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GRÖBNER LATTICE-POINT ENUMERATORS AND SIGNED TILING BY k-IN-LINE POLYOMINOES

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Abstract. Conway and Lagarias observed that a triangular region $T_2(n)$ in a hexagonal lattice admits a *signed tiling* by 3-in-line polyominoes (tribones) if and only if $n \in \{3^2d-1, 3^2d\}_{d\in\mathbb{N}}$. We apply the theory of Gröbner bases over integers to show that $T_3(n)$, a three dimensional lattice tetrahedron of edge-length n, admits a signed tiling by tribones if and only if $n \in \{3^3d-2, 3^3d-1, 3^3d\}_{d\in\mathbb{N}}$. More generally we study *Gröbner lattice-point enumerators* of lattice polytopes and show that they are (modular) quasipolynomials in the case of k-in-line polyominoes. As an example of the "unusual cancelation phenomenon", arising only in signed tilings, we exhibit a configuration of 15 tribones in the 3-space such that exactly one lattice point is covered by an odd number of tiles.

1. Introduction

Following Conway and Lagarias [6], Reid [12], and other authors, we say that a finite region (polyomino) R, in a (hexagonal) lattice tiling of the plane, has a *signed tiling* ($\mathbb{Z}$-tiling), by *prototiles* from a given set Σ, if there exists a (possibly overlapping) placement of a finite number of copies of prototiles in the plane such that:

- the total covering multiplicity of elementary cells (hexagons) in R is $+1$;
- the total covering multiplicity of elementary cells outside of R is 0.

Figures 1 and 2 nicely illustrate these concepts. The set R, depicted in Figure 1 on the left, is a triangular region in the hexagonal tiling of the plane. The prototiles, also exhibited in Figure 1 on the left, are 3-in-line polyominoes, called 3-bones. The

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objective is to cover or more precisely to distribute copies of these prototiles over R, so that they (counted with positive or negative multiplicity) form a covering of R.

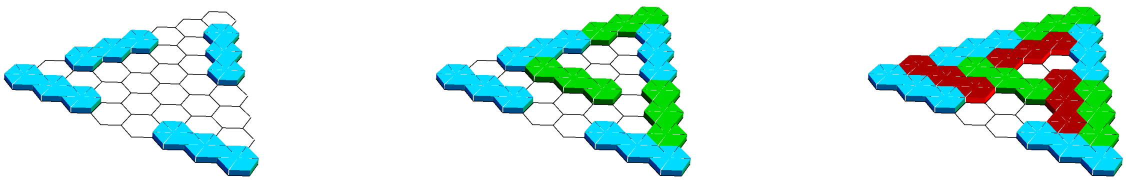

FIGURE 1

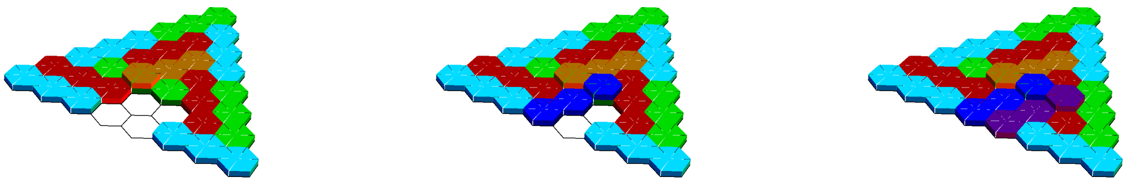

FIGURE 2

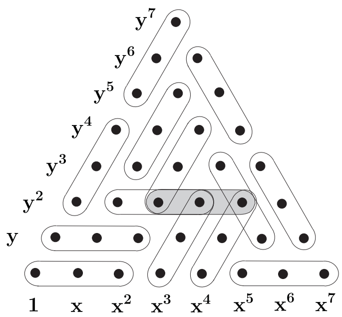

FIGURE 3. A signed tiling of a triangular region by 3-bones

We see (Figure 1) how 3-bones are initially added in an attempt to cover R without overlaps. We continue (Figure 2) by allowing overlaps, until R is completely covered with 3-bones. In the rightmost image depicted in Figure 2 we see that each cell (hexagon) has multiplicity $+1$ or $+2$, where precisely three hexagons have multiplicity $+2$. Finally, these three cells can be subtracted by adding a 3-bone of multiplicity -1 (the shaded region depicted in Figure 3).

1.1. **Algebraic method.** In an algebraic reformulation of the problem each cell (lattice point) is associated a monomial $(p, q) = pe_1 + qe_2 \mapsto x^p y^q$ and the signed tiling can be interpreted as an algebraic identity in the ring $\mathbb{Z}[x, y]$.

More explicitly the basic 3-bones are interpreted as quadratic polynomials $b_1 = x^2 + x + 1, b_2 = y^2 + y + 1, b_3 = x^2 + xy + y^2$, the region R is represented by the polynomial $T_2(8)$, where

$$T_2(n) = \sum_{\substack{0\leq i,j\leq n-1 \\ i+j\leq n-1}} x^i y^j \,, \tag{1.1}$$

the shaded region in Figure 3 is recorded as the polynomial $x^2y^2b_1$ and the algebraic equivalent of the signed tiling described in Figures 1, 2, 3 is the identity

$$T_2(8) = (1+y+x^5-x^2y^2)b_1 + (x^3+x^4+y^2+x^2y^2+xy^3+y^5)b_2 + (x^5y+x^6y+x^3y^4)b_3\,.$$

1.2. **Ideal membership problem and Gröbner bases.** As demonstrated in the previous section, the existence of a signed tiling in general can be reduced to the *Ideal membership problem* [7, Chapter 2], which can be often successfully treated by the method of Gröbner basis [7, 8].

The approach to signed polyomino tilings via Gröbner bases was originally proposed by Bodini and Nouvel [5]. We independently discovered this idea and, inspired by [12], applied it in [10] to the calculation of *tile homology groups* (originally introduced in [12]) and in [9] for the study of $\mathbb{Z}$-tilings with symmetries [9].

Since we apply the general theory to polynomials with integer coefficients, we work with *strong Gröbner bases* [1, 11] (called a D-Gröbner base in [4]), see also [10, Section 5] or our Section 6 for a brief introduction.

1.3. **Summary of new results.** Conway and Lagarias proved [6, Theorem 1.4] that a triangular region $T_2(n)$ in a hexagonal lattice admits a signed tiling by 3-in-line polyominoes (called tribones in [15]) if and only if $n \in \{9d-1, 9d\}_{d\in\mathbb{N}}$. In particular the $\mathbb{Z}$-tiling exhibited in Figure 3 is discovered by these authors.

By applying the theory of Gröbner bases over integers, we extended in [10] this result to k-bones (k-in-line polyominoes) for all $k \geq 2$. More explicitly we showed that the triangular region in the hexagonal tiling of the plane associated to the polynomial $T_2(n)$ admits a signed tiling by k-bones if and only if

$$n \in \{k^2d-1, k^2d\}_{d\in\mathbb{N}}\,.$$

In this paper we address the general problem of $\mathbb{Z}$-tiling by k-bones in d-dimensional lattices, with the emphasis on the tiling of three dimensional polytopes with 3-bones.

We proved (Theorem 2.1) that the lattice tetrahedron associated to the polynomial $T_3(n)$ admits a $\mathbb{Z}$-tiling by all six tribones in the 3-dimensional lattice if and only if

$$n \in \{3^3d-2, 3^3d-1, 3^3d\}_{d\in\mathbb{N}}\,.$$

A new phenomenon, characteristic for $\mathbb{Z}$-tiling with tribones in dimension 3, is the appearance of a constant polynomial 9 in the associated Gröbner basis. As a consequence we construct in Section 3 a "tribone star", that is a configuration of tribones with integer weights such that the total weight is non-zero only at the center of the star.

We call this a "cancelation phenomenon" and, as another consequence, we exhibit (Corollary 3.1) a configuration of 15 tribones in the 3-space where exactly one lattice point (the center of the star) is covered by an odd number of tiles.

Motivated by the ideas used in the proof of Theorem 2.1, we introduce *Gröbner lattice-point enumerators* in Section 4.1, as a proper setting for studying general d-dimensional, $\mathbb{Z}$-polyomino tilings. We demonstrate how the general theory can be considerably simplified in the case of k-in-line prototiles (k-bones) by introducing *cyclotomic ideals* (Section 4.4).

As a first step in developing the associated "Ehrhart theory", we show in Section 5 (Theorem 5.2) that Gröbner lattice-point enumerators for k-bones are (modular) quasipolynomials. In other words they behave similarly as the classical lattice-point enumerators of rational polytopes, a fact that considerably simplifies their calculation.

2. Signed Tiling of the Lattice Tetrahedron $T_3(n)$

2.1. The tribone ideal I_3^3 in variables x, y, z. A three-in-line polyomino or a tribone, in a cubical integer lattice, is a translate of one of the six types of trominoes, associated with the following quadratic polynomials:

$$b_1 = x^2 + x + 1, \qquad b_2 = y^2 + y + 1, \qquad b_3 = x^2 + xy + y^2,$$
$$b_4 = x^2 + xz + z^2, \qquad b_5 = y^2 + yz + z^2, \qquad b_6 = z^2 + z + 1.$$

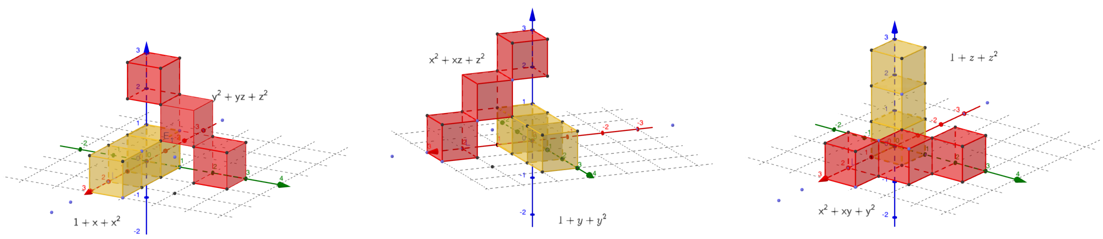

Figure 4. Tribones $b_1, b_2, b_3, b_4, b_5, b_6$.

Let $I_3^3 = \langle b_1, b_2, b_3, b_4, b_5, b_6 \rangle$ be the ideal generated by tribones and GBI the strong Gröbner bases of the ideal I with respect to the lexicographical term order,

$$\begin{aligned} GBI = &\{x^2 + x + 1, xy - y - x - 2, xz - x - z - 2, 3x - 3, \\ &y^2 + y + 1, yz - y - z - 2, 3y - y, z^2 + z + 1, 3z - 3, 9\}. \end{aligned}$$

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In[•]:= IdealTr = {1 + x + x^2, 1 + y + y^2, 1 + z + z^2,
          x^2 + x * y + y^2, y^2 + y * z + z^2, x^2 + x * z + z^2}
Out[•]= {1 + x + x^2, 1 + y + y^2, 1 + z + z^2, x^2 + x y + y^2, y^2 + y z + z^2, x^2 + x z + z^2}

In[•]:= GroebnerBasis[IdealTr, {x, y, z}, CoefficientDomain → Integers]
Out[•]= {9, -3 + 3 z, 1 + z + z^2, -3 + 3 y, -2 - y - z + y z,
         1 + y + y^2, -3 + 3 x, -2 - x - z + x z, -2 - x - y + x y, 1 + x + x^2}
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FIGURE 5. The tribone ideal and its Gröbner basis (*Wolfram Mathematica* 12.3.1).

Denote the polynomials of the Gröbner bases GBI by:

$$\begin{array}{ll} g_1 = x^2 + x + 1, & g_2 = xy - x - y - 2, \\ g_3 = xz - x - z - 2, & g_4 = 3x - 3, \\ g_5 = y^2 + y + 1, & g_6 = yz - y - z - 2, \\ g_7 = 3y - 3, & g_8 = z^2 + z + 1, \\ g_9 = 3z - 3, & g_{10} = 9. \end{array}$$

2.2. **Signed tiling of the tetrahedron $T_3(n)$.** The 3-dimensional analogue of (1.1) is the tetrahedron $T_3(n)$ in the 3-dimensional integer lattice, associated with the polynomial:

$$T_3(n) = \sum_{\substack{0 \le i,j,k \le n-1 \\ i+j+k \le n-1}} x^i y^j z^k.$$

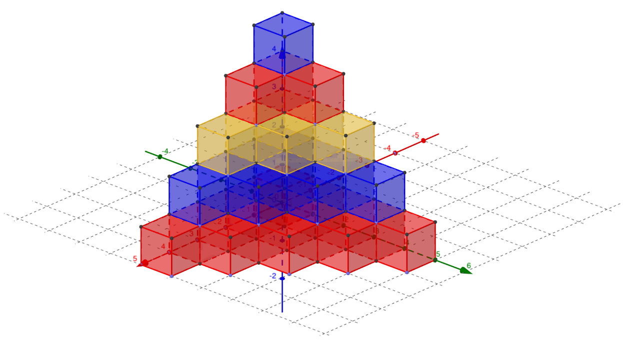

FIGURE 6. The tetrahedron $T_3(5)$.

The goal is to determine for which values of n the tetrahedron $T_3(n)$ admits a $\mathbb{Z}$-tiling by tribones. Following [5] and [10] (see also Section 4) we need to determine when the remainder, obtained by dividing the polynomial $T_3(n)$ by GBI, is equal to zero.

The polynomials $T_3(1) = 1$ and $T_3(2) = 1 + x + y + z$ are already reduced (cannot be further divided by the basis GBI). It follows that they do not admit a $\mathbb{Z}$-tiling with tribones.

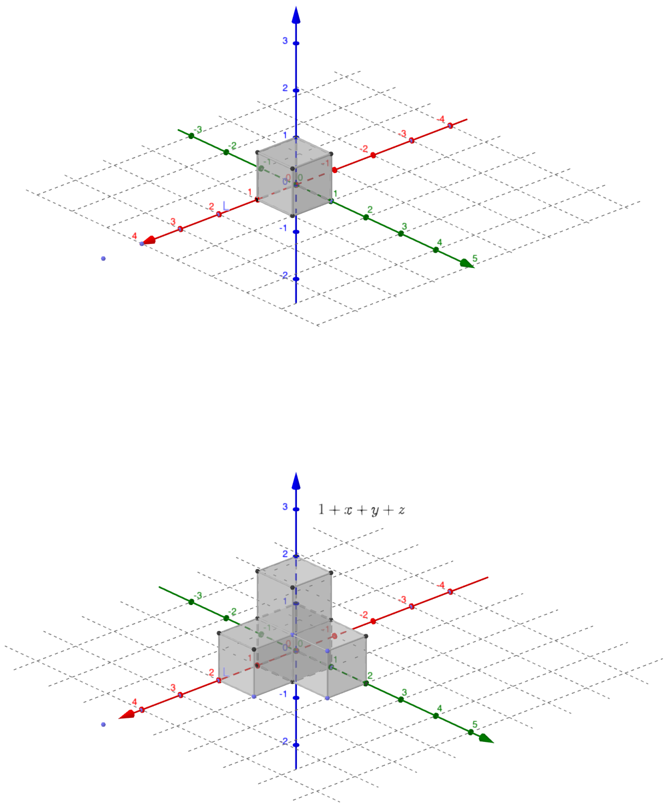

FIGURE 7. Tetrahedron $T_3(n)$ for $n = 1$ and $n = 2$

The remainder on division of the polynomial $T(3)$ by the set GBI is equal to the remainder on division of the region described by the grey cubes (see Figure 8). It follows,

$$T_3(3) \equiv_{GBI} (1 + z)(x + y).$$

Indeed, the region determined by grey cubes is formed by subtracting $1 + z + z^2$ (z-tribone) and $x^2 + xy + y^2$ (xy-tribone) from the region $T_3(3)$.

If $n = 4$, then the remainder on division of the polynomial $T_3(4)$ is congruent with $y^3 + z^3$. Here, $y^3 + z^3$ is a polynomial described by the region of grey cubes formed after subtracting the region determined by the polynomial

$$(z^2 + z + 1)(1 + x + y) + (x + y)(x^2 + xy + y^2).$$

As a consequence we obtain

$$T_3(4) \equiv_{GBI} y^3 + z^3.$$

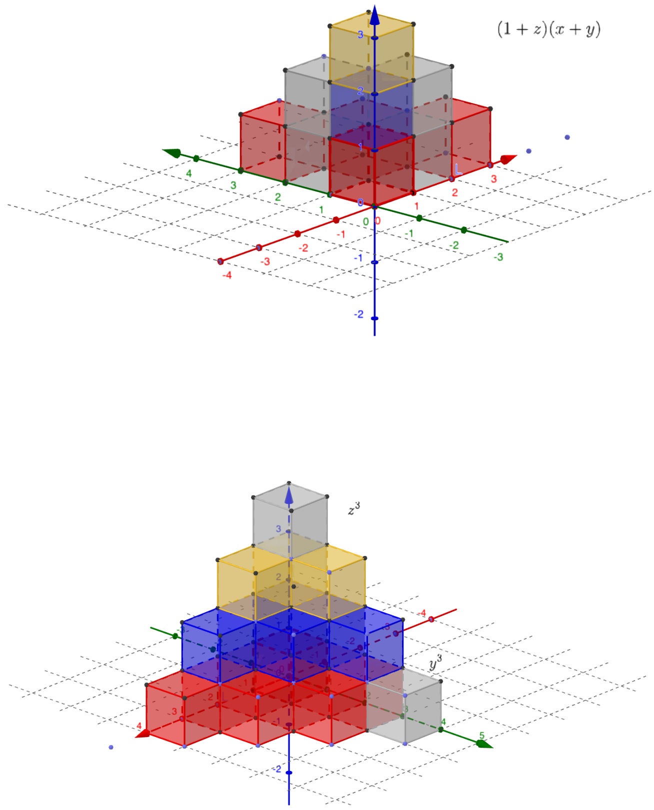

FIGURE 8. Tetrahedron $T_3(n)$ for $n = 3$ and $n = 4$.

The same reasoning applies to the cases $n = 5$ and $n = 6$, which leads to $T_3(5) \equiv_{GBI} (1 + x + y + z)(y^3 + z^3)$ (Figure 9), $T_3(6) \equiv_{GBI} (x + y)(1 + z)(y^3 + z^3)$.

If we proceed with the decomposition of the region $T_3(n)$ in the same manner, we finally conclude

$$T_3(n) \equiv_{GBI} \begin{cases} (x+y)(1+z)f_k(y,z), & n = 3k, \\ f_k(y,z), & n = 3k+1, \\ (1+x+y+z)f_k(y,z), & n = 3k+2, \end{cases} \tag{2.1}$$

where

$$f_k(y,z) = y^{3k} + y^{3(k-1)}z^3 + \cdots + y^3 z^{3(k-1)} + z^{3k} = \sum_{i=0}^{k} y^{3(k-i)} z^{3i}.$$

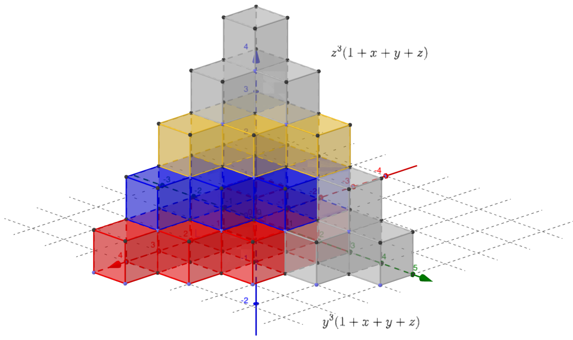

FIGURE 9. Decomposition of the tetrahedron $T_3(5)$.

Lemma 2.1. *For every* $n \in \mathbb{N}$

$$y^3\left(f_0+f_1+\cdots+f_{n-1}\right)=f_0+f_1+\cdots+f_n-\left(z^{3n}+\cdots+z^3+1\right). \tag{2.2}$$

Proof. This is proved by induction on n. The identity is valid for $n=1$ since,

$$y^3 f_0 = y^3 = (1+y^3+z^3)-(z^3+1)=f_0+f_1-(z^3+1).$$

Let us assume that (2.2) is true for $n=k$. Since

$$\begin{aligned}
&y^3\left(f_0+f_1+\cdots+f_{k-1}+f_k\right)\\
=&y^3\left(f_0+f_1+\cdots+f_{k-1}\right)+y^3 f_k\\
=&f_0+\cdots+f_k-\left(z^{3k}+\cdots+z^3+1\right)+y^3\left(y^{3k}+y^{3(k+1)}z^3+\cdots+z^{3k}\right)\\
=&f_0+\cdots+f_k-\left(z^{3k}+\cdots+z^3+1\right)+y^{3(k+1)}+y^{3k}z^3+\cdots+y^3z^{3k}+z^{3(k+1)}\\
&-z^{3(k+1)}\\
=&f_0+\cdots+f_k+f_{k+1}-\left(z^{3(k+1)}+z^{3k}+\cdots+z^3+1\right),
\end{aligned}$$

we conclude that (2.2) holds for $n=k+1$. It follows, by the Principle of mathematical induction, that (2.2) is true for all $n \in \mathbb{N}$. □

Lemma 2.2. *For every* $n \in \mathbb{N}$

(2.3)

$$f_n=(y^3-1)\left(f_0+f_1+\cdots+f_{n-1}\right)+(z^3-1)\left(z^{3(n-1)}+\cdots+(n-1)z^3+n\right)+(n+1).$$

The remainders of the division of polynomial f_n by elements of the basis GBI are periodic, with the period 9.

Proof. If $n=0$ then $f_0=1$. For $n=1$

$$f_1=y^3+z^3=(y^3-1)f_0+(z^3-1)+2,$$

which is in agreement with (2.3). Suppose that the identity (2.3) is valid for some $k \in \mathbb{N}$. Since

$$\begin{aligned} f_{k+1}(y,z) &= y^{3(k+1)} + y^{3k}z^3 + \cdots + y^3z^{3k} + z^{3(k+1)} \\ &= y^3\left(y^{3k} + y^{3(k-1)}z^3 + \cdots + y^3z^{3(k-1)} + z^{3k}\right) + z^{3(k+1)} \\ &= y^3\left((y^3-1)(f_0+\cdots+f_{k-1}) + (z^3-1)(z^{3(k-1)}+\cdots+(k-1)z^3+k)\right. \\ &\quad \left.+(k+1)\right) + z^{3(k+1)} \\ &= (y^3-1)\left(y^3(f_0+\cdots+f_{k-1})\right) + y^3\left((z^3-1)(z^{3(k-1)}+\cdots+\right. \\ &\quad \left.+(k-1)z^3+k)\right) + y^3(k+1) + z^{3(k+1)} \quad \text{(by Lemma 2.1)} \\ &= (y^3-1)\left((f_0+\cdots+f_{k-1}+f_k) - (z^{3k}+\cdots+z^3+1)\right) \\ &\quad + y^3(z^{3k}+\cdots+z^3-k) + y^3(k+1) + z^{3(k+1)} \\ &= (y^3-1)(f_0+\cdots+f_{k-1}+f_k) + z^{3(k+1)} + z^{3k} + \cdots + z^3 + 1 \\ &= (y^3-1)(f_0+\cdots+f_{k-1}+f_k) + (z^3-1)\left(z^{3k}+\cdots+(k+1)\right) \\ &\quad + (k+2), \end{aligned}$$

we conclude that (2.3) holds for $n = k+1$. Therefore, by the Principle of mathematical induction, (2.3) is true for all $n \in \mathbb{N}$.

Since $y^3 - 1 = (y-1)b_2$ i $z^3 - 1 = (z-1)b_6$, we see that

$$(y^3-1)(f_0+f_1+\cdots+f_{n-1}) + (z^3-1)\left(z^{3(n-1)} + 2z^{3(n-2)} + \cdots + (n-1)z^3 + n\right) \in I.$$

From this and (2.3), we conclude that the remainder of the division of f_n by elements of the set GBI equals the remainder of the division $n+1$ by g_{10}. Therefore,

$$\begin{array}{ll} \overline{f_0}^{GBI} = 1, & \overline{f_1}^{GBI} = 2, \\ \overline{f_2}^{GBI} = 3, & \overline{f_3}^{GBI} = 4, \\ \overline{f_4}^{GBI} = -4, & \overline{f_5}^{GBI} = -3, \\ \overline{f_6}^{GBI} = -2, & \overline{f_7}^{GBI} = -1, \\ \overline{f_8}^{GBI} = 0, & \end{array} \tag{2.4}$$

and we see that the remainders are periodic, with period of length 9. For this reason, $f_{9k-1} \equiv_{GBI} 0$, $k \in \mathbb{N}$. □

Theorem 2.1. *The tetrahedron $T_3(n)$ admits a signed tiling by tribones $b_1, b_2, \dots, b_6$ if and only if $n = 3^3k - 2$, $n = 3^3k - 1$ or $n = 3^3k$ for $k \in \mathbb{N}$.*

Proof. The tetrahedron $T_3(n)$ admits a signed tiling by tribones $b_1, \dots, b_6$ if and only if the remainder of the polynomial $T_3(n)$, on division by the Gröbner bases GBI of the ideal I_3^3, is equal to zero.

Since the remainder on division of the polynomial f_n by GBI is periodic with the period 3^2, from (2.1), (2.4) and Table 1, follows that the remainder on division of the polynomial $T_3(n)$ by GBI is periodic with the period 3^3.

Table 1

k	$\overline{T_3(3k-2)}^{GBI}$	$\overline{T_3(3k-1)}^{GBI}$	$\overline{T_3(3k)}^{GBI}$
1	1	x+y+z	4-x-y-z
2	2	2-x-y-z	-1+x+y+z
3	3	3	3
4	4	4+x+y+z	-2-x-y-z
5	-4	-4-x-y-z	2+x+y+z
6	-3	-3	-3
7	-2	-2x+y+z	-1-x-y-z
8	-1	-1-x-y-z	-4+x+y+z
9	0	0	0

From here we finally conclude that the region $T_3(n)$ admits a signed tiling by tribones if and only if $n = 3^3k-2$, $n = 3^3k-1$ or $n = 3^3k$ for some $k \in \mathbb{N}$. $\square$

3. The Role of Number 9 in $\mathbb{Z}$-tiling by Tribones

Let $I_3^3 \subset \mathbb{Z}[x,y,z]$ be the *tribone ideal*, generated by polynomials

$$(3.1)\qquad \begin{array}{lll} A_x = x^2+x+1, & A_y = y^2+y+1, & A_z = z^2+z+1, \\ A_{xy} = x^2+xy+y^2, & A_{xz} = x^2+xz+z^2, & A_{yz} = y^2+yz+z^2, \end{array}$$

renamed to emphasize the symmetry w.r.t. permutations of variables. The Gröbner basis of $I = I_3^3$, with respect to the lexicographic order (Lex) of monomials arising from the order $x > y > z$, is the following:

$$(3.2)\qquad \begin{array}{lll} A_x = x^2+x+1, & A_y = y^2+y+1, & A_z = z^2+y+1, \\ B_{xy} = xy-x-y-2, & B_{xz} = xz-x-z-2, & A_{yz} = yz-y-z-2, \\ C_x = 3x-3, & C_y = 3y-3, & C_z = 3z-3, \\ & D = 9. & \end{array}$$

It follows that there exists a relation

$$(3.3)\qquad 9 = a_1A_x + a_2A_y + a_3A_z + b_1A_{xy} + b_2A_{xz} + b_3A_{yz},$$

for some polynomials $a_i, b_j \in \mathbb{Z}[x,y,z]$. In other words the relation (3.3) guarantees the existence of a signed tiling where the tribones "cancel out" everywhere in the 3-dimensional lattice, except at one point.

Our objective is to make relation (3.3) explicit, for as simple as possible choice of polynomials a_i, b_j.

We essentially apply *Buchberger's Algorithm* (over integers) by iterating the calculation of S-polynomials, beginning with the polynomials from the basis (3.1). Note that, in light of the symmetry of the ideals (3.1) and (3.2), the expression for $3z-3$ in the following proposition can be easily turned in the expression for $3x-3$ (respectively $3y-3$).

Proposition 3.1.

$$\begin{aligned} 9 =& 2\left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right] - \left[RHS(3.9) - 2RHS(3.6)\right], \\ 3z-3 =& \left[RHS(3.9) - 2RHS(3.6)\right] - \left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right]. \end{aligned}$$

Proof. The first row of (3.2) coincides with the first row of (3.1). The second row of (3.2) is obtained by adding and subtracting the polynomials from the first two rows of (3.1), for example

$$B_{xy} = A_{xy} - A_x - A_y\,. \tag{3.4}$$

We continue by computing the S-polynomial of A_x and A_{xy}, and its subsequent reduction

$$\begin{aligned} S[A_x, A_{xy}] =& y(x^2+x+1) - x(xy-x-y-2) = x^2+2xy+2x+y, \\ x^2+2xy+2x+y =& A_x + 2xy + x + y - 1 = A_x + 2B_{xy} + 3(x+y+1). \end{aligned}$$

From here we obtain the relation

$$3(x+y+1) = yA_x - xB_{xy} - A_x - 2B_{xy} = (y-1)A_x - (x+2)B_{xy}, \tag{3.5}$$

which in light of (3.4) produces the relation

$$3(x+y+1) = (y-1)A_x - (x+2)(A_{xy}-A_x-A_y) = (x+y+1)A_x + (x+2)A_y - (x+2)A_{x,y}. \tag{3.6}$$

Similarly, we have the relations

$$3(z+x+1) = (z+x+1)A_z + (z+2)A_x - (z+2)A_{x,z}, \tag{3.7}$$

$$3(y+z+1) = (y+z+1)A_y + (y+2)A_z - (y+2)A_{y,z}, \tag{3.8}$$

and by adding up all three of them we have

$$9+6(x+y+z) = (x+y+z+3)(A_x+A_y+A_z) - (x+2)A_{x,y} - (y+2)A_{y,z} - (z+2)A_{x,z}. \tag{3.9}$$

Let us multiply both sides of (3.6) by 2 and subtract from (3.9). We obtain

$$6z+3 = RHS(3.9) - 2RHS(3.6). \tag{3.10}$$

Note that

$$S(A_z, 6z+3) = 6A_z - z(6z+3) = 3z+6 = 6A_z - zRHS(3.9) + 2zRHS(3.6). \tag{3.11}$$

From (3.10) and (3.11) we finally have

$$\begin{aligned} 9 &= 2(3z+6) - (6z+3) \\ &= 2\left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right] - \left[RHS(3.9) - 2RHS(3.6)\right]. \end{aligned}$$

Note that in passing we obtain an explicit expression for the third row of (3.2) in terms of (3.1). For example

$$\begin{aligned} 3z-3 &= (6z+3) - (3z+6) \\ &= \left[RHS(3.9) - 2RHS(3.6)\right] - \left[6A_z - zRHS(3.9) + 2zRHS(3.6)\right]. \qquad \square \end{aligned}$$

The following corollary is an immediate consequence of the "cancelation phenomenon", exhibited in Proposition 3.1.

Corollary 3.1. *There exists a configuration of* 15 *tribones in the* 3*-space where exactly one lattice point* (*the center of the star*) *is covered by an odd number of tiles.*

Proof. As a consequence of the first relation proved in Proposition 3.1, by reducing modulo 2 we obtain the identity $1 = RHS(3.9)$. By further simplification we obtain the identity

$$1 = (x + y + z + 1)(A_x + A_y + A_z) + xA_{x,y} + yA_{y,z} + zA_{x,z},$$

which completes the proof. □

4. $\mathbb{Z}$-Tiling by k-Bones in d Variables

In this section we address the general problem of the existence of $\mathbb{Z}$-tiling by k-bones in the d-dimensional lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. We use standard abbreviations for monomials (power products) $x^a = x_1^{a_1} x_2^{a_2} \dots x_d^{a_d}$ and rely on standard concepts and terminology used in the theory of lattice-point enumeration in polyhedra, see [2] or [3].

In particular each set $R \subset \mathbb{R}^d$ is associated the *integer-point transform* $\sigma_R = \sum_{a \in R \cap \mathbb{Z}^d} x^a \in \mathbb{Z}[[x_1^{\pm 1}, \dots, x_d^{\pm 1}]]$, which is a Laurent polynomial if and only if R is bounded. Typically R is a convex polytope $Q \subset \mathbb{R}^d_+$ with vertices in $\mathbb{N}^d$ in which case $\sigma_Q \in \mathbb{Z}[x_1, \dots, x_d]$ is simply the sum of all monomials "covered" by Q.

Conversely, for each polynomial $p = \sum_{a \in \mathbb{N}^d} c_a x^a \in \mathbb{Z}[x_1, x_2, \dots, x_d]$ the associated *Newton polytope* is the convex polytope $Newton(p) = \mathrm{Conv}\{a \mid c_a \neq 0\}$.

Let $\Delta = \mathrm{Conv}\{e_i\}_{i=0}^d$ be the standard simplex in $\mathbb{R}^d$, where $e_0 = 0$ and $\{e_i\}_{i=1}^d$ is the standard orthonormal basis of $\mathbb{R}^d$ which generates the lattice $\mathbb{Z}^d$.

Given an integer $k \geq 1$, the k-bones in the d-dimensional lattice are the prototiles associated to the edges $E_{ij} = [ke_i, ke_j]\,(i \neq j)$ of the k^{th} *dilate* $k\Delta = \mathrm{Conv}\{ke_i\}_{i=0}^d$ of the simplex Δ.

More explicitly, the polynomials (integer-point transforms) of the k-bones are

$$\begin{aligned} b_i =& x_i^{k-1} + x_i^{k-2} + \dots + 1, \quad i = 1, \dots, d, \\ b_{ij} =& x_i^{k-1} + x_i^{k-2} x_j + \dots + x_j^{k-2}, \quad 1 \leq i < j \leq d. \end{aligned}$$

The associated ideal

$$I_k^d = \langle b_i, b_{i,j} \rangle \subseteq \mathbb{Z}[x_1, \dots, x_d]$$

is referred to as the k-bone ideal in d-dimensions, or simply as the k-bone ideal.

4.1. **Gröbner lattice-point enumerators.** Our general objective is to study the geometry and combinatorics of $\mathbb{Z}$-tilings of different shapes (convex polytopes) in $\mathbb{R}^d$ by k-bones (or more general prototiles), by methods of combinatorial commutative algebra and Gröbner basis.

The Gröbner basis of the ideal I_k^d with respect to some term order (usually the lexicographic order) is denoted by GBI_k^d (occasionally by GBI or G). We work with $\mathbb{Z}$-coefficients so the Abelian group of all remainders may have torsion and its generators are *reduced monomials* $x^\alpha \notin \langle LM(I_k^d)\rangle$, not contained in the ideal of leading monomials of I_k^d. (The reader is referred to the Appendix (Section 6) for a brief introduction into Gröbner basis theory and a guide to the literature.)

As in Section 2.2 the remainder on division of f by GBI is $\overline{f}^{GBI} = \sum_\alpha c_\alpha x^\alpha$, where x^α are reduced monomials. For improved legibility we sometimes write $Red_G(f)$ instead of $\overline{f}^G$. The coefficient c_α, which takes values in $\mathbb{Z}$ or some quotient $\mathbb{Z}/\nu\mathbb{Z}$, is denoted by

$$[x^\alpha]\left(\overline{f}^{GBI}\right). \tag{4.1}$$

Table 1 (Section 2.2) provides examples of the calculation and illustrates the importance of numerical functions (4.1) for the general polyomino tiling problem.

4.2. **Motivating example.** Here is another point of view which explains why (4.1) are called *Gröbner lattice-point enumerators* (Definition 4.1).

Let Q be a convex polytope with vertices in $\mathbb{N}^d$ and let $\sigma_Q(x) = \sum_{\alpha\in Q\cap\mathbb{N}^d} x^\alpha$ be its "Newton polynomial" (integer-point transform). The usual "discrete volume" (lattice-point enumerator) of Q, defined in [2, 3] as the number of integer points inside Q, is clearly equal to the value of σ_Q at $x = (1, 1, \dots, 1) \in \mathbb{R}^d$.

Moreover, for each polynomial $f(x_1, \dots, x_d) \in \mathbb{Z}[x_1, \dots, x_d]$ there is a relation

$$f(x_1, \dots, x_d) = f_1(x_1 - 1) + \cdots + f_d(x_d - 1) + C, \tag{4.2}$$

where $C = f(1, \dots, 1)$ is the remainder obtained on division of f by the ideal

$$I = \langle x_1 - 1, x_2 - 1, \dots, x_d - 1\rangle.$$

It follows that the number of lattice points in a lattice convex polytope Q can be interpreted as the remainder of σ_Q on division by the ideal I.

4.3. **General research problem.** Division of multivariate polynomials by ideals is in general not unique and in particular the corresponding remainders (such as C in the expression (4.2)) are not uniquely defined. However, the division by the Gröbner basis of an ideal yields a unique remainder (in general a polynomial) which, in agreement with motivating example from Section 4.2, leads to the following research problem.

Let $J \subset \mathbb{Z}[x_1, \dots, x_d]$ be an ideal, say the ideal associated to a set $\mathcal{R}$ of prototiles in $\mathbb{N}^d$. Let $G = G_J$ be the Gröbner basis of J with respect to some term order. It is interesting to ask (for some carefully chosen ideals J) what is the geometric and combinatorial significance of the remainder $\overline{f}_Q^G$ of the integer-point transform σ_Q on division by the Gröbner basis G.

Definition 4.1. The polynomial valued function $Q \mapsto \overline{f}_Q^G$ is referred to as Gröbner or G-discrete volume of Q with respect to the Gröbner basis G. The coefficients (4.1) are called Gröbner lattice-point enumerators of Q.

4.4. **Cyclotomic ideals.** A *cyclotomic ideal* in the ring $\mathbb{Z}[x_1, x_2, \dots, x_d]$ is an ideal of the following form

$$W_k^d = \langle x_1^k - 1, x_2^k - 1, \dots, x_d^k - 1\rangle, \tag{4.3}$$

where d and k are positive integers. In light of the obvious identities

$$x_i^k - 1 = (x_i - 1)(x_i^{k-1} + x_i^{k-2} + \dots + 1), \quad x_i^k - x_j^k = (x_i - x_j)(x_i^{k-1} + x_i^{k-2}x_j + \dots + x_j^{k-1}),$$

W_k^d is contained in the ideal I_k^d generated by k-in-line polyominoes (k-bones) in the d-dimensional lattice.

Proposition 4.1. *The set $S_k^d = \{x_1^k - 1, x_2^k - 1, \dots, x_d^k - 1\}$ is a (strong) Gröbner basis of the ideal W_k^d in the sense of* [11].

Proof. Indeed, the S-polynomial

$$S[x_i^k - 1, x_j^k - 1] = x_j^k(x_i^k - 1) - x_i^k(x_j^k - 1) = (x_i^k - 1) - (x_j^k - 1)$$

is trivially reducible by the basis S_k^d. □

The following criterion for the existence $\mathbb{Z}$-tilings is formulated in [10, Proposition 3.1].

Proposition 4.2. *A polyomino P admits a* signed tiling *by translates of* prototiles $P_1, P_2, \dots, P_k$ *if and only if for some monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with a non-negative exponent $\alpha \in \mathbb{N}^d$ the polynomial $x^\alpha \sigma_P$ is in the ideal generated by polynomials $\sigma_{P_1}, \dots, \sigma_{P_k}$,*

$$x^\alpha \sigma_P \in \langle \sigma_{P_1}, \sigma_{P_2}, \dots, \sigma_{P_k}\rangle. \tag{4.4}$$

Note that $x^\alpha \sigma_P \in J$ implies $x^{\alpha'} \sigma_P \in J$ in any ideal J, provided $x^{\alpha'}$ is divisible by x^α, which allows us to formulate the following simplified criterion for k-bone ideals I_k^d.

Proposition 4.3. *A polyomino P admits a* signed tiling *by translates of k-bones E_{ij}, $0 \le i < j \le d$, if and only if*

$$\sigma_P \in I_k^d\,. \tag{4.5}$$

Proof. If $\sigma_P \in I_k^d$ then obviously P admits a signed tiling by translates of k-bones E_{ij}. Conversely, suppose P admits a signed tiling by translates of k-bones E_{ij}. By Proposition 4.2 there exists a monomial x^α such that $x^\alpha \sigma_P \in I_k^d$. Since for some $\beta \in \mathbb{N}^d$ the vector $\alpha + \beta = k\gamma \in k\mathbb{N}^d$ is divisible by k we conclude that $x^{k\gamma}\sigma_P \in I_k^d$. Since $W_k^d \subset I_k^d$ we know that $x^{k\gamma} \equiv 1 \pmod{I_k^d}$, which in turn implies $\sigma_P \in I_k^d$. □

4.5. **Reduction of monomials x^a modulo W_k^d and I_k^d.** Let $x^a = x_1^{a_1}x_2^{a_2}\dots x_d^{a_d}$ be the monomial with multi-index $a \in \mathbb{Z}_+^d$. Given $z \in \mathbb{Z}_+$, let $\widehat{z} = r(z)$ be the remainder on division of z by k, $r(z) \in \mathbb{Z}_k = \{0, 1, \dots, k-1\}$. The reduced version of the monomial x^a with respect to the ideal W_k^d is the monomial $Red_{W_k^d}(x^a) = x^{\widehat{a}} = x_1^{\widehat{a_1}}x_2^{\widehat{a_2}}\dots x_d^{\widehat{a_d}}$.

Note that $Red_{W_k^d}(x^a)$ is obtained from x^a by successive division (in any order) by elements of the ideal W_k^d.

Our objective is to compute the W_k^d-reduced version of the polynomial $T_k^d(n)$

$$Red_{W_k^d}(T_k^d(n)) = Red_{W_k^d}\left(\sum_{\substack{0\le a\\|a|\le n-1}} x^a\right) := \sum_{\substack{0\le a\\|a|\le n-1}} x^{\widehat{a}}. \tag{4.6}$$

Proposition 4.4. *Let*

$$Red_{W_k^d}(T_k^d(n)) = Red_{W_k^d}\left(\sum_{\substack{0\le a\\|a|\le n-1}} x^a\right) = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)x^r \tag{4.7}$$

be the reduction of the polynomial $T_k^d(n)$ with respect to the ideal W_k^d. Then

$$t_k^d(n,r) = \binom{d+(n|r)}{d},$$

where

$$(n|r) := \left\lfloor \frac{n-1-|r|_1}{k} \right\rfloor$$

and $|r|_1 = |(r_1, r_2, \dots, r_d)|_1 = r_1 + \cdots + r_d$.

Proof. Given a W_k^d-reduced monomial x^r, where $r = (r_1, r_2, \dots, r_d) \in (\mathbb{Z}_k)^d$, we want to calculate the number of solutions of the inequality

$$(kx_1 + r_1) + (kx_2 + r_2) + \cdots + (kx_d + r_d) \le n-1 \tag{4.8}$$

in non-negative integer variables $x_1, \dots, x_d$. Equivalently, we need to calculate the number of non-negative integer solutions of

$$x_1 + x_2 + \cdots + x_d \le \left\lfloor \frac{n-1-|r|_1}{k} \right\rfloor, \tag{4.9}$$

where $\lfloor x \rfloor$ is the integer part of x. Recall the lattice point enumerator [3, Theorem 2.2] of the standard simplex in the positive hyperorthant $\mathbb{R}_+^d$ bounded by the hyperplane $x_1 + \cdots + x_d = m$,

$$L_\Delta(m) = \binom{d+m}{d}.$$

By substitution $m = (n|r)$ we complete the proof of the proposition. □

As a corollary we obtain the following proposition.

Proposition 4.5. *Let $GBI = GBI_k^d$ be a Gröbner basis of the ideal I_k^d with respect to some term order. Then the remainder*

$$\overline{T_k^d(n)}^{GBI}$$

of the polynomial $T_k^d(n)$ on division by $GBI = GBI_k^d$, expressed in terms basic monomials x^α, admits a decomposition

$$\overline{T_k^d(n)}^{GBI} = \sum_\alpha c_\alpha x^\alpha, \tag{4.10}$$

where $c_\alpha = [x^\alpha](\overline{T_k^d(n)}^{GBI})$ is some (finite) $\mathbb{Z}$-linear combination of functions $t_k^d(n,r)$. More explicitly,

$$c_\alpha = [x^\alpha]\left(\overline{T_k^d(n)}^{GBI}\right) = \sum_{r\in(\mathbb{Z}_k)^d} e_\alpha^r t_k^d(n,r),$$

for some integers e_α^r.

Proof. As a consequence of (4.7) we obtain

$$\overline{T_k^d(n)}^{GBI} = Red_{I_k^d}(T_k^d(n)) = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)\overline{x^r}^{GBI} = \sum_{r\in(\mathbb{Z}_k)^d} t_k^d(n,r)\sum_\alpha e_\alpha^r x^\alpha. \tag{4.11}$$

□

5. Ehrhart Theory and Gröbner Bases

Quasipolynomials play a fundamental role in the Ehrhart theory of lattice-point enumerators of polytopes with rational vertices. We demonstrate that they play a similar role in Gröbner lattice-point enumeration with respect to ideals W_k^d and I_k^d.

5.1. **Quasipolynomials.**

A *quasipolynomial* [13, Section 4.4] of degree d is a function $f : \mathbb{N} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{n-1} + \cdots + c_0(n),$$

where each $c_i(n)$ is a periodic function and $c_d(n)$ is not identically equal to zero.

It is not difficult to show that f is a quasipolynomial if and only if there exists an integer $N > 1$ and polynomials $f_0, f_1, \ldots, f_{N-1}$ such that

$$f(n) = f_i(n), \quad \text{if } n \equiv i \pmod N.$$

Quasipolynomials play an exceptionally important role in enumerative combinatorics. For example the *Ehrhart polynomial* $L_Q(n)$, defined as the lattice point enumerator of the n^{th} dilate nQ of a convex polytope Q with rational vertices, is always a quasipolynomial.

It is an easy exercise to check that the function $t_k^d(n,r)$, introduced in Proposition 4.4, is a quasipolynomial in the variable n. In turn, the coefficients c_α (that appear in Proposition 4.5) are also quasipolynomials, being linear combinations of functions $t_k^d(n,r)$.

The functions $c_\alpha = c_\alpha(T_k^d(n))$, being defined essentially as summands of the remainder on division by the ideal I_k^d, are extended in a straightforward way to all convex rational convex polytopes Q. They are referred to as *Gröbner lattice-point enumerators.*

5.2. **Quasipolynomials and generalizations of Pick's theorem.** Here we remind the reader why (quasi)polynomials are important in lattice-point enumeration problems (Ehrhart theory). In the planar case the Ehrhart polynomial is a polynomial $L_Q(n) = a_0n^2 + a_1n + a_2$ where $a_0 = Area(Q)$ and $a_2 = L_Q(0) = 1$. Moreover, $L_Q(1) = a_0 + a_1 + a_2$ is the number of lattice points in Q and, by Ehrhart-Macdonald reciprocity (see [3, Theorem 4.1]),

$$L_Q(1) + L_Q(-1)$$

is the number of lattice points on the boundary of Q.

The four quantities $a_0, L_Q(0), L_Q(1)$ and $L_Q(1)+L_Q(-1)$ can be interpreted as linear forms on the 3-dimensional vector space of all quadratic polynomials and classical *Pick's theorem* is nothing but a non-trivial linear relation

$$\lambda_1 a_0 + \lambda_2 L_Q(0) + \lambda_3 L_Q(1) + \lambda_4(L_Q(1) + L_Q(-1)) = 0. \tag{5.1}$$

Once we know that such a relation exists, the coefficients λ_i are easily evaluated by choosing special polygons Q.

The importance of this proof of Pick's theorem is that it can be easily generalized. For example Reeve's theorem (a 3-dimensional analogue of Pick's theorem) says that in addition to linear forms listed in (5.1) it suffices to take one more, the form $L_Q(2)$ evaluating the number of lattice points in the second dilate of Q.

Similar scheme can be applied to quasipolynomials as well and the following sections should provide a theoretical basis for studying analogues of Pick's theorem for Gröbner basis enumerators of lattice polytopes. (This is the subject of a subsequent publication.)

5.3. **Ehrhart quasipolynomial for Gröbner W_k^d-enumerators.** In this section we prove that Gröbner lattice-point enumerators of lattice polytopes, with respect to the ideal W_k^d, are *quasipolynomials.* We have already calculated (Section 4.5) the W_k^d-reduction of the tetrahedron associated to the polynomial $T_k^d(n)$ and showed (Proposition 4.4) that the result is a quasipolynomial in the variable n. Here we extend this result to the case of a general rational polytope.

Theorem 5.1. *Let σ_Q be the integer-point transform of a rational convex polytope $Q \subset (\mathbb{R}_+)^d$ and $c_\alpha^{W_k^d}(Q) = [x^\alpha](Red_{W_k^d}(\sigma_Q))$ the Gröbner lattice-point enumerator with respect to the ideal W_k^d, associated to a W_k^d-reduced monomial x^α. Then the function*

$$f_\alpha^{W_k^d}(n) = c_\alpha^{W_k^d}(nQ) = [x^\alpha](Red_{W_k^d}(\sigma_{nQ})),$$

computing the Gröbner basis enumerator $c_\alpha^{W_k^d}$ *of the* n^{th} *dilate of the convex polytope* Q*, is a quasipolynomial in the variable* n*.*

As usual in Ehrhart theory [3, Chapter 3], the case of a general rational polytope is reduced to the case of a rational simplex. Moreover the case of general rational simplex (simplicial cone) is treated similarly as the case of a simplex with integral vertices. So the proof of Theorem 5.1 follows from the proof of the following proposition.

Proposition 5.1. *Let* $\Delta \subset (\mathbb{R}_+)^d$ *be a simplex with integral vertices and let* $r = (r_1, \dots, r_d) \in (\mathbb{Z}_k)^d$*. Then a mod-*$k$ *lattice-point enumerator* $L_\Delta^{k,r}(n)$ *of* Δ*, defined as the number of lattice points* $a = (a_1, \dots, a_d) \in \mathbb{Z}^d \cap n\Delta$ *such that* $a_i \equiv r_i \mod k$ *for each* $i \in [d]$*, is a quasipolynomial in* d*.*

Proof. Since $L_\Delta^{k,r}(n) = L_{v+\Delta}^{k,r}(n)$ for each $v \in \mathbb{Z}^d$ we assume, without loss of generality, that $-\frac{r}{k} + \Delta \subset (\mathbb{R}_+)^d$. Let $\Delta = \mathrm{Conv}\{v_i\}_{i=1}^{d+1}$.

By [3, Theorem 3.5] it is known that the integer-point transform σ_{v+K} of a shifted simplicial cone

$$K = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_d w_d \mid \lambda_i \geq 0\} \subseteq \mathbb{R}^d \tag{5.2}$$

is the rational function

$$\sigma_{v+K}(z) = \frac{\sigma_{v+\Pi}(z)}{(1 - z^{w_1})(1 - z^{w_2}) \cdots (1 - z^{w_d})}, \tag{5.3}$$

where

$$\Pi = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_d w_d \mid 0 \leq \lambda_i < 1\}$$

is the associated *fundamental half-open parallelepiped.* Let $w_i = (v_i, 1) \in \mathbb{R}^{d+1}$, $i \in [d+1]$, and let $K \subset \mathbb{R}^{d+1}$ be the associated simplicial cone defined by (5.2), with the associated fundamental parallelepiped Π.

It follows that the integer-point transform $\sigma_K(z, t)$ of K is given by the formula (5.3), where d is replaced by $d + 1$ and the new (vertical) variable is t. Moreover [3, Section 3.3], the n^{th} dilate of Δ is essentially the intersection of K with the horizontal hyperplane $H_n := \{(z, t) \in \mathbb{R}^{d+1} \mid t = n\}$, and the generating function for the Ehrhart polynomial $L_\Delta(n)$, calculating the number of lattice points in $n\Delta$, is given by the formula

$$\sum_{n \geq 0} L_\Delta(n)\, t^n = \sigma_K(\mathbb{1}, t),$$

where the RHS is evaluated at $z = \mathbb{1} = (1, 1, \dots, 1) \in \mathbb{R}^d$.

We want to describe the generating function calculating the lattice points $a \in K$ such that $a = ka' + r$ for some $a' \in \mathbb{Z}^{d+1}$. In other words we need a generating function for the set of lattice points a' in the shifted cone

$$K' = -\frac{r}{k} + \frac{1}{k} K.$$

Again by (5.3), taking into account that K' is scaled down by the factor k, we obtain

$$\sum_{n\geq 0} L_{\Delta}^{k,r}(n)\, t^n = \sigma_{K'}(\mathbb{1}, t) = \frac{g(t)}{(1-t^k)^{d+1}},$$

where $g(t) = \sigma_{\Pi'}(\mathbb{1}, t)$ and $\Pi' = -r/k + \Pi$ is the shifted fundamental parallelepiped of K'.

By assumption $-\frac{r}{k} + \Delta \subset (\mathbb{R}_+)^d$ which implies that

$$(-r/k + \Pi) \cap \mathbb{Z}^{d+1} \subseteq \Pi \cap \mathbb{Z}^{d+1} \subset \mathbb{N}^{d+1}\,.$$

It follows that $\deg(g) < k(d+1)$ and, as a consequence of Proposition 4.4.1 [13, Proposition 4.4.1], we conclude that $L_{\Delta}^{k,r}(n)$ is a quasipolynomial. □

5.4. **Ehrhart theory for Gröbner I_k^d-enumerators.** Here we show that Gröbner lattice-point enumerators of lattice polytopes, with respect to the ideal I_k^d, are (modular reductions of) *quasipolynomials*. Since quasipolynomials naturally appear as lattice points enumerators (Ehrhart theory) for convex polytopes with rational vertices, see [3, Section 3.7], the following result can be interpreted as a first step in the direction of developing Ehrhart theory for Gröbner basis enumerators of rational convex polytopes.

We say that a function $f : \mathbb{N} \to \mathbb{Z}_\nu$ (where $\nu \in \mathbb{Z}_+ \cup \{\infty\}$ and by convention $\mathbb{Z}_\infty = \mathbb{Z}$) is a *modular quasipolynomial*, if there exists and integer valued function $f' : \mathbb{N} \to \mathbb{Z}$ such that $f(n)$ is the mod ν reduction of $f'(n)$ for each $n \in \mathbb{N}$.

Theorem 5.2. *Let σ_Q be the integer-point transform of a rational convex polytope Q in $(\mathbb{R}_+)^d$ and $c_\beta = c_\beta^{I_k^d}(Q) = [x^\alpha](Red_{I_k^d}(\sigma_Q))$ the Gröbner lattice-point enumerator associated to a I_k^d-reduced monomial x^β. Then the function*

$$f_\beta^{I_k^d}(n) = c_\beta^{I_k^d}(nQ) = [x^\beta](Red_{I_k^d}(\sigma_{nQ})),$$

computing the Gröbner lattice-point enumerator c_β of the n^{th} dilate of the convex polytope Q, is a modular quasipolynomial in the variable n.

Proof. Since $W_k^d \subset I_k^d$,

$$Red_{I_k^d}(\sigma_{nQ}) = Red_{I_k^d}(Red_{W_k^d}(\sigma_{nQ})) = Red_{I_k^d}\left(\sum_\alpha c_\alpha x^\alpha\right),$$

where on the right is an expression involving W_k^d-reduced monomials x^α. Since for each I_k^d-reduced monomial x^β

$$[x^\beta](Red_{I_k^d}(\sigma_{nQ})) = [x^\beta]\left(\sum_\alpha c_\alpha Red_{I_k^d}(x^\alpha)\right) = \sum_\alpha c_\alpha [x^\beta]\left(Red_{I_k^d}(x^\alpha)\right),$$

the result is an immediate consequence of Theorem 5.1. □

Remark 5.1. We have shown (Theorems 5.1 and 5.2) that Gröbner lattice-point enumerators of ideals W_k^d and I_k^d are (modular) quasipolynomials. Is this a general phenomenon? In other words is it true that G-enumerators of (polyomino) ideals are (modular) quasipolynomials for any choice of prototiles. We suspect that the answer is negative in general but we don't have an example at hand.

6. Appendix: Gröbner Bases

The reader not familiar with the fundamental concepts and results of Gröbner bases theory is encouraged to use it as black box, after consulting a two page introduction in [14]. Since [14] deals only with polynomials with coefficients in the field here we briefly outline, following [11], how the theory is modified if we work with integer coefficients.

A term is a product $t = cx^\alpha$ where c is the coefficient and $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is the associated monomial (power product). For a given polynomial $f \in \mathbb{Z}[x_1, x_2, \dots, x_k]$ the associated remainder on division by a Gröbner basis G is $\overline{f}^G$ and f reduces to zero $f \xrightarrow{G} 0$ if $\overline{f}^G = 0$. $LM(f)$ and $LC(f)$ are respectively the leading monomial and the leading coefficient with respect to the chosen term order $\preceq$. We write $lcm(a, b)$ and $gcd(a, b)$ respectively for the least common multiple and the greatest common divisor of a and b.

For other basic notions of Gröbner basis theory (over integers), such as S-polynomial, standard representation, etc. the reader is referred to [11] (see also [1, 4] for a more complete exposition of the theory).

6.1. **Gröbner bases over principal ideal domains.** Let $\Lambda = R[x_1, \dots, x_k]$ be the ring of polynomials with coefficients in a principal ideal domain R. For a given ideal $I \subset \Lambda$ the associated *strong Gröbner basis*, called also the D bases in [4], may be introduced as follows (see [1, p. 251] and [4, p. 455]).

Definition 6.1. A finite set $G \subset I$ is a strong Gröbner basis of I (with respect to the chosen term order $\preceq$) if for each $f \in I \setminus \{0\}$ there exists $g \in G$ such that the leading term of f is divisible by the leading term of g, $LT(g)|LT(f)$, meaning that $LT(f) = tLT(g)$ for some term t.

The following theorem provides a useful criterion for testing whether a finite set of polynomials is a Gröbner basis of the ideal generated by them, see [4, Chapter 10, Corollary 10.12].

Theorem 6.1. *Let G be a finite collection of non-zero polynomials which generate an ideal I_G. Suppose that,*

(1) *for each pair $g_1, g_2 \in G$ there exists $h \in G$ such that,*
$$LM(h)|\mathrm{lcm}(LM(g_1), LM(g_2)) \text{ and } LC(h)|\mathrm{gcd}(LC(g_1), LC(g_2));$$

(2) *for each pair $g_1, g_2 \in G$ the associated S-polynomial reduces to zero,*
$$S(g_1, g_2) \xrightarrow{G} 0.$$

Then G is a strong Gröbner basis of I_G.

6.2. **Gröbner bases over Euclidean domains.** The general theory is further simplified if one works with Euclidean domains. Aside from standard references [1, 4] a self-contained account can be found in [11]. In the case of integers one usually chooses the linear ordering,

$$\cdots < 0 < +1 < -1 < +2 < -2 < +3 < -3 < +4 < -4 < +5 < \cdots, \tag{6.1}$$

which allows us to define unambiguously remainders, S-polynomials etc.

Recall that the constant $g_{10} = 9$ is an element of the Gröbner basis GBI of the tribone ideal (Section 2.1). The ordering (6.1) explains why -4 (rather than $+5$) appears in reduced expressions f^{GBI}, for example in Table 1.

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