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NEW RESULTS PARAMETRIC APOSTOL-TYPE FROBENIUS-EULER POLYNOMIALS AND THEIR MATRIX APPROACH

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Abstract. The new algebraic properties of the parametric Apostol-type Frobenius-Euler polynomials and parametric type Frobenius-Euler polynomial have been explained in this research. The researchers have studied the series of the Taylor type and established the relation between the classic Apostol Frobenius-Euler and Frobenius-Euler polynomials. This work has also addressed the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and has shown some of their properties. Finally, this research provided some factorizations of Apostol-type Frobenius-Euler matrix that involves the generalized Pascal matrix, Fibonacci and Lucas matrices, respectively.

1. Introduction

The Apostol type polynomials and numbers, have been used extensively in mathematical analysis and practical applications. For this reason, they have been studied as reported in [1–4, 6, 7, 9, 11, 13–15, 17, 18].
Let $\mathbb{P}$ be the vector space of the polynomials with coefficients in $\mathbb{C}$. Let $\{A_n(x)\}_{n\geq 0}$ be the sequence of polynomials known in the literature as the sequence Appell polynomials if the polynomials $A_n(x)$ are defined by the following generating function: (see, [9, p. 1, (1)]):

$$f(z)e^{xz} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!}, \tag{1.1}$$

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where f is a formal power series in z, these polynomials have found remarkable applications in different branches of mathematics, theoretical physics, and chemistry [1,14]. On the other hand, for a particular case, we have the Apostol Frobenius-Euler polynomials that are generated by choosing in (1.1) the following value of $f(z)$ (see, [5, p. 1, (1)]):

$$f(z) = \frac{1-u}{\lambda e^z - u},$$

from which you get the Apostol Frobenius-Euler polynomials $H_n(x;\lambda;u)$ in variable x, is defined through the generating function (see, [2, p. 2, Definition 2]):

$$\frac{1-u}{\lambda e^z - u} e^{xz} = \sum_{n=0}^{\infty} H_n(x;\lambda;u) \frac{z^n}{n!}, \qquad |z| < \left| \log\left(\frac{\lambda}{u}\right) \right|,$$

where $H_n(\lambda;u)$ denotes the Apostol Frobenius-Euler number. Thus, the Apostol Frobenius-Euler polynomials fulfill the following identities respectively (see, [2, p. 4, Proposition 1 and Proposition 2]):

$$\lambda H_n(x+1;\lambda;u) - u H_n(x;\lambda;u) = (1-u)x^n$$

and

$$\frac{d}{dx}\left[H_n(x;\lambda;u)\right] = nH_{n-1}(x;\lambda;u).$$

Furthermore, if $n \in \mathbb{N}$, then (see, [2, p. 4, Proposition 3]):

$$\int_0^1 H_n(x;\lambda;u) = \frac{u-\lambda}{\lambda} \cdot \frac{H_{n+1}(\lambda;u)}{n+1}.$$

In this paper, the authors will study new properties of the polynomials that are introduced in [10]. The author will also define the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and will show some of their properties. This paper is organized as follows. In Section 2, will be giving some definitions of previous results of parametric type Apostol Frobenius-Euler $H_n^c(x;\lambda;u)$ and $H_n^s(x;\lambda;u)$ polynomials. Section 3, will be obtaining several properties of the parametric Apostol Frobenius-Euler and Frobenius-Euler polynomials. Section 4, will be presenting some new series of the Taylor type involving the Apostol Frobenius-Euler numbers $H_n(\lambda;u)$ and Frobenius-Euler numbers $H_n(u)$. Finally, Section 5 will be addressing the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and show some of their properties.

2. Background an Previous Results

The following standard notions will be used: $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

For real parameters, p and q in [8] was obtained that the Taylor series representation of the following functions $e^{pz}\cos(qz)$ and $e^{pz}\sin(qz)$ is given by

$$e^{pz}\cos(qz) = \sum_{k=0}^{\infty} C_k(p,q)\frac{z^k}{k!},$$
$$e^{pz}\sin(qz) = \sum_{k=0}^{\infty} S_k(p,q)\frac{z^k}{k!},$$

where $C_k(p,q)$ and $S_k(p,q)$ is given by

$$C_k(p,q) = \sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j}p^{k-2j}q^{2j}, \tag{2.1}$$

$$S_k(p,q) = \sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}q^{2j+1}. \tag{2.2}$$

Also it is fulfilled (see, [13, p. 944]):

$$C_k(p,p) = 2^{\frac{k}{2}}p^k\cos\frac{k\pi}{4},$$
$$S_k(p,p) = 2^{\frac{k}{2}}p^k\sin\frac{k\pi}{4},$$
$$C_k(0,q) = q^k\cos\frac{k\pi}{2}, \tag{2.3}$$
$$S_k(0,q) = q^k\sin\frac{k\pi}{2}, \tag{2.4}$$
$$C_k(p,0) = p^k \quad \text{and} \quad S_k(p,0) = 0.$$

Using the definitions of $C_n(p;q)$, $S_n(p;q)$ and the Apostol Frobenius-Euler numbers $H_n(\lambda;u)$ we have, two parametric of Apostol-type Frobenius-Euler polynomials

$$H_{n,c}(p,q;\lambda,u) = H_n(\lambda,u)C_n(p,q),$$
$$H_{n,s}(p,q;\lambda,u) = H_n(\lambda,u)S_n(p,q),$$

which exponential generating of $H_{n,c}(p;q;\lambda;u)$ and $H_{n,s}(p;q;\lambda;u)$ functions are given respectively, by (see, [10, p. 5, (14) and (15)]):

$$\left[\frac{1-u}{\lambda e^z - u}\right]^{(\alpha)} e^{pz}\cos(qz) = \sum_{n=0}^{\infty} H_{n,c}^{[\alpha]}(p,q;\lambda;u)\frac{z^n}{n!}, \tag{2.5}$$

$$\left[\frac{1-u}{\lambda e^z - u}\right]^{(\alpha)} e^{pz}\sin(qz) = \sum_{n=0}^{\infty} H_{n,s}^{[\alpha]}(p,q;\lambda;u)\frac{z^n}{n!}. \tag{2.6}$$

Thus, according to the Cauchy series product, we obtain

$$H_{n,c}(p,q;\lambda;u) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}(\lambda;u)C_k(p,q), \tag{2.7}$$

$$H_{n,s}(p,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(\lambda;u)S_k(p,q). \tag{2.8}$$

Therefore, from equation (2.5) it is observed that when the parameter q takes the value 0 one has $H_{n,c}(p;q;\lambda;u)=H_n(p;\lambda;u)$ and the Apostol Frobenius-Euler polynomials are obtained. On the other hand from (2.5) and (2.6) it is easy to obtain the following statement

$$\begin{aligned} H_{n,c}^{[\alpha+\beta]}(p+q,q;\lambda;u)&=\sum_{k=0}^{n}\binom{n}{k}H_{k,c}^{[\alpha]}(p,q,\lambda;u)H_{n-k,c}^{[\beta]}(q,0,\lambda;u),\\ H_{n,s}^{[\alpha+\beta]}(p+q,q;\lambda;u)&=\sum_{k=0}^{n}\binom{n}{k}H_{k,s}^{[\alpha]}(p,q,\lambda;u)H_{n-k,s}^{[\beta]}\left(q,\frac{\pi}{2z},\lambda;u\right). \end{aligned} \tag{2.9}$$

Below, a list of the first parametric Apostol Frobenius-Euler polynomials for $H_{n,c}(p,q;\lambda;u)$ and $H_{n,s}(p,q;\lambda;u)$ are shown:

$$\begin{aligned} H_{0,c}(p,q;\lambda;u)=&\frac{1-u}{\lambda-u},\\ H_{1,c}(p,q;\lambda;u)=&\frac{1-u}{\lambda-u}p-\frac{\lambda(1-u)}{(\lambda-u)^2},\\ H_{2,c}(p,q;\lambda;u)=&\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right](1-u)-\frac{1-u}{\lambda-u}q^2+\frac{1-u}{(\lambda-u)^2}p^2\\ &+\left[\frac{2(1-u)\lambda}{(\lambda-u)^2}\right]p,\\ H_{3,c}(p,q;\lambda;u)=&\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]p-3\frac{1-u}{\lambda-u}pq^2+3\lambda\frac{1-u}{(\lambda-u)^2}q^2\\ &+(1-u)\left[-\frac{6\lambda^3}{(\lambda-u)^4}+\frac{6\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right],\\ H_{4,c}(p,q;\lambda;u)=&\frac{1-u}{\lambda-u}p^4-\frac{4\lambda(1-u)}{(\lambda-u)^2}p^3+6(1-u)\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]p^2\\ &+4(1-u)\left[\frac{-6\lambda^3}{(\lambda-u)^4}+\frac{6\lambda^2}{(\lambda-u)^3}+\frac{\lambda}{(\lambda-u)^2}\right]p-\frac{6(1-u)}{\lambda-u}p^2q^2\\ &+12\lambda\frac{1-u}{(\lambda-u)^2}pq^2\\ &-6\lambda(1-u)\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right]q^2+\frac{1-u}{\lambda-u}q^4\\ &+(1-u)\left[\frac{24\lambda^4}{(\lambda-u)^4}-\frac{36\lambda^3}{(\lambda-u)^4}+\frac{14\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right],\\ H_{0,s}(p,q;\lambda;u)=&0, \end{aligned}$$

$$\begin{aligned}
H_{1,s}(p,q;\lambda;u) =& \frac{1-u}{\lambda-u}q,\\
H_{2,s}(p,q;\lambda;u) =& -2\lambda\frac{1-u}{(\lambda-u)^2}q+2\frac{1-u}{\lambda-u}pq,\\
H_{3,s}(p,q;\lambda;u) =& -\lambda\frac{1-u}{\lambda-u}q^3+2\frac{1-u}{\lambda-u}p^2q-6\lambda\frac{1-u}{\lambda-u}pq\\
&+3(1-u)\lambda\left[\frac{2\lambda}{(\lambda-u)^3}-\frac{1}{(\lambda-u)^2}\right]q.
\end{aligned}$$

Let p be any nonzero real number. The generalized Pascal matrix of first kind $P[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [16, Definition 1]):

$$p_{i,j}(p) := \begin{cases} \binom{i}{j}(p)^{i-j}, & i \geq j,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\{F_n\}_{n\geq 1}$ be the Fibonacci sequence, i.e., $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci matrix $\mathscr{F} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [19]):

$$f_{i,j} := \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0,\\ 0, & \text{if } i-j+1 < 0.\end{cases}$$

Let $\{L_n\}_{n\geq 1}$ be the Lucas numbers sequence, i.e., $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$. The Lucas matrix $\mathscr{L} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [20]):

$$l_{i,j} := \begin{cases} L_{i-j+1}, & \text{if } i-j \geq 0,\\ 0, & \text{if } i-j < 0.\end{cases}$$

3. The Parametric of Apostol-type Frobenius-Euler Polynomials and their Properties of $H_n^c(p,q;\lambda;u)$ and $H_n^s(p,q;\lambda;u)$

In this section, some properties of the parametric Apostol-type Frobenius-Euler polynomials $H_{n,c}(p,q;\lambda;u)$ and $H_{n,s}(p,q;\lambda;u)$, will be presented.

Proposition 3.1. *For every $n \in \mathbb{N}$, the parametric Apostol-type Frobenius-Euler $H_{n,c}(p;q;\lambda;u)$ and $H_{n,s}(p;q;\lambda;u)$ polynomials meet the following identity*

$$\lambda H_{n,c}(1+p,q;\lambda;u) - uH_{n,c}(p;q;\lambda;u) =(1-u)C_n(p,q), \tag{3.1}$$

$$\lambda H_{n,s}(1+p,q;\lambda;u) - uH_{n,s}(p;q;\lambda;u) =(1-u)S_n(p,q). \tag{3.2}$$

Proof.

$$\begin{aligned}
(\lambda+u+1)\sum_{n=0}^{\infty}H_{n,c}(1+p,q;\lambda;u)\frac{z^n}{n!} =& \frac{1-u}{\lambda e^z-u}e^{(1+p)z}\cos(qz)(\lambda+u+1)\\
=& e^{pz}\left(\lambda e^z+ue^z+e^z-u+u\right)\cos(qz)\frac{1-u}{\lambda e^z-u}
\end{aligned}$$

$$
\begin{aligned}
&=(1-u)e^{pz}\cos(qz)+ue^{pz}\cos(qz)\frac{1-u}{\lambda e^{z}-u}\\
&\quad+(1+u)e^{(p+1)z}\cos(qz)\frac{1-u}{\lambda e^{z}-u}\\
&=(1+u)\sum_{n=0}^{\infty}H_{n,c}(1+p,q;\lambda;u)\frac{z^n}{n!}\\
&\quad+(1-u)\sum_{n=0}^{\infty}C_n(p,q)\frac{z^n}{n!}\\
&\quad+u\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}. \qquad \square
\end{aligned}
$$

So, the first statement given in (3.1) was demonstrated. The proof of (3.2) is obtained analogously.

Corollary 3.1. *If in Proposition* 3.1 *the relationships* (3.1) *and* (3.2) *take a value of* $p=0$, *then it is true*

$$\lambda H_{2n,c}(1,q;\lambda;u)-uH_{2n,c}(q;\lambda;u)=(1-u)(-1)^n q^{2n}$$

and

$$\lambda H_{2n+1,s}(1,q;\lambda;u)-uH_{2n+1,s}(q;\lambda;u)=(1-u)(-1)^n q^{2n+1}.$$

Proposition 3.2. *For every* $n\in\mathbb{Z}^+$, *the parametric Apostol-type Frobenius-Euler* $H_{n,c}(p;q;\lambda;u)$ *and* $H_{n,s}(p;q;\lambda;u)$ *polynomials meet the following identity*

$$H_{n,c}(p+l,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{k,c}(p,q;\lambda;u)l^{n-k}, \tag{3.3}$$

$$H_{n,s}(p+l,q;\lambda;u)=\sum_{k=0}^{n}\binom{n}{k}H_{k,s}(p,q;\lambda;u)l^{n-k}. \tag{3.4}$$

Proof. Using (2.5) one obtained

$$
\begin{aligned}
\sum_{n=0}^{\infty}H_{n,c}(p+l,q;\lambda;u)\frac{z^n}{n!}&=\left(\frac{1-u}{\lambda e^{z}-u}e^{pz}\cos(qz)\right)e^{lz}\\
&=\left(\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}l^n\frac{z^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}H_{k,c}(p,q;\lambda;u)l^{n-k}.
\end{aligned}
$$

The first affirmation obtained in (3.3) has been proven. The other result (3.4) can be demonstrated similarly. $\square$

Corollary 3.2. *The following statements are valid*

$$H_{n,c}(p+1,q;\lambda;u)-H_{n,c}(p,q;\lambda;u)=\sum_{k=0}^{n-1}\binom{n}{k}H_{k,c}(p,q;\lambda;u)$$

and

$$H_{n,s}(p+1,q;\lambda;u)-H_{n,s}(p,q;\lambda;u)=\sum_{k=0}^{n-1}\binom{n}{k}H_{k,s}(p,q;\lambda;u).$$

Using the Corollary 3.2 and the Proposition 3.1, the following recurrence formulas are obtained:

$$\begin{aligned}H_{n,c}(p,q;\lambda;u)=&\frac{1}{\lambda-u}\left[(1-u)C_n(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}H_{k,c}(p,q;\lambda;u)\right],\\ H_{n,s}(p,q;\lambda;u)=&\frac{1}{\lambda-u}\left[(1-u)S_{n-1}(p,q)-\lambda\sum_{k=0}^{n-1}\binom{n}{k}H_{k,s}(p,q;\lambda;u)\right],\end{aligned}\tag{3.5}$$

where $H_{0,c}(p,q;\lambda;u)=\dfrac{1-u}{\lambda-u}$ and $H_{0,s}(p,q;\lambda;u)=0$.

Proposition 3.3. *For every $n\in\mathbb{N}$, the following partial derivative identities are correct*

$$\frac{\partial}{\partial p}\left[H_{n,c}(p,q;\lambda;u)\right]=nH_{n-1,c}(p,q;\lambda;u),\tag{3.6}$$

$$\frac{\partial}{\partial p}\left[H_{n,s}(p,q;\lambda;u)\right]=nH_{n-1,s}(p,q;\lambda;u),\tag{3.7}$$

$$\frac{\partial}{\partial q}\left[H_{n,c}(p,q;\lambda;u)\right]=-nH_{n-1,s}(p,q;\lambda;u),\tag{3.8}$$

$$\frac{\partial}{\partial q}\left[H_{n,s}(p,q;\lambda;u)\right]=nH_{n-1,c}(p,q;\lambda;u).\tag{3.9}$$

It will be shown (3.6), the proof of (3.7), (3.8) and (3.9) are similar.

Proof.

$$\begin{aligned}\frac{\partial}{\partial p}\left[\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^n}{n!}\right]&=\sum_{k=0}^{\infty}\frac{\partial}{\partial p}\left[H_{n,c}(p,q;\lambda;u)\right]\frac{z^n}{n!}\\ &=\frac{1-u}{\lambda e^z-u}ze^{pz}\cos(qz)\\ &=\sum_{n=0}^{\infty}H_{n,c}(p,q;\lambda;u)\frac{z^{n+1}}{n!}\\ &=\sum_{n=1}^{\infty}H_{n-1,c}(p,q;\lambda;u)\frac{z^n}{(n-1)!}\\ &=\sum_{n=1}^{\infty}nH_{n-1,c}(p,q;\lambda;u)\frac{z^n}{(n)!},\end{aligned}$$

by comparing the coefficients of the series, one has the result. □

Proposition 3.4. *The polynomials* $H_{n,c}(p,q;\lambda;u)$ *and* $H_{n,s}(p,q;\lambda;u)$ *are, respectively, of degrees* n *and* $n-1$ *in the variable* p*, it is also asserted that*

$$H_{n,c}(p;q;\lambda;u) = \frac{1-u}{\lambda-u}p^n - n\frac{1-u}{(\lambda-u)^2}p^{n-1} + \cdots, \tag{3.10}$$

$$H_{n,s}(p;q;\lambda;u) = \frac{n(1-u)q}{\lambda-u}p^{n-1} - \frac{n(n-1)(1-u)\lambda q}{(\lambda-u)^2}p^{n-2} + \cdots. \tag{3.11}$$

Proof. First, the result given in (3.10) is shown using the method of mathematical induction on n. On the other hand of (3.5) it has

$$\begin{aligned} H_{0,c}(p,q;\lambda;u) &= \frac{1-u}{\lambda-u}, \\ H_{1,c}(p,q;\lambda;u) &= \frac{(1-u)p}{\lambda-u} - \frac{\lambda(1-u)}{(\lambda-u)^2} \end{aligned}$$

and

$$H_{2,c}(p,q;\lambda;u) = \frac{(1-u)p^2}{(\lambda-u)^3} - 2\frac{(1-u)\lambda p}{(\lambda-u)^2} - \frac{1-u}{\lambda-u}q^2 + \frac{2\lambda^2(1-u)-\lambda(\lambda-u)}{(\lambda-u)^3}.$$

Therefore, the statement given in (3.10) is valid for $n = 0, 1, 2$. It will be assumed that it is correct for $n-1$. Using (3.6), we get

$$\frac{\partial}{\partial p}\left[H_{n,c}(p,q;\lambda;u)\right] = n\frac{1-u}{\lambda+1}p^{n-1} - n(n-1)\frac{1-u}{(\lambda-u)^2}p^{n-2} + \cdots.$$

To obtain the final result given in (3.10) it is necessary to integrate with respect to variable p. The results (3.9) and (3.11) are obtained analogously. □

Proposition 3.5. *If* $n \in \mathbb{N}$*,* $\lambda > 0$*,* $u \neq \lambda$ *and* m *is an odd positive integer, then*

$$H_{n,c}(mp,q;\lambda^{\frac{1}{m}};u^{\frac{1}{m}}) = m^n \sum_{k=0}^{m-1} u^{\frac{m-1}{m}} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p+\frac{k}{m},\frac{q}{m};\lambda;u\right) \tag{3.12}$$

and

$$H_{n,s}(mp,q;\lambda^{\frac{1}{m}};u^{\frac{1}{m}}) = m^n \sum_{k=0}^{m-1} u^{\frac{m-1}{m}} \left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,s}\left(p+\frac{k}{m},\frac{q}{m};\lambda;u\right). \tag{3.13}$$

Proof. To prove (3.12), it avails to consider the following relation:

$$\sum_{k=0}^{\infty}\left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p+\frac{k}{m},\frac{q}{m};\lambda;u\right) = \frac{1-u}{\lambda e^z-u}\left(\frac{\lambda}{u}\right)^{\frac{k}{m}} e^{\left(p+\frac{k}{m}\right)z}\cos\left(\frac{qz}{m}\right),$$

take a sum over k from 0 to $m-1$, one has

$$\begin{aligned} \sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^{\frac{k}{m}} H_{n,c}\left(p+\frac{k}{m},\frac{q}{m};\lambda;u\right) &= \frac{1-u}{\lambda e^z-u}\left(\frac{\lambda}{u}\right)^{\frac{k}{m}} e^{pz}\cos\left(\frac{qz}{m}\right) \\ &= \frac{(1-u)e^{mp\frac{z}{m}}\cos\left(\frac{qz}{m}\right)u^{\frac{1-m}{m}}}{\lambda^{\frac{1}{m}}e^{\frac{z}{m}}-u^{\frac{1}{m}}} \end{aligned}$$

$$= \sum_{n=0}^{\infty} m^{-n} u^{\frac{1-m}{m}} H_{n,c}\left(mp, q; \lambda^{\frac{1}{m}}; u^{\frac{1}{m}}\right) \frac{z^n}{n!}.$$

To the test (3.13) the proof is similarly. □

New results are presented below for parametric Frobenius-Euler polynomials.

Proposition 3.6. *For every $n \in \mathbb{N}$, the parametric Frobenius-Euler $H_n^c(p; q; u)$ and $H_{n,s}(p; q; u)$ polynomials meet the following identity*

$$H_{n,c}(1+p, q; u) - uH_{n,c} = (1-u)C_n(p, q), \tag{3.14}$$

$$H_{n,s}(1+p, q; u) - uH_{n,s} = (1-u)S_n(p, q). \tag{3.15}$$

Proof.

$$\begin{aligned}
(1+u+1)\sum_{n=0}^{\infty} H_{n,c}(1+p, q; u) =& \frac{1-u}{e^z-u} e^{(1+p)z}\cos(qz)(1+u+1)\\
=& e^{pz}\left(e^z + ue^z + e^z - u + u\right)\cos(qz)\frac{1-u}{e^z-u}\\
=& (1-u)e^{pz}\cos(qz) + (1+u)e^{(p+1)z}\cos(qz)\frac{1-u}{e^z-u}\\
&+ ue^{pz}\cos(qz)\frac{1-u}{e^z-u}\\
=& (1-u)\sum_{n=0}^{\infty} C_n(p, q)\frac{z^n}{n!}\\
&+ (1+u)\sum_{n=0}^{\infty} H_{n,c}(1+p, q; u)\frac{z^n}{n!}\\
&+ u\sum_{n=0}^{\infty} H_{n,c}(p, q; u)\frac{z^n}{n!},
\end{aligned}$$

which proves the first assertion (3.14). The proof of the second assertion (3.15) is similar. □

Corollary 3.3. *For every $n \in \mathbb{N}$, the following identities hold true*

$$\begin{aligned}
H_{n,c}(1+p, q; u) - uH_{n,c}(p, q, u) =& (1-u)C_n(p, q),\\
H_{n,s}(1+p, q; u) - uH_{n,s}(p; q; u) =& (1-u)S_n(p, q).
\end{aligned}$$

Corollary 3.4. *If in Proposition 3.6 the relationships (3.14) and (3.15) take a value of $p = 0$, then it is true*

$$H_{2n,c}(1, q; u) - uH_{2n,c}(q; u) = (1-u)(-1)^n q^{2n}$$

and

$$H_{2n+1,s}(1, q; \lambda; u) - uH_{2n+1,s}(q; u) = (1-u)(-1)^n q^{2n+1}.$$

Proposition 3.7. *For every $n \in \mathbb{Z}^+$, the following identities hold true*

$$H_{n,c}(p+l,q;u) = \sum_{k=0}^{n} \binom{n}{k} H_{k,c}(p,q;u) l^{n-k}$$

and

$$H_{n,s}(p+l,q;u) = \sum_{k=0}^{n} \binom{n}{k} H_{k,s}(p,q;u) l^{n-k}.$$

Corollary 3.5. *The following statements are valid*

$$H_{n,c}(p+1,q;u) - H_{n,c}(p,q;u) = \sum_{k=0}^{n-1} \binom{n}{k} H_{k,c}(p,q;u)$$

and

$$H_{n,s}(p+1,q;u) - H_{n,s}(p,q;\lambda;u) = \sum_{k=0}^{n-1} \binom{n}{k} H_{k,s}(p,q;\lambda;u).$$

Using Corollary 3.5 and Proposition 3.6, the following recurrences are obtained:

$$H_{n,c}(p,q;u) = \frac{1}{1-u}\left[(1-u)C_n(p,q) - \sum_{k=0}^{n-1} \binom{n}{k} H_{k,c}(p,q;u)\right]$$

and

$$H_{n,s}(p,q;u) = \frac{1}{1-u}\left[(1-u)S_{n-1}(p,q) - \sum_{k=0}^{n-1} \binom{n}{k} H_{k,s}(p,q;u)\right],$$

where $H_{0,c}(p,q;u) = 1$ and $H_{0,s}(p,q;u) = 0$.

Proposition 3.8. *For every $n \in \mathbb{N}$, the following identities hold true*

$$\frac{\partial}{\partial p}\left[H_{n,c}(p,q;u)\right] = nH_{n-1,c}(p,q;u), \tag{3.16}$$

$$\frac{\partial}{\partial p}\left[H_{n,s}(p,q;u)\right] = nH_{n-1,s}(p,q;u) \tag{3.17}$$

and

$$\frac{\partial}{\partial q}\left[H_{n,c}(p,q;u)\right] = -nH_{n-1,s}(p,q;u), \tag{3.18}$$

$$\frac{\partial}{\partial q}\left[H_{n,s}(p,q;u)\right] = nH_{n-1,c}(p,q;u). \tag{3.19}$$

It will be shown (3.16), the demonstrations of (3.17), (3.18) and (3.19) are similar.

Proof.

$$\begin{aligned}\frac{\partial}{\partial p}\left[\sum_{n=0}^{\infty} H_{n,c}(p,q;u)\frac{z^n}{n!}\right] &= \sum_{k=0}^{\infty} \frac{\partial}{\partial p}\left[H_{n,c}(p,q;u)\right]\frac{z^n}{n!} \\ &= \frac{1-u}{e^z-u} z e^{pz}\cos(qz)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} H_{n,c}(p,q;u) \frac{z^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty} H_{n-1,c}(p,q;u) \frac{z^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} n H_{n-1,c}(p,q;u) \frac{z^n}{(n)!}. \qquad \square
\end{aligned}
$$

Proposition 3.9. *The polynomials $H_{n,c}(p,q;u)$ and $H_{n,s}(p,q;u)$ are, respectively, of degrees n and $n-1$ in the variable p it is also asserted that*

$$
\begin{aligned}
H_{n,c}(p;q;u) &= p^n - \frac{1}{1-u} p^{n-1} + \cdots , \\
H_{n,s}(p;q;u) &= nqp^{n-1} - \frac{n(n-1)q}{1-u} p^{n-2} + \cdots .
\end{aligned}
$$

4. Taylor Type Series Involving the Apostol-type Frobenius-Euler Numbers and Frobenius-Euler Numbers $H_n(\lambda;u)$ and $H_n(u)$

An important fact of relationships (2.5) and (2.6) is that one can trace them as the expansion in Taylor series of some functions on the point $z=0$ and relate it to Apostol-Frobenius-Euler and Frobenius-Euler numbers. So, replacing (2.7) and (2.8) in (2.5) and (2.6), one has

$$
\begin{aligned}
f^c_{H,\lambda;u}(z;p,q) &= \frac{1-u}{\lambda e^z - u} e^{pz} \cos(qz) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} H_{n-k}(\lambda;u) C_k(p,q) \right] \frac{z^n}{n!}, \quad (4.1) \\
f^s_{H,\lambda;u}(z;p,q) &= \frac{1-u}{\lambda e^z - u} e^{pz} \sin(qz) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} H_{n-k}(\lambda;u) S_k(p,q) \right] \frac{z^n}{n!}, \\
f^c_{H,u}(z;p,q) &= \frac{1-u}{e^z - u} e^{pz} \cos(qz) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} H_{n-k}(u) C_k(p,q) \right] \frac{z^n}{n!}, \\
f^s_{H,u}(z;p,q) &= \frac{1-u}{e^z - u} e^{pz} \sin(qz) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} H_{n-k}(u) S_k(p,q) \right] \frac{z^n}{n!},
\end{aligned}
$$

where $C_k(p,q)$ and $S_k(p,q)$ are defined in (2.1) and (2.2). Some particular cases will be shown using result previously known in Section 6 of [13].

Example 4.1. In (4.1), taking $p=0$ and $q=1$, and using (2.3) and (2.4), one obtain

$$
\begin{aligned}
f^c_{H,\lambda;u}(z;0,1) &= \frac{1}{\lambda e^z - u} \cos(z) \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{1}{1-u} \binom{n}{k} H_{n-k}(\lambda;u) \cos\frac{k\pi}{2} \right] \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{1}{1-u} \binom{n}{k} H_{n-k}(\lambda;u) \right] \frac{z^n}{n!}.
\end{aligned}
$$

Therefore, one has

$$\frac{1}{\lambda e^z - u}\cos(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{1}{1-u}\binom{n}{2k} H_{n-2k}(\lambda;u)\right]\frac{z^n}{n!},$$

as well as

$$\frac{1}{\lambda e^z - u}\sin(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k \frac{1}{1-u}\binom{n}{2k+1} H_{n-2k-1}(\lambda;u)\right]\frac{z^n}{n!},$$

$$\frac{1}{e^z - u}\cos(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k \frac{1}{1-u}\binom{n}{2k} H_{n-2k}(u)\right]\frac{z^n}{n!}$$

and

$$\frac{1}{e^z - u}\sin(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k \frac{1}{1-u}\binom{n}{2k+1} H_{n-2k-1}(u)\right]\frac{z^n}{n!}.$$

Example 4.2. Putting $p = q = 1$ in (4.1), one gets

$$\frac{e^z}{\lambda e^z - u}\cos(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k} H_{n-k}(\lambda;u)\ \cos\frac{k\pi}{4}\right]\frac{z^n}{n!},$$

$$\frac{e^z}{\lambda e^z - u}\sin(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k} H_{n-k}(\lambda;u)\ \sin\frac{k\pi}{4}\right]\frac{z^n}{n!},$$

$$\frac{e^z}{e^z - u}\cos(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k} H_{n-k}(u)\ \cos\frac{k\pi}{4}\right]\frac{z^n}{n!}$$

and

$$\frac{e^z}{e^z - u}\sin(z) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{2^{\frac{k}{2}}}{1-u}\binom{n}{k} H_{n-k}(u)\ \sin\frac{k\pi}{4}\right]\frac{z^n}{n!}.$$

5. Parametric Apostol-type Frobenius-Euler Polynomials Matrix

Inspired by [11,12,16], this section will address the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and will show some of their properties.

Definition 5.1. The generalized $(n+1)\times(n+1)$ parametric Apostol-type Frobenius-Euler polynomials matrices $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ and $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ are defined by

$$\mathscr{H}_{i,j,c}^{(\alpha)}(p,q;\lambda;u) = \begin{cases} \binom{i}{j} H_{i-j,c}^{(\alpha)}(p,q;\lambda;u), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathscr{H}_{i,j,s}^{(\alpha)}(p,q;\lambda;u)=\begin{cases}\binom{i}{j}H_{i-j,s}^{(\alpha)}(p,q;\lambda;u), & i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

Since, $H_{n,c}^{(0)}(p;0;\lambda;u)=p^n$ and $H_{n,s}^{(0)}\left(p;\frac{\pi}{2z};\lambda;u\right)=p^n$, we obtain

$$\mathscr{H}_c^{(0)}(p;0;\lambda;u)=P[p],\quad \mathscr{H}_s^{(0)}\left(p;\frac{\pi}{2z};\lambda;u\right)=P[p].$$

Theorem 5.1. *The generalized parametric Apostol-type Frobenius-Euler polynomials matrices $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ and $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ satisfies the following product formulae*

$$\begin{aligned}\mathscr{H}_c^{(\alpha+\beta)}(p+q;q;\lambda;u)&=\mathscr{H}_c^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\beta)}(q;0;\lambda;u)\\&=\mathscr{H}_c^{(\beta)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\alpha)}(q;0;\lambda;u)\\&=\mathscr{H}_c^{(\alpha)}(q;0;\lambda;u)\,\mathscr{H}_c^{(\beta)}(p;q;\lambda;u),\end{aligned}\tag{5.1}$$

$$\begin{aligned}\mathscr{H}_s^{(\alpha+\beta)}(p+q;q;\lambda;u)&=\mathscr{H}_s^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_s^{(\beta)}\left(q;\frac{\pi}{2z};\lambda;u\right)\\&=\mathscr{H}_s^{(\beta)}(p;q;\lambda;u)\,\mathscr{H}_s^{(\beta)}\left(q;\frac{\pi}{2z};\lambda;u\right)\\&=\mathscr{H}_s^{(\alpha)}\left(q;\frac{\pi}{2z};\lambda;u\right)\,\mathscr{H}_s^{(\beta)}(p;q;\lambda;u).\end{aligned}\tag{5.2}$$

Proof. Let $D_{i,j,c}^{[\alpha,\beta]}(\lambda;u)(p,q)$ be the (i,j)-th entry of the matrix product $\mathscr{H}_c^{(\alpha)}(p;q;\lambda;u)\,\mathscr{H}_c^{(\beta)}(q;0;\lambda;u)$, then by the addition formula (2.9) we have

$$\begin{aligned}D_{i,j,c}^{[\alpha,\beta]}(\lambda;u)(p,q)&=\sum_{k=0}^{n}\binom{i}{k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)\binom{k}{j}H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\sum_{k=j}^{i}\binom{i}{k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)\binom{k}{j}H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}H_{i-k,c}^{[\alpha]}(p;q;\lambda;u)H_{k-j,c}^{[\beta]}(q;0;\lambda;u)\\&=\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}H_{i-j-k,c}^{[\alpha]}(p;q,\lambda;u)H_{k,c}^{[\beta]}(q;0;\lambda;u)\\&=\binom{i}{j}H_{i-j,c}^{[\alpha+\beta]}(p+q;q;\lambda;u),\end{aligned}$$

which implies (5.1). The second and third equalities of the theorem and (5.2), can be derived in a similar way. □

Corollary 5.1. *The generalized parametric Apostol-type Frobenius-Euler polynomials matrices* $\mathscr{H}_c^{(\alpha)}(p,q;\lambda;u)$ *and* $\mathscr{H}_s^{(\alpha)}(p,q;\lambda;u)$ *satisfy the following relations*

$$\begin{aligned}\mathscr{H}_c^{[\alpha]}(p+q;q;\lambda;u) &= \mathscr{H}_c^{[\alpha]}(p;q;\lambda;u)P[q] = P[p]\mathscr{H}_c^{[\alpha]}(q;q;\lambda;u)\\ &= \mathscr{H}_c^{[\alpha]}(q;q;\lambda;u)P[p],\\ \mathscr{H}_s^{[\alpha]}(p+q;q;\lambda;u) &= \mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)P[q] = P[p]\mathscr{D}_s^{[\alpha]}(q;q;\lambda;u)\\ &= \mathscr{D}_s^{[\alpha]}(q;q;\lambda;u)P[p].\end{aligned}$$

In particular,

$$\mathscr{H}_c(p+q;q;\lambda;u) = P[p]\mathscr{H}_c(q;q;\lambda;u) = P[q]\mathscr{H}_c(p;q,\lambda;u).$$

Example 5.1. For $\alpha=1$ the first three polynomials $\mathscr{H}_{k,c}^{[\alpha]}(p;q;\lambda;u)$, $k=0,1,2$, are

$$\begin{aligned}H_0^c(p,q;\lambda;u) =&\frac{1-u}{\lambda-u},\\ H_1^c(p,q;\lambda;u) =&\frac{1-u}{\lambda-u}p-\frac{\lambda(1-u)}{(\lambda-u)^2},\\ H_2^c(p,q;\lambda;u) =&\left[\frac{2\lambda^2}{(\lambda-u)^3}-\frac{\lambda}{(\lambda-u)^2}\right](1-u)-\frac{1-u}{\lambda-u}q^2+\frac{1-u}{(\lambda-u)^2}p^2\\ &+\left[\frac{2(1-u)\lambda}{(\lambda-u)^2}\right]p.\end{aligned}$$

Hence, for $n=2$ we have

$$\mathscr{H}_c^{[1]}(p;q,\lambda,u) = \begin{bmatrix} H_0^c(p,q;\lambda;u) & 0 & 0 \\ H_1^c(p,q;\lambda;u) & H_0^c(p,q;\lambda;u) & 0 \\ H_2^c(p,q;\lambda;u) & 2H_1^c(p,q;\lambda;u) & H_0^c(p,q;\lambda;u) \end{bmatrix}.$$

For $\alpha,u,\lambda\in\mathbb{C}$, $0\le i,j\le n$, let $\mathbb{K}_c^{[\alpha]}(p;q;\lambda,u)$ and $\mathbb{K}_s^{[\alpha]}(p;q;\lambda,u)$ be the matrices whose entries are defined by

$$\begin{aligned}\tilde{r}_{i,j,c}^{[\alpha]}(p;q;\lambda;u) =&\binom{i}{j}H_{i-j,c}^{[\alpha]}(p;q;\lambda;u)-\binom{i-1}{j}H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)\\ &-\binom{i-2}{j}H_{i-j-2,c}^{[\alpha]}(p;q;\lambda;u),\\ \tilde{r}_{i,j,s}^{[\alpha]}(p;q;\lambda;u) =&\binom{i}{j}H_{i-j,s}^{[\alpha]}(p;q;\lambda;u)-\binom{i-1}{j}H_{i-j-1,s}^{[\alpha]}(p;q;\lambda;u)\\ &-\binom{i-2}{j}H_{i-j-2,s}^{[\alpha]}(p;q;\lambda;u).\end{aligned}$$

On the other hand, $\mathscr{J}_c^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{J}_s^{[\alpha]}(p;q;\lambda;u)$ are the matrices whose entries are given by

$$\tilde{s}_{i,j,c}^{[\alpha]}(p;q;\lambda;u) =\binom{i}{j}H_{i-j,c}^{[\alpha]}(p;q;\lambda;u)-\binom{i}{j+1}H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)$$

$$
\begin{aligned}
&-\binom{i}{j+2}H^{[\alpha]}_{i-j-2,c}(p;q;\lambda;u),\\
\tilde{s}^{[\alpha]}_{i,j,s}(p;q;\lambda;u)=&\binom{i}{j}H^{[\alpha]}_{i-j,s}(p;q;\lambda;u)-\binom{i}{j+1}H^{[\alpha]}_{i-j-1,s}(p;q;\lambda;u)\\
&-\binom{i}{j+2}H^{[\alpha]}_{i-j-2,s}(p;q;\lambda;u).
\end{aligned}
$$

Using the definitions of $\mathbb{K}^{[\alpha]}_c(p;q;\lambda;u)$, $\mathbb{K}^{[\alpha]}_s(p;q;\lambda;u)$, $\mathscr{J}^{[\alpha]}_c(p;q;\lambda;u)$ and $\mathscr{J}^{[\alpha]}_s(p;q;\lambda;u)$, it is observed that

$$
\begin{aligned}
\tilde{r}^{[\alpha]}_{0,0,c}(p;q;\lambda;u)=&\tilde{r}^{[\alpha]}_{1,1,c}(p;q;\lambda;u)=\tilde{s}^{[\alpha]}_{0,0,c}(p;q;\lambda;u)=\tilde{s}^{[\alpha]}_{1,1,c}(p;q;\lambda;u)=H^{[\alpha]}_{0,c}(p,q,\lambda;u),\\
\tilde{r}^{[\alpha]}_{0,j,c}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{0,j,c}(p;q;\lambda;u)=0,\quad j\geq 1,\\
\tilde{r}^{[\alpha]}_{1,0,c}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{1,0,c}(p;q;\lambda;u)=H^{[\alpha]}_{1,c}(p;q;\lambda;u)-H^{[\alpha]}_{0,c}(p;q;\lambda;u),\\
\tilde{r}^{[\alpha]}_{1,j,c}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{1,j,c}(p;q;\lambda;u)=0,\quad j\geq 2,\\
\tilde{r}^{[\alpha]}_{i,0,c}(p;q;\lambda;u)=&H^{[\alpha]}_{i,c}(p;q;\lambda;u)-H^{[\alpha]}_{i-1,c}(p;q;\lambda;u)-H^{[\alpha]}_{i-2,c}(p;q;\lambda;u),\quad i\geq 2,\\
\tilde{s}^{[\alpha]}_{i,0,c}(p;q;\lambda;u)=&H^{[\alpha]}_{i,c}(p;q;\lambda;u)-2H^{[\alpha]}_{i-1,c}(p;q;\lambda;u)-H^{[\alpha]}_{i-2,c}(p;q;\lambda;u),\quad i\geq 2,\\
\tilde{r}^{[\alpha]}_{0,0,s}(p;q;\lambda;u)=&\tilde{r}^{[\alpha]}_{1,1,s}(p;q;\lambda;u)=\tilde{s}^{[\alpha]}_{0,0,s}(p;q;\lambda;u)=\tilde{s}^{[\alpha]}_{1,1,s}(p;q;\lambda;u)=H^{[\alpha]}_{0,s}(p,q,\lambda;u),\\
\tilde{r}^{[\alpha]}_{0,j,s}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{0,j,s}(p;q;\lambda;u)=0,\quad j\geq 1,\\
\tilde{r}^{[\alpha]}_{1,0,s}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{1,0,s}(p;q;\lambda;u)=H^{[\alpha]}_{1,s}(p;q;\lambda;u)-H^{[\alpha]}_{0,s}(p;q;\lambda;u),\\
\tilde{r}^{[\alpha]}_{1,j,s}(p;q;\lambda;u)=&\tilde{s}^{[\alpha]}_{1,j,s}(p;q,\lambda;u)=0,\quad j\geq 2,\\
\tilde{r}^{[\alpha]}_{i,0,s}(p;q;\lambda;u)=&H^{[\alpha]}_{i,s}(p;q;\lambda;u)-H^{[\alpha]}_{i-1,s}(p;q;\lambda;u)-H^{[\alpha]}_{i-2,s}(p;q;\lambda;u),\quad i\geq 2,\\
\tilde{s}^{[\alpha]}_{i,0,s}(p;q;\lambda;u)=&H^{[\alpha]}_{i,s}(p;q;\lambda;u)-2H^{[\alpha]}_{i-1,s}(p;q;\lambda;u)-H^{[\alpha]}_{i-2,s}(p;q;\lambda;u),\quad i\geq 2.
\end{aligned}
$$

For $\alpha,\lambda,u\in\mathbb{C}$, $0\leq i,j\leq n$, let $\mathscr{L}^{[\alpha]}_{1,c}(p;q;\lambda;u)$ and $\mathscr{L}^{[\alpha]}_{1,s}(p;q;\lambda;u)$ be the matrices whose entries are given by

$$
\begin{aligned}
\hat{l}^{[\alpha]}_{i,j,1,c}(p;q;\lambda;u)=&\binom{i}{j}H^{[\alpha]}_{i-j,c}(p;q;\lambda;u)-3\binom{i-j}{j}H^{[\alpha]}_{i-j-1,c}(p;q;\lambda;u)\\
&+5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}H^{[\alpha]}_{k-j,c}(p;q;\lambda;u),\\
\hat{l}^{[\alpha]}_{i,j,1,s}(p;q;\lambda;u)=&\binom{i}{j}H^{[\alpha]}_{i-j,s}(p;q;\lambda;u)-3\binom{i-j}{j}H^{[\alpha]}_{i-j-1,s}(p;q;\lambda;u)\\
&+5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}H^{[\alpha]}_{k-j,s}(p;q;\lambda;u).
\end{aligned}
$$

Similarly, let $\mathscr{L}_{2,c}^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{L}_{2,s}^{[\alpha]}(p;q;\lambda;u)$, $(n+1)\times(n+1)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,2,c}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j} H_{i-j,c}^{[\alpha]}(p;q;\lambda;u) - 3\binom{i}{j+1} H_{i-j-1,c}^{[\alpha]}(p;q;\lambda;u)\\ &+ 5\sum_{k=j+1}^{i} (-1)^{k-j} 2^{k-j-2}\binom{i}{k} H_{i-k,c}^{[\alpha]}(p;q;\lambda;u),\\ \hat{l}_{i,j,2,s}^{[\alpha]}(p;q;\lambda;u) =& \binom{i}{j} H_{i-j,s}^{[\alpha]}(p;q;\lambda;u) - 3\binom{i}{j+1} H_{i-j-1,s}^{[\alpha]}(p;q;\lambda;u)\\ &+ 5\sum_{k=j+1}^{i} (-1)^{k-j} 2^{k-j-2}\binom{i}{k} H_{i-k,s}^{\alpha}(p;q;\lambda;u).\end{aligned}$$

Next we will show factorizations of the matrices $\mathscr{H}_c^{[,\alpha]}(p;q;\lambda;u)$ and $\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)$ involving the Fibonacci and Lucas matrices, respectively.

Theorem 5.2. *The parametric Apostol-type Frobenius-Euler polynomials matrix $\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)$ can be factored in terms of the Fibonacci matrix $\mathscr{F}$ as follows*

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{F}\mathbb{K}_c^{[\alpha]}(p;q;\lambda;u), \tag{5.3}$$

$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u) = \mathscr{F}\mathbb{K}_s^{[\alpha]}(p;q;\lambda;u), \tag{5.4}$$

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{J}_c^{[\alpha]}(p;q;\lambda;u)\mathscr{F}, \tag{5.5}$$

$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u) = \mathscr{J}_s^{[\alpha]}(p;q;\lambda;u)\mathscr{F}. \tag{5.6}$$

Proof. The relation (5.3) is equivalent to

$$\mathscr{F}^{-1}\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathbb{K}_c^{[\alpha]}(p;q;\lambda;u),$$

following the ideas of [11] or [19, Theorem 4.1], and making the corresponding modifications, (5.3) is obtained. □

In addition, the relations (5.3), (5.4), (5.5) and (5.6) allow us to deduce the following identities:

$$\begin{aligned}\mathbb{K}_c^{[\alpha]}(p;q;\lambda;u) =& \mathscr{F}^{-1}\,\mathscr{J}_c^{[\alpha]}(p;q,\lambda;u)\,\mathscr{F},\\ \mathbb{K}_s^{[\alpha]}(p;q;\lambda;u) =& \mathscr{F}^{-1}\,\mathscr{J}_s^{[\alpha]}(p;q,\lambda;u)\,\mathscr{F}.\end{aligned}$$

An analogous reasoning used in the proof of Theorem 5.2, allows us to prove the results below.

Example 5.2. For $\alpha=1$, the matrices, for $n=2$, $K_c^{[1]}(p;q,\lambda;u)$ and $\mathscr{F}$ are

$$\mathbb{K}_c^{[1]}(p;q,\lambda;u) = \begin{bmatrix} a & 0 & 0\\ b-a & a & 0\\ c-b-a & 2b-a & a\end{bmatrix},$$

where $a = H_{0,c}(p,q;\lambda;u)$, $b = H_{1,c}(p,q;\lambda;u)$ and $c = H_{2,c}(p,q;\lambda;u)$,

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Hence,

$$\mathscr{F}\mathbb{K}_c^{[1]} = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix} \quad \text{and} \quad \mathscr{H}_c^{[1]}(p,q,\lambda;u) = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix}.$$

This is a particular case of Theorem 5.2 affirmation (5.3).

Example 5.3. For $\alpha = 1$, the matrices, for $n = 2$, $\mathscr{J}^{[1]}(p;q;\lambda;u)$ and $\mathscr{F}$ are

$$\mathscr{J}^{[1]}(p;q\lambda;u) = \begin{bmatrix} a & 0 & 0 \\ b-a & a & 0 \\ c-2b-a & 2b-a & a \end{bmatrix}, \quad \mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Then

$$\mathscr{J}^{[1]}(p;q;\lambda;u)\mathscr{F} = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix} \quad \text{and} \quad \mathscr{H}_c^{[1]}(p;q;\lambda;u) = \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 2b & a \end{bmatrix}.$$

This is a particular case of Theorem 5.2 affirmation (5.5).

Theorem 5.3. *The parametric Apostol-type Frobenius-Euler polynomials matrix $\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u)$ and $\mathscr{H}_s^{[\alpha]}(p;q;\lambda;u)$ can be factored in terms of the Lucas matrix $\mathscr{L}$ of the following form*

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{L}\mathscr{L}_{1,c}^{[\alpha]}(x;y;a) \tag{5.7}$$

or

$$\mathscr{H}_c^{[\alpha]}(p;q;\lambda,u)) = \mathscr{L}_{2,c}^{[\alpha]}(p;q;\lambda,u))\mathscr{L},$$
$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda,u) = \mathscr{L}\mathscr{L}_{1,s}^{[\alpha]}(p;q;\lambda,u)$$

or

$$\mathscr{H}_s^{[\alpha]}(p;q;\lambda,u) = \mathscr{L}_{2,s}^{[m-1,\alpha]}(p;q;\lambda,u))\mathscr{L}.$$

Proof. The relation (5.7) is equivalent to

$$\mathscr{L}^{-1}\mathscr{H}_c^{[\alpha]}(p;q;\lambda;u) = \mathscr{L}_{1,c}^{[\alpha]}(x;y;a),$$

following the ideas of [11, Theorem 9], and making the corresponding modifications, (5.7) is obtained. ☐

Example 5.4. For $\alpha = 1$, the matrices, for $n = 2$, $\mathscr{L}_{1,c}^{[1]}(p;q;\lambda,u)$ and $\mathscr{L}$ are

$$\mathscr{L}_{1,c}^{[1]}(p;q,\lambda,u) = \begin{bmatrix} a & 0 & 0 \\ b-3a & a & 0 \\ c-3b+5a & 2b-3a & a \end{bmatrix}, \quad \mathscr{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix},$$

$$\mathscr{L}\mathscr{L}_{1,c}^{[1]}(x;y;a)=\begin{bmatrix}a & 0 & 0\\ b & a & 0\\ c & 2b & a\end{bmatrix}\quad\text{and}\quad \mathscr{H}_{c}^{[1]}(p,q,\lambda,u)=\begin{bmatrix}a & 0 & 0\\ b & a & 0\\ c & 2b & a\end{bmatrix}.$$

This is a particular case of Theorem 5.3 affirmation (5.7).

6. Conclusions

The paper aims to present the study of new properties of the polynomials that are introduced in [10]. Certain expressions, representations, and summations of these polynomials are derived in terms of well-known classical special functions. The results we have considered in this paper indicate the usefulness of the series rearrangement technique used to deal with the theory of special functions. we have obtained new series of the Taylor type involving the Apostol Frobenius-Euler numbers and Frobenius-Euler numbers. Finally, they addressed the generalized parametric Apostol-type Frobenius-Euler polynomials matrices and show some of their properties.

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