Kragujevac Journal of Mathematics
Volume 49(3) (2025), Pages 365–373.

POLYNOMIAL WEIGHTED APPROXIMATION BY SZÁSZ-MIRAKYAN OPERATORS OF MAX-PRODUCT TYPE

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Abstract. In this paper, we study approximation of Szász-Mirakyan operators of max-product type in polynomial weighted spaces. We reckon the rate of approximation in terms of some exponential weighted spaces for obtain a better rate of approximation than the corresponding positive linear operators.

1. Introduction

In [9], the Szász-Mirakjan operators were defined as below

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0,\infty), n \geq 1. \tag{1.1}$$

Many studies have been done about the approximation results for this operators and estimates of the rate of convergence. These studies are mainly using positive linear operators. However, nonlinear operators of max-product type were studied in the papers [2–4] and the conclusion is that they have the same order of approximation as in the case of positive linear operators and even better for some subclasses of functions. In [3], the authors investigated the nonlinear operators of Favard-Szász-Mirakjan of max-product type defined by

$$F_n(f,x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}, \quad x \in [0,\infty), \tag{1.2}$$

Key words and phrases. Szász-Mirakyan operators, weighted modulus of continuity, weighted approximation

2020 *Mathematics Subject Classification.* Primary: 41A10, Secondary: 41A25, 41A36.

DOI 10.46793/KgJMat2503.365A

Received: January 03, 2020.

Accepted: April 08, 2022.

where $\bigvee$ indicates the supremum. Moreover, they studied these operators for continuous and bounded functions defined on $x \geq 0$ in [3, 4]. In [5], A. Holhoş studied the approximation properties of F_n in weighted spaces with the weight $w(x) = e^{\alpha\varphi(x)}$, where $\varphi(x) = \sqrt{x}$ and $\alpha > 0$ is constant independent of x (see [1]). In [8], he introduced a new modulus of continuity. In [5], the author estimated the rate of convergence of these operators to the identity operator . Firstly, he introduce some general notations to obtain the results.

The function $\varphi : I \to J$ is defined on a noncompact interval $I \subseteq \mathbb{R}$. The interval $J \subseteq \mathbb{R}$ is just $\varphi(I)$. The space of continuous functions is defined as

$$C_{\varphi,\alpha} = \left\{ f \in C(I) \text{ there is } M > 0 \text{ such that } \frac{|f(x)|}{e^{\alpha\varphi(x)}} \leq M \text{ for every } x \in I \right\}. \tag{1.3}$$

This space can be endowed with the norm

$$\|f\|_{\varphi,\alpha} = \sup_{x \in I} e^{-\alpha\varphi(x)} |f(x)|. \tag{1.4}$$

The modulus of continuity $\omega_{\varphi,\alpha}(f; \cdot)$ is given for every $f \in C_{\varphi,\alpha}$ and $\delta \geq 0$ as follows

$$\omega_{\varphi,\alpha}(f; \delta) = \frac{|f(t) - f(x)|}{\max\left(e^{\alpha\varphi(t)}, e^{\alpha\varphi(x)}\right)}, \tag{1.5}$$

which the supremum is taken for all $x, t \in I$ such that $\varphi(t) \in (\varphi(x) - \delta, \varphi(x) + \delta) \cap \varphi(I)$. For $\alpha = 0$ and $\varphi(x) = x$, we obtain the usual modulus of continuity $\omega(f; \delta)$.

In this paper, our main problem is that the operators F_n can be used for approximation with polynomial weight $w(x) = (1 + x)^\alpha$ by taking $\varphi(x) = \ln(1 + x)$ and we estimate the rate of convergence of these operators to the identity operator. Hence, we show approximation of Szász-Mirakyan operators of max-product type in polynomial weighted spaces.

2. Polynomial Weighted by Szász-Mirakyan operators

In this section, we prove some auxiliary results to obtain some estimates of the rate of approximation of functions given by (1.1) and (1.2).

Remark 2.1. For $n \in \mathbb{N}$ take the intervals

$$I_0 = \left[0, \left(\frac{n}{n+1}\right)^\alpha\right), \quad I_k = \left[k\left(\frac{n+k-1}{n+k}\right)^\alpha, (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha\right]. \tag{2.1}$$

The intervals are nonempty, disjoint and their union is the positive half line. Indeed,

$$l_k = (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha - k\left(\frac{n+k-1}{n+k}\right)^\alpha \geq 0.$$

Lemma 2.1. *If* $nx \in I_j$, *then* $\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha = \frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha$.

Proof. Let us denote $a_k = \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha$. We get

$$0 \le a_{k+1} \le a_k \text{ if and only if } nx \in \left[0, (k+1)\left(\frac{n+k}{n+k+1}\right)^\alpha\right).$$

Let us take $k = 0, 1, \ldots$ We obtain

$$\begin{aligned} a_1 \le a_0 &\text{ if and only if } nx \in \left[0, \left(\frac{n}{n+1}\right)^\alpha\right), \\ a_2 \le a_1 &\text{ if and only if } nx \in \left[0, 2\left(\frac{n+1}{n+2}\right)^\alpha\right), \\ a_3 \le a_2 &\text{ if and only if } nx \in \left[0, 3\left(\frac{n+2}{n+3}\right)^\alpha\right), \end{aligned}$$

and so on. From all these inequalities, we get

$$\begin{aligned} &\text{if } nx \in I_0, \text{ then } a_k \le a_0, \quad \text{for all } k = 0, 1, \ldots, \\ &\text{if } nx \in I_1, \text{ then } a_k \le a_1, \quad \text{for all } k = 0, 1, \ldots, \\ &\text{if } nx \in I_2, \text{ then } a_k \le a_2, \quad \text{for all } k = 0, 1, \ldots, \end{aligned}$$

and so on. Generally, if $nx \in I_j$, then $a_k \le a_j$, for all $k = 0, 1, \ldots$, that proves the lemma. □

Lemma 2.2. *For every $x \ge 0$ we obtain $F_n\left((1+t)^\alpha, x\right) \le (1+x)^\alpha \left(1 + \frac{\alpha}{n}\right)^\alpha$.*

Proof. Let us take $nx \in I_j$. By using Lemma 2.1, we obtain

$$F_n\left((1+t)^\alpha, x\right) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} = \frac{\frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha}{\frac{(nx)^m}{m!}}.$$

Let us take $m = \lfloor nx \rfloor$. So, we have $m \le nx < (j+1)\left(\frac{n+k}{n+k+1}\right)^\alpha < j+1$ and we can say that $nx \le \lfloor nx \rfloor + 1 = m+1$, hence we obtain $\frac{nx}{m+1} \le 1$. By using Bernoulli inequality, we have

$$j - nx \le j - j\left(\frac{n+j-1}{n+j}\right)^\alpha = j\left(1 - \left(1 - \frac{1}{n+j}\right)^\alpha\right) \le j\alpha\frac{1}{n+j} \le \alpha.$$

Hence, we get

$$\begin{aligned} e^{-\alpha\ln(1+x)} F_n\left((1+t)^\alpha, x\right) &= \frac{1}{(1+t)^\alpha} \frac{\frac{(nx)^j}{j!}\left(\frac{n+j}{n}\right)^\alpha}{\frac{(nx)^m}{m!}} \\ &\le \left(\frac{nx}{m+1}\right)^{j-m}\left(\frac{n+j}{n(1+x)}\right)^\alpha \le \left(\frac{n+j}{n+nx}\right)^\alpha \\ &= \left(\frac{n+j+nx-nx}{n+nx}\right)^\alpha = \left(1 + \frac{j-nx}{n+nx}\right)^\alpha \\ &\le \left(1 + \frac{\alpha}{n}\right)^\alpha. \end{aligned}$$

□

Remark 2.2. For every $x \geq 0$, we have

$$F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right) \leq (1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha.$$

Indeed,

$$\begin{aligned} F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right) &= \max\left\{F_n\left((1+t)^\alpha,x\right),F_n\left((1+x)^\alpha,x\right)\right\} \\ &\leq \max\left\{(1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha,(1+x)^\alpha\right\} \\ &= (1+x)^\alpha\left(1+\frac{\alpha}{n}\right)^\alpha. \end{aligned}$$

Remark 2.3. For $\varphi(x)=\ln(1+x)$, for every function f belonging to $C_{\varphi,\alpha}$ the functions $F_n f$ also belonging to $C_{\varphi,\alpha}$. Indeed,

$$\begin{aligned} |F_n(f,x)| \leq F_n(|f|,x) &\leq F_n\left(\|f\|_{\varphi,\alpha}(1+x)^\alpha,x\right) \\ &= \|f\|_{\varphi,\alpha} F_n\left((1+x)^\alpha,x\right) = \|f\|_{\varphi,\alpha}\left(1+\frac{\alpha}{n}\right)^\alpha(1+x)^\alpha. \end{aligned}$$

Lemma 2.3. *For every $x \geq 0$ and $n \in \mathbb{N}$, the following inequality is obtained*

$$\frac{\bigvee_{k\leq nx}\frac{(nx)^k}{k!}\left(\ln(1+(nx))-\ln(1+k)\right)}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}} \leq \frac{1}{2}.$$

Proof. For $x=0$, we obtain equality. By taking $m=\lfloor nx\rfloor$, we proved that $\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}=\frac{(nx)^m}{m!}$. Let us consider the inequality $\ln(1+x)\leq x$, then we get

$$\ln(1+(nx))-\ln(1+k)=\ln\left(\frac{1+(nx)}{1+k}\right) \leq \frac{(nx)-k}{1+k}.$$

Let us take $b_k=\frac{(nx)^k}{k!}\cdot\frac{(nx)-k}{1+k}$. Firstly, to evaluate the maximum of b_k, we observe that

$$\frac{b_k}{b_{k-1}}=\frac{nx}{k+1}\cdot\frac{nx-k}{nx-k+1} \leq 1$$

if and only if $(nx)^2-(2k+1)(nx)+(k^2-1) \leq 0$. This inequality's solution is equivalent to $nx \in [p_k,q_k]$ which is

$$p_k=\frac{(2k+1)-\sqrt{4k+5}}{2}, \quad q_k=\frac{(2k+1)+\sqrt{4k+5}}{2}.$$

We can write the following inequality

$$0 \leq p_k < p_{k+1} < \frac{2k+1}{2} < \frac{2k+3}{2} < q_k < q_{k+1} \leq 1, \quad \text{for all } k \geq 0.$$

After some computations, we get $nx \in \left[\frac{2m-1}{2}, \frac{2m+1}{2}\right)$. We deduce that if $nx \in I_j$, then $b_k \le b_j$ for every $k \ge 1$. We obtain

$$\frac{\bigvee_{1\le k\le nx} b_k}{\frac{(nx)^m}{m!}} \le \left(\frac{2j+1}{2} - j\right) = \frac{1}{2}. \qquad \square$$

Lemma 2.4. *For $x \ge 0$, the following inequality holds true*

$$\frac{\bigvee_{k>nx} \frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha (\ln(1+k) - \ln(1+nx))}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}} \le \left(1 + \frac{2\alpha}{n}\right)^\alpha (1+x)^\alpha .$$

Proof. For $x = 0$ we have equality. Let us take $x > 0$. Consider $m \ge 0$ the integer with the property that $nx \in \left[\frac{2m-1}{2}, \frac{2m+1}{2}\right)$. Using the inequality we get

$$\ln(1+k) - \ln(1+nx) \le \frac{k-nx}{1+nx},$$

and denoting

$$c_k = \frac{\frac{(nx)^k}{k!}}{\frac{(nx)^m}{m!}}\left(\frac{n+k}{n(1+x)}\right)^\alpha \frac{k-nx}{1+nx},$$

it remains to prove that

$$\bigvee_{k=m+1}^{\infty} c_k \le \left(1 + \frac{2\alpha}{n}\right)^\alpha .$$

Let us take the inequality

$$\frac{c_{k+1}}{c_k} = \frac{nx}{k+1}\left(\frac{n+k+1}{n+k}\right)^\alpha \frac{k+1-nx}{k-nx} \ge 1$$

if and only if

$$\alpha_k (nx)^2 - (\alpha_k(k+1) + k + 1)(nx) + k(k+1) \le 0,$$

where $\alpha_k = \left(\frac{n+k+1}{n+k}\right)^\alpha$. Hence, $c_{k+1} \ge c_k$ is true if and only if $nx \in [r_k, s_k]$, where

$$r_k = \frac{k+1}{2} + \frac{k+1}{2\alpha_k} - \sqrt{E_k}, \quad s_k = \frac{k+1}{2} + \frac{k+1}{2\alpha_k} + \sqrt{E_k}$$

and

$$E_k = \frac{(\alpha_k(k+1) + k + 1)^2 - 4\alpha_k k(k+1)}{4\alpha_k^2}.$$

Now, we prove below that

$$0 < r_k < r_{k+1} < \frac{k+2}{2} < s_k. \tag{2.2}$$

By using (2.2) we deduce that $r_m < \frac{m}{2} \le nx$. Let us take the unique $j \ge m$ such that $x \in [r_j, r_{j+1})$. For every $k \ge j+1$, we get $nx \in [r_k, s_k]$, so $c_{k+1} \le c_k$. We obtain that $c_k \le c_{j+1}$ for every $k \ge j+1$. Now consider $k \in \{m, \ldots, j\}$. Using (2.2) again, we get $r_k \le r_j \le nx < \frac{m+2}{2} < \frac{k+2}{2} < s_k$. Since $nx \in [r_k, s_k]$, we obtain $c_{k+1} \ge c_k$ and so $c_{j+1} \ge c_k$ for every $k \in \{m, \ldots, j\}$.

Now, we need some estimates to evaluate the maximum of c_k for $k \geq m+1$. We have

$$\begin{aligned} j+1-r_j =& j+1-\frac{j+1}{2}-\frac{j}{2\alpha_j}+\sqrt{E_k} \\ =& \frac{\alpha_j(j+1)-j}{2\alpha_j}+\sqrt{\frac{(\alpha_k(k+1)+k)^2-4\alpha_k k^2}{4\alpha_k^2}} \\ \leq & \frac{n^2+1}{2n}, \end{aligned}$$

because

$$\alpha_j(j+1)=(j+1)\left(e^{\alpha\ln\left(1+\frac{1}{n+j}\right)}\right)\leq (j+1)\alpha\frac{1}{n+j}\leq\alpha.$$

Consequently,

$$\begin{aligned} \bigvee_{k=m+1}^{\infty} c_k =& c_{j+1} \\ =& \frac{\frac{(nx)^{j+1}}{(j+1)!}}{\frac{(nx)^m}{m!}}\left(\frac{n+j+1}{n+nx}\right)^{\alpha}\frac{(j+1-nx)}{1+nx} \\ \leq & \left(\frac{n+j+1}{n+r_j}\right)^{\alpha}\frac{j+1-r_j}{1+r_j}\leq\left(1+\frac{2\alpha}{n}\right)^{\alpha}. \end{aligned}$$

Let us consider the inequality (2.2). The most difficult to prove is the inequality $r_{k+1} > r_k$, other statuses as in [6]. We have

$$\begin{aligned} r_{k+1}-r_k =& \frac{k+2}{2}+\frac{k+2}{2\alpha_{k+1}}-\sqrt{E_{k+1}}-\frac{k+1}{2}+\frac{k+1}{2\alpha_k}-\sqrt{E_k} \\ =& \frac{1}{2}\cdot\frac{(k+2)\alpha_k-(k+1)(\alpha_{k+1})}{2\alpha_k\alpha_{k+1}}+\frac{E_k-E_{k+1}}{\sqrt{E_k}-\sqrt{E_{k+1}}} \\ =& \frac{\alpha_k(k+2)\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)-(k+1)\alpha_{k+1}\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)}{4\alpha_k\alpha_{k+1}} \\ &+\frac{(E_k-E_{k+1})\,4\alpha_k\alpha_{k+1}}{4\alpha_k\alpha_{k+1}\left(\sqrt{E_k}+\sqrt{E_{k+1}}\right)}. \end{aligned}$$

This equality's first half is positive, it is clear. For positivity of the second part of the equality, let us take $\frac{k+2}{2} < s_k$ and $r_{k+1} < \frac{k+2}{2}$ from (2.2), then we get

$$\sqrt{E_k} > \frac{\alpha_k-k}{2\alpha_k},\quad \sqrt{E_{k+1}} > \frac{k+1}{2\alpha_{k+1}}.$$

Hence, we proved the lemma. □

Lemma 2.5. *For every $x \geq 0$ and $n \in \mathbb{N}$, we get*

$$F_n\left(\max\left\{(1+t)^{\alpha},(1+x)^{\alpha}\right\}\left|\ln(1+t)-\ln(1+x)\right|,x\right)\leq\max\left\{\left(1+\frac{2\alpha}{n}\right)^{\alpha},\frac{1}{2}\right\}(1+x)^{\alpha}.$$

Proof. We have

$$F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}\left|\ln(1+t)-\ln(1+x)\right|,x\right)=\max\left\{A_n,B_n\right\},$$

where

$$\begin{aligned}A_n=&\frac{\bigvee_{k>nx}\frac{(nx)^k}{k!}\left(\frac{n+k}{n}\right)^\alpha(\ln(1+k)-\ln(1+nx))}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}},\\ B_n=&\frac{\bigvee_{k\le nx}\frac{(nx)^k}{k!}(1+x)^\alpha(\ln(1+nx)-\ln(1+k))}{\bigvee_{k=0}^{\infty}\frac{(nx)^k}{k!}}.\end{aligned}$$

By Lemma 2.4 we have $\left(1+\frac{2\alpha}{n}\right)^\alpha$ and by Lemma 2.3, $\frac{1}{2}(1+x)^\alpha$. □

Theorem 2.1. *For $\varphi(x)$, for every $f\in C_{\varphi,\alpha}$ the estimation of the error of uniform approximation by F_n is bounded by*

$$\|F_nf-f\|_{\varphi,\alpha}\le\left(\left(1+\frac{2\alpha}{n}\right)^\alpha(1+x)^\alpha+\frac{1}{2}(1+x)^\alpha\right)\omega_{\varphi,\alpha}\left(f,\frac{1}{\sqrt{n}}\right),$$

for every $n\in\mathbb{N}$.

Proof. Because $F_n(1,x)=1$, using ([3], Lemma 2.1) we get

$$\begin{aligned}|F_n(f;x)-f(x)|\le&F_n\left(|f(t)-f(x)|,x\right)\\ \le&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}\left(1+\frac{|\varphi(t)-\varphi(x)|}{\delta_n}\right),x\right)\omega_{\varphi,\alpha}(f,\delta_n)\\ \le&\left(C_n+\frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f,\delta_n),\end{aligned}$$

which

$$\begin{aligned}C_n(x)=&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\},x\right),\\ D_n(x)=&F_n\left(\max\left\{(1+t)^\alpha,(1+x)^\alpha\right\}|\varphi(t)-\varphi(x)|,x\right).\end{aligned}$$

Using Remark 2.2 and Lemma 2.5 and choosing $\delta_n=\frac{1}{\sqrt{n}}$, we have

$$\frac{1}{(1+x)^\alpha}|F_n(f;x)-f(x)|\le\left(\left(1+\frac{2\alpha}{n}\right)^\alpha+\frac{1}{2}\right)\omega_{\varphi,\alpha}\left(f,\frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

Remark 2.4. Let us take consideration that for polynomial weighted of the operator given in (1.2) the order of approximation is better that $\frac{1}{\sqrt{n}}$. From [3], we deduce that the estimate

$$|F_n(f,x)-f(x)|\le\frac{M}{n},\quad n\ge1,$$

is true for a positive, increasing, concave and Lipschitz function f, which is not necessarily bounded.

Theorem 2.2. *For $f \in C_{\varphi,\alpha}$ we have*

$$\|S_n f - f\|_{\varphi,\alpha} \leq C_\alpha \cdot \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $n \in \mathbb{N}$, where $C > 0$ is a constant.

Proof. We know that $S_n(e^{\alpha\sqrt{t}}, x) \leq M_\alpha \cdot e^{\alpha\sqrt{x}}$, which $M_\alpha > 0$ is a constant depending only on α in [1], Lemma 3.1 and in [7], similar inequality given for $\alpha = 0$. We get

$$\begin{aligned}
|S_n(f,x) - f(x)| \leq & S_n\left(|f(t) - f(x)|, x\right) \\
\leq & S_n\left(((1+t)^\alpha + (1+x)^\alpha)\left(1 + \frac{\varphi(t) - \varphi(x)}{\delta_n}\right), x\right)\omega_{\varphi,\alpha}(f, \delta_n) \\
\leq & \left(C_n(x) + \frac{D_n(x)}{\delta_n}\right)\omega_{\varphi,\alpha}(f, \delta_n),
\end{aligned}$$

where

$$\begin{aligned}
C_n(x) =& S_n\left((1+t)^\alpha + (1+x)^\alpha, x\right) \leq (M+1)(1+x)^\alpha, \\
D_n(x) =& S_n\left(((1+t)^\alpha + (1+x)^\alpha)\left|\varphi(t) - \varphi(x)\right|, x\right) \\
=& S_n\left((1+t)^\alpha\left|\varphi(t) - \varphi(x)\right|, x\right) + (1+x)^\alpha S_n\left(\left|\varphi(t) - \varphi(x)\right|, x\right).
\end{aligned}$$

Using the Cauchy-Schwarz inequality $|S_n(fg, x)| \leq \sqrt{S_n(f^2, x)} \cdot \sqrt{S_n(g^2, x)}$ and the estimation $S_n\left(\left|\varphi(t) - \varphi(x)\right|^2, x\right) \leq \frac{1}{n}$ (see the proof of Corollary 3.2 and Remark 3.3 from [1]) we get

$$\begin{aligned}
D_n(x) \leq & \sqrt{S_n\left((1+t)^{2\alpha}, x\right)}\sqrt{S_n\left(\left|\varphi(t) - \varphi(x)\right|^2, x\right)} + (1+x)^\alpha S_n\left(\left|\varphi(t) - \varphi(x)\right|, x\right) \\
\leq & \sqrt{(M_{2\alpha}(1+t))^{2\alpha}}\frac{1}{\sqrt{n}} + (1+x)^\alpha\frac{1}{\sqrt{n}} = \left(\sqrt{M_{2\alpha}} + 1\right)(1+x)^\alpha\frac{1}{\sqrt{n}}.
\end{aligned}$$

Choosing $\delta_n = \frac{1}{\sqrt{n}}$

$$\frac{1}{(1+x)^\alpha}|S_n(f,x) - f(x)| \leq C_\alpha \omega_{\varphi,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

which proves the theorem. □

Acknowledgements. The authors thank to the referees for their careful reading and their excellent suggestions.

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