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CONCERNING MULTIVARIATE BERNSTEIN POLYNOMIALS AND STOCHASTIC LOGIC

YAMILET QUINTANA[1,2]

Abstract. Among the applications of the Bernstein polynomials in one variable is their use in solving problems associated with stochastic computing. Taking as a starting point the notion of stochastic logic in the sense of Qian-Riedel-Rosenberg, the aim of this paper is to investigate some necessary and sufficient conditions for guaranteeing whether polynomial operations can be implemented with stochastic logic based on multivariate Bernstein polynomials with coefficients in the unit interval.

1. Introduction

Stochastic computing (SC) arises as a collection of techniques to represent analog quantities by probabilities of discrete events, or represent continuous values by means of random bit-streams, so that complex operations can be performed by simple bitwise operations on random pulse trains [1, 7–10, 27]. The analogy between probability algebras and Boolean algebras [12, 13, 25] is used to obtain very simple processing units and an adequate arithmetic. The basic operations described in the literature are the addition and the multiplication since these are the fundamental operations involved in neural networks and in the design of stochastic circuitry (fields in which fertile ground has been found for applications of SC). Also, SC has been applied to division and square-rooting [10, 33], matrix operations and decoding of low-density parity check (LDPC) codes [11, 21, 23], and polynomial arithmetic [28, 29].

A stochastic number can be defined as a pair (x, p_x), where x is a finite binary sequence, i.e., $x \in \{0,1\}^N$, for some $N \in \mathbb{N}$ and $p_x \in [0,1]$ is the probability of

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observing a 1 at an arbitrary position of x [1, 9, 10, 24]. So, a stochastic number is represented by a finite binary sequence (or bit-stream) in such a way that the probability (ratio) of '1' in the binary sequence is interpreted as the number itself. The probability p_x is sometimes called value of the stochastic number (see, e.g., [24]).

For instance, if (x, p_x) is a stochastic number whose binary sequence x has N components, of which m are equal to 1 and $N - m$ are equal to 0, then $p_x = \frac{m}{N}$ and, clearly, the representation of the pair (x, p_x) is not unique. SC uses a redundant number system in which there are $\binom{N}{m}$ possible representations for each value $p_x = \frac{m}{N}$. Furthermore, a binary sequence x can only has associated probabilities in the set $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$, so only a small subset of the real numbers in $[0, 1]$ can be expressed exactly in SC.

The main idea behind the combinational circuits design with polynomial arithmetic of Qian et al. [28, 29] consist of the following.

(1) Take advantage -in a suitable way- of the redundancy provided by SC for choosing binary sequences $x \in \{0, 1\}^N$ corresponding to the value p_x, in order to make an association between x and a certain N-tuple of independent random variables $X = (X_1, \ldots, X_N)$, where each component X_k has Bernoulli distribution with some parameter $p_k \in [0, 1]$.

(2) Given a Boolean function $y = f(x_1, \ldots, x_N)$ implementing a combinational circuit, use the association aforementioned for inducing a stochastic circuit implemented by a function of the form $Y = F(X_1, \ldots, X_N)$ (see for instance, [25]).

The passage of the Boolean function $y = f(x_1, \ldots, x_N)$ to the function $Y = F(X_1, \ldots, X_N)$ is called stochastic logic or stochastic logic in the sense of Qian-Riedel-Rosenberg [28, 29] and the following property holds.

Theorem 1.1. ([28, Theorem 1]). *Given a Boolean function $f : \{0, 1\}^N \to \{0, 1\}$. Stochastic logic yields a polynomial in N variables $\hat{F}$ given by*

$$\hat{F}(a_1, \ldots, a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1 \ldots i_N}$ are integers. Moreover, for each $y = f(x_1, \ldots, x_N)$ we have

$$p_Y = \hat{F}(p_{X_1}, p_{X_2}, \ldots, p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N} \prod_{k=1}^{N} p_{X_k}^{i_k} \right).$$

It is worth pointing out that to the best of our knowledge, the treatment or implementation by use of some stochastic logic of Qian-Riedel-Rosenberg type has not been considered for Boolean maps of the form $f : \{0, 1\}^N \to \{0, 1\}^N$. Thus, the following questions related to Theorem 1.1 arise: Can Theorem 1.1 be extended in this setting? In negative case, what is the difficult for finding such an extension? In affirmative case, how do we characterize such an extension? In this paper, we are interested in the theoretical issues concern stochastic logic of Qian-Riedel-Rosenberg type. In particular, we focus our attention on the theoretical connection between a stochastic logic of Qian-Riedel-Rosenberg type and certain class of multivariate Bernstein polynomials related with combinational circuits. So, some of aforementioned questions will be answer in the present paper.

The outline of the paper is as follows. Section 2 contains some relevant properties of the induced multivariate Bernstein polynomials. In Section 3 the notion of stochastic logic of Qian-Riedel-Rosenberg type is introduced, its connection with induced multivariate Bernstein polynomials is given and our main results are stated and proved. Finally, Section 4 is devoted to a brief additional remark on a model of stochastic logic based on the so-called degenerate Bernstein polynomials. Throughout this paper, we only consider combinational circuitry.

2. Multivariate Bernstein Polynomials

This section is devoted to introduce a class of multivariate Bernstein polynomials and recall some of their structural properties. We adopt the way of writing multivariate Bernstein polynomials used in [3]. For more details the reader can see [3], [22, § 2.9, p. 51] and the references thereof. However, before we look at this class we will recall the definition and some algebraic and analytic properties of the Bernstein polynomials in one variable (cf., [22, 29]).

Given $n \in \mathbb{N}$, for $f : [0,1] \to \mathbb{R}$ a continuous function and $t \in [0,1]$, the nth Bernstein polynomial of f is given by

$$B_n(t) = B_n(f;t) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k(1-t)^{n-k}. \tag{2.1}$$

The polynomials $B_n(t)$ converge uniformly to f on $[0,1]$ and this fact is the key piece for the Bernstein constructive demonstration of Weierstrass approximation theorem [22, 26].

The polynomials appearing in the formula on the right hand side of (2.1), namely;

$$b_k(t) = b_{k,n}(t) := \binom{n}{k} t^k(1-t)^{n-k}, \quad k = 0, \dots, n,$$

form a basis for the space of polynomials of degree at most n with real coefficients and the set $\{b_{k,n}(t) : k = 0, \dots, n\}$ and it is usually called Bernstein basis [5,30]. Also, it is clear that $\deg(b_{k,n}(t)) = n$, for each $k = 0, \dots, n$.

We call Bernstein polynomial to the representation in terms of the Bernstein basis of any polynomial $P(t)$ of degree at most n and real coefficients. So, there exists a unique vector $(\beta_{0,0}, \beta_{1,n}, \dots, \beta_{n,n}) \in \mathbb{R}^{n+1}$ such that

$$P(t) = \underbrace{\sum_{k=0}^{n} \beta_{k,n} b_{k,n}(t)}_{\text{Bernstein polynomial}} . \tag{2.2}$$

The name Bernstein polynomial for the expression on the right hand side of (2.2) was coined by Qian et al. (cf., [28, 29]), although Farouki and Goodman [5] have preferred to use the term *Bernstein form of* $P(t)$ to refer to the same expression. By (2.2) we have that the nth Bernstein polynomial of the function $f \in C[0,1]$ given by (2.1) becomes in a particular case of Bernstein polynomial, for which $\beta_{k,n} = f\left(\frac{k}{n}\right)$, $k = 0, 1, \ldots, n$.

The following results show some pertinent properties of the Bernstein basis and polynomials.

Proposition 2.1. *The Bernstein basis* $\{b_{k,n}(t) : k = 0, \ldots, n\}$ *satisfies the following algebraic and analytic properties* [6, 29].

(i) *Partition of unity property.*

$$\sum_{k=0}^{n} b_{k,n}(t) = 1, \qquad \text{for all } t \in \mathbb{R}.$$

(ii) *Non-negativity property.*

$$b_{k,n}(t) \geq 0, \qquad \text{for all } t \in [0,1].$$

(iii) *Symmetry property.*

$$b_{k,n}(t) = b_{n-k,n}(1-t), \qquad \text{for all } t \in [0,1].$$

(iv) *Recurrence formula.*

$$b_{k,n+1}(t) = t b_{k-1,n}(t) + (1-t) b_{k,n}(t), \qquad \text{for all } t \in [0,1].$$

(v) *Unimodality or extremal property. For* $n \geq 1$, $b_{k,n}(t)$ *attains a relative maximum at* $t = \frac{k}{n}$, $k = 0, \ldots, n$.

(vi) *Degree elevation property. For* $k = 0, \ldots, n$, *we have*

$$b_{k,n}(t) = \frac{n+1-k}{n+1} b_{k,n+1}(t) + \frac{k+1}{n+1} b_{k+1,n+1}(t),$$

for all $t \in [0,1]$.

(vii) *Representation in terms of the canonical basis of the space of polynomials of degree at most* n *with real coefficients.*

$$b_{k,n}(t) = \sum_{j=k}^{n} (-1)^{j-k} \binom{n}{j} \binom{j}{k} t^j.$$

Proposition 2.2. *Let* $P(t) = \sum_{k=0}^{n} \beta_{k,n} b_{k,n}(t)$ *be a Bernstein polynomial. Then the following properties hold* [6, 29].

(i) $P(0) = \beta_{0,n}$ *and* $P(1) = \beta_{n,n}$.

(ii) *Inversion formula. For each* $0 \leq j \leq n$, *we have*

$$t^j = \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} b_{k,n}(t).$$

(iii) *Change of basis. If $P(t)$ has the following representation in terms of the canonical basis of the space of polynomials of degree at most N with real coefficients:*

$$P(t) = \sum_{k=0}^{n} a_{k,n} t^k,$$

then

$$\beta_{k,n} = \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{n}{j}} a_{j,n}, \quad k = 0, \ldots, n.$$

(iv) *Lower and upper bounds.*

$$\min_{0 \le k \le n} \beta_{k,n} \le P(t) \le \max_{0 \le k \le n} \beta_{k,n}.$$

(v) *Degree elevation procedure. For any $m \ge n$, it is always possible to represent $P(t)$ in terms of the Bernstein basis $\{b_{k,m+1}(t) : k = 0, \ldots, m+1\}$ as follows*

$$P(t) = \sum_{k=0}^{m+1} \beta_{k,m+1} b_{k,m+1}(t),$$

where the Bernstein coefficients $\beta_{k,m+1}$ are given by

$$\beta_{k,m+1} = \begin{cases} \beta_{0,m}, & \text{for } k = 0, \\ \frac{k}{m+1} \beta_{k-1,m} + \left(1 - \frac{k}{m+1}\right) \beta_{k,m}, & \text{for } k = 1, \ldots, m, \\ \beta_{m,m}, & \text{for } k = m+1. \end{cases}$$

(vi) (*cf.* [29, Theorem 1]) *Uniform approximation of the Bernstein coefficients. Let $g(t)$ be a polynomial of degree $n \ge 0$. For any $\epsilon > 0$, there exists a positive integer $M \ge n$ such that for all integer $m \ge M$ and $k = 0, 1, \ldots, m$, we have*

$$\left| \beta_{k,m} - g\left(\frac{k}{m}\right) \right| < \epsilon,$$

where $\beta_{0,m}, \beta_{1,m}, \ldots, \beta_{m,m}$ satisfy that $g(t) = \sum_{k=0}^{m} \beta_{k,m} b_{k,m}(t)$.

Given $N \in \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, to deal with multivariate polynomials we recall the standard multi-index notation. A multi-index is denoted by $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{N}_0^N$. For two given multi-indices $\alpha, \nu \in \mathbb{N}_0^N$ we write $\alpha \le \nu$ if and only if $\alpha_j \le \nu_j$, $j = 1, \ldots, N$. The multi-index $\alpha + \nu$ is defined by $\alpha + \nu = (\alpha_1 + \nu_1, \ldots, \alpha_N + \nu_N)$. If $\alpha \le \nu$, the multi-index $\nu - \alpha$ is defined by $\nu - \alpha = (\nu_1 - \alpha_1, \ldots, \nu_N - \alpha_N)$. We write $\binom{\nu}{\alpha}$ for the multiplication $\binom{\nu_1}{\alpha_1} \cdots \binom{\nu_N}{\alpha_N}$, whenever $\alpha \le \nu$. For $\nu \in \mathbb{N}_0^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ a monomial in variables $x_1, \ldots, x_N$ of index ν is defined by

$$x^\nu = x_1^{\nu_1} \cdots x_N^{\nu_N}.$$

We denote by $\mathbb{P}^N = \mathbb{R}[x_1, \ldots, x_N]$ the space of all polynomials of N variables with real coefficients. Let $p(x) = p(x_1, \ldots, x_N) \in \mathbb{P}^N$. We say that a multi-index $\kappa = (\kappa_1, \ldots, \kappa_N)$ is the multi-index of maximum degree of $p(x)$ if κ_j is the maximum degree of x_j in $p(x)$, $j = 1, \ldots, N$ (cf. [3]).

So, the set $S = \{\nu \in \mathbb{N}_0^N : \nu \leq \kappa\}$ contains all the combinations from $\mathbb{N}_0^N$ which are smaller than or equal to the multi-index κ of maximum degree. Hence, $p(x)$ can be expressed as

$$p(x) = p(x_1, \ldots, x_N) = \sum_{\nu \in S} a_{\nu,\kappa} x^{\nu}, \tag{2.3}$$

where $a_{\nu,\kappa} \in \mathbb{R}$. The multivariate polynomial appearing on the right hand side of (2.3) is called the power form of $p(x)$.

An N-dimensional generalization of the Bernstein polynomials can be defined as follows. Let $f : [0,1]^N \to \mathbb{R}$ be a bounded function. The N-dimensional Bernstein of f is given by

$$B_{n_1,\ldots,n_N}(f; (x_1, \ldots, x_N)) := \sum_{\nu \in S^*} f\left(\frac{\nu_1}{n_1}, \ldots, \frac{\nu_N}{n_N}\right) B_{\nu,\mathbf{N}}(x_1, \ldots, x_N), \tag{2.4}$$

where $S^* = \{\nu \in \mathbb{N}_0^N : 0 \leq \nu \leq \mathbf{N}\}$, $\mathbf{N} = (n_1, \ldots, n_N)$ and $B_{\nu,\mathbf{N}}(x_1, \ldots, x_N) = \prod_{j=1}^N b_{\nu_j,n_j}(x_j)$.

It is well known that the N-dimensional Bernstein $B_{n_1,\ldots,n_N}(f; (x_1, \ldots, x_N))$ converges to $f((x_1, \ldots, x_N))$ at any point of continuity of this function, as all $n_k \to \infty$ (cf., [4,14]), and from (2.4) it is possible to induce a multivariate Bernstein polynomial as follows:

$$P(x) = P(x_1, \ldots, x_N) := \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x_1, \ldots, x_N) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \tag{2.5}$$

$x \in [0,1]^N$, $c_{\nu,\mathbf{N}} \in \mathbb{R}$. We call to the polynomial $P(x_1, \ldots, x_N)$ induced multivariate Bernstein polynomial.

Furthermore, if $x \in [0,1]^N$ and $p(x)$ is a multivariate polynomial which is written by means of a power form (2.3), then $p(x)$ can be expressed in terms of an induced multivariate Bernstein polynomial as follows.

$$p(x) = \sum_{\nu \in S} c_{\nu,\kappa} B_{\nu,\kappa}(x), \tag{2.6}$$

where the Bernstein coefficients $c_{\nu,\kappa}$ are given by

$$c_{\nu,\kappa} = \sum_{\alpha \leq \nu} \frac{\binom{\nu}{\alpha}}{\binom{\kappa}{\alpha}} a_{\alpha,\kappa}, \quad \nu \in S. \tag{2.7}$$

The multivariate polynomial appearing on the right hand side of (2.6) is called Bernstein form of $p(x)$.

3. Main Results

Given $N \in \mathbb{N}$ and (x, p_x) a stochastic number with $x \in \{0,1\}^N$. For each $k = 1, 2, \ldots, N$ we choose $p_k \in [0,1]$ and consider discrete and independent random variables X_k having Bernoulli distribution with parameter p_k, i.e., $X_k \sim Be(p_k)$

(cf. [25,31]). Since $x_k \in \{0,1\}$, each probability density function is given by

$$P\{X_k = x_k\} = p_k^{x_k}(1-p_k)^{1-x_k}. \tag{3.1}$$

We define

$$p_{X_k} := P\{X_k = 1\} = p_k \quad \text{and} \quad 1 - p_{X_k} := P\{X_k = 0\} = 1 - p_k, \quad k = 1, 2, \ldots, N.$$

Assume that a combinational circuit implements the Boolean map $f : \{0,1\}^N \to \{0,1\}^N$. Let $(f_1, \ldots, f_N)$ be the component functions of f. Thus, each Boolean function $f_j : \{0,1\}^N \to \{0,1\}$ can be assumed as a subcircuit associated to f, $j = 1, 2, \ldots, N$.

Given (x, p_x) a stochastic number with $x = (x_1, \ldots, x_N) \in \{0,1\}^N$, choose an N-tuple of discrete and independent random variables $X = (X_1, \ldots, X_N)$ such that $X_k \sim Be(p_k)$ for some $p_k \in [0,1]$ and satisfying (3.1). We can associate to each component function $f_j : \{0,1\}^N \to \{0,1\}$, a discrete random variable Y_j using that its probability density function is uniquely determined by the given N-tuple $X = (X_1, \ldots, X_N)$. More precisely, for determining $p_{Y_j} := P\{Y_j = 1\}$, $j = 1, \ldots, N$, we proceed as follows (cf. [25]). Since each $y_j = f_j(x) = f_j(x_1, \ldots, x_N) \in \{0,1\}$, for $j = 1, \ldots, N$, we have

$$\begin{aligned} p_{Y_j} = P\{Y_j = 1\} &= \sum_{\substack{x_1,\ldots,x_N: \\ f_j(x_1,\ldots,x_N)=1}} P\{X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N\} \\ &= \sum_{\substack{x_1,\ldots,x_N: \\ f_j(x_1,\ldots,x_N)=1}} \left(\prod_{k=1}^{N} P\{X_k = x_k\} \right). \end{aligned} \tag{3.2}$$

The identity (3.2) is consequence of the independence of X_k, and since $P\{X_k = x_k\}$ is either p_{X_k} or $1 - p_{X_k}$, depending on the value of x_k in the given combination. Thus, the Boolean function $f_j : \{0,1\}^N \to \{0,1\}$ induces a function F_j acting on the discrete and independent random variables $X_1, \ldots, X_N$ such that for each $Y_j = F_j(X_1, \ldots, X_N)$ the identity (3.2) holds. Furthermore, the random variable Y_j has Bernoulli distribution with parameter p_{Y_j}.

It is easily seen that p_{Y_j} is a multivariate polynomial with arguments $p_{X_1}, \ldots, p_{X_N}$, and if we expand (3.2) into a power form, each product term has an integer coefficient and the degree of each variable in that term is less than or equal to 1. Hence, applying Theorem 1.1 we have that the stochastic logic yields a polynomial in N variables $\hat{F}_j$ given by

$$\hat{F}_j(a_1, \ldots, a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N; j} \prod_{k=1}^{N} a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1 \ldots i_N; j}$ are integers. Moreover, for each $y_j = f_j(x_1, \ldots, x_N)$, $j = 1, \ldots, N$, we have

$$p_{Y_j} = \hat{F}_j(p_{X_1}, p_{X_2}, \ldots, p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1 \ldots i_N; j} \prod_{k=1}^{N} p_{X_k}^{i_k} \right).$$

We call stochastic logic of Qian-Riedel-Rosenberg type to the passage of the Boolean map $f:\{0,1\}^N \to \{0,1\}^N$ to the map $F=(F_1,\dots,F_N)$ acting on the discrete and independent random variables $X_1,\dots,X_N$ such that for each $Y_j = F_j(X_1,\dots,X_N)$ the identity (3.2) holds.

We summarize the previous ideas in the following theorem.

Theorem 3.1. *Given a Boolean map $f:\{0,1\}^N \to \{0,1\}^N$. The stochastic logic of Qian-Riedel-Rosenberg type yields a map $\hat{F}=(\hat{F}_1,\dots,\hat{F}_N)$ acting on the discrete and independent random variables $X_1,\dots,X_N$, whose component functions are multivariate polynomials of the form*

$$\hat{F}_j(a_1,\dots,a_N) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1\dots i_N;j} \prod_{k=1}^{N} a_k^{i_k} \right),$$

where the coefficients $\alpha_{i_1\dots i_N;j}$ are integers. Moreover, for each $y_j = f_j(x_1,\dots,x_N)$, $j=1,\dots,N$, we have

$$p_{Y_j} = \hat{F}_j(p_{X_1},p_{X_2},\dots,p_{X_N}) = \sum_{i_1=0}^{1} \cdots \sum_{i_N=0}^{1} \left(\alpha_{i_1\dots i_N;j} \prod_{k=1}^{N} p_{X_k}^{i_k} \right). \tag{3.3}$$

Example 3.1. Consider the Boolean map $f:\{0,1\}^3 \to \{0,1\}^3$ given by

$$f(x_1,x_2,x_3) = ((x_1 \wedge x_3) \vee (x_2 \wedge (\neg x_3)), x_1 \wedge x_3, x_2 \wedge (\neg x_1)),$$

where $\wedge$ means logical AND, $\vee$ means logical OR, and $\neg$ means logical negation. Choose $p_1,p_2,p_3 \in [0,1]$ and let X_1,X_2,X_3 be three discrete and independent random variables such that $X_1 \sim Be(p_1)$, $X_2 \sim Be(p_2)$, $X_3 \sim Be(p_3)$ whose probability density functions satisfy (3.1). It is clear that

$$\begin{aligned}
f_1(x_1,x_2,x_3) &= (x_1 \wedge x_3) \vee (x_2 \wedge (\neg x_3)),\\
f_2(x_1,x_2,x_3) &= x_1 \wedge x_3,\\
f_3(x_1,x_2,x_3) &= x_2 \wedge (\neg x_1).
\end{aligned}$$

By the definition of p_{Y_j}, $j=1,2,3$, we have

$$\begin{aligned}
p_{Y_1} &= P\{X_1=1,X_2=0,X_3=1\} + P\{X_1=1,X_2=1,X_3=1\}\\
&\quad + P\{X_1=0,X_2=1,X_3=0\} + P\{X_1=1,X_2=1,X_3=0\}\\
&= p_{X_1}(1-p_{X_2})p_{X_3} + p_{X_1}p_{X_2}p_{X_3} + (1-p_{X_1})p_{X_2}(1-p_{X_3}) + p_{X_1}p_{X_2}(1-p_{X_3})\\
&= p_{X_2} + p_{X_1}p_{X_3} - p_{X_2}p_{X_3},\\
p_{Y_2} &= P\{X_1=1,X_2=0,X_3=1\} + P\{X_1=1,X_2=1,X_3=1\}\\
&= p_{X_1}(1-p_{X_2})p_{X_3} + p_{X_1}p_{X_2}p_{X_3} = p_{X_1}p_{X_3},\\
p_{Y_3} &= P\{X_1=0,X_2=1,X_3=0\} + P\{X_1=0,X_2=1,X_3=1\}\\
&= (1-p_{X_1})p_{X_2}(1-p_{X_3}) + (1-p_{X_1})p_{X_2}p_{X_3} = p_{X_2} - p_{X_1}p_{X_2},
\end{aligned}$$

and the random variables Y_1, Y_2 and Y_3 are given by

$$\begin{aligned} Y_1 &= F_1(X_1, X_2, X_3) = X_2 + X_1X_3 - X_2X_3, \\ Y_2 &= F_2(X_1, X_2, X_3) = X_1X_3, \\ Y_3 &= F_3(X_1, X_2, X_3) = X_2 - X_1X_2, \end{aligned} \tag{3.4}$$

which confirms that (3.4) induces a map $\hat{F} = (\hat{F}_1, \hat{F}_2, \hat{F}_3)$ acting on the discrete and independent random variables X_1, X_2, X_3, whose component functions are polynomials in the variables (a, b, c) with integer coefficients:

$$\begin{aligned} \hat{F}_1(a, b, c) &= b + ac - bc, \\ \hat{F}_2(a, b, c) &= ac, \\ \hat{F}_3(a, b, c) &= b - ab. \end{aligned}$$

We now come to the second part of the main results of this section: the connection between stochastic logic of Qian-Riedel-Rosenberg type and induced multivariate Bernstein polynomials. Suppose that we have a combinational circuit $w = g(x_1, x_2, \ldots, x_N)$ consisting of N combinational subcircuits $y_j = f_j(x_1, x_2, \ldots, x_N)$, $j = 1, \ldots, N$, and only an N-input AND gate. Each combinational subcircuit $y_j = f_j(x_1, x_2, \ldots, x_N)$ consists of a decoding block and a multiplexing block, which transform the N inputs $\{x_1, \ldots, x_N\} \in \{0, 1\}$ as follows: If k out of the inputs $\{x_1, \ldots, x_N\}$ of the jth decoding block are logical 1, then s_{kj} is set to 1 and the other outputs are set to 0, $(0 \leq k \leq N)$. So, the output of the jth decoding block is $s^j = (s_{0j}, \ldots, s_{Nj})$. The outputs of the jth decoding block are fed into the jth multiplexing block, as shown in Figure 1, and they act as the selecting signals (control inputs). The data signals (inputs) of the jth multiplexing block consist of $N + 1$ inputs $z_{0j}, \ldots, z_{Nj} \in \{0, 1\}$.

Once the jth multiplexing block is used, the Boolean function $y_j = f_j(x_1, x_2, \ldots, x_N)$ takes the form

$$y_j = \bigvee_{k=0}^{N} (z_{kj} \wedge s_{kj}), \quad j = 0, \ldots, N, \tag{3.5}$$

which means that the output of the jth multiplexing block y_j is set to be the input z_{kj} if $s_{kj} = 1$.

Next, the inputs of the N-input AND gate are $y_1, \ldots, y_N \in \{0, 1\}$ and the Boolean function $w = g(x_1, x_2, \ldots, x_N)$ can be expressed as

$$w = \bigwedge_{j=1}^{N} y_j = \bigwedge_{j=1}^{N} \left[\bigvee_{k=0}^{N} (z_{kj} \wedge s_{kj}) \right]. \tag{3.6}$$

Using the association (3.1) for $(x_1, \ldots, x_N)$, $(s_{0j}, \ldots, s_{Nj})$ and $(z_{0j}, \ldots, z_{Nj})$ we can choose discrete and independent random variables $(X_1, \ldots, X_n)$, $(S_{0j}, \ldots, S_{Nj})$ and $(Z_{0j}, \ldots, Z_{Nj})$, such that $X_j \sim Be(p_j)$, $S_{kj} \sim Be(\hat{p}_{kj})$ and $Z_{kj} \sim Be(\hat{\hat{p}}_{kj})$,

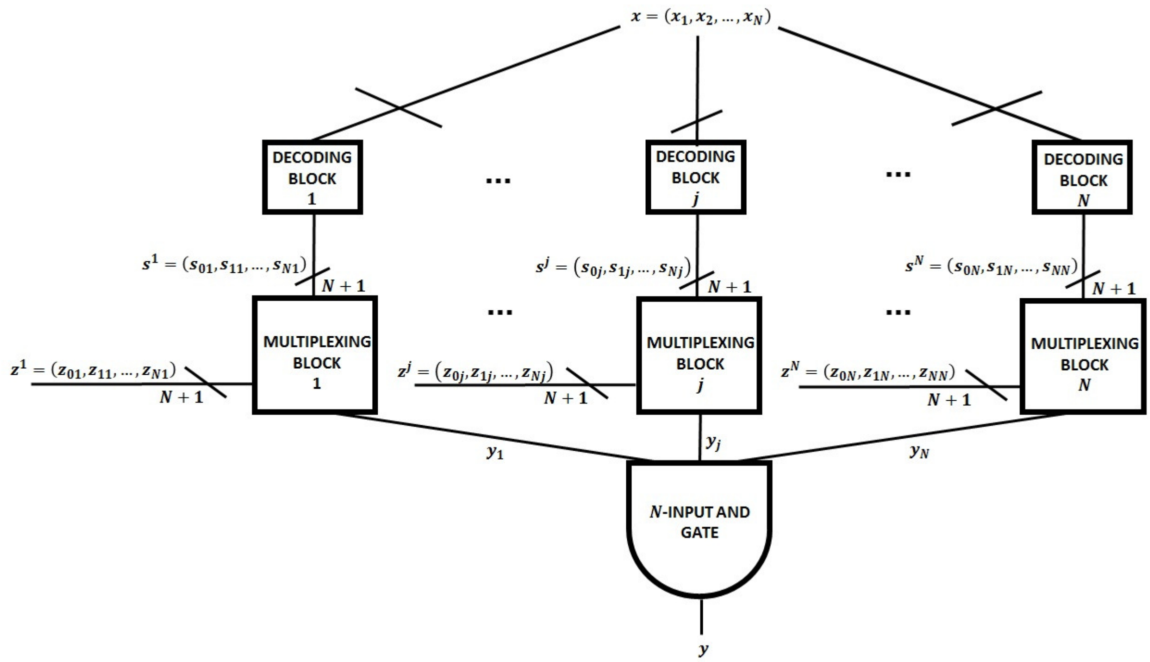

FIGURE 1. Combinational circuit associated to a multivariate Bernstein polynomial with coefficients in $[0, 1]$.

$k = 0, \ldots, N$, $j = 1, \ldots, N$. Similarly, we define

$$p_{X_k} := P\{X_k = 1\} = p_k \quad \text{and} \quad 1 - p_{X_k} := P\{X_k = 0\} = 1 - p_k,$$
$$p_{S_{kj}} := P\{S_{kj} = 1\} = \hat{p}_{kj} \quad \text{and} \quad 1 - p_{S_{kj}} := P\{S_{kj} = 0\} = 1 - \hat{p}_{kj},$$
$$p_{Z_{kj}} := P\{Z_{kj} = 1\} = \hat{\hat{p}}_{kj} \quad \text{and} \quad 1 - p_{Z_{kj}} := P\{Z_{kj} = 0\} = 1 - \hat{\hat{p}}_j,$$

for $k = 0, \ldots, N$, $j = 1, \ldots, N$.

Applying Theorem 3.1 to the Boolean map $f : \{0,1\}^N \to \{0,1\}^N$ given by $f(x_1, \ldots, x_N) = (y_1, \ldots, y_N)$, we have that the stochastic logic of Qian-Riedel-Rosenberg type yields N multivariate polynomials as in (3.3), such that $p_{Y_j} = \hat{F}_j(p_{X_1}, \ldots, p_{X_N})$, $j = 1, \ldots, N$.

Let W be the discrete random variable associated to Boolean function $w = g(x_1, x_2, \ldots, x_N)$ by means of

$$P\{W = 1\} = \sum_{\substack{x_1, \ldots, x_N: \\ g(x_1, \ldots, x_N) = 1}} P\{X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N\}.$$

We define $p_W := P\{W = 1\}$ and $1 - p_W := P\{W = 0\}$. According to (3.6) and Theorem 3.1 we have

$$p_W = \prod_{j=1}^{N} p_{Y_j} = \prod_{j=1}^{N} \hat{F}_j(p_{X_1}, \ldots, p_{X_N}). \tag{3.7}$$

Let us consider the polynomial $q_j(t)$ given by

$$q_j(t) = \hat{F}_j(\underbrace{t, t, \dots, t}_{N-\text{times}}), \quad j = 0, \dots, N.$$

Assume that $p_{X_1} = \cdots = p_{X_N} = t_0$, since s_{kj} is set to 1 if and only if k out of N inputs of the jth decoding block are 1, the probability that S_{kj} is 1 is (see, e.g., [1, pp. 10–11]):

$$p_{S_{kj}} = P\{S_{kj} = 1\} = \binom{N}{k} t_0^k (1 - t_0)^{N-k} = b_{k,N}(t_0), \quad k = 0, \dots, N.$$

Now, assume that $p_{Z_{kj}} = \beta^j_{k,N}$. Then

$$p_{Y_j} = P\{Y_j = 1\} = \sum_{k=0}^{N} P\{Y_j = 1 | S_{kj} = 1\} P\{S_{kj} = 1\}, \tag{3.8}$$

but from (3.5) is deduced that $S_{kj} = 1$ implies $Y_j = Z_{kj}$, so

$$P\{Y_j = 1 | S_{kj} = 1\} = P\{Z_{kj} = 1\} = p_{Z_{kj}} = \beta^j_{k,N}. \tag{3.9}$$

By (3.8) and (3.9) we obtain

$$q_j(t_0) = p_{Y_j} = \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t_0), \quad j = 1, \dots, N,$$

and (3.7) becomes

$$p_W = \prod_{j=1}^{N} \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t_0).$$

Therefore, under the constrains imposed by us, each combinational subcircuit $y_j = f_j(x_1, x_2, \dots, x_N)$ would require that $q_j(t)$ be a Bernstein polynomial whose coefficients $\beta^j_{k,N}$ belong to $[0, 1]$, (cf., [28–30]). Consequently, the combinational circuit $w = g(x_1, x_2, \dots, x_N)$ would require an induced multivariate Bernstein polynomial $P(x_1, \dots, x_N)$ such that

$$P(x_1, \dots, x_N) = \sum_{k=0}^{N} \cdots \sum_{k=0}^{N} c_{k,\mathbf{N}} B_{k,\mathbf{N}}(x_1, \dots, x_N),$$

where $c_{k,\mathbf{N}} = \prod_{j=1}^{N} \beta^j_{k,N}$, $B_{k,\mathbf{N}}(x_1, \dots, x_N) = \prod_{j=1}^{N} b_{k,N}(x_j)$, for $k = 0, \dots, N$ and multi-index of maximum degree $\mathbf{N} = (N, \dots, N)$. That is, the combinational circuit $w = g(x_1, x_2, \dots, x_N)$ would require that $P(x_1, \dots, x_N)$ be an induced multivariate Bernstein polynomial whose coefficients $c_{k,\mathbf{N}}$ are expressed as a product of N real numbers belonging to $[0, 1]$. Thus, we have the following theorem.

Theorem 3.2. *Let $P(x_1, \dots, x_N)$ be an induced multivariate Bernstein as in* (2.5) *such that*

(i) *the components of its multi-index of maximum degree $\boldsymbol{N}$ are equal;*

(ii) *its Bernstein coefficients satisfy that* $c_{\nu,\boldsymbol{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j}$ *with* $\nu = (k, \ldots, k)$, $k = 0, \ldots, N$.

If all the factors $\beta_{k,N}^{j}$ *belong to* $[0,1]$*, then we can design a stochastic logic of Qian-Riedel-Rosenberg type to compute* $P(x_1, \ldots, x_N)$.

A multivariate polynomial can be represented in a power form as (2.3). If it can be converted into an induced multivariate Bernstein polynomial satisfying the hypothesis of Theorem 3.2, then the preceding arguments show us how to implement it with stochastic logic of Qian-Riedel-Rosenberg type. The following result describes such a class of induced multivariate Bernstein polynomials.

Theorem 3.3. *Let* N *be any fixed positive integer. If* $q_j(t)$ *is a polynomial such that some of the following conditions is satisfied:*

(i) $q_j(t)$ *is identically equal to* 0 *or to* 1, $j = 1, \ldots, N$;

(ii) *for any* $t \in (0,1)$ *we have* $0 < q_j(t) < 1$, *with* $q_j(0) \geq 0$ *and* $q_j(1) \leq 1$, *for all* $j = 1, \ldots, N$.

Then for $x \in [0,1]^N$ *the multivariate polynomial* $q(x)$ *given by*

$$q(x) = q(x_1, \ldots, x_N) = \prod_{j=1}^{N} q_j(x_j) \tag{3.10}$$

can be converted into an induced multivariate Bernstein polynomial as in Theorem 3.2 *with Bernstein coefficients expressed as a product of* N *real numbers belonging to* $[0,1]$.

Reciprocally, if $q(x)$ *can be converted into an induced multivariate Bernstein polynomial as in* (2.5) *with Bernstein coefficients expressed as a product of* N *real numbers belonging to* $[0,1]$*, then the polynomials* $q_j(t)$ *satisfy* (i) *or* (ii)*,* $j = 1, \ldots, N$.

Proof. We begin by noting if $q_j(t) = 0$ for every $t \in [0,1]$, $j = 1, \ldots, N$ then taking

$$\beta_{k,N}^{j} = 0, \quad \text{for } k = 0, \ldots, N, j = 1, \ldots, N,$$

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j} = 0, \quad \text{with } \nu = (k, \ldots, k), k = 0, \ldots, N, \text{ and } \mathbf{N} = (N, \ldots, N),$$

it follows that

$$q_j(t) = \sum_{k=0}^{N} \beta_{k,N}^{j} b_{k,N}(t) = 0, \quad \text{for every } t \in [0,1],$$

$$q(x) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x) = 0, \quad \text{for every } x \in [0,1]^N.$$

Analogously, if $q_j(t)) = 1$ for every $t \in [0,1]$ then taking

$$\beta_{k,N}^{j} = 1, \quad \text{for } k = 0, \ldots, N, j = 1, \ldots, N,$$

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{N} \beta_{k,N}^{j} = 1, \quad \text{with } \nu = (k, \ldots, k), k = 0, \ldots, N, \text{ and } \mathbf{N} = (N, \ldots, N),$$

and using part (i) of Proposition 2.1, it follows that

$$q_j(t) = \sum_{k=0}^{N} \beta^j_{k,N} b_{k,N}(t) = 1, \qquad \text{for every } t \in [0,1],$$
$$q(x) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x) = 1, \qquad \text{for every } x \in [0,1]^N.$$

Now consider any polynomials $q_j(t)$ such that $q_j(t) \neq 0$ and $q_j(t) \neq 1$ for every $t \in [0,1]$, and $q_j(t)$ satisfy (ii) for all $j = 1, \ldots, n$. We distinguish four possible cases according to the inequalities satisfied by $q_j(0)$ and $q_j(1)$, for all $j = 1, \ldots, N$:

Case I: $0 \leq q_j(0)$ and $q_j(1) < 1$, for all $j = 1, \ldots, N$. For the sake of clarity and readability, we have decided to include the details of the proof of this case. However, one can check that it suffices to follow the reasoning in [29, Theorem 4], making the appropriate modifications.

Since $q_j(t)$ is a continuous function on the compact interval $[0,1]$, it attains its maximum value M_{q_j} on $[0,1]$. Thus $M_{q_j} < 1$, because $q_j(t) < 1$ for all $t \in [0,1]$. Let $\epsilon_j = 1 - M_{q_j} > 0$, by part (vi) of Proposition 2.2 there exists a positive integer $M_j \geq N$ such that for all $m \geq M_j$ and $k = 0, \ldots, m$, we have

$$\left| \beta^j_{k,m} - q_j\left(\frac{k}{m}\right) \right| < \epsilon_j, \quad j = 1, \ldots, N,$$

where $\beta^j_{0,m}, \ldots, \beta^j_{m,m}$ satisfy that $q_j(t) = \sum_{k=0}^{m} \beta^j_{k,m} b_{k,m}(t)$, $j = 1, \ldots, N$. Thus, for all $m \geq M_j$ and $k = 0, \ldots, m$,

$$\beta^j_{k,m} < q_j\left(\frac{k}{m}\right) + \epsilon_j \leq M_{q_j} + 1 - M_{q_j} = 1. \tag{3.11}$$

Denote by r_j the multiplicity of 0 as root of $q_j(t)$ (where $r_j = 0$ if $q_j(0) > 0$) and by s_j the multiplicity of 0 as root of 1 as root of $q_j(t)$ (where $s_j = 0$ if $q_j(1) \neq 0$). We can factorize each $q_j(t)$ as

$$q_j(t) = t^{r_j}(1-t)^{s_j} h_j(t), \tag{3.12}$$

where $h_j(t)$ is a polynomial satisfying that $h_j(0) \neq 0$ and $h_j(1) \neq 1$, $j = 1, \ldots, N$.

It is clear that $h_j(0) > 0$, since if we suppose, contrary of our claim, that $h_j(0) \leq 0$, using that $h_j(0) \neq 0$ we have necessarily $h_j(0) < 0$, and by the continuity of $h_j(t)$, there exists $t^*_j \in (0,1)$ such that $h_j(t^*_j) < 0$. Hence, $q_j(t^*_j) = t^{r_j}(1-t)^{s_j} h_j(t^*_j) < 0$. This contradicts the fact that $q_j(t) > 0$ for all $t \in (0,1)$. Similarly, we have $h_j(1) > 0$.

Consequently, $h_j(t) > 0$ for all $t \in [0,1]$. Since $h_j(t)$ is a continuous function on the compact interval $[0,1]$, it attains its minimum value m_{h_j} on $[0,1]$, and clearly, $m_{h_j} > 0$.

Let $\varepsilon_j = m_{h_j} > 0$, by part (vi) of Proposition 2.2 there exists a positive integer $K_j \geq N - r_j - s_j$ such that for all $d \geq K_j$ and $k = 0, \ldots, d$, we have

$$\left| \gamma^j_{k,d} - h_j\left(\frac{k}{d}\right) \right| < \varepsilon_j, \quad j = 1, \ldots, N,$$

where $\gamma^j_{0,d},\dots,\gamma^j_{d,d}$ satisfy that

$$h_j(t)=\sum_{k=0}^{d}\gamma^j_{k,d}b_{k,d}(t),\quad j=1,\dots,N. \tag{3.13}$$

Thus, for all $d\geq K_j$ and $k=0,\dots,d$, we have

$$\gamma^j_{k,d}>h_j\left(\frac{k}{d}\right)-\varepsilon_j\geq m_{h_j}-m_{h_j}=0.$$

Combining and (3.12) (3.13), we get

$$\begin{aligned}q_j(t)&=t^{r_j}(1-t)^{s_j}\sum_{k=0}^{d}\gamma^j_{k,d}b_{k,d}(t)=\sum_{k=0}^{d}\frac{\gamma^j_{k,d}\binom{d}{k}}{\binom{d+r_j+s_j}{k+r_j}}\binom{d+r_j+s_j}{k+r_j}b_{k,d+r_j+s_j}(t)\\&=\sum_{k=0}^{d+r_j+s_j}\beta^j_{k,d+r_j+s_j}b_{k,d+r_j+s_j}(t),\end{aligned}$$

where $\beta_{k,d+r_j+s_j}$ are the coefficients of the Bernstein polynomial of degree $d+r_j+s_j$ of $q_j(t)$, and

$$\beta^j_{k,d+r_j+s_j}=\begin{cases}0, & \text{for } 0\leq k<r_j \text{ and } d+r_j<k\leq d+r_j+s_j,\\ \frac{\gamma^j_{k,d}\binom{d}{k}}{\binom{d+r_j+s_j}{k+r_j}}>0, & \text{for } r_j\leq k\leq d+r_j.\end{cases}$$

Thus, taking $r=\max_{1\leq N}\{r_j\}$, $s=\max_{1\leq N}\{s_j\}$ and $K=\max_{1\leq N}\{K_j\}$ when $m\geq d+r+s\geq K+r+s$, we have

$$\beta^j_{k,m}\geq 0,\quad k=0,\dots,m. \tag{3.14}$$

According to (3.11) and (3.14) if we take $M=\max\{M_j\}$ and choose an $m_0\geq\max\{M,K+r+s\}$, then $q_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$q_j(t)=\sum_{k=0}^{m_0}\beta^j_{k,m_0}b_{k,m_0}(t),$$

with $0\leq\beta^j_{k,m_0}\leq 1$, for all $k=0,\dots,m_0$ and $j=1\dots,N$. Now, taking

$$c_{\nu,\mathbf{N}}=\prod_{j=1}^{m_0}\beta^j_{k,m_0},\qquad \text{with } \nu=(k,\dots,k), k=0,\dots,m_0, \text{ and } \mathbf{N}=(m_0,\dots,m_0),$$

it follows that

$$q(x)=q(x_1,\dots,x_N)=\prod_{j=1}^{N}q_j(x_j)=\sum_{\nu\in S^*}c_{\nu,\mathbf{N}}B_{\nu,\mathbf{N}}(x),\qquad \text{for every } x\in[0,1]^N.$$

Case II: $q_j(0)=0$ and $q_j(1)=1$, for all $j=1,\dots,N$. It suffices to combine a reasoning similar to that in the proof of *Case I* with the reasoning in [29, Theorem 5], making the appropriate modifications.

Case III: $0<q_j(0)$ and $q_j(1)\leq 1$, for all $j=1,\dots,N$. Consider the polynomials $g_j(t)=1-q_j(t)$, for all $t\in[0,1]$, $j=1,\dots,N$. Then $0<g_j(t)<1$, for all $t\in(0,1)$

with $0 \le g_j(0)$ and $g_j(1) < 1$, for all $j = 1, \dots, N$. Then in view of *Case I* we can choose an $m_0 \ge N$, then $g_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$g_j(t) = \sum_{k=0}^{m_0} \beta^j_{k,m_0} b_{k,m_0}(t),$$

with $0 \le \beta^j_{k,m_0} \le 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Hence, using part (i) of Proposition 2.1, it follows that

$$q_j(t) = 1 - g_j(t) = \sum_{k=0}^{m_0} (1 - \beta^j_{k,m_0}) b_{k,m_0}(t) = \sum_{k=0}^{m_0} \gamma^j_{k,m_0} b_{k,m_0}(t),$$

where $\gamma^j_{k,m_0} = 1 - \beta^j_{k,m_0}$, with $0 \le \gamma^j_{k,m_0} \le 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Now, taking

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{m_0} \gamma^j_{k,m_0}, \qquad \text{with } \nu = (k, \dots, k), k = 0, \dots, m_0, \text{ and } \mathbf{N} = (m_0, \dots, m_0),$$

it follows that

$$q(x) = q(x_1, \dots, x_N) = \prod_{j=1}^{N} q_j(x_j) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \qquad \text{for every } x \in [0,1]^N.$$

Case IV: $q_j(0) = 1$ and $q_j(1) = 0$, for all $j = 1, \dots, N$. Consider the polynomials $g_j(t) = 1 - q_j(t)$, for all $t \in [0,1]$, $j = 1, \dots, N$. Then $0 < g_j(t) < 1$, for all $t \in (0,1)$ with $0 \le g_j(0)$ and $g_j(1) < 1$, for all $j = 1, \dots, N$. Then in view of *Case II* we can choose an $m_0 \ge N$, then $g_j(t)$ can be expressed as a Bernstein polynomial of degree m_0:

$$g_j(t) = \sum_{k=0}^{m_0} \beta^j_{k,m_0} b_{k,m_0}(t),$$

with $0 \le \beta^j_{k,m_0} \le 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$. Hence, using part (i) of Proposition 2.1, it follows that

$$q_j(t) = 1 - g_j(t) = \sum_{k=0}^{m_0} (1 - \beta^j_{k,m_0}) b_{k,m_0}(t) = \sum_{k=0}^{m_0} \gamma^j_{k,m_0} b_{k,m_0}(t),$$

where $\gamma^j_{k,m_0} = 1 - \beta^j_{k,m_0}$, with $0 \le \gamma^j_{k,m_0} \le 1$, for all $k = 0, \dots, m_0$ and $j = 1 \dots, N$.

Therefore, if we take

$$c_{\nu,\mathbf{N}} = \prod_{j=1}^{m_0} \gamma^j_{k,m_0}, \qquad \text{with } \nu = (k, \dots, k), k = 0, \dots, m_0, \text{ and } \mathbf{N} = (m_0, \dots, m_0),$$

it follows that

$$q(x) = q(x_1, \dots, x_N) = \prod_{j=1}^{N} q_j(x_j) = \sum_{\nu \in S^*} c_{\nu,\mathbf{N}} B_{\nu,\mathbf{N}}(x), \qquad \text{for every } x \in [0,1]^N.$$

Finally, it can be shown that if $q_j(t)$ is not identically equal to 0 or to 1 for some j and there exists a $t_0 \in (0,1)$ such that $q_j(t_0) = 0$ or 1, then we cannot express the polynomial $q_j(t)$ as a Bernstein polynomial with coefficients in the unit interval (cf. [28]). Consequently, $q(x)$ cannot be converted into an induced multivariate Bernstein polynomial as in (2.5) with Bernstein coefficients expressed as a product of N real numbers belonging to $[0,1]$.

This completes the proof. $\square$

Notice that the multi-index of maximum degree of the induced multivariate Bernstein polynomial with coefficients in the unit interval may be greater than the multi-index of maximum degree of the original polynomial.

Example 3.2. Consider the polynomial $q(x,y) = q_1(x)q_1(y)$, where $q_1(x) = 3x - 8x^2 + 6x^3$ and $q_2(y) = y$, for all $x, y \in [0,1]$. The polynomial $q(x,y)$ has multi-index of maximum degree $\kappa = (3,1)$, and the polynomials $q_1(x)$ and $q_2(y)$ satisfy the conditions

$$\begin{array}{llll} 0 < q_1(x) < 1, & \text{whenever } x \in (0,1), & q_1(0) = 0, & q_1(1) = 1, \\ 0 < q_2(y) < 1, & \text{whenever } y \in (0,1), & q_2(0) = 0, & q_2(1) = 1. \end{array}$$

Using (2.7) and part (v) of Proposition 2.2 we have

$$\begin{aligned} q(x,y) &= \left(b_{1,3}(x) - \frac{2}{3}b_{2,3}(x) + b_{3,3}(x)\right) b_{1,1}(y) \\ &= \left(\frac{3}{4}b_{1,4}(x) + \frac{1}{6}b_{2,4}(x) - \frac{1}{4}b_{3,4}(x) + b_{4,4}(x)\right) b_{1,1}(y) \\ &= \left(\frac{3}{5}b_{1,5}(x) + \frac{2}{5}b_{2,5}(x) + b_{5,5}(x)\right) b_{1,1}(y) \\ &= \frac{3}{5}B_{((1,1),(5,1))}(x,y) + \frac{2}{5}B_{((2,1),(5,1))}(x,y) + B_{((5,1),(5,1))}(x,y), \end{aligned}$$

and the induced multivariate Bernstein polynomial of $q(x,y)$:

$$P(x,y) = \frac{3}{5}B_{((1,1),(5,1))}(x,y) + \frac{2}{5}B_{((2,1),(5,1))}(x,y) + B_{((5,1),(5,1))}(x,y)$$

has multi-index of maximum degree $\mathbf{N} = (5,1)$.

The following example show a polynomial $q(x,y)$ which can be converted into an induced multivariate Bernstein polynomial, however it cannot be implemented with stochastic logic of Qian-Riedel-Rosenberg type.

Example 3.3. Consider the polynomial $q(x,y) = 3xy - 8x^2y^2 + 6x^3y^3$ with multi-index of maximum degree $\kappa = (3,3)$, satisfying the conditions $0 < q(x,y) < 1$, whenever $(x,y) \in (0,1)^2$, $q(0,0) = 0$ and $q(1,1) = 1$. Since

$$q(x,y) = P_1(x)b_{1,3}(y) + P_2(x)b_{2,3}(y) + P_3(x)b_{3,3}(y) + P_3(y)b_{3,3}(x),$$

where $P_1(x) = \frac{1}{3}b_{1,3}(x) + \frac{2}{3}b_{2,3}(x)$, $P_2(x) = \frac{2}{3}b_{1,3}(x) + \frac{4}{9}b_{2,3}(x)$, $P_3(x) = b_{1,3}(x) - \frac{2}{3}b_{2,3}(x) + \frac{1}{2}b_{3,3}(x)$, and the coefficients of $P_3(x)$ do not all belong to the interval $[0,1]$, using part (v) of Proposition 2.2 we see that

$$\begin{aligned}P_3(x) = {} & \frac{3}{13}b_{1,13}(x) + \frac{14}{39}b_{2,13}(x) + \frac{21}{52}b_{3,13}(x) + \frac{5}{13}b_{4,13}(x) + \frac{25}{78}b_{5,13}(x) + \frac{3}{13}b_{6,13}(x)\\ & + \frac{7}{52}b_{7,13}(x) + \frac{2}{39}b_{8,13}(x) + \frac{11}{156}b_{11,13}(x) + \frac{3}{13}b_{12,13}(x) + \frac{1}{2}b_{13,13}(x).\end{aligned}$$

From (2.6) and (2.7) it follows that

$$q(x,y) = r_1(x,y) + r_2(x,y) + r_3(x,y) + r_4(x,y), \tag{3.15}$$

where

$$\begin{aligned}
r_1(x,y) = {} & \frac{1}{13}B_{((1,1),(13,3))}(x,y) + \frac{2}{13}B_{((2,1),(13,3))}(x,y) + \frac{5}{22}B_{((3,1),(13,3))}(x,y)\\
& + \frac{42}{143}B_{((4,1),(13,3))}(x,y) + \frac{50}{143}B_{((5,1),(13,3))}(x,y) + \frac{56}{143}B_{((6,1),(13,3))}(x,y)\\
& + \frac{119}{286}B_{((7,1),(13,3))}(x,y) + \frac{60}{143}B_{((8,1),(13,3))}(x,y) + \frac{57}{143}B_{((9,1),(13,3))}(x,y)\\
& + \frac{50}{143}B_{((10,1),(13,3))}(x,y) + \frac{7}{26}B_{((11,1),(13,3))}(x,y) + \frac{2}{13}B_{((12,1),(13,3))}(x,y),\\
r_2(x,y) = {} & \frac{2}{13}B_{((1,2),(13,3))}(x,y) + \frac{32}{117}B_{((2,2),(13,3))}(x,y) + \frac{155}{429}B_{((3,2),(13,3))}(x,y)\\
& + \frac{60}{143}B_{((4,2),(13,3))}(x,y) + \frac{580}{1287}B_{((5,2),(13,3))}(x,y) + \frac{196}{429}B_{((6,2),(13,3))}(x,y)\\
& + \frac{63}{143}B_{((7,2),(13,3))}(x,y) + \frac{40}{99}B_{((8,2),(13,3))}(x,y) + \frac{50}{143}B_{((9,2),(13,3))}(x,y)\\
& + \frac{40}{143}B_{((10,2),(13,3))}(x,y) + \frac{23}{117}B_{((11,2),(13,3))}(x,y) + \frac{4}{39}B_{((12,2),(13,3))}(x,y),\\
r_3(x,y) = {} & \frac{3}{13}B_{((1,3),(13,3))}(x,y) + \frac{14}{39}B_{((2,3),(13,3))}(x,y) + \frac{21}{52}B_{((3,3),(13,3))}(x,y)\\
& + \frac{5}{13}B_{((3,3),(13,3))}(x,y) + \frac{25}{78}B_{((5,3),(13,3))}(x,y) + \frac{3}{13}B_{((6,3),(13,3))}(x,y)\\
& + \frac{7}{52}B_{((7,3),(13,3))}(x,y) + \frac{2}{39}B_{((8,3),(13,3))}(x,y) + \frac{11}{156}B_{((11,3),(13,3))}(x,y)\\
& + \frac{3}{13}B_{((12,3),(13,3))}(x,y) + \frac{1}{2}B_{((13,3),(13,3))}(x,y),\\
r_4(x,y) = {} & r_3(y,x).
\end{aligned}$$

Hence, the induced multivariate Bernstein polynomial on the right hand side of (3.15) has multi-index of maximum degree $\mathbf{N} = (13,3)$. However, $q(x,y)$ cannot be factorized as (3.10).

As a consequence of Theorems 3.2 and 3.3 we obtain the following result.

Corollary 3.1. *Let N be any fixed positive integer. If $q_j(t)$ is a polynomial such that some of the following conditions is satisfied:*

(i) *$q_j(t)$ is identically equal to 0 or to 1, $j = 1,\dots,N$;*

(ii) *for any* $t \in (0,1)$ *we have* $0 < q_j(t) < 1$*, with* $q_j(0) \geq 0$ *and* $q_j(1) \leq 1$*, for all* $j = 1, \ldots, N$*,*

then we can design a stochastic logic of Qian-Riedel-Rosenberg type to compute the multivariate polynomial $q(x)$ *given by*

$$q(x) = q(x_1, \ldots, x_N) = \prod_{j=1}^{N} q_j(x_j), \quad x \in [0,1]^N.$$

Reciprocally, if $q(x)$ *can be implemented with stochastic logic of Qian-Riedel-Rosenberg type, then the polynomials* $q_j(t)$ *satisfy* (i) *or* (ii)*,* $j = 1, \ldots, N$*.*

4. A Further Remark

In recent years, extensive researches have been done for various degenerate versions of some special polynomials and numbers and have yielded many interesting arithmetical and combinatorial results. These include the degenerate Stirling numbers of the first and second kinds, degenerate central factorial numbers of the second kind, degenerate Bernoulli numbers of the second kind, degenerate Bernstein polynomials, degenerate Bell numbers and polynomials, degenerate central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, and so on.

Degenerate versions of some special polynomials have been shown to play an important role in various areas. However, not much is known about the properties of these polynomials (cf., e.g, [15–20, 32] and references thereof). In particular, as a degenerate version of Bernstein polynomials, the degenerate Bernstein polynomials were introduced recently by Kim and Kim in [16].

In this regard, the remarkable papers [16, 19] suggest that the fundamental properties and identities satisfied by the degenerate Bernoulli polynomials could be used to define a special model of stochastic logic. Thus, one of our future projects is to explore such a model.

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[1]Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés, Madrid, Spain
Email address: yaquinta@math.uc3m.es

[2]Departamento de Matemáticas Puras y Aplicadas
Universidad Simón Bolívar
Edificio Matemáticas y Sistemas (MYS)
Apartado Postal: 89000, Caracas 1080 A, Venezuela