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PERMUTING TRI-DERIVATIONS ON POSETS

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Abstract. Let P be a partially ordered set (poset). The main objective of the present paper is to introduce and study the idea of permuting tri-derivations of posets. Several characterization theorems involving permuting tri-derivations are given. In particular, we prove that if d_1 and d_2 are two permuting tri-derivations of P with traces ϕ_1 and ϕ_2, then $\phi_1 \leq \phi_2$ if and only if $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$.

1. Introduction

Motivated by the ideas of derivations and related maps in rings and algebras (see [1, 2, 7, 9] and references therein), the notions of derivation on lattices were introduced and studied in [10] and [11], respectively. Recently, several authors have studied and verified a lot of meaningful conclusions by applying derivations and its generalized forms to lattices (see [3] for more details). In see of over mentioned development, it is very common to exchange the idea of derivations to partially ordered sets. In this direction some progress have already been made (see [14]). In the year 2009, Öztürk et al. [8] brought about the idea of permuting tri-derivations to lattices and investigated some related properties (for more information see also [4] and [13]).

In the present paper, the notion of permuting tri-derivation of a partially ordered sets is introduced and some related properties are investigated. Precisely, in Section 2, the notion of permuting tri-derivations of partially ordered sets is presented and concentrate their essential properties. Further, the fixed sets (for more information about fixed sets see [12]) are examined in light of the permuting tri-derivations. Finally,

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Section 3 is devoted to the study of the properties of ideals and the operations related with the permuting tri-derivations.

Throughout this paper, $(P, \leq)$ always denotes a partially ordered set (poset). We additionally utilize the shorthand P to indicate a poset. According to [14], for $z \in P$, we write, $\downarrow z = \{p \in P : p \leq z\}$ and $\uparrow z = \{p \in P : z \leq p\}$. For $W \subseteq P$, we denote $l(W) = \{p \in P : p \leq w, \text{ for all } w \in W\}$ the lower cone of W and $u(W) = \{p \in P : w \leq p, \text{ for all } w \in W\}$ the upper cone of W dually. It is quickly clear that both are antitone and their compositions $l(u(\cdot))$ and $u(l(\cdot))$ are monotone. Also, we have $l(u(l(\cdot))) = l(\cdot)$, $u(l(u(\cdot))) = u(\cdot)$ from [5]. If $W = \{w_1, w_2, \dots, w_n\}$ is a finite subset, then we write $l(W) = l(w_1, w_2, \dots, w_n)$ and $u(W) = u(w_1, w_2, \dots, w_n)$ simply. Moreover, for $W_1 \subseteq P$ and $W_2 \subseteq P$, we will denote $l(W_1, W_2)$ for $l(W_1 \cup W_2)$ and $u(W_1, W_2)$ for $u(W_1 \cup W_2)$. For $A \subseteq P$, we write $\downarrow A = \{p \in P : p \leq a \text{ for some } a \in A\}$. From [6], we find that if $A = \downarrow A$, then A is said to be a lower set. A is directed if it is nonempty and every finite subset of A has an upper bound in A. From nonemptiness, it is ample to expect each combine of components in A has an upper bound in A. A subset J of P is called an ideal if it is a directed lower set.

2. Permuting Tri-Derivations on Posets

The following notions are essential in our discussions.

Definition 2.1. ([14, Definition 2.1]) Let $(P, \leq)$ be a poset and $d : P \to P$ be a function. We call d a derivation on P if it satisfies the following conditions:

(i) $d(l(x, y)) = l(u(l(d(x), y), l(x, d(y))))$ for all $x, y \in P$;
(ii) $l(d(u(x, y))) = l(u(d(x), d(y)))$ for all $x, y \in P$.

Let $(P, \leq)$ be a poset. A mapping $f : P \times P \times P \to P$ is called permuting if $f(x, y, z) = f(x, z, y) = f(y, x, z) = f(y, z, x) = f(z, x, y) = f(z, y, x)$ for all $x, y, z \in P$. A mapping $d : P \to P$ defined by $d(x) = f(x, x, x)$ for all $x \in P$, is called the trace of f where f is a permuting mapping.

Inspired by the notion permuting tri-derivations on rings [2, 7] and lattices [8, 13] the following notion on posets is introduced.

Definition 2.2. Let $(P, \leq)$ be a poset and $d : P \times P \times P \to P$ be a permuting mapping. Nextly, d is called a permuting tri-derivation on P if for all $x, y, z, w \in P$ the following conditions hold:

(i) $d(l((x, w), y, z) = l(u(l(d(x, y, z), w), l(x, d(w, y, z))))$ for all $x, y, z, w \in P$;
(ii) $l(d(u(x, w), y, z))) = l(u(d(x, y, z), d(w, y, z))$ for all $x, y, z, w \in P$.

Remark 2.1. Note that, a permuting tri-derivation on P satisfies the following conditions:

(i) $d(x, l(y, w), z) = l(u(l(d(x, y, z), w), l(y, d(x, w, z))))$ for all $x, y, z, w \in P$;
(ii) $l(d(x, u(y, w), z))) = l(u(d(x, y, z)), d(x, w, z)))$ for all $x, y, z, w \in P$;
(iii) $d(x, y, l(z, w)) = l(u(l(d(x, y, z), w), l(z, d(x, y, w))))$ for all $x, y, z, w \in P$;

(iv) $l(d(x, y, u(z, w)))) = l(u(d(x, y, z)), d(x, y, w)))$ for all $x, y, z, w \in P$.

Example 2.1. Let $(P, \leq) = (\mathbb{N}, \leq)$. Define the function $d : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $d(x, y, z) = \min\{x, y, z\}$ for all $x, y, z \in P$. It is straightforward to check that d is a permuting tri-derivation on P.

Proposition 2.1. *Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. Then the followings hold:*

(1) $d(x, y, z) \leq x$, $d(x, y, z) \leq y$ *and* $d(x, y, z) \leq z$ *for all* $x, y, z \in P$;
(2) $d(x, y, z) \in l(x, y, z)$, *for all* $x, y, z \in P$;
(3) *if* $x_1 \leq x_2$ *and* $y, z \in P$, *then* $d(x_1, y, z) \leq d(x_2, y, z)$;
(4) *if* $y_1 \leq y_2$ *and* $x, z \in P$, *then* $d(x, y_1, z) \leq d(x, y_2, z)$;
(5) *if* $z_1 \leq z_2$ *and* $x, y \in P$, *then* $d(x, y, z_1) \leq d(x, y, z_2)$;
(6) $\phi(x) \leq x$, *for all* $x \in P$;
(7) $\phi(l(x)) \subseteq l(\phi(x))$, *for all* $x \in P$;
(8) *if* $x \leq y$, *then* $\phi(x) \leq \phi(y)$;
(9) $\phi^2(x) = \phi(x)$, *for all* $x \in P$.

Proof. (1) Let d be a permuting tri-derivation on P. Then

$$\begin{aligned} d(l(x, x), y, z) &= l(u(l(d(x, y, z), x), l(x, d(x, y, z)))) \\ &= l(u(l(x, d(x, y, z)))) \\ &= l(x, d(x, y, z)), \end{aligned}$$

for all $x, y, z \in P$. Since $d(x, y, z) \in d(l(x, x), y, z)$, the above relation gives $d(x, y, z) \in l(x, d(x, y, z))$ for all $x, y, z \in P$. In this way, we conclude that $d(x, y, z) \leq x$ for all $x, y, z \in P$. Similarly, we can prove $d(x, y, z) \leq y$ and $d(x, y, z) \leq z$. Hence, $d(x, y, z) \leq x$, $d(x, y, z) \leq y$ and $d(x, y, z) \leq z$ for all $x, y, z \in P$.

(2) It is obvious from (1).

(3) Let $x_1 \leq x_2$ and $y, z \in P$. Then

$$l(d(u(x_1, x_2)), y, z) = l(d(u(x_2), y, z)) = l(u(d(x_1, y, z), d(x_2, y, z))),$$

for all $x_1, x_2, y, z \in P$. Since $d(x_1, y, z) \in l(u(d(x_1, y, z), d(x_2, y, z)))$, we find that $d(x_1, y, z) \in l(d(u(x_2), y, z))$ for all $x_1, x_2, y, z \in P$. Hence, $d(x_1, y, z) \leq d(x_2, y, z)$ for all $x_1, x_2, y, z \in P$.

(4), (5) Proofs run on comparable lines as in (3).

(6) By the definition,

$$\begin{aligned} d(l(x, x), x, x) &= l(u(l(d(x, x, x), x), l(x, d(x, x, x)))) \\ &= l(u(l(x, d(x, x, x)))) \\ &= l(x, d(x, x, x)), \end{aligned}$$

for all $x \in P$. Since $d(x, x, x) \in d(x, l(x, x), x))$, the last relation gives

$$\phi(x) = d(x, x, x) \in l(x, d(x, x, x)), \quad \text{for all } x \in P.$$

Consequently, we get $\phi(x) = d(x, x, x) \leq x$ for all $x \in P$.

(7) Let $x \in P$. Then

$$\begin{aligned}\phi(l(x)) &= \{d(y,y,y) : y \in P \text{ and } y \leq x\},\\ &\subseteq d(l(y,y),y,y)\\ &= l(u(l(d(y,y,y),y),l(y,d(y,y,y))))\\ &= l(u(l(d(y,y,y),y)))\\ &= l(u(l(d(y,y,y))))\\ &= l(d(y,y,y))\\ &= l(\phi(y)), \quad \text{for all } y \in P \text{ and } y \leq x.\end{aligned}$$

This implies that $\phi(l(x)) \subseteq l(\phi((x)))$ for all $x \in P$.

(8) Let $x, y \in P$ such that $x \leq y$. Then, applications of part (7) we get $\phi(l(y)) \subseteq l(\phi(y))$. Since $\phi(x) \in \phi(l(y))$, we find that $\phi(x) \in l(\phi(y))$ for all $x, y \in P$. Hence, we conclude that $\phi(x) \leq \phi(y)$ for all $x, y \in P$.

(9) In view of part (5), we get $\phi^2(x) = \phi(\phi(x)) \leq \phi(x) \leq x$ for all $x \in P$. Then for all $x \in P$

$$\begin{aligned}\phi(l(x)) &\subseteq l(\phi(x)),\\ &\subseteq d(l(x),y,y)\\ &= d(l(x,x),y,y)\\ &= l(u(l(d(x,y,y),x),l(d(x,y,y)),x))\\ &= l(u(l(d(x,y,y)),l(d(x,y,y))))\\ &= l(u(l(d(x,y,y)))\\ &= l(d(x,y,y))\\ &\subseteq l(x,y)\\ &= l(y), \quad \text{for all } y \in P \text{ and } y \leq x.\end{aligned}$$

Then for all $x \in P$ we have $\phi(l(x)) \subseteq l(y)$ for all $y \in P$ such that $y \leq x$. Since $\phi^2(x) \leq x$ for all $x \in P$, we observe that $\phi(l(x)) \subseteq l(\phi^2(x))$ for all $x \in P$. Since $\phi(x) \in \phi(l(x))$ for all $x \in P$, so $\phi(x) \in l(\phi^2(x))$ for all $x \in P$. This implies that $\phi(x) \leq \phi^2(x)$ for all $x \in P$. Hence, finally, $\phi^2(x) = \phi(x)$ for all $x \in P$. $\square$

Example 2.2. Let $(P, \leq) = (\mathbb{N}, \leq)$. Define the function $d : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $d(x,y,z) = \max\{x,y,z\}$ for all $x, y, z \in P$. Then, d is not a permuting tri-derivation on P.

Corollary 2.1. *Let P be a poset with the least element 0 and let d be a permuting tri-derivation on P. Then $d(0,y,z) = 0$ for all $y, z \in P$.*

Lemma 2.1. *Let P be a poset and I be an ideal of P. Next, let d be a permuting tri-derivation on P. Then $d(x,y,z) \in I$ for all $x, y, z \in I$.*

Proof. Let $x, y, z \in I$. Then in view of Proposition 2.1 (1), we get $d(x, y, z) \leq x$ for all $x, y, z \in I$. The last expression yields $d(x, y, z) \in I$, since $x \in I$. Hence, the result holds. □

Lemma 2.2. *Let d be a permuting tri-derivation on P with trace ϕ. Then the following statements hold:*

(1) *If $d(l(x), x, x) = l(y)$, then $\phi(x) = y$ for all $x, y \in P$;*
(2) *If $d(u(x), x, x) = u(y)$, then $\phi(x) = y$ for all $x, y \in P$.*

Proof. (1) Let $x, y \in P$ such that $d(l(x), x, x) = l(y)$. Then, by the definition of $l(\cdot)$, we get $y \in l(y)$ for all $y \in P$. This gives $y \in d(l(x), x, x)$. Hence, there exists $z \in l(x)$ such that $d(z, x, x) = y$. Application of Proposition 2.1(3) yields $y = d(z, x, x) \leq d(x, x, x) = \phi(x)$ for $x \in P$. Therefore, the above relation forces that $y \leq \phi(x)$ for all $x, y \in P$. On the other hand if $\phi(x) \in d(l(x), x, x) = l(y)$, then we obtain $\phi(x) \leq y$. Hence $\phi(x) = y$ for all $x, y \in P$.

(2) By using comparable approach with fundamental variety, we can prove (2). □

Theorem 2.1. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. Then $\phi(1) = 1$ if and only if $d(x, 1, 1) = x$ for all $x \in P$.*

Proof. By the assumption, $\phi(1) = d(1, 1, 1) = 1$. In view of Proposition 2.1(1), it is easy to see that $d(x, 1, 1) \leq x$ for all $x \in P$. Secondly, to prove that $x \leq d(x, 1, 1)$ for all $x \in P$. Let $x \in P$. Then, we have

$$\begin{aligned} d(l(x), 1, 1) &= d(l(x, 1), 1, 1) \\ &= l(u(l(d(x, 1, 1), 1), l(x, d(1, 1, 1))) \\ &= l(u(l(d(x, 1, 1), 1), l(x, 1)) \\ &= l(u(l(d(x, 1, 1)), l(x)) \\ &= l(u(l(x))) \quad (\text{since } d(x, 1, 1) \leq x) \\ &= l(x). \end{aligned}$$

By another way, observe that

$$\begin{aligned} d(l(x), 1, 1) &= d(l(x, x), 1, 1) \\ &= l(u(l(d(x, 1, 1), x), l(x, d(x, 1, 1))) \\ &= l(u(l(d(x, 1, 1))), l(d(x, 1, 1))) \\ &= l(u(l(d(x, 1, 1)))) \\ &= l(d(x, 1, 1). \end{aligned}$$

On comparing the above two expressions, we get $l(x) = l(d(x, 1, 1))$ for all $x \in P$. Hence $d(x, 1, 1) = x$ for all $x \in P$. The converse part is clear. □

Theorem 2.2. *Let P be a poset with a least element 0 and a greatest element 1. Next, let d be a permuting tri-derivation on P. Then $d(1, 0, 0) = 0$ if and only if $d(x, 0, 0) = 0$ for all $x \in P$.*

Proof. Suppose that $d(1,0,0) = 0$ and $x \in P$. Then

$$\begin{aligned} d(l(1),0,0) &= d(l(1,1),0,0) \\ &= l(u(l(d(1,0,0),1),l(1,d(1,0,0)))) \\ &= l(u(l(0,1),l(1,0)) \\ &= l(u(l(0),l(0)) = l(u(l(0))) \\ &= l(0) = \{0\}. \end{aligned}$$

But $l(1) = P$ and $x \in P$, the above relation gives $d(x,0,0) \in d(l(1),0) = l(0) = \{0\}$. Hence, $d(x,0,0) = 0$ for all $x \in P$. For the converse part, proof is obvious. □

Theorem 2.3. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. If $x \leq \phi(1)$, then $d(x,1,1) = x$ for all $x \in P$.*

Proof. Let $x \leq \phi(1) = d(1,1,1)$ for all $x \in P$. Then for all $x \in P$, we have

$$\begin{aligned} d(l(x),1,1) &= d(l(x,1),1,1) \\ &= l(u(l(d(x,1,1),1),l(x,d(1,1,1)))) \\ &= l(u(l(d(x,1,1)),l(x))) \\ &= l(u(l(x))) \quad (\text{since } d(x,1,1) \leq x) \\ &= l(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(l(x),1,1) &= d(l(x,x),1,1) \\ &= l(u(l(d(x,1,1),x),l(x,d(x,1,1)))) \\ &= l(u(l(d(x,1,1)),l(d(x,1,1)))) \\ &= l(u(l(d(x,1,1)))) \\ &= l(d(x,1,1)). \end{aligned}$$

By comparing the above two expressions, we infer that $l(d(x,1,1)) = l(x)$. Hence, $d(x,1,1) = x$ for all $x \in P$. This proves the theorem completely. □

Corollary 2.2. *Let P be a poset with a greatest element 1 and d be a permuting tri-derivation on P with trace ϕ. Then $\phi(1) = 1$ if and only if $\phi = id_P$ (identity map on P).*

Proof. Assume that $\phi(1) = d(1,1,1) = 1$. Now we prove that $x = \phi(x) = d(x,x,x)$ for all $x \in P$. Let $x \in P$. Then, we have

$$\begin{aligned} d(l(x),x,x) &= d(l(x,1),x,x) \\ &= l(u(l(d(x,x,x),1),l(x,d(1,x,x))) \\ &= l(u(l(d(x,x,x)),l(d(1,x,x)) \\ &= l(u(l(d(1,x,x))) \quad (\text{since } d(x,x,x) \leq d(1,x,x)) \\ &= l(d(1,x,x)). \end{aligned}$$

By another way, observe that

$$\begin{aligned} d(l(x),x,x) &= d(l(x,x),x,x) \\ &= l(u(l(d(x,x,x),x),l(x,d(x,x,x))) \\ &= l(u(l(d(x,x,x))),l(d(x,x,x))) \\ &= l(u(l(d(x,x,x)))) \\ &= l(d(x,x,x). \end{aligned}$$

On comparing the above two expressions, we get $l(d(x,x,x)) = l(d(1,x,x))$ for all $x \in P$. Hence $d(x,x,x) = d(1,x,x)$ for all $x \in P$. Again

$$\begin{aligned} d(l(x),x,1) &= d(l(x,1),x,1) \\ &= l(u(l(d(x,x,1),1),l(x,d(1,x,1))) \\ &= l(u(l(d(x,x,1)),l(d(1,x,1)) \\ &= l(u(l(d(1,x,1))) \quad \text{(since } d(x,x,1) \leq d(1,x,1)) \\ &= l(d(1,x,1)). \end{aligned}$$

Similarly we observe that

$$\begin{aligned} d(l(x),x,1) &= d(l(x,x),x,1) \\ &= l(u(l(d(x,x,1),x),l(x,d(x,x,1))) \\ &= l(u(l(d(x,x,1))),l(d(x,x,1))) \\ &= l(u(l(d(x,x,1)))) \\ &= l(d(x,x,1). \end{aligned}$$

From the above two expressions, we get $l(d(1,x,1)) = l(d(x,x,1))$ for all $x \in P$. So $d(1,x,1) = d(x,x,1)$ for all $x \in P$. Since d is permuting map then $d(1,x,1) = d(x,1,1) = d(1,x,x) = d(x,x,1)$ for all $x \in P$. Hence, $\phi(x) = d(x,x,x) = d(x,1,1)$ for all $x \in P$. Applications of Theorem 2.3 gives $\phi(x) = x$ for all $x \in P$, i.e., $\phi = id_P$. The converse part is obvious. □

Theorem 2.4. *Let P be a poset and $d : P \times P \times P \to P$ be a permuting map. Then, d is a permuting tri-derivation on P if and only if*

(1) *$d(l(x,y),z,w)) = l(d(x,z,w),y)) = l(x,d(y,z,w))$ for all $x,y,z,w \in P$;*
(2) *$l(d(u(x,y),z,w))) = l(u(d(x,z,w),d(y,z,w)))$ for all $x,y,z,w \in P$.*

Proof. Essentially ought to appear that the condition (1) in Definition 2.1 is identical to the one (1) in this hypothesis. First, we suppose that the condition in this hypothesis holds. Then

$$\begin{aligned} d(l(x,y),z,w) &= l(d(x,z,w),y) \\ &= l(u(l(d(x,z,w),y))) \\ &= l(u(l(d(x,z,w),y),l(x,d(y,z,w)))), \end{aligned}$$

for all $x, y, z, w \in P$. Secondly, suppose that d is a permuting tri-derivation on P. Then

$$\begin{aligned} l(d(x,y,z),w) &= l(u(l(d(x,y,z),w))) \\ &\subseteq l(u(l(d(x,y,z),w), l(x,d(y,z,w)))) \\ &= d(l(x,w),y,z)), \end{aligned}$$

for all $x, y, z, w \in P$. On the other hand, suppose that $v \in d(l(x,y),z,w))$, then there exists $t \in l(x,y)$ satisfying the relation $d(t,z,w) = v$. By using Proposition 2.1 (1) and (3), it is easy to see that $d(t,z,w) \leq d(x,z,w)$, $d(t,z,w) \leq d(y,z,w) \leq y$. This shows that $v = d(t,z,w) \in l(d(x,z,w),y)$. Thus $d(l(x,y),z,w)) \subseteq l(d(x,z,w),y)$. Hence,

$$d(l(x,y),z,w)) = l(d(x,z,w),y), \quad \text{for all } x,y,z,w \in P.$$

Similarly, the case $d(l(x,y),z,w)) = l(x,d(y,z,w))$ for all $x, y, z, w \in P$. This proves the theorem. □

Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. Put $Fix_\phi(P) = \{x \in P : \phi(x) = x\}$. If P has a least element 0, then $0 \in Fix_\phi(P)$. In view of Proposition 2.1, it is easy to get $Fix_\phi(P) \neq \emptyset$.

Proposition 2.2. *Let d, t be two permuting tri-derivations on P with traces ϕ_1, ϕ_2, respectively. Then $\phi_1 = \phi_2$ if and only if $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$.*

Proof. It is clear that if $\phi_1 = \phi_2$, then $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$. Conversely, assume that $Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$, and $x \in P$. Then by Proposition 2.1 (9), obtain $\phi_1(x) \in Fix_{\phi_1}(P) = Fix_{\phi_2}(P)$. This implies that $\phi_2(\phi_1(x)) = \phi_1(x)$. By a similar way we get $\phi_1(\phi_2(x)) = \phi_2(x)$ for all $x \in P$. Application of Proposition 2.1 (6), (8) yields that $\phi_1(x) \leq \phi_2(x)$ and $\phi_2(x) \leq \phi_1(x)$ for all $x \in P$. Consequently, $\phi_1 = \phi_2$. □

Proposition 2.3. *Let P be a poset with a least element 0 and d be a permuting tri-derivation on P with trace ϕ. Then the followings hold.*

(1) *$Fix_\phi(P) \neq \emptyset$.*
(2) *If $x \in Fix_\phi(P)$, and $y \leq x$ then $y \in Fix_\phi(P)$.*
(3) *If P is directed, then, for any $x, y \in Fix_\phi(P)$, there exists $z \in Fix_\phi(P)$ satisfying $x \leq z$, $y \leq z$.*

Proof. (1) Since $\phi(0) = d(0,0,0) = 0$, then $0 \in Fix_\phi(P)$. Thus, $Fix_\phi(P) \neq \emptyset$.

(2) Assume that $x \in Fix_\phi(P)$, and $y \leq x$ then $\phi(x) = d(x,x,x) = x$. Then using Proposition 2.1 (6) implies that $\phi(y) \leq y$. Now prove that $y \leq \phi(y)$. Using Theorem 2.4 (1), to get $d(l(y),x,x) = d(l(x,y),x,x) = l(d(x,x,x),y) = l(x,y) = l(y)$. Since $y \in l(y)$, so $y \in d(l(y),x,x)$ and this leads to $y \leq d(y,x,x)$. Hence $d(y,x,x) = y$. Again by using Theorem 2.4 (1) get $d(l(y),y,y) = d(l(x,y),y,y) = l(d(x,y,y),y) = l(d(x,y,y)))$. Application of Lemma 2.2 (1) yields that $\phi(y) = d(x,y,y)$. Thus, using

Theorem 2.4 (2) implies that

$$\begin{aligned} l(d(u(y), y, x)) &= l(d(u(y, y), y, x)) \\ &= l(u(d(y, y, x), d(y, y, x))) \\ &= l(u(d(y, y, x))). \end{aligned}$$

Since $d(y, y, x) = d(x, y, y)$, $d(y, x, x) = d(x, y, x) \in l(d(u(y), y, x))$, and this leads to $d(y, x, x) \in l(u(d(x, y, y)))$. Thus, $y = d(y, x, x) \leq d(x, y, y) = \phi(y)$. Hence, $y \in Fix_\phi(P)$.

(3) Assume that P is directed. Then for any $x, y \in P$, there exists $v \in P$ such that $x \leq v$ and $y \leq v$. Since $x, y \in Fix_\phi(P)$, then $\phi(x) = x$ and $\phi(y) = y$. Since $\phi(x) = x \leq \phi(v)$ and $\phi(y) = y \leq \phi(v)$. Put $z = \phi(v)$, hence by Proposition 2.1 (7) we get $z \in Fix_\phi(P)$. □

Corollary 2.3. *Let P be a directed poset with the least element 0. Then $Fix_\phi(P)$ is an ideal of P.*

3. Structural Properties of Posets Including Permuting Tri-Derivations

In this section, P is a poset with the least element 0.

Theorem 3.1. *Let P be a poset with the least element 0 and d be a permuting tri-derivation on P with trace ϕ. Then $\ker \phi = \{x \in P : \phi(x) = 0\}$ is a nonempty lower set of P.*

Proof. In view of Proposition 2.1, $\phi(0) = d(0, 0, 0) = 0$. Thus, $0 \in \ker \phi$, and hence $\ker \phi \neq \emptyset$. Suppose that $x \in \ker \phi$ and $y \in P$ such that $y \leq x$. Then $\phi(x) = 0$ and $y \leq x$. Using Proposition 2.1 (8) to get $\phi(y) \leq \phi(x) = 0$. Thus, $\phi(y) = 0$ for all $y \in P$. This shows that $y \in \ker \phi$. Hence, $\ker \phi = \{x \in P : \phi(x) = 0\}$ is a nonempty lower set of P. □

Proposition 3.1. *Let P be a poset with the least element 0. Next, let d be a permuting tri-derivation on P with trace ϕ and I be an ideal of P. Then, $\phi^{-1}(I)$ is an ideal of P such that $\ker \phi \subseteq \phi^{-1}(I)$.*

Proof. Since $\phi(0) = 0$, $0 \in \phi^{-1}(I)$. Then, $\phi^{-1}(I) \neq \emptyset$. Suppose $x \in \phi^{-1}(I)$ and $y \leq x$. Then $\phi(x) \in I$. Thus, using Proposition 2.1 (8), to obtain $\phi(y) \leq \phi(x) \in I$. Since I is an ideal, hence $\phi(y) \in I$, and this leads to $y \in \phi^{-1}(I)$. Hence, $\phi^{-1}(I)$ is an ideal of P. On the other hand, note that $\ker \phi = \phi^{-1}(\{0\}) \subseteq \phi^{-1}(I)$. □

Proposition 3.2. *Let P be a poset and d be a permuting tri-derivation on P with trace ϕ. If I, J are two ideals of P such that $I \subseteq J$, then $\phi(I) \subseteq \phi(J)$.*

Proof. Assume that $x \in \phi(I)$, then there exists $y \in I \subseteq J$ such that $x = \phi(y)$. Hence, $x \in \phi(J)$. This implies that $\phi(I) \subseteq \phi(J)$. □

Theorem 3.2. *Let P be a poset and d_1, d_2 be two permuting tri-derivations on P with traces ϕ_1, ϕ_2, respectively. Then $\phi_1(x) \leq \phi_2(x)$ for all $x \in P$ if and only if $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$.*

Proof. Let d_1, d_2 be two permuting tri-derivations on P, with traces ϕ_1, ϕ_2, respectively, such that $\phi_1 \leq \phi_2$. Then, for any $x \in P$, $\phi_1(x) \in Fix_{\phi_1}(P)$, i.e., $\phi_1(x) = \phi_1(\phi_1(x)) \leq \phi_2(\phi_1(x))$. Proposition 2.1 (6) gives that $\phi_2(\phi_1(x)) \leq \phi_1(x)$. Thus, $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$. This shows that $\phi_2(\phi_1(x)) = \phi_1(x)$ for all $x \in P$. On the other hand we find that $\phi_1(x) = \phi_2(\phi_1(x)) \leq \phi_2(x)$, for any $x \in P$, from Proposition 2.1 (6), (8). This implies that $\phi_1(x) \leq \phi_2(x)$ for all $x \in P$. This completes the proof of the theorem. ☐

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References

[1] M. Ashraf, S. Ali and C. Haetinger, *On derivations in rings and their applications*, Aligarh Bull. Math. **2**5(2) (2006), 79–107.
[2] M. Ashraf and M. A. Siddeeque, *On permuting n-derivations in near-rings*, Commun. Korean Math. Soc. **2**8(4) (2013), 697–707. `https://doi.org/10.4134/CKMS.2013.28.4.697`
[3] T. Albu and M. L. Teply, *Generalized derivation of posets and modular lattices*, Discrete Math. **2**14 (2000), 1–19. `https://doi.org/10.1016/S0012-365X(99)00194-6`
[4] M. Asci, O. Kecilioglu and S. Ceran, *Permuting tri-(f, g)-derivations on lattices*, Ann. Fuzzy Math. Inform. **1**(2) (2011), 189–196.
[5] I. Chajda and J. Rachünek, *Forbidden configurations for distributive and modular ordered sets*, Nathematics **5** (1989), 407–423. `https://doi.org/10.1007/BF00353659`
[6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge United Kingdom, 2003.
[7] M. A. Öztürk, *Permuting tri-derivations in prime and semi-prime rings*, East Asian Math. J. **1**5(2) (1999), 177–190.
[8] M. A. Öztürk, H. Yazarli and K. H. Kim, *Permuting tri-derivations in lattices*, Quaest. Math. **3**2 (2009), 415–425. `https://doi.org/10.2989/QM.2009.32.3.10.911`
[9] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
[10] G. Szasz, *Derivations of lattices*, Acta Sci. Math. **37** (1975), 149–154.
[11] X. L. Xin, T. Y. Li and J. H. Lu, *On derivations of lattices*, Inform. Sci. **1**78 (2008), 307–316. `https://doi.org/10.1016/j.ins.2007.08.018`
[12] X. L. Xin, *The fixed set of a derivation in lattices*, J. Fixed Point Theory Appl. **1** (2012), 1–12.
[13] H. Yazarli and M. A. Öztürk, *Permuting tri-f-derivations in lattices*, Commun. Korean Math. Soc. **2**6(1) (2011), 13–21. `https://doi.org/10.4134/CKMS.2011.26.1.013`
[14] H. Zhang and Q. Li, *On derivations of partially ordered sets*, Math. Slovaca **6**7(1) (2017), 17–22. `https://doi.org/10.1515/ms-2016-0243`

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