequently, $\phi_1 = \phi_2$. □

Proposition 2.3. *Let P be a poset with a least element 0 and d be a permuting tri-derivation on P with trace ϕ. Then the followings hold.*

(1) *$Fix_\phi(P) \neq \emptyset$.*
(2) *If $x \in Fix_\phi(P)$, and $y \leq x$ then $y \in Fix_\phi(P)$.*
(3) *If P is directed, then, for any $x, y \in Fix_\phi(P)$, there exists $z \in Fix_\phi(P)$ satisfying $x \leq z$, $y \leq z$.*

Proof. (1) Since $\phi(0) = d(0,0,0) = 0$, then $0 \in Fix_\phi(P)$. Thus, $Fix_\phi(P) \neq \emptyset$.

(2) Assume that $x \in Fix_\phi(P)$, and $y \leq x$ then $\phi(x) = d(x,x,x) = x$. Then using Proposition 2.1 (6) implies that $\phi(y) \leq y$. Now prove that $y \leq \phi(y)$. Using Theorem 2.4 (1), to get $d(l(y),x,x) = d(l(x,y),x,x) = l(d(x,x,x),y) = l(x,y) = l(y)$. Since $y \in l(y)$, so $y \in d(l(y),x,x)$ and this leads to $y \leq d(y,x,x)$. Hence $d(y,x,x) = y$. Again by using Theorem 2.4 (1) get $d(l(y),y,y) = d(l(x,y),y,y) = l(d(x,y,y),y) = l(d(x,y,y)))$. Application of Lemma 2.2 (1) yields that $\phi(y) = d(x,y,y)$. Thus, using

Theorem 2.4 (2) implies that

$$\begin{aligned} l(d(u(y), y, x)) &= l(d(u(y, y), y, x)) \\ &= l(u(d(y, y, x), d(y, y, x))) \\ &= l(u(d(y, y, x))). \end{aligned}$$

Since $d(y, y, x) = d(x, y, y)$, $d(y, x, x) = d(x, y, x) \in l(d(u(y), y, x))$, and this leads to $d(y, x, x) \in l(u(d(x, y, y)))$. Thus, $y = d(y, x, x) \leq d(x, y, y) = \phi(y)$. Hence, $y \in Fix_\phi(P)$.

(3) Assume that P is directed. Then for any $x, y \in P$, there exists $v \in P$ such that $x \leq v$ and $y \leq v$. Since $x, y \in Fix_\phi(P)$, then $\phi(x) = x$ and $\phi(y) = y$. Since $\phi(x) = x \leq \phi(v)$ and $\phi(y) = y \leq \phi(v)$. Put $z = \phi(v)$, hence by Proposition 2.1 (7) we get $z \in Fix_\phi(P)$. □

Corollary 2.3. *Let P be a directed poset with the least element 0. Then $Fix_\phi(P)$ is an ideal of P.*

3. Structural Properties of Posets Including Permuting Tri-Derivations

In this section, P is a poset with the least element 0.

Theorem 3.1. *Let P be a poset with the least element 0 and d be a permuting tri-derivation on P with trace ϕ. Then $\ker\phi = \{x \in P : \phi(x) = 0\}$ is a nonempty lower set of P.*

Proof. In view of Proposition 2.1, $\phi(0) = d(0, 0, 0) = 0$. Thus, $0 \in \ker\phi$, and hence $\ker\phi \neq \emptyset$. Suppose that $x \in \ker\phi$ and $y \in P$ such that $y \leq x$. Then $\phi(x) = 0$ and $y \leq x$. Using Proposition 2.1 (8) to get $\phi(y) \leq \phi(x) = 0$. Thus, $\phi(y) = 0$ for all $y \in P$. This shows that $y \in \ker\phi$. Hence, $\ker\phi = \{x \in P : \phi(x) = 0\}$ is a nonempty lower set of P. □

Proposition 3.1. *Let P be a poset with the least element 0. Next, let d be a permuting tri-derivation on P with trace ϕ and I be an ideal of P. Then, $\phi^{-1}(I)$ is an ideal of P such that $\ker\phi \subseteq \phi^{-1}(I)$.*

Proof. Since $\phi(0) = 0$, $0 \in \phi^{-1}(I)$. Then, $\phi^{-1}(I) \neq \emptyset$. Suppose $x \in \phi^{-1}(I)$ and $y \leq x$. Then $\phi(x) \in I$. Thus, using Proposition 2.1 (8), to obtain $\phi(y) \leq \phi(x) \in I$. Since I is an ideal, hence $\phi(y) \in I$, and this leads to $y \in \phi^{-1}(I)$. Hence, $\phi^{-1}(I)$ is an ideal of P. On the other hand, note that $\ker\phi = \phi^{-1}(\{0\}) \subseteq \phi^{-1}(I)$. □

Proposition 3.2. *Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. If I, J are two ideals of P such that $I \subseteq J$, then $\phi(I) \subseteq \phi(J)$.*

Proof. Assume that $x \in \phi(I)$, then there exists $y \in I \subseteq J$ such that $x = \phi(y)$. Hence, $x \in \phi(J)$. This implies that $\phi(I) \subseteq \phi(J)$. □

Theorem 3.2. *Let P be a poset and d_1, d_2 be two permuting tri-derivations on P with traces ϕ_1, ϕ_2, respectively. Then $\phi_1(x) \leq \phi_2(x)$ for all $x \in P$ if and only if $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$.*

Proof. Let d_1, d_2 be two permuting tri-derivations on P, with traces ϕ_1, ϕ_2, respectively, such that $\phi_1 \leq \phi_2$. Then, for any $x \in P$, $\phi_1(x) \in Fix_{\phi_1}(P)$, i.e., $\phi_1(x) = \phi_1(\phi_1(x)) \leq \phi_2(\phi_1(x))$. Proposition 2.1 (6) gives that $\phi_2(\phi_1(x)) \leq \phi_1(x)$. Thus, $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$. This shows that $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$. On the other hand we find that $\phi_1(x) = \phi_2(\phi_1(x)) \leq \phi_2(x)$, for any $x \in P$, from Proposition 2.1 (6), (8). This implies that $\phi_1(x) \leq \phi_2(x)$ for all $x \in P$. This completes the proof of the theorem. □

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POLYNOMIAL WEIGHTED APPROXIMATION BY SZÁSZ-MIRAKYAN OPERATORS OF MAX-PRODUCT TYPE

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Abstract. In this paper, we study approximation of Szász-Mirakyan operators of max-product type in polynomial weighted spaces. We reckon the rate of approximation in terms of some exponential weighted spaces for obtain a better rate of approximation than the corresponding positive linear operators.

1. Introduction

In [9], the Szász-Mirakjan operators were defined as below

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\quad x\in[0,\infty),n\geq 1. \tag{1.1}$$

Many studies have been done about the approximation results for this operators and estimates of the rate of convergence. These studies are mainly using positive linear operators. However, nonlinear operators of max-product type were studied in the papers [2–4] and the conclusion is that they have the same order of approximation as in the case of positive linear operators and even better for some subclasses of functions. In [3], the authors investigated the nonlinear operators of Favard-Szász-Mirakjan of max-product type defined by

$$F_n(f,x)=\frac{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}},\quad x\in[0,\infty), \tag{1.2}$$

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where $\bigvee$ indicates the supremum. Moreover, they studied these operators for continuous and bounded functions defined on $x \geq 0$ in [3, 4]. In [5], A. Holhoş studied the approximation properties of F_n in weighted spaces with the weight $w(x) = e^{\alpha\varphi(x)}$, where $\varphi(x) = \sqrt{x}$ and $\alpha > 0$ is constant independent of x (see [1]). In [8], he introduced a new modulus of continuity. In [5], the author estimated the rate of convergence of these operators to the identity operator . Firstly, he introduce some general notations to obtain the results.

The function $\varphi : I \to J$ is defined on a noncompact interval $I \subseteq \mathbb{R}$. The interval $J \subseteq \mathbb{R}$ is just $\varphi(I)$. The space of continuous functions is defined as

$$C_{\varphi,\alpha} = \left\{ f \in C(I) \text{ there is } M > 0 \text{ such that } \frac{|f(x)|}{e^{\alpha\varphi(x)}} \leq M \text{ for every } x \in I \right\}. \tag{1.3}$$

This space can be endowed with the norm

$$\|f\|_{\varphi,\alpha} = \sup_{x\in I} e^{-\alpha\varphi(x)} |f(x)|. \tag{1.4}$$

The modulus of continuity $\omega_{\varphi,\alpha}(f; \cdot)$ is given for every $f \in C_{\varphi,\alpha}$ and $\delta \geq 0$ as follows

$$\omega_{\varphi,\alpha}(f; \delta) = \frac{|f(t) - f(x)|}{\max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right)}, \tag{1.5}$$

which the supremum is taken for all $x, t \in I$ such that $\varphi(t) \in (\varphi(x) - \delta, \varphi(x) + \delta) \cap \varphi(I)$. For $\alpha = 0$ and $\varphi(x) = x$, we obtain the usual modulus of continuity $\omega(f; \delta)$.

In this paper, our main problem is that the operators F_n can be used for approximation with polynomial weight $w(x) = (1 + x)^\alpha$ by taking $\varphi(x) = \ln(1 + x)$ and we estimate the rate of convergence of these operators to the identity operator. Hence, we show approximation of Szász-Mirakyan operators of max-product type in polynomial weighted spaces.

2. Polynomial Weighted by Szász-Mirakyan operators

In this section, we prove some auxiliary results to obtain some estimates of the rate of approximation of functions given by (1.1) and (1.2).

Remark 2.1. For $n \in \mathbb{N}$ take the intervals

$$I_0 = \left[0, \left(\frac{n}{n+1}\right)^\alpha\right), \quad I_k = \left[k\left(\frac{n+k-1}{n+k}\right)^\alpha, (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha\right]. \tag{2.1}$$

The intervals are nonempty, disjoint and their union is the positive half line. Indeed,

$$l_k = (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha - k\left(\frac{n+k-1}{n+k}\right)^\alpha \geq 0.$$

Lemma 2.1. *If $nx \in I_j$, then* $\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha = \frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha$.

Proof. Let us denote $a_k = \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha$. We get

$$0 \le a_{k+1} \le a_k \text{ if and only if } nx \in \left[0, (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha\right).$$

Let us take $k = 0, 1, \ldots$ We obtain

$$\begin{aligned} a_1 &\le a_0 \text{ if and only if } nx \in \left[0, \left(\frac{n}{n+1}\right)^\alpha\right), \\ a_2 &\le a_1 \text{ if and only if } nx \in \left[0, 2\left(\frac{n+1}{n+2}\right)^\alpha\right), \\ a_3 &\le a_2 \text{ if and only if } nx \in \left[0, 3\left(\frac{n+2}{n+3}\right)^\alpha\right), \end{aligned}$$

and so on. From all these inequalities, we get

$$\begin{aligned} &\text{if } nx \in I_0, \text{ then } a_k \le a_0, \quad \text{for all } k = 0, 1, \ldots, \\ &\text{if } nx \in I_1, \text{ then } a_k \le a_1, \quad \text{for all } k = 0, 1, \ldots, \\ &\text{if } nx \in I_2, \text{ then } a_k \le a_2, \quad \text{for all } k = 0, 1, \ldots, \end{aligned}$$

and so on. Generally, if $nx \in I_j$, then $a_k \le a_j$, for all $k = 0, 1, \ldots$, that proves the lemma. □

Lemma 2.2. *For every $x \ge 0$ we obtain $F_n\left((1+t)^\alpha, x\right) \le (1+x)^\alpha \left(1 + \frac{\alpha}{n}\right)^\alpha$.*

Proof. Let us take $nx \in I_j$. By using Lemma 2.1, we obtain

$$F_n\left((1+t)^\alpha, x\right) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = \frac{\frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha}{\frac{(nx)^m}{m!}}.$$

Let us take $m = \lfloor nx \rfloor$. So, we have $m \le nx < (j+1)\left(\frac{n+k}{n+k+1}\right)^\alpha < j+1$ and we can say that $nx \le \lfloor nx \rfloor + 1 = m+1$, hence we obtain $\frac{nx}{m+1} \le 1$. By using Bernoulli inequality, we have

$$j - nx \le j - j\left(\frac{n+j-1}{n+j}\right)^\alpha = j\left(1 - \left(1 - \frac{1}{n+j}\right)^\alpha\right) \le j\alpha\frac{1}{n+j} \le \alpha.$$

Hence, we get

$$\begin{aligned} e^{-\alpha\ln(1+x)} F_n\left((1+t)^\alpha, x\right) &= \frac{1}{(1+t)^\alpha} \frac{\frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha}{\frac{(nx)^m}{m!}} \\ &\le \left(\frac{nx}{m+1}\right)^{j-m}\left(\frac{n+j}{n(1+x)}\right)^\alpha \le \left(\frac{n+j}{n+nx}\right)^\alpha \\ &= \left(\frac{n+j+nx-nx}{n+nx}\right)^\alpha = \left(1 + \frac{j-nx}{n+nx}\right)^\alpha \\ &\le \left(1 + \frac{\alpha}{n}\right)^\alpha. \end{aligned}$$

□

Remark 2.2. For every $x \geq 0$, we have

$$F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right) \leq (1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha.$$

Indeed,

$$\begin{aligned}F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right) =& \max\left\{F_n\left((1+t)^\alpha,x\right),F_n\left((1+x)^\alpha,x\right)\right\}\\ \leq& \max\left\{(1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha,(1+x)^\alpha\right\}\\ =&(1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha.\end{aligned}$$

Remark 2.3. For $\varphi(x)=\ln(1+x)$, for every function f belonging to $C_{\varphi,\alpha}$ the functions F_nf also belonging to $C_{\varphi,\alpha}$. Indeed,

$$\begin{aligned}\left|F_n\left(f,x\right)\right| \leq F_n\left(|f|,x\right) \leq& F_n\left(\|f\|_{\varphi,\alpha}(1+x)^\alpha,x\right)\\ =& \|f\|_{\varphi,\alpha}F_n\left((1+x)^\alpha,x\right) = \|f\|_{\varphi,\alpha}\left(1+\frac{\alpha}{n}\right)^\alpha(1+x)^\alpha.\end{aligned}$$

Lemma 2.3. *For every $x \geq 0$ and $n \in \mathbb{N}$, the following inequality is obtained*

$$\frac{\bigvee_{k\leq nx}\frac{(nx)^k}{k!}\left(\ln(1+(nx))-\ln(1+k)\right)}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}} \leq \frac{1}{2}.$$

Proof. For $x=0$, we obtain equality. By taking $m=\lfloor nx\rfloor$, we proved that $\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}=\frac{(nx)^m}{m!}$. Let us consider the inequality $\ln(1+x)\leq x$, then we get

$$\ln(1+(nx))-\ln(1+k)=\ln\left(\frac{1+(nx)}{1+k}\right)\leq\frac{(nx)-k}{1+k}.$$

Let us take $b_k=\frac{(nx)^k}{k!}\cdot\frac{(nx)-k}{1+k}$. Firstly, to evaluate the maximum of b_k, we observe that

$$\frac{b_k}{b_{k-1}}=\frac{nx}{k+1}\cdot\frac{nx-k}{nx-k+1}\leq 1$$

if and only if $(nx)^2-(2k+1)(nx)+(k^2-1)\leq 0$. This inequality's solution is equivalent to $nx\in[p_k,q_k]$ which is

$$p_k=\frac{(2k+1)-\sqrt{4k+5}}{2},\quad q_k=\frac{(2k+1)+\sqrt{4k+5}}{2}.$$

We can write the following inequality

$$0\leq p_k<p_{k+1}<\frac{2k+1}{2}<\frac{2k+3}{2}<q_k<q_{k+1}\leq 1,\quad \text{for all } k\geq 0.$$

After some computations, we get $nx \in \left[\frac{2m-1}{2}, \frac{2m+1}{2}\right)$. We deduce that if $nx \in I_j$, then $b_k \leq b_j$ for every $k \geq 1$. We obtain

$$\frac{\bigvee_{1\leq k\leq nx} b_k}{\frac{(nx)^m}{m!}} \leq \left(\frac{2j+1}{2} - j\right) = \frac{1}{2}. \qquad \square$$

Lemma 2.4. *For $x \geq 0$, the following inequality holds true*

$$\frac{\bigvee_{k>nx} \frac{(nx)^k}{k!} \left(\frac{n+k}{n}\right)^{\alpha} (\ln(1+k) - \ln(1+nx))}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} \leq \left(1 + \frac{2\alpha}{n}\right)^{\alpha} (1+x)^{\alpha}.$$

Proof. For $x = 0$ we have equality. Let us take $x > 0$. Consider $m \geq 0$ the integer with the property that $nx \in \left[\frac{2m-1}{2}, \frac{2m+1}{2}\right)$. Using the inequality we get

$$\ln(1+k) - \ln(1+nx) \leq \frac{k-nx}{1+nx},$$

and denoting

$$c_k = \frac{\frac{(nx)^k}{k!}}{\frac{(nx)^m}{m!}} \left(\frac{n+k}{n(1+x)}\right)^{\alpha} \frac{k-nx}{1+nx},$$

it remains to prove that

$$\bigvee_{k=m+1}^{\infty} c_k \leq \left(1 + \frac{2\alpha}{n}\right)^{\alpha}.$$

Let us take the inequality

$$\frac{c_{k+1}}{c_k} = \frac{nx}{k+1} \left(\frac{n+k+1}{n+k}\right)^{\alpha} \frac{k+1-nx}{k-nx} \geq 1$$

if and only if

$$\alpha_k (nx)^2 - (\alpha_k(k+1) + k + 1)(nx) + k(k+1) \leq 0,$$

where $\alpha_k = \left(\frac{n+k+1}{n+k}\right)^{\alpha}$. Hence, $c_{k+1} \geq c_k$ is true if and only if $nx \in [r_k, s_k]$, where

$$r_k = \frac{k+1}{2} + \frac{k+1}{2\alpha_k} - \sqrt{E_k}, \quad s_k = \frac{k+1}{2} + \frac{k+1}{2\alpha_k} + \sqrt{E_k}$$

and

$$E_k = \frac{(\alpha_k(k+1) + k + 1)^2 - 4\alpha_k k(k+1)}{4\alpha_k^2}.$$

Now, we prove below that

$$0 < r_k < r_{k+1} < \frac{k+2}{2} < s_k. \tag{2.2}$$

By using (2.2) we deduce that $r_m < \frac{m}{2} \leq nx$. Let us take the unique $j \geq m$ such that $x \in [r_j, r_{j+1})$. For every $k \geq j+1$, we get $nx \in [r_k, s_k]$, so $c_{k+1} \leq c_k$. We obtain that $c_k \leq c_{j+1}$ for every $k \geq j+1$. Now consider $k \in \{m, \ldots, j\}$. Using (2.2) again, we get $r_k \leq r_j \leq nx < \frac{m+2}{2} < \frac{k+2}{2} < s_k$. Since $nx \in [r_k, s_k]$, we obtain $c_{k+1} \geq c_k$ and so $c_{j+1} \geq c_k$ for every $k \in \{m, \ldots, j\}$.

Now, we need some estimates to evaluate the maximum of c_k for $k \geq m+1$. We have

$$\begin{aligned} j+1-r_j =& j+1-\frac{j+1}{2}-\frac{j}{2\alpha_j}+\sqrt{E_k} \\ =& \frac{\alpha_j(j+1)-j}{2\alpha_j}+\sqrt{\frac{(\alpha_k(k+1)+k)^2-4\alpha_k k^2}{4\alpha_k^2}} \\ \leq& \frac{n^2+1}{2n}, \end{aligned}$$

because

$$\alpha_j(j+1)=(j+1)\left(e^{\alpha\ln\left(1+\frac{1}{n+j}\right)}\right)\leq (j+1)\alpha\frac{1}{n+j}\leq\alpha.$$

Consequently,

$$\begin{aligned} \bigvee_{k=m+1}^{\infty} c_k =& c_{j+1} \\ =& \frac{\frac{(nx)^{j+1}}{(j+1)!}}{\frac{(nx)^m}{m!}}\left(\frac{n+j+1}{n+nx}\right)^{\alpha}\frac{(j+1-nx)}{1+nx} \\ \leq& \left(\frac{n+j+1}{n+r_j}\right)^{\alpha}\frac{j+1-r_j}{1+r_j}\leq\left(1+\frac{2\alpha}{n}\right)^{\alpha}. \end{aligned}$$

Let us consider the inequality (2.2). The most difficult to prove is the inequality $r_{k+1}>r_k$, other statuses as in [6]. We have

$$\begin{aligned} r_{k+1}-r_k =& \frac{k+2}{2}+\frac{k+2}{2\alpha_{k+1}}-\sqrt{E_{k+1}}-\frac{k+1}{2}+\frac{k+1}{2\alpha_k}-\sqrt{E_k} \\ =& \frac{1}{2}\cdot\frac{(k+2)\alpha_k-(k+1)(\alpha_{k+1})}{2\alpha_k\alpha_{k+1}}+\frac{E_k-E_{k+1}}{\sqrt{E_k}-\sqrt{E_{k+1}}} \\ =& \frac{\alpha_k(k+2)\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)-(k+1)\alpha_{k+1}\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)}{4\alpha_k\alpha_{k+1}} \\ &+\frac{(E_k-E_{k+1})\,4\alpha_k\alpha_{k+1}}{4\alpha_k\alpha_{k+1}\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)}. \end{aligned}$$

This equality's first half is positive, it is clear. For positivity of the second part of the equality, let us take $\frac{k+2}{2}<s_k$ and $r_{k+1}<\frac{k+2}{2}$ from (2.2), then we get

$$\sqrt{E_k}>\frac{\alpha_k-k}{2\alpha_k},\quad \sqrt{E_{k+1}}>\frac{k+1}{2\alpha_{k+1}}.$$

Hence, we proved the lemma. □

Lemma 2.5. *For every $x \geq 0$ and $n \in \mathbb{N}$, we get*

$$F_n\left(\max\left\{(1+t)^{\alpha},(1+x)^{\alpha}\right\}\left|\ln(1+t)-\ln(1+x)\right|,x\right)\leq\max\left\{\left(1+\frac{2\alpha}{n}\right)^{\alpha},\frac{1}{2}\right\}(1+x)^{\alpha}.$$

Proof. We have

$$F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}\left|\ln(1+t)-\ln(1+x)\right|,x\right)=\max\left\{A_n,B_n\right\},$$

where

$$\begin{aligned}A_n=&\frac{\bigvee_{k>nx}\frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha(\ln(1+k)-\ln(1+nx))}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}},\\ B_n=&\frac{\bigvee_{k\le nx}\frac{(nx)^k}{k!}(1+x)^\alpha(\ln(1+nx)-\ln(1+k))}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}}.\end{aligned}$$

By Lemma 2.4 we have $\left(1+\frac{2\alpha}{n}\right)^\alpha$ and by Lemma 2.3, $\frac{1}{2}(1+x)^\alpha$. □

Theorem 2.1. *For $\varphi(x)$, for every $f\in C_{\varphi,\alpha}$ the estimation of the error of uniform approximation by F_n is bounded by*

$$\|F_nf-f\|_{\varphi,\alpha}\le\left(\left(1+\frac{2\alpha}{n}\right)^\alpha(1+x)^\alpha+\frac{1}{2}(1+x)^\alpha\right)\omega_{\varphi,\alpha}\left(f,\frac{1}{\sqrt{n}}\right),$$

for every $n\in\mathbb{N}$.

Proof. Because $F_n(1,x)=1$, using ([3], Lemma 2.1) we get

$$\begin{aligned}|F_n(f;x)-f(x)|\le&F_n\left(|f(t)-f(x)|,x\right)\\ \le&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}\left(1+\frac{|\varphi(t)-\varphi(x)|}{\delta_n}\right),x\right)\omega_{\varphi,\alpha}(f,\delta_n)\\ \le&\left(C_n+\frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f,\delta_n),\end{aligned}$$

which

$$\begin{aligned}C_n(x)=&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right),\\ D_n(x)=&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}|\varphi(t)-\varphi(x)|,x\right).\end{aligned}$$

Using Remark 2.2 and Lemma 2.5 and choosing $\delta_n=\frac{1}{\sqrt{n}}$, we have

$$\frac{1}{(1+x)^\alpha}|F_n(f;x)-f(x)|\le\left(\left(1+\frac{2\alpha}{n}\right)^\alpha+\frac{1}{2}\right)\omega_{\varphi,\alpha}\left(f,\frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

Remark 2.4. Let us take consideration that for polynomial weighted of the operator given in (1.2) the order of approximation is better that $\frac{1}{\sqrt{n}}$. From [3], we deduce that the estimate

$$|F_n(f,x)-f(x)|\le\frac{M}{n},\quad n\ge1,$$

is true for a positive, increasing, concave and Lipschitz function f, which is not necessarily bounded.

Theorem 2.2. *For $f \in C_{\varphi,\alpha}$ we have*

$$\|S_n f - f\|_{\varphi,\alpha} \leq C_\alpha \cdot \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $n \in \mathbb{N}$, where $C > 0$ is a constant.

Proof. We know that $S_n(e^{\alpha\sqrt{t}}, x) \leq M_\alpha \cdot e^{\alpha\sqrt{x}}$, which $M_\alpha > 0$ is a constant depending only on α in [1], Lemma 3.1 and in [7], similar inequality given for $\alpha = 0$. We get

$$\begin{aligned}|S_n(f,x) - f(x)| \leq & S_n\left(|f(t) - f(x)|, x\right)\\ \leq & S_n\left(((1+t)^\alpha + (1+x)^\alpha)\left(1 + \frac{\varphi(t) - \varphi(x)}{\delta_n}\right), x\right)\omega_{\varphi,\alpha}(f, \delta_n)\\ \leq & \left(C_n(x) + \frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f, \delta_n),\end{aligned}$$

where

$$\begin{aligned}C_n(x) =& S_n\left((1+t)^\alpha + (1+x)^\alpha, x\right) \leq (M+1)(1+x)^\alpha,\\ D_n(x) =& S_n\left(((1+t)^\alpha + (1+x)^\alpha)\left|\varphi(t) - \varphi(x)\right|, x\right)\\ =& S_n\left((1+t)^\alpha\left|\varphi(t) - \varphi(x)\right|, x\right) + (1+x)^\alpha S_n\left(\left|\varphi(t) - \varphi(x)\right|, x\right).\end{aligned}$$

Using the Cauchy-Schwarz inequality $|S_n(fg,x)| \leq \sqrt{S_n(f^2,x)} \cdot \sqrt{S_n(g^2,x)}$ and the estimation $S_n\left(\left|\varphi(t) - \varphi(x)\right|^2, x\right) \leq \frac{1}{n}$ (see the proof of Corollary 3.2 and Remark 3.3 from [1]) we get

$$\begin{aligned}D_n(x) \leq & \sqrt{S_n\left((1+t)^{2\alpha}, x\right)}\sqrt{S_n\left(\left|\varphi(t) - \varphi(x)\right|^2, x\right)} + (1+x)^\alpha S_n\left(\left|\varphi(t) - \varphi(x)\right|, x\right)\\ \leq & \sqrt{(M_{2\alpha}(1+t))^{2\alpha}}\frac{1}{\sqrt{n}} + (1+x)^\alpha\frac{1}{\sqrt{n}} = \left(\sqrt{M_{2\alpha}} + 1\right)(1+x)^\alpha\frac{1}{\sqrt{n}}.\end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{n}}$

$$\frac{1}{(1+x)^\alpha}|S_n(f,x) - f(x)| \leq C_\alpha \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

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CRITICAL EXPONENTS CURVE FOR SEMILINEAR SYSTEM OF WEAKLY COUPLED EFFECTIVELY DAMPED WAVES WITH DIFFERENT POWER NONLINEARITIES

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Abstract. In this paper we prove a blow-up result for the semi linear system of weakly coupled effectively damped waves with different power nonlinearities

$$u_{tt} - \Delta u + b(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b(t)v_t = |u|^q,$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x),$$

where $b(t)$ will be explained in detail in the next sections. We apply the so called "test function method" to determine the range for the exponents $p, q > 0$ in the nonlinear terms in which local in time existence may not globally prolonged with respect to the t variable under suitable integral sign assumptions for the Cauchy data u_0, u_1, v_0, v_1. Since we prove the blow-up in a complementary range for powers of the nonlinear terms to that for the global existence of small data solutions (see [7]), the main blow-up of this paper is optimal.

1. Introduction

The sharpness of the results for the global (in time) existence of small data solutions or the notion of "blow-up of local (in time) solutions" means that if the pivotal condition for the global (in time) existence is not satisfied, then the solution does, in general, not exist globally (in time) regardless of the size of the data. Among several methods to prove blow-up results, the test function method is an important method which was introduced in the paper [19] and applied by Zhang for damped waves in [28].

Key words and phrases. Weakly coupled hyperbolic systems, damped wave equations, Cauchy problem, blow up, effective dissipation.

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A fundamental step to use this method consists in the modification of the choice of a suitable scaling for the test function with respect to the time and space variables. In particular, the scaling with respect to t is given by the function $F(R)$, introduced in [3, Definition 2.2] which is strongly related to the coefficient $b(t)$.

Let us consider the Cauchy problem for the classical damped wave equation with power nonlinearity

$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

The nonexistence result for $p = p_{Fuj}(n)$ has been established in [28]. Todorova and Yordanov proved in [26] that $p_{Fuj}(n) = 1 + \frac{2}{n}$ is critical.

In the following we recall an important result which the reader can find in the book of Ebert and Reissig [8]. The proof of Theorem 1.1 explains the basics and the philosophy of the test function method.

Theorem 1.1. *Let* $(u_0, u_1) \in \mathcal{A}_{1,1} = (H^1 \cap L^1) \times (L^2 \cap L^1)$ *satisfy the assumption*

$$\int_{\mathbb{R}^n} \Big(u_0(x) + u_1(x)\Big) dx > 0, \tag{1.2}$$

with $n \geq 1$ *and* $p \in (1, 1 + \frac{2}{n}]$. *Then there exists a unique locally* (*in time*) *defined energy solution* u *to* (1.1) *in* $\mathcal{C}([0,T), H^1) \cap \mathcal{C}^1([0,T), L^2)$ *for some* $T > 0$. *This solution cannot be continued to the interval* $[0,\infty)$ *in time.*

The Cauchy problem (1.1) has also been investigated by many authors [9–17, 20–23, 28, 29].

Let us now consider the weakly coupled system of semilinear classical damped waves

$$\begin{aligned} &u_{tt} - \Delta u + u_t = |v|^p, \quad v_{tt} - \Delta v + v_t = |u|^q, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{1.3}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $p, q \geq 1$ and $pq > 1$. Motivated by some previous papers concerned with the case of the Cauchy problem for a semilinear single equation, the authors in [24] and [25] studied the blow-up behavior of solutions of the system (1.3). In the following theorem we will recall the result of F. Sun and M. Wang published in [25].

Theorem 1.2. *Let* $n \geq 1$. *Assume that* $q \geq p \geq 1$ *and* $\frac{n}{2} \leq \frac{q+1}{pq-1}$. *If the data satisfy*

$$(u_i, v_i) \in [W^{1-i,1}(\mathbb{R}^n) \cap W^{1-i,\infty}(\mathbb{R}^n)]^2, \quad \textit{for } i = 0, 1,$$

and

$$\int_{\mathbb{R}^n} u_i(x) dx > 0, \quad \int_{\mathbb{R}^n} v_i(x) dx > 0, \quad \textit{for } i = 0, 1,$$

then the Sobolev solution (u, v) *of the Cauchy problem* (1.3) *does not exist globally* (*in time*).

2. Blow-up Result for Weakly Coupled Systems of Semilinear Damped Waves with Different Coefficients in the Dissipation Terms

Firstly, let us consider the Cauchy problem for a semilinear classical damped wave equation, namely

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{2.1}$$

where the dissipation term $b(t)u_t$ is supposed to be effective in the sense of Wirth [27]. The damping term $b(t)u_t$ is called effective in the model (2.1) if $b = b(t)$ satisfies the following properties:

- b is a positive and monotonic function with $tb(t) \to \infty$ as $t \to \infty$;
- $((1+t)^2 b(t))^{-1} \in L^1(0,\infty)$;
- $b \in \mathcal{C}^3[0,\infty)$ and $|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}$ for $k = 1,2,3$;
- $\frac{1}{b} \notin L^1(0,\infty)$ and there exists a constant $a \in [0,1)$ such that $tb'(t) \leq ab(t)$.

Typical examples are

$$b(t) = \frac{\mu}{(1+t)^r}, \quad b(t) = \frac{\mu}{(1+t)^r}(\log(e+t))^\gamma, \quad b(t) = \frac{\mu}{(1+t)^r(\log(e+t))^\gamma},$$

for some $\mu > 0$, $\gamma > 0$ and $r \in (-1,1)$.

We introduce for $m \in [1,2)$ the function space

$$\mathcal{A}_{m,1} := (H^1 \cap L^m) \times (L^2 \cap L^m),$$

with the norm

$$\|(u,v)\|_{\mathcal{A}_{m,1}} := \|u\|_{H^1} + \|u\|_{L^m} + \|v\|_{L^2} + \|v\|_{L^m}.$$

We denote by $B(t,0)$ the primitive of $1/b(t)$ which vanishes at $t = 0$, that is,

$$B(t,0) := \int_0^t \frac{1}{b(r)} dr.$$

In [2] the authors determined the critical exponent $p = p_{Fuj}(n) := 1 + \frac{2}{n}$. That means after proving the global existence for some admissible range $p > p_{Fuj}(n)$, the authors proved also that, in general, the solution cannot be globally defined for $1 < p \leq p_{Fuj}(n)$ under suitable sign assumptions for the Cauchy data. In other words, we have, in general, only local solutions (in time). The case $b(t) = \frac{\mu}{(1+t)^r}$ with $\mu > 0$ and $r > 0$ was studied in [18].

Let us consider now the Cauchy problem for the following system:

$$\begin{aligned} &u_{tt} - \Delta u + b(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b(t)v_t = |u|^q, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{2.2}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^n$. As we already remarked during the treatment of the models (1.3) and (1.1) the test function method is not influenced by higher regularity of the data. We restrict ourselves to prove the sharpness of our results for the Cauchy problem (2.2), where the data are supposed to belong to the energy space $\mathcal{A}_{1,1} := (H^1 \cap L^1) \times (L^2 \cap L^1)$.

In [7] the authors proved the global (in time) existence of small data solution to (2.2), which means that the solution exists globally for

$$\frac{n}{2} > \frac{\max\{p;q\}+1}{pq-1}.$$

Theorem 2.1 ([7]). *Let $n \leq \frac{2m^2}{2-m}$ and $n < \frac{2m}{m-1}$. The data (u_0,u_1), (v_0,v_1) are supposed to belong to $\mathcal{A}_{m,1}\times\mathcal{A}_{m,1}$ with $m\in[1,2)$. Finally, the exponents p and q satisfy the assumptions*

$$\frac{2}{m} \leq \min\{p;q\} < p_{Fuj,m}(n) < \max\{p;q\}, \quad \text{if } n \leq 2, \tag{2.3}$$

$$\frac{2}{m} \leq \min\{p;q\} < p_{Fuj,m}(n) < \max\{p;q\} \leq p_{GN}(n), \quad \text{if } n > 2, \tag{2.4}$$

and

$$m\left(\frac{\max\{p;q\}+1}{pq-1}\right) < \frac{n}{2}.$$

Then there exists a small constant ϵ_0 such that if

$$\|(u_0,u_1)\|_{\mathcal{A}_{m,1}} + \|(v_0,v_1)\|_{\mathcal{A}_{m,1}} \leq \epsilon_0,$$

then there exists a uniquely determined globally (in time) energy solution to (2.2) in

$$\Big(\mathcal{C}([0,\infty),H^1)\cap\mathcal{C}^1([0,\infty),L^2)\Big)^2.$$

In the following we will prove the optimality of our results from Theorem 2.1. That means, if

$$\frac{n}{2} \leq \frac{\max\{p;q\}+1}{pq-1},$$

then, under suitable integral sign assumptions on the initial data, the local (in time) energy solution cannot be extended globally. The ideas of the proof of the following theorem are based on the paper [3] which is devoted to study a general case of model (2.1).

Theorem 2.2. *Let $b=b(t)$ such that $b(t)u_t$, $b(t)v_t$ are effective dissipation terms. Moreover, let*

$$\liminf_{t\longrightarrow\infty}\frac{b'(t)}{b(t)^2} > -1, \quad \limsup_{t\longrightarrow\infty}\frac{tb'(t)}{b(t)} < 1,$$

and let p,q such that

$$\frac{n}{2} \leq \frac{\max\{p,q\}+1}{pq-1},$$

where $pq>1$. Then there exists no global classical solution $(u,v)\in\left(\mathcal{C}^2([0,\infty)\times\mathbb{R}^n)\right)^2$ to (2.2) with initial data $((u_0,u_1),(v_0,v_1))\in\mathcal{A}_{1,1}\times\mathcal{A}_{1,1}$ such that

$$\begin{aligned}\int_{\mathbb{R}^n} u_0(x)+\hat{b}_1^{-1}u_1(x)dx &> 0,\\ \int_{\mathbb{R}^n} v_0(x)+\hat{b}_1^{-1}v_1(x)dx &> 0,\end{aligned} \tag{2.5}$$

where $\hat{b}_1$ *is defined in* (2.6).

Before proving this theorem we show the following lemma which will be used later in the proof.

Lemma 2.1. *Let* $g = g(t) \in \mathcal{C}([0,\infty))$ *be a solution of the following initial value problem for an ordinary differential equation*

$$-g'(t) + g(t)b(t) = 1, \quad g(0) = \frac{1}{\hat{b}_1}. \tag{2.6}$$

If $b = b(t)$ *satisfies the assumptions of Theorem* 2.2, *then it holds* $g(t) \approx \frac{1}{b(t)}$ *and*

$$|g'(t) - 1| \leq C. \tag{2.7}$$

The proof of Lemma 2.1 can be concluded from [3] and [18].

Proof. For the sake of brevity we assume that $q > p$. We multiply (2.2) by the positive function $g = g(t)$ which is defined in Lemma 2.1. In this way we obtain

$$\begin{aligned}(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + (-g'(t) + g(t)b(t))u_t &= g(t)|v|^p,\\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + (-g'(t) + g(t)b(t))v_t &= g(t)|u|^q.\end{aligned}$$

From the definition of $g = g(t)$ we may conclude

$$\begin{aligned}(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + u_t &= g(t)|v|^p,\\ (g(t)v)_{tt} - \Delta(g(t)v) - (g'(t)v)_t + v_t &= g(t)|u|^q.\end{aligned}$$

We introduce the test functions $\eta \in \mathcal{C}_0^\infty[0,\infty)$ with $0 \leq \eta(t) \leq 1$, where

$$\eta(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \frac{1}{2},\\ 0, & \text{for } t \geq 1,\end{cases}$$

$\phi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with $0 \leq \phi(x) \leq 1$, where

$$\phi(x) = \begin{cases} 1, & \text{for } 0 \leq |x| \leq \frac{1}{2},\\ 0, & \text{for } |x| \geq 1.\end{cases}$$

Moreover, one can choose test functions η, ϕ and $1 < \alpha, \beta, \alpha', \beta' < p$ such that

$$\max\left\{\frac{|\eta'(t)|^\beta}{\eta(t)}; \frac{|\eta''(t)|^\alpha}{\eta(t)}\right\} \leq C, \quad \text{for } \frac{1}{2} \leq t \leq 1,$$

and

$$\max\left\{\frac{|\nabla \phi(x)|^{\beta'}}{\phi(x)}; \frac{|\Delta\phi(x)|^{\alpha'}}{\phi(x)}\right\} \leq C, \quad \text{for } \frac{1}{2} < |x| < 1,$$

where we choose $1 < \alpha, \beta, \alpha', \beta' < \min\{p; q\}$. Let R be a large parameter in $[0,\infty)$ and

$$Q_R := [0, F(R)] \times B_R, \quad B_R := \{x \in \mathbb{R}^n : |x| \leq R\}.$$

We define the test function

$$\psi_R(t,x) := \eta_R(t)\phi_R(x) = \eta\Big(\frac{t}{F(R)}\Big)\phi\Big(\frac{x}{R}\Big),$$

where $F(R) = B^{-1}(R^2, 0)$ and $B^{-1}(t, 0)$ is the inverse function of $B(t, 0)$. It follows that $F : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $F(0) = 0$ and $\lim_{R\to\infty} F(R) = \infty$. Moreover, we have $R \lesssim F(R)$ as a result of $b(t) \gtrsim (1 + t)^{-1}$.

We have after integrating by parts

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) = -\int_{B_R} (u_0 + \hat{b}_1^{-1}u_1)\psi_R dx + \int_{Q_R} \Big(g(t)u\partial_t^2\psi_R + (g'(t)-1)u\partial_t\psi_R + g(t)u\Delta\psi_R\Big)d(t,x)$$

and

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) = -\int_{B_R} (v_0 + \hat{b}_1^{-1}v_1)\psi_R dx + \int_{Q_R} \Big(g(t)v\partial_t^2\psi_R + (g'(t)-1)v\partial_t\psi_R + g(t)v\Delta\psi_R\Big)d(t,x).$$

For sufficiently large R, thanks to (2.5), this implies

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim \int_{Q_R} \Big|g(t)u\partial_t^2\psi_R + (g'(t)-1)u\partial_t\psi_R + g(t)u\Delta\psi_R\Big|\, d(t,x)$$

and

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim \int_{Q_R} \Big|g(t)v\partial_t^2\psi_R + (g'(t)-1)v\partial_t\psi_R + g(t)v\Delta\psi_R\Big|\, d(t,x).$$

Using Lemma 2.1, Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$ and (2.7) we get

$$\int_{Q_R} |ug(t)\partial_t^2\psi_R|d(t,x) \le \Big(\int_{Q_R} |u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}} \Big(\int_{Q_R} \psi_R^{-\frac{q'}{q}} g(t)|\partial_t^2\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}}, \tag{2.8}$$

$$\int_{Q_R} |u(g'(t)-1)\partial_t\psi_R|d(t,x) \le \Big(\int_{Q_R} |u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}} \Big(\int_{Q_R} g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}} |\partial_t\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}}, \tag{2.9}$$

$$\int_{Q_R} |ug(t)\Delta\psi_R|d(t,x) \le \Big(\int_{Q_R} |u|^q g(t)\psi_R d(t,x)\Big)^{\frac{1}{q}} \Big(\int_{Q_R} \psi_R^{-\frac{q'}{q}} g(t)|\Delta\psi_R|^{q'} d(t,x)\Big)^{\frac{1}{q'}}. \tag{2.10}$$

We apply a change of variables $t = F(R)\tau$ and $x = Ry$. Then we have

$$d(t,x) = F(R)R^n d(\tau,y), \quad \partial_t\psi_R = F(R)^{-1}\partial_\tau\psi_R, \quad \partial_t^2\psi_R = F(R)^{-2}\partial_\tau^2\psi_R,$$

and

$$\Delta_x\psi_R = R^{-2}\Delta_y\psi_R, \frac{F(R)}{2} \le t \le F(R), \frac{R}{2} \le |x| \le R \Leftrightarrow \frac{1}{2} \le \tau, |y| \le 1.$$

With this change of variables we get for (2.8) the chain of inequalities

$$\begin{aligned}
&\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\\
=&\Big(\int_{\frac12}^1\int_{\frac12}^1\psi_R^{-\frac{q'}{q}}g(t)(F(R)\tau)F(R)^{-2q'}|\psi_R|^{\frac{q'}{\alpha}}F(R)R^n d\tau dy\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}g(t)dt\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}\frac{1}{b(t)}dt\Big)^{\frac{1}{q'}}\\
\lesssim&\Big(F(R)^{-2q'}R^nB(F(R),0)\Big)^{\frac{1}{q'}}\\
\lesssim&F(R)^{\frac{n+2-2q'}{q'}}.
\end{aligned}$$

Consequently, we arrive at

$$\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\partial_t^2\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.11}$$

In the same way we can prove for (2.10) the estimate

$$\Big(\int_{Q_R}\psi_R^{-\frac{q'}{q}}g(t)|\Delta\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.12}$$

Finally, let us turn to (2.9). We have

$$\begin{aligned}
\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}&\lesssim\Big(F(R)^{-q'}\int_{Q_R}b(t)^{q'-1}\psi_R^{-\frac{q'}{q}}|\psi_R|^{\frac{q'}{\beta}}d(t,x)\Big)^{\frac{1}{q'}}\\
&\lesssim\Big(F(R)^{-q'}R^n\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{q'-1}dt\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Since $F(0)=0$ and

$$F'(R)=(B^{-1}(R^2,0))'=\frac{2R}{B'(F(R))}=2Rb(F(R)),$$

using $b(t)\approx b(\frac{t}{2})$ and $B(t,0)-B(\frac{t}{2},0)\approx B(t,0)$ from [2, Remark 4.1], we get

$$\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{q'-1}dt\approx(b(F(R)))^{q'}\int_{\frac{F(R)}{2}}^{F(R)}b(t)^{-1}dt\approx(b(F(R)))^{q'}R^2.$$

Moreover, we have

$$\frac{b(F(R))}{F(R)}\approx\frac{1}{B(F(R),0)}=R^{-2}.$$

Finally, we obtain

$$\Big(\int_{Q_R}g(t)b(t)^{q'}\psi_R^{-\frac{q'}{q}}|\partial_t\psi_R|^{q'}d(t,x)\Big)^{\frac{1}{q'}}\lesssim F(R)^{\frac{n+2-2q'}{q'}}. \tag{2.13}$$

Consequently, from (2.11) to (2.13) we get

$$\int_{Q_R} g(t)|v|^p\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2q'}{q'}}\Big(\int_{Q_R}|u|^q g\psi_R d(t,x)\Big)^{\frac{1}{q}}. \tag{2.14}$$

Analogously, one can get also

(2.15)

$$\int_{Q_R} g(t)|u|^q\psi_R d(t,x) \lesssim F(R)^{\frac{n+2-2p'}{p'}}\Big(\int_{Q_R}|v|^p g\psi_R d(t,x)\Big)^{\frac{1}{p}}, \quad \text{where } \frac{1}{p}+\frac{1}{p'}=1.$$

From (2.14) and (2.15) we obtain

$$\Big(\int_{Q_R} g(t)|v|^p\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \le F(R)^{s_1},$$

$$\Big(\int_{Q_R} g(t)|u|^q\psi_R d(t,x)\Big)^{\frac{pq-1}{pq}} \le F(R)^{s_2}, \tag{2.16}$$

where

$$s_1 = \frac{n+2}{q'}-2+\Big(\frac{n+2}{p'}-2\Big)\frac{1}{q} \quad \text{and} \quad s_2 = \frac{n+2}{p'}-2+\Big(\frac{n+2}{q'}-2\Big)\frac{1}{p}.$$

The assumption $\frac{n}{2} \le \frac{q+1}{pq-1}$ implies that $s_2 \le 0$. We consider two cases.

- If $s_2 < 0$, then letting $R \to \infty$ in the inequality (2.16) we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} g(t)|u|^q d(t,x) = 0.$$

  This implies $u \equiv 0$. This is a contradiction to the assumptions.
- If $s_2 = 0$, then there exists a positive number R_0 such that

$$\int_\Omega g(t)|u|^q\psi_R d(t,x) \le R_0,$$

  where $\Omega = \{(t,x) \in [0,\infty)\times\mathbb{R}^n : \frac{F(R)}{2} \le t \le F(R), \frac{R}{2} \le |x| \le R\}$. From $\partial_t\psi_R = \partial_{tt}\psi_R = \Delta\psi_R = 0$ for $(t,x) \in Q_R \setminus \Omega$, one can prove similarly to (2.14) and (2.15) the following estimates:

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|v|^p\psi_R d(t,x) + \int_{B_R}(u_0+\hat{b}_1^{-1}u_1)\psi_R dx \lesssim F(R)^{\frac{n+2-2q'}{q'}}\Big(\int_\Omega |u|^q g\psi_R d(t,x)\Big)^{\frac{1}{q}},$$

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u|^q\psi_R d(t,x) + \int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\psi_R dx \lesssim F(R)^{\frac{n+2-2p'}{p'}}\Big(\int_\Omega |v|^p g\psi_R d(t,x)\Big)^{\frac{1}{p}}.$$

  Last estimates for $s_2 = 0$ leads to

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u|^q d(t,x) + \int_{B_R}(v_0+\hat{b}_1^{-1}v_1)\psi_R dx \lesssim 0,$$

  for $R \to \infty$. This is also a contradiction. The proof is completed. □

3. Concluding Remarks

Recently, in [1] the author proved the blow-up of solutions for a model with constant coefficients considering the additional regularity L^m by taking a lower bound for the initial data $u_0(x) \in L^1_{loc}$ and $u_0(x) \geq \epsilon|x|^{-\frac{n}{m}} \log|x|$. Assuming a similar condition in our case by mixing additional regularities, we get from $\int_{B_R}(u_0 + \hat{b}_1^{-1}u_1)\psi_R(0,x)dx$ and $\int_{B_R}(v_0 + \hat{b}_1^{-1}v_1)\psi_R(0,x)dx$ a lower bound with respect to $R \lesssim F(R)$ after using $\psi_R(0,x) = \phi_R(x)$. This generated R cannot leads to the requested contraction. Finally, this means that the mentioned approach is not suitable for our model.

Assuming the weakly coupled system of semilinear damped waves (2.2) with different coefficients in the dissipation terms $b_1(t)u_t$ and $b_2(t)u_t$.

$$\begin{aligned} &u_{tt} - \Delta u + b_1(t)u_t = |v|^p, \quad v_{tt} - \Delta v + b_2(t)v_t = |u|^q, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{3.1}$$

The global existence (in time) of solutions of this Cauchy problem was treated in [4–7], where the data are defined in different classes of regularity which are the followings: low regular data, data from energy space, data from Sobolev spaces with suitable regularity and, finally, large regular data. The blow-up of (3.1) where $b_1(t) = \frac{\mu}{(1+t)^{r_1}}$, $b_2(t) = \frac{\mu}{(1+t)^{r_2}}$, $r_1, r_2 \in (-1, 1)$, with data from energy space can be treated in a separated forthcoming project.

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INVESTIGATIONS ON A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION AND GRADIENT SOLITONS

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Abstract. This article carries out the investigation of a three-dimensional Riemannian manifold N^3 endowed with a semi-symmetric type non-metric connection. Firstly, we construct a non-trivial example to prove the existence of a semi-symmetric type non-metric connection on N^3. It is established that a N^3 with the semi-symmetric type non-metric connection, whose metric is a gradient Ricci soliton, is a manifold of constant sectional curvature with respect to the semi-symmetric type non-metric connection. Moreover, we prove that if the Riemannian metric of N^3 with the semi-symmetric type non-metric connection is a gradient Yamabe soliton, then either N^3 is a manifold of constant scalar curvature or the gradient Yamabe soliton is trivial with respect to the semi-symmetric type non-metric connection. We also characterize the manifold N^3 with a semi-symmetric type non-metric connection whose metrics are Einstein solitons and m-quasi Einstein solitons of gradient type, respectively.

1. Introduction

In this paper, on a Riemannian manifold N^3, we carry out an investigation of gradient solitons with a semi-symmetric type non-metric connection (briefly, $SSNMC$). Many years ago, on a differentiable manifold, Friedman and Schouten [11] presented the concept of semi-symmetric linear connection. After that in 1932, on a Riemannian manifold, Hayden [15] introduced the notion of metric connection with torsion. In 1970, a systematic investigation of semi-symmetric metric connection which plays a

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significant role in the study of Riemannian manifolds, was conducted by Yano [23]. In this connection, we may mention the work of Zengin et al. [24, 25].

On N^3, a linear connection $\widehat{\nabla}$ is named semi-symmetric if $\widehat{T}$, the torsion tensor defined by

$$\widehat{T}(U,V) = \widehat{\nabla}_U V - \widehat{\nabla}_V U - [U,V] \tag{1.1}$$

obeys

$$\widehat{T}(U,V) = \psi(V)U - \psi(U)V, \tag{1.2}$$

where ψ is a 1-form defined by $\psi(U) = g(U,\xi)$, for a fixed vector field ξ (the associated vector field of $\widehat{\nabla}$). If in the right side of the equation (1.2) we substitute the independent vector fields U and V, respectively, by ϕU and ϕV, where ϕ is a $(1,1)$-tensor field [12], then the connection $\widehat{\nabla}$ transforms into a quarter-symmetric connection.

Again, if a semi-symmetric connection $\widehat{\nabla}$ on N^3 obeys

$$(\widehat{\nabla}_U g)(V,Y) = 0, \tag{1.3}$$

then $\widehat{\nabla}$ is called metric [23]. If $\widehat{\nabla} g \neq 0$, then it is called non-metric [15]. Here, we choose the $SSNMC$, that is, $\widehat{\nabla} g \neq 0$ and the connection $\widehat{\nabla}$ obeys the equation (1.2). The concept of the $SSNMC$ on a Riemannian manifold was investigated in [1]. After that, several researchers investigated the properties of $SSNMC$ on manifolds with different structures (see [6, 10, 18, 19]).

Hamilton [14] introduced the concept of Ricci flow as a solution to the challenge of obtaining a canonical metric on a smooth manifold. Ricci flow occurs when the metric of a Riemannian manifold N^3 is fulfilled by the evolution equation $\frac{\partial}{\partial t} g_{ij}(t) = -2\mathcal{S}_{ij}$, where $\mathcal{S}_{ij}$ and g_{ij} are the components of the Ricci tensor and the metric tensor, respectively. Ricci solitons were created via self-similar solutions to the Ricci flow.

A metric of N^3 is named a Ricci soliton [13] if it fulfills

$$\mathfrak{L}_W g + 2\lambda g + 2\widehat{\mathcal{S}} = 0, \tag{1.4}$$

for some $\lambda \in \mathbb{R}$, the set of real numbers. Here, $\mathfrak{L}$ being the Lie derivative operator and $\widehat{\mathcal{S}}$ is the Ricci tensor with respect to the non-metric connection $\widehat{\nabla}$. W is a complete vector field known as a potential vector field. The Ricci soliton is considered to be shrinking, expanding or steady depending on whether λ is negative, positive, or zero. If W is Killing or zero, the Ricci soliton is trivial and N^3 is Einstein. Also, if $W = Df$ for some smooth function f, then equation (1.4) turns into

$$\widehat{\nabla}^2 f + \widehat{\mathcal{S}} + \lambda g = 0, \tag{1.5}$$

where $\widehat{\nabla}^2$ and D indicate the Hessian and the gradient operator of g, respectively. The metric obeying the equation (1.5) is called a gradient Ricci soliton. Here, f is said to be the potential function of the gradient Ricci soliton.

On a complete Riemannian manifold N^3, Hamilton [14] proposed the idea of Yamabe flow, which was inspired by Yamabe's conjecture ("metric of a complete Riemannian manifold is conformally connected to a metric with constant scalar curvature"). A

Riemannian manifold N^3 equipped with a Riemannian metric g is called a Yamabe flow if it obeys:

$$\frac{\partial}{\partial t}g(t) + rg(t) = 0, \quad g_0 = g(t), \tag{1.6}$$

where t indicates the time and r being the scalar curvature of N^3. A Riemannian manifold N^3 equipped with a Riemannian metric g is named a Yamabe soliton if it fulfills

$$\mathfrak{L}_W g - 2(\widehat{r} - \lambda)g = 0, \tag{1.7}$$

for real constant $\lambda : M \to \mathbb{R}$ and $\widehat{r}$ is the scalar curvature with respect to the non-metric connection $\widehat{\nabla}$. Here, W is called the potential vector field. In N^3, with the condition $W = Df$, the Yamabe soliton reduces to the gradient Yamabe soliton. Thus, (1.7) takes the form

$$\widehat{\nabla}^2 f - (\widehat{r} - \lambda)g = 0. \tag{1.8}$$

If f is constant (or, W is Killing) on M, then the soliton becomes trivial. The 3-Kenmotsu manifolds and almost co-Kähler manifolds with Yamabe solitons have been characterized by Wang [21] and Suh and De [20], respectively. Chen and Deshmukh [5, 9] studied the Yamabe solitons on Riemannian manifolds. Some interesting outcomes on this solitons have been investigated in [2, 3, 7, 8, 17] and also by others.

The notion of gradient Einstein soliton was presented in [4] and obeys

$$\widehat{\mathcal{S}} - \frac{1}{2}\widehat{r}g + \widehat{\nabla}^2 f + \lambda g = 0, \tag{1.9}$$

where $\lambda \in \mathbb{R}$ is a constant and f indicates a smooth function.

A Riemannian manifold N^3 endowed with the Riemannian metric g is named a gradient m-quasi Einstein metric [4] if there exists a constant λ, a smooth function $f : N^3 \to \mathbb{R}$ and obeys

$$\widehat{\mathcal{S}} - \lambda g + \widehat{\nabla}^2 f - \frac{1}{m} df \otimes df = 0, \tag{1.10}$$

where $\otimes$ indicate the tensor product and m is an integer. In this case f being the m-quasi Einstein potential function [4]. Here, the gradient m-quasi Einstein soliton is expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$. If $m = \infty$, the foregoing equation represents a gradient Ricci soliton and the metric represents almost gradient Ricci soliton if it obeys the condition $m = \infty$ and λ is a smooth function. Few characterizations of the above metrics were characterized by He et al. [16].

The foregoing investigations motivate us to study the Riemannian manifold N^3 endowed with a $SSNMC$.

The content of the paper is laid out as: In Section 2, we produce the preliminary ideas of $SSNMC$. The existence of a $SSNMC$ on a Riemannian manifold are established in Section 3. The gradient Ricci soliton on N^3 equipped with a $SSNMC$ is investigated in Section 4. Section 5 concerns with gradient Yamabe soliton on N^3 with a $SSNMC$. We study the properties of N^3 with a $SSNMC$ whose metrics are gradient Einstein solitons and gradient m-quasi Einstein solitons, in Section 6 and Section 7, respectively.

2. Semi-Symmetric Non-Metric Connection

A linear connection $\widehat{\nabla}$ on N, defined by

$$\widehat{\nabla}_U V = \nabla_U V + \psi(V)U, \tag{2.1}$$

∇ being the Levi-Civita connection, is a $SSNMC$. It also obeys

$$(\widehat{\nabla}_U g)(V, Y) = -\psi(V)g(U, Y) - \psi(Y)g(U, V). \tag{2.2}$$

Then $\widehat{R}$, the curvature tensor with respect to the $SSNMC$, $\widehat{\nabla}$, and R, the Riemannian curvature tensor are related by [1]

$$\widehat{R}(U, V)Y = R(U, V)Y - \alpha^*(V, Y)U + \alpha^*(U, Y)V, \tag{2.3}$$

for all U, V, Y on N^3, where α^* is a $(0, 2)$-tensor field defined by

$$\alpha^*(U, V) = (\nabla_U \psi)(V) - \psi(U)\psi(V). \tag{2.4}$$

Throughout this article, we choose that the vector field ξ is a unit parallel vector field with respect to the Levi-Civita connection ∇. Then $\nabla_U \xi = 0$, which immediately implies

$$R(U, V)\xi = 0 \tag{2.5}$$

and

$$\mathcal{S}(U, \xi) = 0. \tag{2.6}$$

Also, using $\nabla_U \xi = 0$, we obtain

$$(\nabla_U \psi)V = 0. \tag{2.7}$$

Hence, by the preceding equation, we get from (2.3)

$$\widehat{R}(U, V)Y = R(U, V)Y + \psi(Y)[\psi(V)U - \psi(U)V]. \tag{2.8}$$

From the foregoing equation, we can easily have

$$\widehat{\mathcal{S}}(U, V) = \mathcal{S}(U, V) + 2\psi(U)\psi(V). \tag{2.9}$$

Contracting the above equation, we lead

$$\widehat{r} = r - 2, \tag{2.10}$$

since $\psi(\xi) = g(\xi, \xi) = 1$. Making use of (2.5), we infer from (2.8)

$$\widehat{R}(U, V)\xi = \psi(V)U - \psi(U)V. \tag{2.11}$$

Therefore, we obtain the subsequent relations

$$\psi(\widehat{R}(U, V)Y) = 0, \tag{2.12}$$

$$\widehat{\mathcal{S}}(U, \xi) = 2\psi(U), \quad \widehat{Q}\xi = 2\xi. \tag{2.13}$$

We first establish the subsequent lemma.

Lemma 2.1. *Let N^3 be a Riemannian manifold with a SSNMC, $\widehat{\nabla}$. Then we have*

$$\xi\widehat{r} = 0. \tag{2.14}$$

Proof. In N^3, the Riemannian curvature tensor is expressed by

$$\begin{aligned} R(U,V)Y =& g(V,Y)QU - g(U,Y)QV + \mathcal{S}(V,Y)U - \mathcal{S}(U,Y)V \\ & - \frac{r}{2}[g(V,Y)U - g(U,Y)V]. \end{aligned} \tag{2.15}$$

Making use of (2.8) and (2.9), we acquire

$$\begin{aligned} &\widehat{R}(U,V)Y - \psi(Y)[\psi(V)U - \psi(U)V] \\ =& g(V,Y)[\widehat{Q}U - 2\xi\psi(U)] - g(U,Y)[\widehat{Q}V - 2\xi\psi(V)] + [\widehat{\mathcal{S}}(V,Y) - 2\psi(V)\psi(Y)]U \\ & - [\widehat{\mathcal{S}}(U,Y) - 2\psi(U)\psi(Y)]V - \frac{r}{2}[g(V,Y)U - g(U,Y)V]. \end{aligned} \tag{2.16}$$

Putting $V = Y = \xi$, the foregoing equation yields

$$\widehat{Q}U = \left(\frac{\widehat{r}}{2} + 1\right)U - \left(\frac{\widehat{r}}{2} - 1\right)\psi(U)\xi. \tag{2.17}$$

Taking covariant derivative along V, we write

$$(\nabla_V\widehat{Q})U = \frac{(V\widehat{r})}{2}[U - \psi(U)\xi]. \tag{2.18}$$

Contracting the foregoing equation we acquire the desired result. □

The projective curvature tensor $\widehat{P}$ of N^3 with respect to $\widehat{\nabla}$ is defined by

$$\widehat{P}(U,V)Y = \widehat{R}(U,V)Y - \frac{1}{2}[\widehat{\mathcal{S}}(V,Y)U - \widehat{\mathcal{S}}(U,Y)V]. \tag{2.19}$$

Making use of (2.8) and (2.9), (2.19) reduces to

$$\widehat{P}(U,V)Y = P(U,V)Y, \tag{2.20}$$

where P represents the projective curvature tensor with respect to the Levi-Civita connection ∇ defined by

$$P(U,V)Y = R(U,V)Y - \frac{1}{2}[\mathcal{S}(V,Y)U - \mathcal{S}(U,Y)V]. \tag{2.21}$$

Theorem 2.1. *If N^3 is endowed with a SSNMC $\widehat{\nabla}$, then the projective curvature tensor with respect to $\widehat{\nabla}$ and ∇, respectively, coincide on N^3.*

In differential geometry, the investigation of conformal curvature tensor performs a significant role. Also, it has various applications in applied physics and the other branches of modern sciences. Motivated by the above facts we investigate the properties of the conformal curvature tensor C. With respect to $\widehat{\nabla}$, the conformal curvature tensor $\widehat{C}$ is defined by

$$\begin{aligned}\widehat{C}(U,V)Y =&\widehat{R}(U,V)Y - [\widehat{\mathcal{S}}(V,Y)U - \widehat{\mathcal{S}}(U,Y)V + g(V,Y)\widehat{Q}U \\ &- g(U,Y)\widehat{Q}V] + \frac{\widehat{r}}{2}[g(V,Y)U - g(U,Y)V],\end{aligned} \tag{2.22}$$

for all U, V and Y on N^3 [22]. Utilizing (2.8) and (2.9) in (2.22), we obtain

$$\begin{aligned}\widehat{C}(U,V)Y =&C(U,V)Y - \widehat{\psi}(V)\widehat{\psi}(Y)U + \widehat{\psi}(U)\widehat{\psi}(Y)V \\ &+ 2\xi g(V,Y)\widehat{\psi}(U) - 2\xi g(U,Y)\widehat{\psi}(V) + g(V,Y)U - g(U,Y)V,\end{aligned} \tag{2.23}$$

where C represents the conformal curvature tensor with respect to the Levi-Civita connection ∇ defined by

$$\begin{aligned}C(U,V)Y =&R(U,V)Y - [\mathcal{S}(V,Y)U - \mathcal{S}(U,Y)V + g(V,Y)QU - g(U,Y)QV] \\ &+ \frac{r}{2}[g(V,Y)U - g(U,Y)V].\end{aligned} \tag{2.24}$$

Putting $Y = \xi$ in (2.23), we get

$$\widehat{C}(U,V)\xi = C(U,V)\xi. \tag{2.25}$$

Hence, we have the subsequent theorem.

Theorem 2.2. *If N^3 is equipped with a SSNMC $\widehat{\nabla}$, then the the conformal curvature tensor with respect to $\widehat{\nabla}$ and ∇, satisfy the relation* (2.25)*.*

3. Existence of a Semi-Symmetric Type Non-Metric Connection

Here we construct a non-trivial example of semi-symmetric type non-metric connection on a Riemannian manifold.

Example 3.1. Let us consider a three-dimensional differentiable manifold $N^3 = \{(u,v,w) \in \mathbb{R}^3, w \neq 0\}$, where (u,v,w) indicates the standard coordinate of $\mathbb{R}^3$. Let us choose

$$k_1 = e^w \frac{\partial}{\partial u}, \quad k_2 = e^w \frac{\partial}{\partial v}, \quad k_3 = \frac{\partial}{\partial w}. \tag{3.1}$$

At each point of N^3 the preceding vector fields are linearly independent. Here we define the Riemannian metric g as

$$\begin{aligned}g(k_1,k_3) =&g(k_1,k_2) = g(k_2,k_3) = 0,\\ g(k_1,k_1) =&g(k_2,k_2) = g(k_3,k_3) = 1,\end{aligned}$$

ψ indicates a 1-form defined by $\psi(U) = g(U, \xi)$, where $\xi = k_3$. Hence, (N^3, g) is a three-dimensional Riemannian manifold. The Lie brackets are calculated as

$$\begin{aligned}[k_1, k_3] &= k_1 k_3 - k_3 k_1 \\ &= e^w \frac{\partial}{\partial u}\left(\frac{\partial}{\partial w}\right) - \left(\frac{\partial}{\partial w}\right)\left(e^w \frac{\partial}{\partial u}\right) \\ &= e^w \frac{\partial^2}{\partial u \partial w} - e^w \frac{\partial^2}{\partial w \partial u} - e^w \frac{\partial}{\partial u} \\ &= -k_1. \end{aligned} \tag{3.2}$$

Similarly,

$$[k_1, k_2] = 0 \quad \text{and} \quad [k_2, k_3] = -k_2. \tag{3.3}$$

∇, the Levi-Civita connection with respect to g, is obtained by

$$\begin{aligned} 2g(\nabla_U V, Y) =& Ug(V, Y) + Vg(Y, U) - Yg(U, V) \\ & - g(U, [V, Y]) - g(V, [U, Y]) + g(Y, [U, V]), \end{aligned} \tag{3.4}$$

which is termed as Koszul's formula.

Making use of (3.4) we have

$$2g(\nabla_{k_1} k_3, k_1) = -2g(k_1, k_1). \tag{3.5}$$

Again by (3.4)

$$2g(\nabla_{k_1} k_3, k_2) = 0 = -2g(k_1, k_2) \tag{3.6}$$

and

$$2g(\nabla_{k_1} k_3, k_3) = 0 = -2g(k_1, k_3). \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get

$$2g(\nabla_{k_1} k_3, U) = -2g(k_1, U),$$

for all $U \in \mathfrak{X}(N)$.

Thus, $\nabla_{k_1} k_3 = -k_1$. Therefore, (3.4) further gives

$$\begin{gathered} \nabla_{k_1} k_2 = 0, \quad \nabla_{k_1} k_1 = k_3, \\ \nabla_{k_2} k_3 = -k_2, \quad \nabla_{k_2} k_2 = k_3, \quad \nabla_{k_2} k_1 = 0, \\ \nabla_{k_3} k_3 = 0, \quad \nabla_{k_3} k_2 = 0, \quad \nabla_{k_3} k_1 = 0. \end{gathered} \tag{3.8}$$

We know that

$$R(U, V)Y = \nabla_U \nabla_V Y - \nabla_V \nabla_U Y - \nabla_{[U,V]} Y, \tag{3.9}$$

where R is the Riemann curvature tensor. Utilizing the foregoing results and with the help of (3.9), we acquire

$$\begin{gathered} R(k_1, k_2)k_3 = 0, \quad R(k_1, k_3)k_3 = -k_1, \\ R(k_1, k_2)k_2 = -k_1, \quad R(k_2, k_3)k_2 = k_3, \quad R(k_1, k_3)k_2 = 0, \end{gathered}$$

$$R(k_1,k_2)k_1 = k_2, \quad R(k_2,k_3)k_1 = 0, \quad R(k_1,k_3)k_1 = k_3.$$

Using the above expressions, the Ricci tensor can be obtained as

$$\mathcal{S}(k_1,k_1) = g(R(k_1,k_2)k_2,k_1) + g(R(k_1,k_3)k_3,k_1) = -2. \tag{3.10}$$

Similarly, we get

$$\mathcal{S}(k_2,k_2) = \mathcal{S}(k_3,k_3) = -2. \tag{3.11}$$

Therefore, the scalar curvature r is calculated as

$$r = \mathcal{S}(k_1,k_1) + \mathcal{S}(k_2,k_2) + \mathcal{S}(k_3,k_3) = -6. \tag{3.12}$$

Making use of the above expressions and using the equation (2.1), we have

$$\widehat{\nabla}_{k_1}k_3 = 0, \quad \widehat{\nabla}_{k_1}k_2 = 0, \quad \widehat{\nabla}_{k_1}k_1 = k_3,$$
$$\widehat{\nabla}_{k_2}k_3 = 0, \quad \widehat{\nabla}_{k_2}k_2 = k_3, \quad \widehat{\nabla}_{k_2}k_1 = 0,$$
$$\widehat{\nabla}_{k_3}k_3 = k_3, \quad \widehat{\nabla}_{k_3}k_2 = 0, \quad \widehat{\nabla}_{k_3}k_1 = 0. \tag{3.13}$$

From the last equation and using (1.2), we obtain $\widehat{T}(k_1,k_3) = k_1$ and $\psi(k_3)k_1 - \psi(k_1)k_3 = k_1$. Similarly, other components can be verified. Therefore, the linear connection $\widehat{\nabla}$ defined on (N^3,g) as (2.1), is a semi-symmetric connection. Also, we have

$$(\widehat{\nabla}_{k_1}g)(k_1,k_3) = -1 \neq 0. \tag{3.14}$$

Thus, the linear connection $\widehat{\nabla}$ is non-metric on (N^3,g).

4. Gradient Ricci Solitons on N^3 with a $SSNMC$

This section carries out the study of gradient Ricci solitons in N^3 with a $SSNMC$.

Let us choose that the soliton vector W of the Ricci soliton (g,W,λ) in N^3 with a $SSNMC$ is a gradient of some smooth function f. Then using (1.5), we infer

$$\widehat{\nabla}_U Df = -\widehat{Q}U - \lambda U, \tag{4.1}$$

for all $U \in \mathfrak{X}(N)$. Making use of the above equation, the subsequent relation

$$\widehat{R}(U,V)Df = \widehat{\nabla}_U\widehat{\nabla}_V Df - \widehat{\nabla}_V\widehat{\nabla}_U Df - \widehat{\nabla}_{[U,V]}Df \tag{4.2}$$

yields

$$\widehat{R}(U,V)Df = (\widehat{\nabla}_U\widehat{Q})(V) - (\widehat{\nabla}_V\widehat{Q})(U). \tag{4.3}$$

The contraction of the preceding equation gives

$$\widehat{\mathcal{S}}(U,Df) = -\frac{1}{2}(U\widehat{r}). \tag{4.4}$$

Again, from (2.17) we obtain

$$\widehat{\mathcal{S}}(U,Df) = \left(\frac{\widehat{r}}{2}+1\right)(Uf) - \left(\frac{\widehat{r}}{2}-1\right)\psi(U)(\xi f). \tag{4.5}$$

Comparing the equations (4.4) and (4.5)

$$-\frac{1}{2}(U\widehat{r}) = \left(\frac{\widehat{r}}{2}+1\right)(Uf) - \left(\frac{\widehat{r}}{2}-1\right)\psi(U)(\xi f). \tag{4.6}$$

Now, putting $U=\xi$ in (4.6), we find

$$\xi f = 0, \tag{4.7}$$

since $\xi\widehat{r}=0$.

Equation (4.3) gives

$$g(\widehat{R}(U,V)\xi, Df) = 0. \tag{4.8}$$

Again, from equation (2.11) we infer that

$$g(\widehat{R}(U,V)\xi, Df) = \psi(V)(Uf) - \psi(U)(Vf). \tag{4.9}$$

Comparing last two equations and putting $V=\xi$ and using $\xi f = 0$, we lead

$$Uf = 0, \tag{4.10}$$

which shows that $f=$ constant. Making use of the fact that f is constant, equation (4.1) infers that the manifold is an Einstein manifold. Hence, the Riemannian manifold N^3 is of constant sectional curvature.

Theorem 4.1. *Let the soliton vector field W of the Ricci soliton (g, W, λ) in N^3 with a $SSNMC$ be a gradient Ricci soliton. Then N^3 is a manifold of constant sectional curvature with respect to the $SSNMC$.*

5. Gradient Yamabe Solitons on N^3 with a $SSNMC$

From equation (1.8), we find

$$\widehat{\nabla}_V Df = (\widehat{r}-\lambda)V. \tag{5.1}$$

Differentiating (5.1) covariantly along the vector field U, we obtain

$$\widehat{\nabla}_U\widehat{\nabla}_V Df = (U\widehat{r})V + (\widehat{r}-\lambda)\widehat{\nabla}_U V. \tag{5.2}$$

Interchanging U and V in the above equation and then utilizing the preceding equation in $\widehat{R}(U,V)Df = \widehat{\nabla}_U\widehat{\nabla}_V Df - \widehat{\nabla}_V\widehat{\nabla}_U Df - \widehat{\nabla}_{[U,V]}Df$, we lead

$$\widehat{R}(U,V)Df = (U\widehat{r})V - (V\widehat{r})U. \tag{5.3}$$

Contracting the previous equation over U, we get

$$\widehat{\mathcal{S}}(V, Df) = -2(V\widehat{r}). \tag{5.4}$$

Combining the last equation and (4.5), we infer

$$-2(U\widehat{r}) = \left(\frac{\widehat{r}}{2}+1\right)(Uf) - \left(\frac{\widehat{r}}{2}-1\right)\psi(U)(\xi f). \tag{5.5}$$

Putting $U=\xi$ in the foregoing equation, we have

$$\xi f = 0, \tag{5.6}$$

since $\xi\widehat{r} = 0$. Thus, from (5.5), we obtain

$$-2(U\widehat{r}) = \left(\frac{\widehat{r}}{2} + 1\right)(Uf). \tag{5.7}$$

Now, from equation (5.3) we find that

$$g(\widehat{R}(U,V)\xi, Df) = \psi(U)(V\widehat{r}) - \psi(V)(U\widehat{r}). \tag{5.8}$$

Combining equation (4.9) and (5.8), we have

$$\psi(V)(Uf) - \psi(U)(Vf) = \psi(U)(V\widehat{r}) - \psi(V)(U\widehat{r}). \tag{5.9}$$

Setting $V = \xi$ in the previous equation gives

$$(U\widehat{r}) = -(Uf). \tag{5.10}$$

Utilizing (5.10) in (5.7) we infer that

$$\left(\frac{\widehat{r}}{2} - 1\right)(Uf) = 0, \tag{5.11}$$

which entails that either $\widehat{r} = 2$ or $\widehat{r} \neq 2$.

If $\widehat{r} = 2$, then from (2.10) we infer that $r = 4$. Therefore, N^3 is of constant scalar curvature.

Next, we suppose that $\widehat{r} \neq 2$, that is, $(Uf) = 0$, which implies f is a constant. Therefore, the gradient Yamabe soliton is trivial.

Hence, we state the result.

Theorem 5.1. *Let the Riemannian metric of N^3 with a SSNMC be the gradient Yamabe soliton. Then, either N^3 is a manifold of constant scalar curvature or the gradient Yamabe soliton is trivial with respect to the SSNMC.*

Also, if $\widehat{r} = 2$, then using the equation (2.17) we acquires that the manifold is an Einstein manifold. Hence, the Riemannian manifold N^3 is of constant sectional curvature.

Corollary 5.1. *Let the Riemannian metric of N^3 with a SSNMC be the gradient Yamabe soliton. Then, either N^3 is a manifold of constant sectional curvature or the gradient Yamabe soliton is trivial with respect to the SSNMC.*

6. Gradient Einstein Solitons on N^3 with a $SSNMC$

Making use of (1.9), we have

$$\widehat{\nabla}_V Df = -\widehat{Q}V + \frac{\widehat{r}}{2}V - \lambda V. \tag{6.1}$$

Differentiating (6.1) covariantly along U, we find

$$\widehat{\nabla}_U \widehat{\nabla}_V Df = -\widehat{\nabla}_U \widehat{Q}V + \frac{1}{2}(U\widehat{r})V + \left(\frac{\widehat{r}}{2} - \lambda\right)\widehat{\nabla}_U V. \tag{6.2}$$

Interchanging U and V and then making use of the above equation in $\widehat{R}(U,V)Df = \widehat{\nabla}_U\widehat{\nabla}_V Df - \widehat{\nabla}_V\widehat{\nabla}_U Df - \widehat{\nabla}_{[U,V]}Df$, we infer

$$\widehat{R}(U,V)Df = \frac{1}{2}[(U\widehat{r})V - (V\widehat{r})U] - (\widehat{\nabla}_U\widehat{Q})(V) + (\widehat{\nabla}_V\widehat{Q})(U). \tag{6.3}$$

Contracting the foregoing equation over U, we obtain

$$\widehat{\mathbb{S}}(V,Df) = -\frac{1}{2}(V\widehat{r}). \tag{6.4}$$

Combining the last equation and (4.5), we get

$$-\frac{1}{2}(U\widehat{r}) = \left(\frac{\widehat{r}}{2}+1\right)(Uf) - \left(\frac{\widehat{r}}{2}-1\right)\psi(U)(\xi f). \tag{6.5}$$

Setting $U = \xi$ in (6.5), we have

$$(\xi f) = 0, \tag{6.6}$$

since $\xi\widehat{r} = 0$. Thus, from (6.5), we acquire

$$-\frac{1}{2}(U\widehat{r}) = \left(\frac{\widehat{r}}{2}+1\right)(Uf). \tag{6.7}$$

Now, from equation (6.3) we obtain that

$$g(\widehat{R}(U,V)\xi, Df) = -\frac{1}{2}[\psi(U)(V\widehat{r}) - \psi(V)(U\widehat{r})]. \tag{6.8}$$

Combining equation (4.9) and (6.8), we lead

$$\psi(V)(Uf) - \psi(U)(Vf) = -\frac{1}{2}[\psi(U)(V\widehat{r}) - \psi(V)(U\widehat{r})]. \tag{6.9}$$

Putting $V = \xi$ in the last equation yields

$$(Uf) = -\frac{1}{2}(U\widehat{r}). \tag{6.10}$$

Using (6.10) in (6.7) we find that

$$\frac{\widehat{r}}{2}(Uf) = 0. \tag{6.11}$$

Hence, either $\widehat{r} = 0$ or $\widehat{r} \neq 0$.

If $\widehat{r} = 0$, then from (2.10) we acquire that $r = 2$. Therefore, N^3 is of constant scalar curvature.

Next, we suppose that $\widehat{r} \neq 0$, that is, $(Uf) = 0$, which implies f is a constant. Then, equation (6.1) reveals that N^3 is an Einstein manifold. Hence, N^3 is of constant sectional curvature, since the manifold is of dimension 3.

Thus, we state the subsequent.

Theorem 6.1. *If the Riemannian metric of N^3 with a SSNMC is a gradient Einstein soliton, then N^3 is either a manifold of constant scalar curvature or a manifold of constant sectional curvature with respect to the SSNMC.*

7. Gradient m-Quasi Einstein Solitons on N^3 with a $SSNMC$

Here, we investigate the m-quasi Einstein metric on N^3 with a $SSNMC$. Initially, we prove the following lemma.

Lemma 7.1. *In N^3, we have the following:*

$$\widehat{R}(U,V)Df = (\widehat{\nabla}_V\widehat{Q})U - (\widehat{\nabla}_U\widehat{Q})V + \frac{\lambda}{m}\{(Vf)U - (Uf)V\} + \frac{1}{m}\{(Uf)\widehat{Q}V - (Vf)\widehat{Q}U\}, \tag{7.1}$$

for all $U, V \in \mathfrak{X}(M)$.

Proof. Let the Riemannian metric of N^3 with a $SSNMC$ be a m-quasi Einstein metric. Therefore, the equation (1.10) can be represented as

$$\widehat{\nabla}_U Df = -\widehat{Q}U + \frac{1}{m}g(U,Df)Df + \lambda U. \tag{7.2}$$

Covariant derivative of (7.2) along V yields

$$\widehat{\nabla}_V\widehat{\nabla}_U Df = -\widehat{\nabla}_V\widehat{Q}U + \frac{1}{m}\widehat{\nabla}_V g(U,Df)Df + \frac{1}{m}g(U,Df)\widehat{\nabla}_V Df + \lambda\widehat{\nabla}_V U. \tag{7.3}$$

Exchanging U and V in (7.3), we obtain

$$\widehat{\nabla}_U\widehat{\nabla}_V Df = -\widehat{\nabla}_U\widehat{Q}V + \frac{1}{m}\widehat{\nabla}_U g(V,Df)Df + \frac{1}{m}g(V,Df)\widehat{\nabla}_U Df + \lambda\widehat{\nabla}_U V \tag{7.4}$$

and

$$\widehat{\nabla}_{[U,V]} Df = -\widehat{Q}[U,V] + \frac{1}{m}g([U,V],Df)Df + \lambda[U,V]. \tag{7.5}$$

Utilizing (7.2)–(7.5) and the relation $\widehat{R}(U,V)Df = \widehat{\nabla}_U\widehat{\nabla}_V Df - \widehat{\nabla}_V\widehat{\nabla}_U Df - \widehat{\nabla}_{[U,V]}Df$, we have

$$\widehat{R}(U,V)Df = (\widehat{\nabla}_V\widehat{Q})U - (\widehat{\nabla}_U\widehat{Q})V + \frac{\lambda}{m}\{(Vf)U - (Uf)V\} + \frac{1}{m}\{(Uf)\widehat{Q}V - (Vf)\widehat{Q}U\}.$$

□

Now contracting the equation (7.1) over U, we obtain

$$\widehat{\mathcal{S}}(V,Df) = \frac{1}{2}(V\widehat{r}) + \frac{2\lambda}{m}(Vf) - \frac{1}{m}\left\{\left(\frac{\widehat{r}}{2}+3\right)(Vf) + \left(\frac{\widehat{r}}{2}-1\right)(\xi f)\psi(V)\right\}. \tag{7.6}$$

Combining (7.6) and (4.5), we have

$$\frac{1}{2}(V\widehat{r}) + \frac{2\lambda}{m}(Vf) - \frac{1}{m}\left\{\left(\frac{\widehat{r}}{2}+3\right)(Vf) + \left(\frac{\widehat{r}}{2}-1\right)(\xi f)\psi(V)\right\} = \left(\frac{\widehat{r}}{2}+1\right)(Vf) - \left(\frac{\widehat{r}}{2}-1\right)\psi(V)(\xi f). \tag{7.7}$$

Setting $V = \xi$ in (7.7), we obtain

$$(2m + \widehat{r} - 2\lambda + 2)(\xi f) = 0, \tag{7.8}$$

since $\xi\widehat{r} = 0$.

Now, from equation (7.1) we have

$$g(\widehat{R}(U,V)\xi, Df) = \left(\frac{\lambda}{m} - \frac{2}{m}\right)[\psi(V)(Uf) - \psi(U)(Vf)]. \tag{7.9}$$

Combining equations (4.9) and (7.9), we find that

$$\psi(V)(Uf) - \psi(U)(Vf) = \left(\frac{\lambda}{m} - \frac{2}{m}\right)[\psi(V)(Uf) - \psi(U)(Vf)]. \tag{7.10}$$

Putting $V = \xi$ in the foregoing equation yields

$$(\lambda - m - 2)(Uf) = 0, \tag{7.11}$$

where we have used $\xi f = 0$.

Hence, either $(\lambda - m - 2) = 0$ or $(\lambda - m - 2) \neq 0$.

If $(\lambda - m - 2) = 0$, then we get $\lambda = m + 2 =$ positive integer. Hence, the gradient m-quasi Einstein soliton is expanding.

If we suppose that $(\lambda - m - 2) \neq 0$, then $(Uf) = 0$, which implies f is a constant. Then, equation (7.1) reveals that N^3 is an Einstein manifold. Hence, N^3 is of constant sectional curvature, since the manifold is of dimension 3,.

Hence, we state the following.

Theorem 7.1. *If the Riemannian metric of N^3 with a $SSNMC$ is a gradient m-quasi Einstein soliton, then either the soliton is expanding or it is a manifold of constant sectional curvature with respect to the $SSNMC$, provided* $(2m + \widehat{r} - 2\lambda + 2) \neq 0$.

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NOTE ON HAMILTONIAN GRAPHS IN ABELIAN 2-GROUPS

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Abstract. We analyze a graph G whose vertices are subgroups of $\mathbb{Z}_2^k$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Two vertices are joined if their respective subgroups have nontrivial intersection. We prove that such a graph is $6(2^{k-2}-1)$-regular. If a graph is regular, a classical theorem by Ore claims that a graph is Hamiltonian if the degree of any vertex is at least one half of the number of vertices. Using Ore's theorem, we show that G is Hamiltonian for $k \in \{3,4\}$. Ore's theorem cannot be applied when $k \geq 5$. Nevertheless, we manage to construct a Hamiltonian cycle for $k=5$. Our construction uses orbits of one $\mathbb{Z}_2^4$ group under an action of an automorphism of order 31. It is highly likely that this approach could be generalized for $k>5$.

1. Introduction and notation

Many algebraic structures, including groups, have nice interpretations in graph theory (see for example [1,3] and [4]). Readers can find more on groups and graphs in [5]. If there is a cycle in a graph that visits every vertex, then the graph is Hamiltonian. In this paper we are interested in Hamiltonian graphs defined on Abelian groups of exponent 2. For some classical results on Hamiltonian graphs see [5]. The main tool in our analysis will be the application of various group rings, for example see [2]. An elementary Abelian group of order 2^k is denoted by E_{2^k}. If $x_1, x_2, \ldots, x_k$ are generators, then we can write $E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle$. Additionally, $x_i^2 = 1$ for all $i \in [k] = \{1, 2, \ldots, k\}$. With $E_{2^l}[H]$ we denote a collection of all subgroups of order 2^l that are contained in $H \leq E_{2^k}$.

We introduce a set $E_{2^s}[T,H]^{-1} = \{S \mid T \leq S \leq H,\ S \cong E_{2^s}\}$ of all E_{2^s}-subgroups that contain T and that are also contained in H. One can see that if $t \leq s \leq m$,

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$H \cong E_{2^m}$, and $T \cong E_{2^t}$, then $|E_{2^s}[T,H]^{-1}| = |E_{2^{s-t}}[H/T]| = |E_{2^{s-t}}[E_{2^{m-t}}]| = \begin{bmatrix} m-t \\ s-t \end{bmatrix}_2$, where H/T is a quotient group isomorphic to $E_{2^{m-t}}$ and $\begin{bmatrix} a \\ b \end{bmatrix}_2$ is a Gaussian coefficient.

Let $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ be a graph with vertices $T \leq E_{2^k}$, where $T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Edges $\mathcal{E}_k$ are defined as follows:

$$\{T_1, T_2\} \in \mathcal{E}_k \Leftrightarrow T_1 \cap T_2 \cong \mathbb{Z}_2.$$

This means that two E_{2^2} groups are joined if and only if they have a common involution (nontrivial intersection). Our main goal is to see when such graphs are Hamiltonian. We will show that Ore's Theorem immediately yields that $(E_{2^2}[E_{2^3}], \mathcal{E}_3)$ and $(E_{2^2}[E_{2^4}], \mathcal{E}_4)$ are Hamiltonian.

We will use $\deg(u)$ to denote the degree of a vertex.

Theorem 1.1 (Ore)**.** *Let G be a connected graph with $n > 3$ vertices. If $\deg(x) + \deg(y) > n$ for all non-adjacent vertices x and y, then G is Hamiltonian.*

A graph $G = (V, E)$ is a r-regular graph if $\deg(x) = r$ for all vertices $x \in V$. As an immediate consequence of Theorem 1.1 we have the following.

Corollary 1.1. *If $G = (V, E)$ is r-regular graph and if $\deg(x) > \frac{1}{2}|V|$, then G is Hamiltonian.*

2. Regularity

In this section we will prove that $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is a regular graph. This means that we need to show that for any $T \in E_{2^2}[E_{2^k}]$ there is a constant number of $S \in E_{2^2}[E_{2^k}]$ such that $|T \cap S| = 2$.

From this point on, we will assume that $k > 2$. Furthermore, we will show that if $k \in \{3, 4\}$, then a graph $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is Hamiltonian.

Theorem 2.1. *A graph $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is $6(2^{k-2} - 1)$-regular. The inequality*

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) < 0$$

holds for all $V \in E_{2^2}[E_{2^k}]$ if any only if $k < 5$.

Proof. Let V be a vertex of $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$. Put $V^* = V \setminus \{1\}$. Let us denote with $n(V)$ the collection of all vertices adjacent to V. If $P \in n(V)$, then $P \cong E_{2^2}$ and $P \cap V = \langle g \rangle$ for some $g \in E_{2^k}^*$. Also, $P \in E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}$. Hence,

$$n(V) = \left[\bigcup_{g \in V^*} E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \right] \setminus \{V\}.$$

On the other hand, we have

$$|E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| = |E_2[E_{2^k}/\langle g \rangle]| = |E_2[E_{2^{k-1}}]| = 2^{k-1} - 1.$$

If $g, h \in V^*$ and $g \neq h$, then

$$|E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| = |E_{2^2}[E_{2^k}] \cap \{V\}| = 1.$$

Also, for three mutually different $g, h, k \in T^*$ we get

$$|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k\rangle, E_{2^k}]^{-1}| = 1.$$

Using the inclusion-exclusion formula, the following holds

$$\begin{aligned}\deg(V) = &\sum_{g\in V^*} |E_{2^2}[\langle g\rangle, E_{2^k}]^{-1}| - \sum_{g\neq h,\ g,h\in V^*} \left|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1}\right| \\ &+ \sum_{g\neq h\neq k\neq g,\ g,h,k\in V^*} \left|E_{2^2}[\langle g\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h\rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k\rangle, E_{2^k}]^{-1}\right| - 1 \\ =&\binom{3}{1}(2^{k-1}-1) - \binom{3}{2}\cdot 1 + 1 - 1 \\ =&6(2^{k-2}-1).\end{aligned}$$

Notice that $|E_{2^2}[E_{2^k}]| = \begin{bmatrix} k \\ 2\end{bmatrix}_2 = \frac{1}{3}(2^k-1)(2^{k-1}-1)$. Put $t = 2^{k-2}$. Therefore,

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) = \frac{1}{6}(4t-1)(2t-1) - 6(t-1) = \frac{1}{6}(8t^2 - 42t + 37).$$

For $k = 3$ and $k = 4$ we get $8t^2 - 42t + 37 < 0$. For $k \geq 5$ we have $8t^2 - 42t + 37 > 0$. This proves our claim. □

Now, using Corollary 1.1, we see that the following holds.

Corollary 2.1. *Graphs* $(E_{2^2}[E_{2^3}], \mathcal{E}_3)$ *and* $(E_{2^2}[E_{2^4}], \mathcal{E}_4)$ *are Hamiltonian. Furthermore, necessary conditions for application of Ore's theorem are not satisfied for* $k \geq 5$.

3. Hamiltonian Cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$

Let $E_{2^5} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle = \langle a, b, c, d, e\rangle$, where a, b, c, d, e are generators of E_{2^5}. Any automorphism $\alpha \in \mathrm{Aut}(E_{2^5})$ is represented by its action on generators. We can denote any $\alpha \in \mathrm{Aut}(E_{2^5})$ by

$$\alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix},$$

for some $g_i \in E_{2^5}^*$. This means $\alpha(a) = g_1,\ \alpha(b) = g_2$ and so on. The order of an automorphism $o(\alpha)$ is the smallest nonnegative integer n such that α^n is an identity map. If $X \subseteq E_{2^5}$ and $\alpha \in \mathrm{Aut}(E_{2^5})$, then with $X^{\langle\alpha\rangle}$ we will denote one α-orbit of X. If α is of order n, then an orbit $X^{\langle\alpha\rangle}$ can be represented in a group ring $\mathbb{Z}[E_{2^5}]$ like this:

$$X^{\langle\alpha\rangle} = X + X^{\alpha} + \cdots + X^{\alpha^{n-1}}.$$

The following lemma will be crucial for a construction of a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$.

Lemma 3.1. *Let* $E_{2^5} = \langle a, b, c, d, e\rangle$ *and let* $\alpha \in Aut(E_{2^5})$ *be given by*

$$\alpha = \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix},$$

then $o(\alpha) = 31$ *and* $H^{\langle\alpha\rangle} = E_{2^4}[E_{2^5}]$, *where* $H = \langle a,b,c,d\rangle$. *If* $T = \langle a,b,c\rangle$ *and* $\Delta_i = T \cap T^{\alpha^i}$ *for* $i \in \mathbb{Z}_{31}$, *then*

$$\Delta_i = \begin{cases} \langle b,c\rangle, & \text{if } i = 1, 14,\\ \langle a,bc\rangle, & \text{if } i = 13, 30,\\ \langle ab,c\rangle, & \text{if } i = 17, 18,\\ \cong \mathbb{Z}_2, & \text{otherwise.} \end{cases}$$

Proof. We can rewrite an automorphism α in a simplified form like this: $\alpha = (bc, cd, bcd, de, a)$. For the purpose of finding α^i we represent α in a matrix form over $\mathbb{Z}_2$

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Rows and columns are indexed by a,b,c,d,e. After calculating powers of α over $\mathbb{Z}_2$, we get that α^{31} is an identity matrix. Furthermore, α^i is not an identity matrix for all $i < 31$. Therefore, $o(\alpha) = 31$. For example, using the same approach, we get $\alpha^{13} = (de, abcde, bc, abde, d)$ and $\alpha^{14} = (ade, acde, b, abe, de)$. Hence, $T^{\alpha^{13}} = \langle de, abcde, bc\rangle = \langle de, abc, bc\rangle = \langle de, a, bc\rangle$ and $\Delta_{13} = T \cap T^{\alpha^{13}} = \langle a, bc\rangle$. Furthermore, $T^{\alpha^{14}} = \langle ade, acde, b\rangle = \langle ade, c, b\rangle$ and $\Delta_{14} = \langle b,c\rangle$. Also, $\alpha^{17} = (ae, c, ab, acd, acde)$, $\alpha^{18} = (abc, bcd, bd, e, ae)$ and $\alpha^{30} = (e, bc, abc, ac, acd)$. For all other cases Δ_i is a group of order 2. In the Appendix, one can find all powers α^i together with the images T^{α^i}.

Assume that $H^{\alpha^i} = H$ for some power $i < 31$. Then $\Delta_i = T \cong E_{2^3}$. This is a contradiction with $|\Delta_i| \leq 4$, hence $H^{\alpha^i} \neq H$. Since the number of all E_{2^4} subgroups of E_{2^5} is $|E_{2^4}[E_{2^5}]| = \begin{bmatrix}5\\4\end{bmatrix}_2 = 2^5 - 1 = 31$, this means that an α-orbit of H contains all E_{2^4} subgroups of E_{2^5}. Therefore, $H^{\langle\alpha\rangle} = E_{2^4}[E_{2^5}]$. $\square$

Throughout the rest of the paper the subgroup $\langle a,b,c\rangle \leq E_{2^5} = \langle a,b,c,d,e\rangle$ shall be denoted by T and α shall be the automorphism defined in the Lemma 3.1. We are now ready to sketch the main idea for a construction of a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$. A main building block will be an α-orbit of T. There are 7 vertices or subgroups of order 4 in T^{α^i}, $i \in \mathbb{Z}_{31}$. We will show, in Theorem 3.4, that a collection of all vertices from $\bigcup_{i=0}^{31} E_{2^2}[T^{\alpha^i}]$ is in fact the set of all vertices $E_{2^2}[E_{2^5}]$. Also $T \cap T^{\alpha} \cong E_{2^2}$ is a vertex. The same holds for all other $T^{\alpha^i} \cap T^{\alpha^{i+1}}$. As we will see from Theorem 3.5, vertices $T^{\alpha^i} \cap T^{\alpha^{i+1}}$ are all mutually different. As a final step, we will introduce a recursive procedure that will enable us to choose vertices from each $E_{2^2}[T^{\alpha^i}]$ so that they all together form a Hamiltonian cycle.

Motivated by the previous lemma we introduce slightly different notation:

$$\Delta_{\Omega_1} = \langle b,c\rangle, \quad \Omega_1 = \{1, 14\},$$

$$\Delta_{\Omega_2} = \langle a, bc\rangle, \quad \Omega_2 = \{13, 30\},$$
$$\Delta_{\Omega_3} = \langle ab, c\rangle, \quad \Omega_3 = \{17, 18\}.$$

Lemma 3.2. *Groups* $\Delta_{\Omega_i}^{\alpha^k}$ *and* Δ_{Ω_i} *are distinct for all* $i \in [3]$ *and* $k \in [30]$.

Proof. Assume the opposite. Let $i \in [3]$ and $k \in [30]$ such that $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_i}$. Since $o(\alpha) = 31$ is a prime, then α^k generate entire $\langle\alpha\rangle$. Hence $\langle\alpha\rangle = \langle\alpha^k\rangle$. Let $H = \langle a, b, c, d\rangle$. Lemma 3.1 implies that $H^{\langle\alpha^k\rangle} = E_{2^4}[E_{2^5}]$. There is $s \in \mathbb{Z}_{31}$ such that $\Delta_{\Omega_i} \le H^{(\alpha^k)^s}$. Since $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_i}$, then $\Delta_{\Omega_i} = \Delta_{\Omega_i}^{(\alpha^k)^t} \le (H^{(\alpha^k)^s})^{(\alpha^k)^t} = H^{(\alpha^k)^{s+t}}$ for all $t \in \mathbb{Z}_{31}$. A mapping $t \mapsto s + t$ is one-to-one map on $\mathbb{Z}_{31}$. Hence, we can write in a group ring $\mathbb{Z}[E_{2^4}[E_{2^5}]]$ the following:

$$\sum_{t=0}^{30} H^{(\alpha^k)^{s+t}} = \sum_{t\in\mathbb{Z}_{31}} ((H)^{\alpha^k})^t = E_{2^4}[E_{2^5}].$$

From $\Delta_{\Omega_i} \le H^{(\alpha^k)^{s+t}}$ for all $t \in \mathbb{Z}_{31}$ it follows $|E_{2^4}[\Delta_{\Omega_i}, E_{2^5}]^{-1}| \ge 31$. This is a contradiction with

$$|E_{2^4}[\Delta_{\Omega_i}, E_{2^5}]^{-1}| = |E_{2^2}[E_{2^5}/\Delta_{\Omega_i}]| = |E_{2^2}[E_{2^3}]| = \begin{bmatrix}3\\2\end{bmatrix}_2 = 2^3 - 1 = 7. \qquad \square$$

Corollary 3.1. *If* $\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_j}$, *then* α^k *is a unique element from* $\langle\alpha\rangle$.

Proof. Suppose that k_1 and k_2 are integers such that $\Delta_{\Omega_i}^{\alpha^{k_1}} = \Delta_{\Omega_i}^{\alpha^{k_2}} = \Delta_{\Omega_j}$. It follows that $\Delta_{\Omega_i}^{\alpha^{k_1-k_2}} = \Delta_{\Omega_i}$. By Lemma 3.2, a map $\alpha^{k_1-k_2}$ is an identity map. Thus $k_1 = k_2$. $\square$

Lemma 3.3. *Subgroups* Δ_{Ω_i}, $i \in [3]$, *satisfy the following:* $\Delta_{\Omega_1}^{\alpha^{30}} = \Delta_{\Omega_2}$, $\Delta_{\Omega_2}^{\alpha^{18}} = \Delta_{\Omega_3}$, $\Delta_{\Omega_3}^{\alpha^{14}} = \Delta_{\Omega_1}$.

Proof. From Lemma 3.1 we have $\Delta_{\Omega_1}^{\alpha^{30}} = (T \cap T^{\alpha})^{\alpha^{30}} = T^{\alpha^{30}} \cap T = \Delta_{\Omega_2}$. Hence $\Delta_{\Omega_1}^{\alpha^{30}} = \Delta_{\Omega_2}$. Furthermore, $\Delta_{\Omega_1}^{\alpha^{17}} = (T \cap T^{\alpha^{14}})^{\alpha^{17}} = T^{\alpha^{17}} \cap T = \Delta_{\Omega_3}$. Now we have $\Delta_{\Omega_3}^{\alpha^{14}} = \Delta_{\Omega_1}$. Moreover $\Delta_{\Omega_3}^{\alpha^{13}} = \Delta_{\Omega_2}$ and $\Delta_{\Omega_2}^{\alpha^{18}} = \Delta_{\Omega_3}$. This proves our claim. $\square$

Theorem 3.1. *For* T *and* α *the following holds*

$$\sum_{0\le i<j\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}]| = 31 \cdot 3.$$

Proof. Take some i and j such that $T^{\alpha^i} \cap T^{\alpha^j} \cong E_{2^2}$. Then

$$T^{\alpha^i} \cap T^{\alpha^j} = (T \cap T^{\alpha^{j-i}})^{\alpha^i} = (\Delta_{j-i})^{\alpha^i} = (\Delta_{i-j})^{\alpha^j} \cong E_{2^2}.$$

This means that $\Delta_{j-i} = \Delta_{i-j} \cong E_{2^2}$. Thus, by Lemma 3.1, we get $\{i-j, j-i\} \in \{\{1, 30\}, \{13, 18\}, \{14, 17\}\}$. Since $i \in \mathbb{Z}_{31}$, each $\{i, j\}$ contributes 31 to the sum $\sum_{0\le i<j\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}]|$. Therefore, the final number is $31 \cdot 3$. This proves our assertion. $\square$

Theorem 3.2. *For T and α the following holds*

$$\sum_{0\le i<j<k\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}] \cap E_{2^2}[T^{\alpha^k}]| = 31.$$

Proof. Let $A = T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cong E_{2^2}$ for some $0 \le i < j < k \le 31$. Then $A = (T^{\alpha^i} \cap T^{\alpha^j}) \cap (T^{\alpha^i} \cap T^{\alpha^k})$. This means $A = (T \cap T^{\alpha^{j-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{k-i}})^{\alpha^i} = (\Delta_{j-i} \cap \Delta_{k-i})^{\alpha^i}$. Hence $\Delta_{j-i} \cap \Delta_{k-i} \cong E_{2^2}$. Since $|\Delta_t| \le 4$ we get $\Delta_{j-i} = \Delta_{k-i} \cong E_{2^2}$. Since $j-i \ne k-i$, we get $\{j-i, k-i\} = \Omega_s$ for some $s \in [3]$.

If $s = 1$, then $\{j-i, k-i\} = \{1, 14\}$. This implies that $\{i, j, k\}$ can be represented as $\{i, i+1, i+14\}$ where $i \in \mathbb{Z}_{31}$.

The case $s = 2$ gives us $\{j-i, k-i\} = \{13, 30\}$. Hence, $\{i, j, k\}$ can be represented as $\{i, i+13, i+30\}$ where $i \in \mathbb{Z}_{31}$. However, we get

$$\{\{i, i+13, i+30\} \mid i \in \mathbb{Z}_{31}\} = \{\{(i-1)+1, (i-1)+1+13, (i-1)+1+30\} \mid i \in \mathbb{Z}_{31}\},$$

and this set is equal to $= \{\{j, j+1, j+14\} \mid j \in \mathbb{Z}_{31}\}$ where $j = i-1$ in $\mathbb{Z}_{31}$. Therefore, the previous two cases are in fact the same.

If $s = 3$, then $\{j-i, k-i\} = \{17, 18\}$. Now we get $\{i, j, k\}$ is of the form $\{i, i+17, i+18\}$ where $i \in \mathbb{Z}_{31}$. Notice that

$$\{\{i, i+17, i+18\} \mid i \in \mathbb{Z}_{31}\} = \{\{(i+17)-17, i+17, (i+17)+1\} \mid i \in \mathbb{Z}_{31}\}.$$

It follows

$$\{\{j-17, j, j+1\} \mid j \in \mathbb{Z}_{31}\} = \{\{j+14, j, j+1\} \mid j \in \mathbb{Z}_{31}\},$$

where $j = i+17$ in $\mathbb{Z}_{31}$. Thus, all the three cases are the same and so we have one representative.

This means that we have one representative of a triple $\{i, j, k\}$ such that $T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cong E_{2^2}$ where $i \in \mathbb{Z}_{31}$. This proves the claim of the theorem. ☐

Theorem 3.3. *For T and α the following holds*

$$\sum_{0\le i<j<k<s\le 30} |E_{2^2}[T^{\alpha^i}] \cap E_{2^2}[T^{\alpha^j}] \cap E_{2^2}[T^{\alpha^k}] \cap E_{2^2}[T^{\alpha^s}]| = 0.$$

Proof. Assume that $A = T^{\alpha^i} \cap T^{\alpha^j} \cap T^{\alpha^k} \cap T^{\alpha^s} \cong E_{2^2}$ for some $0 \le i < j < k < s \le 30$. It implies that

$$A = (T \cap T^{\alpha^{j-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{k-i}})^{\alpha^i} \cap (T \cap T^{\alpha^{s-i}})^{\alpha^i} = (\Delta_{j-i} \cap \Delta_{k-i} \cap \Delta_{s-i})^{\alpha^i}.$$

This means that $\Delta_{j-i} = \Delta_{k-i} = \Delta_{s-i} \cong E_{2^2}$. Since T^{α^i}, T^{α^j}, T^{α^k}, T^{α^s} are mutually different, we get $|\{j-i, k-i, s-i\}| = 3$. Also, $\Delta_{j-i} = \Delta_{k-i} = \Delta_{s-i} \cong E_{2^2}$ implies $\{j-i, k-i, s-i\} \subseteq \Omega_i$ for some i. That is a contradiction since $|\Omega_i| = 2$. ☐

The next result finally shows that orbit $T^{\langle\alpha\rangle}$ contains all E_{2^2} subgroups of E_{2^5}.

Theorem 3.4. *For T and α the following holds*

$$\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}] = E_{2^2}[E_{2^5}].$$

Proof. The total number of all E_{2^2} subgroups of E_{2^5} is $|E_{2^2}[E_{2^5}]| = \begin{bmatrix}5\\2\end{bmatrix}_2 = 31\cdot 5$. Using the inclusion-exclusion formula and Theorems 3.1, 3.2 and 3.3 we get

$$\begin{aligned}\left|\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]\right| =&\sum_{i=0}^{30}\left|E_{2^2}[T^{\alpha^i}]\right| - \sum_{0\le i<j\le 30}\left|E_{2^2}[T^{\alpha^i}]\cap E_{2^2}[T^{\alpha^j}]\right| \\ &+ \sum_{0\le i<j<k\le 30}\left|E_{2^2}[T^{\alpha^i}]\cap E_{2^2}[T^{\alpha^j}]\cap E_{2^2}[T^{\alpha^k}]\right| + \cdots + \\ =&31\cdot 7 - 31\cdot 3 + 31 - 0 + 0 - \cdots \\ =&31\cdot 5.\end{aligned}$$

Therefore, every group from $E_{2^2}[E_{2^5}]$ is contained in $\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]$. □

Theorem 3.5. *A graph $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$ is Hamiltonian.*

Proof. Since $T \cong E_{2^3}$ and $AB = T$, where $A, B \in E_{2^2}[T^{\alpha^i}]$, it follows that $|A\cap B| = \frac{|A|\cdot|B|}{|E_{2^3}|} = 2$. Hence, A and B are adjacent. Therefore, the vertices in $E_{2^2}[T^{\alpha^i}] \cong K_7$ induce a complete graph on 7 vertices denoted by K_7. Thus, if we delete some vertices together with the edges incident to them from $E_{2^2}[T^{\alpha^i}]$, there will be a path in a remaining graph that visits each remaining vertex.

The subgraphs $E_{2^2}[T^{\alpha^{i-1}}]$, $E_{2^2}[T^{\alpha^i}]$ and $E_{2^2}[T^{\alpha^{i+1}}]$ have common vertices $T^{\alpha^i}\cap T^{\alpha^{i-1}}$ and $T^{\alpha^i}\cap T^{\alpha^{i+1}}$. Let $L(T^{\alpha^i}) = \{T^{\alpha^i}\cap T^{\alpha^{i-1}}, T^{\alpha^i}\cap T^{\alpha^{i+1}}\}$. Notice that $L(T^{\alpha^i}) = \{\Delta_1^{\alpha^{i-1}}, \Delta_1^{\alpha^i}\}$ (since $T\cap T^\alpha = \Delta_1$). We may look at vertices $L(T^{\alpha^i})$ as links between neighboring graphs $E_{2^2}[T^{\alpha^{i-1}}]$, $E_{2^2}[T^{\alpha^i}]$ and $E_{2^2}[T^{\alpha^{i+1}}]$.

Suppose that there are at least two equal vertices in $\bigcup_{i=0}^{30} L(T^{\alpha^i})$. Let $T^{\alpha^i}\cap T^{\alpha^{i+1}} = T^{\alpha^s}\cap T^{\alpha^{s+1}}$ for some $i\neq s$. Thus, $(T\cap T^\alpha)^{\alpha^i} = (T\cap T^\alpha)^{\alpha^s}$. Hence, $\Delta_1^{\alpha^i} = \Delta_1^{\alpha^s}$ and $\Delta_1^{\alpha^{i-s}} = \Delta_1$ for $\alpha^{i-s}\neq id$. This is a contradiction with Lemma 3.2. Therefore, all vertices in $\bigcup_{i=0}^{30} L(T^{\alpha^i})$ are mutually different.

As the initial step of a recursive construction of a Hamiltonian cycle, we define $E_{2^2}[T^{\alpha^i}]_0 = E_{2^2}[T^{\alpha^i}]$ for all $i\in\mathbb{Z}_{31}$. Assume that we have formed a sequence $\left(E_{2^2}[T^{\alpha^i}]_{m_i}\right)_{i\in\mathbb{Z}_{31}}$, where m_i is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within $E_{2^2}[T^{\alpha^i}]$.

If there is a vertex A and $j\neq i$ such that $A\in\left(E_{2^2}[T^{\alpha^i}]_{m_i}\backslash L(T^{\alpha^i})\right)\cap E_{2^2}[T^{\alpha^j}]_{m_j}$, then A is not a link, but it is a vertex in graphs $E_{2^2}[T^{\alpha^i}]_{m_i}$ and $E_{2^2}[T^{\alpha^j}]_{m_j}$. Then, we delete a vertex A and the edges incident to it. In this case let $E_{2^2}[T^{\alpha^i}]_{m_i+1} = E_{2^2}[T^{\alpha^i}]_{m_i}\setminus\{A\}$.

If such a vertex A does not exist, we leave $E_{2^2}[T^{\alpha^i}]_{m_i}$ unchanged and denote that by $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$. Now, continue the same procedure with $E_{2^2}[T^{\alpha^{i+1}}]_{m_{i+1}}$. Following this process, after finite number of steps, we will construct a sequence $\left(\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}\right)_{i\in\mathbb{Z}_{31}}$. Using a notation in a group ring $\mathbb{Z}[E_{2^2}[E_{2^5}]]$, we have the following:

$$\bigcup_{i\in\mathbb{Z}_{31}}\ \bigcup_{A\in\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}} A = E_{2^2}[E_{2^5}].$$

Note that by Theorem 3.4, $\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}]$ contains all edges in E_{2^5}. From $|E_{2^2}[T^{\alpha^i}]| = 7$ and the fact that we do not delete links in this procedure, we get $m_i \leq 5$ and $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i} \cong K_{7-m_i}$.

Therefore, there is always a path through each vertex of $\widetilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$. Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$. □

4. Appendix

We list here all the powers α^i together with the images T^{α^i}:

$$\begin{aligned}
\alpha =&(bc, cd, bcd, de, a), \quad T^{\alpha} = \langle bc, cd, bcd\rangle,\\
\alpha^2 =&(b, bce, bde, ade, bc), \quad T^{\alpha^2} = \langle b, bce, bde\rangle,\\
\alpha^3 =&(bc, ab, ace, abcde, b), \quad T^{\alpha^3} = \langle bc, ab, ace\rangle,\\
\alpha^4 =&(bce, bd, ad, acde, cd), \quad T^{\alpha^4} = \langle bce, bd, ad\rangle,\\
\alpha^5 =&(ab, ce, bcde, ae, bce), \quad T^{\alpha^5} = \langle ab, ce, bcde, ae, bce\rangle,\\
\alpha^6 =&(bd, abcd, abde, abc, ab), \quad T^{\alpha^6} = \langle bd, abcd, abde\rangle,\\
\alpha^7 =&(ce, cde, abe, c, bd), \quad T^{\alpha^7} = \langle ce, cde, abe\rangle,\\
\alpha^8 =&(abcd, abce, abd, abc, ce), \quad T^{\alpha^8} = \langle abcd, abce, abd\rangle,\\
\alpha^9 =&(cde, ac, be, bde, abcd), \quad T^{\alpha^9} = \langle cde, ac, be\rangle,\\
\alpha^{10} =&(abce, d, acd, ace, cde), \quad T^{\alpha^{10}} = \langle abce, d, acd\rangle,\\
\alpha^{11} =&(ac, de, e, ad, abce), \quad T^{\alpha^{11}} = \langle ac, de, e\rangle,\\
\alpha^{12} =&(d, ade, a, bcde, ad), \quad T^{\alpha^{12}} = \langle d, ade, a\rangle,\\
\alpha^{13} =&(de, abcde, bc, abde, d), \quad T^{\alpha^{13}} = \langle de, abcde, bc\rangle,\\
\alpha^{14} =&(ade, acde, b, abe, de), \quad T^{\alpha^{14}} = \langle ade, acde, b\rangle,\\
\alpha^{15} =&(abcde, ae, cd, abd, ade), \quad T^{\alpha^{15}} = \langle abcde, ae, cd\rangle,\\
\alpha^{16} =&(acde, abc, bce, be, abcde), \quad T^{\alpha^{16}} = \langle acde, abc, bce\rangle,\\
\alpha^{17} =&(ae, c, ab, acd, acde), \quad T^{\alpha^{17}} = \langle ae, c, ab\rangle,\\
\alpha^{18} =&(abc, bcd, bd, e, ae), \quad T^{\alpha^{18}} = \langle abc, bcd, bd\rangle,\\
\alpha^{19} =&(c, bde, ce, a, abc), \quad T^{\alpha^{19}} = \langle c, bde, ce\rangle,\\
\alpha^{20} =&(bcd, ace, abcd, bc, c), \quad T^{\alpha^{20}} = \langle bcd, ace, abcd\rangle,\\
\alpha^{21} =&(bde, ad, cde, b, bcd), \quad T^{\alpha^{21}} = \langle bde, ad, cde\rangle,\\
\alpha^{22} =&(ace, bcde, abce, cd, bde), \quad T^{\alpha^{22}} = \langle ace, bcde, abce\rangle,
\end{aligned}$$

$$\alpha^{23} = (ad, abde, ac, bc, ace), \quad T^{\alpha^{23}} = \langle ad, abde, ac \rangle,$$
$$\alpha^{24} = (bcde, abe, d, ab, ad), \quad T^{\alpha^{24}} = \langle bcde, abe, d \rangle,$$
$$\alpha^{25} = (abde, abd, de, bd, bcde), \quad T^{\alpha^{25}} = \langle abde, abd, de \rangle,$$
$$\alpha^{26} = (abe, be, ade, ce, abde), \quad T^{\alpha^{26}} = \langle abe, be, ade \rangle,$$
$$\alpha^{27} = (abd, acd, abcde, abcd, abe), \quad T^{\alpha^{27}} = \langle abd, acd, abcde \rangle,$$
$$\alpha^{28} = (be, e, acde, cde, abd), \quad T^{\alpha^{28}} = \langle be, e, acde \rangle,$$
$$\alpha^{29} = (ace, a, ae, abce, be), \quad T^{\alpha^{29}} = \langle ace, a, ae \rangle,$$
$$\alpha^{30} = (e, bc, abc, ac, acd), \quad T^{\alpha^{30}} = \langle e, bc, abc \rangle,$$
$$\alpha^{31} = (bc, cd, bcd, de, a), \quad T^{\alpha^{31}} = \langle bc, cd, bcd \rangle.$$

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NEW RESULTS PARAMETRIC APOSTOL-TYPE FROBENIUS-EULER POLYNOMIALS AND THEIR MATRIX APPROACH

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Abstract. The new algebraic properties of the parametric Apostol-type Frobenius-Euler polynomials and parametric type Frobenius-Euler polynomial have been explained in this research. The researchers have studied the series of the Taylor type and established the relation between the classic Apostol Frobenius-Euler and Frobenius-Euler polynomials. This work has also addressed the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and has shown some of their properties. Finally, this research provided some factorizations of Apostol-type Frobenius-Euler matrix that involves the generalized Pascal matrix, Fibonacci and Lucas matrices, respectively.

1. Introduction

The Apostol type polynomials and numbers, have been used extensively in mathematical analysis and practical applications. For this reason, they have been studied as reported in [1–4, 6, 7, 9, 11, 13–15, 17, 18].
Let $\mathbb{P}$ be the vector space of the polynomials with coefficients in $\mathbb{C}$. Let $\{A_n(x)\}_{n\geq 0}$ be the sequence of polynomials known in the literature as the sequence Appell polynomials if the polynomials $A_n(x)$ are defined by the following generating function: (see, [9, p. 1, (1)]):

$$f(z)e^{xz} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!}, \tag{1.1}$$

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where f is a formal power series in z, these polynomials have found remarkable applications in different branches of mathematics, theoretical physics, and chemistry [1,14]. On the other hand, for a particular case, we have the Apostol Frobenius-Euler polynomials that are generated by choosing in (1.1) the following value of $f(z)$ (see, [5, p. 1, (1)]):

$$f(z) = \frac{1-u}{\lambda e^z - u},$$

from which you get the Apostol Frobenius-Euler polynomials $H_n(x;\lambda;u)$ in variable x, is defined through the generating function (see, [2, p. 2, Definition 2]):

$$\frac{1-u}{\lambda e^z - u} e^{xz} = \sum_{n=0}^{\infty} H_n(x;\lambda;u)\frac{z^n}{n!}, \quad |z| < \left|\log\left(\frac{\lambda}{u}\right)\right|,$$

where $H_n(\lambda;u)$ denotes the Apostol Frobenius-Euler number. Thus, the Apostol Frobenius-Euler polynomials fulfill the following identities respectively (see, [2, p. 4, Proposition 1 and Proposition 2]):

$$\lambda H_n(x+1;\lambda;u) - uH_n(x;\lambda;u) = (1-u)x^n$$

and

$$\frac{d}{dx}\left[H_n(x;\lambda;u)\right] = nH_{n-1}(x;\lambda;u).$$

Furthermore, if $n \in \mathbb{N}$, then (see, [2, p. 4, Proposition 3]):

$$\int_0^1 H_n(x;\lambda;u) = \frac{u-\lambda}{\lambda} \cdot \frac{H_{n+1}(\lambda;u)}{n+1}.$$

In this paper, the authors will study new properties of the polynomials that are introduced in [10]. The author will also define the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and will show some of their properties. This paper is organized as follows. In Section 2, will be giving some definitions of previous results of parametric type Apostol Frobenius-Euler $H_n^c(x;\lambda;u)$ and $H_n^s(x;\lambda;u)$ polynomials. Section 3, will be obtaining several properties of the parametric Apostol Frobenius-Euler and Frobenius-Euler polynomials. Section 4, will be presenting some new series of the Taylor type involving the Apostol Frobenius-Euler numbers $H_n(\lambda;u)$ and Frobenius-Euler numbers $H_n(u)$. Finally, Section 5 will be addressing the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and show some of their properties.

2. Background an Previous Results

The following standard notions will be used: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

For real parameters, p and q in [8] was obtained that the Taylor series representation of the following functions $e^{pz}\cos(qz)$ and $e^{pz}\sin(qz)$ is given by

$$e^{pz}\cos(qz) = \sum_{k=0}^{\infty} C_k(p,q)\frac{z^k}{k!},$$
$$e^{pz}\sin(qz) = \sum_{k=0}^{\infty} S_k(p,q)\frac{z^k}{k!},$$

where $C_k(p,q)$ and $S_k(p,q)$ is given by

$$C_k(p,q) = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k}{2j} p^{k-2j} q^{2j}, \tag{2.1}$$

$$S_k(p,q) = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} (-1)^j \binom{k}{2j+1} p^{k-2j-1} q^{2j+1}. \tag{2.2}$$

Also it is fulfilled (see, [13, p. 944]):

$$C_k(p,p) = 2^{\frac{k}{2}} p^k \cos\frac{k\pi}{4},$$
$$S_k(p,p) = 2^{\frac{k}{2}} p^k \sin\frac{k\pi}{4},$$
$$C_k(0,q) = q^k \cos\frac{k\pi}{2}, \tag{2.3}$$
$$S_k(0,q) = q^k \sin\frac{k\pi}{2}, \tag{2.4}$$
$$C_k(p,0) = p^k \quad \text{and} \quad S_k(p,0) = 0.$$

Using the definitions of $C_n(p;q)$, $S_n(p;q)$ and the Apostol Frobenius-Euler numbers $H_n(\lambda;u)$ we have, two parametric of Apostol-type Frobenius-Euler polynomials

$$H_{n,c}(p,q;\lambda,u) = H_n(\lambda,u)C_n(p,q),$$
$$H_{n,s}(p,q;\lambda,u) = H_n(\lambda,u)S_n(p,q),$$

which exponential generating of $H_{n,c}(p;q;\lambda;u)$ and $H_{n,s}(p;q;\lambda;u)$ functions are given respectively, by (see, [10, p. 5, (14) and (15)]):

$$\left[\frac{1-u}{\lambda e^z - u}\right]^{(\alpha)} e^{pz}\cos(qz) = \sum_{n=0}^{\infty} H_{n,c}^{[\alpha]}(p,q;\lambda;u)\frac{z^n}{n!}, \tag{2.5}$$

$$\left[\frac{1-u}{\lambda e^z - u}\right]^{(\alpha)} e^{pz}\sin(qz) = \sum_{n=0}^{\infty} H_{n,s}^{[\alpha]}(p,q;\lambda;u)\frac{z^n}{n!}. \tag{2.6}$$

Thus, according to the Cauchy series product, we obtain

$$H_{n,c}(p,q;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(\lambda;u) C_k(p,q), \tag{2.7}$$

$$H_{n,s}(p,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(\lambda;u)S_k(p,q). \tag{2.8}$$

Therefore, from equation (2.5) it is observed that when the parameter q takes the value 0 one has $H_{n,c}(p;q;\lambda;u)=H_n(p;\lambda;u)$ and the Apostol Frobenius-Euler polynomials are obtained. On the other hand from (2.5) and (2.6) it is easy to obtain the following statement

$$\begin{aligned} H^{[\alpha+\beta]}_{n,c}(p+q,q;\lambda;u) &=\sum_{k=0}^{n}\binom{n}{k}H^{[\alpha]}_{k,c}(p,q,\lambda;u)H^{[\beta]}_{n-k,c}(q,0,\lambda;u),\\ H^{[\alpha+\beta]}_{n,s}(p+q,q;\lambda;u) &=\sum_{k=0}^{n}\binom{n}{k}H^{[\alpha]}_{k,s}(p,q,\lambda;u)H^{[\beta]}_{n-k,s}\left(q,\frac{\pi}{2z},\lambda;u\right). \end{aligned} \tag{2.9}$$

Below, a list of the first parametric Apostol Frobenius-Euler polynomials for $H_{n,c}(p,q;\lambda;u)$ and $H_{n,s}(p,q;\lambda;u)$ are shown:

$$\begin{aligned} H_{0,c}(p,q;\lambda;u) =&\frac{1-u}{\lambda-u},\\ H_{1,c}(p,q;\lambda;u) =&\frac{1-u}{\lambda-u}p-\frac{\lambda(1-u)}{(\lambda-u)^2},\\ H_{2,c}(p,q;\lambda;u) =&\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right](1-u)-\frac{1-u}{\lambda-u}q^2+\frac{1-u}{(\lambda-u)^2}p^2\\ &+\left[\frac{2(1-u)\lambda}{(\lambda-u)^2}\right]p,\\ H_{3,c}(p,q;\lambda;u) =&\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]p-3\frac{1-u}{\lambda-u}pq^2+3\lambda\frac{1-u}{(\lambda-u)^2}q^2\\ &+(1-u)\left[-\frac{6\lambda^3}{(\lambda-u)^4}+\frac{6\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right],\\ H_{4,c}(p,q;\lambda;u) =&\frac{1-u}{\lambda-u}p^4-\frac{4\lambda(1-u)}{(\lambda-u)^2}p^3+6(1-u)\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]p^2\\ &+4(1-u)\left[\frac{-6\lambda^3}{(\lambda-u)^4}+\frac{6\lambda^2}{(\lambda-u)^3}+\frac{\lambda}{(\lambda-u)^2}\right]p-\frac{6(1-u)}{\lambda-u}p^2q^2\\ &+12\lambda\frac{1-u}{(\lambda-u)^2}pq^2\\ &-6\lambda(1-u)\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]q^2+\frac{1-u}{\lambda-u}q^4\\ &+(1-u)\left[\frac{24\lambda^4}{(\lambda-u)^4}-\frac{36\lambda^3}{(\lambda-u)^4}+\frac{14\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right],\\ H_{0,s}(p,q;\lambda;u) =&0, \end{aligned}$$

$$
\begin{aligned}
H_{1,s}(p,q;\lambda;u) =& \frac{1-u}{\lambda-u}q,\\
H_{2,s}(p,q;\lambda;u) =& -2\lambda\frac{1-u}{(\lambda-u)^2}q + 2\frac{1-u}{\lambda-u}pq,\\
H_{3,s}(p,q;\lambda;u) =& -\lambda\frac{1-u}{\lambda-u}q^3 + 2\frac{1-u}{\lambda-u}p^2q - 6\lambda\frac{1-u}{\lambda-u}pq\\
&+3(1-u)\lambda\left[\frac{2\lambda}{(\lambda-u)^3} - \frac{1}{(\lambda-u)^2}\right]q.
\end{aligned}
$$

Let p be any nonzero real number. The generalized Pascal matrix of first kind $P[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [16, Definition 1]):

$$
p_{i,j}(p) := \begin{cases} \binom{i}{j}(p)^{i-j}, & i \geq j,\\ 0, & \text{otherwise.}\end{cases}
$$

Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence, i.e., $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci matrix $\mathscr{F} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [19]):

$$
f_{i,j} := \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0,\\ 0, & \text{if } i-j+1 < 0.\end{cases}
$$

Let $\{L_n\}_{n\geq 1}$ be the Lucas numbers sequence, i.e., $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$. The Lucas matrix $\mathscr{L} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [20]):

$$
l_{i,j} := \begin{cases} L_{i-j+1}, & \text{if } i-j \geq 0,\\ 0, & \text{if } i-j < 0.\end{cases}
$$

3. The Parametric of Apostol-type Frobenius-Euler Polynomials and their Properties of $H_n^c(p,q;\lambda;u)$ and $H_n^s(p,q;\lambda;u)$

In this section, some properties of the parametric Apostol-type Frobenius-Euler polynomials $H_{n,c}(p,q;\lambda;u)$ and $H_{n,s}(p,q;\lambda;u)$, will be presented.

Proposition 3.1. *For every $n \in \mathbb{N}$, the parametric Apostol-type Frobenius-Euler $H_{n,c}(p;q;\lambda;u)$ and $H_{n,s}(p;q;\lambda;u)$ polynomials meet the following identity*

$$
\lambda H_{n,c}(1+p,q;\lambda;u) - uH_{n,c}(p;q;\lambda;u) = (1-u)C_n(p,q), \tag{3.1}
$$

$$
\lambda H_{n,s}(1+p,q;\lambda;u) - uH_{n,s}(p;q;\lambda;u) = (1-u)S_n(p,q). \tag{3.2}
$$

Proof.

$$
\begin{aligned}
(\lambda+u+1)\sum_{n=0}^{\infty} H_{n,c}(1+p,q;\lambda;u)\frac{z^n}{n!} =& \frac{1-u}{\lambda e^z - u}e^{(1+p)z}\cos(qz)(\lambda+u+1)\\
=& e^{pz}\left(\lambda e^z + ue^z + e^z - u + u\right)\cos(qz)\frac{1-u}{\lambda e^z - u}
\end{aligned}
$$

$$
\begin{aligned}
&=(1-u)e^{pz}\cos(qz)+ue^{pz}\cos(qz)\frac{1-u}{\lambda e^z-u}\\
&\quad+(1+u)e^{(p+1)z}\cos(qz)\frac{1-u}{\lambda e^z-u}\\
&=(1+u)\sum_{n=0}^{\infty}H_{n,c}(1+p,q;\lambda;u)\frac{z^n}{n!}\\
&\quad+(1-u)\sum_{n=0}^{\infty}C_n(p,q)\frac{z^n}{n!}\\
&\quad+u\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}. \qquad \square
\end{aligned}
$$

So, the first statement given in (3.1) was demonstrated. The proof of (3.2) is obtained analogously.

Corollary 3.1. *If in Proposition* 3.1 *the relationships* (3.1) *and* (3.2) *take a value of* $p=0$*, then it is true*

$$\lambda H_{2n,c}(1,q;\lambda;u)-uH_{2n,c}(q;\lambda;u)=(1-u)(-1)^n q^{2n}$$

and

$$\lambda H_{2n+1,s}(1,q;\lambda;u)-uH_{2n+1,s}(q;\lambda;u)=(1-u)(-1)^n q^{2n+1}.$$

Proposition 3.2. *For every* $n\in\mathbb{Z}^+$*, the parametric Apostol-type Frobenius-Euler* $H_{n,c}(p;q;\lambda;u)$ *and* $H_{n,s}(p;q;\lambda;u)$ *polynomials meet the following identity*

$$H_{n,c}(p+l,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{k,c}(p,q;\lambda;u)l^{n-k}, \tag{3.3}$$

$$H_{n,s}(p+l,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{k,s}(p,q;\lambda;u)l^{n-k}. \tag{3.4}$$

Proof. Using (2.5) one obtained

$$
\begin{aligned}
\sum_{n=0}^{\infty}H_{n,c}(p+l,q;\lambda;u)\frac{z^n}{n!}&=\left(\frac{1-u}{\lambda e^z-u}e^{pz}\cos(qz)\right)e^{lz}\\
&=\left(\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}l^n\frac{z^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}H_{k,c}(p,q;\lambda;u)l^{n-k}.
\end{aligned}
$$

The first affirmation obtained in (3.3) has been proven. The other result (3.4) can be demonstrated similarly. $\square$

Corollary 3.2. *The following statements are valid*

$$H_{n,c}(p+1,q;\lambda;u)-H_{n,c}(p,q;\lambda;u)=\sum_{k=0}^{n-1}\binom{n}{k}H_{k,c}(p,q;\lambda;u)$$

and

$$H_{n,s}(p+1,q;\lambda;u)-H_{n,s}(p,q;\lambda;u)=\sum_{k=0}^{n-1}\binom{n}{k}H_{k,s}(p,q;\lambda;u).$$

Using the Corollary 3.2 and the Proposition 3.1, the following recurrence formulas are obtained:

$$\begin{aligned} H_{n,c}(p,q;\lambda;u) =& \frac{1}{\lambda-u}\left[(1-u)C_n(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}H_{k,c}(p,q;\lambda;u)\right], \\ H_{n,s}(p,q;\lambda;u) =& \frac{1}{\lambda-u}\left[(1-u)S_{n-1}(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}H_{k,s}(p,q;\lambda;u)\right], \end{aligned} \tag{3.5}$$

where $H_{0,c}(p,q;\lambda;u)=\dfrac{1-u}{\lambda-u}$ and $H_{0,s}(p,q;\lambda;u)=0$.

Proposition 3.3. *For every $n\in\mathbb{N}$, the following partial derivative identities are correct*

$$\frac{\partial}{\partial p}\left[H_{n,c}(p,q;\lambda;u)\right] = nH_{n-1,c}(p,q;\lambda;u), \tag{3.6}$$

$$\frac{\partial}{\partial p}\left[H_{n,s}(p,q;\lambda;u)\right] = nH_{n-1,s}(p,q;\lambda;u), \tag{3.7}$$

$$\frac{\partial}{\partial q}\left[H_{n,c}(p,q;\lambda;u)\right] = -nH_{n-1,s}(p,q;\lambda;u), \tag{3.8}$$

$$\frac{\partial}{\partial q}\left[H_{n,s}(p,q;\lambda;u)\right] = nH_{n-1,c}(p,q;\lambda;u). \tag{3.9}$$

It will be shown (3.6), the proof of (3.7), (3.8) and (3.9) are similar.

Proof.

$$\begin{aligned} \frac{\partial}{\partial p}\left[\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}\right] &= \sum_{k=0}^{\infty}\frac{\partial}{\partial p}\left[H_{n,c}(p,q;\lambda;u)\right]\frac{z^n}{n!} \\ &= \frac{1-u}{\lambda e^z-u}ze^{pz}\cos(qz) \\ &= \sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty}H_{n-1,c}(p,q;\lambda;u)\frac{z^n}{(n-1)!} \\ &= \sum_{n=1}^{\infty}nH_{n-1,c}(p,q;\lambda;u)\frac{z^n}{(n)!}, \end{aligned}$$

by comparing the coefficients of the series, one has the result. □

Proposition 3.4. *The polynomials $H_{n,c}(p, q; \lambda; u)$ and $H_{n,s}(p, q; \lambda; u)$ are, respectively, of degrees n and $n-1$ in the variable p, it is also asserted that*

$$H_{n,c}(p; q; \lambda; u) = \frac{1-u}{\lambda - u} p^n - n \frac{1-u}{(\lambda-u)^2} p^{n-1} + \cdots , \tag{3.10}$$

$$H_{n,s}(p; q; \lambda; u) = \frac{n(1-u)q}{\lambda - u} p^{n-1} - \frac{n(n-1)(1-u)\lambda q}{(\lambda-u)^2} p^{n-2} + \cdots . \tag{3.11}$$

Proof. First, the result given in (3.10) is shown using the method of mathematical induction on n. On the other hand of (3.5) it has

$$H_{0,c}(p, q; \lambda; u) = \frac{1-u}{\lambda - u},$$
$$H_{1,c}(p, q; \lambda; u) = \frac{(1-u)p}{\lambda - u} - \frac{\lambda(1-u)}{(\lambda-u)^2}$$

and

$$H_{2,c}(p, q; \lambda; u) = \frac{(1-u)p^2}{(\lambda-u)^3} - 2\frac{(1-u)\lambda p}{(\lambda-u)^2} - \frac{1-u}{\lambda-u} q^2 + \frac{2\lambda^2(1-u) - \lambda(\lambda-u)}{(\lambda-u)^3}.$$

Therefore, the statement given in (3.10) is valid for $n = 0, 1, 2$. It will be assumed that it is correct for $n-1$. Using (3.6), we get

$$\frac{\partial}{\partial p}\left[H_{n,c}(p, q; \lambda; u)\right] = n\frac{1-u}{\lambda+1} p^{n-1} - n(n-1)\frac{1-u}{(\lambda-u)^2} p^{n-2} + \cdots .$$

To obtain the final result given in (3.10) it is necessary to integrate with respect to variable p. The results (3.9) and (3.11) are obtained analogously. □

Proposition 3.5. *If $n \in \mathbb{N}$, $\lambda > 0$, $u \neq \lambda$ and m is an odd positive integer, then*

$$H_{n,c}(mp, q; \lambda^{\frac{1}{m}}; u^{\frac{1}{m}}) = m^n \sum_{k=0}^{m-1} u^{\frac{m-1}{m}} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda; u\right) \tag{3.12}$$

and

$$H_{n,s}(mp, q; \lambda^{\frac{1}{m}}; u^{\frac{1}{m}}) = m^n \sum_{k=0}^{m-1} u^{\frac{m-1}{m}} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,s}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda; u\right). \tag{3.13}$$

Proof. To prove (3.12), it avails to consider the following relation:

$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda; u\right) = \frac{1-u}{\lambda e^z - u} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} e^{\left(p+\frac{k}{m}\right)z} \cos\left(\frac{qz}{m}\right),$$

take a sum over k from 0 to $m-1$, one has

$$\begin{aligned}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p + \frac{k}{m}, \frac{q}{m}; \lambda; u\right) &= \frac{1-u}{\lambda e^z - u} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} e^{pz} \cos\left(\frac{qz}{m}\right) \\ &= \frac{(1-u)e^{mp\frac{z}{m}} \cos\left(\frac{qz}{m}\right) u^{\frac{1-m}{m}}}{\lambda^{\frac{1}{m}} e^{\frac{z}{m}} - u^{\frac{1}{m}}}\end{aligned}$$

$$= \sum_{n=0}^{\infty} m^{-n} u^{\frac{1-m}{m}} H_{n,c}\left(mp, q; \lambda^{\frac{1}{m}}; u^{\frac{1}{m}}\right) \frac{z^n}{n!}.$$

To the test (3.13) the proof is similarly. □

New results are presented below for parametric Frobenius-Euler polynomials.

Proposition 3.6. *For every $n \in \mathbb{N}$, the parametric Frobenius-Euler $H_n^c(p;q;u)$ and $H_{n,s}(p;q;u)$ polynomials meet the following identity*

$$H_{n,c}(1+p,q;u) - uH_{n,c} = (1-u)C_n(p,q), \tag{3.14}$$

$$H_{n,s}(1+p,q;u) - uH_{n,s} = (1-u)S_n(p,q). \tag{3.15}$$

Proof.

$$\begin{aligned}(1+u+1)\sum_{n=0}^{\infty} H_{n,c}(1+p,q;u) =& \frac{1-u}{e^z-u} e^{(1+p)z}\cos(qz)(1+u+1)\\
=& e^{pz}\left(e^z + ue^z + e^z - u + u\right)\cos(qz)\frac{1-u}{e^z-u}\\
=& (1-u)e^{pz}\cos(qz) + (1+u)e^{(p+1)z}\cos(qz)\frac{1-u}{e^z-u}\\
&+ ue^{pz}\cos(qz)\frac{1-u}{e^z-u}\\
=& (1-u)\sum_{n=0}^{\infty} C_n(p,q)\frac{z^n}{n!}\\
&+ (1+u)\sum_{n=0}^{\infty} H_{n,c}(1+p,q;u)\frac{z^n}{n!}\\
&+ u\sum_{n=0}^{\infty} H_{n,c}(p,q;u)\frac{z^n}{n!},\end{aligned}$$

which proves the first assertion (3.14). The proof of the second assertion (3.15) is similar. □

Corollary 3.3. *For every $n \in \mathbb{N}$, the following identities hold true*

$$\begin{aligned}H_{n,c}(1+p,q;u) - uH_{n,c}(p,q,u) =& (1-u)C_n(p,q),\\
H_{n,s}(1+p,q;u) - uH_{n,s}(p;q;u) =& (1-u)S_n(p,q).\end{aligned}$$

Corollary 3.4. *If in Proposition* 3.6 *the relationships* (3.14) *and* (3.15) *take a value of $p = 0$, then it is true*

$$H_{2n,c}(1,q;u) - uH_{2n,c}(q;u) = (1-u)(-1)^n q^{2n}$$

and

$$H_{2n+1,s}(1,q;\lambda;u) - uH_{2n+1,s}(q;u) = (1-u)(-1)^n q^{2n+1}.$$

Proposition 3.7. *For every $n \in \mathbb{Z}^+$, the following identities hold true*

$$H_{n,c}(p+l,q;u) = \sum_{k=0}^{n} \binom{n}{k} H_{k,c}(p,q;u) l^{n-k}$$

and

$$H_{n,s}(p+l,q;u) = \sum_{k=0}^{n} \binom{n}{k} H_{k,s}(p,q;u) l^{n-k}.$$

Corollary 3.5. *The following statements are valid*

$$H_{n,c}(p+1,q;u) - H_{n,c}(p,q;u) = \sum_{k=0}^{n-1} \binom{n}{k} H_{k,c}(p,q;u)$$

and

$$H_{n,s}(p+1,q;u) - H_{n,s}(p,q;\lambda;u) = \sum_{k=0}^{n-1} \binom{n}{k} H_{k,s}(p,q;\lambda;u).$$

Using Corollary 3.5 and Proposition 3.6, the following recurrences are obtained:

$$H_{n,c}(p,q;u) = \frac{1}{1-u}\left[(1-u)C_n(p,q) - \sum_{k=0}^{n-1} \binom{n}{k} H_{k,c}(p,q;u)\right]$$

and

$$H_{n,s}(p,q;u) = \frac{1}{1-u}\left[(1-u)S_{n-1}(p,q) - \sum_{k=0}^{n-1} \binom{n}{k} H_{k,s}(p,q;u)\right],$$

where $H_{0,c}(p,q;u) = 1$ and $H_{0,s}(p,q;u) = 0$.

Proposition 3.8. *For every $n \in \mathbb{N}$, the following identities hold true*

$$\frac{\partial}{\partial p}\left[H_{n,c}(p,q;u)\right] = nH_{n-1,c}(p,q;u), \tag{3.16}$$

$$\frac{\partial}{\partial p}\left[H_{n,s}(p,q;u)\right] = nH_{n-1,s}(p,q;u) \tag{3.17}$$

and

$$\frac{\partial}{\partial q}\left[H_{n,c}(p,q;u)\right] = -nH_{n-1,s}(p,q;u), \tag{3.18}$$

$$\frac{\partial}{\partial q}\left[H_{n,s}(p,q;u)\right] = nH_{n-1,c}(p,q;u). \tag{3.19}$$

It will be shown (3.16), the demonstrations of (3.17), (3.18) and (3.19) are similar.

Proof.

$$\begin{aligned}\frac{\partial}{\partial p}\left[\sum_{n=0}^{\infty} H_{n,c}(p,q;u)\frac{z^n}{n!}\right] &= \sum_{k=0}^{\infty} \frac{\partial}{\partial p}\left[H_{n,c}(p,q;u)\right]\frac{z^n}{n!} \\ &= \frac{1-u}{e^z-u} z e^{pz}\cos(qz)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} H_{n,c}(p,q;u)\frac{z^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty} H_{n-1,c}(p,q;u)\frac{z^{n}}{(n-1)!} \\
&= \sum_{n=1}^{\infty} nH_{n-1,c}(p,q;u)\frac{z^{n}}{(n)!}.
\end{aligned}
$$

□

Proposition 3.9. *The polynomials $H_{n,c}(p,q;u)$ and $H_{n,s}(p,q;u)$ are, respectively, of degrees n and $n-1$ in the variable p it is also asserted that*

$$
\begin{aligned}
H_{n,c}(p;q;u) =& p^n - \frac{1}{1-u}p^{n-1} + \cdots, \\
H_{n,s}(p;q;u) =& nqp^{n-1} - \frac{n(n-1)q}{1-u}p^{n-2} + \cdots.
\end{aligned}
$$

4. Taylor Type Series Involving the Apostol-type Frobenius-Euler Numbers and Frobenius-Euler Numbers $H_n(\lambda;u)$ and $H_n(u)$

An important fact of relationships (2.5) and (2.6) is that one can trace them as the expansion in Taylor series of some functions on the point $z=0$ and relate it to Apostol-Frobenius-Euler and Frobenius-Euler numbers. So, replacing (2.7) and (2.8) in (2.5) and (2.6), one has

$$
\begin{aligned}
f^c_{H,\lambda;u}(z;p,q) =& \frac{1-u}{\lambda e^z - u}e^{pz}\cos(qz) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(\lambda;u)C_k(p,q)\right]\frac{z^n}{n!}, \\
f^s_{H,\lambda;u}(z;p,q) =& \frac{1-u}{\lambda e^z - u}e^{pz}\sin(qz) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(\lambda;u)S_k(p,q)\right]\frac{z^n}{n!}, \\
f^c_{H,u}(z;p,q) =& \frac{1-u}{e^z - u}e^{pz}\cos(qz) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(u)C_k(p,q)\right]\frac{z^n}{n!}, \\
f^s_{H,u}(z;p,q) =& \frac{1-u}{e^z - u}e^{pz}\sin(qz) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(u)S_k(p,q)\right]\frac{z^n}{n!},
\end{aligned}
\tag{4.1}
$$

where $C_k(p,q)$ and $S_k(p,q)$ are defined in (2.1) and (2.2). Some particular cases will be shown using result previously known in Section 6 of [13].

Example 4.1. In (4.1), taking $p=0$ and $q=1$, and using (2.3) and (2.4), one obtain

$$
\begin{aligned}
f^c_{H,\lambda;u}(z;0,1) &= \frac{1}{\lambda e^z - u}\cos(z) \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{1-u}\binom{n}{k}H_{n-k}(\lambda;u)\cos\frac{k\pi}{2}\right]\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{1}{1-u}\binom{n}{k}H_{n-k}(\lambda;u)\right]\frac{z^n}{n!}.
\end{aligned}
$$

Therefore, one has

$$\frac{1}{\lambda e^z-u}\cos(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{1}{1-u}\binom{n}{2k}H_{n-2k}(\lambda;u)\right]\frac{z^n}{n!},$$

as well as

$$\frac{1}{\lambda e^z-u}\sin(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\frac{1}{1-u}\binom{n}{2k+1}H_{n-2k-1}(\lambda;u)\right]\frac{z^n}{n!},$$

$$\frac{1}{e^z-u}\cos(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{1}{1-u}\binom{n}{2k}H_{n-2k}(u)\right]\frac{z^n}{n!}$$

and

$$\frac{1}{e^z-u}\sin(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\frac{1}{1-u}\binom{n}{2k+1}H_{n-2k-1}(u)\right]\frac{z^n}{n!}.$$

Example 4.2. Putting $p=q=1$ in (4.1), one gets

$$\frac{e^z}{\lambda e^z-u}\cos(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k}H_{n-k}(\lambda;u)\ \cos\frac{k\pi}{4}\right]\frac{z^n}{n!},$$

$$\frac{e^z}{\lambda e^z-u}\sin(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k}H_{n-k}(\lambda;u)\ \sin\frac{k\pi}{4}\right]\frac{z^n}{n!},$$

$$\frac{e^z}{e^z-u}\cos(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k}H_{n-k}(u)\ \cos\frac{k\pi}{4}\right]\frac{z^n}{n!}$$

and

$$\frac{e^z}{e^z-u}\sin(z)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k}H_{n-k}(u)\ \sin\frac{k\pi}{4}\right]\frac{z^n}{n!}.$$

5. Parametric Apostol-type Frobenius-Euler Polynomials Matrix

Inspired by [11,12,16], this section will address the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and will show some of their properties.

Definition 5.1. The generalized $(n+1)\times(n+1)$ parametric Apostol-type Frobenius-Euler polynomials matrices $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ and $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ are defined by

$$\mathscr{H}_{i,j,c}^{(\alpha)}(p,q;\lambda;u)=\begin{cases}\binom{i}{j}H_{i-j,c}^{(\alpha)}(p,q;\lambda;u), & i\geq j,\\ 0, & \text{otherwise},\end{cases}$$

and

$$\mathscr{H}_{i,j,s}^{(\alpha)}(p,q;\lambda;u)=\begin{cases}\binom{i}{j}H_{i-j,s}^{(\alpha)}(p,q;\lambda;u), & i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

Since, $H_{n,c}^{(0)}(p;0;\lambda;u)=p^n$ and $H_{n,s}^{(0)}\left(p;\frac{\pi}{2z};\lambda;u\right)=p^n$, we obtain

$$\mathscr{H}_{c}^{(0)}(p;0;\lambda;u)=P[p],\quad \mathscr{H}_{s}^{(0)}\left(p;\frac{\pi}{2z};\lambda;u\right)=P[p].$$

Theorem 5.1. *The generalized parametric Apostol-type Frobenius-Euler polynomials matrices* $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ *and* $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ *satisfies the following product formulae*

$$\begin{aligned}\mathscr{H}_c^{(\alpha+\beta)}(p+q;q;\lambda;u)&=\mathscr{H}_c^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\beta)}(q;0;\lambda;u)\\&=\mathscr{H}_c^{(\beta)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\alpha)}(q;0;\lambda;u)\\&=\mathscr{H}_c^{(\alpha)}(q;0;\lambda;u)\,\mathscr{H}_c^{(\beta)}(p;q;\lambda;u),\end{aligned}\tag{5.1}$$

$$\begin{aligned}\mathscr{H}_s^{(\alpha+\beta)}(p+q;q;\lambda;u)&=\mathscr{H}_s^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_s^{(\beta)}\left(q;\frac{\pi}{2z};\lambda;u\right)\\&=\mathscr{H}_s^{(\beta)}(p;q;\lambda;u)\,\mathscr{H}_s^{(\beta)}\left(q;\frac{\pi}{2z};\lambda;u\right)\\&=\mathscr{H}_s^{(\alpha)}\left(q;\frac{\pi}{2z};\lambda;u\right)\,\mathscr{H}_s^{(\beta)}(p;q;\lambda;u).\end{aligned}\tag{5.2}$$

Proof. Let $D_{i,j,c}^{[\alpha,\beta]}(\lambda;u)(p,q)$ be the (i,j)-th entry of the matrix product $\mathscr{H}_c^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\beta)}(q;0;\lambda;u)$, then by the addition formula (2.9) we have

$$\begin{aligned}D_{i,j,c}^{[\alpha,\beta]}(\lambda;u)(p,q)&=\sum_{k=0}^{n}\binom{i}{k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)\binom{k}{j}H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\sum_{k=j}^{i}\binom{i}{k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)\binom{k}{j}H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}H_{i-j-k,c}^{[\alpha]}(p;q,\lambda;u)H_{k,c}^{[\beta]}(q;0;\lambda;u)\\&=\binom{i}{j}H_{i-j,c}^{[\alpha+\beta]}(p+q;q;\lambda;u),\end{aligned}$$

which implies (5.1). The second and third equalities of the theorem and (5.2), can be derived in a similar way. □

Corollary 5.1. *The generalized parametric Apostol-type Frobenius-Euler polynomials matrices* $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ *and* $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ *satisfy the following relations*

$$\begin{aligned}\mathscr{H}_c^{[\alpha]}(p+q;q;\lambda;u) &= \mathscr{H}_c^{[\alpha]}(p;q;\lambda;u)P[q] = P[p]\mathscr{H}_c^{[\alpha]}(q;q;\lambda;u)\\ &= \mathscr{H}_c^{[\alpha]}(q;q;\lambda;u)P[p],\\ \mathscr{H}_s^{[\alpha]}(p+q;q;\lambda;u) &= \mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)P[q] = P[p]\mathscr{D}_s^{[\alpha]}(q;q;\lambda;u)\\ &= \mathscr{D}_s^{[\alpha]}(q;q;\lambda;u)P[p].\end{aligned}$$

In particular,

$$\mathscr{H}_c(p+q;q;\lambda;u) = P[p]\mathscr{H}_c(q;q;\lambda;u) = P[q]\mathscr{H}_c(p;q,\lambda;u).$$

Example 5.1. For $\alpha = 1$ the first three polynomials $\mathscr{H}_{k,c}^{[\alpha]}(p;q;\lambda;u)$, $k = 0,1,2$, are

$$\begin{aligned}H_0^c(p,q;\lambda;u) =& \frac{1-u}{\lambda-u},\\ H_1^c(p,q;\lambda;u) =& \frac{1-u}{\lambda-u}p - \frac{\lambda(1-u)}{(\lambda-u)^2},\\ H_2^c(p,q;\lambda;u) =& \left[\frac{2\lambda^2}{(\lambda-u)^3} - \frac{\lambda}{(\lambda-u)^2}\right](1-u) - \frac{1-u}{\lambda-u}q^2 + \frac{1-u}{(\lambda-u)^2}p^2\\ &+ \left[\frac{2(1-u)\lambda}{(\lambda-u)^2}\right]p.\end{aligned}$$

Hence, for $n = 2$ we have

$$\mathscr{H}_c^{[1]}(p;q,\lambda,u) = \begin{bmatrix} H_0^c(p,q;\lambda;u) & 0 & 0\\ H_1^c(p,q;\lambda;u) & H_0^c(p,q;\lambda;u) & 0\\ H_2^c(p,q;\lambda;u) & 2H_1^c(p,q;\lambda;u) & H_0^c(p,q;\lambda;u) \end{bmatrix}.$$

For $\alpha, u, \lambda \in \mathbb{C}$, $0 \le i,j \le n$, let $\mathbb{K}_c^{[\alpha]}(p;q;\lambda,u)$ and $\mathbb{K}_s^{[\alpha]}(p;q;\lambda,u)$ be the matrices whose entries are defined by

$$\begin{aligned}\tilde{r}_{i,j,c}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j}H_{i-j,c}^{[\alpha]}(p;q;\lambda;u) - \binom{i-1}{j}H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)\\ &- \binom{i-2}{j}H_{i-j-2,c}^{[\alpha]}(p;q;\lambda;u),\\ \tilde{r}_{i,j,s}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j}H_{i-j,s}^{[\alpha]}(p;q;\lambda;u) - \binom{i-1}{j}H_{i-j-1,s}^{[\alpha]}(p;q;\lambda;u)\\ &- \binom{i-2}{j}H_{i-j-2,s}^{[\alpha]}(p;q;\lambda;u).\end{aligned}$$

On the other hand, $\mathscr{J}_c^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{J}_s^{[\alpha]}(p;q;\lambda;u)$ are the matrices whose entries are given by

$$\tilde{s}_{i,j,c}^{[\alpha]}(p;q;\lambda;u) = \binom{i}{j}H_{i-j,c}^{[\alpha]}(p;q;\lambda;u) - \binom{i}{j+1}H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)$$

$$- \binom{i}{j+2} H^{[\alpha]}_{i-j-2,c}(p;q;\lambda;u),$$

$$\tilde{s}^{[\alpha]}_{i,j,s}(p;q;\lambda;u) = \binom{i}{j} H^{[\alpha]}_{i-j,s}(p;q;\lambda;u) - \binom{i}{j+1} H^{[\alpha]}_{i-j-1,s}(p;q;\lambda;u) - \binom{i}{j+2} H^{[\alpha]}_{i-j-2,s}(p;q;\lambda;u).$$

Using the definitions of $\mathbb{K}^{[\alpha]}_c(p;q;\lambda;u)$, $\mathbb{K}^{[\alpha]}_s(p;q;\lambda;u)$, $\mathscr{J}^{[\alpha]}_c(p;q;\lambda;u)$ and $\mathscr{J}^{[\alpha]}_s(p;q;\lambda;u)$, it is observed that

$$\begin{aligned}
\tilde{r}^{[\alpha]}_{0,0,c}(p;q;\lambda;u) =& \tilde{r}^{[\alpha]}_{1,1,c}(p;q;\lambda;u) = \tilde{s}^{[\alpha]}_{0,0,c}(p;q;\lambda;u) = \tilde{s}^{[\alpha]}_{1,1,c}(p;q;\lambda;u) = H^{[\alpha]}_{0,c}(p,q,\lambda;u),\\
\tilde{r}^{[\alpha]}_{0,j,c}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{0,j,c}(p;q;\lambda;u) = 0, \quad j \geq 1,\\
\tilde{r}^{[\alpha]}_{1,0,c}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{1,0,c}(p;q;\lambda;u) = H^{[\alpha]}_{1,c}(p;q;\lambda;u) - H^{[\alpha]}_{0,c}(p;q;\lambda;u),\\
\tilde{r}^{[\alpha]}_{1,j,c}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{1,j,c}(p;q;\lambda;u) = 0, \quad j \geq 2,\\
\tilde{r}^{[\alpha]}_{i,0,c}(p;q;\lambda;u) =& H^{[\alpha]}_{i,c}(p;q;\lambda;u) - H^{[\alpha]}_{i-1,c}(p;q;\lambda;u) - H^{[\alpha]}_{i-2,c}(p;q;\lambda;u), \quad i \geq 2,\\
\tilde{s}^{[\alpha]}_{i,0,c}(p;q;\lambda;u) =& H^{[\alpha]}_{i,c}(p;q;\lambda;u) - 2H^{[\alpha]}_{i-1,c}(p;q;\lambda;u) - H^{[\alpha]}_{i-2,c}(p;q;\lambda;u), \quad i \geq 2,\\
\tilde{r}^{[\alpha]}_{0,0,s}(p;q;\lambda;u) =& \tilde{r}^{[\alpha]}_{1,1,s}(p;q;\lambda;u) = \tilde{s}^{[\alpha]}_{0,0,s}(p;q;\lambda;u) = \tilde{s}^{[\alpha]}_{1,1,s}(p;q;\lambda;u) = H^{[\alpha]}_{0,s}(p,q,\lambda;u),\\
\tilde{r}^{[\alpha]}_{0,j,s}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{0,j,s}(p;q;\lambda;u) = 0, \quad j \geq 1,\\
\tilde{r}^{[\alpha]}_{1,0,s}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{1,0,s}(p;q;\lambda;u) = H^{[\alpha]}_{1,s}(p;q;\lambda;u) - H^{[\alpha]}_{0,s}(p;q;\lambda;u),\\
\tilde{r}^{[\alpha]}_{1,j,s}(p;q;\lambda;u) =& \tilde{s}^{[\alpha]}_{1,j,s}(p;q,\lambda;u) = 0, \quad j \geq 2,\\
\tilde{r}^{[\alpha]}_{i,0,s}(p;q;\lambda;u) =& H^{[\alpha]}_{i,s}(p;q;\lambda;u) - H^{[\alpha]}_{i-1,s}(p;q;\lambda;u) - H^{[\alpha]}_{i-2,s}(p;q;\lambda;u), \quad i \geq 2,\\
\tilde{s}^{[\alpha]}_{i,0,s}(p;q;\lambda;u) =& H^{[\alpha]}_{i,s}(p;q;\lambda;u) - 2H^{[\alpha]}_{i-1,s}(p;q;\lambda;u) - H^{[\alpha]}_{i-2,s}(p;q;\lambda;u), \quad i \geq 2.
\end{aligned}$$

For $\alpha, \lambda, u \in \mathbb{C}$, $0 \leq i, j \leq n$, let $\mathscr{L}^{[\alpha]}_{1,c}(p;q;\lambda;u)$ and $\mathscr{L}^{[\alpha]}_{1,s}(p;q;\lambda;u)$ be the matrices whose entries are given by

$$\hat{l}^{[\alpha]}_{i,j,1,c}(p;q;\lambda;u) = \binom{i}{j} H^{[\alpha]}_{i-j,c}(p;q;\lambda;u) - 3\binom{i-j}{j} H^{[\alpha]}_{i-j-1,c}(p;q;\lambda;u) + 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j} H^{[\alpha]}_{k-j,c}(p;q;\lambda;u),$$

$$\hat{l}^{[\alpha]}_{i,j,1,s}(p;q;\lambda;u) = \binom{i}{j} H^{[\alpha]}_{i-j,s}(p;q;\lambda;u) - 3\binom{i-j}{j} H^{[\alpha]}_{i-j-1,s}(p;q;\lambda;u) + 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j} H^{[\alpha]}_{k-j,s}(p;q;\lambda;u).$$

Similarly, let $\mathscr{L}_{2,c}^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{L}_{2,s}^{[\alpha]}(p;q;\lambda;u)$, $(n+1)\times(n+1)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,2,c}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j} H_{i-j,c}^{[\alpha]}(p;q;\lambda;u) - 3\binom{i}{j+1} H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)\\ &+5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k} H_{i-k,c}^{[\alpha]}(p;q;\lambda;u),\\ \hat{l}_{i,j,2,s}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j} H_{i-j,s}^{[\alpha]}(p;q;\lambda;u) - 3\binom{i}{j+1} H_{i-j-1,s}^{[\alpha]}(p;q;\lambda;u)\\ &+5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k} H_{i-k,s}^{\alpha}(p;q;\lambda;u).\end{aligned}$$

Next we will show factorizations of the matrices $\mathscr{H}_c^{[,\alpha]}(p;q;\lambda;u)$ and $\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)$ involving the Fibonacci and Lucas matrices, respectively.

Theorem 5.2. *The parametric Apostol-type Frobenius-Euler polynomials matrix $\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)$ can be factored in terms of the Fibonacci matrix $\mathscr{F}$ as follows*

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{F}\mathbb{K}_c^{[\alpha]}(p;q;\lambda;u), \tag{5.3}$$

$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u) = \mathscr{F}\mathbb{K}_s^{[\alpha]}(p;q;\lambda;u), \tag{5.4}$$

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{J}_c^{[\alpha]}(p;q;\lambda;u)\mathscr{F}, \tag{5.5}$$

$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u) = \mathscr{J}_s^{[\alpha]}(p;q;\lambda;u)\mathscr{F}. \tag{5.6}$$

Proof. The relation (5.3) is equivalent to

$$\mathscr{F}^{-1}\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathbb{K}_c^{[\alpha]}(p;q;\lambda;u),$$

following the ideas of [11] or [19, Theorem 4.1], and making the corresponding modifications, (5.3) is obtained. □

In addition, the relations (5.3), (5.4), (5.5) and (5.6) allow us to deduce the following identities:

$$\begin{aligned}\mathbb{K}_c^{[\alpha]}(p;q;\lambda;u) =& \mathscr{F}^{-1}\,\mathscr{J}_c^{[\alpha]}(p;q,\lambda;u)\,\mathscr{F},\\ \mathbb{K}_s^{[\alpha]}(p;q;\lambda;u) =& \mathscr{F}^{-1}\,\mathscr{J}_s^{[\alpha]}(p;q,\lambda;u)\,\mathscr{F}.\end{aligned}$$

An analogous reasoning used in the proof of Theorem 5.2, allows us to prove the results below.

Example 5.2. For $\alpha=1$, the matrices, for $n=2$, $K_c^{[1]}(p;q,\lambda;u)$ and $\mathscr{F}$ are

$$\mathbb{K}_c^{[1]}(p;q,\lambda;u) = \begin{bmatrix} a & 0 & 0\\ b-a & a & 0\\ c-b-a & 2b-a & a\end{bmatrix},$$

where $a = H_{0,c}(p, q; \lambda; u)$, $b = H_{1,c}(p, q; \lambda; u)$ and $c = H_{2,c}(p, q; \lambda; u)$,

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Hence,

$$\mathscr{F}\mathbb{K}_c^{[1]} = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix} \quad \text{and} \quad \mathscr{H}_c^{[1]}(p, q, \lambda; u) = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix}.$$

This is a particular case of Theorem 5.2 affirmation (5.3).

Example 5.3. For $\alpha = 1$, the matrices, for $n = 2$, $\mathscr{J}^{[1]}(p; q; \lambda; u)$ and $\mathscr{F}$ are

$$\mathscr{J}^{[1]}(p; q\lambda; u) = \begin{bmatrix} a & 0 & 0 \\ b-a & a & 0 \\ c-2b-a & 2b-a & a \end{bmatrix}, \quad \mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Then

$$\mathscr{J}^{[1]}(p; q; \lambda; u)\mathscr{F} = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix} \quad \text{and} \quad \mathscr{H}_c^{[1]}(p; q; \lambda; u) = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix}.$$

This is a particular case of Theorem 5.2 affirmation (5.5).

Theorem 5.3. *The parametric Apostol-type Frobenius-Euler polynomials matrix $\mathscr{H}_c^{[\alpha]}(p; q; \lambda; u)$ and $\mathscr{H}_s^{[\alpha]}(p; q; \lambda; u)$ can be factored in terms of the Lucas matrix $\mathscr{L}$ of the following form*

$$\mathscr{H}_c^{[\alpha]}(p; q; \lambda; u) = \mathscr{L}\mathscr{L}_{1,c}^{[\alpha]}(x; y; a) \tag{5.7}$$

or

$$\mathscr{H}_c^{[\alpha]}(p; q; \lambda, u)) = \mathscr{L}_{2,c}^{[\alpha]}(p; q; \lambda, u))\mathscr{L},$$
$$\mathscr{H}_s^{[\alpha]}(p; q; \lambda, u) = \mathscr{L}\mathscr{L}_{1,s}^{[\alpha]}(p; q; \lambda, u)$$

or

$$\mathscr{H}_s^{[\alpha]}(p; q; \lambda, u) = \mathscr{L}_{2,s}^{[m-1,\alpha]}(p; q; \lambda, u))\mathscr{L}.$$

Proof. The relation (5.7) is equivalent to

$$\mathscr{L}^{-1}\mathscr{H}_c^{[\alpha]}(p; q; \lambda; u) = \mathscr{L}_{1,c}^{[\alpha]}(x; y; a),$$

following the ideas of [11, Theorem 9], and making the corresponding modifications, (5.7) is obtained. □

Example 5.4. For $\alpha = 1$, the matrices, for $n = 2$, $\mathscr{L}_{1,c}^{[1]}(p; q; \lambda, u)$ and $\mathscr{L}$ are

$$\mathscr{L}_{1,c}^{[1]}(p; q, \lambda, u) = \begin{bmatrix} a & 0 & 0 \\ b-3a & a & 0 \\ c-3b+5a & 2b-3a & a \end{bmatrix}, \quad \mathscr{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix},$$

$$\mathscr{L}\mathscr{L}_{1,c}^{[1]}(x;y;a)=\begin{bmatrix} a & 0 & 0\\ b & a & 0\\ c & 2b & a\end{bmatrix} \quad \text{and} \quad \mathscr{H}_c^{[1]}(p,q,\lambda,u)=\begin{bmatrix} a & 0 & 0\\ b & a & 0\\ c & 2b & a\end{bmatrix}.$$

This is a particular case of Theorem 5.3 affirmation (5.7).

6. Conclusions

The paper aims to present the study of new properties of the polynomials that are introduced in [10]. Certain expressions, representations, and summations of these polynomials are derived in terms of well-known classical special functions. The results we have considered in this paper indicate the usefulness of the series rearrangement technique used to deal with the theory of special functions. we have obtained new series of the Taylor type involving the Apostol Frobenius-Euler numbers and Frobenius-Euler numbers. Finally, they addressed the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and show some of their properties.

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GIELIS TRANSFORMATION OF THE ARCHIMEDEAN SPIRAL

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Abstract. The article shows that the Archimedean spiral, usually described as a smooth spiral, can be transformed in many different shapes. The main part of the article concentrates on the curvature of the transformed spirals. It will also be shown that the shape some of them is an approximation of spiral antennas.

1. Introduction

Gielis transformations of curves were originally introduced in connection with the modelling of shapes of various biological objects, e.g., flowers, fruits, an arrangement of leaves, shapes of shells, and so on [1–4]. Gradually, studies have appeared pointing to the possibility of using transformed curves also in technical applications, e.g., [7–12].

This article aims to continue in theoretical studies in the area of the so-called Gielis' superformula and Gielis curves. In the early 19th century, a French mathematician Gabriel Lamé introduced a generalized equation of the ellipse

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1, \tag{1.1}$$

where $a, b, n \in \mathbb{Q}^+$. The equation (1.1) can generate different types of curves, such as asteroids ($n = 2/3$), parallelograms ($n = 1$), circles and ellipses ($n = 2$), squares and rectangles ($n \to \infty$). All these curves are called Lamé curves or superellipses (Figure 1), e.g., [1–4].

Key words and phrases. Gielis transformation, Archimedean spiral, Gielis curves, curvature, antennas.
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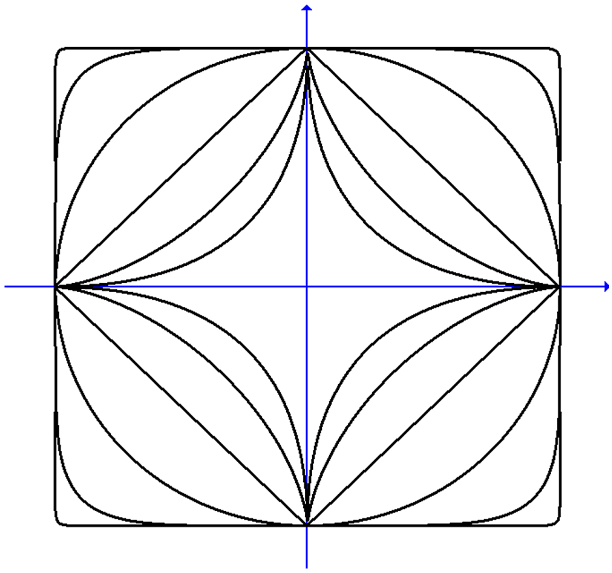

FIGURE 1. Lamé curves for $n = \frac{1}{2}, \frac{2}{3}, 1, 2, 6, 50$, whereas $a = b$ (curves for other cases are obtained by changing the scale on the axes)

The curve (1.1) can also be expressed in polar coordinates (ρ, θ)

$$\rho = \left(\left| \frac{\cos\theta}{a} \right|^n + \left| \frac{\sin\theta}{b} \right|^n \right)^{-\frac{1}{n}}. \tag{1.2}$$

In the late 20th century, Belgium botanist Johan Gielis generalized (1.2) to the form

$$\rho = \left(\left| \frac{1}{a} \cos \frac{m\theta}{4} \right|^{n_1} + \left| \frac{1}{b} \sin \frac{m\theta}{4} \right|^{n_2} \right)^{-\frac{1}{q}}, \tag{1.3}$$

where $a, b, m, n_1, n_2, q \in \mathbb{R}^+$. As can be seen from the equation (1.3), Gielis replaced the exponent n by three independent exponents n_1, n_2, q and inserted an extra parameter $\frac{m}{4}$ into the argument of both trigonometric functions. The Gielis transformation consists in replacing the plane curve expressed in polar coordinates (ρ, θ) with a curve

$$\rho = f(\theta) \left(\left| \frac{1}{a} \cos \frac{m\theta}{4} \right|^{n_1} + \left| \frac{1}{b} \sin \frac{m\theta}{4} \right|^{n_2} \right)^{-\frac{1}{q}}. \tag{1.4}$$

Gielis called the transformation (1.3) and (1.4) as a superformula. Without loss of generality, in (1.3), we focus on the case $a = b = 1$ and $n_1 = n_2 = p$ and put [13]

$$g_{m,p,q}(\theta) = \left(\left| \cos \frac{m\theta}{4} \right|^p + \left| \sin \frac{m\theta}{4} \right|^p \right)^{-\frac{1}{q}}. \tag{1.5}$$

The curve defined by the equation $\rho = g_{m,p,q}(\theta)$ can be interpreted as the Gielis transformation of a unit circle centered at the origin for various choices of the parameters m, p, q. Figure 2 shows that Gielis curves can provide far more complicated shapes than Lamé curves. There are plenty of examples of natural shapes similar to Gielis curves [2, 3, 15, 16].

In this article, the properties of the curves generated by the Gielis transformation of the Archimedean spirals will be investigated. There are two available approaches to what the Archimedean spirals are. The first one considers the general equation in polar coordinates (ρ, θ) of the form

$$\rho = a\theta^{1/n} + b, \tag{1.6}$$

where a, b and n are real constants. Several special cases can be described, depending on the value of n: the arithmetic spiral ($n = 1$), the hyperbolic spiral ($n = -1$), the Fermat spiral ($n = 2$), and lituus ($n = -2$) [4, 5, 14]. The second approach considers the terms the arithmetic spiral and the Archimedean spiral as synonyms (Archimedean spiral, Wikipedia, The Free Encyclopedia, Available from: `https://en.wikipedia.org/w/index.php?title=Archimedean_spiral&oldid=949421005`). In the next parts of this article, the second approach will be followed, and the equation (1.6) will be of the form

$$\rho = a\theta + b. \tag{1.7}$$

The equation (1.7) describes the trajectory of a point moving at a constant speed along a ray spinning around the origin at a constant angular velocity. Changing the parameter b moves the center of the spiral outward from the origin (for the option $b > 0$ toward $\theta = 0$ and for the option $b < 0$ toward $\theta = \pi$). The parameter a changes the distance between loops of the spiral.

Without loss of generality, in the equation (1.7), we focus on the case $b = 0$ and put

$$\rho = a\theta. \tag{1.8}$$

The Archimedean spirals have a variety of real-world applications. Scroll compressors, made from two members (one of them fixed and the other rotating), each of them in the shape of an Archimedean spiral, are used for compressing gases (H. Sakata, O. Masayuki, Fluid compressing device having coaxial spiral members, United States Patent 5603614. `http://www.freepatentsonline.com/5603614.html`). The Archimedean spirals have a constant distance between successive coils and they appear naturally in such systems as a roll of paper, the grooves of a gramophone record, and so on [4, 5]. In food microbiology, the Archimedean spirals are used to quantify bacterial concentration through a spiral platter [6].

There are also plenty of types of Archimedean spiral shaped antennas. Some of them are in the shape of the smooth Archimedean spiral [11] and the others, as it will be shown latter, are in the shape of transformed Archimedean spirals, e.g., [7–10].

In the article [13], Matsuura discusses the mathematical structure of the curves given by the equation $\rho = g_{m,p,q}(\theta)$. Matsuura also introduces the concept of Gielis regular polygons, which he further compares with regular polygons. The substantial part of the article deals with the curvature of Gielis curves. In the article [15], the

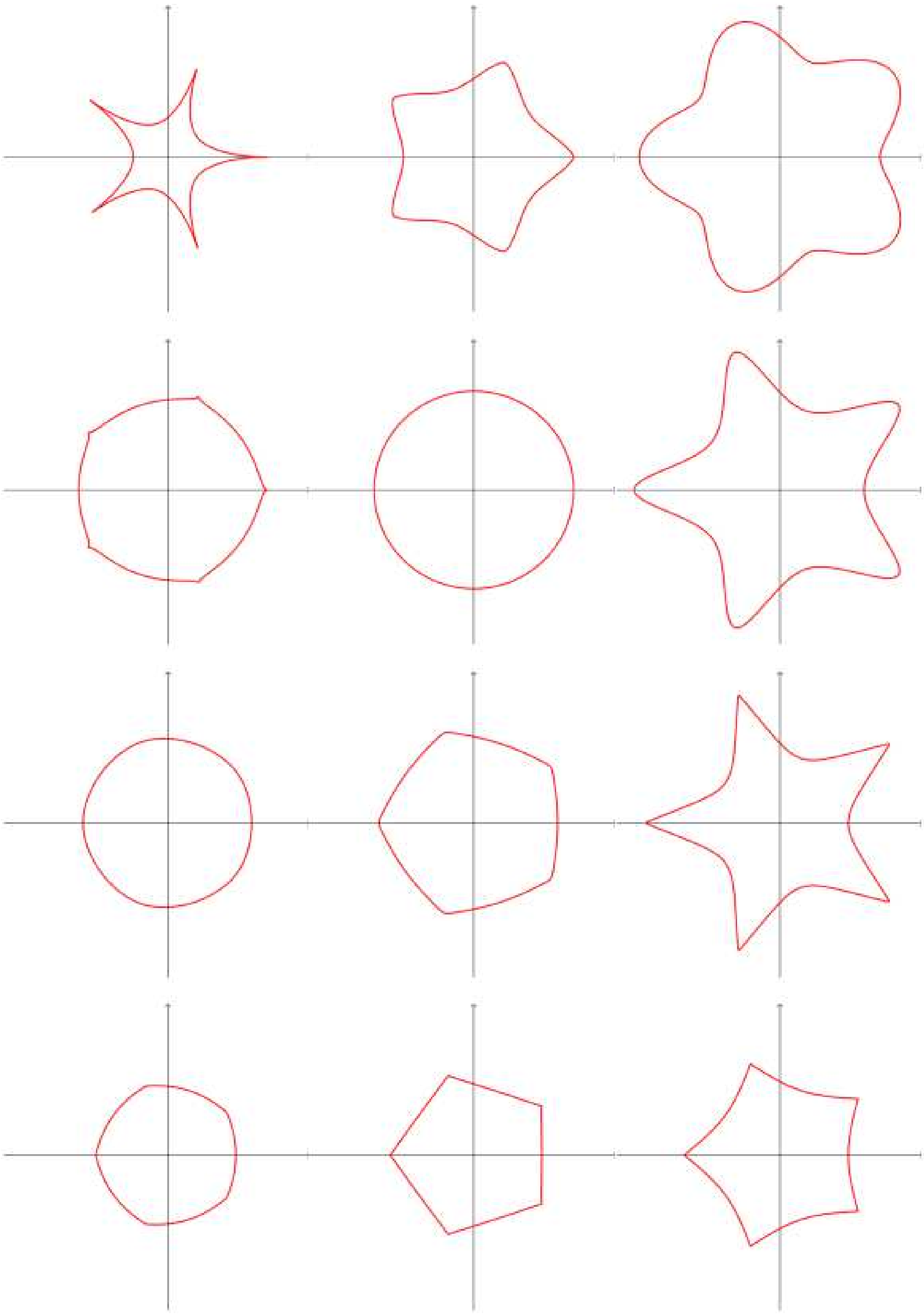

FIGURE 2. Gielis curves defined by the equation $\rho = g_{m,p,q}(\theta)$ ($m = 5$): first row $q = 0{,}5$ ($p = 0{,}5$, $p = 1{,}5$, $p = 2{,}5$); second row $q = 5$ ($p = 0{,}5$, $p = 2$, $p = 10$); third row $q = 50$ ($p = 5$, $p = 20$, $p = 100$); fourth row $q = 500$ ($p = 100$, $p = 300$, $p = 500$)

properties of the transformed logarithmic spirals were investigated and compared with similarly shaped objects.

The rest of this paper is organized as follows. Firstly, we summarize the known facts about the transformations of Gielis curves [13] and the logarithmic spirals [15] and compare them with the Gielis transformation of the Archimedean spiral. Subsequently, we investigate the curvature of the subspiral ($p < 2$) and superspiral ($p > 2$) at the anchor points and the vertices of the curves. We also discuss the influence of the value of the parameter m (integer or non-integer) on the shape of spirals. Finally, we point out objects and shapes, which could be modelled with transformed spirals.

2. Gielis Transformation of the Archimedean Spiral

Using equations (1.5) and (1.8) we obtain the equation

$$g_{a,m,p,q}(\theta) = a\theta\left(\left|\cos\frac{m\theta}{4}\right|^p + \left|\sin\frac{m\theta}{4}\right|^p\right)^{-\frac{1}{q}}, \tag{2.1}$$

which determines Gielis transformation of the Archimedean spiral. Throughout the rest of this paper we will be using the following notation and terms.

(i) We denote the planar curves obtain according to the equation (2.1) by the symbol $G_{a,m,p,q}$, i.e. $G_{a,m,p,q}(\theta) = g_{a,m,p,q}(\theta)(\cos\theta, \sin\theta)$, the Archimedean spiral by the symbol G_a, i.e., $G_a(\theta) = a\theta(\cos\theta, \sin\theta)$. Figures 3, 5 and 6 show some examples of transformations of the Archimedean spiral. In Figure 3 one can see that the coils of the spiral intersect only for rational values of m.

(ii) The pole of the spiral is the point which spiral approaches for $\theta \to -\infty$. In the case of the non-shifted spiral, this point lies at the origin of the Cartesian coordinate system.

(iii) The anchor point of $G_{a,m,p,q}$ means such a point of G_a, whose position does not change during the transformation, i.e., $G_{a,m,p,q}(\theta) = G_a(\theta)$.

(iv) The vertex of $G_{a,m,p,q}$ means the point of $G_{a,m,p,q}$ corresponding to the value of θ (Fig. 4), where $g_{m,p,q}$ has a local maximum (later we will show that for $p < 2$ the vertices are identical with anchor points).

(v) The coil of the spiral means the part of the curve where $\theta \in [2k\pi, 2(k+1)\pi)$ for given $k \in \mathbb{Z}$.

The following statements summarize some properties of transformed spirals, the proofs are routine.

Lemma 2.1. *The parameter m determines the number of anchor points in one spiral coil of $G_{a,m,p,q}$ as follows.*

(*i*) *For $m \in N$, the spiral has exactly m anchor points in one coil.*

(*ii*) *For $m \notin N$, the number of anchor points in one coil corresponds to $\lceil m \rceil$, i.e., the next higher integer.*

Lemma 2.2. *The function $g_{m,p,q}$ satisfies the following properties ($k \in \mathbb{Z}$).*

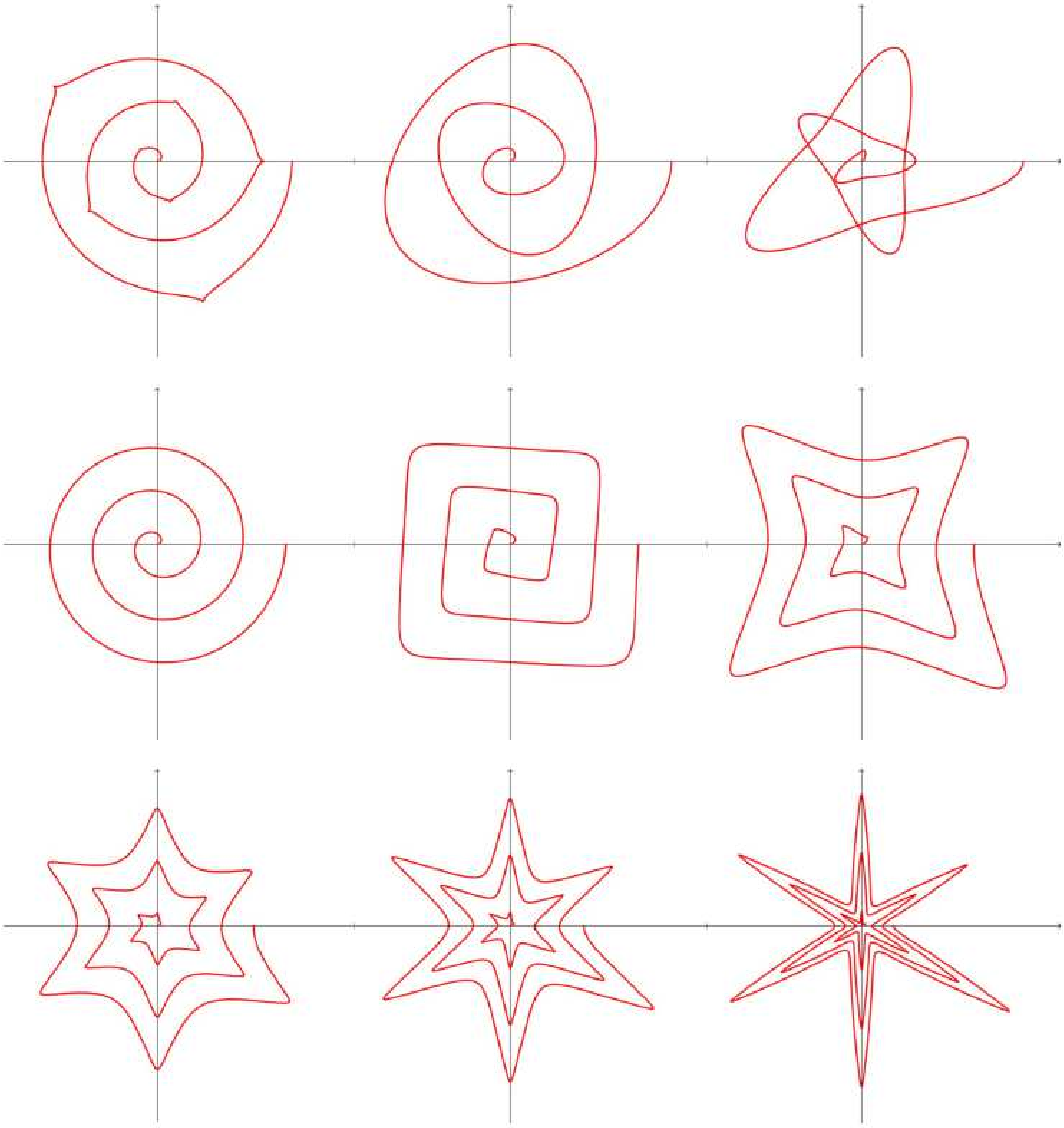

FIGURE 3. Gielis transformation of the Archimedean spiral ($\theta \in [0, 6\pi]$): first row $m = 2{,}5, q = 3$ ($p = 0{,}5$, $p = 4$, $p = 10$); second row $m = 4, q = 10$ ($p = 2$, $p = 10$, $p = 20$); third row $m = 6, q = 20$ ($p = 30$, $p = 50$, $p = 100$)

(*i*) *For* $p < 2$ *it is increasing on* $\left[\frac{(2k-1)\pi}{m}, \frac{2k\pi}{m}\right]$ *and decreasing on* $\left[\frac{2k\pi}{m}, \frac{(2k+1)\pi}{m}\right]$.
(*ii*) *For* $p = 2$ *it is constant on the whole real axis.*
(*iii*) *For* $p > 2$ *it is increasing on* $\left[\frac{2k\pi}{m}, \frac{(2k+1)\pi}{m}\right]$ *and decreasing on* $\left[\frac{(2k+1)\pi}{m}, \frac{2(k+1)\pi}{m}\right]$.
(*iv*) *For all* $\theta = \frac{2k\pi}{m}$ ($k \in \mathbb{Z}$) *it is* $g_{m,p,q}(\theta) = \theta$.

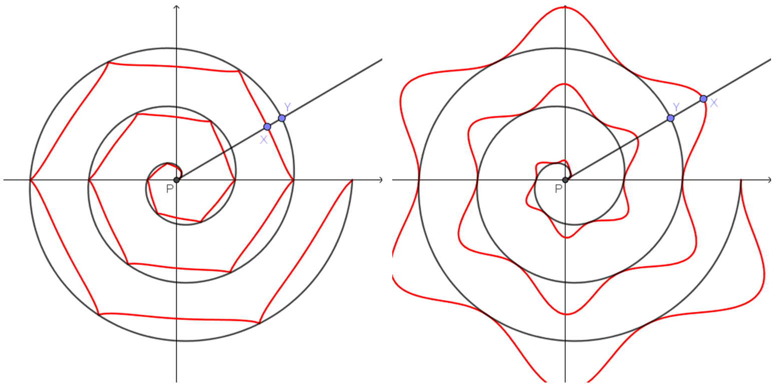

FIGURE 4. Gielis transformation of the Archimedean spiral ($\theta \geq 0$, left $p < 2$, right $p > 2$)

Corollary 2.1. (i) If $p = 2$, then $G_{a,m,p,q}$ is the Archimedean spiral G_a. Let the points X, Y lie on the same coils of $G_{a,m,p,q}$ and G_a, and at the same time on the same half-line starting from the pole P of the spiral. If

- $p < 2$, then $|PX| \leq |PY|$;
- $p > 2$, then $|PX| \geq |PY|$.

(ii) If $p < 2$, then the anchor points and vertices of $G_{a,m,p,q}$ correspond to the choice $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$. If $p > 2$, then the anchor points of $G_{a,m,p,q}$ correspond to the choice $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$, and the vertices to the choice $\theta = \frac{(2k+1)\pi}{m}$ $(k \in \mathbb{Z})$.

Theorem 2.2. *The function $g_{a,m,p,q}$ satisfies the following properties*

(i) $g_{a,m,p,q}(\theta + \frac{2\pi}{m}) = \frac{2\pi a}{m} g_{a,m,p,q}(\theta)$;
(ii) $\lim_{q \to \infty} g_{a,m,p,q}(\theta) = a\theta$.

Proof. The claims (i) and (ii) follow directly from the definition of the function $g_{a,m,p,q}$. □

Remark 2.1. For $p < 2$ we call the curve $G_{a,m,p,q}$ a subspiral of the Archimedean spiral, for $p > 2$ is the curve $G_{a,m,p,q}$ a superspiral of the Archimedean spiral.

3. CURVATURE OF SUBSPIRAL AND SUPERSPIRAL

The aim of this section is to examine the curvature of subspiral and superspiral. The curvature can generally be characterized as an amount by which a curve deviates from being a straight line whose curvature is zero. If we consider, that spirals are given with (2.1), and we use the relation for the curvature of the curve given in polar coordinates, then we obtain

$$\kappa_{a,m,p,q}(\theta) = \frac{g_{a,m,p,q}(\theta)^2 + 2g'_{a,m,p,q}(\theta)^2 - g_{a,m,p,q}(\theta)g''_{a,m,p,q}(\theta)}{\left\{g_{a,m,p,q}(\theta)^2 + g'_{a,m,p,q}(\theta)^2\right\}^{\frac{3}{2}}}, \tag{3.1}$$

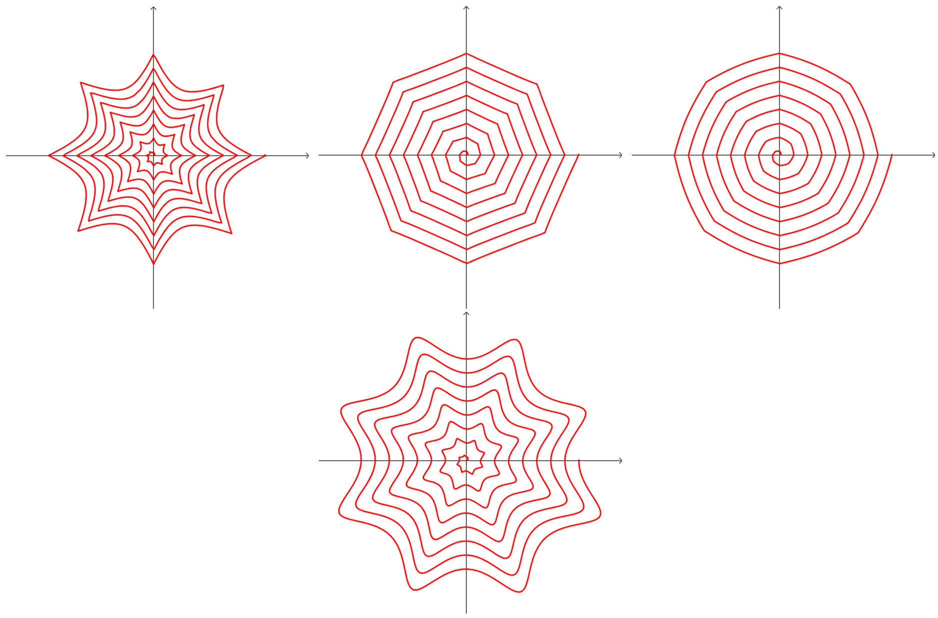

FIGURE 5. Curvature at the anchor points (vertices) of the subspiral (upper row: $p = 1$ and successively $q = 1$, $q = (m/4)^2 p$, $q = 8$), and at the vertices of the superspiral (lower row: $p = q = 8$), $m = 8$

where $\kappa_{a,m,p,q}(\theta)$ denotes the curvatute of $G_{a,m,p,q}$. For $p < 2$ the function $x \mapsto |x|^p$ does not have the second derivative in zero, therefore $g(\theta)$ does not have the second derivative at the points $\frac{2k\pi}{m}$ $(k \in \mathbb{Z})$ and the curvature is not defined there. If we substitute in (3.1) the formula $a\theta g_{m,p,q}(\theta)$ for $g_{a,m,p,q}(\theta)$, then after simplifying we obtain

$$\kappa_{a,m,p,q}(\theta) = \frac{1}{a} \cdot \frac{\theta^2 g(\theta)^2 + 2\Big(g(\theta) + \theta g'(\theta)\Big)^2 - \theta g(\theta)\Big(2g'(\theta) + \theta g''(\theta)\Big)}{\Big\{\theta^2 g(\theta)^2 + \Big(g(\theta) + \theta g'(\theta)\Big)^2\Big\}^{\frac{3}{2}}}, \tag{3.2}$$

where $g(\theta)$ is a shortcut for $g_{m,p,q}(\theta)$.

Since the second fraction in formula (3.2) represents $\frac{2\pi}{m}$-periodic function is sufficient to examine the curvature on the interval $[0, \frac{2\pi}{m})$. When investigating the curvature of transformed spirals we focus on the anchor points, i.e., we determine the curvature for $\theta = \frac{(2k+1)\pi}{m}$ $(k \in \mathbb{Z})$, and the vertices of the spirals, i.e., we determine the curvature for $\theta = \frac{2k\pi}{m}$ $(k \in \mathbb{Z})$. Because of the above, it will be sufficient to do the calculations in case of anchor points for $\theta = \frac{\pi}{m}$, and in the case of vertices for $\theta = 0$.

Theorem 3.1. *The curvature $\kappa_{a,m,p,q}(\frac{\pi}{m})$ satisfies the following properties.*

(*i*) $\kappa_{a,m,p,q}(\frac{\pi}{m}) = \frac{1}{a g_{m,p,q}(\frac{\pi}{m})} \cdot \frac{1}{\{1+(\frac{\pi}{m})^2\}^{\frac{3}{2}}} \Big\{2 + (\frac{\pi}{m})^2 + \frac{\pi^2 p(p-2)}{16q}\Big\}$

(*ii*) *Let $p < 2$. Then*

- $\kappa_{a,m,p,q}(\frac{\pi}{m}) < 0$ *if* $q < \frac{\pi^2 (m/4)^2 p(2-p)}{2m^2+\pi^2}$,

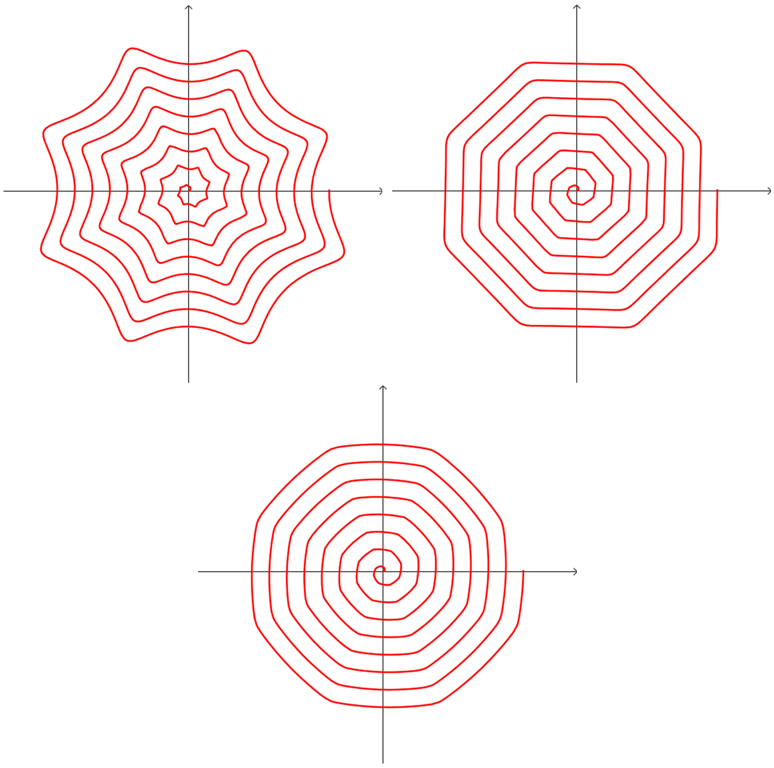

FIGURE 6. Curvature at the anchor points of the superspiral ($p = 10$ and successively $q = 15$, $q = (m/4)^2 p$, $q = 75$), $m = 8$

- $\kappa_{a,m,p,q}(\frac{\pi}{m}) = 0$ *if* $q = \frac{\pi^2 (m/4)^2 p(2-p)}{2m^2+\pi^2}$,
- $\kappa_{a,m,p,q}(\frac{\pi}{m}) > 0$ *if* $q > \frac{\pi^2 (m/4)^2 p(2-p)}{2m^2+\pi^2}$.

(*iii*) *If* $p \geq 2$, *then* $\kappa_{a,m,p,q}(\frac{\pi}{m}) > 0$.

Proof. To prove (*i*), it is sufficient to substitute into formula (3.2)

$$g\left(\frac{\pi}{m}\right) = 2^{\frac{p-2}{2q}}, \quad g'\left(\frac{\pi}{m}\right) = 0, \quad g''\left(\frac{\pi}{m}\right) = \frac{(m/4)^2 p(2-p)}{q} 2^{\frac{p-2}{2q}}.$$

The claims (*ii*) and (*iii*) follow directly from (*i*). □

In the claim (*ii*) of the previous theorem is for the choice $p < 2$ mentioned the dependence of curvature on the value of the parameter q. Examples of the curves with the negative, zero and positive curvature at points that are "halfway" between the anchor points are shown in Figure 5.

Theorem 3.2. *Let* $p > 2$. *The curvature* $\kappa_{a,m,p,q}(0)$ *satisfies the following properties.*

(*i*) $\kappa_{a,m,p,q}(0) = \frac{1}{a} \cdot \frac{1}{\left\{1+(\frac{\pi}{m})^2\right\}^{\frac{3}{2}}} \left\{2 + (\frac{\pi}{m})^2 - \frac{\pi^2 p}{16q}\right\}$.

(*ii*) *If*

- $q < \frac{\pi^2 (m/4)^2 p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) < 0$,

- $q = \frac{\pi^2(m/4)^2 p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) = 0$,
- $q > \frac{\pi^2(m/4)^2 p}{2m^2+\pi^2}$, *then* $\kappa_{a,m,p,q}(0) > 0$.

Proof. To prove (i), it is sufficient to substitute into formula (3.2)

$$g(0) = 1, \quad g'(0) = 0, \quad g''(0) = \frac{(m/4)^2 p}{q}.$$

The claim (ii) follows directly from (i). □

Examples of the curves with the negative, zero and positive curvature at the anchor points of the superspirals are shown in Figure 6.

Remark 3.1. For $p = 0$ or $p = 2$ is

$$\kappa_{a,m,p,q}(\theta) = \frac{1}{g_a(\theta)} \cdot \frac{2+\theta^2}{(1+\theta^2)^{3/2}}$$

the curvature of the Archimedean spiral G_a.

4. Transformed Archimedean Spirals as Approximations of Spiral Antennas

The requirement for miniaturizing the antennas led to looking for specific transformed shapes. There are many types of planar spiral antennas whose design is based mainly on the use of the Archimedean or logarithmic geometry. The antennas operate in different configurations, e.g., the circular, the rectangular, the polygonal, sinuous meander or log-periodic. The mentioned configurations have their advantages and disadvantages but generally allow to reach frequency independent antennas.

Although there are different types of antennas with a different configuration, it can be shown that it is possible to approximate many of them in terms of Gielis transformation (Figure 7).

On the other hand, relative simplicity and flexibility of transformation might be used when looking for an advance or novel construction of the antennas.

5. Conclusion

In this paper, some properties of the Gielis transformation of the Archimedean spiral were analyzed. We focused in particular on the curvature in anchor points and the vertices of the transformed curves. In the end, we showed that the Gielis transformation might be handy when one looks for the appropriate shape of the Archimedean spiral-like antennas.

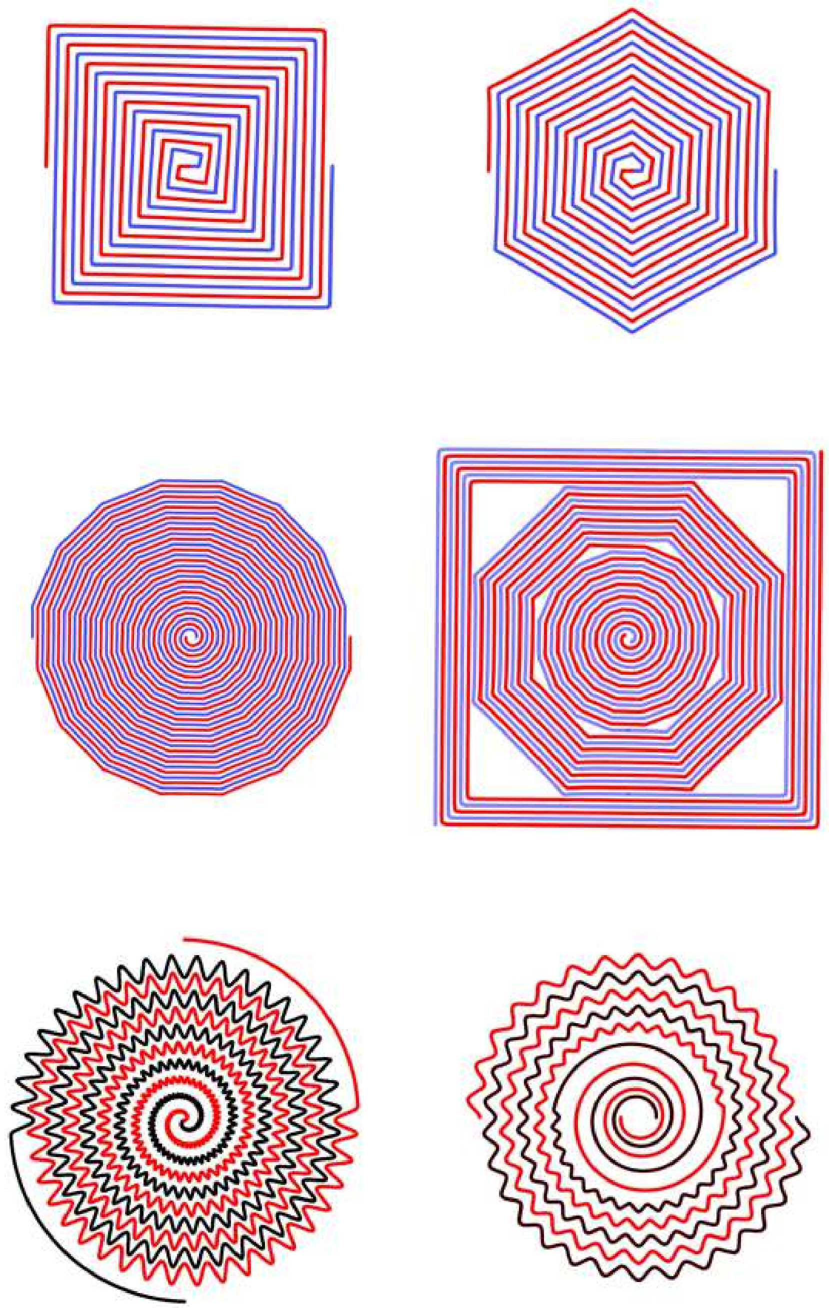

FIGURE 7. Models of spiral antennas with different configuration approximated via Gielis transformation of the Archimedean and logarithmic spiral

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GRÖBNER LATTICE-POINT ENUMERATORS AND SIGNED TILING BY k-IN-LINE POLYOMINOES

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Abstract. Conway and Lagarias observed that a triangular region $T_2(n)$ in a hexagonal lattice admits a *signed tiling* by 3-in-line polyominoes (tribones) if and only if $n \in \{3^2d-1, 3^2d\}_{d\in\mathbb{N}}$. We apply the theory of Gröbner bases over integers to show that $T_3(n)$, a three dimensional lattice tetrahedron of edge-length n, admits a signed tiling by tribones if and only if $n \in \{3^3d-2, 3^3d-1, 3^3d\}_{d\in\mathbb{N}}$. More generally we study *Gröbner lattice-point enumerators* of lattice polytopes and show that they are (modular) quasipolynomials in the case of k-in-line polyominoes. As an example of the "unusual cancelation phenomenon", arising only in signed tilings, we exhibit a configuration of 15 tribones in the 3-space such that exactly one lattice point is covered by an odd number of tiles.

1. Introduction

Following Conway and Lagarias [6], Reid [12], and other authors, we say that a finite region (polyomino) R, in a (hexagonal) lattice tiling of the plane, has a *signed tiling* ($\mathbb{Z}$-tiling), by *prototiles* from a given set Σ, if there exists a (possibly overlapping) placement of a finite number of copies of prototiles in the plane such that:

- the total covering multiplicity of elementary cells (hexagons) in R is $+1$;
- the total covering multiplicity of elementary cells outside of R is 0.

Figures 1 and 2 nicely illustrate these concepts. The set R, depicted in Figure 1 on the left, is a triangular region in the hexagonal tiling of the plane. The prototiles, also exhibited in Figure 1 on the left, are 3-in-line polyominoes, called 3-bones. The

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objective is to cover or more precisely to distribute copies of these prototiles over R, so that they (counted with positive or negative multiplicity) form a covering of R.

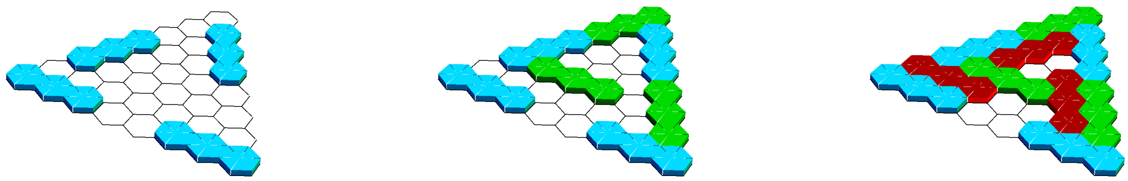

FIGURE 1

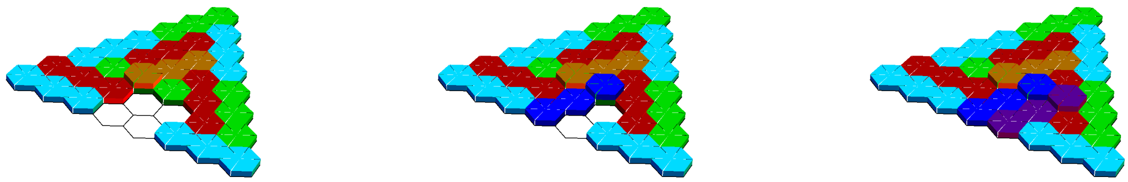

FIGURE 2

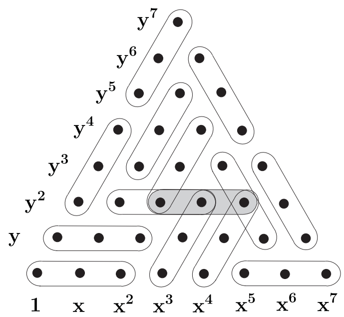

FIGURE 3. A signed tiling of a triangular region by 3-bones

We see (Figure 1) how 3-bones are initially added in an attempt to cover R without overlaps. We continue (Figure 2) by allowing overlaps, until R is completely covered with 3-bones. In the rightmost image depicted in Figure 2 we see that each cell (hexagon) has multiplicity $+1$ or $+2$, where precisely three hexagons have multiplicity $+2$. Finally, these three cells can be subtracted by adding a 3-bone of multiplicity -1 (the shaded region depicted in Figure 3).

1.1. **Algebraic method.** In an algebraic reformulation of the problem each cell (lattice point) is associated a monomial $(p, q) = pe_1 + qe_2 \mapsto x^p y^q$ and the signed tiling can be interpreted as an algebraic identity in the ring $\mathbb{Z}[x, y]$.

More explicitly the basic 3-bones are interpreted as quadratic polynomials $b_1 = x^2 + x + 1, b_2 = y^2 + y + 1, b_3 = x^2 + xy + y^2$, the region R is represented by the polynomial $T_2(8)$, where

$$T_2(n) = \sum_{\substack{0\leq i,j\leq n-1\\ i+j\leq n-1}} x^i y^j \,, \tag{1.1}$$

the shaded region in Figure 3 is recorded as the polynomial $x^2y^2b_1$ and the algebraic equivalent of the signed tiling described in Figures 1, 2, 3 is the identity

$$T_2(8) = (1+y+x^5-x^2y^2)b_1+(x^3+x^4+y^2+x^2y^2+xy^3+y^5)b_2+(x^5y+x^6y+x^3y^4)b_3\,.$$

1.2. **Ideal membership problem and Gröbner bases.** As demonstrated in the previous section, the existence of a signed tiling in general can be reduced to the *Ideal membership problem* [7, Chapter 2], which can be often successfully treated by the method of Gröbner basis [7,8].

The approach to signed polyomino tilings via Gröbner bases was originally proposed by Bodini and Nouvel [5]. We independently discovered this idea and, inspired by [12], applied it in [10] to the calculation of *tile homology groups* (originally introduced in [12]) and in [9] for the study of $\mathbb{Z}$-tilings with symmetries [9].

Since we apply the general theory to polynomials with integer coefficients, we work with *strong Gröbner bases* [1,11] (called a D-Gröbner base in [4]), see also [10, Section 5] or our Section 6 for a brief introduction.

1.3. **Summary of new results.** Conway and Lagarias proved [6, Theorem 1.4] that a triangular region $T_2(n)$ in a hexagonal lattice admits a signed tiling by 3-in-line polyominoes (called tribones in [15]) if and only if $n \in \{9d-1, 9d\}_{d\in\mathbb{N}}$. In particular the $\mathbb{Z}$-tiling exhibited in Figure 3 is discovered by these authors.

By applying the theory of Gröbner bases over integers, we extended in [10] this result to k-bones (k-in-line polyominoes) for all $k \geq 2$. More explicitly we showed that the triangular region in the hexagonal tiling of the plane associated to the polynomial $T_2(n)$ admits a signed tiling by k-bones if and only if

$$n \in \{k^2d-1, k^2d\}_{d\in\mathbb{N}}\,.$$

In this paper we address the general problem of $\mathbb{Z}$-tiling by k-bones in d-dimensional lattices, with the emphasis on the tiling of three dimensional polytopes with 3-bones.

We proved (Theorem 2.1) that the lattice tetrahedron associated to the polynomial $T_3(n)$ admits a $\mathbb{Z}$-tiling by all six tribones in the 3-dimensional lattice if and only if

$$n \in \{3^3d-2, 3^3d-1, 3^3d\}_{d\in\mathbb{N}}\,.$$

A new phenomenon, characteristic for $\mathbb{Z}$-tiling with tribones in dimension 3, is the appearance of a constant polynomial 9 in the associated Gröbner basis. As a consequence we construct in Section 3 a "tribone star", that is a configuration of tribones with integer weights such that the total weight is non-zero only at the center of the star.

We call this a "cancelation phenomenon" and, as another consequence, we exhibit (Corollary 3.1) a configuration of 15 tribones in the 3-space where exactly one lattice point (the center of the star) is covered by an odd number of tiles.

Motivated by the ideas used in the proof of Theorem 2.1, we introduce *Gröbner lattice-point enumerators* in Section 4.1, as a proper setting for studying general d-dimensional, $\mathbb{Z}$-polyomino tilings. We demonstrate how the general theory can be considerably simplified in the case of k-in-line prototiles (k-bones) by introducing *cyclotomic ideals* (Section 4.4).

As a first step in developing the associated "Ehrhart theory", we show in Section 5 (Theorem 5.2) that Gröbner lattice-point enumerators for k-bones are (modular) quasipolynomials. In other words they behave similarly as the classical lattice-point enumerators of rational polytopes, a fact that considerably simplifies their calculation.

2. Signed Tiling of the Lattice Tetrahedron $T_3(n)$

2.1. The tribone ideal I_3^3 in variables x, y, z. A three-in-line polyomino or a tribone, in a cubical integer lattice, is a translate of one of the six types of trominoes, associated with the following quadratic polynomials:

$$b_1 = x^2 + x + 1, \qquad b_2 = y^2 + y + 1, \qquad b_3 = x^2 + xy + y^2,$$
$$b_4 = x^2 + xz + z^2, \qquad b_5 = y^2 + yz + z^2, \qquad b_6 = z^2 + z + 1.$$

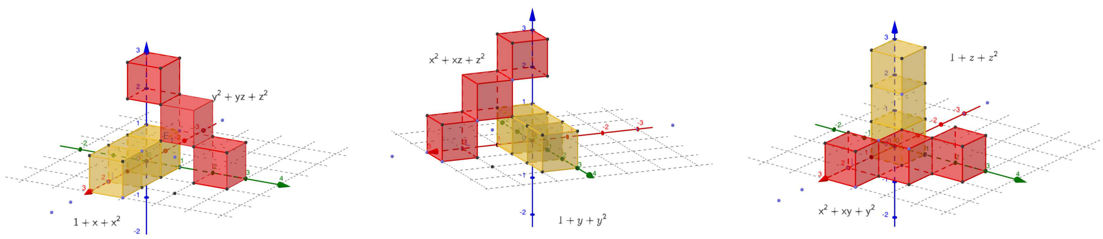

Figure 4. Tribones $b_1, b_2, b_3, b_4, b_5, b_6$.

Let $I_3^3 = \langle b_1, b_2, b_3, b_4, b_5, b_6 \rangle$ be the ideal generated by tribones and GBI the strong Gröbner bases of the ideal I with respect to the lexicographical term order,

$$\begin{aligned} GBI =&\{x^2 + x + 1, xy - y - x - 2, xz - x - z - 2, 3x - 3, \\ &y^2 + y + 1, yz - y - z - 2, 3y - y, z^2 + z + 1, 3z - 3, 9\}. \end{aligned}$$

```
In[•]:= IdealTr = {1 + x + x^2, 1 + y + y^2, 1 + z + z^2,
          x^2 + x * y + y^2, y^2 + y * z + z^2, x^2 + x * z + z^2}
Out[•]= {1 + x + x², 1 + y + y², 1 + z + z², x² + x y + y², y² + y z + z², x² + x z + z²}

In[•]:= GroebnerBasis[IdealTr, {x, y, z}, CoefficientDomain → Integers]
Out[•]= {9, -3 + 3 z, 1 + z + z², -3 + 3 y, -2 - y - z + y z,
         1 + y + y², -3 + 3 x, -2 - x - z + x z, -2 - x - y + x y, 1 + x + x²}
```

FIGURE 5. The tribone ideal and its Gröbner basis (*Wolfram Mathematica* 12.3.1).

Denote the polynomials of the Gröbner bases GBI by:

$$\begin{array}{ll} g_1 = x^2 + x + 1, & g_2 = xy - x - y - 2, \\ g_3 = xz - x - z - 2, & g_4 = 3x - 3, \\ g_5 = y^2 + y + 1, & g_6 = yz - y - z - 2, \\ g_7 = 3y - 3, & g_8 = z^2 + z + 1, \\ g_9 = 3z - 3, & g_{10} = 9. \end{array}$$

2.2. **Signed tiling of the tetrahedron $T_3(n)$.** The 3-dimensional analogue of (1.1) is the tetrahedron $T_3(n)$ in the 3-dimensional integer lattice, associated with the polynomial:

$$T_3(n) = \sum_{\substack{0 \le i,j,k \le n-1 \\ i+j+k \le n-1}} x^i y^j z^k.$$

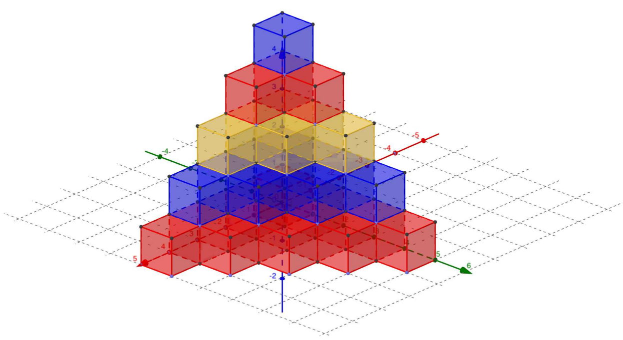

FIGURE 6. The tetrahedron $T_3(5)$.

The goal is to determine for which values of n the tetrahedron $T_3(n)$ admits a $\mathbb{Z}$-tiling by tribones. Following [5] and [10] (see also Section 4) we need to determine when the remainder, obtained by dividing the polynomial $T_3(n)$ by GBI, is equal to zero.

The polynomials $T_3(1) = 1$ and $T_3(2) = 1 + x + y + z$ are already reduced (cannot be further divided by the basis GBI). It follows that they do not admit a $\mathbb{Z}$-tiling with tribones.

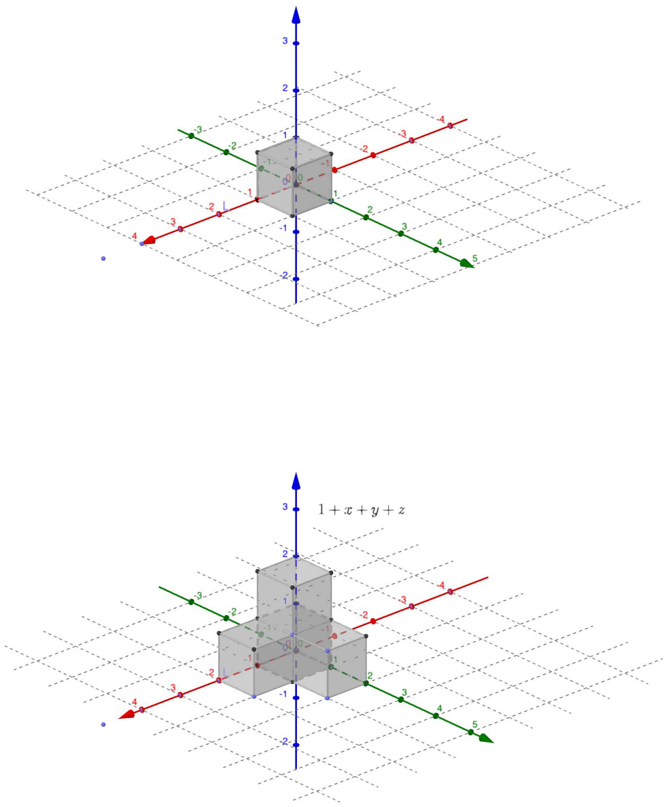

FIGURE 7. Tetrahedron $T_3(n)$ for $n = 1$ and $n = 2$

The remainder on division of the polynomial $T(3)$ by the set GBI is equal to the remainder on division of the region described by the grey cubes (see Figure 8). It follows,

$$T_3(3) \equiv_{GBI} (1 + z)(x + y).$$

Indeed, the region determined by grey cubes is formed by subtracting $1 + z + z^2$ (z-tribone) and $x^2 + xy + y^2$ (xy-tribone) from the region $T_3(3)$.

If $n = 4$, then the remainder on division of the polynomial $T_3(4)$ is congruent with $y^3 + z^3$. Here, $y^3 + z^3$ is a polynomial described by the region of grey cubes formed after subtracting the region determined by the polynomial

$$(z^2 + z + 1)(1 + x + y) + (x + y)(x^2 + xy + y^2).$$

As a consequence we obtain

$$T_3(4) \equiv_{GBI} y^3 + z^3.$$

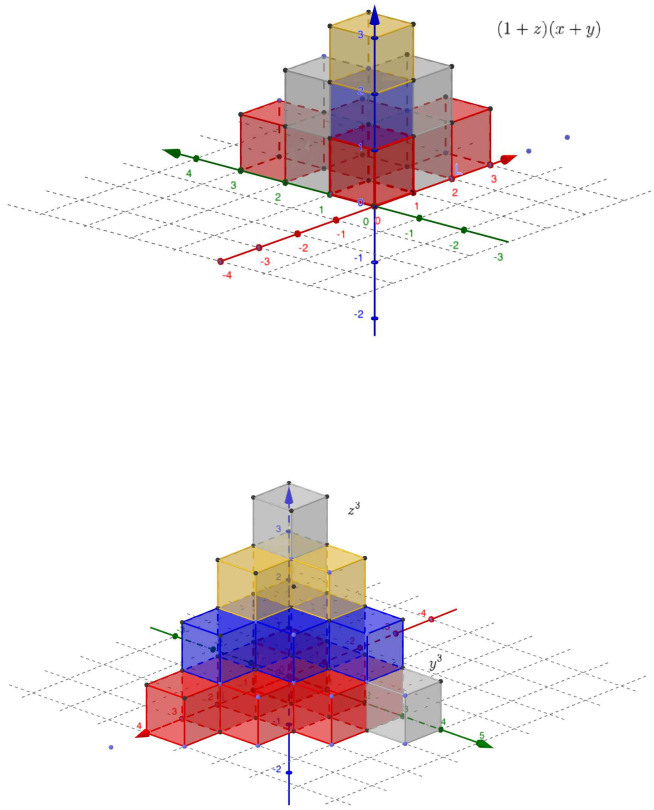

FIGURE 8. Tetrahedron $T_3(n)$ for $n = 3$ and $n = 4$.

The same reasoning applies to the cases $n = 5$ and $n = 6$, which leads to $T_3(5) \equiv_{GBI} (1 + x + y + z)(y^3 + z^3)$ (Figure 9), $T_3(6) \equiv_{GBI} (x + y)(1 + z)(y^3 + z^3)$.

If we proceed with the decomposition of the region $T_3(n)$ in the same manner, we finally conclude

$$T_3(n) \equiv_{GBI} \begin{cases} (x+y)(1+z)f_k(y,z), & n = 3k, \\ f_k(y,z), & n = 3k+1, \\ (1+x+y+z)f_k(y,z), & n = 3k+2, \end{cases} \tag{2.1}$$

where

$$f_k(y,z) = y^{3k} + y^{3(k-1)}z^3 + \cdots + y^3 z^{3(k-1)} + z^{3k} = \sum_{i=0}^{k} y^{3(k-i)} z^{3i}.$$

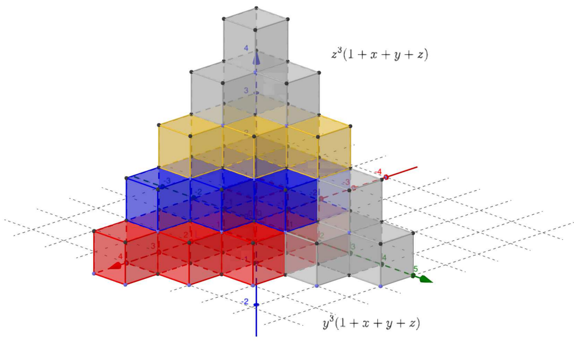

FIGURE 9. Decomposition of the tetrahedron $T_3(5)$.

Lemma 2.1. *For every $n \in \mathbb{N}$*

$$y^3\left(f_0 + f_1 + \cdots + f_{n-1}\right) = f_0 + f_1 + \cdots + f_n - \left(z^{3n} + \cdots + z^3 + 1\right). \tag{2.2}$$

Proof. This is proved by induction on n. The identity is valid for $n = 1$ since,

$$y^3 f_0 = y^3 = (1 + y^3 + z^3) - (z^3 + 1) = f_0 + f_1 - (z^3 + 1).$$

Let us assume that (2.2) is true for $n = k$. Since

$$\begin{aligned}
&y^3\left(f_0 + f_1 + \cdots + f_{k-1} + f_k\right)\\
=&y^3\left(f_0 + f_1 + \cdots + f_{k-1}\right) + y^3 f_k\\
=&f_0 + \cdots + f_k - \left(z^{3k} + \cdots + z^3 + 1\right) + y^3\left(y^{3k} + y^{3(k+1)}z^3 + \cdots + z^{3k}\right)\\
=&f_0 + \cdots + f_k - \left(z^{3k} + \cdots + z^3 + 1\right) + y^{3(k+1)} + y^{3k}z^3 + \cdots + y^3 z^{3k} + z^{3(k+1)}\\
&- z^{3(k+1)}\\
=&f_0 + \cdots + f_k + f_{k+1} - \left(z^{3(k+1)} + z^{3k} + \cdots + z^3 + 1\right),
\end{aligned}$$

we conclude that (2.2) holds for $n = k+1$. It follows, by the Principle of mathematical induction, that (2.2) is true for all $n \in \mathbb{N}$. □

Lemma 2.2. *For every $n \in \mathbb{N}$*

(2.3)

$$f_n = (y^3 - 1)\left(f_0 + f_1 + \cdots + f_{n-1}\right) + (z^3 - 1)\left(z^{3(n-1)} + \cdots + (n-1)z^3 + n\right) + (n+1).$$

The remainders of the division of polynomial f_n by elements of the basis GBI are periodic, with the period 9.

Proof. If $n = 0$ then $f_0 = 1$. For $n = 1$

$$f_1 = y^3 + z^3 = (y^3 - 1)f_0 + (z^3 - 1) + 2,$$

which is in agreement with (2.3). Suppose that the identity (2.3) is valid for some $k \in \mathbb{N}$. Since

$$\begin{aligned}
f_{k+1}(y,z) &= y^{3(k+1)} + y^{3k}z^3 + \cdots + y^3 z^{3k} + z^{3(k+1)} \\
&= y^3\left(y^{3k} + y^{3(k-1)}z^3 + \cdots + y^3 z^{3(k-1)} + z^{3k}\right) + z^{3(k+1)} \\
&= y^3\left((y^3-1)(f_0+\cdots+f_{k-1}) + (z^3-1)(z^{3(k-1)} + \cdots + (k-1)z^3 + k)\right. \\
&\quad \left. +(k+1)\right) + z^{3(k+1)} \\
&= (y^3-1)\left(y^3(f_0+\cdots+f_{k-1})\right) + y^3\left((z^3-1)(z^{3(k-1)}+\cdots+\right. \\
&\quad \left. +(k-1)z^3+k)\right) + y^3(k+1) + z^{3(k+1)} \quad \text{(by Lemma 2.1)} \\
&= (y^3-1)\left((f_0+\cdots+f_{k-1}+f_k) - (z^{3k}+\cdots+z^3+1)\right) \\
&\quad + y^3(z^{3k}+\cdots+z^3-k) + y^3(k+1) + z^{3(k+1)} \\
&= (y^3-1)(f_0+\cdots+f_{k-1}+f_k) + z^{3(k+1)} + z^{3k} + \cdots + z^3 + 1 \\
&= (y^3-1)(f_0+\cdots+f_{k-1}+f_k) + (z^3-1)\left(z^{3k}+\cdots+(k+1)\right) \\
&\quad + (k+2),
\end{aligned}$$

we conclude that (2.3) holds for $n = k+1$. Therefore, by the Principle of mathematical induction, (2.3) is true for all $n \in \mathbb{N}$.

Since $y^3 - 1 = (y-1)b_2$ i $z^3 - 1 = (z-1)b_6$, we see that

$$(y^3-1)(f_0+f_1+\cdots+f_{n-1}) + (z^3-1)\left(z^{3(n-1)} + 2z^{3(n-2)} + \cdots + (n-1)z^3 + n\right) \in I.$$

From this and (2.3), we conclude that the remainder of the division of f_n by elements of the set GBI equals the remainder of the division $n+1$ by g_{10}. Therefore,

$$\begin{array}{llll}
\overline{f_0}^{GBI} = 1, & \overline{f_1}^{GBI} = 2, \\
\overline{f_2}^{GBI} = 3, & \overline{f_3}^{GBI} = 4, \\
\overline{f_4}^{GBI} = -4, & \overline{f_5}^{GBI} = -3, \\
\overline{f_6}^{GBI} = -2, & \overline{f_7}^{GBI} = -1, \\
\overline{f_8}^{GBI} = 0, &
\end{array} \tag{2.4}$$

and we see that the remainders are periodic, with period of length 9. For this reason, $f_{9k-1} \equiv_{GBI} 0$, $k \in \mathbb{N}$. □

Theorem 2.1. *The tetrahedron $T_3(n)$ admits a signed tiling by tribones $b_1, b_2, \ldots, b_6$ if and only if $n = 3^3k - 2$, $n = 3^3k - 1$ or $n = 3^3k$ for $k \in \mathbb{N}$.*

Proof. The tetrahedron $T_3(n)$ admits a signed tiling by tribones $b_1, \ldots, b_6$ if and only if the remainder of the polynomial $T_3(n)$, on division by the Gröbner bases GBI of the ideal I_3^3, is equal to zero.

Since the remainder on division of the polynomial f_n by GBI is periodic with the period 3^2, from (2.1), (2.4) and Table 1, follows that the remainder on division of the polynomial $T_3(n)$ by GBI is periodic with the period 3^3.

Table 1

k	$\overline{T_3(3k-2)}^{GBI}$	$\overline{T_3(3k-1)}^{GBI}$	$\overline{T_3(3k)}^{GBI}$
1	1	x+y+z	4-x-y-z
2	2	2-x-y-z	-1+x+y+z
3	3	3	3
4	4	4+x+y+z	-2-x-y-z
5	-4	-4-x-y-z	2+x+y+z
6	-3	-3	-3
7	-2	-2x+y+z	-1-x-y-z
8	-1	-1-x-y-z	-4+x+y+z
9	0	0	0

From here we finally conclude that the region $T_3(n)$ admits a signed tiling by tribones if and only if $n = 3^3k - 2$, $n = 3^3k - 1$ or $n = 3^3k$ for some $k \in \mathbb{N}$. □

3. The Role of Number 9 in $\mathbb{Z}$-tiling by Tribones

Let $I_3^3 \subset \mathbb{Z}[x, y, z]$ be the *tribone ideal*, generated by polynomials

$$\begin{array}{lll} A_x = x^2 + x + 1, & A_y = y^2 + y + 1, & A_z = z^2 + z + 1, \\ A_{xy} = x^2 + xy + y^2, & A_{xz} = x^2 + xz + z^2, & A_{yz} = y^2 + yz + z^2, \end{array} \tag{3.1}$$

renamed to emphasize the symmetry w.r.t. permutations of variables. The Gröbner basis of $I = I_3^3$, with respect to the lexicographic order (Lex) of monomials arising from the order $x > y > z$, is the following:

$$\begin{array}{lll} A_x = x^2 + x + 1, & A_y = y^2 + y + 1, & A_z = z^2 + y + 1, \\ B_{xy} = xy - x - y - 2, & B_{xz} = xz - x - z - 2, & A_{yz} = yz - y - z - 2, \\ C_x = 3x - 3, & C_y = 3y - 3, & C_z = 3z - 3, \\ & D = 9. & \end{array} \tag{3.2}$$

It follows that there exists a relation

$$9 = a_1 A_x + a_2 A_y + a_3 A_z + b_1 A_{xy} + b_2 A_{xz} + b_3 A_{yz}, \tag{3.3}$$

for some polynomials $a_i, b_j \in \mathbb{Z}[x, y, z]$. In other words the relation (3.3) guarantees the existence of a signed tiling where the tribones "cancel out" everywhere in the 3-dimensional lattice, except at one point.

Our objective is to make relation (3.3) explicit, for as simple as possible choice of polynomials a_i, b_j.

We essentially apply *Buchberger's Algorithm* (over integers) by iterating the calculation of S-polynomials, beginning with the polynomials from the basis (3.1). Note that, in light of the symmetry of the ideals (3.1) and (3.2), the expression for $3z-3$ in the following proposition can be easily turned in the expression for $3x-3$ (respectively $3y-3$).

Proposition 3.1.

$$9 = 2\left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right] - \left[RHS(3.9) - 2RHS(3.6)\right],$$
$$3z - 3 = \left[RHS(3.9) - 2RHS(3.6)\right] - \left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right].$$

Proof. The first row of (3.2) coincides with the first row of (3.1). The second row of (3.2) is obtained by adding and subtracting the polynomials from the first two rows of (3.1), for example

$$B_{xy} = A_{xy} - A_x - A_y\,. \tag{3.4}$$

We continue by computing the S-polynomial of A_x and A_{xy}, and its subsequent reduction

$$S[A_x, A_{xy}] = y(x^2 + x + 1) - x(xy - x - y - 2) = x^2 + 2xy + 2x + y,$$
$$x^2 + 2xy + 2x + y = A_x + 2xy + x + y - 1 = A_x + 2B_{xy} + 3(x + y + 1).$$

From here we obtain the relation

$$3(x + y + 1) = yA_x - xB_{xy} - A_x - 2B_{xy} = (y - 1)A_x - (x + 2)B_{xy}, \tag{3.5}$$

which in light of (3.4) produces the relation

$$3(x+y+1) = (y-1)A_x - (x+2)(A_{xy} - A_x - A_y) = (x+y+1)A_x + (x+2)A_y - (x+2)A_{x,y}. \tag{3.6}$$

Similarly, we have the relations

$$3(z + x + 1) = (z + x + 1)A_z + (z + 2)A_x - (z + 2)A_{x,z}, \tag{3.7}$$
$$3(y + z + 1) = (y + z + 1)A_y + (y + 2)A_z - (y + 2)A_{y,z}, \tag{3.8}$$

and by adding up all three of them we have

$$9 + 6(x+y+z) = (x+y+z+3)(A_x + A_y + A_z) - (x+2)A_{x,y} - (y+2)A_{y,z} - (z+2)A_{x,z}. \tag{3.9}$$

Let us multiply both sides of (3.6) by 2 and subtract from (3.9). We obtain

$$6z + 3 = RHS(3.9) - 2RHS(3.6). \tag{3.10}$$

Note that

$$S(A_z, 6z + 3) = 6A_z - z(6z + 3) = 3z + 6 = 6A_z - zRHS(3.9) + 2zRHS(3.6). \tag{3.11}$$

From (3.10) and (3.11) we finally have

$$\begin{aligned} 9 &= 2(3z + 6) - (6z + 3) \\ &= 2\left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right] - \left[RHS(3.9) - 2RHS(3.6)\right]. \end{aligned}$$

Note that in passing we obtain an explicit expression for the third row of (3.2) in terms of (3.1). For example

$$\begin{aligned} 3z - 3 &= (6z + 3) - (3z + 6) \\ &= \left[RHS(3.9) - 2RHS(3.6)\right] - \left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right]. \end{aligned}$$

□

The following corollary is an immediate consequence of the "cancelation phenomenon", exhibited in Proposition 3.1.

Corollary 3.1. *There exists a configuration of* 15 *tribones in the* 3*-space where exactly one lattice point* (*the center of the star*) *is covered by an odd number of tiles.*

Proof. As a consequence of the first relation proved in Proposition 3.1, by reducing modulo 2 we obtain the identity $1 = RHS(3.9)$. By further simplification we obtain the identity

$$1 = (x + y + z + 1)(A_x + A_y + A_z) + xA_{x,y} + yA_{y,z} + zA_{x,z},$$

which completes the proof. □

4. $\mathbb{Z}$-Tiling by k-Bones in d Variables

In this section we address the general problem of the existence of $\mathbb{Z}$-tiling by k-bones in the d-dimensional lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. We use standard abbreviations for monomials (power products) $x^a = x_1^{a_1} x_2^{a_2} \dots x_d^{a_d}$ and rely on standard concepts and terminology used in the theory of lattice-point enumeration in polyhedra, see [2] or [3].

In particular each set $R \subset \mathbb{R}^d$ is associated the *integer-point transform* $\sigma_R = \sum_{a \in R \cap \mathbb{Z}^d} x^a \in \mathbb{Z}[[x_1^{\pm 1}, \dots, x_d^{\pm 1}]]$, which is a Laurent polynomial if and only if R is bounded. Typically R is a convex polytope $Q \subset \mathbb{R}_+^d$ with vertices in $\mathbb{N}^d$ in which case $\sigma_Q \in \mathbb{Z}[x_1, \dots, x_d]$ is simply the sum of all monomials "covered" by Q.

Conversely, for each polynomial $p = \sum_{a \in \mathbb{N}^d} c_a x^a \in \mathbb{Z}[x_1, x_2, \dots, x_d]$ the associated *Newton polytope* is the convex polytope $Newton(p) = \mathrm{Conv}\{a \mid c_a \neq 0\}$.

Let $\Delta = \mathrm{Conv}\{e_i\}_{i=0}^d$ be the standard simplex in $\mathbb{R}^d$, where $e_0 = 0$ and $\{e_i\}_{i=1}^d$ is the standard orthonormal basis of $\mathbb{R}^d$ which generates the lattice $\mathbb{Z}^d$.

Given an integer $k \geq 1$, the k-bones in the d-dimensional lattice are the prototiles associated to the edges $E_{ij} = [ke_i, ke_j]\,(i \neq j)$ of the k^{th} *dilate* $k\Delta = \mathrm{Conv}\{ke_i\}_{i=0}^d$ of the simplex Δ.

More explicitly, the polynomials (integer-point transforms) of the k-bones are

$$\begin{aligned} b_i =& x_i^{k-1} + x_i^{k-2} + \dots + 1, \quad i = 1, \dots, d, \\ b_{ij} =& x_i^{k-1} + x_i^{k-2} x_j + \dots + x_j^{k-2}, \quad 1 \leq i < j \leq d. \end{aligned}$$

The associated ideal

$$I_k^d = \langle b_i, b_{i,j} \rangle \subseteq \mathbb{Z}[x_1, \dots, x_d]$$

is referred to as the k-bone ideal in d-dimensions, or simply as the k-bone ideal.

4.1. **Gröbner lattice-point enumerators.** Our general objective is to study the geometry and combinatorics of $\mathbb{Z}$-tilings of different shapes (convex polytopes) in $\mathbb{R}^d$ by k-bones (or more general prototiles), by methods of combinatorial commutative algebra and Gröbner basis.

The Gröbner basis of the ideal I_k^d with respect to some term order (usually the lexicographic order) is denoted by GBI_k^d (occasionally by GBI or G). We work with $\mathbb{Z}$-coefficients so the Abelian group of all remainders may have torsion and its generators are *reduced monomials* $x^\alpha \notin \langle LM(I_k^d)\rangle$, not contained in the ideal of leading monomials of I_k^d. (The reader is referred to the Appendix (Section 6) for a brief introduction into Gröbner basis theory and a guide to the literature.)

As in Section 2.2 the remainder on division of f by GBI is $\overline{f}^{GBI} = \sum_\alpha c_\alpha x^\alpha$, where x^α are reduced monomials. For improved legibility we sometimes write $Red_G(f)$ instead of $\overline{f}^G$. The coefficient c_α, which takes values in $\mathbb{Z}$ or some quotient $\mathbb{Z}/\nu\mathbb{Z}$, is denoted by

$$[x^\alpha]\left(\overline{f}^{GBI}\right). \tag{4.1}$$

Table 1 (Section 2.2) provides examples of the calculation and illustrates the importance of numerical functions (4.1) for the general polyomino tiling problem.

4.2. **Motivating example.** Here is another point of view which explains why (4.1) are called *Gröbner lattice-point enumerators* (Definition 4.1).

Let Q be a convex polytope with vertices in $\mathbb{N}^d$ and let $\sigma_Q(x) = \sum_{\alpha\in Q\cap\mathbb{N}^d} x^\alpha$ be its "Newton polynomial" (integer-point transform). The usual "discrete volume" (lattice-point enumerator) of Q, defined in [2, 3] as the number of integer points inside Q, is clearly equal to the value of σ_Q at $x = (1, 1, \ldots, 1) \in \mathbb{R}^d$.

Moreover, for each polynomial $f(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d]$ there is a relation

$$f(x_1, \ldots, x_d) = f_1(x_1 - 1) + \cdots + f_d(x_d - 1) + C, \tag{4.2}$$

where $C = f(1, \ldots, 1)$ is the remainder obtained on division of f by the ideal

$$I = \langle x_1 - 1, x_2 - 1, \ldots, x_d - 1\rangle.$$

It follows that the number of lattice points in a lattice convex polytope Q can be interpreted as the remainder of σ_Q on division by the ideal I.

4.3. **General research problem.** Division of multivariate polynomials by ideals is in general not unique and in particular the corresponding remainders (such as C in the expression (4.2)) are not uniquely defined. However, the division by the Gröbner basis of an ideal yields a unique remainder (in general a polynomial) which, in agreement with motivating example from Section 4.2, leads to the following research problem.

Let $J \subset \mathbb{Z}[x_1, \ldots, x_d]$ be an ideal, say the ideal associated to a set $\mathcal{R}$ of prototiles in $\mathbb{N}^d$. Let $G = G_J$ be the Gröbner basis of J with respect to some term order. It is interesting to ask (for some carefully chosen ideals J) what is the geometric and combinatorial significance of the remainder $\overline{f}_Q^G$ of the integer-point transform σ_Q on division by the Gröbner basis G.

Definition 4.1. The polynomial valued function $Q \mapsto \overline{f}_Q^G$ is referred to as Gröbner or G-discrete volume of Q with respect to the Gröbner basis G. The coefficients (4.1) are called Gröbner lattice-point enumerators of Q.

4.4. **Cyclotomic ideals.** A *cyclotomic ideal* in the ring $\mathbb{Z}[x_1, x_2, \ldots, x_d]$ is an ideal of the following form

$$W_k^d = \langle x_1^k - 1, x_2^k - 1, \ldots, x_d^k - 1\rangle, \tag{4.3}$$

where d and k are positive integers. In light of the obvious identities

$$x_i^k - 1 = (x_i - 1)(x_i^{k-1} + x_i^{k-2} + \cdots + 1), \quad x_i^k - x_j^k = (x_i - x_j)(x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1}),$$

W_k^d is contained in the ideal I_k^d generated by k-in-line polyominoes (k-bones) in the d-dimensional lattice.

Proposition 4.1. *The set* $S_k^d = \{x_1^k - 1, x_2^k - 1, \ldots, x_d^k - 1\}$ *is a* (*strong*) *Gröbner basis of the ideal* W_k^d *in the sense of* [11].

Proof. Indeed, the S-polynomial

$$S[x_i^k - 1, x_j^k - 1] = x_j^k(x_i^k - 1) - x_i^k(x_j^k - 1) = (x_i^k - 1) - (x_j^k - 1)$$

is trivially reducible by the basis S_k^d. □

The following criterion for the existence $\mathbb{Z}$-tilings is formulated in [10, Proposition 3.1].

Proposition 4.2. *A polyomino P admits a* signed tiling *by translates of* prototiles $P_1, P_2, \ldots, P_k$ *if and only if for some monomial* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *with a non-negative exponent* $\alpha \in \mathbb{N}^d$ *the polynomial* $x^\alpha \sigma_P$ *is in the ideal generated by polynomials* $\sigma_{P_1}, \ldots, \sigma_{P_k}$,

$$x^\alpha \sigma_P \in \langle \sigma_{P_1}, \sigma_{P_2}, \ldots, \sigma_{P_k}\rangle. \tag{4.4}$$

Note that $x^\alpha \sigma_P \in J$ implies $x^{\alpha'} \sigma_P \in J$ in any ideal J, provided $x^{\alpha'}$ is divisible by x^α, which allows us to formulate the following simplified criterion for k-bone ideals I_k^d.

Proposition 4.3. *A polyomino P admits a* signed tiling *by translates of k-bones E_{ij}, $0 \leq i < j \leq d$, if and only if*

$$\sigma_P \in I_k^d\,. \tag{4.5}$$

Proof. If $\sigma_P \in I_k^d$ then obviously P admits a signed tiling by translates of k-bones E_{ij}. Conversely, suppose P admits a signed tiling by translates of k-bones E_{ij}. By Proposition 4.2 there exists a monomial x^α such that $x^\alpha \sigma_P \in I_k^d$. Since for some $\beta \in \mathbb{N}^d$ the vector $\alpha + \beta = k\gamma \in k\mathbb{N}^d$ is divisible by k we conclude that $x^{k\gamma}\sigma_P \in I_k^d$. Since $W_k^d \subset I_k^d$ we know that $x^{k\gamma} \equiv 1 \pmod{I_k^d}$, which in turn implies $\sigma_P \in I_k^d$. □

4.5. **Reduction of monomials x^a modulo W_k^d and I_k^d.** Let $x^a = x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}$ be the monomial with multi-index $a \in \mathbb{Z}_+^d$. Given $z \in \mathbb{Z}_+$, let $\widehat{z} = r(z)$ be the remainder on division of z by k, $r(z) \in \mathbb{Z}_k = \{0, 1, \dots, k-1\}$. The reduced version of the monomial x^a with respect to the ideal W_k^d is the monomial $Red_{W_k^d}(x^a) = x^{\widehat{a}} = x_1^{\widehat{a_1}}x_2^{\widehat{a_2}}\dots x_d^{\widehat{a_d}}$.

Note that $Red_{W_k^d}(x^a)$ is obtained from x^a by successive division (in any order) by elements of the ideal W_k^d.

Our objective is to compute the W_k^d-reduced version of the polynomial $T_k^d(n)$

$$Red_{W_k^d}(T_k^d(n)) = Red_{W_k^d}\left(\sum_{\substack{0\le a\\ |a|\le n-1}} x^a\right) := \sum_{\substack{0\le a\\ |a|\le n-1}} x^{\widehat{a}}. \tag{4.6}$$

Proposition 4.4. *Let*

$$Red_{W_k^d}(T_k^d(n)) = Red_{W_k^d}\left(\sum_{\substack{0\le a\\ |a|\le n-1}} x^a\right) = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)x^r \tag{4.7}$$

be the reduction of the polynomial $T_k^d(n)$ with respect to the ideal W_k^d. Then

$$t_k^d(n,r) = \binom{d+(n|r)}{d},$$

where

$$(n|r) := \left\lfloor \frac{n-1-|r|_1}{k} \right\rfloor$$

and $|r|_1 = |(r_1, r_2, \dots, r_d)|_1 = r_1 + \cdots + r_d$.

Proof. Given a W_k^d-reduced monomial x^r, where $r = (r_1, r_2, \dots, r_d) \in (\mathbb{Z}_k)^d$, we want to calculate the number of solutions of the inequality

$$(kx_1 + r_1) + (kx_2 + r_2) + \cdots + (kx_d + r_d) \le n-1 \tag{4.8}$$

in non-negative integer variables $x_1, \dots, x_d$. Equivalently, we need to calculate the number of non-negative integer solutions of

$$x_1 + x_2 + \cdots + x_d \le \left\lfloor \frac{n-1-|r|_1}{k} \right\rfloor, \tag{4.9}$$

where $\lfloor x \rfloor$ is the integer part of x. Recall the lattice point enumerator [3, Theorem 2.2] of the standard simplex in the positive hyperorthant $\mathbb{R}_+^d$ bounded by the hyperplane $x_1 + \cdots + x_d = m$,

$$L_\Delta(m) = \binom{d+m}{d}.$$

By substitution $m = (n|r)$ we complete the proof of the proposition. □

As a corollary we obtain the following proposition.

Proposition 4.5. *Let $GBI = GBI_k^d$ be a Gröbner basis of the ideal I_k^d with respect to some term order. Then the remainder*

$$\overline{T_k^d(n)}^{GBI}$$

of the polynomial $T_k^d(n)$ on division by $GBI = GBI_k^d$, expressed in terms basic monomials x^α, admits a decomposition

$$\overline{T_k^d(n)}^{GBI} = \sum_\alpha c_\alpha x^\alpha, \tag{4.10}$$

where $c_\alpha = [x^\alpha](\overline{T_k^d(n)}^{GBI})$ is some (finite) $\mathbb{Z}$-linear combination of functions $t_k^d(n,r)$. More explicitly,

$$c_\alpha = [x^\alpha]\left(\overline{T_k^d(n)}^{GBI}\right) = \sum_{r\in(\mathbb{Z}_k)^d} e_\alpha^r t_k^d(n,r),$$

for some integers e_α^r.

Proof. As a consequence of (4.7) we obtain

$$\overline{T_k^d(n)}^{GBI} = Red_{I_k^d}(T_k^d(n)) = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)\overline{x^r}^{GBI} = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)\sum_\alpha e_\alpha^r x^\alpha. \tag{4.11}$$

□

5. Ehrhart Theory and Gröbner Bases

Quasipolynomials play a fundamental role in the Ehrhart theory of lattice-point enumerators of polytopes with rational vertices. We demonstrate that they play a similar role in Gröbner lattice-point enumeration with respect to ideals W_k^d and I_k^d.

5.1. **Quasipolynomials.** A *quasipolynomial* [13, Section 4.4] of degree d is a function $f : \mathbb{N} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{n-1} + \cdots + c_0(n),$$

where each $c_i(n)$ is a periodic function and $c_d(n)$ is not identically equal to zero.

It is not difficult to show that f is a quasipolynomial if and only if there exists an integer $N > 1$ and polynomials $f_0, f_1, \dots, f_{N-1}$ such that

$$f(n) = f_i(n), \quad \text{if } n \equiv i \pmod N.$$

Quasipolynomials play an exceptionally important role in enumerative combinatorics. For example the *Ehrhart polynomial* $L_Q(n)$, defined as the lattice point enumerator of the n^{th} dilate nQ of a convex polytope Q with rational vertices, is always a quasipolynomial.

It is an easy exercise to check that the function $t_k^d(n,r)$, introduced in Proposition 4.4, is a quasipolynomial in the variable n. In turn, the coefficients c_α (that appear in Proposition 4.5) are also quasipolynomials, being linear combinations of functions $t_k^d(n,r)$.

The functions $c_\alpha = c_\alpha(T_k^d(n))$, being defined essentially as summands of the remainder on division by the ideal I_k^d, are extended in a straightforward way to all convex rational convex polytopes Q. They are referred to as *Gröbner lattice-point enumerators.*

5.2. **Quasipolynomials and generalizations of Pick's theorem.** Here we remind the reader why (quasi)polynomials are important in lattice-point enumeration problems (Ehrhart theory). In the planar case the Ehrhart polynomial is a polynomial $L_Q(n) = a_0n^2 + a_1n + a_2$ where $a_0 = Area(Q)$ and $a_2 = L_Q(0) = 1$. Moreover, $L_Q(1) = a_0 + a_1 + a_2$ is the number of lattice points in Q and, by Ehrhart-Macdonald reciprocity (see [3, Theorem 4.1]),

$$L_Q(1) + L_Q(-1)$$

is the number of lattice points on the boundary of Q.

The four quantities $a_0, L_Q(0), L_Q(1)$ and $L_Q(1)+L_Q(-1)$ can be interpreted as linear forms on the 3-dimensional vector space of all quadratic polynomials and classical *Pick's theorem* is nothing but a non-trivial linear relation

$$\lambda_1 a_0 + \lambda_2 L_Q(0) + \lambda_3 L_Q(1) + \lambda_4(L_Q(1) + L_Q(-1)) = 0. \tag{5.1}$$

Once we know that such a relation exists, the coefficients λ_i are easily evaluated by choosing special polygons Q.

The importance of this proof of Pick's theorem is that it can be easily generalized. For example Reeve's theorem (a 3-dimensional analogue of Pick's theorem) says that in addition to linear forms listed in (5.1) it suffices to take one more, the form $L_Q(2)$ evaluating the number of lattice points in the second dilate of Q.

Similar scheme can be applied to quasipolynomials as well and the following sections should provide a theoretical basis for studying analogues of Pick's theorem for Gröbner basis enumerators of lattice polytopes. (This is the subject of a subsequent publication.)

5.3. **Ehrhart quasipolynomial for Gröbner W_k^d-enumerators.** In this section we prove that Gröbner lattice-point enumerators of lattice polytopes, with respect to the ideal W_k^d, are *quasipolynomials.* We have already calculated (Section 4.5) the W_k^d-reduction of the tetrahedron associated to the polynomial $T_k^d(n)$ and showed (Proposition 4.4) that the result is a quasipolynomial in the variable n. Here we extend this result to the case of a general rational polytope.

Theorem 5.1. *Let σ_Q be the integer-point transform of a rational convex polytope $Q \subset (\mathbb{R}_+)^d$ and $c_\alpha^{W_k^d}(Q) = [x^\alpha](Red_{W_k^d}(\sigma_Q))$ the Gröbner lattice-point enumerator with respect to the ideal W_k^d, associated to a W_k^d-reduced monomial x^α. Then the function*

$$f_\alpha^{W_k^d}(n) = c_\alpha^{W_k^d}(nQ) = [x^\alpha](Red_{W_k^d}(\sigma_{nQ})),$$

computing the Gröbner basis enumerator $c_\alpha^{W_k^d}$ *of the* n^{th} *dilate of the convex polytope* Q*, is a quasipolynomial in the variable* n*.*

As usual in Ehrhart theory [3, Chapter 3], the case of a general rational polytope is reduced to the case of a rational simplex. Moreover the case of general rational simplex (simplicial cone) is treated similarly as the case of a simplex with integral vertices. So the proof of Theorem 5.1 follows from the proof of the following proposition.

Proposition 5.1. *Let* $\Delta \subset (\mathbb{R}_+)^d$ *be a simplex with integral vertices and let* $r = (r_1, \dots, r_d) \in (\mathbb{Z}_k)^d$*. Then a mod-*$k$ *lattice-point enumerator* $L_\Delta^{k,r}(n)$ *of* Δ*, defined as the number of lattice points* $a = (a_1, \dots, a_d) \in \mathbb{Z}^d \cap n\Delta$ *such that* $a_i \equiv r_i \mod k$ *for each* $i \in [d]$*, is a quasipolynomial in* d*.*

Proof. Since $L_\Delta^{k,r}(n) = L_{v+\Delta}^{k,r}(n)$ for each $v \in \mathbb{Z}^d$ we assume, without loss of generality, that $-\frac{r}{k} + \Delta \subset (\mathbb{R}_+)^d$. Let $\Delta = \mathrm{Conv}\{v_i\}_{i=1}^{d+1}$.

By [3, Theorem 3.5] it is known that the integer-point transform σ_{v+K} of a shifted simplicial cone

$$K = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_d w_d \mid \lambda_i \geq 0\} \subseteq \mathbb{R}^d \tag{5.2}$$

is the rational function

$$\sigma_{v+K}(z) = \frac{\sigma_{v+\Pi}(z)}{(1 - z^{w_1})(1 - z^{w_2}) \cdots (1 - z^{w_d})}, \tag{5.3}$$

where

$$\Pi = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_d w_d \mid 0 \leq \lambda_i < 1\}$$

is the associated *fundamental half-open parallelepiped.* Let $w_i = (v_i, 1) \in \mathbb{R}^{d+1}$, $i \in [d+1]$, and let $K \subset \mathbb{R}^{d+1}$ be the associated simplicial cone defined by (5.2), with the associated fundamental parallelepiped Π.

It follows that the integer-point transform $\sigma_K(z, t)$ of K is given by the formula (5.3), where d is replaced by $d + 1$ and the new (vertical) variable is t. Moreover [3, Section 3.3], the n^{th} dilate of Δ is essentially the intersection of K with the horizontal hyperplane $H_n := \{(z, t) \in \mathbb{R}^{d+1} \mid t = n\}$, and the generating function for the Ehrhart polynomial $L_\Delta(n)$, calculating the number of lattice points in $n\Delta$, is given by the formula

$$\sum_{n \geq 0} L_\Delta(n)\, t^n = \sigma_K(\mathbb{1}, t),$$

where the RHS is evaluated at $z = \mathbb{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$.

We want to describe the generating function calculating the lattice points $a \in K$ such that $a = ka' + r$ for some $a' \in \mathbb{Z}^{d+1}$. In other words we need a generating function for the set of lattice points a' in the shifted cone

$$K' = -\frac{r}{k} + \frac{1}{k} K.$$

Again by (5.3), taking into account that K' is scaled down by the factor k, we obtain

$$\sum_{n\geq 0} L_{\Delta}^{k,r}(n)\, t^n = \sigma_{K'}(\mathbb{1},t) = \frac{g(t)}{(1-t^k)^{d+1}},$$

where $g(t)=\sigma_{\Pi'}(\mathbb{1},t)$ and $\Pi' = -r/k+\Pi$ is the shifted fundamental parallelepiped of K'.

By assumption $-\frac{r}{k}+\Delta \subset (\mathbb{R}_+)^d$ which implies that

$$(-r/k+\Pi)\cap \mathbb{Z}^{d+1} \subseteq \Pi\cap\mathbb{Z}^{d+1} \subset \mathbb{N}^{d+1}\,.$$

It follows that $\deg(g) < k(d+1)$ and, as a consequence of Proposition 4.4.1 [13, Proposition 4.4.1], we conclude that $L_{\Delta}^{k,r}(n)$ is a quasipolynomial. □

5.4. **Ehrhart theory for Gröbner I_k^d-enumerators.** Here we show that Gröbner lattice-point enumerators of lattice polytopes, with respect to the ideal I_k^d, are (modular reductions of) *quasipolynomials*. Since quasipolynomials naturally appear as lattice points enumerators (Ehrhart theory) for convex polytopes with rational vertices, see [3, Section 3.7], the following result can be interpreted as a first step in the direction of developing Ehrhart theory for Gröbner basis enumerators of rational convex polytopes.

We say that a function $f:\mathbb{N}\to\mathbb{Z}_\nu$ (where $\nu\in\mathbb{Z}_+\cup\{\infty\}$ and by convention $\mathbb{Z}_\infty=\mathbb{Z}$) is a *modular quasipolynomial*, if there exists and integer valued function $f':\mathbb{N}\to\mathbb{Z}$ such that $f(n)$ is the mod ν reduction of $f'(n)$ for each $n\in\mathbb{N}$.

Theorem 5.2. *Let σ_Q be the integer-point transform of a rational convex polytope Q in $(\mathbb{R}_+)^d$ and $c_\beta = c_\beta^{I_k^d}(Q) = [x^\alpha](Red_{I_k^d}(\sigma_Q))$ the Gröbner lattice-point enumerator associated to a I_k^d-reduced monomial x^β. Then the function*

$$f_\beta^{I_k^d}(n) = c_\beta^{I_k^d}(nQ) = [x^\beta](Red_{I_k^d}(\sigma_{nQ})),$$

computing the Gröbner lattice-point enumerator c_β of the n^{th} dilate of the convex polytope Q, is a modular quasipolynomial in the variable n.

Proof. Since $W_k^d\subset I_k^d$,

$$Red_{I_k^d}(\sigma_{nQ}) = Red_{I_k^d}(Red_{W_k^d}(\sigma_{nQ})) = Red_{I_k^d}\left(\sum_\alpha c_\alpha x^\alpha\right),$$

where on the right is an expression involving W_k^d-reduced monomials x^α. Since for each I_k^d-reduced monomial x^β

$$[x^\beta](Red_{I_k^d}(\sigma_{nQ})) = [x^\beta]\left(\sum_\alpha c_\alpha Red_{I_k^d}(x^\alpha)\right) = \sum_\alpha c_\alpha [x^\beta]\left(Red_{I_k^d}(x^\alpha)\right),$$

the result is an immediate consequence of Theorem 5.1. □

Remark 5.1. We have shown (Theorems 5.1 and 5.2) that Gröbner lattice-point enumerators of ideals W_k^d and I_k^d are (modular) quasipolynomials. Is this a general phenomenon? In other words is it true that G-enumerators of (polyomino) ideals are (modular) quasipolynomials for any choice of prototiles. We suspect that the answer is negative in general but we don't have an example at hand.

6. Appendix: Gröbner Bases

The reader not familiar with the fundamental concepts and results of Gröbner bases theory is encouraged to use it as black box, after consulting a two page introduction in [14]. Since [14] deals only with polynomials with coefficients in the field here we briefly outline, following [11], how the theory is modified if we work with integer coefficients.

A term is a product $t = cx^\alpha$ where c is the coefficient and $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is the associated monomial (power product). For a given polynomial $f \in \mathbb{Z}[x_1, x_2, \dots, x_k]$ the associated remainder on division by a Gröbner basis G is $\overline{f}^G$ and f reduces to zero $f \xrightarrow{G} 0$ if $\overline{f}^G = 0$. $LM(f)$ and $LC(f)$ are respectively the leading monomial and the leading coefficient with respect to the chosen term order $\preceq$. We write $lcm(a, b)$ and $gcd(a, b)$ respectively for the least common multiple and the greatest common divisor of a and b.

For other basic notions of Gröbner basis theory (over integers), such as S-polynomial, standard representation, etc. the reader is referred to [11] (see also [1, 4] for a more complete exposition of the theory).

6.1. Gröbner bases over principal ideal domains.

Let $\Lambda = R[x_1, \dots, x_k]$ be the ring of polynomials with coefficients in a principal ideal domain R. For a given ideal $I \subset \Lambda$ the associated *strong Gröbner basis*, called also the D bases in [4], may be introduced as follows (see [1, p. 251] and [4, p. 455]).

Definition 6.1. A finite set $G \subset I$ is a strong Gröbner basis of I (with respect to the chosen term order $\preceq$) if for each $f \in I \setminus \{0\}$ there exists $g \in G$ such that the leading term of f is divisible by the leading term of g, $LT(g)|LT(f)$, meaning that $LT(f) = tLT(g)$ for some term t.

The following theorem provides a useful criterion for testing whether a finite set of polynomials is a Gröbner basis of the ideal generated by them, see [4, Chapter 10, Corollary 10.12].

Theorem 6.1. *Let G be a finite collection of non-zero polynomials which generate an ideal I_G. Suppose that,*

(1) *for each pair $g_1, g_2 \in G$ there exists $h \in G$ such that,*

$$LM(h)|\mathrm{lcm}(LM(g_1), LM(g_2)) \text{ and } LC(h)|\mathrm{gcd}(LC(g_1), LC(g_2));$$

(2) *for each pair $g_1, g_2 \in G$ the associated S-polynomial reduces to zero,*

$$S(g_1, g_2) \xrightarrow{G} 0.$$

Then G is a strong Gröbner basis of I_G.

6.2. **Gröbner bases over Euclidean domains.** The general theory is further simplified if one works with Euclidean domains. Aside from standard references [1, 4] a self-contained account can be found in [11]. In the case of integers one usually chooses the linear ordering,

$$\cdots < 0 < +1 < -1 < +2 < -2 < +3 < -3 < +4 < -4 < +5 < \cdots, \tag{6.1}$$

which allows us to define unambiguously remainders, S-polynomials etc.

Recall that the constant $g_{10} = 9$ is an element of the Gröbner basis GBI of the tribone ideal (Section 2.1). The ordering (6.1) explains why -4 (rather than $+5$) appears in reduced expressions f^{GBI}, for example in Table 1.

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CONCERNING MULTIVARIATE BERNSTEIN POLYNOMIALS AND STOCHASTIC LOGIC

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Abstract. Among the applications of the Bernstein polynomials in one variable is their use in solving problems associated with stochastic computing. Taking as a starting point the notion of stochastic logic in the sense of Qian-Riedel-Rosenberg, the aim of this paper is to investigate some necessary and sufficient conditions for guaranteeing whether polynomial operations can be implemented with stochastic logic based on multivariate Bernstein polynomials with coefficients in the unit interval.

1. Introduction

Stochastic computing (SC) arises as a collection of techniques to represent analog quantities by probabilities of discrete events, or represent continuous values by means of random bit-streams, so that complex operations can be performed by simple bitwise operations on random pulse trains [1, 7–10, 27]. The analogy between probability algebras and Boolean algebras [12, 13, 25] is used to obtain very simple processing units and an adequate arithmetic. The basic operations described in the literature are the addition and the multiplication since these are the fundamental operations involved in neural networks and in the design of stochastic circuitry (fields in which fertile ground has been found for applications of SC). Also, SC has been applied to division and square-rooting [10, 33], matrix operations and decoding of low-density parity check (LDPC) codes [11, 21, 23], and polynomial arithmetic [28, 29].

A stochastic number can be defined as a pair (x, p_x), where x is a finite binary sequence, i.e., $x \in \{0,1\}^N$, for some $N \in \mathbb{N}$ and $p_x \in [0,1]$ is the probability of

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observing a 1 at an arbitrary position of x [1, 9, 10, 24]. So, a stochastic number is represented by a finite binary sequence (or bit-stream) in such a way that the probability (ratio) of '1' in the binary sequence is interpreted as the number itself. The probability p_x is sometimes called value of the stochastic number (see, e.g., [24]).

For instance, if (x, p_x) is a stochastic number whose binary sequence x has N components, of which m are equal to 1 and $N - m$ are equal to 0, then $p_x = \frac{m}{N}$ and, clearly, the representation of the pair (x, p_x) is not unique. SC uses a redundant number system in which there are $\binom{N}{m}$ possible representations for each value $p_x = \frac{m}{N}$. Furthermore, a binary sequence x can only has associated probabilities in the set $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$, so only a small subset of the real numbers in $[0, 1]$ can be expressed exactly in SC.

The main idea behind the combinational circuits design with polynomial arithmetic of Qian et al. [28, 29] consist of the following.

(1) Take advantage -in a suitable way- of the redundancy provided by SC for choosing binary sequences $x \in \{0, 1\}^N$ corresponding to the value p_x, in order to make an association between x and a certain N-tuple of independent random variables $X = (X_1, \ldots, X_N)$, where each component X_k has Bernoulli distribution with some parameter $p_k \in [0, 1]$.

(2) Given a Boolean function $y = f(x_1, \ldots, x_N)$ implementing a combinational circuit, use the association aforementioned for inducing a stochastic circuit implemented by a function of the form $Y = F(X_1, \ldots, X_N)$ (see for instance, [25]).

The passage of the Boolean function $y = f(x_1, \ldots, x_N)$ to the function $Y = F(X_1, \ldots, X_N)$ is called stochastic logic or stochastic logic in the sense of Qian-Riedel-Rosenberg [28, 29] and the following property holds.

Theorem 1.1. ([28, Theorem 1]). *Given a Boolean function $f : \{0, 1\}^N \to \{0, 1\}$. Stochastic logic yields a polynomial in N variables $\hat{F}$ given by*

$$\hat{F}(a_1, \ldots, a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1 \ldots i_N}$ are integers. Moreover, for each $y = f(x_1, \ldots, x_N)$ we have

$$p_Y = \hat{F}(p_{X_1}, p_{X_2}, \ldots, p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} p_{X_k}^{i_k} \right).$$

It is worth pointing out that to the best of our knowledge, the treatment or implementation by use of some stochastic logic of Qian-Riedel-Rosenberg type has not been considered for Boolean maps of the form $f : \{0, 1\}^N \to \{0, 1\}^N$. Thus, the following questions related to Theorem 1.1 arise: Can Theorem 1.1 be extended in this setting? In negative case, what is the difficult for finding such an extension? In affirmative case, how do we characterize such an extension? In this paper, we are interested in the theoretical issues concern stochastic logic of Qian-Riedel-Rosenberg type. In particular, we focus our attention on the theoretical connection between a stochastic logic of Qian-Riedel-Rosenberg type and certain class of multivariate Bernstein polynomials related with combinational circuits. So, some of aforementioned questions will be answer in the present paper.

The outline of the paper is as follows. Section 2 contains some relevant properties of the induced multivariate Bernstein polynomials. In Section 3 the notion of stochastic logic of Qian-Riedel-Rosenberg type is introduced, its connection with induced multivariate Bernstein polynomials is given and our main results are stated and proved. Finally, Section 4 is devoted to a brief additional remark on a model of stochastic logic based on the so-called degenerate Bernstein polynomials. Throughout this paper, we only consider combinational circuitry.

2. Multivariate Bernstein Polynomials

This section is devoted to introduce a class of multivariate Bernstein polynomials and recall some of their structural properties. We adopt the way of writing multivariate Bernstein polynomials used in [3]. For more details the reader can see [3], [22, § 2.9, p. 51] and the references thereof. However, before we look at this class we will recall the definition and some algebraic and analytic properties of the Bernstein polynomials in one variable (cf., [22, 29]).

Given $n \in \mathbb{N}$, for $f : [0,1] \to \mathbb{R}$ a continuous function and $t \in [0,1]$, the nth Bernstein polynomial of f is given by

$$B_n(t) = B_n(f;t) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}. \tag{2.1}$$

The polynomials $B_n(t)$ converge uniformly to f on $[0,1]$ and this fact is the key piece for the Bernstein constructive demonstration of Weierstrass approximation theorem [22, 26].

The polynomials appearing in the formula on the right hand side of (2.1), namely;

$$b_k(t) = b_{k,n}(t) := \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, \dots, n,$$

form a basis for the space of polynomials of degree at most n with real coefficients and the set $\{b_{k,n}(t) : k = 0, \dots, n\}$ and it is usually called Bernstein basis [5,30]. Also, it is clear that $\deg(b_{k,n}(t)) = n$, for each $k = 0, \dots, n$.

We call Bernstein polynomial to the representation in terms of the Bernstein basis of any polynomial $P(t)$ of degree at most n and real coefficients. So, there exists a unique vector $(\beta_{0,0}, \beta_{1,n}, \dots, \beta_{n,n}) \in \mathbb{R}^{n+1}$ such that

$$P(t) = \underbrace{\sum_{k=0}^{n} \beta_{k,n} b_{k,n}(t)}_{\text{Bernstein polynomial}} . \tag{2.2}$$

The name Bernstein polynomial for the expression on the right hand side of (2.2) was coined by Qian et al. (cf., [28, 29]), although Farouki and Goodman [5] have preferred to use the term *Bernstein form of* $P(t)$ to refer to the same expression. By (2.2) we have that the nth Bernstein polynomial of the function $f \in C[0,1]$ given by (2.1) becomes in a particular case of Bernstein polynomial, for which $\beta_{k,n} = f\left(\frac{k}{n}\right)$, $k = 0, 1, \ldots, n$.

The following results show some pertinent properties of the Bernstein basis and polynomials.

Proposition 2.1. *The Bernstein basis* $\{b_{k,n}(t) : k = 0, \ldots, n\}$ *satisfies the following algebraic and analytic properties* [6, 29].

(i) *Partition of unity property.*

$$\sum_{k=0}^{n} b_{k,n}(t) = 1, \quad \text{for all } t \in \mathbb{R}.$$

(ii) *Non-negativity property.*

$$b_{k,n}(t) \geq 0, \quad \text{for all } t \in [0,1].$$

(iii) *Symmetry property.*

$$b_{k,n}(t) = b_{n-k,n}(1-t), \quad \text{for all } t \in [0,1].$$

(iv) *Recurrence formula.*

$$b_{k,n+1}(t) = t b_{k-1,n}(t) + (1-t) b_{k,n}(t), \quad \text{for all } t \in [0,1].$$

(v) *Unimodality or extremal property. For* $n \geq 1$, $b_{k,n}(t)$ *attains a relative maximum at* $t = \frac{k}{n}$, $k = 0, \ldots, n$.

(vi) *Degree elevation property. For* $k = 0, \ldots, n$, *we have*

$$b_{k,n}(t) = \frac{n+1-k}{n+1} b_{k,n+1}(t) + \frac{k+1}{n+1} b_{k+1,n+1}(t),$$

for all $t \in [0,1]$.

(vii) *Representation in terms of the canonical basis of the space of polynomials of degree at most* n *with real coefficients.*

$$b_{k,n}(t) = \sum_{j=k}^{n} (-1)^{j-k} \binom{n}{j} \binom{j}{k} t^j.$$

Proposition 2.2. *Let* $P(t) = \sum_{k=0}^{n} \beta_{k,n} b_{k,n}(t)$ *be a Bernstein polynomial. Then the following properties hold* [6, 29].

(i) $P(0) = \beta_{0,n}$ *and* $P(1) = \beta_{n,n}$.

(ii) *Inversion formula. For each* $0 \leq j \leq n$, *we have*

$$t^j = \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} b_{k,n}(t).$$

(iii) *Change of basis. If $P(t)$ has the following representation in terms of the canonical basis of the space of polynomials of degree at most N with real coefficients:*

$$P(t) = \sum_{k=0}^{n} a_{k,n} t^k,$$

then

$$\beta_{k,n} = \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{n}{j}} a_{j,n}, \quad k = 0, \ldots, n.$$

(iv) *Lower and upper bounds.*

$$\min_{0 \leq k \leq n} \beta_{k,n} \leq P(t) \leq \max_{0 \leq k \leq n} \beta_{k,n}.$$

(v) *Degree elevation procedure. For any $m \geq n$, it is always possible to represent $P(t)$ in terms of the Bernstein basis $\{b_{k,m+1}(t) : k = 0, \ldots, m+1\}$ as follows*

$$P(t) = \sum_{k=0}^{m+1} \beta_{k,m+1} b_{k,m+1}(t),$$

where the Bernstein coefficients $\beta_{k,m+1}$ are given by

$$\beta_{k,m+1} = \begin{cases} \beta_{0,m}, & \text{for } k = 0, \\ \frac{k}{m+1}\beta_{k-1,m} + \left(1 - \frac{k}{m+1}\right)\beta_{k,m}, & \text{for } k = 1, \ldots, m, \\ \beta_{m,m}, & \text{for } k = m+1. \end{cases}$$

(vi) (*cf.* [29, Theorem 1]) *Uniform approximation of the Bernstein coefficients. Let $g(t)$ be a polynomial of degree $n \geq 0$. For any $\epsilon > 0$, there exists a positive integer $M \geq n$ such that for all integer $m \geq M$ and $k = 0, 1, \ldots, m$, we have*

$$\left| \beta_{k,m} - g\left(\frac{k}{m}\right) \right| < \epsilon,$$

where $\beta_{0,m}, \beta_{1,m}, \ldots, \beta_{m,m}$ satisfy that $g(t) = \sum_{k=0}^{m} \beta_{k,m} b_{k,m}(t)$.

Given $N \in \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, to deal with multivariate polynomials we recall the standard multi-index notation. A multi-index is denoted by $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{N}_0^N$. For two given multi-indices $\alpha, \nu \in \mathbb{N}_0^N$ we write $\alpha \leq \nu$ if and only if $\alpha_j \leq \nu_j$, $j = 1, \ldots, N$. The multi-index $\alpha + \nu$ is defined by $\alpha + \nu = (\alpha_1 + \nu_1, \ldots, \alpha_N + \nu_N)$. If $\alpha \leq \nu$, the multi-index $\nu - \alpha$ is defined by $\nu - \alpha = (\nu_1 - \alpha_1, \ldots, \nu_N - \alpha_N)$. We write $\binom{\nu}{\alpha}$ for the multiplication $\binom{\nu_1}{\alpha_1} \cdots \binom{\nu_N}{\alpha_N}$, whenever $\alpha \leq \nu$. For $\nu \in \mathbb{N}_0^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ a monomial in variables $x_1, \ldots, x_N$ of index ν is defined by

$$x^\nu = x_1^{\nu_1} \cdots x_N^{\nu_N}.$$

We denote by $\mathbb{P}^N = \mathbb{R}[x_1, \ldots, x_N]$ the space of all polynomials of N variables with real coefficients. Let $p(x) = p(x_1, \ldots, x_N) \in \mathbb{P}^N$. We say that a multi-index $\kappa = (\kappa_1, \ldots, \kappa_N)$ is the multi-index of maximum degree of $p(x)$ if κ_j is the maximum degree of x_j in $p(x)$, $j = 1, \ldots, N$ (cf. [3]).

So, the set $S = \{\nu \in \mathbb{N}_0^N : \nu \leq \kappa\}$ contains all the combinations from $\mathbb{N}_0^N$ which are smaller than or equal to the multi-index κ of maximum degree. Hence, $p(x)$ can be expressed as

$$p(x) = p(x_1, \dots, x_N) = \sum_{\nu \in S} a_{\nu,\kappa} x^\nu, \tag{2.3}$$

where $a_{\nu,\kappa} \in \mathbb{R}$. The multivariate polynomial appearing on the right hand side of (2.3) is called the power form of $p(x)$.

An N-dimensional generalization of the Bernstein polynomials can be defined as follows. Let $f : [0,1]^N \to \mathbb{R}$ be a bounded function. The N-dimensional Bernstein of f is given by

$$B_{n_1,\dots,n_N}(f; (x_1, \dots, x_N)) := \sum_{\nu \in S^*} f\left(\frac{\nu_1}{n_1}, \dots, \frac{\nu_N}{n_N}\right) B_{\nu,\mathbf{N}}(x_1, \dots, x_N), \tag{2.4}$$

where $S^* = \{\nu \in \mathbb{N}_0^N : 0 \leq \nu \leq \mathbf{N}\}$, $\mathbf{N} = (n_1, \dots, n_N)$ and $B_{\nu,\mathbf{N}}(x_1, \dots, x_N) = \prod_{j=1}^N b_{\nu_j,n_j}(x_j)$.

It is well known that the N-dimensional Bernstein $B_{n_1,\dots,n_N}(f; (x_1, \dots, x_N))$ converges to $f((x_1, \dots, x_N))$ at any point of continuity of this function, as all $n_k \to \infty$ (cf., [4,14]), and from (2.4) it is possible to induce a multivariate Bernstein polynomial as follows:

$$P(x) = P(x_1, \dots, x_N) := \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x_1, \dots, x_N) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \tag{2.5}$$

$x \in [0,1]^N$, $c_{\nu,\mathbf{N}} \in \mathbb{R}$. We call to the polynomial $P(x_1, \dots, x_N)$ induced multivariate Bernstein polynomial.

Furthermore, if $x \in [0,1]^N$ and $p(x)$ is a multivariate polynomial which is written by means of a power form (2.3), then $p(x)$ can be expressed in terms of an induced multivariate Bernstein polynomial as follows.

$$p(x) = \sum_{\nu \in S} c_{\nu,\kappa} B_{\nu,\kappa}(x), \tag{2.6}$$

where the Bernstein coefficients $c_{\nu,\kappa}$ are given by

$$c_{\nu,\kappa} = \sum_{\alpha \leq \nu} \frac{\binom{\nu}{\alpha}}{\binom{\kappa}{\alpha}} a_{\alpha,\kappa}, \quad \nu \in S. \tag{2.7}$$

The multivariate polynomial appearing on the right hand side of (2.6) is called Bernstein form of $p(x)$.

3. Main Results

Given $N \in \mathbb{N}$ and (x, p_x) a stochastic number with $x \in \{0,1\}^N$. For each $k = 1, 2, \dots, N$ we choose $p_k \in [0,1]$ and consider discrete and independent random variables X_k having Bernoulli distribution with parameter p_k, i.e., $X_k \sim Be(p_k)$

(cf. [25,31]). Since $x_k \in \{0,1\}$, each probability density function is given by

$$P\{X_k = x_k\} = p_k^{x_k}(1-p_k)^{1-x_k}. \tag{3.1}$$

We define

$$p_{X_k} := P\{X_k = 1\} = p_k \quad \text{and} \quad 1 - p_{X_k} := P\{X_k = 0\} = 1 - p_k, \quad k = 1, 2, \dots, N.$$

Assume that a combinational circuit implements the Boolean map $f : \{0,1\}^N \to \{0,1\}^N$. Let $(f_1, \dots, f_N)$ be the component functions of f. Thus, each Boolean function $f_j : \{0,1\}^N \to \{0,1\}$ can be assumed as a subcircuit associated to f, $j = 1, 2, \dots, N$.

Given (x, p_x) a stochastic number with $x = (x_1, \dots, x_N) \in \{0,1\}^N$, choose an N-tuple of discrete and independent random variables $X = (X_1, \dots, X_N)$ such that $X_k \sim Be(p_k)$ for some $p_k \in [0,1]$ and satisfying (3.1). We can associate to each component function $f_j : \{0,1\}^N \to \{0,1\}$, a discrete random variable Y_j using that its probability density function is uniquely determined by the given N-tuple $X = (X_1, \dots, X_N)$. More precisely, for determining $p_{Y_j} := P\{Y_j = 1\}$, $j = 1, \dots, N$, we proceed as follows (cf. [25]). Since each $y_j = f_j(x) = f_j(x_1, \dots, x_N) \in \{0,1\}$, for $j = 1, \dots, N$, we have

$$\begin{aligned} p_{Y_j} = P\{Y_j = 1\} &= \sum_{\substack{x_1,\dots,x_N: \\ f_j(x_1,\dots,x_N)=1}} P\{X_1 = x_1, X_2 = x_2, \dots, X_N = x_N\} \\ &= \sum_{\substack{x_1,\dots,x_N: \\ f_j(x_1,\dots,x_N)=1}} \left(\prod_{k=1}^N P\{X_k = x_k\} \right). \end{aligned} \tag{3.2}$$

The identity (3.2) is consequence of the independence of X_k, and since $P\{X_k = x_k\}$ is either p_{X_k} or $1 - p_{X_k}$, depending on the value of x_k in the given combination. Thus, the Boolean function $f_j : \{0,1\}^N \to \{0,1\}$ induces a function F_j acting on the discrete and independent random variables $X_1, \dots, X_N$ such that for each $Y_j = F_j(X_1, \dots, X_N)$ the identity (3.2) holds. Furthermore, the random variable Y_j has Bernoulli distribution with parameter p_{Y_j}.

It is easily seen that p_{Y_j} is a multivariate polynomial with arguments $p_{X_1}, \dots, p_{X_N}$, and if we expand (3.2) into a power form, each product term has an integer coefficient and the degree of each variable in that term is less than or equal to 1. Hence, applying Theorem 1.1 we have that the stochastic logic yields a polynomial in N variables $\hat{F}_j$ given by

$$\hat{F}_j(a_1, \dots, a_N) = \sum_{i_1=0}^1 \cdots \sum_{i_N=0}^1 \left(\alpha_{i_1 \dots i_N; j} \prod_{k=1}^N a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1 \dots i_N; j}$ are integers. Moreover, for each $y_j = f_j(x_1, \dots, x_N)$, $j = 1, \dots, N$, we have

$$p_{Y_j} = \hat{F}_j(p_{X_1}, p_{X_2}, \dots, p_{X_N}) = \sum_{i_1=0}^1 \cdots \sum_{i_N=0}^1 \left(\alpha_{i_1 \dots i_N; j} \prod_{k=1}^N p_{X_k}^{i_k} \right).$$

We call stochastic logic of Qian-Riedel-Rosenberg type to the passage of the Boolean map $f : \{0,1\}^N \to \{0,1\}^N$ to the map $F = (F_1, \dots, F_N)$ acting on the discrete and independent random variables $X_1, \dots, X_N$ such that for each $Y_j = F_j(X_1, \dots, X_N)$ the identity (3.2) holds.

We summarize the previous ideas in the following theorem.

Theorem 3.1. *Given a Boolean map $f : \{0,1\}^N \to \{0,1\}^N$. The stochastic logic of Qian-Riedel-Rosenberg type yields a map $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N)$ acting on the discrete and independent random variables $X_1, \dots, X_N$, whose component functions are multivariate polynomials of the form*

$$\hat{F}_j(a_1, \dots, a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \dots i_N; j} \prod_{k=1}^{N} a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1 \dots i_N; j}$ are integers. Moreover, for each $y_j = f_j(x_1, \dots, x_N)$, $j = 1, \dots, N$, we have

$$p_{Y_j} = \hat{F}_j(p_{X_1}, p_{X_2}, \dots, p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \dots i_N; j} \prod_{k=1}^{N} p_{X_k}^{i_k} \right). \tag{3.3}$$

Example 3.1. Consider the Boolean map $f : \{0,1\}^3 \to \{0,1\}^3$ given by

$$f(x_1, x_2, x_3) = ((x_1 \wedge x_3) \vee (x_2 \wedge (\neg x_3)), x_1 \wedge x_3, x_2 \wedge (\neg x_1)),$$

where $\wedge$ means logical AND, $\vee$ means logical OR, and $\neg$ means logical negation. Choose $p_1, p_2, p_3 \in [0,1]$ and let X_1, X_2, X_3 be three discrete and independent random variables such that $X_1 \sim Be(p_1)$, $X_2 \sim Be(p_2)$, $X_3 \sim Be(p_3)$ whose probability density functions satisfy (3.1). It is clear that

$$\begin{aligned} f_1(x_1, x_2, x_3) &= (x_1 \wedge x_3) \vee (x_2 \wedge (\neg x_3)), \\ f_2(x_1, x_2, x_3) &= x_1 \wedge x_3, \\ f_3(x_1, x_2, x_3) &= x_2 \wedge (\neg x_1). \end{aligned}$$

By the definition of p_{Y_j}, $j = 1, 2, 3$, we have

$$\begin{aligned} p_{Y_1} &= P\{X_1 = 1, X_2 = 0, X_3 = 1\} + P\{X_1 = 1, X_2 = 1, X_3 = 1\} \\ &\quad + P\{X_1 = 0, X_2 = 1, X_3 = 0\} + P\{X_1 = 1, X_2 = 1, X_3 = 0\} \\ &= p_{X_1}(1 - p_{X_2})p_{X_3} + p_{X_1}p_{X_2}p_{X_3} + (1 - p_{X_1})p_{X_2}(1 - p_{X_3}) + p_{X_1}p_{X_2}(1 - p_{X_3}) \\ &= p_{X_2} + p_{X_1}p_{X_3} - p_{X_2}p_{X_3}, \\ p_{Y_2} &= P\{X_1 = 1, X_2 = 0, X_3 = 1\} + P\{X_1 = 1, X_2 = 1, X_3 = 1\} \\ &= p_{X_1}(1 - p_{X_2})p_{X_3} + p_{X_1}p_{X_2}p_{X_3} = p_{X_1}p_{X_3}, \\ p_{Y_3} &= P\{X_1 = 0, X_2 = 1, X_3 = 0\} + P\{X_1 = 0, X_2 = 1, X_3 = 1\} \\ &= (1 - p_{X_1})p_{X_2}(1 - p_{X_3}) + (1 - p_{X_1})p_{X_2}p_{X_3} = p_{X_2} - p_{X_1}p_{X_2}, \end{aligned}$$

and the random variables Y_1, Y_2 and Y_3 are given by

$$\begin{aligned} Y_1 &= F_1(X_1, X_2, X_3) = X_2 + X_1X_3 - X_2X_3, \\ Y_2 &= F_2(X_1, X_2, X_3) = X_1X_3, \\ Y_3 &= F_3(X_1, X_2, X_3) = X_2 - X_1X_2, \end{aligned} \tag{3.4}$$

which confirms that (3.4) induces a map $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ acting on the discrete and independent random variables X_1, X_2, X_3, whose component functions are polynomials in the variables (a, b, c) with integer coefficients:

$$\begin{aligned} \hat{F}_1(a, b, c) &= b + ac - bc, \\ \hat{F}_2(a, b, c) &= ac, \\ \hat{F}_3(a, b, c) &= b - ab. \end{aligned}$$

We now come to the second part of the main results of this section: the connection between stochastic logic of Qian-Riedel-Rosenberg type and induced multivariate Bernstein polynomials. Suppose that we have a combinational circuit $w = g(x_1, x_2, \ldots, x_N)$ consisting of N combinational subcircuits $y_j = f_j(x_1, x_2, \ldots, x_N)$, $j = 1, \ldots, N$, and only an N-input AND gate. Each combinational subcircuit $y_j = f_j(x_1, x_2, \ldots, x_N)$ consists of a decoding block and a multiplexing block, which transform the N inputs $\{x_1, \ldots, x_N\} \in \{0, 1\}$ as follows: If k out of the inputs $\{x_1, \ldots, x_N\}$ of the jth decoding block are logical 1, then s_{kj} is set to 1 and the other outputs are set to 0, $(0 \leq k \leq N)$. So, the output of the jth decoding block is $s^j = (s_{0j}, \ldots, s_{Nj})$. The outputs of the jth decoding block are fed into the jth multiplexing block, as shown in Figure 1, and they act as the selecting signals (control inputs). The data signals (inputs) of the jth multiplexing block consist of $N + 1$ inputs $z_{0j}, \ldots, z_{Nj} \in \{0, 1\}$.

Once the jth multiplexing block is used, the Boolean function $y_j = f_j(x_1, x_2, \ldots, x_N)$ takes the form

$$y_j = \bigvee_{k=0}^{N} (z_{kj} \wedge s_{kj}), \quad j = 0, \ldots, N, \tag{3.5}$$

which means that the output of the jth multiplexing block y_j is set to be the input z_{kj} if $s_{kj} = 1$.

Next, the inputs of the N-input AND gate are $y_1, \ldots, y_N \in \{0, 1\}$ and the Boolean function $w = g(x_1, x_2, \ldots, x_N)$ can be expressed as

$$w = \bigwedge_{j=1}^{N} y_j = \bigwedge_{j=1}^{N} \left[\bigvee_{k=0}^{N} (z_{kj} \wedge s_{kj}) \right]. \tag{3.6}$$

Using the association (3.1) for $(x_1, \ldots, x_N)$, $(s_{0j}, \ldots, s_{Nj})$ and $(z_{0j}, \ldots, z_{Nj})$ we can choose discrete and independent random variables $(X_1, \ldots, X_n)$, $(S_{0j}, \ldots, S_{Nj})$ and $(Z_{0j}, \ldots, Z_{Nj})$, such that $X_j \sim Be(p_j)$, $S_{kj} \sim Be(\hat{p}_{kj})$ and $Z_{kj} \sim Be(\hat{\hat{p}}_{kj})$,

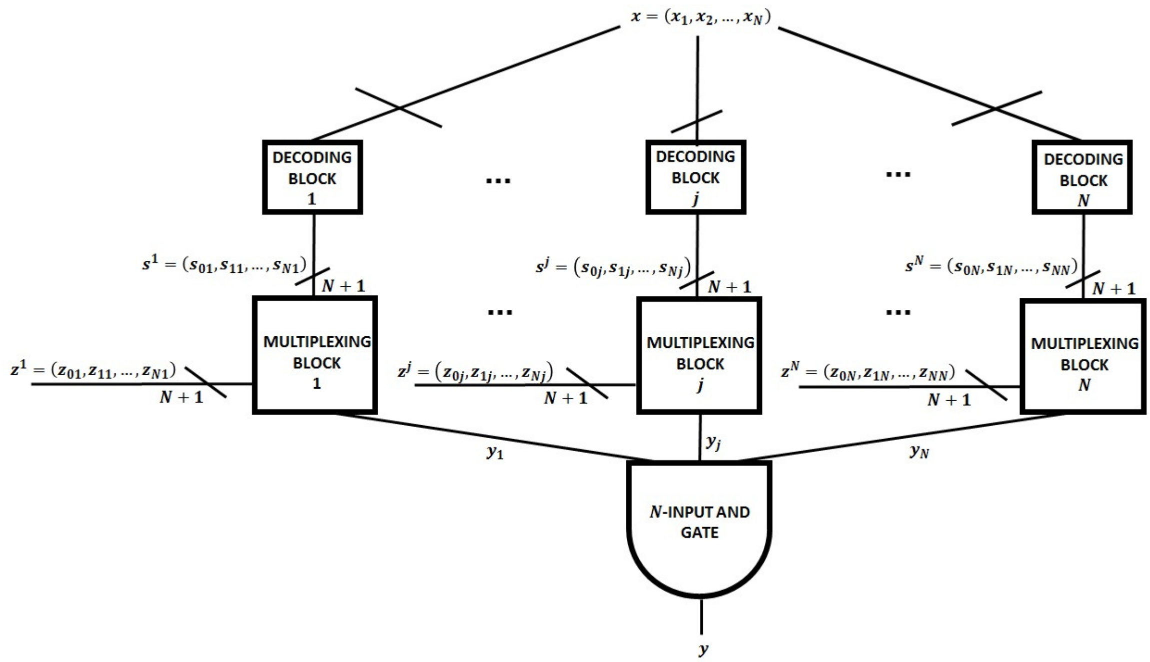

FIGURE 1. Combinational circuit associated to a multivariate Bernstein polynomial with coefficients in $[0, 1]$.

$k = 0, \dots, N$, $j = 1, \dots, N$. Similarly, we define

$$p_{X_k} := P\{X_k = 1\} = p_k \quad \text{and} \quad 1 - p_{X_k} := P\{X_k = 0\} = 1 - p_k,$$
$$p_{S_{kj}} := P\{S_{kj} = 1\} = \hat{p}_{kj} \quad \text{and} \quad 1 - p_{S_{kj}} := P\{S_{kj} = 0\} = 1 - \hat{p}_{kj},$$
$$p_{Z_{kj}} := P\{Z_{kj} = 1\} = \hat{\hat{p}}_{kj} \quad \text{and} \quad 1 - p_{Z_{kj}} := P\{Z_{kj} = 0\} = 1 - \hat{\hat{p}}_{j},$$

for $k = 0, \dots, N$, $j = 1, \dots, N$.

Applying Theorem 3.1 to the Boolean map $f : \{0,1\}^N \to \{0,1\}^N$ given by $f(x_1, \dots, x_N) = (y_1, \dots, y_N)$, we have that the stochastic logic of Qian-Riedel-Rosenberg type yields N multivariate polynomials as in (3.3), such that $p_{Y_j} = \hat{F}_j(p_{X_1}, \dots, p_{X_N})$, $j = 1, \dots, N$.

Let W be the discrete random variable associated to Boolean function $w = g(x_1, x_2, \dots, x_N)$ by means of

$$P\{W = 1\} = \sum_{\substack{x_1, \dots, x_N : \\ g(x_1, \dots, x_N) = 1}} P\{X_1 = x_1, X_2 = x_2, \dots, X_N = x_N\}.$$

We define $p_W := P\{W = 1\}$ and $1 - p_W := P\{W = 0\}$. According to (3.6) and Theorem 3.1 we have

$$p_W = \prod_{j=1}^{N} p_{Y_j} = \prod_{j=1}^{N} \hat{F}_j(p_{X_1}, \dots, p_{X_N}). \tag{3.7}$$

Let us consider the polynomial $q_j(t)$ given by

$$q_j(t) = \hat{F}_j(\underbrace{t, t, \dots, t}_{N-\text{times}}), \quad j = 0, \dots, N.$$

Assume that $p_{X_1} = \dots = p_{X_N} = t_0$, since s_{kj} is set to 1 if and only if k out of N inputs of the jth decoding block are 1, the probability that S_{kj} is 1 is (see, e.g., [1, pp. 10–11]):

$$p_{S_{kj}} = P\{S_{kj} = 1\} = \binom{N}{k} t_0^k (1 - t_0)^{N-k} = b_{k,N}(t_0), \quad k = 0, \dots, N.$$

Now, assume that $p_{Z_{kj}} = \beta^j_{k,N}$. Then

$$p_{Y_j} = P\{Y_j = 1\} = \sum_{k=0}^{N} P\{Y_j = 1 | S_{kj} = 1\} P\{S_{kj} = 1\}, \tag{3.8}$$

but from (3.5) is deduced that $S_{kj} = 1$ implies $Y_j = Z_{kj}$, so

$$P\{Y_j = 1 | S_{kj} = 1\} = P\{Z_{kj} = 1\} = p_{Z_{kj}} = \beta^j_{k,N}. \tag{3.9}$$

By (3.8) and (3.9) we obtain

$$q_j(t_0) = p_{Y_j} = \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t_0), \quad j = 1, \dots, N,$$

and (3.7) becomes

$$p_W = \prod_{j=1}^{N} \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t_0).$$

Therefore, under the constrains imposed by us, each combinational subcircuit $y_j = f_j(x_1, x_2, \dots, x_N)$ would require that $q_j(t)$ be a Bernstein polynomial whose coefficients $\beta^j_{k,N}$ belong to $[0, 1]$, (cf., [28–30]). Consequently, the combinational circuit $w = g(x_1, x_2, \dots, x_N)$ would require an induced multivariate Bernstein polynomial $P(x_1, \dots, x_N)$ such that

$$P(x_1, \dots, x_N) = \sum_{k=0}^{N} \cdots \sum_{k=0}^{N} c_{k,\mathbf{N}} B_{k,\mathbf{N}}(x_1, \dots, x_N),$$

where $c_{k,\mathbf{N}} = \prod_{j=1}^{N} \beta^j_{k,N}$, $B_{k,\mathbf{N}}(x_1, \dots, x_N) = \prod_{j=1}^{N} b_{k,N}(x_j)$, for $k = 0, \dots, N$ and multi-index of maximum degree $\mathbf{N} = (N, \dots, N)$. That is, the combinational circuit $w = g(x_1, x_2, \dots, x_N)$ would require that $P(x_1, \dots, x_N)$ be an induced multivariate Bernstein polynomial whose coefficients $c_{k,\mathbf{N}}$ are expressed as a product of N real numbers belonging to $[0, 1]$. Thus, we have the following theorem.

Theorem 3.2. *Let $P(x_1, \dots, x_N)$ be an induced multivariate Bernstein as in* (2.5) *such that*

(i) *the components of its multi-index of maximum degree $\boldsymbol{N}$ are equal;*

(ii) *its Bernstein coefficients satisfy that* $c_{\nu,\boldsymbol{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j}$ *with* $\nu = (k,\ldots,k)$, $k = 0,\ldots,N$.

If all the factors $\beta_{k,N}^{j}$ *belong to* $[0,1]$*, then we can design a stochastic logic of Qian-Riedel-Rosenberg type to compute* $P(x_1,\ldots,x_N)$.

A multivariate polynomial can be represented in a power form as (2.3). If it can be converted into an induced multivariate Bernstein polynomial satisfying the hypothesis of Theorem 3.2, then the preceding arguments show us how to implement it with stochastic logic of Qian-Riedel-Rosenberg type. The following result describes such a class of induced multivariate Bernstein polynomials.

Theorem 3.3. *Let* N *be any fixed positive integer. If* $q_j(t)$ *is a polynomial such that some of the following conditions is satisfied:*

(i) $q_j(t)$ *is identically equal to* 0 *or to* 1, $j = 1,\ldots,N$;

(ii) *for any* $t \in (0,1)$ *we have* $0 < q_j(t) < 1$*, with* $q_j(0) \geq 0$ *and* $q_j(1) \leq 1$*, for all* $j = 1,\ldots,N$.

Then for $x \in [0,1]^N$ *the multivariate polynomial* $q(x)$ *given by*

$$q(x) = q(x_1,\ldots,x_N) = \prod_{j=1}^{N} q_j(x_j) \tag{3.10}$$

can be converted into an induced multivariate Bernstein polynomial as in Theorem 3.2 *with Bernstein coefficients expressed as a product of* N *real numbers belonging to* $[0,1]$.

Reciprocally, if $q(x)$ *can be converted into an induced multivariate Bernstein polynomial as in* (2.5) *with Bernstein coefficients expressed as a product of* N *real numbers belonging to* $[0,1]$*, then the polynomials* $q_j(t)$ *satisfy* (i) *or* (ii), $j = 1,\ldots,N$.

Proof. We begin by noting if $q_j(t) = 0$ for every $t \in [0,1]$, $j = 1,\ldots,N$ then taking

$$\beta_{k,N}^{j} = 0, \quad \text{for } k = 0,\ldots,N, j = 1,\ldots,N,$$

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j} = 0, \quad \text{with } \nu = (k,\ldots,k), k = 0,\ldots,N, \text{ and } \mathbf{N} = (N,\ldots,N),$$

it follows that

$$q_j(t) = \sum_{k=0}^{N} \beta_{k,N}^{j} b_{k,N}(t) = 0, \quad \text{for every } t \in [0,1],$$

$$q(x) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x) = 0, \quad \text{for every } x \in [0,1]^N.$$

Analogously, if $q_j(t)) = 1$ for every $t \in [0,1]$ then taking

$$\beta_{k,N}^{j} = 1, \quad \text{for } k = 0,\ldots,N, j = 1,\ldots,N,$$

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j} = 1, \quad \text{with } \nu = (k,\ldots,k), k = 0,\ldots,N, \text{ and } \mathbf{N} = (N,\ldots,N),$$

and using part (i) of Proposition 2.1, it follows that

$$q_j(t) = \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t) = 1, \qquad \text{for every } t \in [0,1],$$
$$q(x) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x) = 1, \qquad \text{for every } x \in [0,1]^N.$$

Now consider any polynomials $q_j(t)$ such that $q_j(t) \neq 0$ and $q_j(t) \neq 1$ for every $t \in [0,1]$, and $q_j(t)$ satisfy (ii) for all $j = 1, \dots, n$. We distinguish four possible cases according to the inequalities satisfied by $q_j(0)$ and $q_j(1)$, for all $j = 1, \dots, N$:

Case I: $0 \leq q_j(0)$ and $q_j(1) < 1$, for all $j = 1, \dots, N$. For the sake of clarity and readability, we have decided to include the details of the proof of this case. However, one can check that it suffices to follow the reasoning in [29, Theorem 4], making the appropriate modifications.

Since $q_j(t)$ is a continuous function on the compact interval $[0,1]$, it attains its maximum value M_{q_j} on $[0,1]$. Thus $M_{q_j} < 1$, because $q_j(t) < 1$ for all $t \in [0,1]$. Let $\epsilon_j = 1 - M_{q_j} > 0$, by part (vi) of Proposition 2.2 there exists a positive integer $M_j \geq N$ such that for all $m \geq M_j$ and $k = 0, \dots, m$, we have

$$\left| \beta^j_{k,m} - q_j\left(\frac{k}{m}\right) \right| < \epsilon_j, \quad j = 1, \dots, N,$$

where $\beta^j_{0,m}, \dots, \beta^j_{m,m}$ satisfy that $q_j(t) = \sum_{k=0}^{m} \beta^j_{k,m} b_{k,m}(t)$, $j = 1, \dots, N$. Thus, for all $m \geq M_j$ and $k = 0, \dots, m$,

$$\beta^j_{k,m} < q_j\left(\frac{k}{m}\right) + \epsilon_j \leq M_{q_j} + 1 - M_{q_j} = 1. \tag{3.11}$$

Denote by r_j the multiplicity of 0 as root of $q_j(t)$ (where $r_j = 0$ if $q_j(0) > 0$) and by s_j the multiplicity of 0 as root of 1 as root of $q_j(t)$ (where $s_j = 0$ if $q_j(1) \neq 0$). We can factorize each $q_j(t)$ as

$$q_j(t) = t^{r_j}(1-t)^{s_j} h_j(t), \tag{3.12}$$

where $h_j(t)$ is a polynomial satisfying that $h_j(0) \neq 0$ and $h_j(1) \neq 1$, $j = 1, \dots, N$.

It is clear that $h_j(0) > 0$, since if we suppose, contrary of our claim, that $h_j(0) \leq 0$, using that $h_j(0) \neq 0$ we have necessarily $h_j(0) < 0$, and by the continuity of $h_j(t)$, there exists $t^*_j \in (0,1)$ such that $h_j(t^*_j) < 0$. Hence, $q_j(t^*_j) = t^{r_j}(1-t)^{s_j} h_j(t^*_j) < 0$. This contradicts the fact that $q_j(t) > 0$ for all $t \in (0,1)$. Similarly, we have $h_j(1) > 0$.

Consequently, $h_j(t) > 0$ for all $t \in [0,1]$. Since $h_j(t)$ is a continuous function on the compact interval $[0,1]$, it attains its minimum value m_{h_j} on $[0,1]$, and clearly, $m_{h_j} > 0$.

Let $\varepsilon_j = m_{h_j} > 0$, by part (vi) of Proposition 2.2 there exists a positive integer $K_j \geq N - r_j - s_j$ such that for all $d \geq K_j$ and $k = 0, \dots, d$, we have

$$\left| \gamma^j_{k,d} - h_j\left(\frac{k}{d}\right) \right| < \varepsilon_j, \quad j = 1, \dots, N,$$

where $\gamma^j_{0,d}, \dots, \gamma^j_{d,d}$ satisfy that

$$h_j(t) = \sum_{k=0}^{d} \gamma^j_{k,d} b_{k,d}(t), \quad j = 1, \dots, N. \tag{3.13}$$

Thus, for all $d \geq K_j$ and $k = 0, \dots, d$, we have

$$\gamma^j_{k,d} > h_j\left(\frac{k}{d}\right) - \varepsilon_j \geq m_{h_j} - m_{h_j} = 0.$$

Combining and (3.12) (3.13), we get

$$\begin{aligned} q_j(t) &= t^{r_j}(1-t)^{s_j} \sum_{k=0}^{d} \gamma^j_{k,d} b_{k,d}(t) = \sum_{k=0}^{d} \frac{\gamma^j_{k,d}\binom{d}{k}}{\binom{d+r_j+s_j}{k+r_j}} \binom{d+r_j+s_j}{k+r_j} b_{k,d+r_j+s_j}(t) \\ &= \sum_{k=0}^{d+r_j+s_j} \beta^j_{k,d+r_j+s_j} b_{k,d+r_j+s_j}(t), \end{aligned}$$

where $\beta_{k,d+r_j+s_j}$ are the coefficients of the Bernstein polynomial of degree $d + r_j + s_j$ of $q_j(t)$, and

$$\beta^j_{k,d+r_j+s_j} = \begin{cases} 0, & \text{for } 0 \leq k < r_j \text{ and } d + r_j < k \leq d + r_j + s_j, \\ \frac{\gamma^j_{k,d}\binom{d}{k}}{\binom{d+r_j+s_j}{k+r_j}} > 0, & \text{for } r_j \leq k \leq d + r_j. \end{cases}$$

Thus, taking $r = \max_{1 \leq N}\{r_j\}$, $s = \max_{1 \leq N}\{s_j\}$ and $K = \max_{1 \leq N}\{K_j\}$ when $m \geq d + r + s \geq K + r + s$, we have

$$\beta^j_{k,m} \geq 0, \quad k = 0, \dots, m. \tag{3.14}$$

According to (3.11) and (3.14) if we take $M = \max\{M_j\}$ and choose an $m_0 \geq \max\{M, K + r + s\}$, then $q_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$q_j(t) = \sum_{k=0}^{m_0} \beta^j_{k,m_0} b_{k,m_0}(t),$$

with $0 \leq \beta^j_{k,m_0} \leq 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Now, taking

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{m_0} \beta^j_{k,m_0}, \quad \text{with } \nu = (k, \dots, k), k = 0, \dots, m_0, \text{ and } \mathbf{N} = (m_0, \dots, m_0),$$

it follows that

$$q(x) = q(x_1, \dots, x_N) = \prod_{j=1}^{N} q_j(x_j) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \quad \text{for every } x \in [0,1]^N.$$

Case II: $q_j(0) = 0$ and $q_j(1) = 1$, for all $j = 1, \dots, N$. It suffices to combine a reasoning similar to that in the proof of *Case I* with the reasoning in [29, Theorem 5], making the appropriate modifications.

Case III: $0 < q_j(0)$ and $q_j(1) \leq 1$, for all $j = 1, \dots, N$. Consider the polynomials $g_j(t) = 1 - q_j(t)$, for all $t \in [0,1]$, $j = 1, \dots, N$. Then $0 < g_j(t) < 1$, for all $t \in (0,1)$

with $0 \leq g_j(0)$ and $g_j(1) < 1$, for all $j = 1, \dots, N$. Then in view of *Case I* we can choose an $m_0 \geq N$, then $g_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$g_j(t) = \sum_{k=0}^{m_0} \beta^j_{k,m_0} b_{k,m_0}(t),$$

with $0 \leq \beta^j_{k,m_0} \leq 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Hence, using part (i) of Proposition 2.1, it follows that

$$q_j(t) = 1 - g_j(t) = \sum_{k=0}^{m_0} (1 - \beta^j_{k,m_0}) b_{k,m_0}(t) = \sum_{k=0}^{m_0} \gamma^j_{k,m_0} b_{k,m_0}(t),$$

where $\gamma^j_{k,m_0} = 1 - \beta^j_{k,m_0}$, with $0 \leq \gamma^j_{k,m_0} \leq 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Now, taking

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{m_0} \gamma^j_{k,m_0}, \qquad \text{with } \nu = (k, \dots, k), k = 0, \dots, m_0, \text{ and } \mathbf{N} = (m_0, \dots, m_0),$$

it follows that

$$q(x) = q(x_1, \dots, x_N) = \prod_{j=1}^{N} q_j(x_j) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \qquad \text{for every } x \in [0,1]^N.$$

Case IV: $q_j(0) = 1$ and $q_j(1) = 0$, for all $j = 1, \dots, N$. Consider the polynomials $g_j(t) = 1 - q_j(t)$, for all $t \in [0,1]$, $j = 1, \dots, N$. Then $0 < g_j(t) < 1$, for all $t \in (0,1)$ with $0 \leq g_j(0)$ and $g_j(1) < 1$, for all $j = 1, \dots, N$. Then in view of *Case II* we can choose an $m_0 \geq N$, then $g_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$g_j(t) = \sum_{k=0}^{m_0} \beta^j_{k,m_0} b_{k,m_0}(t),$$

with $0 \leq \beta^j_{k,m_0} \leq 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Hence, using part (i) of Proposition 2.1, it follows that

$$q_j(t) = 1 - g_j(t) = \sum_{k=0}^{m_0} (1 - \beta^j_{k,m_0}) b_{k,m_0}(t) = \sum_{k=0}^{m_0} \gamma^j_{k,m_0} b_{k,m_0}(t),$$

where $\gamma^j_{k,m_0} = 1 - \beta^j_{k,m_0}$, with $0 \leq \gamma^j_{k,m_0} \leq 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$.

Therefore, if we take

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{m_0} \gamma^j_{k,m_0}, \qquad \text{with } \nu = (k, \dots, k), k = 0, \dots, m_0, \text{ and } \mathbf{N} = (m_0, \dots, m_0),$$

it follows that

$$q(x) = q(x_1, \dots, x_N) = \prod_{j=1}^{N} q_j(x_j) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \qquad \text{for every } x \in [0,1]^N.$$

Finally, it can be shown that if $q_j(t)$ is not identically equal to 0 or to 1 for some j and there exists a $t_0 \in (0,1)$ such that $q_j(t_0) = 0$ or 1, then we cannot express the polynomial $q_j(t)$ as a Bernstein polynomial with coefficients in the unit interval (cf. [28]). Consequently, $q(x)$ cannot be converted into an induced multivariate Bernstein polynomial as in (2.5) with Bernstein coefficients expressed as a product of N real numbers belonging to $[0,1]$.

This completes the proof. $\square$

Notice that the multi-index of maximum degree of the induced multivariate Bernstein polynomial with coefficients in the unit interval may be greater than the multi-index of maximum degree of the original polynomial.

Example 3.2. Consider the polynomial $q(x,y)=q_1(x)q_1(y)$, where $q_1(x)=3x-8x^2+6x^3$ and $q_2(y)=y$, for all $x,y\in[0,1]$. The polynomial $q(x,y)$ has multi-index of maximum degree $\kappa=(3,1)$, and the polynomials $q_1(x)$ and $q_2(y)$ satisfy the conditions

$$\begin{aligned} &0<q_1(x)<1, \quad \text{whenever } x\in(0,1), \quad q_1(0)=0, \quad q_1(1)=1,\\ &0<q_2(y)<1, \quad \text{whenever } y\in(0,1), \quad q_2(0)=0, \quad q_2(1)=1. \end{aligned}$$

Using (2.7) and part (v) of Proposition 2.2 we have

$$\begin{aligned} q(x,y) &= \left(b_{1,3}(x)-\frac{2}{3}b_{2,3}(x)+b_{3,3}(x)\right)b_{1,1}(y)\\ &= \left(\frac{3}{4}b_{1,4}(x)+\frac{1}{6}b_{2,4}(x)-\frac{1}{4}b_{3,4}(x)+b_{4,4}(x)\right)b_{1,1}(y)\\ &= \left(\frac{3}{5}b_{1,5}(x)+\frac{2}{5}b_{2,5}(x)+b_{5,5}(x)\right)b_{1,1}(y)\\ &= \frac{3}{5}B_{((1,1),(5,1))}(x,y)+\frac{2}{5}B_{((2,1),(5,1))}(x,y)+B_{((5,1),(5,1))}(x,y), \end{aligned}$$

and the induced multivariate Bernstein polynomial of $q(x,y)$:

$$P(x,y)=\frac{3}{5}B_{((1,1),(5,1))}(x,y)+\frac{2}{5}B_{((2,1),(5,1))}(x,y)+B_{((5,1),(5,1))}(x,y)$$

has multi-index of maximum degree $\mathbf{N}=(5,1)$.

The following example show a polynomial $q(x,y)$ which can be converted into an induced multivariate Bernstein polynomial, however it cannot be implemented with stochastic logic of Qian-Riedel-Rosenberg type.

Example 3.3. Consider the polynomial $q(x,y)=3xy-8x^2y^2+6x^3y^3$ with multi-index of maximum degree $\kappa=(3,3)$, satisfying the conditions $0<q(x,y)<1$, whenever $(x,y)\in(0,1)^2$, $q(0,0)=0$ and $q(1,1)=1$. Since

$$q(x,y)=P_1(x)b_{1,3}(y)+P_2(x)b_{2,3}(y)+P_3(x)b_{3,3}(y)+P_3(y)b_{3,3}(x),$$

where $P_1(x)=\frac{1}{3}b_{1,3}(x)+\frac{2}{3}b_{2,3}(x)$, $P_2(x)=\frac{2}{3}b_{1,3}(x)+\frac{4}{9}b_{2,3}(x)$, $P_3(x)=b_{1,3}(x)-\frac{2}{3}b_{2,3}(x)+\frac{1}{2}b_{3,3}(x)$, and the coefficients of $P_3(x)$ do not all belong to the interval $[0,1]$, using part (v) of Proposition 2.2 we see that

$$P_3(x) = \frac{3}{13}b_{1,13}(x) + \frac{14}{39}b_{2,13}(x) + \frac{21}{52}b_{3,13}(x) + \frac{5}{13}b_{4,13}(x) + \frac{25}{78}b_{5,13}(x) + \frac{3}{13}b_{6,13}(x)$$
$$+ \frac{7}{52}b_{7,13}(x) + \frac{2}{39}b_{8,13}(x) + \frac{11}{156}b_{11,13}(x) + \frac{3}{13}b_{12,13}(x) + \frac{1}{2}b_{13,13}(x).$$

From (2.6) and (2.7) it follows that

$$q(x,y) = r_1(x,y) + r_2(x,y) + r_3(x,y) + r_4(x,y), \tag{3.15}$$

where

$$r_1(x,y) = \frac{1}{13}B_{((1,1),(13,3))}(x,y) + \frac{2}{13}B_{((2,1),(13,3))}(x,y) + \frac{5}{22}B_{((3,1),(13,3))}(x,y)$$
$$+ \frac{42}{143}B_{((4,1),(13,3))}(x,y) + \frac{50}{143}B_{((5,1),(13,3))}(x,y) + \frac{56}{143}B_{((6,1),(13,3))}(x,y)$$
$$+ \frac{119}{286}B_{((7,1),(13,3))}(x,y) + \frac{60}{143}B_{((8,1),(13,3))}(x,y) + \frac{57}{143}B_{((9,1),(13,3))}(x,y)$$
$$+ \frac{50}{143}B_{((10,1),(13,3))}(x,y) + \frac{7}{26}B_{((11,1),(13,3))}(x,y) + \frac{2}{13}B_{((12,1),(13,3))}(x,y),$$

$$r_2(x,y) = \frac{2}{13}B_{((1,2),(13,3))}(x,y) + \frac{32}{117}B_{((2,2),(13,3))}(x,y) + \frac{155}{429}B_{((3,2),(13,3))}(x,y)$$
$$+ \frac{60}{143}B_{((4,2),(13,3))}(x,y) + \frac{580}{1287}B_{((5,2),(13,3))}(x,y) + \frac{196}{429}B_{((6,2),(13,3))}(x,y)$$
$$+ \frac{63}{143}B_{((7,2),(13,3))}(x,y) + \frac{40}{99}B_{((8,2),(13,3))}(x,y) + \frac{50}{143}B_{((9,2),(13,3))}(x,y)$$
$$+ \frac{40}{143}B_{((10,2),(13,3))}(x,y) + \frac{23}{117}B_{((11,2),(13,3))}(x,y) + \frac{4}{39}B_{((12,2),(13,3))}(x,y),$$

$$r_3(x,y) = \frac{3}{13}B_{((1,3),(13,3))}(x,y) + \frac{14}{39}B_{((2,3),(13,3))}(x,y) + \frac{21}{52}B_{((3,3),(13,3))}(x,y)$$
$$+ \frac{5}{13}B_{((3,3),(13,3))}(x,y) + \frac{25}{78}B_{((5,3),(13,3))}(x,y) + \frac{3}{13}B_{((6,3),(13,3))}(x,y)$$
$$+ \frac{7}{52}B_{((7,3),(13,3))}(x,y) + \frac{2}{39}B_{((8,3),(13,3))}(x,y) + \frac{11}{156}B_{((11,3),(13,3))}(x,y)$$
$$+ \frac{3}{13}B_{((12,3),(13,3))}(x,y) + \frac{1}{2}B_{((13,3),(13,3))}(x,y),$$

$$r_4(x,y) = r_3(y,x).$$

Hence, the induced multivariate Bernstein polynomial on the right hand side of (3.15) has multi-index of maximum degree $\mathbf{N} = (13,3)$. However, $q(x,y)$ cannot be factorized as (3.10).

As a consequence of Theorems 3.2 and 3.3 we obtain the following result.

Corollary 3.1. *Let N be any fixed positive integer. If $q_j(t)$ is a polynomial such that some of the following conditions is satisfied:*

(i) *$q_j(t)$ is identically equal to 0 or to 1, $j = 1,\ldots,N$;*

(ii) *for any* $t \in (0,1)$ *we have* $0 < q_j(t) < 1$, *with* $q_j(0) \geq 0$ *and* $q_j(1) \leq 1$, *for all* $j = 1, \ldots, N$,

then we can design a stochastic logic of Qian-Riedel-Rosenberg type to compute the multivariate polynomial $q(x)$ *given by*

$$q(x) = q(x_1, \ldots, x_N) = \prod_{j=1}^{N} q_j(x_j), \quad x \in [0,1]^N.$$

Reciprocally, if $q(x)$ *can be implemented with stochastic logic of Qian-Riedel-Rosenberg type, then the polynomials* $q_j(t)$ *satisfy* (i) *or* (ii), $j = 1, \ldots, N$.

4. A Further Remark

In recent years, extensive researches have been done for various degenerate versions of some special polynomials and numbers and have yielded many interesting arithmetical and combinatorial results. These include the degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, and so on.

Degenerate versions of some special polynomials have been shown to play an important role in various areas. However, not much is known about the properties of these polynomials (cf., e.g, [15–20, 32] and references thereof). In particular, as a degenerate version of Bernstein polynomials, the degenerate Bernstein polynomials were introduced recently by Kim and Kim in [16].

In this regard, the remarkable papers [16, 19] suggest that the fundamental properties and identities satisfied by the degenerate Bernoulli polynomials could be used to define a special model of stochastic logic. Thus, one of our future projects is to explore such a model.

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NUMERICAL METHOD FOR SOLUTION OF FOURTH-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY GREEN'S FUNCTION

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Abstract. In this paper, we generalize Picard-Green's Embedded method for solving fourth-order Volterra integro-differential equations. We prove the existence and uniqueness theorems. Moreover, we illustrate some numerical examples to present the better approximation with a minimum error. We use MATLAB for numerical solutions.

1. Introduction

Several authors have been interested in differential equations since they are widely used in applications in the technical field as well as in the science and engineering sciences. Particularly elastic theory, biomechanics, electromagnetics, fluids models in physics and biology such as dynamics, heat transfer, population dynamics, and the spread of infectious diseases are frequently encountered.

Studies for the solution of integral and integro-differential equations (IDEs) have continued since Volterra [1, 9, 19]. Although studies on these equations include linear equations, it is often not possible to find their analytical solutions to these equations. For this reason, numerical approaches [2] find more place in the literature. Various algorithms for finding the approximate numerical values are introduced and implemented to find the best results.

Some of these are Wavelet-Galerkin method [6], monotone iterative methods [5, 20], homotopy perturbation method reproducing kernel [4], Adomian decomposition

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method [8], Picard-Green's method [7, 18], Tau method [11], spectral collocation methods [12], Taylor polynomials [14], Lagrange interpolation [16], exponential spline method [17] and the references therein. Furthermore, higher-order boundary value problems (BVPs) for IDEs have been researched by Agarwal [3] and Morchalo [15].

Consider the following boundary value problem

$$\begin{aligned} L[y] &= p_0(t)y''''(t) + p_1(t)y'''(t) + p_2(t)y''(t) + p_3(t)y'(t) + p_4(t)y(t) \\ &= f(t) + \int_0^t K(t,s)g(y(s))ds, \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} B_a[y] &= \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y''(a) + \alpha_4 y'''(a) = \zeta_1, \\ B_b[y] &= \beta_1 y(b) + \beta_2 y'(b) + \beta_3 y''(b) + \beta_4 y'''(b) = \zeta_2, \\ B_c[y] &= \gamma_1 y(c) + \gamma_2 y'(c) + \gamma_3 y''(c) + \gamma_4 y'''(c) = \zeta_3, \\ B_d[y] &= \omega_1 y(d) + \omega_2 y'(d) + \omega_3 y''(d) + \omega_4 y'''(d) = \zeta_4, \end{aligned} \tag{1.2}$$

where $t \in (a,b)$, ζ_i, $i = 1,\dots,4$, are constants and either $c = a$ or $c = b$ and either $d = a$ or $d = b$. The existence and uniqueness results for (1.1)–(1.2) are given in [10].

The Green's function $G(t,s)$ of problem (1.1) and (1.2) is;

$$G(t,s) = \begin{cases} a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4, & a < t < s, \\ b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4, & s < t < b, \end{cases}$$

where $t \neq s$, y_i are linearly independent solutions of $L[y]$ and a_i, b_i are constants for $i = 1,\dots,4$.

To implement the proposed methodology, we denote the linear integral operator

$$T[y] = y_h + \int_a^b G(t,s)(p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s))ds, \tag{1.3}$$

where y_h is the homogeneous solution of (1.1)–(1.2). From (1.3), we get

$$\begin{aligned} T[y] =& y_h + \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ & - f(s) - \int_0^s K(t,s)g(y(t))dt\Big]ds + \int_a^b G(t,s)\left(f(s) + \int_0^s K(t,s)g(y(t))dt\right)ds. \end{aligned} \tag{1.4}$$

Let y_p be the particular solution of (1.1), then

$$y_p = \int_a^b G(t,s)\left(f(s) + \int_0^s K(t,s)g(y(t))dt\right)ds. \tag{1.5}$$

By applying $y = y_p + y_h$, from (1.4) and (1.5), we obtain

$$\begin{aligned} T[y] =& y + \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ & - f(s) - \int_0^s K(t,s)g(y(t))dt\Big]ds. \end{aligned} \tag{1.6}$$

Let the starting function y_0 be the homogeneous solution of $L[y] = 0$ and $y_{n+1} = T[y_n]$, for all $n \geq 0$, then Picard-Green's fixed point iteration method for (1.1) is defined as

$$\begin{aligned} y_{n+1} =& y_n + \int_a^b G(t,s)\Big[p_0(s){y_n}''''(s) + p_1(s){y_n}'''(s) + p_2(s){y_n}''(s) + p_3(s){y_n}'(s) \\ &+ p_4(s)y_n(s) - f(s) - \int_0^s K(t,s)g(y_n(t))dt\Big]ds. \end{aligned} \tag{1.7}$$

In this paper, we generalize Picard-Green's Embedding method (PGEM) for the fourth-order BVPs of Volterra IDEs. We show convergence and prove the convergence theorem. We demonstrate that the developed method offers a better approach than the existing methods by numerical examples.

2. Convergence Analysis and Convergence Rate

In this section, we will introduce convergence analysis using nonlinear differential equations and the contraction principle and determine the convergence rate.

Consider the fourth-order BVP

$$y^{iv}(t) = f(t, y(t), y'(t), y''(t), y'''(t)) + \int_0^t K(t,s)g(y(s))ds, \tag{2.1}$$

with the boundary conditions

$$y(0) = y'(0) = y(1) = y'(1) = 0. \tag{2.2}$$

The solution of the problem (2.1)–(2.2) is as follows

$$y_p = \int_0^1 G(t,s)\left[f(s, y_p, {y'}_p, {y''}_p, {y'''}_p) + \int_0^t K(t,s)g(y_p(s))ds\right]ds \tag{2.3}$$

and

$$\begin{aligned} T[y_p] = \int_0^1 G(t,s)\Big[&p_0(s){y_p}''''(s) + p_1(s){y_p}'''(s) \\ &+ p_2(s){y_p}''(s) + p_3(s){y_p}'(s) + p_4(s)y_p(s)\Big]ds, \end{aligned} \tag{2.4}$$

where $G(t,s)$ is

$$G(t,s) = \begin{cases} t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1. \end{cases}$$

From (2.3) and (2.4), we get

$$\begin{aligned} T[y_p] =& y_p \\ &+ \int_a^b G(t,s)\Big[p_0(s)y''''(s) + p_1(s)y'''(s) + p_2(s)y''(s) + p_3(s)y'(s) + p_4(s)y(s) \\ &- f(s, y_p, {y'}_p, {y''}_p, {y'''}_p) + \int_0^s K(t,s)g(y_p(t)dt\Big]ds. \end{aligned}$$

By applying PGEM to the problem (2.1)–(2.2), we obtain the following iterative scheme.

$$\begin{aligned} &y_{n+1}\\ &= y_n + \int_a^b G^*(t,s)\Big[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\Big]\,ds. \end{aligned}$$

In particular, we have

$$\begin{aligned} y_{n+1} &= y_n - \int_0^t \left(s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right)\right) \quad (2.5)\\ &\quad\times\Big[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\Big]\,ds \quad (2.6)\\ &\quad - \int_t^1 \left(t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right)\right)\\ &\quad\times\Big[{y_n}''''(s) - f(s, y_n(s), {y_n}'(s), {y_n}''(s), {y_n}'''(s)) - \int_0^s K(t,s)g(y_n(t))dt\Big]\,ds. \end{aligned}$$

Theorem 2.1. *Let $X = C[0,1]$ be a Banach space with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, $x \in X$. Assume that the function g satisfies the Lipschitz condition such that $|g(y) - g(v)| \le L|y - v|$, $L \in (0,1]$. Then operator T defined in* (1.6) *is a Banach's contraction and the sequence y_n converges strongly to the solution of the problem* (2.1) *and* (2.2) *under the following conditions*

$$Q = \left(\frac{1}{98}\right) A < 1,$$

where

$$A = \max_{[0,1]\times R^4} \left|\frac{\partial f(t, y, y', y'', y''')}{\partial y}\right| + \|K\| L \left(\frac{1}{2}\right).$$

Proof. Integrating (2.5) by parts, we get

$$y_{n+1} = y_n(t) + \int_0^1 G^*(t,s)\left[f(s, y_n, {y_n}', {y_n}'', {y_n}''') + \int_0^s K(t,s)g(y_n(t))dt\right] ds. \quad (2.7)$$

Let $T_G : [0,1] \to [0,1]$ be the right side of (2.7), then

$$\begin{aligned} \|T_G(y_n) - T_G(y_m)\| = \Big\|\int_0^1 G^*(t,s)\Big[& f(s, y, y', y'', y''') + \int_0^s K(t,s)g(y_n(t))dt)\\ & - f(s, y_m, y'_m, y''_m, y'''_m) + \int_0^s K(t,s)g(y_m(t))dt\Big]\,ds\Big\|. \end{aligned}$$

By using the fact that

$$\|G\| = \max_{0\le t,s\le 1} |G^*(t,s)| = \frac{1}{98},$$

we get

$$\|T_G(y_n) - T_G(y_m)\| \leq \frac{1}{98}\int_0^1 \Big\|\Big[f(s, y_n, y_n', y_n'', y_n''') + \int_0^s K(t,s)g(y_n(t))dt \\ -f(s, y_m, y_m', y_m'', y_m''') + \int_0^t K(t,s)g(y_m(s))dt\Big]\Big\| ds.$$

Implementing Mean Value Theorem, we obtain

$$\|T_G(y_n) - T_G(y_m)\| \leq \frac{1}{98}A\|y_n - y_m\|.$$

Therefore, we get

$$\|T_G(y_n) - T_G(y_m)\| \leq Q\|y_n(t) - y_m(t)\|, \tag{2.8}$$

where $Q \in (0,1)$. From (2.8) we have

$$\begin{aligned} \|y_n - y_m\| &= \|(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_{m+1} - y_m)\| \\ &\leq \|y_n - y_{n-1}\| + \|y_{n-1} - y_{n-2}\| + \cdots + \|y_{m+1} - y_m\| \\ &\leq (Q^{n-1} + Q^{n-2} + \cdots + Q^m)\|y_1 - y_0\| \\ &\leq Q^m(1 + Q + Q^2 + \cdots + Q^{n-m-1})\|y_1 - y_0\| \\ &= Q^m\left(\frac{1 - Q^{n-m}}{1 - Q}\right)\|y_1 - y_0\|. \end{aligned}$$

Since $Q \in (0,1)$, we have

$$\|y_n - y_m\| \leq \frac{Q^m}{1-Q}\|y_1 - y_0\|, \tag{2.9}$$

which converges to zero, i.e., $\|y_n - y_m\| \to 0$, while $m \to 0$. Thus, $T_G(y)$ is a contraction mapping. □

Let y^* be the solution of problem (2.1) and (2.2). Then $T(y^*) = y^*$. From (2.8) and (2.9), we have

$$\|y_{n+1} - y^*\| = \|T(y_n) - y^*\| \leq Q\|y_n - y^*\| \leq \cdots \leq Q^{n+1}\|y_0 - y^*\|.$$

Since $0 < Q < 1$, it concludes that y_n converges strongly to y^*. The rest proof can be completed from the proof of [13, Proposition 1].

3. Numerical Examples

In this section, we give numerical examples to confirm the applicability of the main results.

Example 3.1. Consider the fourth order BVP

$$y^{iv}(t) = f(t) + \int_0^t y(s)ds, \tag{3.1}$$

with the boundary conditions

$$y(0) = y'(0) = 1, \quad y(1) = 1 + e, \quad y'(1) = 2e, \tag{3.2}$$

where $f(t) = -t + 5e^t - 1$ and the exact solution $y(t) = 1 + te^t$ and the Green's function is

$$G(t,s) = \begin{cases} t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1. \end{cases}$$

By applying PGEM, we get

$$\begin{aligned} y_{n+1} = & y_n - \int_0^t \left[s^3\left(\frac{-2t^3+3t^2-1}{6}\right) + s^2\left(\frac{t^3-2t^2+t}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) + s - 5e^s + 1 - \int_0^s y_n(t)dt\right] ds \\ & - \int_t^1 \left[t^3\left(\frac{-2s^3+3s^2-1}{6}\right) + t^2\left(\frac{s^3-2s^2+s}{2}\right)\right] \\ & \times \left[y_n^{iv}(s) + s - 5e^s + 1 - \int_0^s y_n(t)dt\right] ds, \end{aligned} \tag{3.3}$$

where the starting function is $y_0 = t^3 + (e-2)t^2 + t + 1$. The absolute error of the problem is estimated by

$$Err = |y(t) - y_n(t)|.$$

Table 1 gives the maximum errors of the problem (3.1)–(3.2) to demonstrate the high accuracy of the proposed method. Considering the values in the table, the margin of error decreases considerably and approaches zero as the number of iterations increases.

TABLE 1. The maximum errors of Example 1

No. of iterations	6	8	10	12
Max Error(n)	2.96E-18	1.23E-24	5.13E-31	2.13E-37

Table 2 shows the absolute errors for the second and third iterations solved by two different methods. The table shows that PGEM has a better convergence rate than Adomian Decomposition Method (MADM). Meanwhile, the chart 1 represents the line graphs of the absolute errors of both methods for the third iteration. Therefore, it is clear that PGEM approaches 0 faster than MADM.

TABLE 2. The absolute errors (n) of Example 1

t	Numerical Solution	PGEM Error (2)	PGEM Error(3)	MADM Error(2)	MADM Error(3)
0.1	1.1111924502842667426690545607791	1.68E-06	1.08E-09	4.54E-05	2.29E-08
0.3	1.4086348815636588244756957461284	1.05E-05	6.75E-09	4.23E-05	4.74E-07
0.5	1.8295725879184859388029560326160	1.70E-05	1.10E-08	6.63E-05	3.37E-07
0.7	2.4133047634097381621411544034260	1.35E-05	8.81E-09	6.92E-05	4.78E-07
0.9	3.2143183796746349483923129747728	2.74E-06	1.80E-09	7.97E-06	5.81E-08

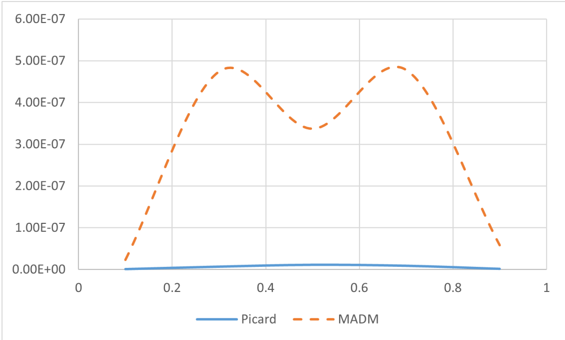

FIGURE 1. The relative absolute errors of Example 1

Example 3.2. Consider non-linear BVP

$$y^{iv} = 1 + \int_0^t e^{-s} y^2(s) ds, \tag{3.4}$$

corresponding to boundary conditions

$$y(0) = y'(0) = 1, \quad y(1) = y'(1) = e. \tag{3.5}$$

The exact solution of the problem given above is $y(t) = e^x$, and the Green's function of (3.4)–(3.5) is

$$G(t,s) = \begin{cases} t^3 \left(\frac{-2s^3+3s^2-1}{6}\right) + t^2 \left(\frac{s^3-2s^2+s}{2}\right), & 0 < t < s, \\ s^3 \left(\frac{-2t^3+3t^2-1}{6}\right) + s^2 \left(\frac{t^3-2t^2+t}{2}\right), & s < t < 1, \end{cases}$$

where the starting function is $y_0 = (-e-3)t^3 + (2e-5)t^2 + t + 1$.

By applying PGEM, we get

$$\begin{aligned} y_{n+1} = & y_n - \int_0^t \left[s^3 \left(\frac{-2t^3 + 3t^2 - 1}{6} \right) + s^2 \left(\frac{t^3 - 2t^2 + t}{2} \right) \right] \\ & \times \left[y_n^{iv}(s) - 1 - \int_0^s e^{-s} y_n^2(t) dt \right] ds \\ & - \int_t^1 \left[t^3 \left(\frac{-2s^3 + 3s^2 - 1}{6} \right) + t^2 \left(\frac{s^3 - 2s^2 + s}{2} \right) \right] \\ & \times \left[y_n^{iv}(s) - 1 - \int_0^s e^{-s} y_n^2(t) dt \right] ds. \end{aligned} \tag{3.6}$$

Table 3 demonstrates the high accuracy of the proposed method for the problem given in Example 2. It presents second iteration errors for PGEM, MADM, and MDMGF (Modified Decomposition Method with Green function). The results of recommended method PGEM converge to the exact solution faster.

Table 4 shows the third iteration errors for the methods discussed in Table 3. When we examine these results, it is clear that the results of the PGEM method decrease faster as the number of iterations increases and converge to zero faster than the other

Table 3. The absolute errors (n) of Example 2

t	Numerical Solution Error(2)	PGEM	MADM	MDMGF
0.1	1.1051709173255609245824473770908	5.85E-07	8.48E-05	1.43E-05
0.3	1.3498588028660124806131472805192	3.66E-06	9.16E-05	9.17E-05
0.5	1.6487212630265689433471827882745	5.93E-06	3.66E-04	1.56E-04
0.7	2.0137527013290129690745570786289	4.71E-06	4.54E-04	1.34E-04
0.9	2.4596031099034457029111001120299	9.55E-07	3.00E-05	3.00E-05

Table 4. The other absolute errors (n) of Example 2

t	Numerical Solution Error(3)	MADM	PGEM	MDMGF
0.1	1.1051709173255609245824473770908	2.32E-06	7.50E-10	4.32E-08
0.3	1.3498588028660124806131472805192	7.72E-05	4.71E-09	2.75E-07
0.5	1.6487212630265689433471827882745	7.52E-05	7.67E-09	4.54E-07
0.7	2.0137527013290129690745570786289	4.72E-05	6.14E-09	3.72E-07
0.9	2.4596031099034457029111001120299	7.61E-06	1.25E-09	7.61E-08

methods, as in Table 3. These results clearly show that PGEM is more effective, as we tried to demonstrate.

While the Figure 2 shows the comparisons of the values in the tables 3 and 4, Fig. 3 depicts the comparisons between the exact solutions and the numerical solutions obtained in the third iteration. Overall, it is clear from the first graph that the values obtained via PGEM tend to approach zero faster than other methods. Moreover, as shown by the second graph, the numerical solutions got by PGEM are very close to the exact values.

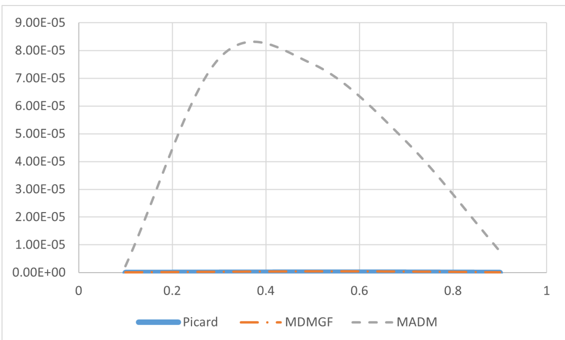

Figure 2. The absolute errors of Example 2

4. Conclusion

In this study, we generalize Picard-Green's fixed-point iteration method, one of the most popular methods for fourth-order nonlinear and linear IVPs, by embedding

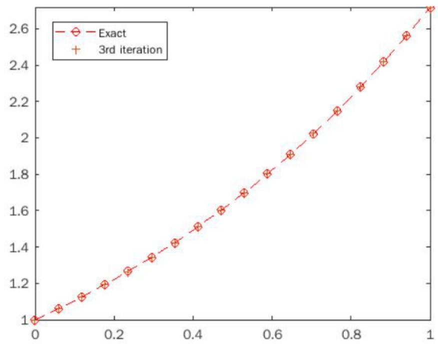

Figure 3. Exact and numerical solutions

Green's function. We proved the convergence and got the convergence rate. We solve some examples to show the correctness and generality of the proposed scheme. We compared the numerical results obtained by the determined method with the results of the methods well known in the literature. For comparison, we considered the MADM and MDMGF methods. We used MATLAB to calculate numerical results. We presented the obtained results with the help of tables and figures. Our method gives better results than other methods when comparing numerical results, exact results, and calculated values. Therefore, the aim of our study has been revealed.

There are many iteration methods in the literature to find the best approach. This study compared the results obtained for the fourth-order Volterra integro-differential equations with the Adomian decomposition methods. However, solving higher order linear and nonlinear differential and integro-differential equations with a better approach than other existing methods is still a problem to be developed. We believe its solution will lead to many studies.

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