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CRITICAL POINT APPROACHES FOR A CLASS OF DIFFERENTIAL EQUATIONS WITH STURM-LIOUVILLE TYPE NONHOMOGENEOUS BOUNDARY CONDITIONS

SHAPOUR HEIDARKHANI AND FARAHNAZ AYAZI

Abstract. A class of p-Laplacian equations with Sturm-Liouville type nonhomogeneous boundary value problem with nonlinear derivative depending on two control parameters is investigated. Existence and multiplicity of solutions are discussed by means of variational methods and critical point theory. Two examples supporting our theoretical results are also presented.

1. Introduction

Various generalizations of classical Sturm-Liouville problems for ordinary linear differential equations have attracted a lot of attention because of appearance of new important applications in physical sciences and applied mathematics. Sturm-Liouville boundary value problems have received a lot of attention in recent years. There have been many papers studying the existence of solutions for boundary value problems, for a small sample of recent work, we refer the reader to [1,7,8,11,13,16–18] that authors have studied the existence of solutions of Sturm-Liouville boundary value problem by using critical point theorem and fixed point theorem. For example, Bonanno and Riccobono in [8] have established the existence of multiple solutions for the second order Sturm-Liouville boundary value problem

$$\begin{cases}(\rho\phi_p(x'))' + s\phi_p(x) = \lambda f(t,x), & t \in [a,b],\\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B, & \end{cases}$$

Key words and phrases. Multiple solutions, p-Laplacian equation, Sturm-Liouville type nonhomogeneous boundary condition, variational methods.

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where $p > 1$, $\phi_p(x) = |x|^{p-2}x$, $\rho, s \in L^{\infty([a,b])}$ with $\mathrm{essinf}_{[a,b]}\rho > 0$ and $\mathrm{essinf}_{[a,b]}s > 0$, $A, B \in \mathbb{R}$, $\alpha, \beta, \gamma, \sigma > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and λ is a positive real parameter. In [18] Tian and Ge, applying a three critical point theorem due to Averna and Bonanno discussed the existence of three solutions for a Sturm-Liouville boundary value problem depending upon the parameter λ, while in [17] using lower and upper solutions approach and variational methods they proved the existence of multiple solutions for second order Sturm-Liouville boundary value problem

$$\begin{cases} -Lu = f(x,u), & x \in [0,1], \\ R_1(u) = 0,\ R_2(u) = 0, \end{cases}$$

where $Lu = (p(x)u')' - q(x)u$ is a Sturm-Liouville operator $R_1(u) = \alpha u'(0) - \beta u(0)$, $R_2(u) = \gamma u'(1) + \sigma u(1)$. In [13] using critical point theory and Ricceri's variational principle, the existence of infinitely many classical solutions to a boundary value system with Sturm-Liouville boundary conditions was obtained.

In the present paper, we investigate the existence of solutions for the Sturm-Liouville type nonhomogeneous boundary value problem

(1.1)
$$\begin{cases} -(\phi_p(u'))' = \left(\lambda f(x,u(x)) + \displaystyle\int_0^{u'(x)} \frac{\partial}{\partial x}\left(\frac{(p-1)|\tau|^{p-2}}{h(x,\tau)}\right) d\tau\right) h(x,u'(x)), & x \in (a,b), \\ \alpha u(a) - \beta u'(a) = A, \quad \gamma u(b) + \sigma u'(b) = B, \end{cases}$$

where $p > 1$, $\phi_p(t) = |t|^{p-1}t$, $\lambda > 0$, is a parameter, $\alpha, \gamma, \beta, \sigma > 0$ and A, B are arbitrary constants. The function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions

- (i) $0 < m := \inf_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t) \le M := \sup_{(x,t)\in[a,b]\times\mathbb{R}} h(x,t)$;
- (ii) the function $t \to h(x,t)$ is continuous for all $x \in [a,b]$ and the function $x \to h(x,t)$ is in $C^1([a,b])$ for all $t \in \mathbb{R}$.

We also assume that the function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

In [14] Sun et al. established the new criteria for the existence of infinitely many solutions for a class of one-dimensional p-Laplacian equations with Sturm-Liouville type nonhomogeneous boundary problem (1.1) with the perturbation term $\mu g(x, u(x))$.

We also refer the interested reader to the papers [3, 12] in which using variational methods and critical point theory, the existence of solutions for boundary value problems with nonlinear derivative dependence have been discussed. A second-order impulsive differential inclusion with Sturm-Liouville boundary conditions is studied. By using a nonsmooth version of a three critical point theorem of Ricceri, the existence of three solutions is obtained in [15]. In [4] utilizing variational methods the existence of at least one weak solution for elliptic problems on the real line was discused.

Here, we study the existence of multiple solutions for the problem (1.1). In Theorem 3.1 we prove the existence of at least two solutions for the problem (1.1). As a special case of Theorem 3.1, we investigate the existence of at least two solutions, when $w(x) = d$, that d is a constant; see Corollary 3.1. In Theorem 3.2 we show that the problem (1.1) has at least three solutions. We also show that for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero if $f(x,0)=0$ for all $x\in[a,b]$, the solutions are nontrivial; see Remark 3.1. Moreover, we deduce the existence of solutions for small positive values of the parameter λ such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see Remark 3.2. Finally, we give two examples to show the application of our results.

2. Preliminaries

Let X be a real Banach space and for two functions $\Phi,\Psi: X\to\mathbb{R}$ for all $r,r_1,r_2>\inf_X\Psi$, with $r_1<r_2$ we define the following functions

$$\varphi_1(r)=\inf_{u\in\Psi^{-1}(]-\infty,r[)}\frac{\Phi(u)-\inf_{u\in\overline{\Psi^{-1}(]-\infty,r[)}^{\omega}}\Phi(u)}{r-\Psi(u)}, \tag{2.1}$$

$$\varphi_2(r_1,r_2)=\inf_{u\in\Psi^{-1}(]-\infty,r_1[)}\sup_{v\in\Psi^{-1}([r_1,r_2[)}\frac{\Phi(u)-\Phi(v)}{\Psi(v)-\Psi(u)}, \tag{2.2}$$

where $\overline{\Psi^{-1}(]-\infty,r[)}^{\omega}$ is the closure $\Psi^{-1}(]-\infty,r[)$ in the weak topology.

Theorem 2.1. ([5, Theorem 1.1.]) *Let X be a reflexive real Banach space, and let $\Phi,\Psi: X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(a_1) $\inf_X\Psi<r_1<r_2$;
(a_2) $\varphi_1(r_1)<\varphi_2(r_1,r_2)$;
(a_3) $\varphi_1(r_2)<\varphi_2(r_1,r_2)$.

Then, there exists a positive real number σ such that, for each

$$\lambda\in\left]\frac{1}{\varphi_2(r_1,r_2)},\min\left\{\frac{1}{\varphi_1(r_1)},\frac{1}{\varphi_1(r_2)}\right\}\right[,$$

the equation $\Psi'+\lambda\Phi'$ admits at least two solutions whose norms are less than σ.

For all $r_1,r_2,r_3>\inf_X\Psi$ we define

$$\varphi_3(r_1,r_2,r_3)=\inf_{u\in\Psi^{-1}([r_1,r_2[)}\sup_{v\in\Psi^{-1}([r_2,r_3[)}\frac{\Phi(u)-\Phi(v)}{\Psi(v)-\Psi(u)}. \tag{2.3}$$

Clearly, $\varphi_2(r_2,r_3)\le\varphi_3(r_1,r_2,r_3)$.

Theorem 2.2. ([5, Theorem 2.2.]) *Let X be a reflexive real Banach space, and let $\Phi,\Psi: X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$. Assume also that there exist two constants r_1,r_3 and r_3 such that*

(b_1) $\inf_X\Psi<r_1<r_2<r_3$;

(b_2) $\max\{\varphi_1(r_1), \varphi_1(r_2), \varphi_1(r_3)\} < \min\{\varphi_2(r_1, r_2), \varphi_3(r_1, r_2, r_3)\}$.

Then there exists a positive real number σ such that for each

$$\lambda \in \left] \max\left\{\frac{1}{\varphi_2(r_1, r_2)}, \frac{1}{\varphi_3(r_1, r_2, r_3)}\right\}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}, \frac{1}{\varphi_1(r_3)}\right\}\right[,$$

the equation $\Psi' + \lambda\Phi' = 0$ admits at least three solutions whose norms are less than σ.

Theorems 2.1 and 2.2 have been used to the existence of multiple solutions for a two point boundary value problem driven by one-dimensional p-Laplacian and a second-order Sturm-Liouville boundary value problem in [5, 16], respectively. The present paper paper is a continuation for the application of the critical point theorems.

Let X be the Sobolev space $W^{1,p}([a, b])$ equipped with norm

$$\|u\| := \left(\int_a^b |u(t)|^p + |u'(t)|^p dt\right)^{\frac{1}{p}}, \quad \text{for all } u \in X.$$

Then, the space $(X, \|.\|)$ is a real reflexive Banach space and $\max\{\|u\|_{L^p}, \|u'\|_{L^p}\} \leq \|u\|$ for each $u \in X$. By the Sobolev embedding theorem (see [9]), X is compactly embedded into $C([a, b])$. We also denote $\|\cdot\|_\infty$ as the usual norm of $L^\infty([a, b])$.

For all $x \in [a, b]$ and $s \in \mathbb{R}$, define the functions

$$J_x(s) = J(x, s) := \int_0^s \frac{(p-1)|\delta|^{p-2}}{h(x, \delta)} d\delta$$

and

$$H_x(s) = H(x, s) := \int_0^s J(x, \tau) d\tau.$$

For any fixed $x \in [a, b]$, the fact that $H_x''(s) = J_x'(s) = \frac{(p-1)|s|^{p-2}}{h(x,s)} \geq 0$ implies that H_x is a strictly convex C^2 function and J_x is a strictly increasing C^1 function. Simple calculation shows that for every $x \in [a, b]$, $s \in \mathbb{R}$,

$$\frac{|s|^{p-1}}{M} \leq |J(x, s)| \leq \frac{|s|^{p-1}}{m}, \quad \frac{|s|^p}{pM} \leq |H(x, s)| \leq \frac{|s|^p}{pm}. \tag{2.4}$$

For each $u \in X$, let the functionals $\Psi, \Phi : X \to \mathbb{R}$ be as follows

$$\Psi(u) = \int_a^b H(x, u'(x))dx + \frac{\beta}{\alpha} H\left(a, \frac{\alpha}{\beta} u(a) - \frac{1}{\beta} A\right) + \frac{\sigma}{\gamma} H\left(b, -\frac{\gamma}{\sigma} u(b) + \frac{1}{\sigma} B\right) \tag{2.5}$$

and

$$\Phi(u) = \int_a^b F(x, u(x))dx, \tag{2.6}$$

where

$$F(x, t) := \int_0^t f(x, s)ds, \quad \text{for all } (x, t) \in [a, b] \times \mathbb{R}.$$

In view of (2.4), one has

$$\begin{aligned}&\frac{1}{Mp}\left(\|u'\|_{L^p}^p+\frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a)-\frac{1}{\alpha}A\right|^p+\frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b)-\frac{1}{\gamma}B\right|^p\right)\\ &\le\Psi(u)\le\frac{1}{mp}\left(\|u'\|_{L^p}^p+\frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a)-\frac{1}{\alpha}A\right|^p+\frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b)-\frac{1}{\gamma}B\right|^p\right).\end{aligned}\tag{2.7}$$

Lemma 2.1. ([14, Lemma 2.1]) *Assume that $u\in X$ and there exists $r>0$ such that $\Phi(u)\le r$, then, we have*

$$\|u\|_\infty\le(Mpr)^{\frac{1}{p}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{1}{q}}+(b-a)^{\frac{1}{q}}\right)+\frac{1}{\alpha}|A|,$$

where q is the conjugate of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$.

Definition 2.1. We say that u is a classical solution to (1.1) if $u\in C^1([a,b])$, $|u'|^{p-2}u'\in AC^1([a,b])$, $\alpha u(a)-\beta u'(a)=\int_a^b\xi(x)u(x)dx$, $\gamma u(b)-\sigma u'(b)=\int_a^{b\eta(x)}u(x)dx$ and

$$-(\phi_p(u'(x)))'=\left(\lambda f(x,u(x))+\int_0^{u'(x)}\frac{\partial}{\partial x}\left(\frac{(p-1)|\tau|^{p-2}}{h(x,\tau)}d\tau\right)\right)h(x,u'(x)),$$

for almost every complete $x\in[a,b]$, where $AC^1([a,b])$ denotes the space of those functions whose first derivatives along with themselves are absolutely continuous on $[a,b]$.

Definition 2.2. We say that u is a weak solution to (1.1) if $u\in X$ and

$$\begin{aligned}&\int_a^b J(x,u'(x))v'(x)dx+J\left(a,\frac{\alpha}{\beta}u(a)-\frac{1}{\beta}A\right)v(a)-J\left(b,-\frac{\gamma}{\sigma}u(b)+\frac{1}{\sigma}B\right)v(b)\\ &-\lambda\int_a^b f(x,u(x))v(x)dx=0,\end{aligned}$$

for any $v\in X$.

Lemma 2.2. ([14, Lemma 2.4]) *Weak solutions of* (1.1) *coincide with classical solutions of* (1.1).

Lemma 2.3. ([14, Lemma 2.5]) *Assume that the functional $\Psi:X\to\mathbb{R}$ is defined by* (2.5). *Then Ψ is sequentially weakly lower semicontinuous, continuous, $\lim_{\|u\|\to+\infty}\Psi(u)=+\infty$ and its Gâteaux derivative $u\in X$ is the functional $\Psi'(u)$ given by*

$$\begin{aligned}\Psi'(u)(v)=&\int_a^b J(x,u'(x))v'(x)dx+J\left(a,\frac{\alpha}{\beta}u(a)-\frac{1}{\beta}A\right)v(a)\\ &-J\left(b,-\frac{\gamma}{\sigma}u(b)+\frac{1}{\sigma}B\right)v(b),\end{aligned}$$

for every $v\in X$.

Remark 2.1. If $u \in X$ is a critical point of $I_\lambda = \Psi + \lambda\Phi$ in view of Definition 2.2, then, u is a classical solution of the problem (1.1).

3. Main Results

For any $\nu > 0$, we define

$$Q(\nu) := \left\{ t \in \mathbb{R} : |t| \le \nu \left(\left(\frac{\beta}{\alpha} \right)^{\frac{1}{q}} + (b-a)^{\frac{1}{q}} \right) + \frac{1}{\alpha}|A| \right\}.$$

We formulate our first main result as an application of Theorem 2.1 as follows.

Theorem 3.1. *Assume there exist two positive constants $c_1 < c_2$ and a function $w \in X$ such that*

(A_1) $c_1^p \le K_w \le \frac{m}{M} c_2^p$, *where*

$$K_w := \left(\|w'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}} \left| w(a) - \frac{1}{\alpha} A \right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}} \left| w(b) - \frac{1}{\gamma} B \right|^p \right);$$

(A_2) $A_{c_i} Mp < \dfrac{\int_a^b F(x, w(x))dx - \int_a^b \sup_{t \in Q(c_1)} F(x,t)dx}{\Psi(w)}$ *for* $i = 1, 2$.

Then, for each

$$\lambda \in \left] \frac{\Psi(w)}{\int_a^b F(x, w(x))dx - \int_a^b \sup_{t \in Q(c_1)} F(x,t)dx}, \frac{\min\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\}}{Mp} \right[,$$

the problem (1.1) *has at least two classical solutions whose norms in* $C([a,b])$ *are less than* c_2 *where* $A_{c_i} = \frac{1}{c_i^p} \int_a^b \sup_{t \in Q(c_i)} F(x,t)dx$.

Proof. Let Ψ, Φ be as given by (2.5) and (2.6), respectively. By Lemma 2.3 we observe that $\Psi, \Phi : X \to \mathbb{R}$ are two sequentially weakly lower semicontinuous and Gâteaux differentiable functions and Ψ is continuous and satisfies $\lim_{\|u\| \to +\infty} \Psi(u) = +\infty$. We want to obtain at least two critical points of $I_\lambda = \Psi + \lambda\Phi$ by applying Theorem 2.1. It remains to verify condition (a_1), (a_2) and (a_3) in Theorem 2.1. Let $r_i = \frac{c_i^p}{Mp}$, $i = 1, 2$. By (2.7) and (A_1) we have

$$r_1 < \frac{1}{Mp} K_w \le \Psi(w) \le \frac{1}{mp} K_w < r_2.$$

It is easy to see that (a_1) holds since $r_1, r_2 > 0$. Now we will show that (a_2) in Theorem 2.1 is satisfied. Taking into account that the function $u \equiv 0$ on $[a,b]$ obviously belongs to $\Psi^{-1}(]-\infty, r[)$ and that $\Psi(0) = \Phi(0) = 0$, we get

$$\varphi_1(r) = \inf_{u \in \Psi^{-1}(]-\infty, r[)} \frac{\Phi(u) - \inf_{u \in \overline{\Psi^{-1}(]-\infty, r[)}^w} \Phi(x)}{r - \Psi(u)} \le -\frac{1}{r} \inf_{u \in \overline{\Psi^{-1}(]-\infty, r[)}^w} \Phi(u). \tag{3.1}$$

Noticing $\overline{\Psi^{-1}(]-\infty, r[)}^{w} = \Psi^{-1}(]-\infty, r[)$ by Lemma 2.1 we obtain

$$\begin{aligned}\Psi^{-1}(-\infty, r) &= \{u \in X : \Psi(u) < r\} \\ &\subseteq \left\{u \in X : \|u\|_\infty \leq (Mpr)^{\frac{1}{p}}\left(\left(\frac{\beta}{\alpha}\right)^{\frac{1}{q}} + (b-a)^{\frac{1}{q}}\right) + \frac{1}{\alpha}|A|\right\} \\ &= \left\{u \in X : \max_{x\in[a,b]} |u(x)| \in Q(c)\right\}.\end{aligned}$$

Then

$$\begin{aligned}\varphi_1(r) &\leq \frac{\sup_{u\in\Psi^{-1}(-\infty,r)} \int_a^b F(x, u(x))dx}{r} \\ &\leq \frac{\int_a^b \sup_{t\in Q(c)} F(x,t)dx}{r},\end{aligned}$$

and therefore, we have

$$\varphi_1(r_i) \leq \frac{Mp}{c_i^p} \int_a^b \sup_{t\in Q(c_i)} F(x,t)dx, \quad i = 1, 2.$$

On the one hand, by Lemma 2.1 and $r_1 \leq \Psi(w) \leq r_2$ we have

$$\begin{aligned}\varphi_2(r_1, r_2) &= \inf_{u\in\Psi^{-1}(]-\infty,r_1[)} \sup_{v\in\Psi^{-1}([r_1,r_2[)} \frac{\Phi(u) - \Phi(v)}{\Psi(v) - \Psi(u)} \\ &\geq \inf_{u\in\Psi^{-1}(]-\infty,r_1[)} \frac{\Phi(u) - \Phi(w)}{\Psi(w) - \Psi(u)} \\ &\geq \inf_{u\in\Psi^{-1}(]-\infty,r_1[)} \frac{1}{\Psi(w) - \Psi(u)}\left(\int_a^b F(x, w(x))dx - \int_a^b F(x, u(x))dx\right) \\ &\geq \frac{\int_a^b F(x, w(x))dx - \int_a^b \sup_{t\in Q(c_1)} F(x,t)dx}{\Psi(w) - \Psi(u)}.\end{aligned}$$

By (A_2) we have that $\int_a^b F(x, w(x))dx - \int_a^b F(x, u(x))dx > 0$, so

$$\varphi_2(r_1, r_2) \geq \frac{\int_a^b F(x, w(x))dx - \int_a^b \sup_{t\in Q(c_1)} F(x,t)dx}{\Psi(w)}.$$

Then, from (A_2), (a_2) and (a_3) in Theorem 2.1 are fulfilled. By choosing $\sigma = r_2$, the conclusion follows. Therefore, it follows that the functional I_λ has two critical points which are the weak solutions of the problem (1.1), and since from Lemma 2.3 the weak solutions coincide with the classical solutions, we have the desired result. □

In Theorem 3.1, the condition (A_2) is related to the function $w \in W^{1,p}$. A different function $w \in W^{1,p}$ would lead to a different condition, which is similar to (A_2). For example, we let $w(x) = d$ where d is a constant. We have the following result.

Corollary 3.1. *Assume there exist three positive constants c_1, d, c_2 such that*

(A_1') $c_1^p < K_d < \frac{m}{M}c_2^p$, *where*

$$K_d := \left(\frac{\alpha}{\beta}\right)^{p-1}\left|d - \frac{1}{\alpha}A\right|^p + \left(\frac{\gamma}{\sigma}\right)^{p-1}\left|d - \frac{1}{\gamma}B\right|^p;$$

(A_2') $A_{c_i}\frac{M}{m} < \frac{B(d,c_1)}{K_d}$,

where A_{c_i} *is defined in Theorem* 3.1 *and*

$$B(d,c_1) = \int_a^b F(x,d)dx - \int_a^b \sup_{t\in Q(c_1)} F(x,t)dx.$$

Then, for every

$$\lambda \in \left]\frac{K_d}{mpB(d,c_1)}, \frac{\min\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\}}{Mp}\right[,$$

the problem (1.1) *has at least two classical solutions whose norms in* $C([a,b])$ *are less than* c_2.

Next, we state our second main result as an application of Theorem 2.2 as follows.

Theorem 3.2. *Assume that there exist five constants* c_1, d_1, c_2, d_2, c_3 *with*

$$c_i^p < K_{d_i} \leq c_{i+1}^p\frac{m}{M}, \quad i = 1, 2,$$

such that

$$\frac{M}{m}\ A^*(c_1,c_2,c_3) \leq B^*_{c_1,c_2}(d_1,d_2), \tag{3.2}$$

where

$$A^*(c_1,c_2,c_3) = \max\{A_{c_i} : i = 1,2,3\}$$

and

$$B^*_{c_1,c_2}(d_1,d_2) = \min\left\{\frac{B(d_1,c_1)}{K_{d_1}}, \frac{B(d_2,c_2)}{K_{d_2}}\right\}.$$

Then, for each

$$\lambda \in \left]\frac{1}{mpB^*_{c_1,c_2}(d_1,d_2)}, \frac{1}{Mp\ A^*(c_1,c_2,c_3)}\right[,$$

the problem (1.1) *admits at least three classical solutions whose norms in* $C([a,b])$ *are less than* c_3.

Proof. Take the Banach space X and the functionals Ψ, Φ on X are defined by (2.5) and (2.6). Let $r_i = \frac{c_i^p}{Mp}$ and $w_1 = d_1$, $w_2 = d_2$. By the same arguing as given in the proof of Theorem 3.1 one has

$$r_1 < \Psi(w_1) < r_2 < \Psi(w_2) < r_3,$$

$$\varphi_2(r_1,r_2) \geq \frac{mp}{K_{d_1}}B(d_1,c_1),$$

$$\varphi_2(r_1,r_2,r_3) \geq \varphi_2(r_2,r_3) \geq \frac{mp}{K_{d_2}}B(d_2,c_2)$$

and

$$\varphi(r_i) \leq Mp\ A_{c_i}, \quad i = 1, 2, 3.$$

Therefore, taking into account (3.2), there exist at least three classical solutions. Not taking into account the zero solution, there are at least three nonzero classical solutions whose norms in $C([a, b])$ are less than c_3. Then, taking into account the fact that the weak solutions of the problem (1.1) are exactly critical points of the functional I_λ, also by using Lemma 2.3, we know the weak solutions coincide with the classical solutions, so we have the desired conclusion. □

Remark 3.1. If $f(x, 0) \neq 0$ for some $x \in [a, b]$, then the ensured solutions in Theorem 3.1 are non-trivial. On the other hand, the non-triviality of the solution can be achieved also in $f(x, 0) = 0$ for some $x \in [a, b]$, requiring the extra condition at zero, and there are a non-empty open set $D \subseteq (a, b)$ and $B \subset D$ such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{x \in B} F(x, \xi)}{|\xi|^p} = +\infty$$

and

$$\liminf_{\xi \to 0^+} \frac{\inf_{x \in D} F(x, \xi)}{|\xi|^p} > -\infty.$$

Indeed, let $0 < \overline{\lambda} < \lambda^*$, where

$$\lambda^* = \frac{\min\left\{\frac{1}{Ac_1}, \frac{1}{Ac_2}\right\}}{Mp}.$$

Let Φ and Ψ be as given in (2.5) and (2.6), respectively. Due to Corollary 3.1 for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \overline{\lambda}\right)$ there exists a critical point of $I_\lambda = \Psi + \lambda\Phi$ such that $u_\lambda \in \Psi^{-1}(-\infty, r)$, where $r_\lambda = \frac{c_\lambda^p}{Mp}$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Psi^{-1}(-\infty, r)$. We will prove that u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\| \to 0^+} \frac{\Phi(u)}{\Psi(u)} = +\infty. \tag{3.3}$$

Thanks to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that for every $\xi \in [0, \sigma]$

$$\lim_{\xi \to 0^+} \frac{\inf_{x \in B} F(x, \xi_n)}{|\xi_n|^p} = +\infty \tag{3.4}$$

and

$$\inf_{x \in D} F(x, \xi) > \kappa|\xi|^p.$$

We consider a set $G \subset B$ of positive measure and a function $v \in X$ such that

- (k_1) $v(t) \in [0, 1]$ for every $t \in (a, b)$;
- (k_2) $v(t) = 1$ for every $t \in G$;
- (k_3) $v(t) = 0$ for every $t \in (a, b)\backslash D$.

Finally, fix $M > 0$ and consider a real positive number η with

$$M < \frac{mp\eta \text{ meas}(G) + mp\kappa \int_{D\setminus G} |v(t)|dt}{K_u},$$

where

$$K_u = \frac{1}{mp}\left(\|u'\|_{L^p}^p + \frac{\alpha^{p-1}}{\beta^{p-1}}\left|u(a) - \frac{1}{\alpha}A\right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u(b) - \frac{1}{\gamma}B\right|^p\right).$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$\inf_{x\in B} F(x, \xi_n) \geq \kappa|\xi_n|^p,$$

for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n large enough), one has

$$\begin{aligned}\frac{\Phi(\xi_n v)}{\Psi(\xi_n v)} =& \frac{\int_G F(t, \xi_n)dt + \int_{D\setminus G} F(t, \xi_n v(t))dt}{\Psi(\xi_n v)}\\ >& \frac{mp\eta \text{ meas}(G) + mp\kappa \int_{D\setminus G} |v(t)|dt}{K_u} > M.\end{aligned}$$

Since M could be arbitrarily large, it yields

$$\lim_{n\to\infty} \frac{\Phi(\xi_n v)}{\Psi(\xi_n v)} = +\infty$$

from which (3.3) clearly follows. Hence, there exists $\{\omega_n\} \subset X$ strongly converging to zero such that, $\omega_n \in \Psi^{-1}(-\infty, r)$ and

$$I_\lambda(\omega_n) = \Psi(\omega_n) + \lambda\Phi(\omega_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Psi^{-1}(-\infty, r)$, we conclude that

$$I_\lambda(u_\lambda) < 0. \tag{3.5}$$

Remark 3.2. From (3.5) we easily observe that the map

$$\left(\frac{K_d}{mpB(d, c_1)}, \lambda^*\right) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.6}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, bearing in mind that Ψ is coercive and for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$ the solution $u_\lambda \in \Psi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Then, there exists a positive constant N such that

$$\left|\int_a^b f(x, u(x))v(x)dx\right| \leq N\|u_\lambda\| \leq NL, \tag{3.7}$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in X$ and every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Psi'(u_\lambda)(u_\lambda) = -\lambda \int_a^b f(x, u_\lambda(x))u_\lambda(x)dx,$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Then, it follows

$$\begin{aligned}
0 &\leq \frac{1}{Mp}\left(\|u'_\lambda\|^p_{L^p} + \frac{\alpha^{p-1}}{\beta^{p-1}}\left|u_\lambda(a) - \frac{1}{\alpha}A\right|^p + \frac{\gamma^{p-1}}{\sigma^{p-1}}\left|u_\lambda(b) - \frac{1}{\gamma}B\right|^p\right) \\
&\leq \Psi'(u_\lambda)(u_\lambda) \\
&= -\lambda \int_a^b f(x, u_\lambda(x))u_\lambda(x)dx,
\end{aligned}$$

for every $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. Letting $\lambda \to 0^+$ by (3.7), we get

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. Finally, we have to show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. We see that for any $u \in X$ one has

$$I_\lambda = \lambda\left(\frac{\Psi(u)}{\lambda} + \Phi(u)\right). \tag{3.8}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Psi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Psi(u_{\lambda_i})}{\lambda_i} + \Phi(u_{\lambda_i})\right) = \inf_{v \in \Psi^{-1}(-\infty, r)} \left(\frac{\Psi(v)}{\lambda_i} + \Phi(v)\right),$$

for every $i = 1, 2$. Clearly, (3.6) together with (3.8) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \tag{3.9}$$

Moreover

$$m_{\lambda_2} < m_{\lambda_1}, \tag{3.10}$$

due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.8)–(3.10) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1},$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$. The arbitrariness of $\lambda < \lambda^*$ shows that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in \left(\frac{K_d}{mpB(d,c_1)}, \lambda^*\right)$.

We now present the following example to illustrate Corollary 3.1.

Example 3.1. Let $a = 0$, $b = 1$, $\alpha = \beta = 1$, $\gamma = 1$, $\sigma = 2$, $A = 0$, $B = 10$, $p = 2$, $h(x,t) = 1 + x + |\sin t|$ for every $(x,t) \in [0,1] \times \mathbb{R}$ and $f(x,t) = \frac{1}{10^6}\Big(t^9 e^{-t}(10-t)\sin x\Big)$ for every $t \in \mathbb{R}$. By the expression of f, we have $F(x,t) = \frac{1}{10^6}\Big(t^{10}e^{-t}\sin x\Big)$ for every $t \in \mathbb{R}$. We observe that $m = 1$, and $M = 3$. Choosing $d = 10$, $c_1 = \frac{1}{10}$, $c_2 = 10^2$, since $Q(c_1) = \frac{2}{10}$, $Q(c_2) = 2 \times 10^2$, $K_d = 10^2$, we see that all conditions in Corollary 3.1 are satisfied. Therefore, taking Remark 3.2 it follows that for each

$$\lambda \in \left] \frac{10^2}{9180e^{-10}}, \frac{2^{13} \times 57375e^{-200}}{6} \right[,$$

the problem

$$\begin{cases} -(\phi_p(u'))' = \left(\lambda f(u) + \displaystyle\int_0^{u'(x)} \frac{\partial}{\partial x}\Big(\frac{(p-1)|\tau|^{p-2}}{1+x+|\sin\tau|}\Big)d\tau\right) \\ \times(1 + x + |\sin u'(x)|), & x \in (0,1), \\ u(0) - u'(0) = 0, \quad u(1) + 2u'(1) = 10, \end{cases}$$

has at least two nontrivial solutions $u_{1\lambda}$ and $u_{2\lambda}$ in X such that

$$\lim_{\lambda \to 0^+} \|u_{i\lambda}\| = 0$$

and the real function

$$\begin{aligned} \lambda \to &\int_a^b H(x, u'_{i\lambda}(x))dx + H\Big(0, u_{i\lambda}(0)\Big) + \frac{\sigma}{\gamma}H\Big(1, -\frac{1}{2}u_{i\lambda}(1) + \frac{1}{2}10\Big) \\ &+ \frac{\lambda}{10^6}\int_0^1 t^{10}e^{-t}\sin u_{i\lambda}(x)dx, \end{aligned}$$

for $i = 1, 2$.

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SOME REMARKS ON THE RANDIĆ ENERGY OF GRAPHS

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Abstract. Let G be a graph of order n. The Randić energy of G is defined as $RE(G) = \sum_{i=1}^{n} |\rho_i|$, where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the Randić eigenvalues of G. In this study, we present improved bounds for $RE(G)$ as well as a relationship between (ordinary) graph energy and $RE(G)$.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph of order n and size m, with vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$. Denote by $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. If vertices v_i and v_j are adjacent in G, it will be denoted as $i \sim j$.

Let $A = (a_{ij})$, be the $(0,1)$ adjacency matrix of G. The eigenvalues of matrix A, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are the (ordinary) eigenvalues of G [4]. Some well known properties of these eigenvalues are [4]:

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A) = 0, \quad \sum_{i=1}^{n} \lambda_i^2 = \operatorname{tr}(A^2) = \sum_{i=1}^{n} d_i = 2m, \quad \prod_{i=1}^{n} \lambda_i = \det A.$$

Denote with $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*|$ the non-increasing arrangement of the absolute values of eigenvalues of G. The notion of (ordinary) graph energy was introduced in [12]. It is defined to be

$$E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} |\lambda_i^*|.$$

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The Randić matrix of G [2] is defined as

$$R = R(G) = D^{-1/2} A D^{-1/2}.$$

The eigenvalues of matrix R , $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, form the Randić spectrum of G. Some properties of Randić eigenvalues are (see, e.g., [2]):

$$\sum_{i=1}^{n} \rho_i = \operatorname{tr}(R) = 0, \quad \sum_{i=1}^{n} \rho_i^2 = \operatorname{tr}(R^2) = 2R_{-1}(G),$$

where $R_{-1}(G)$ is a vertex–degree based graph invariant introduced in [3] defined as

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j}.$$

It is known as general Randić index R_{-1}, as well as modified second Zagreb index [24].

In [14] it was proven that the following identity is valid

$$\det R = \frac{\det A}{\prod_{i=1}^{n} d_i}. \tag{1.1}$$

The other two vertex–degree based topological indices that are of interest for the present paper are the first Zagreb index [17]

$$M_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{i \sim j} (d_i + d_j),$$

and the inverse degree index [9] defined as

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i} = \sum_{i \sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right).$$

Denote with $|\rho_1^*| \geq |\rho_2^*| \geq \cdots \geq |\rho_n^*|$ the non-increasing arrangement of the absolute values of Randić eigenvalues of G. The Randić energy of G is defined as [2]

$$RE(G) = \sum_{i=1}^{n} |\rho_i| = \sum_{i=1}^{n} |\rho_i^*|.$$

More on its mathematical properties can be found in [1–3, 5, 7, 14, 20, 22].

In this paper, we obtain improved bounds for $RE(G)$ as well as a relationship between $E(G)$ and $RE(G)$.

2. Preliminaries

In this section we recall some results from spectral graph theory and analytical inequalities that are of interest for the present paper.

Lemma 2.1 ([20]). *The Randić spectral radius is $\rho_1 = 1$.*

Remark 2.1. In [14] it was observed that when $G \cong \overline{K}_n$ then $\rho_1 = 0$. Therefore, if G has at least one edge, then $\rho_1 = 1$.

Let $G_1 \bigvee G_2$ denote the complete product of two graphs G_1 and G_2. This graph is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

Lemma 2.2 ([8]). *Let G be a connected graph of order n with maximum vertex degree $\Delta = n-1$. Then $|\rho_2| = |\rho_3| = \cdots = |\rho_n|$ if and only if $G \cong K_n$, or $G \cong K_1 \bigvee r\,K_2$, with $n = 2r+1$ $(r \geq 2)$.*

Lemma 2.3 ([20]). *Let G be a connected graph of order n. Then*

$$RE(G) \leq 1 + \sqrt{(n-1)(2R_{-1}(G)-1)}. \tag{2.1}$$

Remark 2.2. The inequality (2.1) was also proved in [19, 21], as well as in [5] as a special case of one more general result. In [8] it was proved that when $\Delta = n-1$, equality in (2.1) holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee r\,K_2$, with $n = 2r+1$ $(r \geq 2)$.

Lemma 2.4 ([1]). *Let G be a connected bipartite graph of order $n \geq 2$. Then*

$$RE(G) \leq 2 + \sqrt{(n-2)(2R_{-1}(G)-2)}. \tag{2.2}$$

Remark 2.3. The inequality (2.2) was also proved in [21]. In [8] it was proven that equality in (2.2) holds if and only if $G \cong K_{p,q}$, $p+q=n$, for odd n.

Lemma 2.5 ([11]). *Let G be a connected bipartite graph of order $n \geq 3$ with Randić eigenvalues $\rho_1 = 1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} \geq \rho_n = -1$ and let $\rho = \max_{2 \leq i \leq n-1}\{|\rho_i|\}$. Then, for any real k, $\rho \geq k \geq \sqrt{\frac{2R_{-1}(G)-2}{n-2}}$, holds*

$$RE(G) \leq 2 + k + \sqrt{(n-3)(2R_{-1}(G)-2-k^2)}. \tag{2.3}$$

Equality holds if G is a complete bipartite graph, in which case $k=0$.

Remark 2.4. In [18, Theorem 3.4] it was claimed that when

$$\frac{1}{\Delta} \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}, \tag{2.4}$$

then

$$RE(G) \leq 1 + \frac{1}{\Delta} + \sqrt{(n-2)\left(2R_{-1}(G) - 1 - \frac{1}{\Delta^2}\right)}, \tag{2.5}$$

which would mean that (2.5) is stronger than (2.1). However, if (2.4) is true, then $\Delta \geq n-1$, which is not possible. Therefore, the inequality (2.5) is not correct.

Lemma 2.6 ([6, 18]). *Let G be a connected graph of order n. Then*

$$RE(G) \geq 1 + (n-1)\left(\frac{|\det A|}{\prod_{i=1}^{n} d_i}\right)^{\frac{1}{n-1}}. \tag{2.6}$$

The following analytical inequality would be used in proofs of theorems in the present paper.

Lemma 2.7 ([23])**.** *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of positive real numbers and $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, two similarly ordered sequences of non–negative real numbers. Then*

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{2.7}$$

When $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n$, are of different monotonicity, then opposite inequality is valid. Equality holds if and only if $a_1 = \cdots = a_n$ or $b_1 = \cdots = b_n$.

3. Main Results

In the next theorem we establish a lower bound on $RE(G)$.

Theorem 3.1. *Let G be a connected graph of order n. Then, for any real k, $|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, holds*

$$RE(G) \geq 1 + k + (n-2) \left(\frac{|\det A|}{k \prod_{i=1}^{n} d_i} \right)^{\frac{1}{n-2}}. \tag{3.1}$$

Equality holds if and only if G is a graph with the property $\rho_1 = |\rho_1^*| = 1$, $|\rho_2^*| = k$, and $|\rho_i^*| = \sqrt{\frac{2R_{-1}(G)-1-k^2}{n-2}}$, for $i = 3, 4, \dots, n$.

Proof. Using arithmetic-geometric mean inequality (see, e.g., [23]), Lemma 2.1 and (1.1) we obtain

$$\begin{aligned} RE(G) =& \sum_{i=1}^{n} |\rho_i^*| = 1 + |\rho_2^*| + \sum_{i=3}^{n} |\rho_i^*| \\ \geq& 1 + |\rho_2^*| + (n-2) \left(\frac{|\det R|}{|\rho_2^*|} \right)^{\frac{1}{n-2}} \\ =& 1 + |\rho_2^*| + (n-2) \left(\frac{|\det A|}{|\rho_2^*| \prod_{i=1}^{n} d_i} \right)^{\frac{1}{n-2}}. \end{aligned} \tag{3.2}$$

Let us consider the following function defined by

$$f(k) = x + (n-2) \left(\frac{|\det A|}{x \prod_{i=1}^{n} d_i} \right)^{\frac{1}{n-2}}.$$

Observe that f is increasing for $x \geq \left(\frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{\frac{1}{n-1}}$. Considering Lemmas 2.1 and 2.3 together with (1.1), for any real k, $|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, we have that

$$|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}} \geq \frac{RE(G)-1}{n-1} = \frac{\sum_{i=2}^{n} |\rho_i^*|}{n-1} \geq \left(\prod_{i=2}^{n} |\rho_i^*| \right)^{\frac{1}{n-1}} = \left(\frac{|\det A|}{\prod_{i=1}^{n} d_i} \right)^{\frac{1}{n-1}}.$$

Then, we deduce that $f(|\rho_2^*|) \geq f(k)$. Combining this with (3.2), the inequality (3.1) is obtained. The equality in (3.1) holds if and only if

$$|\rho_2^*| = k \quad \text{and} \quad |\rho_3^*| = \cdots = |\rho_n^*|.$$

Since $\sum_{i=2}^n |\rho_i^*|^2 = 2R_{-1}(G) - 1$, the above conditions imply that $|\rho_3^*| = \cdots = |\rho_n^*| = \sqrt{\frac{2R_{-1}(G)-1-k^2}{n-2}}$. This completes the proof. □

Corollary 3.1. *Let G be a connected graph of order n. Then*

$$RE(G) \geq 1 + \sqrt{\frac{2R_{-1}(G)-1}{n-1}} + (n-2)\left(\frac{|\det A|}{\sqrt{\frac{2R_{-1}(G)-1}{n-1}}\prod_{i=1}^n d_i}\right)^{\frac{1}{n-2}}. \tag{3.3}$$

If the maximum vertex degree Δ is equal to $n-1$, the equality in (3.3) *holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee r\,K_2$, with $n = 2r+1$ $(r \geq 2)$.*

Proof. The inequality (3.3) is obtained from (3.1) for $k = \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$. Now, assume that equality in (3.3) holds. Then

$$|\rho_2^*| = \sqrt{\frac{2R_{-1}(G)-1}{n-1}} \quad \text{and} \quad |\rho_3^*| = \cdots = |\rho_n^*|.$$

Since $\sum_{i=2}^n |\rho_i^*| = 2R_{-1}(G) - 1$, we get

$$|\rho_3^*| = \cdots = |\rho_n^*| = \sqrt{\frac{2R_{-1}(G)-1}{n-1}}.$$

The above results state that $|\rho_2^*| = |\rho_3^*| = \cdots = |\rho_n^*|$, that is $|\rho_2| = |\rho_3| = \cdots = |\rho_n|$. Then, by Lemma 2.2 if $\Delta = n-1$, the equality in (3.3) holds if and only if $G \cong K_n$, or $G \cong K_1 \bigvee r\,K_2$, with $n = 2r+1$ $(r \geq 2)$. □

Remark 3.1. The lower bounds (3.1) and (3.3) are stronger than the lower bound (2.6). Moreover, by Theorem 3.1, it is possible to derive stronger lower bound than (3.3) using any real k such that $|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$.

In the next theorem we establish a relationship between Randić energy and general Randić index R_{-1}.

Theorem 3.2. *Let G be a connected graph of order $n \geq 3$. Then, for any real k, such that $|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, we have*

$$RE(G) \leq 1 + k + \sqrt{(n-2)(2R_{-1}(G) - 1 - k^2)}. \tag{3.4}$$

Equality holds if and only if G is a graph with the property $\rho_1 = |\rho_1^*| = 1$, $|\rho_2^*| = k$ and $|\rho_i^*| = \sqrt{\frac{2R_{-1}(G)-1-k^2}{n-2}}$, for $i = 3, 4, \ldots, n$.

Proof. By the Cauchy–Schwarz inequality (see, e.g., [23]), we have that

$$\sum_{i=3}^n |\rho_i^*| \leq \left(\sum_{i=3}^n 1\right)^{1/2} \left(\sum_{i=3}^n |\rho_i^*|^2\right)^{1/2},$$

that is

$$RE(G) \leq |\rho_1^*| + |\rho_2^*| + \sqrt{(n-2)(2R_{-1}(G) - |\rho_1^*|^2 - |\rho_2^*|^2)}.$$

By Lemma 2.1, we have that $\rho_1 = |\rho_1^*| = 1$. Considering this fact with the above inequality, we get

$$RE(G) \leq 1 + |\rho_2^*| + \sqrt{(n-2)(2R_{-1}(G) - 1 - |\rho_2^*|^2)}. \tag{3.5}$$

Now, observe the function

$$f(x) = x + \sqrt{(n-2)(2R_{-1}(G) - 1 - x^2)}, \quad x \geq 0.$$

This function is monotone decreasing for $x \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$. Therefore for any $k \geq 0$ with the property $|\rho_2^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, holds that $f(|\rho_2^*|) \leq f(k)$. From this inequality and (3.5) we obtain (3.4).

The equality case for (3.4) can be proved similarly as in case of Theorem 3.1. □

Remark 3.2. When $k = \sqrt{\frac{2R_{-1}(G)-1}{n-1}}$, from (3.4) the inequality (2.1) is obtained, which means that inequality (3.4) is stronger than (2.1).

Remark 3.3. Recall that the Randić spectrum of a bipartite graph is symmetric with respect to the origin, that is, $\rho_i = -\rho_{n-i+1}$, for $i = 1, 2, \ldots, n$ [10]. In this case, $|\rho_1^*| = \rho_1 = 1 = |\rho_n| = |\rho_2^*|$. On the other hand, $\rho = |\rho_3^*| = |\rho_4^*|$.

Having in mind the above remark, by a similar procedure as in Theorem 3.2, the following result can be proven.

Theorem 3.3. *Let G be a connected bipartite graph of order $n \geq 5$. Then, for any real k such that $|\rho_3^*| \geq k \geq \sqrt{\frac{2R_{-1}(G)-2}{n-2}}$, we have*

$$RE(G) \leq 2 + 2k + \sqrt{(n-4)(2R_{-1}(G) - 2 - 2k^2)}. \tag{3.6}$$

Equality holds if and only if G is a graph with the property $\rho_1 = |\rho_1^*| = |\rho_2^*| = 1$, $|\rho_3^*| = |\rho_4^*| = k$ and $|\rho_i^*| = \sqrt{\frac{2R_{-1}(G)-2-2k^2}{n-4}}$, for $i = 5, \ldots, n$.

Remark 3.4. When $k = \sqrt{\frac{2R_{-1}(G)-2}{n-2}}$, from (3.6) the inequality (2.2) is obtained. Furthermore, the inequality (3.6) is stronger than (2.2) and (2.3).

Theorem 3.4. *Let G be a connected graph of order $n \geq 2$. Then*

$$RE(G) \leq 1 + \sqrt{(n-1)\left(2R_{-1}(G) - 1 - \frac{1}{2}(|\rho_2^*| - |\rho_n^*|)^2\right)}. \tag{3.7}$$

Equality holds if and only if G is a graph with the property $\rho_1 = |\rho_1^*| = 1$ and $|\rho_3^*| = \cdots = |\rho_{n-1}^*| = \frac{|\rho_2^*|+|\rho_n^*|}{2}$.

Proof. Based on the Lagrange's identity (see e.g. [23]), we have that

$$
\begin{aligned}
(n-1)\sum_{i=2}^{n}|\rho_i^*|^2-\left(\sum_{i=2}^{n}|\rho_i^*|\right)^2 &= \sum_{2\le i<j\le n}(|\rho_i^*|-|\rho_j^*|)^2\\
&\ge (|\rho_2^*|-|\rho_n^*|)^2+\sum_{i=3}^{n-1}\Big((|\rho_i^*|-|\rho_n^*|)^2+(|\rho_2^*|-|\rho_i^*|)^2\Big)\\
&\ge (|\rho_2^*|-|\rho_n^*|)^2+\frac{1}{2}\sum_{i=3}^{n-1}(|\rho_2^*|-|\rho_n^*|)^2 \qquad (3.8)\\
&= \frac{n-1}{2}(|\rho_2^*|-|\rho_n^*|)^2.
\end{aligned}
$$

Since

$$(n-1)\sum_{i=2}^{n}|\rho_i^*|^2-\left(\sum_{i=2}^{n}|\rho_i^*|\right)^2=(n-1)(2R_{-1}(G)-1)-(RE(G)-1)^2,$$

from (3.8) the inequality (3.7) is obtained.

Equality in (3.8) holds if and only if $|\rho_3^*|=\cdots=|\rho_{n-1}^*|$ and $|\rho_i^*|-|\rho_n^*|=|\rho_2^*|-|\rho_i^*|$, for $i=3,\ldots,n-1$, which implies that equality in (3.7) holds if and only if $\rho_1=|\rho_1^*|=1$ and $|\rho_3^*|=\cdots=|\rho_{n-1}^*|=\frac{|\rho_2^*|+|\rho_n^*|}{2}$. ☐

Remark 3.5. Let us note that the inequality (3.7) is stronger than (2.1).

The proof of the next theorem is analogous to that of Theorem 3.4, hence omitted.

Theorem 3.5. *Let G be a connected bipartite graph of order $n\ge 4$. Then*

$$RE(G)\le 2+\sqrt{(n-2)\left(2R_{-1}(G)-2-\frac{1}{2}(|\rho_3^*|-|\rho_n^*|)^2\right)}. \qquad (3.9)$$

Equality holds if and only if G is a graph with the property $\rho_1=|\rho_1^*|=|\rho_2^*|=1$ and $|\rho_4^*|=\cdots=|\rho_{n-1}^*|=\frac{|\rho_3^*|+|\rho_n^*|}{2}$.

Remark 3.6. Notice that the inequality (3.9) is stronger than (2.2).

We now give a relationship between $E(G)$ and $RE(G)$.

Theorem 3.6. *Let G be a graph of order $n\ge 2$ and size m, without isolated vertices. Then we have*

$$E(G)RE(G)\le 2n\sqrt{mR_{-1}(G)}. \qquad (3.10)$$

Equality holds if and only if $G\cong\frac{n}{2}K_2$, for even n.

Proof. For $p_i=1$, $a_i=|\lambda_i^*|$, $b_i=|\rho_i^*|$, $i=1,2,\ldots,n$, the inequality (2.7) becomes

$$\sum_{i=1}^{n}1\sum_{i=1}^{n}|\lambda_i^*|\,|\rho_i^*|\ge\sum_{i=1}^{n}|\lambda_i^*|\,\sum_{i=1}^{n}|\rho_i^*|,$$

that is

$$E(G)RE(G) \leq n\sum_{i=1}^{n} |\lambda_i^*|\,|\rho_i^*|. \tag{3.11}$$

On the other hand, having in mind Cauchy–Schwarz inequality, we have that

$$n\sum_{i=1}^{n} |\lambda_i^*|\,|\rho_i^*| \leq n\left(\sum_{i=1}^{n} |\lambda_i^*|^2\right)^{1/2}\left(\sum_{i=1}^{n} |\rho_i^*|^2\right)^{1/2},$$

that is

$$n\sum_{i=1}^{n} |\lambda_i^*|\,|\rho_i^*| \leq 2n\sqrt{mR_{-1}(G)}. \tag{3.12}$$

Now, from (3.11) and (3.12) we arrive at (3.10).

Equality in (3.11) holds if and only if $|\lambda_1^*| = \cdots = |\lambda_n^*|$, or $|\rho_1^*| = \cdots = |\rho_n^*|$. Equality in (3.12) holds if and only if $|\lambda_i^*| = C\,|\rho_i^*|$, $C = Const$, for $i = 1, 2, \ldots, n$. Thus, equalities in both (3.11) and (3.12) hold if and only if $|\lambda_1^*| = \cdots = |\lambda_n^*|$ and $|\rho_1^*| = \cdots = |\rho_n^*|$. Since G has no isolated vertices, equality in (3.10) holds if and only if $G \cong \frac{n}{2}K_2$, for even n. □

Recall that the Sombor energy of a graph G, denoted by $E_{SO}(G)$, is introduced as the sum of the absolute values of the eigenvalues of its Sombor matrix [13]. The following relationship between graph energy and Sombor energy can be found in [15].

Theorem 3.7 ([15])**.** *If G is a bipartite graph whose all cycles (if any) have size not divisible by 4, then*

$$E_{SO}(G) \leq \sqrt{2}\Delta E(G).$$

From Theorem 3.6 and Theorem 3.7, we directly have the following.

Corollary 3.2. *If G is a bipartite graph whose all cycles (if any) have size not divisible by 4, then*

$$E_{SO}(G)\, RE(G) \leq 2\Delta n\sqrt{2mR_{-1}(G)}.$$

Corollary 3.3. *Let G be a graph of order $n \geq 2$ and size m, without isolated vertices. Then we have*

$$E(G) \leq n\sqrt{mR_{-1}(G)}.$$

Proof. In [3] it was proven that $RE(G) \geq 2$. Considering this with inequality (3.10) we obtain the required result. □

Corollary 3.4. *Let G be a graph of order $n \geq 2$ and size m, without isolated vertices. Then we have*

$$E(G) \leq n\sqrt{\frac{M_2(G)}{m}}, \tag{3.13}$$

where $M_2(G) = \sum_{i\sim j} d_i d_j$ is the second Zagreb index [16].

Proof. In [3] it was also proven that

$$RE(G) \geq 2R_{-1}(G).$$

From the above and inequality (3.10) we obtain

$$E(G) \leq n\sqrt{\frac{m}{R_{-1}(G)}}. \tag{3.14}$$

On the other hand, by the inequality between arithmetic and harmonic means (see, e.g., [23]), we have that

$$M_2(G)R_{-1}(G) \geq m^2.$$

Combining the above and inequality (3.14) we arrive at (3.13). □

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$k-$FRACTIONAL OSTROWSKI TYPE INEQUALITIES VIA $(s,r)-$CONVEX

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Abstract. We introduce the generalized class named it the class of $(s,r)-$convex in mixed kind, this class includes $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex, quasi convex and the class of ordinary convex. Also, we state the generalization of the classical Ostrowski inequality via $k-$fractional integrals, which is obtained for functions whose first derivative in absolute values is $(s,r)-$convex in mixed kind. Moreover, we establish some Ostrowski type inequalities via $k-$fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(s,r)-$convex in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means are given.

1. Introduction

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality which is stated as follows.

Key words and phrases. Ostrowski inequality, convex function, power mean inequality, Hölder's inequality.

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Theorem 1.1 ([14])**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on (a, b), $|f'(t)| \leq M$, for all $t \in (a, b)$. Then*

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \leq M(b-a)\left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right), \tag{1.1}$$

for all $x \in (a, b)$.

Also, one can see the numerous variants and applications in [5]-[11]. Nowadays, with the increasing demand of researchers for the study of natural phenomena, the use of fractional differential operators and fractional differential equations has become an effective means to achieve this goal. Compared with integer order operators, fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. This kind of operators have expanded and have been widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission and control.

On other hand convexity is very simple and ordinary concept. Due to its massive applications in industry and business, convexity has a great influence on our daily life. In the solution of many real world problems the concept of convexity is very decisive. The problems faced in constrained control and estimation are convex. Geometrically, a real valued function is said to be convex if the line segment segment joining any two of its points lies on or above the graph of the function in Euclidean space. First we present the important classes of convex functions from literature.

Definition 1.1 ([3])**.** The function $g : I \to \mathbb{R}$, $I \subset (0, \infty)$, is convex, if

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y),$$

for all $x, y \in I, t \in [0, 1]$.

Definition 1.2 ([15])**.** Let function $s \in (0, 1]$, the $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is $s-$convex in 1^{st} kind, if

$$g(tx + (1-t)y) \leq t^s g(x) + (1-t^s)g(y),$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.3 ([3])**.** The $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is quasi convex, if

$$g(tx + (1-t)y) \leq \max\{g(x), g(y)\},$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.4 ([15])**.** Let $s \in (0, 1]$, the function $g : I \to [0, \infty)$, $I \subset (0, \infty)$, is $s-$convex in 2^{nd} kind, if

$$g(tx + (1-t)y) \leq t^s g(x) + (1-t)^s g(y),$$

for all $x, y \in I$, $t \in [0, 1]$.

Definition 1.5 ([3]). The function $g : I \to [0,\infty)$, $I \subset (0,\infty)$, is a $P-$convex, if $g(x) \geq 0$ and for all $x, y \in I$ and $t \in [0,1]$,

$$g\left(tx + (1-t)y\right) \leq g(x) + g(y).$$

An important area in the field of applied and pure mathematics is the integral inequality. As it is known, inequalities aim to develop different mathematical methods. Nowadays, we need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. The concept of convexity plays a strong role in the field of inequalities due to the behavior of its definition and its properties. Furthermore, there is a strong correlation between convexity and symmetry concepts.

Definition 1.6 ([12]). The Riemann-Liouville integrals $I^{\varepsilon}_{a^+}f$ and $I^{\varepsilon}_{b^-}f$ of $f \in L_1([a,b])$ having order $\varepsilon > 0$ with $0 \leq a < b$ are defined by

$$I^{\varepsilon}_{a^+}f(x) = \frac{1}{\Gamma(\varepsilon)}\int_a^x \frac{f(t)}{(x-t)^{1-\varepsilon}}dt, \quad x > a,$$

and

$$I^{\varepsilon}_{b^-}f(x) = \frac{1}{\Gamma(\varepsilon)}\int_x^b \frac{f(t)}{(t-x)^{1-\varepsilon}}dt, \quad x < b,$$

respectively. Here $\Gamma(\varepsilon) = \int_0^\infty e^{-u}u^{\varepsilon-1}du$ is the Gamma function and $I^0_{a^+}f(x) = I^0_{b^-}f(x) = f(x)$. We also make use of Euler's beta function, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 1.7 ([12]). The $k-$fractional integrals ${}^kJ^{\varepsilon}_{a^+}f$ and ${}^kJ^{\varepsilon}_{b^-}f$ of $f \in L_1([a,b])$ having order $\varepsilon > 0$ with $0 \leq a < b$, $k > 0$ are defined by

$${}^kJ^{\varepsilon}_{a^+}f(x) = \frac{1}{k\Gamma_k(\varepsilon)}\int_a^x \frac{f(t)}{(x-t)^{1-\frac{\varepsilon}{k}}}dt, \quad x > a,$$

and

$${}^kJ^{\varepsilon}_{b^-}f(x) = \frac{1}{k\Gamma_k(\varepsilon)}\int_x^b \frac{f(t)}{(t-x)^{1-\frac{\varepsilon}{k}}}dt, \quad x < b,$$

respectively. Here $\Gamma_k(\varepsilon) = \int_0^\infty e^{-\frac{u^k}{k}}u^{\varepsilon-1}du$ is the generalized gamma function and ${}^1J^0_{a^+}f(x) = {}^1J^0_{b^-}f(x) = f(x)$.

Throught this paper, we denote

$$Y_f(\varepsilon,k,a,x,b) = \left(\frac{(x-a)^{\frac{\varepsilon}{k}} + (b-x)^{\frac{\varepsilon}{k}}}{(b-a)}\right)f(x) - \frac{k\Gamma_k(\varepsilon+1)}{b-a}\left({}^kJ^{\varepsilon}_{x^-}f(a) + {}^kJ^{\varepsilon}_{x^+}f(b)\right),$$

$$Z_f(\varepsilon,x,a,b) = \left(\frac{(x-a)^{\varepsilon} + (b-x)^{\varepsilon}}{b-a}\right)f(x) - \frac{\Gamma(\varepsilon+1)}{b-a}\left(I^{\varepsilon}_{x^-}f(a) + I^{\varepsilon}_{x^+}f(b)\right),$$

$${}^{\varepsilon}\kappa^b_a(x) = \left(\frac{(x-a)^{\varepsilon+1} + (b-x)^{\varepsilon+1}}{b-a}\right).$$

In order to prove our main results we need the following lemma.

Lemma 1.1 ([12]). *Let $f : I \to \mathbb{R}$, $I \subset (0, \infty)$, be an absolutely continuous function and $a, b \in I, a < b$. If $f' \in L_1[a, b]$, $\varepsilon, k > 0$, then*

$$\begin{aligned} Y_f(\varepsilon, k, a, x, b) =& \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \int_0^1 t^{\frac{\varepsilon}{k}} f'(tx + (1-t)a)\, dt \\ &- \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \int_0^1 t^{\frac{\varepsilon}{k}} f'(tx + (1-t)b)\, dt. \end{aligned}$$

Theorem 1.2 ([12]). *Let $f : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a, b \in I, a < b$, $f' \in L_1[a, b]$ and for $\varepsilon, k > 1$, Montgomery identity for $k-$fractional integrals holds:*

$$f(x) = \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon} f(b) - {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b)) + {}^kJ_a^{\varepsilon}(P_1(x,b)f'(b)),$$

where $P_1(x, t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a} \cdot \frac{k\Gamma_k(\varepsilon)}{(b-x)^{\frac{\varepsilon}{k}-1}}, & \text{if } t \in [a, x], \\ \frac{t-b}{b-a} \cdot \frac{k\Gamma_k(\varepsilon)}{(b-x)^{\frac{\varepsilon}{k}-1}}, & \text{if } t \in (x, b]. \end{cases}$$

Let $[a, b] \subseteq (0, +\infty)$, we may define special means as follows

(a) the arithmetic mean

$$A = A(a,b) := \frac{a+b}{2};$$

(b) the geometric mean

$$G = G(a,b) := \sqrt{ab};$$

(c) the harmonic mean

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}};$$

(d) the logarithmic mean

$$L = L(a,b) := \begin{cases} a, & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b; \end{cases}$$

(e) the identric mean

$$I = I(a,b) := \begin{cases} a, & \text{if } a = b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b; \end{cases}$$

(f) the $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b, \\ \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases}$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

2. $k-$Fractional Ostrowski Type Inequalities via $(s,r)-$Convex

In this section, we introduce the concept of $(s,r)-$convex in mixed kind. This class contains many classes of convex from literature of convex analysis. The main aim of this study is to reveal new generalized-Ostrowski-type inequalities via $(s,r)-$convex using $k-$fractional operator which generalizes Riemann-Liouville integral operator.

Definition 2.1. Let $(s,r) \in (0,1]^2$, the function $g : I \to [0,\infty)$, $I \subset (0,\infty)$, is $(s,r)-$convex in mixed kind, if

$$g\left(tx+(1-t)y\right) \leq t^{rs}g(x) + (1-t^r)^s g(y), \tag{2.1}$$

for all $x, y \in I$, $t \in [0,1]$.

Remark 2.1. In Definition 2.1, we can see the following.

(a) If $s = 1$ and $r \in [0,1]$ in (2.1), we get $r-$convex in 1^{st} kind.
(b) If $r \to 0$ and $s = 1$, in (2.1), we get quasi convex.
(c) If $r = 1$ and $s \in [0,1]$ in (2.1), we get $s-$convex in 2^{nd} kind.
(d) If $s \to 0$ and $r = 1$ in (2.1), we get $P-$convex.
(e) If $s = r = 1$ in (2.1), gives us ordinary convex.

Now, we will generalize the Ostrowski type inequalities via $(s,r)-$convex by using $k-$fractional integral operator.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b), $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $g : I \subset \mathbb{R} \to \mathbb{R}$, be an $(s,r)-$convex function in mixed sense, then we have the inequalities*

$$\begin{aligned} & g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon}f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\ \leq & \frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^{rs}}\left((x-a)^{rs-1}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt \right. \\ & \left. + \frac{((b-a)^r-(x-a)^r)^s}{b-x}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right), \end{aligned} \tag{2.2}$$

for all $x \in (a,b)$.

Proof. Utilizing the Theorem 1.2, we get

$$\begin{aligned}
& f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}} \; {}^{k}J_a^{\varepsilon} f(b) + \; {}^{k}J_a^{\varepsilon-1}(P_1(x,b)f(b)) \\
=& {}^{k}J_a^{\varepsilon}(P_1(x,b)f'(b)) \\
=& \frac{1}{k\Gamma_k(\varepsilon)} \int_a^b P_1(x,t)\frac{f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}dt \\
=& \left(\frac{x-a}{b-a}\right)\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{x-a}\int_a^x \frac{(t-a)\,f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}dt\right) \\
& + \left(1-\left(\frac{x-a}{b-a}\right)\right)\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-x}\int_x^b \frac{(t-b)\,f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}dt\right),
\end{aligned}$$

for all $x \in (a,b)$. Next by using the $(s,r)-$convex function in mixed sense of $g : I \subset [0,\infty) \to \mathbb{R}$, we get

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}} \; {}^{k}J_a^{\varepsilon} f(b) + \; {}^{k}J_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \left(\frac{x-a}{b-a}\right)^{rs} g\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{x-a}\int_a^x \frac{(t-a)\,f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}dt\right) \\
& + \left(1-\left(\frac{x-a}{b-a}\right)^{r}\right)^{s} g\left(\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-x}\int_x^b \frac{(t-b)\,f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}dt\right),
\end{aligned}$$

for all $x \in (a,b)$. Applying Jensen's integral inequality [6], we get (2.2). □

Corollary 2.1. *In Theorem* 2.1, *one can see the following.*

(a) *If* $s = 1$ *and* $r \in (0,1]$ *in* (2.2), *then Ostrowski inequality for* $r-$*convex functions in* 1^{st} *kind:*

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}} \; {}^{k}J_a^{\varepsilon} f(b) + \; {}^{k}J_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^r}\left((x-a)^{r-1}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right. \\
& \left. + \frac{(b-a)^r-(x-a)^r}{(b-x)}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).
\end{aligned}$$

(b) *If* $s = 1$ *and* $r \to 0$ *in* (2.2), *we get quasi-convex function*

$$\begin{aligned}
& g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}} \; {}^{k}J_a^{\varepsilon} f(b) + \; {}^{k}J_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\
\leq & \frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(x-a)}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt.
\end{aligned}$$

(c) *If $r = 1$ and $s \in [0, 1)$ in (2.2), then fractional Ostrowski type inequality for $s-$convex functions in 2^{nd} kind:*

$$\begin{aligned} &g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon}f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\ \le &\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{(b-a)^s}\left((x-a)^{s-1}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt + (b-x)^{s-1}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right). \end{aligned}$$

(d) *If $r = 1$ and $s \to 0$ in (2.2), then fractional Ostrowski type inequality for $P-$convex functions:*

$$\begin{aligned} &g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon}f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\ \le &(b-x)^{1-\frac{\varepsilon}{k}}\left(\frac{1}{x-a}\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt + \frac{1}{b-x}\int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right). \end{aligned}$$

(e) *If $s = r = 1$ in (2.2), then fractional Ostrowski type inequality for convex functions:*

$$\begin{aligned} &g\left(f(x) - \frac{k\Gamma_k(\varepsilon)}{b-a}(b-x)^{1-\frac{\varepsilon}{k}}\ {}^kJ_a^{\varepsilon}f(b) + {}^kJ_a^{\varepsilon-1}(P_1(x,b)f(b))\right) \\ \le &\frac{(b-x)^{1-\frac{\varepsilon}{k}}}{b-a}\left(\int_a^x g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt + \int_x^b g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right). \end{aligned}$$

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$, $[a, b] \subset (0, \infty)$, be an absolutely continuous, and $f' \in L_1[a, b]$. If $|f'|$ is $(s, r)-$convex, $|f'(x)| \le M$, for all $x \in [a, b]$, and $\varepsilon, k > 0$, then*

(2.3)
$$|Y_f(\varepsilon,k,a,x,b)| \le M\left(\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt + \int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)\left(\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} + \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\right).$$

Proof. By using the Lemma 1.1,

$$\begin{aligned} |Y_f(\varepsilon,k,a,x,b)| \le &\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'(tx+(1-t)a)\right|dt \\ &+ \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'(tx+(1-t)b)\right|dt. \end{aligned}$$

Since $|f'|$ is $(s, r)-$convex and by using $|f'(x)| \le M$, we get

$$\begin{aligned} |Y_f(\varepsilon,k,a,x,b)| \le &\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\Big(t^{rs}|f'(x)| + (1-t^r)^s|f'(a)|\Big)dt \\ &+ \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\int_0^1 t^{\frac{\varepsilon}{k}}\Big(t^{rs}|f'(x)| + (1-t^r)^s|f'(b)|\Big)dt. \end{aligned}$$

Therefore,

$$|Y_f(\varepsilon,k,a,x,b)| \leq \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\left(|f'(x)|\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt + |f'(a)|\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)$$
$$+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\left(|f'(x)|\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt + |f'(b)|\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right). \quad \square$$

Remark 2.2. In Theorem 2.2, one can also capture the inequalities for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.2. *In Theorem* 2.2, *one can see for* $k=1$ *the following.*

(a) *The Ostrowski inequality for* $(s,r)-$*convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \leq M\left(\frac{1}{\varepsilon+rs+1}+\frac{B\left(\frac{\varepsilon+1}{r},s+1\right)}{r}\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If* $s=1$ *and* $r\in(0,1]$ *in inequality* (2.3), *then the Ostrowski inequality for* $r-$*convex in* 1^{st} *kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \leq M\left(\frac{1}{\varepsilon+r+1}+\frac{B\left(\frac{\varepsilon+1}{r},2\right)}{r}\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r=1$ *and* $s\in(0,1]$ *in inequality* (2.3), *then the Ostrowski inequality for* $s-$*convex in* 2^{nd} *kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \leq M\left(\frac{1}{\varepsilon+s+1}+B(\varepsilon+1,s+1)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon=r=1$ *and* $s\in(0,1]$ *in inequality* (2.3), *then the inequality* (2.1) *of Theorem* 2 *in* [1].

(e) *If* $r=1$ *and* $s\in(0,1]$ *in inequality* (2.3), *then the inequality* (2.6) *of Theorem* 7 *in* [15].

(f) *If* $s\to 0$ *and* $r=1$, *in inequality* (2.3), *then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \leq M\left(\frac{1}{\varepsilon+1}+B(\varepsilon+1,1)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r=s=1$, *in inequality* (2.3), *then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)| \leq M\left(\frac{1}{\varepsilon+2}+B(\varepsilon+1,2)\right)\ {}^{\varepsilon}\kappa_a^b(x).$$

(h) *If* $\varepsilon=r=s=1$, *in inequality* (2.3), *then the Ostrowski inequality* (1.1) *for convex.*

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$, $[a,b]\subset(0,\infty)$, be an absolutely continuous, and $f'\in L[a,b]$. If $|f'|^q$ is $(s,r)-$convex for $q>1$ and $|f'(x)|\le M$, for all $x\in[a,b]$, and $\varepsilon,k>0$, then*

$$|Y_f(\varepsilon,k,a,x,b)|\le\frac{M}{L^{\frac{1}{q}-1}}\left(\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{(b-a)}\right)\times\left(\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt+\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)^{\frac{1}{q}},\tag{2.4}$$

where

$$L=\int_0^1 t^{\frac{\varepsilon}{k}}dt.$$

Proof. By using the Lemma 1.1, and Power mean inequality,

$$\begin{aligned}|Y_f(\varepsilon,k,a,x,b)|\le{}&\frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\varepsilon}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\&+\frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\varepsilon}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\frac{\varepsilon}{k}}\left|f'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is $(s,r)-$convex and $|f'(x)|\le M$

$$\begin{aligned}|Y_f(\varepsilon,k,a,x,b)|\le{}&\frac{M(x-a)^{\frac{\varepsilon}{k}+1}}{L^{\frac{1}{q}-1}(b-a)}\left(\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt+\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)^{\frac{1}{q}}\\&+\frac{M(b-x)^{\frac{\varepsilon}{k}+1}}{L^{\frac{1}{q}-1}(b-a)}\left(\int_0^1 t^{\frac{\varepsilon}{k}}t^{rs}dt+\int_0^1 t^{\frac{\varepsilon}{k}}(1-t^r)^s dt\right)^{\frac{1}{q}}.\end{aligned}$$

□

Remark 2.3. In Theorem 2.3, one can also capture the inequalities for $s-$convex in 1st and 2nd kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.3. *In Theorem* 2.3, *one can see for $k=1$ the following.*

(a) *The Ostrowski inequality for $(s,r)-$convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)|\le\frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}}\left(\frac{1}{\varepsilon+rs+1}+\frac{B\left(\frac{\varepsilon+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If $s=1$ and $r\in(0,1]$ in inequality* (2.4), *then the Ostrowski inequality for $r-$convex in 1st kind via fractional integrals:*

$$|Z_f(\varepsilon,x,a,b)|\le\frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}}\left(\frac{1}{\varepsilon+s+1}+\frac{B\left(\frac{\varepsilon+1}{s},2\right)}{s}\right)^{\frac{1}{q}}\ {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r = 1$ *and* $s \in (0,1]$ *in inequality* (2.4)*, then the Ostrowski inequality for* $s-$*convex in* 2^{nd} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon + s + 1} + B(\varepsilon+1, s+1) \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon = r = 1$*, and* $s \in (0,1]$ *in inequality* (2.4)*, then the inequality* (2.3) *of Theorem* 4 *in* [1].

(e) *If* $r = 1$ *and* $s \in (0,1]$ *in inequality* (2.4)*, then the inequality* (2.8) *of Theorem* 9 *in* [15].

(f) *If* $r = 1$ *and* $s \to 0$ *in inequality* (2.4)*, then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon + 1} + B(\varepsilon+1, 1) \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r = s = 1$*, in inequality* (2.4)*, then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \leq \frac{M}{(\varepsilon+1)^{1-\frac{1}{q}}} \left(\frac{1}{\varepsilon + 2} + B(\varepsilon+1, 2) \right)^{\frac{1}{q}} \, {}^{\varepsilon}\kappa_a^b(x).$$

Theorem 2.4. *Let* $f : [a,b] \to \mathbb{R}$*,* $[a,b] \subset (0,\infty)$*, be an absolutely continuous,* $f' \in L[a,b]$*. If* $|f'|^q$ *is* $(s,r)-$*convex,* $|f'(x)| \leq M$*, for all* $x \in [a,b]$*,* $\varepsilon, k > 0$*, and* $p, z > 1$ *with* $\frac{1}{z} + \frac{1}{q} = 1$*, then*

(2.5)
$$|Y_f(\varepsilon, k, a, x, b)| \leq \frac{MK^{\frac{1}{z}}}{b-a} \left(\frac{1}{rs+1} + \frac{1}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}} \times \left((x-a)^{\frac{\varepsilon}{k}+1} + (b-x)^{\frac{\varepsilon}{k}+1} \right),$$

where

$$K = \int_0^1 t^{\frac{\varepsilon z}{k}} \, dt.$$

Proof. By using Lemma 1.1, and Hölder's inequality,

$$\begin{aligned} |Y_f(\varepsilon, k, a, x, b)| \leq{}& \frac{(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon z}{k}} dt \right)^{\frac{1}{z}} \left(\int_0^1 |f'(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \\ &+ \frac{(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\int_0^1 t^{\frac{\varepsilon z}{k}} dt \right)^{\frac{1}{z}} \left(\int_0^1 |f'(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^q$ is $(s,r)-$convex and $|f'(x)| \leq M$

$$\begin{aligned} |Y_f(\varepsilon, k, a, x, b)| \leq{}& \frac{K^{\frac{1}{z}}(x-a)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\frac{M^q}{rs+1} + \frac{M^q}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}} \\ &+ \frac{K^{\frac{1}{z}}(b-x)^{\frac{\varepsilon}{k}+1}}{b-a} \left(\frac{M^q}{rs+1} + \frac{M^q}{r} B\left(\frac{1}{r}, s+1\right) \right)^{\frac{1}{q}}. \end{aligned}$$

□

Remark 2.4. In Theorem 2.4, one can also capture the inequalities for $s-$convex in 1^{st} and 2^{nd} kind, $P-$convex and convex via $k-$fractional integrals by using Remark 2.1.

Corollary 2.4. *In Theorem* 2.4, *one can see for* $k = 1$ *the following.*

(a) *The Ostrowski inequality for* $(s, r)-$*convex in mixed kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \le \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{rs + 1} + \frac{B\left(\frac{1}{r}, s + 1\right)}{r} \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(b) *If* $s = 1$ *and* $r \in (0, 1]$ *in inequality* (2.5), *then the Ostrowski inequality for* $r-$*convex in* 1^{st} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \le \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{s + 1} + \frac{B\left(\frac{1}{s}, 2\right)}{s} \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(c) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the Ostrowski inequality for* $s-$*convex in* 2^{nd} *kind via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \le \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \left(\frac{1}{s + 1} + B\left(1, s + 1\right) \right)^{\frac{1}{q}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(d) *If* $\varepsilon = r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the inequality* (2.2) *of Theorem* 3 *in* [1].

(e) *If* $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.5), *then the inequality* (2.7) *of Theorem* 8 *in* [15].

(f) *If* $r = 1$, *and* $s \to 0$ *in inequality* (2.5), *then the Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \le \frac{(2)^{\frac{1}{q}}\, M}{(\varepsilon z + 1)^{\frac{1}{z}}} \ {}^{\varepsilon}\kappa_a^b(x).$$

(g) *If* $r = s = 1$, *in inequality* (2.5), *then the Ostrowski inequality for convex via fractional integrals:*

$$|Z_f(\varepsilon, x, a, b)| \le \frac{M}{(\varepsilon z + 1)^{\frac{1}{z}}} \ {}^{\varepsilon}\kappa_a^b(x).$$

3. Applications to Special Means

If we replace f by $-f$ and $x = \frac{a+b}{2}$ in Theorem 2.1, we get the following.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b), $f':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $g:I\to\mathbb{R}$, $I\subset\mathbb{R}$, be a $(s,r)-$convex function in mixed sense, then*

(3.1)
$$g\left(\frac{k\Gamma_k(\varepsilon)\left(\frac{b-a}{2}\right)^{1-\frac{\varepsilon}{k}}}{b-a}\ {}^{k}J_a^{\varepsilon}f(b)-f\left(\frac{a+b}{2}\right)-\ {}^{k}J_a^{\varepsilon-1}\left(P_1\left(\frac{a+b}{2},b\right)f(b)\right)\right)$$
$$\leq\frac{2^{\varepsilon-1}}{(b-a)^{\varepsilon}}\left(\frac{1}{2^{sr-1}}\int_{\frac{a+b}{2}}^{a}g\left(\frac{(t-a)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_b^{\frac{a+b}{2}}g\left(\frac{(t-b)f'(t)}{(b-t)^{1-\frac{\varepsilon}{k}}}\right)dt\right).$$

Remark 3.1. In Theorem 3.1, if we put $\varepsilon=k=1$ in (3.1), we get
$$g\left(\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\right)$$
$$\leq\frac{1}{b-a}\left(\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g((a-t)f'(t))dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^b g((b-t)f'(t))dt\right).$$

Remark 3.2. Assume that $g:I\to\mathbb{R}$, $I\subset[0,\infty)$, is an $(s,r)-$convex function in mixed kind.

(a) If $\varepsilon=k=1$, $f(t)=\frac{1}{t}$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have
$$(b-a)g\left(\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{t-a}{t^2}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^b g\left(\frac{t-b}{t^2}\right)dt.$$

(b) If $\varepsilon=k=1$, $f(t)=-\ln t$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have
$$(b-a)g\left(\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{t-a}{t}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^b g\left(\frac{t-b}{t}\right)dt.$$

(c) If $\varepsilon=k=1$, $f(t)=t^p$, $p\in\mathbb{R}\setminus\{0,-1\}$ in inequality (3.1), where $t\in[a,b]\subset(0,\infty)$, then we have
$$(b-a)g\left(L_p^p(a,b)-A^p(a,b)\right)$$
$$\leq\frac{1}{2^{sr-1}}\int_a^{\frac{a+b}{2}}g\left(\frac{p\,(a-t)}{t^{1-p}}\right)dt+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{a+b}{2}}^b g\left(\frac{p\,(b-t)}{t^{1-p}}\right)dt.$$

Remark 3.3. In Theorem 2.3, one can see for $\varepsilon=k=1$ the following.

(a) Let $x=\frac{a+b}{2}$, $0<a<b$, $q\geq 1$ and $f:\mathbb{R}\to\mathbb{R}^{+}$, $f(t)=t^{n}$ in (2.4). Then

$$\left|A^{n}\left(a,b\right)-L_{n}^{n}\left(a,b\right)\right|\leq\frac{M\left(b-a\right)}{\left(2\right)^{2-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(b) Let $x=\frac{a+b}{2}$, $0<a<b$, $q\geq 1$ and $f:(0,1]\to\mathbb{R}$, $f(t)=-\ln t$ in (2.4). Then

$$\left|\ln\left(\frac{A\left(a,b\right)}{I\left(a,b\right)}\right)\right|\leq\frac{M\left(b-a\right)}{\left(2\right)^{2-\frac{1}{q}}}\left(\frac{1}{rs+2}+\frac{B\left(\frac{2}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

Remark 3.4. In Theorem 2.4, one can see for $\varepsilon=k=1$ the following.

(a) Let $x=\frac{a+b}{2}$, $0<a<b$, $p^{-1}+q^{-1}=1$ and $f:\mathbb{R}\to\mathbb{R}^{+}$, $f(t)=t^{n}$ in (2.5). Then

$$\left|A^{n}\left(a,b\right)-L_{n}^{n}\left(a,b\right)\right|\leq\frac{M\left(b-a\right)}{2\left(z+1\right)^{\frac{1}{z}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(b) Let $x=\frac{a+b}{2}$, $0<a<b$, $p^{-1}+q^{-1}=1$ and $f:(0,1]\to\mathbb{R}$, $f(t)=-\ln t$ in (2.5). Then

$$\left|\ln\left(\frac{A\left(a,b\right)}{I\left(a,b\right)}\right)\right|\leq\frac{M\left(b-a\right)}{2\left(z+1\right)^{\frac{1}{z}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

4. Conclusion

Ostrowski inequality is one of the most celebrated inequalities. We can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(s,r)-$convex in mixed kind, this class of functions contains many important classes including class of $s-$convex in 1st and 2nd kind, $P-$convex, quasi convex and the class of convex. In this study, theorems are put forward to obtain new upper bounds by $k-$fractional operator for Ostrowski type inequalities. We have stated our first main result in Section 2, the generalization of Ostrowski inequality [14] via $k-$fractional integral and others results obtained by using different techniques including Hölder's inequality and power mean inequality. Also, various established results captured as special cases. Moreover, some applications in terms of special means was presented at the end.

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STRONGLY EXTENDING MODULAR LATTICES

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Abstract. In this paper, our purpose is to initiate the study of the concept of strongly extending modular lattices based on the similar notion of strongly extending modules. We will prove some basic properties of strongly extending modular lattices and employ this results to give applications to the category of modules with a fixed hereditary torsion class and Grothendieck categories.

1. Introduction

The notion of CC or extending for modules and related notions is an interesting topics for several authors that were extensively studied in the literature ([18]). A module M is said to be an *extending* (or a CS) module provided that every submodule of M is contained in a direct summand of M as an essential submodule. A module M is called a *FI-extending* module provided that each of its fully invariant submodule is essential in a direct summand ([12]). Another interesting related concepts of the extending modules is strongly FI-extending ([13, 15]). The strongly FI-extending property of modules has been used for the existence and description of the FI-extending module hull of any finitely generated projective module over a semiprime ring ([14]). A module M is said to be a *strongly FI-extending* module if each fully invariant submodule is essentially contained in a fully invariant direct summand. In [19], a subclass of extending modules, strongly extending modules, introduced and investigated. A module M is said to be *strongly extending* provided that each submodule is essential in a fully invariant direct summand. Recently, the known conditions on modules (extending,

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FI-extending, strongly FI-extending, etc.) have been introduced and considered in lattices, in order to give some interesting results to Grothendieck categories and the category of modules with hereditary torsion theories [5–7, 9, 10].

When we study the classes of extending, FI-extending, strongly FI-extending lattices, it is an ambition to study the notion of strongly extending in lattices. Also one of the motivations to study this topic is the following questions.

(1) If a lattice L is strongly extending, then is it true that every complement is fully invariant?

(2) Is it true that every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L if and only if every complement of L is fully invariant in L?

This paper is allocated to initiate the strongly extending condition for lattices, and investigate their properties that are similar to results on modules introduced and studied in [19]. We will adopt the results from [19] to strongly extending lattices, however it is not always easy because some theoretical tools and techniques in modules do not work in lattices.

In Section 2, we recall some preliminaries and definitions about lattices from [1–11]. We recall the useful notion of linear morphisms between two lattices introduced by Albu and Iosif [5]. This concept is used in our main results. In Section 3, we define the conditions strongly extending and Abelian for lattices, and some of their structural properties are studied. We will answer the previous questions affirmatively. We will show that every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L if and only if $D(L) \subset FI(L)$. Also, it is shown that, a strongly extending lattice L is extending and every idempotent linear endomorphism of a lattice L commutes with another linear endomorphism of L. Moreover, if L is complete and strongly extending, then $D(L)$ is a sublattice of L and every its subset has a greatest lower bound. Further, we prove that the strongly extending condition of lattices is preserved by their complement intervals, and consider when direct joins have this property. In Section 3 and Section 4 we exhibit some usage of the results to Grothendieck categories and the category of modules with a fixed hereditary torsion class.

2. Preliminaries

Throughout this paper, by L, we will indicate a modular lattice $(L, \leq, \wedge, \vee, 0, 1)$ that has least element 0 and greatest element 1. For any $l, k \in L$, where $l \leq k$, let k/l denote the interval $\{x \in L \mid l \leq x \leq k\}$. For basic terminology and notation on lattices, we refer the reader to [4, 16, 17, 20] and [21], but particularly to [4]. For a lattice L, by $D(L)$, $P(L)$, $E(L)$ and $C(L)$, we denote the set of all *complement elements* of L, the set of all *pseudo-complement elements* of L, the set of all *essential elements* of L and the set of all *closed elements* of L, respectively.

A lattice L is said to be *extending* or *CC* if, for any $l \in L$, we have l is essential in $k/0$, for some complement interval $k/0$ in L. Also, L is said to be *quasi-continuous* provided that it is extending and for any two complement elements l_1, l_2 of L with $l_1 \wedge l_2 = 0$, we have $l_1 \vee l_2 \in D(L)$ ([8, Definition 1.2]).

By Albu and Iosif [5], a map $\theta : L \to L'$ between two lattices L with greatest element 1_L, least element 0_L and a lattice L' with greatest element $1_{L'}$, least element $0_{L'}$ is called a *linear morphism* provided that there exist $i \in L'$ and $k \in L$ (k is said to be a kernel of θ) such that $\theta(l) = \theta(l \vee k)$, for each $l \in L$, and f induces a lattice isomorphism:

$$\bar{\theta} : 1_L/k \to i/0_{L'}, \quad \bar{\theta}(l) = \theta(l), \quad \text{for all } l \in 1/k.$$

Assume that L is a lattice. By [6, Examples 0.2 (2)], if $c, d \in L$ and $c \wedge d = 0$, then the mapping

$$p_{d,c} : (c \vee d)/0 \to c/0, \quad p_{d,c}(a) := (a \vee d) \wedge c,$$

is said to be the canonical projection on $c/0$, which is a linear morphism (surjective) and its kernel is d. Notice that if L is a modular lattice, then $p_{d,c}(a) = a$, for all $a \in c/0$. In particular, if $k \in L$ is a complement of $l \in L$, we will use the notation $\widetilde{p}_{l,k}$, the linear endomorphism of L obtained by composing $\widetilde{p}_{l,k}$ with the canonical inclusion mapping $i : k/0 \to L$. If there is not any ambiguity about l, the notation $\widetilde{p}_k$ will be used instead of $\widetilde{p}_{l,k}$.

Throughout this paper, $End(L)$ denotes the collection of all linear endomorphisms of a modular lattice L (it is a monoid, with respect to the composition "$\circ$" of functions). We will use the notation fg for the composition $f \circ g$ of two linear morphisms f, g. An element $l \in L$ is said to be a fully invariant element, provided that $\theta(l) \leq l$ for each $\theta \in End(L)$ ([9]). By $FI(L)$, we will indicate the set $\{l \in L \mid l \text{ is fully invariant in } L\}$. A linear endomorphism θ of a modular lattice L is said to be a *left semicentral idempotent* of $End(L)$ (or L) if $\theta^2 = \theta$ and $\theta\psi = \theta\psi\theta$ for all $\psi \in End(L)$ ([10]). We exhibit by $S_l(L)$ the collection of all left semicentral idempotents of L.

It is assumed throughout this paper that a ring R is an associative ring with unity and all modules are unital right R-modules. The notation $Mod - R$ denotes the category of all unital right R-modules. We denote by M_R a unital right R-module M. Let $L(M_R)$ indicate the lattice of all submodules of a module M_R. For submodules T and H of M, $T \leq H$ will denote that T is a submodule of H.

3. Strongly Extending Lattices

This section is allocated to introduce and investigate our main concept, namely, strongly extending lattices and give some properties of this class of lattices and establish some relations between the notion of strongly extending and the other notions in the literature. We begin with the following lemma which is a quite useful in this note.

Lemma 3.1. *Let θ be an idempotent linear endomorphism of L. Then $\theta(1)$ is a complement of* $\ker(\theta)$ *and $\widetilde{p}_{\theta(1)} = \theta$.*

Proof. Let $k := \ker(\theta)$. We claim $\theta(1) \vee k = 1$ and $\theta(1) \wedge k = 0$. As θ commutes with arbitrary joins ([6, Lemma 06]), $\theta(\theta(1) \vee k) = \theta(\theta(1)) \vee \theta(k) = \theta(1)$. Thus, $\overline{\theta}(\theta(1) \vee k) = \overline{\theta}(1)$. As $\overline{\theta}$ is an isomorphism, $\theta(1) \vee k = 1$.

Since $\overline{\theta}$ is an isomorphism, we have $\theta(1) \wedge k = \overline{\theta}(c)$, for some $c \in 1/k$. Thus, $\theta(1) \wedge k = \theta(c)$. Hence,

$$\theta(\theta(1) \wedge k) = \theta(\theta(c)) = \theta(c).$$

As $\theta(1) \wedge k \leq k$, $\theta(\theta(1) \wedge k) = 0$. Therefore, $0 = \theta(c) = \theta(1) \wedge k$, as desired.

Now we show that $\widetilde{p}_{\theta(1)} = h$. As θ commutes with arbitrary joins and $\theta(k) = 0$, $\theta(x \vee k) = \theta(x) \vee \theta(k) = \theta(x)$. Since θ is idempotent,

$$\begin{aligned}\overline{\theta}(x \vee k) =& \theta(x \vee k) = \theta(x) = \theta(\theta(x)) \\ =& \theta(\theta(x) \vee k) = \overline{\theta}(\theta(x) \vee k).\end{aligned}$$

Thus, $x \vee k = \theta(x) \vee k$, because $\overline{\theta}$ is a lattice isomorphism. As L is modular and $\theta(x) \leq \theta(1)$,

$$\begin{aligned}\widetilde{p}_{\theta(1)}(x) =& (x \vee k) \wedge \theta(1) \\ =& (\theta(x) \vee k) \wedge \theta(1) = \theta(x) \vee (\theta(1) \wedge k) \\ =& \theta(x).\end{aligned}$$

It completes the proof. □

Definition 3.1. A lattice L is said to be Abelian, if any idempotent linear endomorphism of L is central in $End(L)$ (i.e., commute with any linear endomorphism of L).

In the following, we provide a characterization for Abelian lattices.

Proposition 3.1. *Let L be a lattice. Then the following statements are equivalent:*
(1) *$D(L) \subseteq FI(L)$;*
(2) *L is Abelian.*

Proof. (1) ⇒ (2) Let θ be an idempotent linear endomorphism of L. Put $l := \theta(1)$ and $m = \ker(\theta)$. By Lemma 3.1, $l \wedge m = 0$, $l \vee m = 1$ and $\widetilde{p}_l = \theta$. By (1), $l, m \in FI(L)$. Therefore, we have $\widetilde{p}_l, \widetilde{p}_m \in S_l(End(L))$, by [10, Lemma 2.8] (it is known that if $e \in D(L)$, then $\widetilde{p}_e \in S_l(End(L))$ if and only if $e \in FI(L)$ [10, Lemma 2.8]). Let $\psi \in End(L)$. We will show that $\psi\theta = \theta\psi$. Let $x \in L$. Then $\psi(\theta(x)) = \psi(\widetilde{p}_l(x)) = \psi((x \vee m) \wedge l)$. As $(x \vee m) \wedge l \leq l$ and $l \in FI(L)$, we have

$$\psi((x \vee m) \wedge l) \leq \psi(l) \leq l.$$

Moreover, $m \in FI(L)$ and $(x \vee m) \wedge l \leq x \vee m$, hence

$$\psi((x \vee m) \wedge l) \leq \psi(x \vee m) = \psi(x) \vee \psi(m) \leq \psi(x) \vee m.$$

Thus,

$$\psi(\theta(x)) = \psi(\widetilde{p}_l(x)) = \psi((x \vee m) \wedge l) \leq (\psi(x) \vee m) \wedge l = \widetilde{p}_l(\psi(x)) = \theta(\psi(x)).$$

For the reverse, we have $(x \vee l) \wedge m \leq m \leq x \vee m$. Since L is modular,

$$
\begin{aligned}
((x \vee m) \wedge l) \vee ((x \vee l) \wedge m) =& (x \vee m) \wedge (l \vee ((x \vee l) \wedge m)) \\
=& (x \vee m) \wedge ((l \vee m) \wedge (x \vee l)) \\
=& (x \vee m) \wedge (x \vee l).
\end{aligned}
$$

Thus, we have

$$x \leq (x \vee m) \wedge (x \vee l) =((x \vee m) \wedge l) \vee ((x \vee l) \wedge m) = \widetilde{p}_l(x) \wedge \widetilde{p}_m(x).$$

Hence, $\psi(x) \leq \psi((\widetilde{p}_l(x) \wedge \widetilde{p}_m(x)))$ and

$$\theta(\psi(x)) \leq \theta(\psi((\widetilde{p}_l(x) \wedge \widetilde{p}_m(x)))) = \theta(\psi(\widetilde{p}_l(x))) \vee \theta(\psi(\widetilde{p}_m(x))).$$

Since $\widetilde{p}_l = \theta,\ \widetilde{p}_m \in S_l(End(L))$, $\theta\psi\theta = \psi\theta$ and $\widetilde{p}_m\psi\widetilde{p}_m = \psi\widetilde{p}_m$. Therefore,

$$\theta\psi\theta(x) \vee \theta f\widetilde{p}_m(x) = \psi\theta(x) \vee \theta\widetilde{p}_m\psi\widetilde{p}_m(x).$$

As $\theta(\widetilde{p}_m)(c) = 0$, for each $c \in L$, we have $\theta\psi(x) \leq \psi\theta(x)$. Therefore, $e\theta = \psi\theta$, as desired.

(2) $\Rightarrow$ (1) Let $l \in D(L)$. By (2), $\widetilde{p}_l$ is central and so $\widetilde{p}_l \in S_l(End(L))$, by [10, Lemma 2.8]. Therefore, $l \in FI(L)$ and $D(L) \subseteq FI(L)$. □

In the following, we introduce the key definition of this paper.

Definition 3.2. A lattice L is said to be strongly extending, provided that for any $l \in L$, $l \in E(e/0)$ for some $e \in (FI(L) \cap D(L))$.

In the following observation, we give some characterizations of strongly extending lattices.

Theorem 3.1. *Let L be a lattice. Then the following statements are equivalent:*
(1) *L is a strongly extending lattice;*
(2) *L is extending and $C(L) \subseteq FI(L)$;*
(3) *L is extending and $P(L) \subseteq FI(L)$;*
(4) *L is extending and $D(L) \subseteq FI(L)$;*
(5) *L is extending and L is Abelian.*

Proof. (1) $\Rightarrow$ (2) If L is strongly extending, then L is extending. Let $e \in C(L)$. Hence there exists $l \in D(L) \cap FI(L)$ such that $e \in E(l/0)$. Thus, $e = l$, and so $C(L) \subseteq FI(L)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It is clear, because $D(L) \subseteq P(L) \subseteq C(L)$, by [8, Proposition 1.7 (1)].

(4) $\Rightarrow$ (5) It is clear by Proposition 3.1.

(5) $\Rightarrow$ (1) Let $l \in L$. Then $l \in E(k/0)$, for some $k \in D(L)$. By (5), $\widetilde{p}_k \in S_l(End(L))$. Therefore, $k \in FI(L)$, by [10, Lemma 2.8]. Hence, L is strongly extending. □

Corollary 3.1. *If L is a uniform lattice, then L is strongly extending.*

The converse of Corollary 3.1 is true, provided that L is indecomposable.

Theorem 3.2. *Let L be a complete strongly extending lattice. Then $D(L)$ is a sublattice of L. Moreover, every subset of $D(L)$ has a greatest lower bound.*

Proof. Let $e, f \in D(L)$ and $e \vee e' = 1, e \wedge e' = 0,\ f \vee f' = 1$ and $f \wedge f' = 0$. We are going to show $f \vee e \in D(L)$. By Theorem 3.1, $D(L) \subset FI(L)$, and so by [9, Lemma 1.8(4)], we have

$$e = (e \wedge f) \vee (e \vee f').$$

Therefore, $e \wedge f \in D(e)$, and hence $e \wedge f \in D(L)$, by [8, Proposition 1.7(3)].

Now, we will show that $e \vee f \in D(L)$. By [9, Lemma 1.8(4)], we have

$$f = (e \wedge f) \vee (e' \vee f).$$

Therefore,

$$e \vee f = e \vee (f \wedge e) \vee (f \wedge e') = e \vee (f \wedge e').$$

By the previous argument, $f \wedge e' \in D(L)$, hence there exits $t \in L$ such that $1 = (f \wedge e') \vee t$ and $(f \wedge e') \wedge t = 0$. Since $e \in FI(L)$, we have

$$e = (e \wedge t) \vee (e \wedge (f \wedge e')) = e \wedge t,$$

by [9, Lemma 1.8 (4)]. Thus, $e \leq t$ and $e \in D(t/0)$. Let $t = e\dot{\vee}h$. Then $1 = (f \wedge e') \vee e \vee h = (e \vee f)\dot{\vee}h$. Hence, $e \vee f \in D(L)$.

Now, suppose that $\{d_i\}_{i \in I} \subset D(L)$, where I is an arbitrary index set. Then $\bigwedge_{i \in I} d_i \in E(a/0)$, for some $a \in D(L) \cap FI(L)$. Let $d_i' \in D(L)$ be such that $d_i \vee d_i' = 1$ for each $i \in I$. Since $a \in FI(L)$, $a = (a \wedge d_i) \vee (a \wedge d_i')$, by [9, Lemma 1.8 (4)]. Since $\bigwedge_{i \in I} d_i \in E(a/0)$ and $(\bigwedge_{i \in I} d_i) \wedge d_i' = 0$, for each $i \in I$, we have $d_i' \wedge a = 0$. Therefore, $a = a \wedge d_i$, for each $i \in I$, and so $a \leq d_i$, for each $i \in I$. Hence, $a \leq \bigwedge_{i \in I} d_i$ and $a = \bigwedge_{i \in I} d_i \in D(L)$. Hence, every subset of $D(L)$ has a greatest lower bound. □

Corollary 3.2. *Let L be a strongly extending lattice. Then L is quasi-continuous.*

Proof. Assume that L is a strongly extending lattice. Then L satisfies the condition C_1. Moreover, L has C_3 property by Theorem 3.2. □

Next, we give some properties of a strongly extending lattice.

Proposition 3.2. *Let L be a strongly extending lattice. Then the following statements hold.*

(1) *If θ is a linear monomorphism, then $\theta(1) \in E(L)$.*
(2) *If $\theta\psi = 1_{End(L)}$, for some $\psi, \theta \in End(L)$, then $\psi\theta = 1_{End(L)}$.*

Proof. (1) Let θ be a linear monomorphism. Then $\theta(1) \in E(h/0)$, for some $h \in D(L) \cap FI(L)$. Since $h \in D(L)$, $1 = h\dot{\vee}h'$, for some $h' \in L$. Hence, $(\widetilde{p}_{h'} \circ \theta)(1) = 0$. By Theorem 3.1, $\widetilde{p_{h'}}$ is central, therefore $\theta \circ \widetilde{p}_{h'} = \widetilde{p}_{h'} \circ \theta$. Thus, $(\theta \circ \widetilde{p}_{h'})(1) = \theta(h') = 0$. Since θ is a linear monomorphism, $\theta(h') = \theta(0)$ implies that $h' = 0$. Therefore, $h = 1$ and $\theta(1) \in E(L)$.

(2) Let $\theta, \psi \in End(L)$ and $\theta \circ \psi(x) = x$, for each $x \in L$. Then

$$\psi \circ \theta \circ \psi \circ \theta(x) = \psi((\theta \circ \psi)(\theta(x))) = \psi(\theta(x)) = \psi \circ \theta(x).$$

This proves that $\psi\theta$ is an idempotent linear morphism of L. By Theorem 3.1, $\psi\theta$ is central in $End(L)$. Therefore, $\theta \circ (\psi \circ \theta) = (\psi \circ \theta) \circ \theta$. Thus, we have

$$\begin{aligned}\psi \circ \theta(x) =&(\psi \circ \theta)(\theta \circ \psi(x)) = ((\psi \circ \theta) \circ \theta)(\psi(x))\\ =&(\theta \circ (\psi \circ \theta)(\psi(x)) = (\theta \circ \psi)(\theta \circ \psi(x))\\ =&\theta \circ \psi(x) = x.\end{aligned}$$

Therefore, $\psi\theta = 1_{End(L)}$. □

Lemma 3.2. *Let L be a lattice and $1 = c\dot{\vee}d$, for some $c, d \in L$. Then there is not any non-zero linear morphism between $c/0$ and $d/0$ if and only if $c \in FI(L)$.*

Proof. Assume that $c \in FI(L)$. Let $\theta : c/0 \to d/0$ be a linear morphism and λ the composition

$$L \xrightarrow{\widetilde{p}_c} c/0 \xrightarrow{\theta} d/0 \xrightarrow{i} L\,,$$

where $\widetilde{p}_c : L \to c/0$ is the canonical projection $\widetilde{p}_{d,c}$ on $c/0$ and $i : d/0 \to L$ is the mapping of canonical inclusion. Thus, $\lambda \in End(L)$ as a composition of linear morphisms of lattices. Since $c \in FI(L)$, $h(c) \leq c$. It is clear that $\lambda(c) \leq d$. Hence, $\lambda(c) \leq c \wedge d = 0$ and so $\lambda(c) = 0$. This proves $\theta(c) = 0$, and so $\theta = 0$, as desired.

Conversely, assume that there is not any non-zero linear morphism between $c/0$ and $d/0$, for each $i \neq j \in I$. Let $\theta \in End(L)$ and λ be the composition

$$c/0 \xrightarrow{\theta|_{c/0}} L \xrightarrow{p_d} d/0\,,$$

where $\theta|_{c/0}$ is the restriction of θ to $c/0$. Then, by our assumption, $\lambda = 0$. Hence, $p_d(\theta(c)) = 0$. Therefore, $\theta(c) \leq \ker(p_d) = c$, and so c is fully invariant. □

Corollary 3.3. *Let L be a strongly extending lattice and $1 = c\dot{\vee}e$, for some $c, e \in L$. Then there is not any non-zero linear morphism between $c/0$ and $e/0$.*

Proof. It is clear from Theorem 3.1 and Lemma 3.2. □

In the sequel, we show that the strongly extending property of a lattice is preserved by complement intervals and also consider when direct joins have this property.

Proposition 3.3. *Let L be a strongly extending lattice. If $l \in D(L)$, then $l/0$ is strongly extending.*

Proof. Assume that L is strongly extending, $l \in D(L)$ and $x \in l/0$. Then $x \in E(p/0)$, for some $p \in D(L) \cap FI(L)$. As $l, p \in D(L)$, $p \vee q = 1$ and $p \wedge q = 0$, also $l \vee m = 1$ and $l \wedge m = 0$, for some $m, q \in L$. Since $x \in E(p/0)$, $x \in E((p \wedge l)/0)$. We are now going to prove that $p \wedge l \in FI(l/0) \cap D(l/0)$. As $l \vee m = 1$ and $p \in FI(L)$, $p = (p \wedge l) \vee (p \wedge m)$, by [9, Lemma 1.8 (4)]. Therefore, $(p \wedge l) \vee (p \wedge m) \vee q = 1$. By modularity, we have

$$l = l \wedge 1 =d \wedge ((p \wedge l) \vee (p \wedge m) \vee q) = (p \wedge l) \vee (l \wedge ((p \wedge m) \vee q)).$$

Also, $(p \wedge l) \vee (l \wedge ((p \wedge m) \vee q)) \leq l$ and

$$\begin{aligned}(p \wedge l) \wedge (l \wedge ((p \wedge m) \vee q)) =&(p \wedge l) \wedge ((p \wedge m) \vee q)\\ \leq& p \wedge ((p \wedge m) \vee q)\\ =&(p \wedge m) \vee (p \wedge q) = p \wedge m\\ \leq& m.\end{aligned}$$

Therefore,

$$(p \wedge l) \wedge (l \wedge ((p \wedge m) \vee q)) \leq l \wedge m = 0.$$

Hence, we have $p \wedge l \in D(l/0)$. Moreover, $p \wedge l \in FI(l/0)$, by [9, Lemma 1.8 (3)]. This proves that $l/0$ is strongly extending. □

Proposition 3.4. *Let L be a strongly pseudo-complemented lattice and $1 = p\dot{\vee}q$, for some $p, q \in L$. Then the following statements are equivalent:*

(1) *L is strongly extending;*

(2) *each closed element t of L with $t \wedge q = 0$ or $t \wedge p = 0$ is a fully invariant complement.*

Proof. (1) ⇒ (2) It is clear by Theorem 3.1.

(2) ⇒ (1) We will show that, if $t \in C(L)$, then $t \in D(L) \cap FI(L)$. Put $c := t \wedge p$. Then there exists $e \in C(t/0)$ such that $c \in E(e/0)$, because $t/0$ is essentially closed by [8, Lemma 1.6, Lemma 1.14]. As $e \in C(t/0)$ and $t \in C(L)$, we have $e \in C(L)$, by [8, Lemma 1.6, Lemma 1.11]. Since $c \wedge q = 0$ and $c \in E(e/0)$, $e \wedge q = 0$. By (2), $e \in D(L) \cap FI(L)$. Hence,

$$e \vee e' = 1 \quad \text{and} \quad e \wedge e' = 0,$$

for some $e' \in L$. By modularity and $e \leq t$, we have $t = e \vee (e' \wedge t)$. By the previous argument, $e' \wedge t \in C(L)$. Since $c \in E(e/0)$ and $c \wedge e' = 0$, we have $(t \wedge e') \wedge p = 0$. By (2), $t \wedge e' \in D(L) \cap FI(L)$. Hence,

$$1 = (t \wedge e') \vee d \quad \text{and} \quad (t \wedge e') \wedge d = 0,$$

for some $d \in L$. Now, by modularity we have $e' = (t \wedge e') \vee (d \wedge e')$. Therefore,

$$1 = e \vee e' = e \vee (t \wedge e') \vee (d \wedge e') = t \vee (d \wedge e').$$

Moreover,

$$t \wedge (d \wedge e') = (t \wedge e') \wedge d = 0.$$

Thus, $t \in D(L)$. So L is extending by [8, Proposition 1.10 (4)]. Since $e \in FI(L)$ and $e' \wedge t \in FI(L)$, we have $t \in FI(L)$, by [9, Lemma 1.8 (1)]. Therefore, L is strongly extending by Theorem 3.1. □

Theorem 3.3. *Let L be a strongly pseudo-complemented lattice and $1 = m\dot{\vee}n$, for some $m, n \in L$. Then L is strongly extending provided that the following statements hold.*

(1) *$m/0$ and $n/0$ are strongly extending.*

(2) *For each sublattices H_1 of $m/0$, there is not a non-zero linear morphisms from H_1 to $n/0$.*

(3) *For each sublattice H_2 of $n/0$, there is not a non-zero linear morphisms from H_2 to $m/0$.*

Proof. Assume that k is a closed element of L with $k \wedge m = 0$. Let $\widetilde{p}_m : L \to m/0$ and $\widetilde{p}_n : L \to n/0$ be the canonical projections $\widetilde{p}_{n,m}$ and $\widetilde{p}_{m,n}$, respectively. We consider $\widetilde{p}_n|_{k/0} : k/0 \to n/0$, the restriction of $\widetilde{p}_n$ to $k/0$. Let $x = \ker(\widetilde{p}_n|_{k/0})$. Then $x \leq m = \ker(\widetilde{p}_n)$. Therefore, $x = 0$. Thus, $\widetilde{p}_n|_{k/0} : k/0 \to \widetilde{p}_m(k/0)$ is a linear monomorphism by [5, Corollary 1.6]. Therefore, $\widetilde{p}_n|_{k/0} : k/0 \to \widetilde{p}_n(k/0)$ is a lattice isomorphism (by definition of linear monomorphism). Let $\psi : \widetilde{p}_n(k/0) \to k/0$ be the inverse of $\widetilde{p}_n|_{k/0}$. Then we denote by θ the composition

$$\widetilde{p}_n|_{k/0}(k/0) \xrightarrow{\psi} k/0 \xrightarrow{\widetilde{p}_m|_{k/0}} m/0\,.$$

Since $\widetilde{p}_n|_{k/0}(k/0) \subseteq n/0$, we have $\theta = 0$, by our assumption. Therefore,

$$\widetilde{p}_m(\psi(\widetilde{p}_n|_{k/0}(k/0))) = \widetilde{p}_m(k/0) = 0.$$

Hence, $k \leq \ker(\widetilde{p}_m) = n$. Since $k \in C(L)$, $k \in C(n/0)$. Thus by strongly extending property of $n/0$, $k \in FI(n/0) \cap D(n/0)$. By [8, Proposition 1.7 (3)], $k \in D(L)$. By Lemma 3.2, $n \in FI(L)$, therefore $k \in FI(L)$, by [9, Lemma 1.8 (2)]. Hence, by Proposition 3.4, L is strongly extending. □

4. Applications to Grothendieck Categories

This section is allocated to employ the main results in Section 3 to Grothendieck categories. First, we recall some notations and terminology from [1–11]. In this section $\mathcal{G}$ will indicate a Grothendieck category. Let H be an object of $\mathcal{G}$. We will denote by $L(H)$, the upper continuous modular lattice of all subobjects of H ([11], [21, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]). According to [2], for any object H of $\mathcal{G}$, and for each subset $W \subseteq L(H)$, we denote

$$\bigwedge W = \bigcap_{E \in W} E, \quad \bigvee W = \sum_{E \in W} E.$$

We recall the next definition from [2], which is the key definition of this section.

Definition 4.1 ([2])**.** If $\mathbb{P}$ is a condition on lattices, then it is called $H \in \mathcal{G}$ is $\mathbb{P}$, provided that the lattice $L(H)$ satisfies $\mathbb{P}$. Further, a subobject H' of an object $H \in \mathcal{G}$ is $\mathbb{P}$ if the element H' of the lattice $L(H)$ satisfies $\mathbb{P}$.

Now, by Definition 4.1, one can define the concepts of a strongly extending object and fully invariant subobject, etc. Notice that we will use the term direct summand subobject instead of complement subobject.

By [6, Lemma 5.1], it is known that if $H_1, H_2 \in \mathcal{G}$ and $\theta : H_1 \to H_2$ is a morphism, then the canonical mapping $\varphi : L(H_1) \to L(H_2)$ defined by $\varphi(K) := \theta(K)$, for each $K \leq H_1$, is a linear morphism of lattices. Notice that, the notions of linear morphism and morphism are different. For any two objects H_1 and H_2, we denote by $LHom(H_1, H_2)$, the set of all linear morphisms $\psi : L(H_1) \to L(H_2)$.

In the following, we give some results.

Theorem 4.1. *If H is an object of a Grothendieck category $\mathcal{G}$, then H is strongly extending if and only if H is extending and every direct summand of H is fully invariant in H.*

Proposition 4.1. *Let $H = H_1 \oplus H_2$, where $H \in \mathcal{G}$ and H_1, H_2 are subobject of H. If H is strongly extending, then $Hom(H_1, H_2) = 0$ and $Hom(H_2, H_1) = 0$.*

Proof. Assume that $H = H_1 \oplus H_2$ and X is strongly extending. If $\theta : H_1 \to H_2$ is a morphism, then the map $\psi : L(H_1) \to L(H_2)$ defined by $\psi(A) := \theta(A)$, for each $A \leq H_1$, is a linear morphism ([6, Lemma 5.1]). By Corollary 3.3, $\psi = 0$, therefore $\theta = 0$. □

Theorem 4.2. *Assume that H is an object of a Grothendieck category $\mathcal{G}$ and H is strongly extending. Then the intersection of any family of direct summands of H is a direct summand of H.*

Theorem 4.3. *Let $H = H_1 \oplus H_2$, where $H \in \mathcal{G}$ and H_1, H_2 are subobject of H. If H_1 and H_2 are strongly extending and for each subobject K_1 of H_1 and K_2 of H_2, $LHom(K_1, H_2) = 0$ and $LHom(K_2, H_1) = 0$, then H is strongly extending.*

5. Applications to Modules with a Hereditary Torsion Theory

In this section, some applications of the results proved in Sections 3 to the category of modules with a fixed hereditary torsion class are given. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory in $Mod-R$, and $\tau(M)$ the τ-torsion submodule of a module M. We recall some notations and terminology from [1–11]. For an R-module M, by $Sat_\tau(M)$, we will denote the set $\{K \mid K \leq M \text{ and } M/K \in \mathcal{F}\}$. Let $K \leq M$. Then by $\overline{K}$, we will denote the τ-saturation of K (in M) defined by $\overline{K}/K = \tau(M/K)$. Let K be submodule of M. Then K is said to be τ-saturated if $\overline{K} = K$. One can prove that $Sat_{\tau(M)} = \{K \mid K \leq M,\ \overline{K} = K\}$. By [21, Chapter 9, Proposition 4.1], it is known that for a right R-module M, $(Sat_\tau(M), \subseteq, \bigwedge, \bigvee, \tau(M), M)$ is an upper continuous modular lattice (the greatest element is M and the least element is $\tau(M)$) and $\bigvee$ and $\bigwedge$ defined as follows:

$$\bigvee_{i\in J} K_i = \overline{\sum_{i\in J} K_i} \quad \text{and} \quad \bigwedge_{i\in J} K_i = \bigcap_{i\in J} K_i.$$

We refer to [21] the reader for the discussion of torsion theoretical concepts and facts.

We recall the next definition from [2], which is the key definition of this section.

Definition 5.1 ([2])**.** Let $\mathbb{C}$ be a condition on lattices. Then it is called a right R-module M is $\tau - \mathbb{C}$ provided that the lattice $Sat_\tau(M)$ satisfies the condition $\mathbb{C}$.

Moreover, it is called a submodule K of a right R-module M is $\tau - \mathbb{C}$, provided that its τ-saturation $\overline{K}$, which is an element of $Sat_\tau(M)$, satisfies the condition $\mathbb{C}$.

Therefore, we can define the notions of a τ-strongly extending module, τ-Abelian module, etc, based on the Definition 5.1. By [2], we have the concepts of a τ-essential submodule of a module, τ-fully invariant submodules, etc. As $\overline{\overline{K}} = K$, we have K is $\tau - \mathbb{P}$ if and only if $\overline{K}$ is $\tau - \mathbb{P}$. It is known that K is τ-essential in M if and only if $H \cap K \in \mathfrak{T}$ implies that $H \in \mathfrak{T}$, for each $H \leq M$, by [2, Proposition 5.3], moreover, K is a τ-direct summand in M if and only if $M/(K + H) \in \mathfrak{T}$ and $K \cap H \in \mathfrak{T}$, for some $H \leq M$. In [6, Lemma 6.6], it is proved that, if $f : M \to N$ is a morphism of right R-modules, then the canonical mapping $f_\tau : Sat_\tau(M) \to Sat_\tau(N)$ defined by $f_\tau(X) = \overline{f(X)}$, for each $X \in Sat_\tau(M)$ is a linear morphism of lattices.

In the following, we give some results on the strongly τ-extending modules.

Theorem 5.1. *An R-module M is τ-strongly extending if and only if M is τ-CS (τ-extending) and every τ-direct summand of M is τ-fully invariant.*

Proof. Assume that M is τ-strongly extending. It suffices to prove that every τ-direct summand of M is τ-fully invariant. Let N be a τ-direct summand of M. Since M is τ-strongly extending, $Sat_\tau(M)$ is a strongly extending lattice. Hence, $\overline{N}$ is τ-essential in L, where L is fully invariant in lattice $Sat_\tau(M)$. As $\overline{N}$ is closed in $Sat_\tau(M)$, $\overline{N} = L$. Hence, N is τ-fully invariant in M. The converse is clear. □

Proposition 5.1. *Each τ-direct summand a τ-strongly extending module is τ-strongly extending.*

Theorem 5.2. *Suppose that M is a τ-strongly extending R-module and $H_1, H_2 \leq M$ ($H_1, H_2 \notin \mathfrak{T}$) such that $H_1 \cap H_2 \in \mathfrak{T}$, $M = H_1 + H_2$. If $f : H_i \to H_j$ is an R-homomorphism ($1 \leq i \neq j \leq 2$), then $f(H_i) \in \mathfrak{T}$.*

Proof. Since $M = H_1 + H_2$, we have

$$M = H_1 + H_2 \subseteq \overline{H_1} + \overline{H_2} \subseteq \overline{\overline{H_1} + \overline{H_2}}.$$

Therefore, $M = \overline{\overline{H_1} + \overline{H_2}}$. As $H_1 \cap H_2 \in \mathfrak{T}$, we have $\overline{H_1} \wedge \overline{H_2} = \overline{H_1} \cap \overline{H_2} = \tau(M)$. Therefore, $M = \overline{H_1} \dot{\vee} \overline{H_2}$. Let $f : H_1 \to H_2$ be a homomorphism of R-modules H_1 and H_2. Then the canonical mapping $f_\tau : Sat_\tau(H_1) \to Sat_\tau(H_2)$ defined by $f_\tau(X) = \overline{f(X)}$, for each $X \in Sat_\tau(H_1)$ is a linear morphism of lattices. By [3,4], there exist lattice isomorphisms $h : Sat_\tau(H_1) \to Sat_\tau(\overline{H_1})$ and $g : Sat_\tau(H_2) \to Sat_\tau(\overline{H_2})$. By [5, Proposition 2.2(2)], h, g are linear morphisms. Take $\varphi := g \circ f_\tau \circ h^{-1}$. By Corollary 3.3, $\varphi = 0$, thus $\overline{f(H_1)} = 0$, in $Sat_\tau(M)$. Thus, $f(H_1) \in \mathfrak{T}$. Similarly, if $f : H_2 \to H_1$ is a homomorphism between two R-modules H_2 and H_1, then we have $f(H_2) \in \mathfrak{T}$. □

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EXISTENCE RESULTS OF IMPULSIVE HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL AND BOUNDARY HYBRID CONDITIONS

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Abstract. In this paper, we establish sufficient conditions for the existence and uniqueness of solution of impulsive hybrid fractional differential equations with initial and boundary hybrid conditions. The proof of the main result is based on the classical fixed point theorems such as Banach fixed point theorem and Leray-Schauder alternative fixed point theorem. Two examples are included to show the applicability of our results.

1. Introduction

Fractional calculus refers to integration or differentiation of any order. The field has a history as old as calculus itself, which did not attract enough attention for a long time. In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences. For a nice introduction, we refer the reader to [9, 10] and references cited therein.

Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process [8]. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The governing equations of such phenomena may be modeled as impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations as these

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equations provide a natural frame work for mathematical modelling of many real world phenomena, namely in the control theory, physics, chemistry, population dynamics, biotechnology, economics and medical fields.

In [11], Surang Sitho, Sotiris K. Ntouyas and Jessada Tariboon, discussed the existence results for the following hybrid fractional integro-differential equation:

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J = [0,T], \\ x(0) = 0, \end{cases}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha \leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$, with $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ and $h_i(0,0) = 0$, $i = 1, 2, \ldots, m$.

In [4], K. Hilal and A. Kajouni, considered boundary value problems for hybrid differential equations with fractional order (BVPHDEF of short) involving Caputo differential operator of order $0 < \alpha < 1$:

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in C(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

Dhage and Lakshmikantham [2], discussed the following first order hybrid differential equation:

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J = [0,T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and comparison results.

Zhao, Sun, Han and Li [13], are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operator:

$$\begin{cases} D^{q}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J = [0,T], \\ x(0) = 0, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.

Benchohra et al. [1] discussed the following boundary value problems for differential equations with fractional order:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0, T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a, b, c are real constants with $a + b \neq 0$.

Motivated by some recent studies related to the boundary value problem of a class of impulsive hybrid fractional differential equations and by the nice works [12, 14], we consider the following Cauchy problem of hybrid fractional differential equations:

$$(1.1) \qquad \begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) = g(t, u(t)), \quad t \in [0, 1], t \neq t_i, i = 1, 2, \ldots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), \quad t_i \in (0, 1), i = 1, 2, \ldots, n, \\ \frac{u(0)}{f(0,u(0))} = \phi(u), \end{cases}$$

D^α stands for Caputo fractional derivative of order α, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\phi : C([0, 1], \mathbb{R}) \to \mathbb{R}$ are continuous functions such that $\phi(u) = \sum_{i=1}^n \lambda_i u(\xi_i)$, where $\xi_i \in (0, 1)$ for $i = 1, 2, \ldots, n$, and $I_k : \mathbb{R} \to \mathbb{R}$ with $u(t_k^+) = \lim_{\epsilon\to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\epsilon\to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = i$.

In the sequel of this work, we assume that $\sum_{i=1}^n \lambda_i u(\xi_i)^{\alpha-1} < 1$.

This paper is arranged as follows. In Section 2, we recall some tools related to the fractional calculus as well as some needed results. In Section 3, we present the main results. Section 4 is devoted to examples of application of the main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Throughout this paper, let $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{n-1} = (t_{n-1}, t_n], J_n = (t_n, 1]$, $n \in \mathbb{N}$, $n > 1$.

For $t_i \in (0, 1)$ such that $t_1 < t_2 < \cdots < t_n$ we define the following spaces:

$$\begin{aligned} I' =& I \setminus \{t_1, t_2, \ldots, t_n\}, \\ X =& \{u \in C([0, 1], \mathbb{R}) : u \in C(I') \text{ and left } u(t_i^+) \text{ and right limit } u(t_i^-) \\ & \quad \text{exist and } u(t_i^-) = u(t_i), 1 \leq i \leq n\}. \end{aligned}$$

Then, clearly $(X, \|\cdot\|)$ is a Banach space under the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Definition 2.1 ([6]). The fractional integral of the function $h \in L^1([a, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function.

Definition 2.2 ([6]). For a function h defined on the interval $[a, b]$, the Riemann-Liouville fractional-order derivative of h, is defined by

$$({}^cD^\alpha_{a^+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}h(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([6]). For a function h defined on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD^\alpha_{a^+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.1 ([10]). *Let $\alpha, \beta \geq 0$, then the following relations hold:*

1. $I^\alpha t^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}$;
2. ${}^cD^\alpha t^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}$.

Lemma 2.2 ([10]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^\alpha\, {}^cD^\alpha f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}.$$

3. Main Results

In this section, we prove the existence of a solution for Cauchy problem (1.1).

To do so, we will need the following assumptions.

(H_1) The function $u \mapsto \frac{u}{f(t,u)}$ is increasing in $\mathbb{R}$ for every $t \in [0, 1]$.

(H_2) The function f is continuous and bounded, that is, there exists a positive number $L > 0$ such that $|f(t, u)| \leq L$ for all $(t, u) \in [0, 1] \times \mathbb{R}$.

(H_3) There exists a positive number $M_g > 0$, such that

$$|g(t, u) - g(t, \bar{u})| \leq M_g|u - \bar{u}|, \quad \text{for all } u, \bar{u} \in \mathbb{R} \text{ and } t \in [0, 1].$$

(H_4) There exists a constant $A > 0$, such that

$$|I_i(u) - I_i(\bar{u})| \leq A|u - \bar{u}|, \quad i = 1, 2, \ldots, n, \text{ for all } u, \bar{u}, \in \mathbb{R}.$$

(H_5) There exists a constant $K_\phi > 0$, such that

$$|\phi(u) - \phi(v)| \leq K_\phi\|u - v\|, \quad \text{for all } u, v \in C([0, 1], \mathbb{R}).$$

(H_6) There exist constants $M_\phi > 0$ and $N_I > 0$, such that

$$|\phi(u)| \leq M_\phi\|u\|, \quad |I_i(v)| \leq N_I|v|, \quad i = 1, 2, \ldots, n,$$

for all $u \in C([0, 1], \mathbb{R})$ and $v \in \mathbb{R}$.

(H_7) There exists a constant $C > 0$, such that

$$|I_i(u)| \leq C, \quad i = 1, 2, \ldots, n, \text{ for all } u \in \mathbb{R}.$$

(H_8) There exists a constant $\rho > 0$, such that

$$|\phi(u)| \leq \rho, \quad \text{for all } u \in X.$$

(H_9) There exist constants $\rho_0, \rho_1 > 0$, such that

$$|g(t, u(t))| \leq \rho_0 + \rho_1 \|u\|, \quad \text{for all } u \in X \text{ and } t \in [0, 1].$$

For brevity, let us set

$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\right). \tag{3.1}$$

Lemma 3.1. *Let $\alpha \in (0,1)$ and $h : [0,T] \to \mathbb{R}$ be continuous. A function $u \in C([0,T], \mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = u_0 - \int_0^a \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds$$

if and only if u is a solution of the following fractional Cauchy problem:

$$\begin{cases} D^\alpha u(t) = h(t), & t \in [0, T], \\ u(a) = u_0, & a > 0. \end{cases}$$

Lemma 3.2. *Assume that hypotheses (H_1) and (H_2) hold. Let $\alpha \in (0,1)$ and $h : [0,1] \to \mathbb{R}$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t) = f(t, u(t)) \left[\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds\right], \quad t \in [t_i, t_{i+1}], \tag{3.2}$$

where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \notin [t_0, t_1[, \end{cases}$$

if and only if u is a solution of the following impulsive problem:

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) = h(t), & t \in [0,1], t \neq t_i, i = 1, 2, \ldots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), & t_i \in (0,1), i = 1, 2, \ldots, n, \\ \frac{u(0)}{f(0,u(0))} = \phi(u). \end{cases} \tag{3.3}$$

Proof. Assume that u satisfies (3.3). If $t \in [t_0, t_1[$, then

$$D^\alpha\left(\frac{u(t)}{f(t, u(t))}\right) = h(t), \quad t \in [t_0, t_1[, \tag{3.4}$$

$$\frac{u(0)}{f(0, u(0))} = \phi(u). \tag{3.5}$$

Applying I^{α} on both sides of (3.4), we obtain

$$\frac{u(t)}{f(t,u(t))} = \frac{u(0)}{f(0,u(0))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds = \phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

Then we get

$$u(t) = f(t,u(t))\left(\phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

If $t \in [t_1, t_2[$, then

$$D^{\alpha}\Big(\frac{u(t)}{f(t,u(t))}\Big) = h(t), \quad t \in [t_1, t_2[, \tag{3.6}$$

$$u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)). \tag{3.7}$$

According to Lemma 3.1 and the continuity of $t \mapsto f(t,u(t))$, we have

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} &= \frac{u(t_1^+)}{f(t_1,u(t_1))} - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &= \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f(t_1,u(t_1))} - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.\end{aligned}$$

Since

$$u(t_1^-) = f(t_1,u(t_1))\left(\phi(u) + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right),$$

then we get

$$\begin{aligned}\frac{u(t)}{f(t,u(t)))} &= \left(\phi(u) + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &= \phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.\end{aligned}$$

So, one has

$$u(t) = f(t,u(t))\left(\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

For $t \in [t_2, t_3[$, we have

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} &= \frac{u(t_2^+)}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &= \frac{(u(t_2^-) + I_2(u(t_2^-)))}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\end{aligned}$$

and

$$u(t_2^-) = f(t_2,u(t_2))\left(\phi(u) + \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f(t_1,u(t_1))} + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

Therefore, we obtain

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} =&\phi(u) + \frac{(u(t_1^-)+I_1(u(t_1^-)))}{f(t_1,u(t_1))} + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &+ \frac{I_2(u(t_2^-))}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ =&\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \frac{I_2(u(t_2^-))}{f(t_2,u(t_2))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.\end{aligned}$$

Consequently, we get

$$u(t) = f(t,u(t))\left(\phi(u) + \sum_{i=1}^{2} \frac{I_i(u(t_i^-))}{f(t_i,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

By using the same method, for $t \in [t_i, t_{i+1}[$, $i = 3, 4, \dots, n$, one has

$$u(t) = f(t,u(t))\left(\phi(u) + \sum_{i=1}^{k} \frac{I_i(u(t_i^-))}{f(t_i,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

Conversely, assume that u satisfies (3.2). Then for $t \in [t_0, t_1[$, we have

$$u(t) = f(t,u(t))\left(\phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right). \tag{3.8}$$

Then, dividing by $f(t,u(t))$ and applying D^α on both sides of (3.8), we get equation (3.4).

Again, substituting $t = 0$ in (3.8), we obtain $\frac{u(0)}{f(0,u(0))} = \phi(u)$. Since $u \mapsto \frac{u}{f(t,u)}$ is increasing in $\mathbb{R}$ for $t \in [t_0, t_1[$, the map $u \mapsto \frac{u}{f(t,u)}$ is injective in $\mathbb{R}$. Then we get (3.5).

If $t \in [t_1, t_2[$, then we have

$$u(t) = f(t,u(t))\left(\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right). \tag{3.9}$$

Then, dividing by $f(t,u(t))$ and applying D^α on both sides of (3.9), we get equation (3.6). Again by (H_3), substituting $t = t_1$ in (3.8) and taking the limit in (3.9), then (3.9) minus (3.8) gives (3.7).

Similarly, for $t \in [t_i, t_{i+1}[$, $i = 2, 3, \dots, n$, we get

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) = h(t), & t \in [t_k, t_{k+1}[, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)). \end{cases} \tag{3.10}$$

This completes the proof. □

Lemma 3.3. *Let g be continuous, then $u \in X$ is a solution of Cauchy problem* (1.1) *if and only if u is a solution of the integral equation*

(3.11)

$$u(t) = f(t,u(t))\left(\phi(u) + \theta(t)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(t,u(t))ds\right), \quad t \in \big[t_i, t_{i+1}\big],$$

where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \notin [t_0, t_1[. \end{cases}$$

Now we are in a position to present our first result which deals with the existence and uniqueness of solution for Cauchy problem (1.1). This result is based on Banach's fixed point theorem. To do so, we define the operator $\Psi : X \to X$ by

$$\Psi(u)(t) = f(t, u(t)) \left(\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u(s)) ds \right). \tag{3.12}$$

Theorem 3.1. *Assume that conditions* (H_1)-(H_6) *hold and the function* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous. Then Cauchy problem* (1.1) *has an unique solution provided that* $\Delta < 1$, *where* Δ *is the constant given in equation* (3.1).

Proof. Let us set $\sup_{t \in [0,1]} g(t, 0) = \kappa < \infty$, and define a closed ball $\bar{B}$ as follows

$$\bar{B} = \{u \in X : \|u\| \leq r\},$$

where

$$r \geq \frac{L\kappa}{1 - L\left(M_\phi + nN_I + \frac{1}{\Gamma(\alpha+1)} M_g\right)}. \tag{3.13}$$

We show that $\Psi(\bar{B}) \subset \bar{B}$. For $u \in \bar{B}$, we obtain

$$\begin{aligned}
|\Psi(u)(t)| &\leq L \left| \phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u(s)) ds \right| \\
&\leq L \left[M_\phi \|u\| + nN_I \|u\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|g(s, u(s)) - g(s, 0)| + |g(s, 0)|) ds \right] \\
&\leq L \left[M_\phi \|u\| + nN_I \|u\| + \frac{1}{\Gamma(\alpha+1)} \left(M_g \|u\| + \kappa \right) \right] \\
&\leq L \left[(M_\phi + nN_I) r + \frac{1}{\Gamma(\alpha+1)} (M_g r + \kappa) \right].
\end{aligned}$$

Hence, we get

$$\|\Psi(u)\| \leq L \left((M_\phi + nN_I) r + \frac{1}{\Gamma(\alpha+1)} \left(M_g r + \kappa_1 \right) \right). \tag{3.14}$$

From (3.14), it follows that $\|\Psi(u)\| \leq r$.

Next, for $(u, \bar{u}) \in X^2$ and for any $t \in [0,1]$, we have

$$\begin{aligned}|\Psi(u)(t) - \Psi(\bar{u})(t)| =& \Big| f(t,u(t)) \Big[\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds\Big] \\ & - f(t,\bar{u}(t)) \Big[\phi(\bar{u}) + \theta(t) \sum_{i=1}^{n} \frac{I_i(\bar{u}(t_i^-))}{f(t,\bar{u}(t_i))} \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,\bar{u}(s))ds\Big]\Big| \\ \leq & L \left(K_\phi |u-\bar{u}| + nA|u-\bar{u}| + \frac{M_g}{\Gamma(\alpha+1)} |u-\bar{u}| \right),\end{aligned}$$

which implies that

$$\|\Psi(u) - \Psi(\bar{u})\| \leq L\Big(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\Big)\|u-\bar{u}\| = \Delta\|u-\bar{u}\|. \tag{3.15}$$

In view of condition $\Delta < 1$, it follows that Ψ is a contraction operator. So Banach's fixed point theorem applies and hence the operator Ψ has an unique fixed point, which is an unique solution of Cauchy problem (1.1). This completes the proof. □

In our second result, we discuss the existence of solutions for Cauchy problem (1.1) by means of Leray-Schauder alternative.

For brevity, let us set

$$\mu_1 = \frac{L}{\Gamma(\alpha+1)}, \tag{3.16}$$

$$\mu_0 = 1 - \mu_1 \rho_1. \tag{3.17}$$

Lemma 3.4 (Leray-Schauder alternative see [3])**.** *Let $\mathfrak{F} : G \to G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $P(\mathfrak{F}) = \{u \in G : u = \lambda \mathfrak{F} u$ for some $0 < \lambda < 1\}$. Then either the set $P(\mathfrak{F})$ is unbounded or $\mathfrak{F}$ has at least one fixed point.*

Theorem 3.2. *Assume that conditions (H_1)-(H_3) and (H_7)-(H_9) hold. Furthermore, it is assumed that $\mu_1 \rho_1 < 1$, where μ_1 is given by (3.16). Then Cauchy problem (1.1) has at least one solution.*

Proof. We will show that the operator $\Psi : X \to X$ satisfies all the assumptions of Lemma 3.4.

Step 1. We prove that the operator Ψ is completely continuous.

Clearly, it follows from the continuity of functions f and g that the operator Ψ is continuous.

Let $S \subset X$ be bounded. Then we can find a positive constant H such that $|g(t,u(t))| \leq H$, $u \in S$. Thus, for any $u \in S$, we can get

$$\begin{aligned}|\Psi(u)(t)| &\leq L\left(\rho + \sum_{i=1}^{n} C + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} H ds\right)\\ &\leq L\left(\rho + nC + \frac{H}{\Gamma(\alpha+1)}\right),\end{aligned}$$

which yields

$$\|\Psi(u)\| \leq L\left(\rho + nC + \frac{H}{\Gamma(\alpha+1)}\right). \tag{3.18}$$

From the inequality (3.18), we deduce that the operator Ψ is uniformly bounded.

Step 2. Now we show that the operator Ψ is equicontinuous.

For $\tau_1, \tau_2 \in [0,1]$ with $\tau_1 < \tau_2$, we obtain

$$\begin{aligned}&|\Psi(u(\tau_2)) - \Psi(u(\tau_1))|\\ \leq &L\Bigg|\left(\phi(u) + \theta(\tau_2)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + H\int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds\right)\\ &- \left(\phi(u) + \theta(\tau_1)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + H\int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds\right)\Bigg|\\ \leq &L\left(\left|(\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))}\right| + H\left|\int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds - \int_0^{\tau_1} \frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds\right|\right)\\ \leq &L\Bigg(\left|(\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))}\right| + H\Bigg|\int_0^{\tau_1} \frac{(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)} ds\\ &+ \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} ds\Bigg|\Bigg),\end{aligned}$$

which tends to 0 independently of u. This implies that the operator $\Psi(u)$ is equicontinuous. Thus, by the above findings, the operator $\Psi(u)$ is completely continuous.

In the next step, it will be established that the set $P = \{u \in X : u = \lambda\Psi(u), 0 < \lambda < 1\}$ is bounded.

For $u \in P$, we have $u = \lambda\Psi(u)$. Thus, for any $t \in [0,1]$, we can write $u(t) = \lambda\Psi(u)(t)$. Then we obtain

$$\begin{aligned}\|u\| &\leq L\left(\rho + nC + \frac{1}{\Gamma(\alpha+1)}(\rho_0 + \rho_1\|u\|)\right)\\ &\leq L(\rho + nC) + \mu_1(\rho_0 + \rho_1\|u\|).\end{aligned}$$

Hence, we get

$$\|u\| \leq \frac{L(\rho + nC) + \mu_1\rho_0}{\mu_0}.$$

This shows that the set P is bounded. In consequence, all the conditions of Lemma 3.4 are satisfied. Finally, the operator Ψ has at least one fixed point, which is a solution of Cauchy problem (1.1). This completes the proof. □

4. Examples

Example 4.1. Consider the hybrid fractional differential equation:

$$\begin{cases} {}^cD^{\frac{1}{2}}\left(\frac{u(t)}{\frac{e^{-1}+t+\sqrt{u(t)}}{40+t^2}}\right) = \frac{e^{-t}+|\sin u(t)|}{20}, \quad t\in[0,1]\setminus\{t_1\}, \\ u(t_1^+) = u(t_1^-) + (-2u(t_1^-)), \quad t_1 \neq 0,1, \\ \frac{u(0)}{f(0,u(0))} = \sum_{i=1}^{n} \lambda_i u(t_i). \end{cases} \tag{4.1}$$

Here, we have

$$f(t,u(t)) = \frac{e^{-1}+t+\sqrt{u(t)}}{40+t^2},$$

$$g(t,u(t)) = \frac{e^{-t}+|\sin u(t)|}{20},$$

$$|g(t,u_1)-g(t,u_2)| \leq \frac{1}{40}|u_2-u_1|, \quad t\in[0,1] \text{ and } u_1,u_2\in\mathbb{R},$$

$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\right) \simeq 0.0012345687 < 1.$$

Then all the assumptions of Theorem 3.2 are satisfied, thus our results can be applied to Cauchy problem (4.1).

Example 4.2. Consider another example for hybrid fractional differential equations of the following form

$$\begin{cases} {}^cD^{\frac{1}{2}}\left(\frac{v(t)}{\frac{e^{-1}+t^2+\sqrt{v(t)}}{32+t}}\right) = \frac{e^{-2t}+\cos^2(v(t))}{20}, \quad t\in[0,1]\setminus\{t_1\}, \\ v(t_1^+) = v(t_1^-) + (-2v(t_1^-)), \quad t_1 \neq 0,1, \\ \frac{v(0)}{f(0,v(0))} = \sum_{j=1}^{n} \lambda_j v(t_j)). \end{cases} \tag{4.2}$$

Here, we have

$$f(t,v(t)) = \frac{e^{-1}+t^2+\sqrt{v(t)}}{32+t},$$

$$g(t,v(t)) = \frac{e^{-2t}+\cos^2(v(t))}{20},$$

$$|g(t,v_1) - g(t,v_2)| \leq \frac{1}{20}|v_2 - v_1|, \quad t \in [0,1] \text{ and } v_1, v_2 \in \mathbb{R},$$
$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\right) \simeq 0.3354687 < 1.$$

Then all the assumptions of Theorem 3.2 are satisfied, thus our results can be applied to Cauchy problem (4.2).

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QUANTITATIVE UNCERTAINTY PRINCIPLES FOR THE CANONICAL FOURIER BESSEL TRANSFORM

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Abstract. The aim of this paper is to prove new uncertainty principles for the Canonical Fourier Bessel transform. To do so we prove a quantitative uncertainty inequality about the essential supports of a nonzero function for this transformation.

1. Introduction

The classical linear canonical transform (LCT) is considered as a generalization of the Fourier transform, and was first proposed in the 1970s by Collins [5] and Moshinsky and Quesne [26]. Very recently, many works have been devoted the LCT under many different names and in different contexts. Namely, in [22] the LCT is known as the generalized Fresnel transform, in [4] is called ABCD transform and in [1] is also called the special affine Fourier transform. Also, the LCT has been studied by many authors for various Fourier transforms, for examples [11,23,34]. In [11], the authors introduced the Dunkl linear canonical transform (DLCT) which is a generalization of the LCT in the framework of Dunkl transform [7]. DLCT includes many well-known transforms such as the Dunkl transform [7,10] and the canonical Fourier Bessel transform [8,11]. The LCT plays an important role in many fields of optics, radar system analysis, GRIN medium system analysis, filter design, phase retrieval, pattern recognition and many others [3,28,29]. In [8] the authors established some important properties of the Canonical Fourier Bessel transform (QFBT) such as Riemann-Lebesgue lemma, inversion formula, Plancherel theorem and some uncertainty principles.

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On the other hand, the uncertainty principle plays one important role in signal processing. It describes a function and its Fourier transform, which cannot both be simultaneously sharply localized. If we try to limit the behaviour of one we lose control of the other. Many of these uncertainty principles have already been studied from several points of view for the Fourier transform, such as Heisenberg-Pauli-Weyl inequality [6] and local uncertainty inequality [30]. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics, they tell us that a particle's speed and position cannot both be measured with arbitrary precision. In signal analysis, they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. Timelimited functions and bandlimited functions are basic tools of signal and image processing. Unfortunately, the simplest form of the uncertainty principle tells us that a signal cannot be simultaneously time and bandlimited. This leads to the investigation of the set of almost time and almost bandlimited functions, which has been initially carried through Landau, Pollak [24, 25] and then by Donoho, Stark [9]. In recent past, many works have been devoted to establish some uncertainty principles in different setting and for various transforms (see for example [2, 12–21, 31]) and others.

The purpose of this paper is to obtain uncertainty principle similar to Donoho-Stark's principle for the QFBT.

In order to describe our results, we first need to introduce some facts about harmonic analysis related to Canonical Fourier Bessel transform. For more details, see [8].

Throughout this paper, α denotes a real number such that $\alpha \geqslant -\frac{1}{2}$. We use the following notation.

• $C_{e,0}(\mathbb{R})$ denotes the space of even continuous functions on $\mathbb{R}$ and vanishing at infinity. We provide $C_{e,0}(\mathbb{R})$ with the topology of uniform convergence.

• $L^{p,\alpha}$ denotes the Lebesgue space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{p,\alpha} = \left(\int_0^{+\infty} |f(y)|^p y^{2\alpha+1} dy\right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leqslant p < +\infty,$$
$$\|f\|_{\infty,\alpha} = \operatorname*{ess\,sup}_{y\in\mathbb{R}_+} |f(y)| < +\infty, \quad \text{if } p = +\infty.$$

We provide $L^{p,\alpha}$ with the topology defined by the norm $\|\cdot\|_{p,\alpha}$.

• $L^{2,\alpha}$ denotes the Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle f, g\rangle = \int_0^{+\infty} f(y)\overline{g(y)} y^{2\alpha+1} dy.$$

• $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary matrix in $SL(2,\mathbb{R})$, such that $b \neq 0$.

Definition 1.1. The canonical Fourier Bessel transform of a function $f \in L^{1,\alpha}$ is defined by

$$\mathscr{F}^m_\alpha(f)(x) = \frac{c_\alpha}{(ib)^{\alpha+1}} \int_0^{+\infty} K^m_\alpha(x, y) f(y) y^{2\alpha+1} dy,$$

where

$$c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)} \tag{1.1}$$

and

$$K_\alpha^m(x,y) = e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)} j_\alpha\left(\frac{xy}{b}\right).$$

Here j_α denotes the normalized Bessel function of order $\alpha \geqslant -\frac{1}{2}$ and defined by [33]

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+1+k)}\left(\frac{z}{2}\right)^{2k}, \quad z \in \mathbb{C}.$$

Proposition 1.1 ([8]). *We denote by Δ_α^m the differential operator*

$$\Delta_\alpha^m = \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{d}{b}x\right)\frac{d}{dx} - \left(\frac{d^2}{b^2}x^2 + 2i(\alpha+1)\frac{d}{b}\right).$$

(1) *For each $y \in \mathbb{R}$, the kernel $K_\alpha^m(\cdot,y)$ of the canonical Fourier Bessel transform $\mathscr{F}_\alpha^m$ is the unique solution of*

$$\begin{cases} \Delta_\alpha^m K_\alpha^m(\cdot,y) = \frac{-y^2}{b^2}K_\alpha^m(\cdot,y), \\ K_\alpha^m(0,y) = e^{\frac{iay^2}{2b}}, \\ \frac{d}{dx}K_\alpha^m(0,y) = 0. \end{cases}$$

(2) *For each $x, y \in \mathbb{R}$ the kernel K_α^m has the following integral representation*

$$K_\alpha^m(x,y) = \begin{cases} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(\frac{xyt}{b})dt, & \text{if } \alpha > -\frac{1}{2}, \\ e^{\frac{i}{2}\left(\frac{dx^2}{b}+\frac{ay^2}{b}\right)}\cos(\frac{xy}{b}), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

In particular, we have

$$|K_\alpha^m(x,y)| \leqslant 1 \quad \text{for all } x, y \in \mathbb{R}. \tag{1.2}$$

Theorem 1.1 ([8]). (1) (*Plancherel theorem*) *If $f \in L^{1,\alpha} \cap L^{2,\alpha}$, then $\mathscr{F}_\alpha^m(f) \in L^{2,\alpha}$ and*

$$\|\mathscr{F}_\alpha^m(f)\|_{2,\alpha} = \|f\|_{2,\alpha}. \tag{1.3}$$

(2) (*Orthogonality relation*) *For every $f, g \in L^{2,\alpha}$, we have*

$$\langle f, g\rangle = \langle \mathscr{F}_\alpha^m(f), \mathscr{F}_\alpha^m(g)\rangle. \tag{1.4}$$

(3) (*The reversibility property*) *For all $f \in L^{1,\alpha}$, with $\mathscr{F}_\alpha^m \in L^{1,\alpha}$, we have*

$$(\mathscr{F}_\alpha^m \circ \mathscr{F}_\alpha^{m^{-1}})(f) = (\mathscr{F}_\alpha^{m^{-1}} \circ \mathscr{F}_\alpha^m)(f) = f, \quad a.e. \tag{1.5}$$

Babenko-Beckner inequality. Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2,\mathbb{R})$, such that $b \neq 0$. Let p and q be real numbers such that $1 < p \leqslant 2$

and $\frac{1}{p}+\frac{1}{q}=1$. Then, $\mathscr{F}_\alpha^m$ extends to a bounded linear operator on $L^{p,\alpha}$, $\alpha \geqslant -\frac{1}{2}$ and we have

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant |b|^{(\alpha+1)(\frac{2}{q}-1)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha}, \tag{1.6}$$

where c_α is the constant given by (1.1).

Riemann-Lebesgue lemma. For all $f \in L^{1,\alpha}$, the canonical Fourier Bessel transform $\mathscr{F}_\alpha^m(f)$ belongs to $C_{e,0}(\mathbb{R})$ and verifies

$$\|\mathscr{F}_\alpha^m(f)\|_{\infty,\alpha} \leqslant c_\alpha |b|^{-(\alpha+1)} \|f\|_{1,\alpha}. \tag{1.7}$$

2. Donoho-Stark's Uncertainty Principle for the Canonical Fourier Bessel Transform

In this section, based on the techniques of Donoho-Stark [9], we will show uncertainty principle of concentration-type the canonical Fourier Bessel transform.

In the following, we consider a pair of orthogonal projections on $L^{2,\alpha}$. The first is the time-limiting operator defined

$$P_S f = \chi_S f, \tag{2.1}$$

and the second is the frequency-limiting operator defined by

$$\mathscr{F}_\alpha^m(Q_\Sigma f) = \chi_\Sigma \mathscr{F}_\alpha^m(f), \tag{2.2}$$

where S and Σ are two measurable subsets of $\mathbb{R}_+$ and χ_S and χ_Σ denote the characteristic functions of S and Σ.

Definition 2.1. Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^{2,\alpha}$ be a nonzero function.

(1) We say that f is ε_S-concentrated on S if

$$\|P_{S^c} f\|_{2,\alpha} \leqslant \varepsilon_S \|f\|_{2,\alpha}. \tag{2.3}$$

(2) We say that f is ε_Σ-concentrated on Σ for the canonical Fourier Bessel transform if

$$\|Q_{\Sigma^c} f\|_{2,\alpha} \leqslant \varepsilon_\Sigma \|f\|_{2,\alpha}. \tag{2.4}$$

P_S and Q_Σ are projections. Indeed, let $f, g \in L^{2,\alpha}$. By relation (1.4), we have

$$\begin{aligned}
\langle P_S^2 f, g\rangle = \langle P_S f, P_S g\rangle &= \langle \mathscr{F}_\alpha^m(P_S f), \mathscr{F}_\alpha^m(P_S g)\rangle \\
&= \int_0^{+\infty} \mathscr{F}_\alpha^m(P_S f)(y)\overline{\mathscr{F}_\alpha^m(P_S g)(y)} y^{2\alpha+1} dy \\
&= \int_S \mathscr{F}_\alpha^m(f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)} y^{2\alpha+1} dy \\
&= \int_0^{+\infty} \mathscr{F}_\alpha^m(P_S f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)} y^{2\alpha+1} dy \\
&= \langle P_S f, g\rangle.
\end{aligned}$$

Thus, $P_S^2 = P_S$ and hence P_S is a projection.

By the same way,

$$\begin{aligned}\langle Q_\Sigma^2 f, g\rangle = \langle Q_\Sigma f, Q_\Sigma g\rangle &= \langle \mathscr{F}_\alpha^m(Q_\Sigma f), \mathscr{F}_\alpha^m(Q_\Sigma g)\rangle \\ &= \int_0^{+\infty} \mathscr{F}_\alpha^m(Q_\Sigma f)(y)\overline{\mathscr{F}_\alpha^m(Q_\Sigma g)(y)}y^{2\alpha+1}dy \\ &= \int_\Sigma \mathscr{F}_\alpha^m(f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)}y^{2\alpha+1}dy \\ &= \int_0^{+\infty} \mathscr{F}_\alpha^m(Q_\Sigma f)(y)\overline{\mathscr{F}_\alpha^m(g)(y)}y^{2\alpha+1}dy \\ &= \langle Q_\Sigma f, g\rangle.\end{aligned}$$

Thus, $Q_\Sigma^2 = Q_\Sigma f$ and hence $Q_\Sigma f$ is a projection.

For all $f \in L^{2,\alpha}$, given the kernel N which satisfies the following two conditions: $f(\cdot)N(\cdot, y) \in L^{1,\alpha}$ for almost every $y \in \mathbb{R}_+$ and if

$$\mathcal{M}f(x) = \int_0^{+\infty} f(y)N(x,y)y^{2\alpha+1}dy,$$

then $\mathcal{M}f \in L^{2,\alpha}$. Then we define the norm of $\mathcal{M}$ to be

$$\|\mathcal{M}\| = \sup_{f\in L^{2,\alpha}} \frac{\|\mathcal{M}f\|_{2,\alpha}}{\|f\|_{2,\alpha}}, \quad f \neq 0,$$

and the Hilbert-Schmidt norm of $\mathcal{M}$ is given by

$$\|\mathcal{M}\|_{HS} = \left(\int_0^{+\infty}\int_0^{+\infty} |N(x,y)|^2 x^{2\alpha+1}y^{2\alpha+1}dxdy\right)^{\frac{1}{2}}.$$

It is clear that $\|P_S\| = \|Q_\Sigma\| = 1$ (see [9]). If $|\Sigma| < +\infty$, where Σ is a set of finite measure of $\mathbb{R}_+$, we have by [27]

$$|\Sigma| = \int_\Sigma x^{2\alpha+1}dx.$$

Lemma 2.1. *If S and Σ are two measurable sets of $\mathbb{R}_+$ such that $|S| < +\infty$ and $|\Sigma| < +\infty$, then*

$$\|P_S Q_\Sigma\|_{HS} = \|Q_\Sigma P_S\|_{HS}.$$

Proof. From relations (1.5), (2.1) and (2.2), we have

$$\begin{aligned}Q_\Sigma P_S(f)(x) &= \frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\mathscr{F}_\alpha^m(\chi_S f)(y)y^{2\alpha+1}dy \\ &= \frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\left(\frac{c_\alpha}{(ib)^{\alpha+1}}\int_S K_\alpha^m(y,z)f(z)z^{2\alpha+1}dz\right)y^{2\alpha+1}dy \\ &= \frac{c_\alpha^2}{b^{2\alpha+2}}\int_S f(z)\left(\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy\right)z^{2\alpha+1}dz \\ &= \int_S f(z)k(x,z)z^{2\alpha+1}dz,\end{aligned}$$

where

$$k(x,z)=\frac{c_\alpha^2}{b^{2\alpha+2}}\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy,\quad z\in S, x\in\mathbb{R}_+.$$

In the same way, we get

$$\begin{aligned}P_SQ_\Sigma(f)(x)&=\chi_S(x)Q_\Sigma(f)(x)\\&=\chi_S(x)\frac{c_\alpha}{(-ib)^{\alpha+1}}\int_\Sigma \overline{K_\alpha^m(y,x)}\mathscr{F}_\alpha^m(f)(y)y^{2\alpha+1}dy\\&=\chi_S(x)\frac{c_\alpha^2}{b^{2\alpha+2}}\int_\Sigma \overline{K_\alpha^m(y,x)}\left(\int_0^{+\infty}K_\alpha^m(y,z)f(z)z^{2\alpha+1}dz\right)y^{2\alpha+1}dy\\&=\chi_S(x)\frac{c_\alpha^2}{b^{2\alpha+2}}\int_0^{+\infty}f(z)\left(\int_\Sigma \overline{K_\alpha^m(y,x)}K_\alpha^m(y,z)y^{2\alpha+1}dy\right)z^{2\alpha+1}dz\\&=\chi_S(x)\int_0^{+\infty}f(z)k(x,z)z^{2\alpha+1}dz.\end{aligned}$$

Then, from the above results we can easily obtain that

$$\|Q_\Sigma P_S\|_{HS}=\left(\int_S\int_0^{+\infty}|k(x,z)|^2x^{2\alpha+1}z^{2\alpha+1}dxdz\right)^{\frac12}$$

and

$$\|P_SQ_\Sigma\|_{HS}=\left(\int_0^{+\infty}\int_S|k(x,z)|^2x^{2\alpha+1}z^{2\alpha+1}dxdz\right)^{\frac12},$$

which yields the desired result. □

Using Cauchy-Schwarz inequality, we can easily obtain that

$$\|P_SQ_\Sigma\|\leqslant\|P_SQ_\Sigma\|_{HS}. \tag{2.5}$$

Lemma 2.2. *If S and Σ are two measurable subsets of $\mathbb{R}_+$ such that $|S|<+\infty$ and $|\Sigma|<+\infty$, then*

$$\|P_SQ_\Sigma\|\leqslant\frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|},$$

where c_α is the constant given by relation (1.1).

Proof. For $x\in S$, let $g_x(t)=k(x,t)$. Note that

$$\mathscr{F}_\alpha^m(g_x)(y)=\frac{c_\alpha}{(ib)^{\alpha+1}}\chi_\Sigma(y)K_\alpha^m(x,y).$$

By relations (1.3) and (1.2), we have

$$\begin{aligned}\int_0^{+\infty}|g_x(t)|^2t^{2\alpha+1}dt&=\int_0^{+\infty}|\mathscr{F}_\alpha^m(g_x)(y)|^2y^{2\alpha+1}dy\\&=\frac{c_\alpha^2}{|b|^{2\alpha+2}}\int_\Sigma|K_\alpha^m(x,y)|^2y^{2\alpha+1}dy\\&\leqslant\frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma|.\end{aligned}$$

Hence,

$$\int_0^{+\infty}\int_0^{+\infty}|k(x,t)|^2x^{2\alpha+1}t^{2\alpha+1}dxdt \leqslant \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma|\int_S x^{2\alpha+1}dx = \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S|.$$

Therefore,

$$\|P_SQ_\Sigma\|_{HS}^2 \leqslant \frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S|.$$

And the proof is complete by (2.5). □

Proposition 2.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and assume that $\varepsilon_S+\varepsilon_\Sigma<1$, f is ε_S-concentrated on S and $\mathscr{F}_\alpha^m$ is ε_Σ-concentrated on Σ, with $\|f\|_{2,\alpha}=1$. Then*

$$\frac{c_\alpha^2}{|b|^{2\alpha+2}}|\Sigma||S| \geqslant (1-\varepsilon_S-\varepsilon_\Sigma)^2.$$

Proof. Assume that $0<|S|,|\Sigma|<+\infty$. As $\|Q_\Sigma\|=1$, it follows that

$$\begin{aligned}\|f-Q_\Sigma P_S(f)\|_{2,\alpha} &\leqslant \|f-Q_\Sigma(f)\|_{2,\alpha}+\|Q_\Sigma(f)-Q_\Sigma P_S(f)\|_{2,\alpha}\\ &\leqslant \varepsilon_\Sigma+\|Q_\Sigma\|\|f-P_S(f)\|_{2,\alpha}\\ &\leqslant \varepsilon_\Sigma+\varepsilon_S.\end{aligned}$$

The triangle inequality gives

$$\|Q_\Sigma P_S(f)\|_{2,\alpha} \geqslant \|f\|_{2,\alpha}-\|f-Q_\Sigma P_S(f)\|_{2,\alpha} \geqslant 1-\varepsilon_\Sigma-\varepsilon_S.$$

Hence,

$$\|Q_\Sigma P_S\| \geqslant 1-\varepsilon_\Sigma-\varepsilon_S.$$

Then from lemmas 2.1 and 2.2, we get the desired result. □

Theorem 2.1 (Donoho-Stark uncertainty principle-type)**.** *Let $f\in L^{2,\alpha}$ and S,Σ be two measurable subsets of $\mathbb{R}_+$ such that $|S||\Sigma|<\frac{|b|^{2\alpha+2}}{c_\alpha^2}$ and let $\varepsilon_S,\varepsilon_\Sigma>0$ such that $\varepsilon_S^2+\varepsilon_\Sigma^2<1$. If f is ε_S-concentrated on S and ε_Σ-concentrated on Σ for the canonical Fourier Bessel transform, then*

$$\frac{c_\alpha^2}{|b|^{2\alpha+2}}|S||\Sigma| \geqslant \left(1-\sqrt{\varepsilon_S^2+\varepsilon_\Sigma^2}\right)^2.$$

Proof. Since $I=P_S+P_{S^c}=P_SQ_\Sigma+P_SQ_{\Sigma^c}+P_{S^c}$, then, using the orthogonality of P_S and P_{S^c}, we have

$$\begin{aligned}\|f-P_SQ_\Sigma(f)\|_{2,\alpha}^2 &= \|P_SQ_{\Sigma^c}(f)+P_{S^c}(f)\|_{2,\alpha}^2\\ &= \|P_SQ_{\Sigma^c}(f)\|_{2,\alpha}^2+\|P_{S^c}(f)\|_{2,\alpha}^2\\ &\leqslant \|P_S\|^2\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2+\|P_{S^c}(f)\|_{2,\alpha}^2.\end{aligned}$$

From (2.1), we have

$$\|P_S\| \leqslant 1. \tag{2.6}$$

Since P_S is a projection on $L^{2,\alpha}$, then

$$\|P_S\| = \|P_S \circ P_S\| \leqslant \|P_S\|^2. \tag{2.7}$$

By (2.6) and (2.7), we deduce that $\|P_S\| = 1$. Thus,

$$\|f - P_S Q_\Sigma(f)\|_{2,\alpha} \leqslant \sqrt{\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2}. \tag{2.8}$$

On the other hand,

$$\|f - P_S Q_\Sigma(f)\|_{2,\alpha} \geqslant \|f\|_{2,\alpha} - \|P_S Q_\Sigma(f)\|_{2,\alpha} \geqslant (1 - \|P_S Q_\Sigma\|)\|f\|_{2,\alpha}.$$

Then, by (2.8), we have

$$(1 - \|P_S Q_\Sigma\|)\|f\|_{2,\alpha} \leqslant \sqrt{\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2}.$$

Since $\frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|} < 1$, it follows from Lemma 2.2 that

$$\|f\|_{2,\alpha}^2 \leqslant \left(1 - \frac{c_\alpha}{|b|^{\alpha+1}}\sqrt{|S||\Sigma|}\right)^{-2} \left(\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2\right). \tag{2.9}$$

Now, by relations (2.3) and (2.4), we get

$$\|Q_{\Sigma^c}(f)\|_{2,\alpha}^2 + \|P_{S^c}(f)\|_{2,\alpha}^2 \leqslant (\varepsilon_S^2 + \varepsilon_\Sigma^2)\|f\|_{2,\alpha}^2. \tag{2.10}$$

By combining relations (2.9) and (2.10), we obtain the desired result. □

3. $L^{p,\alpha}$-Uncertainty Principles for the Canonical Fourier Bessel Transform

In this section, building on the techniques of Donoho and Stark [9] and Soltani [32], we show a quantitative uncertainty inequality about the essential supports of a nonzero function $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$ and its canonical Fourier Bessel transform.

Proposition 3.1. *Let $f \in L^{1,\alpha} \cap L^{p,\alpha}$, $1 < p \leqslant 2$. Then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} |\operatorname{supp} f|^{\frac{1}{q}} \|f\|_{p,\alpha},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{1,\alpha} \cap L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. Then by Hölder's inequality and (1.7), we get

$$\begin{aligned}
\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} &\leqslant |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} \|\mathscr{F}_\alpha^m(f)\|_{\infty,\alpha} \\
&\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} \|f\|_{1,\alpha} \\
&\leqslant \frac{c_\alpha}{|b|^{\alpha+1}} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{1}{q}} |\operatorname{supp} f|^{\frac{1}{q}} \|f\|_{p,\alpha},
\end{aligned}$$

which gives the desired result. □

Proposition 3.2. *Let* $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$. *Then*

$$1 \leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} |\operatorname{supp} f|^{\frac{2-p}{2p}},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. Then by Hölder's inequality and (1.6), we get

$$\begin{aligned}
\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} &\leqslant |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} \|f\|_{p,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} |\operatorname{supp} \mathscr{F}_\alpha^m(f)|^{\frac{q-2}{2q}} |\operatorname{supp} f|^{\frac{2-p}{2p}} \|f\|_{2,\alpha}.
\end{aligned}$$

Relation (1.3) completes the proof. ☐

Definition 3.1. Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$.

(1) We say that a function $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$ is ε_S-concentrated to S in $L^{p,\alpha}$-norm if and only if

$$\|f - P_S f\|_{p,\alpha} \leqslant \varepsilon_S \|f\|_{p,\alpha}. \tag{3.1}$$

(2) Let $f \in L^{p,\alpha}$, $1 \leqslant p \leqslant 2$. We say that $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentrated on Σ in $L^{q,\alpha}$-norm, $q = \frac{p}{p-1}$ if and only if

$$\|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} \leqslant \varepsilon_\Sigma \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha}. \tag{3.2}$$

Lemma 3.1. *Let* $f \in L^{p,\alpha}$, $1 < p \leqslant 2$. *Then*

$$\|\mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} \leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha},$$

with $q = \frac{p}{p-1}$.

Proof. Let $f \in L^{p,\alpha}$, $1 < p \leqslant 2$ and $q = \frac{p}{p-1}$. From relations (1.6) and (2.2), we get

$$\begin{aligned}
\|\mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha} &= \left(\int_\Sigma |\mathscr{F}_\alpha^m(f)(x)|^q x^{2\alpha+1} dx\right)^{\frac{1}{q}} \\
&\leqslant \|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \\
&\leqslant |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1} \|f\|_{p,\alpha},
\end{aligned}$$

which yields the desired result. ☐

Lemma 3.2. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$, $q = \frac{p}{p-1}$. Then*

$$\|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} \leqslant \frac{c_\alpha}{|b|^{\alpha+1}}|S|^{\frac{1}{q}}|\Sigma|^{\frac{1}{q}}\|f\|_{p,\alpha}.$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. From relation (2.2), we have

$$\|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} = \left(\int_\Sigma |\mathscr{F}_\alpha^m(\chi_S f)(x)|^q x^{2\alpha+1}dx\right)^{\frac{1}{q}}. \tag{3.3}$$

By (1.2) and Hölder's inequality it follows that

$$\begin{aligned}|\mathscr{F}_\alpha^m(\chi_S f)(x)| &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}}\left(\int_S |f(y)|^p y^{2\alpha+1}dy\right)^{\frac{1}{p}}\left(\int_S |K_\alpha^m(x,y)|^q y^{2\alpha+1}dy\right)^{\frac{1}{q}}\\ &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}}|S|^{\frac{1}{q}}\|f\|_{p,\alpha}.\end{aligned}$$

Then from (3.3), we obtain the desired result. □

Theorem 3.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$, $q = \frac{p}{p-1}$. If f is ε_S-concentration to S in $L^{p,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q,\alpha}$-norm, then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant \frac{1}{1-\varepsilon_\Sigma}\left(\frac{c_\alpha}{|b|^{\alpha+1}}|S|^{\frac{1}{q}}|\Sigma|^{\frac{1}{q}} + \varepsilon_S|b|^{(\alpha+1)\left(\frac{2}{q}-1\right)}\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1}\right)\|f\|_{p,\alpha}.$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. From the triangle inequality, relations (1.6), (3.1), (3.2) and Lemma 3.2, we get

$$\begin{aligned}\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha} \leqslant& \|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} + \|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha}\\ \leqslant& \|\mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha} + \|\mathscr{F}_\alpha^m(f) - \mathscr{F}_\alpha^m(Q_\Sigma f)\|_{q,\alpha}\\ &+ \|\mathscr{F}_\alpha^m(Q_\Sigma f) - \mathscr{F}_\alpha^m(Q_\Sigma P_S f)\|_{q,\alpha}\\ \leqslant& \frac{c_\alpha}{|b|^{\alpha+1}}|S|^{\frac{1}{q}}|\Sigma|^{\frac{1}{q}}\|f\|_{p,\alpha} + \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha}\\ &+ |b|^{(\alpha+1)\left(\frac{2}{q}-1\right)}\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1}\|f - P_S f\|_{p,\alpha}\\ \leqslant& \left(\frac{c_\alpha}{|b|^{\alpha+1}}|S|^{\frac{1}{q}}|\Sigma|^{\frac{1}{q}} + \varepsilon_S|b|^{(\alpha+1)\left(\frac{2}{q}-1\right)}\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1}\right)\|f\|_{p,\alpha}\\ &+ \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q,\alpha},\end{aligned}$$

which gives the desired result. □

Theorem 3.2 (Donoho-Stark's uncertainty principle-type)**.** *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 < p_2 \leqslant 2$. If f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q_2,\alpha}$-norm,*

$q_2 = \frac{p_2}{p_2-1}$, *then*

$$\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}|\Sigma|^{\frac{q_1-q_2}{q_1q_2}}}{(1-\varepsilon_\Sigma)(1-\varepsilon_S)}|b|^{(\alpha+1)(\frac{2}{q_1}-1)}\left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}}\right)^{\alpha+1}\|f\|_{p_2,\alpha},$$

where $q_1 = \frac{p_1}{p_1-1}$.

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 < p_2 \leqslant 2$. Since $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{q_2,\alpha}$-norm, then, by Hölder's inequality, we obtain

$$\begin{aligned}\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} &\leqslant \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} + \|\chi_\Sigma\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha}\\ &\leqslant \varepsilon_\Sigma\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} + |\Sigma|^{\frac{q_1-q_2}{q_1q_2}}\|\mathscr{F}_\alpha^m(f)\|_{q_1,\alpha}.\end{aligned}$$

Thus, by (1.6),

$$\|\mathscr{F}_\alpha^m(f)\|_{q_2,\alpha} \leqslant \frac{|\Sigma|^{\frac{q_1-q_2}{q_1q_2}}}{1-\varepsilon_\Sigma}|b|^{(\alpha+1)(\frac{2}{q_1}-1)}\left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}}\right)^{\alpha+1}\|f\|_{p_1,\alpha}. \tag{3.4}$$

On the other hand, since f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm, then by Hölder's inequality, we deduce that

$$\|f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + \|\chi_S f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + |S|^{\frac{p_2-p_1}{p_1p_2}}\|f\|_{p_2,\alpha}.$$

Thus,

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\|f\|_{p_2,\alpha}. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the result of this theorem. □

Corollary 3.1. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$. If f is ε_S-concentration to S in $L^{p,\alpha}$-norm and $\mathscr{F}_\alpha^m(f)$ is ε_Σ-concentration to Σ in $L^{2,\alpha}$-norm, then*

$$(1-\varepsilon_\Sigma)(1-\varepsilon_S) \leqslant |S|^{\frac{2-p}{2p}}|\Sigma|^{\frac{q-2}{2q}}|b|^{(\alpha+1)(\frac{2}{q}-1)}\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}}\right)^{\alpha+1},$$

where $q = \frac{p}{p-1}$.

Let $B^p(\Sigma)$, $1 \leqslant p \leqslant 2$, be the set of functions $g \in L^{p,\alpha}$ that are bandlimited to Σ, i.e., ($g \in B^p(\Sigma)$ implies $Q_\Sigma g = g$).

We say that f is ε_Σ-bandlimited to Σ in $L^{p,\alpha}$-norm if there is a $g \in B^p(\Sigma)$ with

$$\|f-g\|_{p,\alpha} \leqslant \varepsilon_\Sigma\|f\|_{p,\alpha}.$$

In the following, we state an $L^{p_1,\alpha} \cap L^{p_2,\alpha}$ bandlimited uncertainty principle of concentration-type.

Theorem 3.3 (Bandlimited principle-type). *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 \leqslant p_1 < p_2 \leqslant 2$. If f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm and ε_Σ-bandlimited to Σ in $L^{q_2,\alpha}$-norm, $q_2 = \frac{p_2}{p_2-1}$, then*

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\left[(1+\varepsilon_\Sigma)c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}+\varepsilon_\Sigma\right]\|f\|_{p_2,\alpha}.$$

Proof. Assume that $|S| < +\infty$ and $|\Sigma| < +\infty$. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 \leqslant p_1 < p_2 \leqslant 2$. Since f is ε_S-concentration to S in $L^{p_1,\alpha}$-norm, then by Hölder's inequality, we deduce that

$$\|f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + \|P_S f\|_{p_1,\alpha} \leqslant \varepsilon_S\|f\|_{p_1,\alpha} + |S|^{\frac{p_2-p_1}{p_1p_2}}\|P_S f\|_{p_2,\alpha}.$$

Thus,

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\|P_S f\|_{p_2,\alpha}. \tag{3.6}$$

As f is ε_Σ-bandlimited to Σ in $L^{q_2,\alpha}$-norm, there is a $g \in B^{p_2}(\Sigma)$ with

$$\|f-g\|_{p_2,\alpha} \leqslant \varepsilon_\Sigma\|f\|_{p_2,\alpha}.$$

On the other hand, we have

$$\|P_S f\|_{p_2,\alpha} \leqslant \|P_S g\|_{p_2,\alpha} + \|P_S(f-g)\|_{p_2,\alpha} \leqslant \|P_S g\|_{p_2,\alpha} + \varepsilon_\Sigma\|f\|_{p_2,\alpha}.$$

But $g \in B^{p_2}(\Sigma)$, from (2.2), $g(x) = \mathscr{F}_\alpha^{m^{-1}}(\chi_\Sigma\mathscr{F}_\alpha^m(g))(x)$ and by (1.6) and Hölder's inequality, we deduce that

$$\begin{aligned}|g(x)| &\leqslant \frac{c_\alpha}{|b|^{\alpha+1}}|\Sigma|^{\frac{1}{p_2}}\|\mathscr{F}_\alpha^m(g)\|_{q_2,\alpha}\\ &\leqslant c_\alpha|\Sigma|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}\|g\|_{p_2,\alpha}.\end{aligned}$$

Hence,

$$\begin{aligned}\|P_S g\|_{p_2,\alpha} &= \left(\int_S |g(x)|^{p_2}x^{2\alpha+1}dx\right)^{\frac{1}{p_2}}\\ &\leqslant c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}\|g\|_{p_2,\alpha}.\end{aligned}$$

Then by (3.6) and the fact that $\|g\|_{p_2,\alpha} \leqslant (1+\varepsilon_\Sigma)\|f\|_{p_2,\alpha}$, we get

$$\|f\|_{p_1,\alpha} \leqslant \frac{|S|^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_S}\left[(1+\varepsilon_\Sigma)c_\alpha|\Sigma|^{\frac{1}{p_2}}|S|^{\frac{1}{p_2}}|b|^{(\alpha+1)(\frac{2}{q_2}-2)}\left(\frac{(c_\alpha p_2)^{\frac{1}{p_2}}}{(c_\alpha q_2)^{\frac{1}{q_2}}}\right)^{\alpha+1}+\varepsilon_\Sigma\right]\|f\|_{p_2,\alpha}.$$

This completes the desired result. □

Corollary 3.2. *Let S and Σ be two measurable subsets of $\mathbb{R}_+$ and $f \in L^{p,\alpha}$, $1 < p \leqslant 2$. If f is ε_S-concentration to S and ε_Σ-bandlimited to Σ in $L^{p,\alpha}$-norm, then*

$$\frac{1-\varepsilon_S-\varepsilon_\Sigma}{1+\varepsilon_\Sigma} \leqslant c_\alpha |\Sigma|^{\frac{1}{p}} |S|^{\frac{1}{p}} |b|^{(\alpha+1)(\frac{2}{q}-2)} \left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{\alpha+1}.$$

Theorem 3.4 (Matolcsi-Szücs-type inequality)**.** *Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 \leqslant p_2 \leqslant 2$. Then*

$$\|\mathscr{F}^m_\alpha(f)\|_{q_2,\alpha} \leqslant |b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\operatorname{supp} \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} |\operatorname{supp} f|^{\frac{p_2-p_1}{p_1 p_2}} \|f\|_{p_2,\alpha},$$

where $q_1 = \frac{p_1}{p_1-1}$ and $q_2 = \frac{p_2}{p_2-1}$.

Proof. Let $f \in L^{p_1,\alpha} \cap L^{p_2,\alpha}$, $1 < p_1 \leqslant p_2 \leqslant 2$, $q_1 = \frac{p_1}{p_1-1}$ and $q_2 = \frac{p_2}{p_2-1}$. Then, by relation (1.6) and Hölder's inequality, we obtain

$$\begin{aligned} &\|\mathscr{F}^m_\alpha(f)\|_{q_2,\alpha} \\ \leqslant &|\operatorname{supp} \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} \|\mathscr{F}^m_\alpha(f)\|_{q_1,\alpha} \\ \leqslant &|b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\operatorname{supp} \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} \|f\|_{p_1,\alpha} \\ \leqslant &|b|^{(\alpha+1)(\frac{2}{q_1}-1)} \left(\frac{(c_\alpha p_1)^{\frac{1}{p_1}}}{(c_\alpha q_1)^{\frac{1}{q_1}}} \right)^{\alpha+1} |\operatorname{supp} \mathscr{F}^m_\alpha(f)|^{\frac{q_1-q_2}{q_1 q_2}} |\operatorname{supp} f|^{\frac{p_2-p_1}{p_1 p_2}} \|f\|_{p_2,\alpha}, \end{aligned}$$

which yields the desired result. □

Corollary 3.3. *Let $f \in L^{2,\alpha} \cap L^{p,\alpha}$, $1 < p < 2$ and $q = \frac{p}{p-1}$. Then*

$$\left(\frac{(c_\alpha p)^{\frac{1}{p}}}{(c_\alpha q)^{\frac{1}{q}}} \right)^{-(\alpha+1)} |b|^{(\alpha+1)(1-\frac{2}{q})} \leqslant |\operatorname{supp} f|^{\frac{2-p}{2p}} |\operatorname{supp} \mathscr{F}^m_\alpha(f)|^{\frac{q-2}{2q}}.$$

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MP-RESIDUATED LATTICES

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Abstract. This paper is devoted to the study of a fascinating class of residuated lattices, the so-called mp-residuated lattice, in which any prime filter contains a unique minimal prime filter. A combination of algebraic and topological methods is applied to obtain new and structural results on mp-residuated lattices. It is demonstrated that mp-residuated lattices are strongly tied up with the dual hull-kernel topology. Especially, it is shown that a residuated lattice is mp if and only if its minimal prime spectrum, equipped with the dual hull-kernel topology, is Hausdorff if and only if its prime spectrum, equipped with the dual hull-kernel topology, is normal. The class of mp-residuated lattices is characterized by means of pure filters. It is shown that a residuated lattice is mp if and only if its pure filters are precisely its minimal prime filters, if and only if its pure spectrum is homeomorphic to its minimal prime spectrum, equipped with the dual hull-kernel topology.

1. Introduction

Let $\mathfrak{A}$ be a residuated lattice, $\mathscr{F}(\mathfrak{A})$ the lattice of filters, and $\mathscr{PF}(\mathfrak{A})$ the lattice of principal filters of $\mathfrak{A}$. The lattice of coannihilators of $\mathfrak{A}$, say $\Gamma(\mathfrak{A})$, is the skeleton of $\mathscr{F}(\mathfrak{A})$, and the lattice of coannulets of $\mathfrak{A}$, say $\gamma(\mathfrak{A})$, is the skeleton of $\mathscr{PF}(\mathfrak{A})$. So $(\Gamma(\mathfrak{A}); \vee^{\Gamma}, \cap, \{1\}, A)$ is a complete Boolean lattice, in which $\vee^{\Gamma}$ is the join in the skeleton, and $\gamma(\mathfrak{A})$ is a sublattice of $\Gamma(\mathfrak{A})$. $\mathfrak{A}$ is said to be *Baer* provided that $\Gamma(\mathfrak{A})$ is a sublattice of $\mathscr{F}(\mathfrak{A})$, and *Rickart* provided that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathscr{F}(\mathfrak{A})$. Obviously, $\mathfrak{A}$ is Rickart if and only if $\gamma(\mathfrak{A})$ is both Boolean and a sublattice of $\mathscr{F}(\mathfrak{A})$. The latter can be characterized by a property that can be formulated in

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terms of universal algebra, namely that any prime filter contains a unique minimal prime filter.

Historically, this notion is rooted in a query posed by G. Birkhoff [8, Problem 70] inspired by M. H. Stone: "What is the most general pseudocomplemented distributive lattice in which $x^* \vee x^{**} = 1$ identically?" The first solution to this problem belongs to G. Grätzer and E. Schmidt [20] who gave the name "*Stone lattices*" to this class of lattices. They characterized stone lattices as distributive pseudocomplemented lattices in which any pair of incomparable minimal prime ideals is comaximal or equivalently each prime ideal contains a unique minimal prime ideal. Motivated by this characterization, W. Cornish [12] studied distributive lattices with zero in which each prime ideal contains a unique minimal prime ideal under the name of "*normal lattices*". He observed that a distributive lattice with zero, $\mathfrak{A}$, is normal if and only if given $x, y \in A$ such that $x \wedge y = 0$, $x^{\perp}$ and $y^{\perp}$ are comaximal. Cornish used this terminology in light of H. Wallman [36], who proved that the lattice of closed subsets of a T_1 space satisfies the above annihilator condition if and only if the space is normal. G. Artico and U. Marconi [5, Lemma β] showed that in a unitary commutative reduced ring any prime ideal contains a unique minimal prime ideal if and only if the set of its annulets is a sublattice of its ideals. E. Matlis [23, Proposition 2.1] proved that the class of commutative PF rings, i.e., a unitary ring with the property that every principal ideal is flat, introduced by A. Hattori [21, p. 151], is precisely the class of reduced rings in which any prime ideal contains a unique minimal prime ideal. P. Bhattacharjee and W. McGovern [7, Theorem 2.6] tied up the notion of PF rings to the notion of the dual hull-kernel topology. They established that a unitary commutative ring is a PF ring if and only if its minimal prime spectrum, with the dual hull-kernel topology, is Hausdorff. This knot was tightened further by M. Aghajani and A. Tarizadeh [1, Theorem 6.2]. They studied the class of unitary commutative rings which fulfill the above universal property, under the name of "*mp-rings*". They gave a good perspective of mp-rings and asserted that a unitary commutative ring is mp if and only if its prime spectrum, with the dual hull-kernel topology, is normal.

Inspired by the above universal property, many authors have proposed similar notions, under other names, for various structures over the years, see e.g., normal lattices [9, 24], conormal lattices [6, 18, 33], normal residuated lattices [32], mp-rings [1], mp-residuated lattices [31], mp-quantales [16, 17], etc (for a discussion about this terminology, see [33, p. 185] and [22, p. 78]).

It is known that residuated lattices play a critical role in the theory of fuzzy logic. Lots of logical algebras such as MTL-algebras, divisible residuated lattices, BL-algebras, MV-algebras, Heyting algebras, and Boolean algebras are subvarieties of residuated lattices. Residuated lattices are not only important from a logical point of view but also interesting from an algebraic point of view and have some interesting algebraic properties.

Given the above discussions, we decided to take a deeper look at mp-residuated lattices. So the notion of mp-residuated lattices is investigated, and some algebraic and topological characterizations are given. Although, the class of mp-residuated lattices has been investigated by [32], however, here we give some more characterizations for the class of mp-residuated lattices, which seems to give more light to the topological situation. Our findings show that some results obtained by some above papers can also be reproduced via residuated lattices. Also, outcomes show that mp-residuated lattices can be considered as the dual notion of Gelfand residuated lattices, as asserted in [1] for rings. So mp-residuated lattices can be studied both as one of the two main pillars of Rickart residuated lattices (along with quasicomplemented residuated lattices), and as a dual notion of Gelfand residuated lattices.

This paper is organized into four sections as follows. In Section 2, some definitions and facts about residuated lattices are recalled, and some of their propositions extracted. We illustrate this section with some examples of residuated lattices, which will be used in the following sections. Section 3 deals with mp-residuated lattices. Theorem 3.1 shows that a residuated lattice is mp if and only if the bounded distributive lattice of its filters is conormal. Theorem 3.1 (Cornish's characterization) gives an element-wise characterization for mp-residuated lattices. Theorem 3.2 shows that a residuated lattice $\mathfrak{A}$ is mp if and only if $\gamma(\mathfrak{A})$ is a sublattice of $\mathscr{F}(\mathfrak{A})$. Theorem 3.3 (Matlis's characterization) establishes that a residuated lattice $\mathfrak{A}$ is mp if and only if $\mathfrak{A}/D(\mathfrak{p})$ is a domain, for any prime filter $\mathfrak{p}$ of $\mathfrak{A}$. The remaining theorems of this section demonstrate that mp-residuated lattices are strongly tied up with the dual hull-kernel topology. Theorem 3.7 shows that a residuated lattice is mp if and only if its prime spectrum is normal with the dual hull-kernel topology. Section 4 deals with the pure spectrum of an mp-residuated lattice. The pure filters of an mp-residuated lattice are characterized in Theorem 4.4. As an important result in this section in Theorem 4.6 is expressed that a residuated lattice is mp if and only if the set of its minimal prime filters is equal to the its purely-prime filters. Theorem 4.8 verifies that a residuated lattice is mp if and only if the identity map between its pure spectrum and its minimal prime spectrum, equipped with the dual hull-kernel topology, is a homeomorphism. Finally, Corollary 4.2 implies that, like Gelfand residuated lattices, the pure spectrum of an mp-residuated lattice is Hausdorff.

2. Preliminaries

In this section, some definitions, properties, and results relative to residuated lattices, which will be used in the following, recalled.

An algebra $\mathfrak{A} = (A; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is called a *residuated lattice* provided that $\ell(\mathfrak{A}) = (A; \vee, \wedge, 0, 1)$ is a bounded lattice, $(A; \odot, 1)$ is a commutative monoid, and $(\odot, \rightarrow)$ is an adjoint pair. A residuated lattice $\mathfrak{A}$ is called *non-degenerate* if $0 \neq 1$. For a residuated lattice $\mathfrak{A}$, and $a \in A$ we put $\neg a := a \rightarrow 0$ and $a^n := a \odot \cdots \odot a$ (n times), for any integer n. The class of residuated lattices is equational, and so forms a variety. For a survey of residuated lattices, the reader is referred to [15].

Remark 2.1. ([10, Proposition 2.6]). Let $\mathfrak{A}$ be a residuated lattice. The following conditions are satisfied for any $x, y, z \in A$:

(r_1) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(r_2) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Example 2.1 ([34]). Let $A_6 = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is given by Figure 1. Routine calculation shows that $\mathfrak{A}_6 = (A_6; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice in which the commutative operation $\odot$ is given by Table 1 and the operation $\rightarrow$ is given by $x \rightarrow y = \bigvee\{a \in A_6 \mid x \odot a \leq y\}$ for any $x, y \in A_6$.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
	a	a	a	0	a	a
		b	a	0	a	b
			c	c	c	c
				d	d	d
					1	1

TABLE 1. Cayley table for "$\odot$" of $\mathfrak{A}_6$

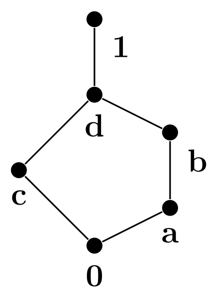

FIGURE 1. Hasse diagram of $\mathfrak{A}_6$

Example 2.2. Let $A_8 = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given by Figure 2. Routine calculation shows that $\mathfrak{A}_8 = (A_8; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice in which the commutative operation $\odot$ is given by Table 2 and the operation $\rightarrow$ is given by $x \rightarrow y = \bigvee\{a \in A_8 \mid x \odot a \leq y\}$ for any $x, y \in A_8$.

Let $\mathfrak{A}$ be a residuated lattice. A non-void subset F of A is called a *filter* of $\mathfrak{A}$ provided that $x, y \in F$ implies $x \odot y \in F$, and $x \vee y \in F$, for any $x \in F$ and $y \in A$. The set of filters of $\mathfrak{A}$ is denoted by $\mathscr{F}(\mathfrak{A})$. A filter F of $\mathfrak{A}$ is called *proper* if $F \neq A$. For any subset X of A, the *filter of* $\mathfrak{A}$ *generated by* X is denoted by $\mathscr{F}(X)$. For each $x \in A$, the filter generated by $\{x\}$ is denoted by $\mathscr{F}(x)$ and said to be *principal*. The set of principal filters is denoted by $\mathscr{PF}(\mathfrak{A})$. Following [19, §5.7], a join-complete lattice $\mathfrak{A}$, is called a *frame* if it satisfies the join infinite distributive law (JID), i.e., for any $a \in A$ and $S \subseteq A$, $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$. A frame $\mathfrak{A}$ is called complete provided

⊙	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
	a	a	0	a	a	a	a	a
		b	0	0	0	0	b	b
			c	c	a	c	a	c
				d	a	a	d	d
					e	c	d	e
						f	f	f
							1	1

TABLE 2. Cayley table for ⊙ of $\mathfrak{A}_8$

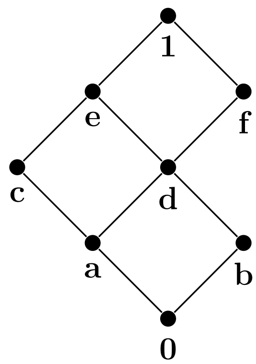

FIGURE 2. Hasse diagram of $\mathfrak{A}_8$

that $\mathfrak{A}$ is a complete lattice. According to [15], $(\mathscr{F}(\mathfrak{A}); \cap, \veebar, \mathbf{1}, A)$ is a complete frame, in which $\veebar\mathfrak{F} = \mathscr{F}(\cup\mathfrak{F})$, for any $\mathfrak{F} \subseteq \mathscr{F}(\mathfrak{A})$.

Example 2.3. Consider the residuated lattice $\mathfrak{A}_6$ from Example 2.1 and the residuated lattice $\mathfrak{A}_8$ from Example 2.2. The sets of their filters are presented in Table 3.

	Filters
$\mathfrak{A}_6$	$\{1\}, \{a, b, d, 1\}, \{c, d, 1\}, \{d, 1\}, A_6$
$\mathfrak{A}_8$	$\{1\}, \{a, c, d, e, f, 1\}, \{c, e, 1\}, \{f, 1\}, A_8$

TABLE 3. The sets of filters of $\mathfrak{A}_6$ and $\mathfrak{A}_8$

The proof of the following proposition has a routine verification, and so it is left to the reader.

Proposition 2.1. *Let $\mathfrak{A}$ be a residuated lattice and F be a filter of $\mathfrak{A}$. The following assertions hold for any $x, y \in A$:*

(1) $\mathscr{F}(x) = \{a \in A \mid x^n \leq a,$ *for some integer* $n\}$*;*
(2) $x \leq y$ *implies* $\mathscr{F}(y) \subseteq \mathscr{F}(x)$*;*

(3) $\mathscr{F}(x) \cap \mathscr{F}(y) = \mathscr{F}(x \vee y)$;
(4) $\mathscr{F}(x) \veebar \mathscr{F}(y) = \mathscr{F}(x \odot y)$;
(5) $\mathscr{PF}(\mathfrak{A})$ *is a sublattice of* $\mathscr{F}(\mathfrak{A})$.

The following proposition gives a characterization for the comaximal filters of a residuated lattice.

Proposition 2.2. *Let* $\mathfrak{A}$ *be a residuated lattice and* F, G *two proper filters of* $\mathfrak{A}$. *The following assertions are equivalent:*

(1) F *and* G *are comaximal, i.e.,* $F \veebar G = A$;
(2) *there exist* $f \in F$ *and* $g \in G$ *such that* $f \odot g = 0$;
(3) *there exists* $a \in A$ *such that* $a \in F$ *and* $\neg a \in G$.

Proof. (1)⇒(2) It is evident by Proposition 2.1.

(2)⇒(3) Let $f \odot g = 0$, for some $f \in F$ and $g \in G$. This implies that $g \leq \neg f$, and the result hold.

(3)⇒(1) It is evident. □

Let $\mathfrak{A}$ be a residuated lattice. A maximal element in the set of proper filters of $\mathfrak{A}$ is called *maximal*, and the set of maximal filters of $\mathfrak{A}$ denoted by $\max(\mathfrak{A})$. A meet-irreducible element in the set of proper filters of $\mathfrak{A}$ is called *prime*, and the set of prime filters of $\mathfrak{A}$ denoted by $Spec(\mathfrak{A})$. Since $\mathscr{F}(\mathfrak{A})$ is a distributive lattice, so $\max(\mathfrak{A}) \subseteq Spec(\mathfrak{A})$. Zorn's lemma verifies that any proper filter is contained in a maximal filter, and so in a prime filter.

A non-empty subset $\mathscr{C}$ of $\mathfrak{A}$ is called $\vee$-*closed* if it is closed under the join operation, i.e $x, y \in \mathscr{C}$ implies $x \vee y \in \mathscr{C}$.

Theorem 2.1. ([25, Theorem 3.18]). *If* $\mathscr{C}$ *is a* $\vee$*-closed subset of* $\mathfrak{A}$ *which does not meet the filter* F, *then* F *is contained in a filter* P *which is maximal with respect to the property of not meeting* $\mathscr{C}$; *furthermore* P *is prime.*

A minimal element in the set of prime filters of a residuated lattice $\mathfrak{A}$ is called *minimal prime*, and the set of minimal prime filters of $\mathfrak{A}$ denoted by $\min(\mathfrak{A})$. For the basic facts concerning prime filters of a residuated lattice, the reader is referred to [25].

Example 2.4. Consider the residuated lattice $\mathfrak{A}_6$ from Example 2.1 and the residuated lattice $\mathfrak{A}_8$ from Example 2.2. The sets of their maximal, prime, and minimal prime filters are presented in Table 4.

Proposition 2.3 ([25]). *Let* $\mathfrak{A}$ *be a residuated lattice. The following assertions hold.*

(1) *A subset* P *of* A *is a minimal prime filter if and only if* $\dot{P} \overset{def.}{=} A \setminus P$ *is a* $\vee$*-closed subset of* $\mathfrak{A}$ *which it is maximal with respect to the property of not containing* 1.
(2) *Any prime filter of a residuated lattice contains a minimal prime filter.*

	Prime filters	
	Maximal filters	Minimal prime filters
$\mathfrak{A}_6$	$\{a,b,d,1\},\{c,d,1\}$	$\{1\}$
$\mathfrak{A}_8$	$\{a,c,d,e,f,1\}$	$\{c,e,1\},\{f,1\}$

TABLE 4. The sets of maximal, prime, and minimal prime filters of $\mathfrak{A}_6$ and $\mathfrak{A}_8$

(3) *A prime filter P of $\mathfrak{A}$ is minimal prime if and only if for any $x \in A$, P contains precisely one of x or $x^{\perp}$.*

Let $\mathfrak{A}$ be a residuated lattice and Π a collection of prime filters of $\mathfrak{A}$. For a subset π of Π we set $k(\pi) = \bigcap \pi$, and for a subset X of A we set $h_\Pi(X) = \{P \in \Pi \mid X \subseteq P\}$ and $d_\Pi(X) = \Pi \setminus h_\Pi(X)$. The collection Π can be topologized by taking the collection $\{h_\Pi(x) \mid x \in A\}$ as a closed (an open) basis, which is called *the* (*dual*) *hull-kernel topology* on Π and denoted by $\Pi_{h(d)}$. The generated topology by $\tau_h \cup \tau_d$ on $Spec(\mathfrak{A})$ is called *the patch topology* and denoted by τ_p. As usual, the Boolean lattice of all clopen subsets of a topological space A_τ shall be denoted by $Clop(A_\tau)$. For a detailed discussion on the (dual) hull-kernel and patch topologies on a residuated lattice, we refer to [29].

Proposition 2.4 ([29])**.** *Let $\mathfrak{A}$ be a residuated lattice. We have:*

$$Clop(Spec_d(\mathfrak{A})) = \{h(e) \mid e \in \beta(\mathfrak{A})\}.$$

Let Π be a collection of prime filters in a residuated lattice $\mathfrak{A}$. In the following, for a given subset π of Π, $cl^{\Pi}_{h(d)}(\pi)$ stands for the closure of π in the topological space $(\Pi, \tau_{h(d)})$. If $\pi = \{P\}$ for some prime filter P of $\mathfrak{A}$, then $cl^{\Pi}_{h(d)}(\{P\})$ is simply denoted by $cl^{\Pi}_{h(d)}(P)$. If Π is understood, it will be dropped.

Lemma 2.1. ([29, Theorem 3.14]). *Let $\mathfrak{A}$ be a residuated lattice, Π a collection of prime filters of $\mathfrak{A}$ and $\mathfrak{p}, \mathfrak{q} \in \Pi$. The following assertions are equivalent:*

(1) $\mathfrak{p} \subseteq \mathfrak{q}$;
(2) $\mathfrak{q} \in cl_h(\mathfrak{p})$;
(3) $\mathfrak{p} \in cl_d(\mathfrak{q})$.

The following proposition characterizes the open sets of the spectrum of a residuated lattice w.r.t the dual hull-kernel topology.

Proposition 2.5. *Let $\mathfrak{A}$ be a residuated lattice. The open sets of $Spec_d(\mathfrak{A})$ are precisely of the form $\{\mathfrak{p} \in Spec(\mathfrak{A}) \mid \mathfrak{p} \cap X \neq \emptyset\}$, where X is a subset of A.*

Proof. Let U be an open set in $Spec_d(\mathfrak{A})$. So $U = \bigcup_{x \in X} h(x)$, for some $X \subseteq A$. It is clear that $\bigcup_{x \in X} h(x) = \{\mathfrak{p} \in Spec(\mathfrak{A}) \mid \mathfrak{p} \cap X \neq \emptyset\}$. □

Remark 2.2. Let $\mathfrak{A}$ be a residuated lattice. By Proposition 2.5, it follows that the closed sets of $Spec_d(\mathfrak{A})$ are precisely of the form $\{\mathfrak{p} \in Spec(\mathfrak{A}) \mid \mathfrak{p} \cap X = \emptyset\}$, where X is a subset of A.

Let Π be a collection of prime filters in a residuated lattice $\mathfrak{A}$. Following G. De Marco [13, p. 290], if π is a subset of Π, its *specialization* (*generalization*) in Π, $\mathscr{S}_{\Pi}(\pi)$ ($\mathscr{G}_{\Pi}(\pi)$), is the set of all primes in Π, which contain (are contained in) some prime belonging to π. One can see that $\mathscr{S}$ and $\mathscr{G}$ are closure operators on the power set of $Spec(\mathfrak{A})$. A fixed point of $\mathscr{S}(\mathscr{G})$ is called $\mathscr{S}_{\Pi}$-stable ($\mathscr{G}_{\Pi}$-stable). If Π is understood, it will be dropped. Notice that for any subset B of A, $\bigcup_{b\in B} h(b)(\bigcup_{b\in B} d(b))$ is $\mathscr{S}(\mathscr{G})$-stable. The following theorem characterizes the closed sets of the (dual) hull-kernel topology.

Theorem 2.2. ([29, Theorem 4.30]). *Let $\mathfrak{A}$ be a residuated lattice and π a subset of $Spec(\mathfrak{A})$. π is closed under the dual hull-kernel topology if and only if it is closed under the patch topology and $\mathscr{G}$-stable.*

For a residuated lattice $\mathfrak{A}$ the hull-kernel topology on $\min(\mathfrak{A})$ is a well-studied structure. For example, it is known that the hull-kernel topology on $\min(\mathfrak{A})$ is totally disconnected [29, Corollary 5.5], and classifications of when $\min(\mathfrak{A})$ is compact [29, Theorem 5.10]. In the sequel, we fucose on the dual hull-kernel topology on $\min(\mathfrak{A})$. In particular, we characterize when $\min_d(\mathfrak{A})$ is Hausdorff.

Proposition 2.6. ([29, Theorem 4.6 (2)]). *Let $\mathfrak{A}$ be a residuated lattice. $Spec_d(\mathfrak{A})$ and $\min_d(\mathfrak{A})$ are compact.*

Let $\mathfrak{A}$ be a residuated lattice. For any subset X of A, we set $X^{\perp} = kd(X)$, $\Gamma(\mathfrak{A}) = \{X^{\perp} \mid X \subseteq A\}$, $\gamma(\mathfrak{A}) = \{x^{\perp} \mid x \in A\}$, and $\lambda(\mathfrak{A}) = \{x^{\perp\perp} \mid x \in A\}$. Elements of $\Gamma(\mathfrak{A})$, $\gamma(\mathfrak{A})$ and $\lambda(\mathfrak{A})$ are called *coannihilators*, *coannulets*, and *dual coannulets* of $\mathfrak{A}$, respectively.

Let $\mathfrak{A}$ be a $\wedge$-semilattice with zero. Recall [19, §I.6.2] that an element $a^* \in A$ is a *pseudocomplement* of $a \in A$ if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. An element can have at most one pseudocomplement. $\mathfrak{A}$ is called *pseudocomplemented* if every element of A has a pseudocomplement. The set $S(\mathfrak{A}) = \{a^* \mid a \in A\}$ is called *the skeleton of* $\mathfrak{A}$ and we have $S(\mathfrak{A}) = \{a \in A \mid a = a^{**}\}$. By [19, Theorem 100], it follows that if $\mathfrak{A}$ is a pseudocomplemented complete $\wedge$-semilattice, then $S(\mathfrak{A})$ is a complete Boolean lattice, where the meet in $S(\mathfrak{A})$ is calculated in $\mathfrak{A}$, the join in $S(\mathfrak{A})$ is given by $\vee X = (\wedge\{x^* \mid x \in X\})^*$, for any $X \subseteq S(\mathfrak{A})$, and $1 \overset{def.}{=} 0^*$.

Applying Proposition 2.11 from [27], it follows that $\Gamma(\mathfrak{A})$ is the skeleton of $\mathscr{F}(\mathfrak{A})$ and $\gamma(\mathfrak{A})$ is the skeleton of $\mathscr{PF}(\mathfrak{A})$. So $(\Gamma(\mathfrak{A}); \vee^{\Gamma}, \cap, \{1\}, A)$ is a complete Boolean lattice, in which $\vee^{\Gamma}$ is the join in the skeleton, and $\gamma(\mathfrak{A})$ is a sublattice of $\Gamma(\mathfrak{A})$. $\mathfrak{A}$ is said to be *Baer* provided that $\Gamma(\mathfrak{A})$ is a sublattice of $\mathscr{F}(\mathfrak{A})$, and *Rickart* provided that $\gamma(\mathfrak{A})$ is a Boolean sublattice of $\mathscr{F}(\mathfrak{A})$. For the basic facts concerning coannihilators and coannulets of residuated lattices we refer to [26].

Let $\mathfrak{A}$ be a residuated lattice. For a $\vee$-closed subset I of $\ell(\mathfrak{A})$, set $\omega(I) = \{a \in A \mid a \vee x = 1, \text{ for some } x \in I\}$, and $\Omega(\mathfrak{A}) = \{\omega(I) \mid I \in id(\ell(\mathfrak{A}))\}$. Using Proposition 3.4 from [32], it follows that $\Omega(\mathfrak{A}) \subseteq \mathscr{F}(\mathfrak{A})$, and so elements of $\Omega(\mathfrak{A})$ are called *ω-filters* of $\mathfrak{A}$. For an ω-filter F of $\mathfrak{A}$, I_F denoted an ideal of $\ell(\mathfrak{A})$, which satisfies $F = \omega(I_F)$. [32, Proposition 3.7] shows that $(\Omega(\mathfrak{A}); \cap, \vee^{\omega}, \{1\}, A)$ is a bounded distributive lattice, in which $F \vee^{\omega} G = \omega(I_F \curlyvee I_G)$, for any $F, G \in \Omega(\mathfrak{A})$ (by $\curlyvee$, we mean the join operation in the lattice of ideals of $\ell(\mathfrak{A})$). For any proper filter H of $\mathfrak{A}$ we set $D(H) = \omega(\dot{H})$. Elements of $D(\{1\})$ shall be called the *unit divisors of* $\mathfrak{A}$. For the basic facts concerning ω-filters of a residuated lattice, interested readers are referred to [32].

Proposition 2.7 ([32]). *Let $\mathfrak{A}$ be residuated lattice. The following assertions hold:*

(1) *$\gamma(\mathfrak{A})$ is a sublattice of $\Omega(\mathfrak{A})$;*
(2) *$D(\mathfrak{p}) = k\mathscr{G}(\mathfrak{p}) = k(\mathscr{G}(\mathfrak{p}) \cap \min(\mathfrak{A}))$, for any prime filter $\mathfrak{p}$ of $\mathfrak{A}$;*
(3) *a prime filter $\mathfrak{p}$ of $\mathfrak{A}$ is minimal prime if and only if $\mathfrak{p} = D(\mathfrak{p})$.*

Definition 2.1. A residuated lattice $\mathfrak{A}$ is said to be *a domain* provided that it has no unit divisors.

The following proposition has a routine verification, and so its proof is left to the reader.

Proposition 2.8. *Let $\mathfrak{A}$ be a residuated lattice and F a filter of $\mathfrak{A}$. The quotient residuated lattice $\mathfrak{A}/F$ is a domain if and only if F is prime.*

3. Mp-Residuated Lattices

In this section, the notion of an mp-residuated lattice is investigated, and some topological characterizations of them are extracted.

Definition 3.1. A residuated lattice $\mathfrak{A}$ is called *mp* provided that any prime filter of $\mathfrak{A}$ contains a unique minimal prime filter of $\mathfrak{A}$.

Example 3.1. One can see that the residuated lattice $\mathfrak{A}_6$ from Example 2.1 is mp and the residuated lattice $\mathfrak{A}_8$ from Example 2.2 is not mp.

Example 3.2. The class of MTL-algebras, and so, MV-algebras, BL-algebras, and Boolean algebras are some subclasses of mp-residuated lattices.

Let $\mathfrak{A}$ be a bounded distributive lattice. $\mathfrak{A}$ is said to be:

- *normal* provided that for all $x, y \in A$, $x \vee y = 1$ implies there exist $u, v \in A$ such that $u \vee x = v \vee y = 1$ and $u \wedge v = 0$;
- *conormal* provided that for all $x, y \in A$, $x \wedge y = 0$ implies there exist $u, v \in L$ such that $u \wedge x = v \wedge y = 0$ and $u \vee v = 1$.

Remark 3.1. In [12] and [24], the above nomenclatures are reversed. We have picked the version of these definitions from [33, Definition 4.3] and [22, p. 67] because of the author's discussion in [22, p. 78].

The following result shows that a residuated lattice is mp if and only if the bounded distributive lattice of its filters is conormal.

Proposition 3.1. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *the bounded distributive lattice* $\mathscr{F}(\mathfrak{A})$ *is conormal;*
(2) *the bounded distributive lattice* $\mathscr{P}\mathscr{F}(\mathfrak{A})$ *is conormal;*
(3) $\mathfrak{A}$ *is mp.*

Proof. (1)⇒(2) Let $x, y \in A$, such that $\mathscr{F}(x) \cap \mathscr{F}(y) = \{1\}$. Then there exist $F, G \in \mathscr{F}(\mathfrak{A})$ such that $F \veebar G = A$ and $F \cap \mathscr{F}(x) = G \cap \mathscr{F}(y) = \{1\}$. Thus there exist $f \in F$ and $g \in G$ such that $f \odot g = 0$. This implies that $\mathscr{F}(f) \veebar \mathscr{F}(g) = A$ and $\mathscr{F}(f) \cap \mathscr{F}(x) = \mathscr{F}(g) \cap \mathscr{F}(y) = \{1\}$.

(2)⇒(3) Using Proposition 2.1, it is straightforward.

(3)⇒(1) Let F and G be two filters of $\mathfrak{A}$ such that $F \cap G = \{1\}$. By distributivity of $\mathfrak{A}$, with a little bit of effort, we can show that $F^{\perp} \veebar G^{\perp} = A$. □

The following theorem gives some algebraic criteria for mp-residuated lattices, inspired by the one obtained for normal lattices [12, Theorem 2.4].

Theorem 3.1 (Cornish's characterization)**.** *Let* $\mathfrak{A}$ *be a residuated lattice. The following assertions are equivalent:*

(1) *any two distinct minimal prime filters are comaximal;*
(2) $\mathfrak{A}$ *is mp;*
(3) *for any prime filter* $\mathfrak{p}$ *of* $\mathfrak{A}$, $D(\mathfrak{p})$ *is a prime filter of* $\mathfrak{A}$*;*
(4) *for any maximal filter* $\mathfrak{m}$ *of* $\mathfrak{A}$, $D(\mathfrak{m})$ *is a prime filter of* $\mathfrak{A}$*;*
(5) *for any pairwise elements* x *and* y *in* A*, i.e,* $x \vee y = 1$, $x^{\perp} \veebar y^{\perp} = A$*;*
(6) *for any pairwise elements* x *and* y *in* A*, there exists* $a \in A$ *such that* $a \in x^{\perp}$ *and* $\neg a \in y^{\perp}$*;*
(7) *for any* $x, y \in A$, $(x \vee y)^{\perp} = x^{\perp} \veebar y^{\perp}$*;*
(8) *for any* $x, y \in A$, $(x \vee y)^{\perp} = A$ *implies* $x^{\perp} \veebar y^{\perp} = A$.

Proof. (1)⇒(2) It is evident.

(2)⇒(3) It follows by Proposition 2.7 (2).

(3)⇒(4) It is evident.

(4)⇒(5) Let x and y be two pairwise elements in A. Assume by absurdum that $x^{\perp} \veebar y^{\perp} \nsubseteq A$. So $x^{\perp} \veebar y^{\perp} \subseteq \mathfrak{m}$, for some maximal filter $\mathfrak{m}$ of $\mathfrak{A}$. Applying Proposition 2.3 (3), it verifies that $x, y \notin D(\mathfrak{m})$; a contradiction.

(5)⇒(6) It follows by Proposition 2.2.

(6)⇒(7) Let $a \in (x \vee y)^{\perp}$. Let $b = a \vee x$. Obviously, b and y are pairwise. There exists $s \in A$ such that $s \in b^{\perp}$ and $\neg s \in y^{\perp}$. By (r_2), it follows that $a \geq (a \vee s) \odot \neg s$. This establishes that $a \in x^{\perp} \veebar y^{\perp}$. The converse inclusion is evident.

(7)⇒(8) It is evident.

(8)⇒(1) Let $\mathfrak{m}$ and $\mathfrak{n}$ be distinct minimal prime filters of $\mathfrak{A}$. Consider $x \in \mathfrak{m} \setminus \mathfrak{n}$ and $y \in \mathfrak{n} \setminus \mathfrak{m}$. Using Proposition 2.3 (3), there exists $z \in x^{\perp} \setminus \mathfrak{m}$. Let $a = y \vee z$. So $(a \vee x)^{\perp} = A$, and this implies that $A = a^{\perp} \veebar x^{\perp} \subseteq \mathfrak{m} \veebar \mathfrak{n}$. □

Theorem 3.2. *Let* $\mathfrak{A}$ *be a residuated lattice. The following assertions are equivalent:*

(1) *for any* $F, G \in \Omega(\mathfrak{A})$, $F \vee^{\omega} G = A$ *implies* $F \veebar G = A$*;*

(2) $\mathfrak{A}$ *is mp;*
(3) *for any* $\mathcal{F} \subseteq \Omega(\mathfrak{A})$, $\veebar\mathcal{F} \in \Omega(\mathfrak{A})$*;*
(4) $(\Omega(\mathfrak{A}); \cap, \veebar)$ *is a frame;*
(5) $(\gamma(\mathfrak{A}); \cap, \veebar)$ *is a lattice.*

Proof. (1)⇒(2) Let $x \vee y = 1$ for some $x, y \in A$. Since $\gamma(\mathfrak{A})$ is a sublattice of $\Omega(\mathfrak{A})$ so we have $x^{\perp} \vee^{\omega} y^{\perp} = x^{\perp} \vee^{\Gamma} y^{\perp} = (x \vee y)^{\perp} = A$.

(2)⇒(3) Let $\{F_i\}_{i\in I}$ be a family of ω-filters of $\mathfrak{A}$. Obviously, we have $\veebar_{i\in I} F_i \subseteq \omega(\curlyvee_{i\in I} I_i)$. Consider $a \in \omega(\curlyvee_{i\in I} I_i)$. Hence, there exists $x \in \curlyvee_{i\in I} I_i$ such that $a \in x^{\perp}$. This states that $x \leq x_{i_1} \vee \cdots \vee x_{i_n}$, for some integer n and $x_{i_j} \in I_{i_j}$. We have the following sequence of formulas:

$$x^{\perp} \subseteq (x_{i_1} \vee \cdots \vee x_{i_n})^{\perp} = x_{i_1}^{\perp} \veebar \cdots \veebar x_{i_n}^{\perp} \subseteq F_{i_1} \veebar \cdots \veebar F_{i_n} \subseteq \veebar_{i\in I} F_i.$$

(3)⇒(4) It is evident.

(4)⇒(5) It follows by Proposition 2.7 (1).

(5)⇒(1) Let $F, G \in \Omega(\mathfrak{A})$ such that $F \vee^{\omega} G = A$. Since $\omega(I_F \curlyvee I_G) = A$, so $1 \in I_F \curlyvee I_G$. This establishes that $f \vee g = 1$, for some $f \in I_F$ and $g \in I_G$. Hence, $A = (f \vee g)^{\perp} = f^{\perp} \vee^{\Gamma} g^{\perp} = f^{\perp} \veebar g^{\perp} \subseteq F \veebar G$. □

E. Matlis [23, Proposition 2.1] gave a criterion for a ring to be PF and showed that a unitary commutative ring $\mathfrak{A}$ is PF if and only if for any maximal ideal $\mathfrak{m}$ of $\mathfrak{A}$, $\mathfrak{A}_{\mathfrak{m}}$ be an integral domain. Motivated by this, the following theorem, which is an immediate consequence of Proposition 2.8 and Theorem 3.1, can be extracted for mp-residuated lattices.

Theorem 3.3 (Matlis's characterization)**.** *Let* $\mathfrak{A}$ *be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $\mathfrak{A}/D(\mathfrak{p})$ *is a domain, for any prime filter* $\mathfrak{p}$ *of* $\mathfrak{A}$*;*
(3) $\mathfrak{A}/D(\mathfrak{m})$ *is a domain, for any maximal filter* $\mathfrak{m}$ *of* $\mathfrak{A}$*.*

The next theorem gives some necessary and sufficient conditions for the collection of minimal prime filters in a residuated lattice to be a Hausdorff space with the dual hull-kernel topology.

Theorem 3.4. *Let* $\mathfrak{A}$ *be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $\min_d(\mathfrak{A})$ *is Hausdorff.*

Proof. (1)⇒(2) Let $\mathfrak{m}$ and $\mathfrak{n}$ be two distinct minimal prime filters of $\mathfrak{A}$. So, there exist $x \in \mathfrak{m}$ and $y \in \mathfrak{n}$ such that $x \odot y = 0$. This follows that $h(x) \cap h(y) = \emptyset$, and the result holds.

(2)⇒(1) Let $\mathfrak{m}$ and $\mathfrak{n}$ be two distinct minimal prime filters of $\mathfrak{A}$. So, there exist $x, y \in A$ such that $\mathfrak{m} \in h(x)$, $\mathfrak{n} \in h(y)$, and $h(x \odot y) = \emptyset$. This shows that $A = x^{\perp\perp} \veebar y^{\perp\perp} \subseteq \mathfrak{m} \veebar \mathfrak{n}$. □

Remark 3.2. By Proposition 2.6 and Theorem 3.4, $\mathfrak{A}$ is an mp-residuated lattice if and only if $\min_d(\mathfrak{A})$ is a T_4 space.

Theorem 3.5. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $h(\mathfrak{m})$ *is closed in $Spec_d(\mathfrak{A})$ for any $\mathfrak{m} \in \min(\mathfrak{A})$.*

Proof. (1)⇒(2) It follows by Proposition 2.3(2) and Theorem 2.2.

(2)⇒(1) Assume by absurdum that there exist two distinct minimal prime filters $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathfrak{A}$ such that $\mathfrak{m} \veebar \mathfrak{n} \neq A$. This implies that there exists a prime filter P containing in $\mathfrak{m}$ and $\mathfrak{n}$, and so $h(\mathfrak{m}) \cap h(\mathfrak{n}) \neq \emptyset$. □

Recall that a *retraction* is a continuous mapping from a topological space into a subspace which preserves the position of all points in that subspace.

Theorem 3.6. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $\min_d(\mathfrak{A})$ *is a retraction of $Spec_d(\mathfrak{A})$.*

Proof. (1)⇒(2) Define $f : Spec(\mathfrak{A}) \to \min(\mathfrak{A})$ by $f(\mathfrak{p}) = \mathfrak{m}_{\mathfrak{p}}$. Set $H = \{\mathfrak{p} \in Spec(\mathfrak{A}) \mid a \notin f(\mathfrak{p})\}$ and $X = (\bigcup H)^c$. Consider $a \in A$. Let $\mathfrak{p} \in f^{-1}(d_m(a))$. This implies that $\mathfrak{p} \in H$, and so $\mathfrak{p} \cap X = \emptyset$. Conversely, suppose that $\mathfrak{p} \cap X = \emptyset$. Let $a \in f(\mathfrak{p})$. So for any $\mathfrak{n} \in d_m(a)$ there exist $x_{\mathfrak{n}} \in \mathfrak{n}$ and $y_{\mathfrak{n}} \in f(\mathfrak{p})$ such that $x_{\mathfrak{n}} \odot y_{\mathfrak{n}} = 0$. Obviously, $d_m(a) \subseteq \bigcup_{\mathfrak{n} \in d_m(a)} h(x_{\mathfrak{n}})$. Since $d_m(a)$ is a compact subspace of $\min_d(\mathfrak{A})$, so $d_m(a) \subseteq \bigcup_{\mathfrak{n} \in \Im} h(x_{\mathfrak{n}}) = h(\bigvee_{\mathfrak{n} \in \Im} x_{\mathfrak{n}})$, where $\Im$ is a finite subset of $d_m(a)$. Letting $y = \bigodot_{\mathfrak{n} \in \Im} y_{\mathfrak{n}}$, we have $y \in f(\mathfrak{p})$ and $x \odot y = 0$. So, there exists a prime filter Q of $\mathfrak{A}$ such that $Q \cap X = \emptyset$ and $y \in Q$. Since $a \notin f(Q)$, so $x \in Q$; a contradiction. This shows that $f^{-1}(d_m(a)) = \{\mathfrak{p} \mid \mathfrak{p} \cap X = \emptyset\}$. So, the result holds by Remark 2.2.

(2)⇒(1) Let $f : Spec_d(\mathfrak{A}) \to \min_d(\mathfrak{A})$ be a retraction and $\mathfrak{m} \in \min(\mathfrak{A})$. Suppose that $\mathfrak{m} \subseteq \mathfrak{p}$, for some $\mathfrak{p} \in Spec(\mathfrak{A})$. By Lemma 2.1, we have $\mathfrak{m} \in cl_d^{Spec(\mathfrak{A})}(\mathfrak{p})$ and by continuity of f and T_1 we obtain that

$$\mathfrak{m} = f(\mathfrak{m}) \in f(cl_d^{Spec(\mathfrak{A})}(\mathfrak{p})) \subseteq cl_d^{\min(\mathfrak{A})}(f(\mathfrak{p})) = \{f(\mathfrak{p})\}.$$

This shows that $\mathfrak{m}$ is the unique minimal prime filter of $\mathfrak{A}$ contained in $\mathfrak{p}$. □

Remark 3.3. By Theorem 3.6, if $\mathfrak{A}$ is an mp-residuated lattice, the map $Spec(\mathfrak{A}) \rightsquigarrow \min(\mathfrak{A})$, which sends any prime filter $\mathfrak{p}$ of $\mathfrak{A}$ to the unique minimal prime filter of $\mathfrak{A}$ containing in it, is the unique retraction from $Spec_d(\mathfrak{A})$ into $\min_d(\mathfrak{A})$.

The next result, which can be compared with Proposition 2.4, characterizes the clopen subsets of $\min_d(\mathfrak{A})$ where $\mathfrak{A}$ is an mp-residuated lattice.

Corollary 3.1. *Let $\mathfrak{A}$ be an mp-residuated lattice. We have:*

$$Clop(\min_d(\mathfrak{A})) = \{h(e) \cap \min(\mathfrak{A}) \mid e \in \beta(\mathfrak{A})\}.$$

Proof. By Theorem 3.6, there exists a retraction $f : Spec_d(\mathfrak{A}) \to \min_d(\mathfrak{A})$. Let $U \in Clop(\min_d(\mathfrak{A}))$. So $f^{\leftarrow}(U) \in Clop(Spec_d(\mathfrak{A}))$. Thus $f^{\leftarrow}(U) = h(e)$, for some $e \in \beta(\mathfrak{A})$, due to Proposition 2.4. This implies that $U = f^{\leftarrow}(U) \cap \min(\mathfrak{A}) = h(e) \cap \min(\mathfrak{A})$. The converse is evident. □

Theorem 3.7. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is mp;*
(2) *$Spec_d(\mathfrak{A})$ is a normal space.*

Proof. (1)⇒(2) Using Theorem 3.6 and Remark 3.3, there exists a retraction $f : Spec_d(\mathfrak{A}) \to \min_d(\mathfrak{A})$, which sends any prime filter of $\mathfrak{A}$ to the unique minimal prime filter of $\mathfrak{A}$ contained in $\mathfrak{p}$, for any prime filter $\mathfrak{p}$ of $\mathfrak{A}$. By Remark 3.2, $\min(\mathfrak{A})$ is a T_4 space, and so f is a closed map. Let C_1 and C_2 be two disjoint closed sets in $Spec_d(\mathfrak{A})$, so $f(C_1)$ and $f(C_2)$ are disjoint closed sets in $\min_d(\mathfrak{A})$. Since $\min_d(\mathfrak{A})$ is normal, there exist disjoint open neighbourhoods N_1 and N_2 of $f(C_1)$ and $f(C_2)$ in $min_d(\mathfrak{A})$, respectively. One can see that $f^{-1}(N_1)$ and $f^{-1}(N_2)$ are disjoint open neighbourhoods of C_1 and C_2 in $Spec_d(\mathfrak{A})$, respectively.

(2)⇒(1) Let $\mathfrak{m} \in \min(\mathfrak{A})$. If $\mathfrak{p} \in Cl_d^{Spec(\mathfrak{A})}(\mathfrak{m})$, $\mathfrak{p} \subseteq \mathfrak{m}$, and this yields that $\mathfrak{p} = \mathfrak{m}$. This shows that $\{\mathfrak{m}\}$ is a closed subset of $Spec_d(\mathfrak{A})$. Now, let $\mathfrak{m}_1, \mathfrak{m}_2 \in \min(\mathfrak{A})$. Thus, there exist $a, b \in A$ such that $h(a)$ and $h(b)$ are disjoint neighborhood of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in $Spec_d(\mathfrak{A})$, respectively. This shows that $h_m(a)$ and $h_m(b)$ are disjoint neighborhood of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in $\min_d(\mathfrak{A})$, respectively. □

Let $\mathfrak{A}$ be a residuated lattice. Consider the following relation $\imath = \{(\mathfrak{p}, \mathfrak{q}) \in X^2 \mid \mathfrak{p} \veebar \mathfrak{q} \neq A\}$ on $X = Spec(\mathfrak{A})$. Obviously, $\imath$ is reflexive and symmetric. Let $\bar{\imath}$ be the transitive closure of $\imath$.

Theorem 3.8. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is mp;*
(2) *for a given minimal prime filter $\mathfrak{m}$ of $\mathfrak{A}$, $\bar{\imath}(\mathfrak{m}) = h(\mathfrak{m})$.*

Proof. (1)⇒(2) Let $\mathfrak{m}$ be a minimal prime filter of $\mathfrak{A}$. Consider $\mathfrak{p} \in \bar{\imath}(\mathfrak{m})$. So, there exists a finite set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ of elements of $Spec(\mathfrak{A})$ with $n \geq 2$ such that $\mathfrak{p}_1 = \mathfrak{p}$, $\mathfrak{p}_n = \mathfrak{m}$, and $(\mathfrak{p}_i, \mathfrak{p}_{i+1}) \in \imath$, for all $1 \leq i \leq n-1$. If $n = 2$, then $\mathfrak{p} \veebar \mathfrak{m} \neq A$, and so $\mathfrak{m} \subseteq \mathfrak{p}$. Assume that $n > 2$. We have $\mathfrak{p}_{n-2} \veebar \mathfrak{p}_{n-1} \neq A$ and $\mathfrak{m} \subseteq \mathfrak{p}_{n-1}$. This verifies that $(\mathfrak{p}_{n-2}, \mathfrak{m}) \in \imath$. Hence, in the equivalency $(\mathfrak{p}, \mathfrak{m}) \in \bar{\imath}$, the number of the involved primes is reduced to $n-1$. Therefore by the induction hypothesis, $\mathfrak{m} \subseteq \mathfrak{p}$. This shows that $\bar{\imath}(\mathfrak{m}) \subseteq h(\mathfrak{m})$. The inverse inclusion is evident.

(2)⇒(1) It is evident. □

Let A_τ be a topological space, and E be an equivalence relation on A. In the following, by A_τ/E we mean the quotient of the space A_τ modulo E. By [14, p. 90], the quotient map $\pi : A_\tau \to A_\tau/E$ is continuous, and a mapping f of the quotient space A_τ/E to a topological space B_ζ is continuous if and only if the composition $f \circ \pi$ is continuous.

Corollary 3.2. *Let $\mathfrak{A}$ be a residuated lattice. $\mathfrak{A}$ is mp if and only if the map $\eta : \min_d(\mathfrak{A}) \to Spec_d(\mathfrak{A})/\bar{\imath}$, given by $\mathfrak{m} \rightsquigarrow \bar{\imath}(\mathfrak{m})$, is a homeomorphism.*

Proof. Let $\min_d(\mathfrak{A})$ is a Hausdorff space. It is evident that $Spec_d(\mathfrak{A})/\bar{\imath} = \{\bar{\imath}(\mathfrak{m}) \mid \mathfrak{m} \in \min(\mathfrak{A})\}$, and this implies that η is a surjection. The injectivity of η follows by Theorem 3.8, and the continuity of it follows by $\eta = \pi \circ i$, where i is the inclusion map. By Remark 3.3 and Theorem 3.6, it follows that $\eta^{-1} \circ \pi$ is a retraction, and this verifies the continuity of η^{-1}, see [14, Proposition 4.2.4]. This shows that η is a homeomorphism. Conversely, let $\eta : \min_d(\mathfrak{A}) \to Spec_d(\mathfrak{A})/\bar{\imath}$ be a homeomorphism. Obviously, $\eta^{-1} \circ \pi$ is a retraction, and so $\min_d(\mathfrak{A})$ is a Hausdorff space due to Theorem 3.6. □

Let $\mathfrak{A}$ be a residuated lattice. Consider the relation $\jmath = \{(\mathfrak{p}, \mathfrak{q}) \in X^2 \mid \dot{\mathfrak{p}} \curlyvee \dot{\mathfrak{q}} \neq A\}$ on $X = Spec(\mathfrak{A})$. Obviously, $\jmath$ is reflexive and symmetric. Let $\bar{\jmath}$ be the transitive closure of $\jmath$.

Remark 3.4. For prime filters $\mathfrak{p}$ and $\mathfrak{q}$ of a residuated lattice $\mathfrak{A}$. One can see that, using [32, Proposition 3.5], $\dot{\mathfrak{p}} \curlyvee \dot{\mathfrak{q}} = A$ if and only if $D(\mathfrak{p}) \veebar D(\mathfrak{q}) = A$.

Theorem 3.9. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) *$\mathfrak{A}$ is mp;*
(2) *for a given minimal prime filter $\mathfrak{m}$ of $\mathfrak{A}$, $\bar{\jmath}(\mathfrak{m}) = h(\mathfrak{m})$.*

Proof. (1)⇒(2) Let $\mathfrak{m}$ be a minimal prime filter of $\mathfrak{A}$. Consider $\mathfrak{p} \in \bar{\jmath}(\mathfrak{m})$. So there exists a finite set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of elements of $Spec(\mathfrak{A})$ with $n \geq 2$ such that $\mathfrak{p}_1 = P$, $\mathfrak{p}_n = \mathfrak{m}$, and $(\mathfrak{p}_i, \mathfrak{p}_{i+1}) \in \jmath$, for all $1 \leq i \leq n-1$. If $n = 2$, then $\dot{\mathfrak{p}} \curlyvee \dot{\mathfrak{m}} \neq A$, and so $\mathfrak{m} \subseteq \mathfrak{p}$ due to Proposition 2.3. Assume that $n > 2$. We have $\dot{\mathfrak{p}_{n-2}} \curlyvee \dot{\mathfrak{p}_{n-1}} \neq A$ and $\mathfrak{m} \subseteq \mathfrak{p}_{n-1}$. Using Zorn's lemma, it verifies that $\dot{\mathfrak{p}_{n-2}} \curlyvee \dot{\mathfrak{p}_{n-1}} \subseteq \mathfrak{c}$, for a maximal $\vee$-closed set of $\mathfrak{A}$. Applying Proposition 2.3 and the hypothesis, it shows that $\mathfrak{m} = \dot{\mathfrak{c}}$. This verifies that $(\mathfrak{p}_{n-2}, \mathfrak{m}) \in \jmath$. Hence, in the equivalency $(\mathfrak{p}, \mathfrak{m}) \in \bar{\jmath}$, the number of the involved primes is reduced to $n-1$. Therefore, by the induction hypothesis, $\mathfrak{m} \subseteq \mathfrak{p}$. This shows that $\bar{\jmath}(\mathfrak{m}) \subseteq h(\mathfrak{m})$. The inverse inclusion is evident.

(2)⇒(1) It is evident. □

The proof of the following corollary is analogous to the proof of Corollary 3.2, and so it is left to the reader.

Corollary 3.3. *Let $\mathfrak{A}$ be a residuated lattice. $\mathfrak{A}$ is mp if and only if the map $\eta : \min_d(\mathfrak{A}) \to Spec_d(\mathfrak{A})/\bar{\jmath}$, given by $\mathfrak{m} \rightsquigarrow \bar{\jmath}(\mathfrak{m})$, is a homeomorphism.*

4. The Pure Spectrum of an mp-Residuated Lattice

This section deals with the pure spectrum of an mp-residuated lattice. For the basic facts concerning pure filters of a residuated lattice, the reader is referred to [28].

For any filter F of a residuated lattice $\mathfrak{A}$, set $\sigma(F) = k\mathscr{G}h(F)$.

Proposition 4.1. ([28, Propositions 5.2 & 5.4]). *Let $\mathfrak{A}$ be a residuated lattice. The following assertions hold:*

(1) $\sigma(F) = \{a \in A \mid F \veebar a^{\perp} = A\}$, *for any filter* F *of* $\mathfrak{A}$;
(2) $F \subseteq G$ *implies* $\sigma(F) \subseteq \sigma(G)$, *for any filters* F *and* G *of* $\mathfrak{A}$;
(3) $\sigma(\mathfrak{m}) = D(\mathfrak{m})$ *for any maximal filter* $\mathfrak{m}$ *of* $\mathfrak{A}$;

Let $\mathfrak{A}$ be a residuated lattice. A filter F of $\mathfrak{A}$ is called *pure* provided that $\sigma(F) = F$. The set of pure filters of $\mathfrak{A}$ is denoted by $\sigma(\mathfrak{A})$. It is obvious that $\{1\}, A \in \sigma(\mathfrak{A})$.

Proposition 4.2. *Let $\mathfrak{A}$ be an mp-residuated lattice and F a filter of $\mathfrak{A}$. $\sigma(F)$ is a pure filter of $\mathfrak{A}$.*

Proof. Let $x \in \sigma(F)$. Applying Proposition 4.1 (1), it follows that $F \veebar x^{\perp} = A$. So $f \odot y = 0$, for some $f \in F$ and $y \in x^{\perp}$. By Proposition 3.1 there exists $a \in A$ such that $a \in x^{\perp}$ and $\neg a \in y^{\perp}$. This implies that $\neg a \in \sigma(\mathfrak{A})$, and so $x \in \sigma(\sigma(F))$. □

The following theorem gives some criteria for mp-residuated lattices by pure filters, inspired by the one obtained for bounded distributive lattices by [11, Theorem 2.11].

Theorem 4.1. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $\Omega(\mathfrak{A}) \subseteq \sigma(\mathfrak{A})$;
(3) $\gamma(\mathfrak{A}) \subseteq \sigma(\mathfrak{A})$.

Proof. (1)⇒(2) Let F be an ω-filter of $\mathfrak{A}$. So, $F = \omega(I)$, for some ideal I of $\ell(\mathfrak{A})$. Consider $x \in F$. So $x \in a^{\perp}$, for some $a \in I$. By Propositions 2.7 (1) and 3.1 (4), it follows that $A = x^{\perp} \veebar a^{\perp} \subseteq x^{\perp} \veebar F$.

(2)⇒(3) By Propositions 2.7 (2), it is evident.

(3)⇒(1) Let $x \vee y = 1$. So $x \in y^{\perp} = \sigma(y^{\perp})$ and this implies that $x^{\perp} \veebar y^{\perp} = A$. Hence, the result holds by Proposition 3.1. □

Remark 4.1. Al-Ezeh in [2, Theorem 1] showed that a unitary commutative ring is a PF ring if and only if any its annulet is a pure ideal. Thus, if we define PF-residuated lattices as those ones in which any coannulet is a pure filter, Theorem 4.1 verifies that the class of PF residuated lattices coincides with the class of mp-residuated lattices.

Lemma 4.1. *Let $\mathfrak{A}$ be a residuated lattice. Any two distinct elements of the set $Spec(\mathfrak{A}) \cap \sigma(\mathfrak{A})$ are comaximal.*

Proof. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two distinct elements of the set $Spec(\mathfrak{A}) \cap \sigma(\mathfrak{A})$. Consider $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. So, $\mathfrak{p}_1 \veebar x^{\perp} = A$ and $x^{\perp} \subseteq \mathfrak{p}_2$. □

Theorem 4.2. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*
(2) $D(\mathfrak{p})$ *is a pure filter of* $\mathfrak{A}$, *for any prime filter* $\mathfrak{p}$ *of* $\mathfrak{A}$;
(3) $D(\mathfrak{m})$ *is a pure filter of* $\mathfrak{A}$, *for any maximal filter* $\mathfrak{m}$ *of* $\mathfrak{A}$;
(4) $\min(\mathfrak{A}) \subseteq \sigma(\mathfrak{A})$.

Proof. (1)⇒(2) It follows by Theorem 4.1.

(2)⇒(3) It is evident.

(3)⇒(4) It follows, with a little bit of effort, by Proposition 2.7 (3).

(4)⇒(1) It follows by Proposition 3.1 and Lemma 4.1. □

Let $\mathfrak{A}$ be a residuated lattice. Recall [28] that a proper pure filter of $\mathfrak{A}$ is called *purely-maximal* provided that it is a maximal element in the set of proper and pure filters of $\mathfrak{A}$. The set of purely-maximal filters of $\mathfrak{A}$ shall be denoted by $\max(\sigma(\mathfrak{A}))$. A proper pure filter $\mathfrak{p}$ of $\mathfrak{A}$ is called *purely-prime* provided that $F_1 \cap F_2 \subseteq \mathfrak{p}$ implies $F_1 \subseteq \mathfrak{p}$ or $F_2 \subseteq \mathfrak{p}$, for any $F_1, F_2 \in \sigma(\mathfrak{A})$. The set of all purely-prime filters of $\mathfrak{A}$ shall be denoted by $Spp(\mathfrak{A})$. It is obvious that $\max(\sigma(\mathfrak{A})) \subseteq Spp(\mathfrak{A})$. Zorn's lemma ensures that any proper pure filter is contained in a purely-maximal filter, and so in a purely-prime filter.

Theorem 4.3. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*

(1) $\mathfrak{A}$ *is mp;*

(2) $\min(\mathfrak{A}) = \max(\sigma(\mathfrak{A}))$.

Proof. (1)⇒(2) Let $\mathfrak{m}$ be a minimal prime filter of $\mathfrak{A}$. By Theorem 4.2. it follows that $\mathfrak{m}$ is a pure filter of $\mathfrak{A}$. Thus there exists $\mathfrak{n} \in \max(\sigma(\mathfrak{A}))$ containing $\mathfrak{m}$. Let $a \in \mathfrak{n}$. So there exists $b \in a^{\perp}$ such that $\neg b \in \mathfrak{n}$. This implies that $b \notin \mathfrak{m}$, and so $a \in \mathfrak{m}$. Conversely, let $\mathfrak{p}$ be a purely-maximal filter of $\mathfrak{A}$. So $\mathfrak{p} \subseteq \mathfrak{n}$, for some $\mathfrak{n} \in \max(\mathfrak{A})$. Using Theorem 3.1, Proposition 4.1 ((2) & (3)), and Theorem 4.2, it shows that $\mathfrak{p} = D(\mathfrak{n}) \in \min(\mathfrak{A})$.

(2)⇒(1) It is evident by Theorem 4.2. □

The following result generalized and improved [4, Theorem 1.8] to residuated lattices.

Proposition 4.3. *Let $\mathfrak{A}$ be an mp-residuated lattice and F a proper pure filter of $\mathfrak{A}$. We have*

$$F = k(\min(\mathfrak{A}) \cap h(F)).$$

Proof. By Theorem 4.3, $\min(\mathfrak{A}) \cap h(F) \neq \emptyset$. Consider $a \in k(\min(\mathfrak{A}) \cap h(F))$. Assume that $a^{\perp} \veebar F$ is proper. Thus, $a^{\perp} \veebar F \subseteq \mathfrak{n}$, for some maximal filter $\mathfrak{n}$ of $\mathfrak{A}$. Let $\mathfrak{m}$ be a minimal prime filter of $\mathfrak{A}$ contained in $\mathfrak{n}$. This implies that $F \subseteq \mathfrak{m}$, and so $\neg b \in \mathfrak{n}$, for some $b \in a^{\perp}$ which is a contradiction. □

The pure ideals of a PF ring are characterized in [3, Theorems 2.4 and 2.5]. These results have been improved and generalized to residuated lattices in Theorem 4.4 and Proposition 4.6.

Theorem 4.4. *Let $\mathfrak{A}$ be an mp-residuated lattice. The pure filters of $\mathfrak{A}$ are precisely of the form $\bigcap_{\mathfrak{m} \in \min(\mathfrak{A}) \cap \mathscr{C}} \mathfrak{m}$, where $\mathscr{C}$ runs over closed subsets of $Spec_d(\mathfrak{A})$.*

Proof. Let $a \in G := \bigcap\{\mathfrak{m} \mid \mathfrak{m} \in \min(\mathfrak{A}) \cap \mathscr{C}\}$, in which $\mathscr{C}$ is a closed subset of $Spec_d(\mathfrak{A})$. So, for any $\mathfrak{m} \in \min(\mathfrak{A}) \cap \mathscr{C}$, we have $\mathfrak{m} \veebar a^{\perp} = A$. By absurdum, assume that $G \veebar a^{\perp} \neq A$. So, $G \veebar a^{\perp}$ is contained in a maximal filter $\mathfrak{n}$. Let $\mathfrak{o}$ be a minimal prime filter of $\mathfrak{A}$ contained in $\mathfrak{m}$. Obviously, $\mathfrak{o} \notin \mathcal{C}$. So for any $\mathfrak{m} \in \min(\mathfrak{A}) \cap \mathcal{C}$, there exist $x_{\mathfrak{m}} \in \mathfrak{m}$ and $y_{\mathfrak{m}} \in \mathfrak{o}$ such that $x_{\mathfrak{m}} \odot y_{\mathfrak{m}} = 0$. Since $\mathcal{C}$ is stable under the generalization, so $\mathcal{C} \subseteq \bigcup_{\mathfrak{m} \in \min(\mathfrak{A}) \cap \mathcal{C}} h(x_{\mathfrak{m}})$. By Proposition 2.6, it follows that $\mathcal{C}$ is compact. So there exist a finite number $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \min(\mathfrak{A}) \cap \mathcal{C}$ such that $\mathcal{C} \subseteq \bigcup_{i=1}^{n} h(x_{\mathfrak{m}_i})$. Set $x = \bigvee_{i=1}^{n} x_{\mathfrak{m}_i}$ and $y = \bigodot_{i=1}^{n} y_{\mathfrak{m}_i}$. Routinely, one can see that $0 = x \odot y \in G \veebar \mathfrak{o}$, which is a contradiction. The converse follows by Proposition 4.3. ☐

Let $\mathfrak{A}$ be a residuated lattice. For any filter F of $\mathfrak{A}$, we set

$$\rho(F) = \underline{\bigvee}\{G \in \sigma(\mathfrak{A}) \mid G \subseteq F\},$$

and it is called *the pure part of F*. Definitely, the pure part of a filter is the largest pure filter contained in it.

Proposition 4.4. *Let $\mathfrak{A}$ be a residuated lattice. Then*

$$\bigcap\{\rho(\mathfrak{m}) \mid \mathfrak{m} \in \max(\mathfrak{A})\} = \{1\}.$$

Proof. It is an immediate consequence of [28, Corollary 4.19]. ☐

Proposition 4.5. *Let $\mathfrak{A}$ be an mp-residuated lattice and $a \in A$. Then $a^{\perp} \cap F_a = \{1\}$, where $F_a = \bigcap_{\mathfrak{m} \in \max(\mathfrak{A}) \cap h(a)} \rho(\mathfrak{m})$.*

Proof. With a little bit of effort, it follows by Theorem 4.1 and Proposition 4.4. ☐

Corollary 4.1. *If $\mathfrak{m}$ is a minimal prime filter of an mp-residuated lattice $\mathfrak{A}$, then $\mathfrak{m} = \underline{\bigvee}_{a \in \mathfrak{m}} F_a$.*

Proof. Let $a \in \mathfrak{m}$. So $b \in a^{\perp}$, for some $b \notin \mathfrak{m}$. This implies that $a \in F_{\neg b}$. The reverse inclusion is deduced from Corollary 4.5. ☐

Proposition 4.6. *Let $\mathfrak{A}$ be an mp-residuated lattice. The pure filters of $\mathfrak{A}$ are precisely of the form $\bigcap_{\mathfrak{m} \in \max(\mathfrak{A}) \cap h(F)} \rho(\mathfrak{m})$, where F is a filter of $\mathfrak{A}$.*

Proof. Let $\mathcal{C} = \{P \in Spec(\mathfrak{A}) \mid P \cap \neg F = \emptyset\}$. One can see that $\max(\mathfrak{A}) \cap h(F) = \min(\mathfrak{A}) \cap \mathcal{C}$. This establishes the result due to Remark 2.2 and Theorem 4.4. ☐

H. Al-Ezeh [3, Theorem 3.5] proved that every purely prime ideal of a PF ring is purely maximal. Now we provide an alternative proof to the following interesting result.

Theorem 4.5. *Let $\mathfrak{A}$ be an mp residuated lattice. Then*

$$Spp(\mathfrak{A}) \subseteq \max(\sigma(\mathfrak{A})).$$

Proof. Let $\mathfrak{p}$ be a purely prime filter of $\mathfrak{A}$. So $\mathfrak{p} \subseteq \mathfrak{m}$, for some $\mathfrak{m} \in \max(\sigma(\mathfrak{A}))$. By Theorem 4.3 we have $\mathfrak{m} \in \min(\mathfrak{A})$. Let $a \in \mathfrak{m}$. By Proposition 2.3 (3) we have $a^{\perp} \nsubseteq \mathfrak{m}$. By Proposition 4.5, it follows that $a^{\perp} \cap F_a \subseteq P$. By Theorem 4.1 and Proposition 4.6, respectively, it follows that $a^{\perp}$ and F_a are pure filters. This implies that $F_a \subseteq \mathfrak{p}$. Hence, by Corollary 4.1, it follows that $\mathfrak{m} = \underline{\bigvee}_{a \in \mathfrak{m}} F_a \subseteq \mathfrak{p}$. ☐

The following theorem is a direct consequence of Theorems 4.2, 4.3 and 4.5. So its proof is left to the reader.

Theorem 4.6. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*
(1) $\mathfrak{A}$ *is mp;*
(2) $\min(\mathfrak{A}) = Spp(\mathfrak{A})$.

For each pure filter F of $\mathfrak{A}$ we set $d_p(F) = \{P \in Spp(\mathfrak{A}) \mid F \nsubseteq P\}$. $Spp(\mathfrak{A})$ can be topologized by taking the set $\{d_p(F) \mid F \in \sigma(\mathfrak{A})\}$ as the open sets. The set $Spp(\mathfrak{A})$ endowed with this topology is called the *pure spectrum* of $\mathfrak{A}$. It is obvious that the closed subsets of the pure spectrum are precisely of the form $h_p(F) = \{P \in Spp(\mathfrak{A}) \mid F \subseteq P\}$, where F runs over pure filters of $\mathfrak{A}$.

The next result, which can be compared with Proposition 2.6, shows that the pure spectrum of a residuated lattice is a compact space.

Theorem 4.7. ([28, Theorem 4.22]). *Let $\mathfrak{A}$ be a residuated lattice. $Spp(\mathfrak{A})$ is a compact space.*

The next theorem gives a criterion for a residuated lattice to be mp, inspired by the one obtained for unitary commutative rings by [35, Theorem 5.6].

Theorem 4.8. *Let $\mathfrak{A}$ be a residuated lattice. The following assertions are equivalent:*
(1) $\mathfrak{A}$ *is mp;*
(2) *the identity map $\iota : Spp(\mathfrak{A}) \to \min_d(\mathfrak{A})$ is a homeomorphism.*

Proof. (1)⇒(2) Consider the identity map $\iota : Spp(\mathfrak{A}) \to \min(\mathfrak{A})$. Using Theorem 4.3, it follows that ι is a well-defined bijection. One can see that $\min(\mathfrak{A}) \cap h(a) = d_p(a^{\perp})$, for any $a \in A$, which implies that ι is continuous. By Theorems 3.4 and 4.7, it follows that $\min_d(\mathfrak{A})$ is Hausdorff, and $Spp(\mathfrak{A})$ is compact, respectively. So, the result holds due to [14, Theorem 3.1.13].

(2)⇒(1) It is evident by Theorem 4.3. □

Using Theorem 4.7, the pure spectrum of a residuated lattice is compact (not necessarily Hausdorff). The following result verifies that the pure spectrum of an mp-residuated lattice is Hausdorff.

Corollary 4.2. *Let $\mathfrak{A}$ be an mp-residuated lattice. $Spp(\mathfrak{A})$ is a Hausdorff space.*

Proof. It is an immediate consequence of Theorems 3.4 and 4.8. □

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BI-PERIODIC HYPER-FIBONACCI NUMBERS

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Abstract. In the present paper, we introduce and study a new generalization of hyper-Fibonacci numbers, called the bi-periodic hyper-Fibonacci numbers. Furthermore, we give a combinatorial interpretation using the weighted tilings approach and prove several identities relating these numbers. Moreover, we derive their generating function and new identities for the classical hyper-Fibonacci numbers.

1. Introduction

The Fibonacci numbers F_n are defined, as usual, by the recurrence relation

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2.$$

The hyper-Fibonacci numbers denoted $F_n^{(r)}$, are introduced by Dil and Mezö [10], for $n, r \in \mathbb{N} \cup \{0\}$, as entries of an infinite matrix arranged such that $F_n^{(r)}$ is the entry of the rth row and nth column, satisfying

$$F_n^{(0)} = F_n,\ F_0^{(r)} = 0 \quad \text{and} \quad F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}, \quad \text{for } n, r \geq 1. \tag{1.1}$$

The sum of the first $n+1$ elements of row $r-1$ is expressed by $F_n^{(r)}$, i.e.,

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}. \tag{1.2}$$

They satisfy many interesting number theoretical and combinatorial properties, see [9]. Belbachir and Belkhir [3] provided a combinatorial interpretation of the hyper-Fibonacci numbers in terms of linear tilings and gave some combinatorial identities.

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They also defined bivariate hyper-Fibonacci polynomials in [4], as

$$F_n^{(r)}(x,y) = xF_{n-1}^{(r)}(x,y) + yF_n^{(r-1)}(x,y), \qquad \text{for } n, r \geq 1, \tag{1.3}$$

with initial conditions $F_n^{(0)}(x,y) = F_n(x,y)$, $F_0^{(r)}(x,y) = 0$, where x, y are real parameters and $F_n(x,y)$ is the nth bivariate Fibonacci polynomial, defined by (see [1,5])

$$F_0(x,y) = 0,\ F_1(x,y) = 1 \quad \text{and} \quad F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y).$$

The bivariate hyper-Fibonacci polynomials are given by the following explicit formula

$$F_{n+1}^{(r)}(x,y) = \sum_{k=r}^{\lfloor n/2 \rfloor + r} \binom{n+2r-k}{k} x^{n+2r-2k} y^k. \tag{1.4}$$

The associated generating function is given as follows

$$\sum_{n \geq 0} F_n^{(r)}(x,y) z^n = \frac{y^r z}{(1 - xz - yz^2)(1 - xz)^r}. \tag{1.5}$$

For $y = 1$, we denote $F_n(x,y)$ by $F_n(x)$.

Edson and Yayenie [12] introduced a new generalization for the Fibonacci sequence, called as bi-periodic Fibonacci sequence, that depends on two real parameters a and b, defined for $n \geqslant 2$, as follows

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}, \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \tag{1.6}$$

with initial values $q_0 = 0$ and $q_1 = 1$. These sequences are found in the study of continued fraction expansion of the quadratic irrational numbers and combinatorics on words or dynamical system theory [18]. Some well-known sequences, such as the Fibonacci sequence, the Pell sequence and the k-Fibonacci sequence for some positive integer k, are special cases of this sequence. For more results related to this sequence, see [8, 11–18]

The generating function of q_n is given by

$$\sum_{n \geq 0} q_n z^n = \frac{z\,(1 + az - z^2)}{1 - (ab+2)z^2 + z^4}. \tag{1.7}$$

Yayenie [18] gave an explicit formula of bi-periodic Fibonacci numbers, as

$$q_{n+1} = a^{\xi(n)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (ab)^{\lfloor n/2 \rfloor - k}, \tag{1.8}$$

where $\xi(n) = n - 2\lfloor n/2 \rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

In this paper, we define a new generalization of hyper-Fibonacci numbers, which we will also call bi-periodic hyper-Fibonacci numbers. We give a combinatorial interpretation of these numbers using a weighted tilings approach and provide several combinatorial proofs of some identities. We also obtain new identities for the classical hyper-Fibonacci numbers. Moreover, by using the generating function of the bivariate hyper-Fibonacci polynomials, we establish the generating function of the bi-periodic hyper-Fibonacci sequence.

Definition 1.1. For any integers $n, r \geq 1$ and nonzero real numbers a and b, the bi-periodic hyper-Fibonacci numbers, denoted by $q_n^{(r)}$, are defined by

$$q_n^{(r)} = \sum_{k=0}^{n} a^{\xi(k)\xi(n+1)} b^{\xi(k+1)\xi(n)} (ab)^{\lfloor (n-k)/2 \rfloor} q_k^{(r-1)}, \tag{1.9}$$

with initial values $q_0^{(r)} = 0$ and $q_n^{(0)} = q_n$, where q_n is the nth bi-periodic Fibonacci number.

The first few generations are as follows in Table 1.

Table 1. Sequence of bi-periodic hyper-Fibonacci numbers in the first few generations

n	0	1	2	3	4	5	6
$q_n^{(0)}$	0	1	a	$ab+1$	a^2b+2a	$a^2b^2+3ab+1$	$a^3b^2+4a^2b+3a$
$q_n^{(1)}$	0	1	$2a$	$3ab+1$	$4a^2b+3a$	$5a^2b^2+6ab+1$	$6a^3b^2+10a^2b+4a$
$q_n^{(2)}$	0	1	$3a$	$6ab+1$	$10a^2b+4a$	$15a^2b^2+10ab+1$	$21a^3b^2+20a^2b+5a$
$q_n^{(3)}$	0	1	$4a$	$10ab+1$	$20a^2b+5a$	$35a^2b^2+15ab+1$	$56a^3b^2+35a^2b+6a$
$q_n^{(4)}$	0	1	$5a$	$15ab+1$	$35a^2b+6a$	$70a^2b^2+21ab+1$	$126a^3b^2+56a^2b+7a$

From the definition, we have the following recurrence relation:

$$q_n^{(r)} = \begin{cases} aq_{n-1}^{(r)} + q_n^{(r-1)}, & \text{if } n \text{ is even,} \\ bq_{n-1}^{(r)} + q_n^{(r-1)}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.10}$$

Note that, for $a = b = 1$, we obtain the classical hyper-Fibonacci sequence (1.1).

2. Combinatorial Identities

The Fibonacci numbers can be interpreted as the number of ways to tile a board of length n (i.e., an n-board) with cells numbered 1 to n from left to right using only squares and dominoes; see [6, 7]. We expand the results to bi-periodic Fibonacci numbers using weighted tilings. We assign a weight to each square in a tiling based on its position. It is assigned a weight a if it is in an odd position and a weight b if it is in an even position. The weight of a tiling of an n-board is defined as the product of the weights of its individual tiles. The sum of all possible weighted tilings is given by q_{n+1}. Furthermore, the total of all possible weighted tilings of an $(n + 2r)$-board with at least r dominoes is given by the bi-periodic hyper-Fibonacci numbers $q_{n+1}^{(r)}$, as shown in Theorem 2.1.

For example, Figure 1 shows the tilings and the sum of their weights of a 5-board. We have $q_6^{(0)} = q_6 = a^3b^2 + 4a^2b + 3a$.

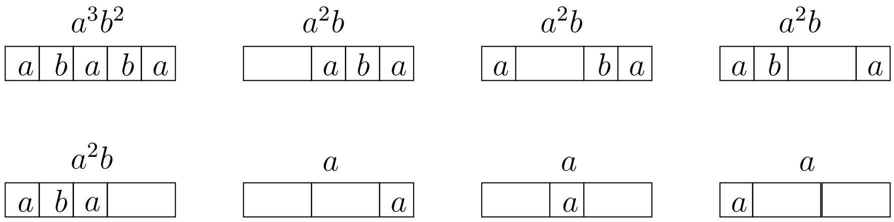

FIGURE 1. Tilings of a 5-board

Figure 2 shows the tilings and the sum of their weights of a 6-board with at least 2 dominoes, there are $q_3^{(2)} = 6ab + 1$ dispositions.

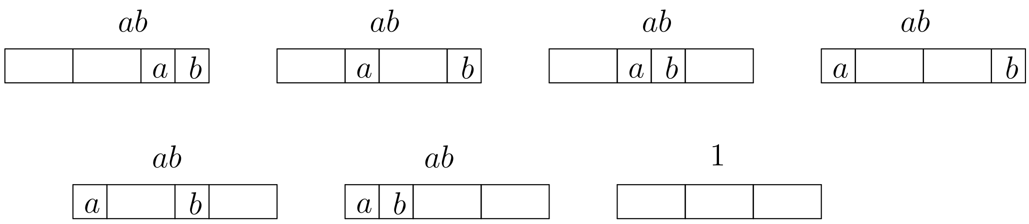

FIGURE 2. Tilings of a 6-board with at least 2 dominos

Therefore, we have the following results.

Theorem 2.1. *For $n, r \geq 0$, $q_{n+1}^{(r)}$ gives the weight of all tilings of an $(n + 2r)$-board having at least r dominoes.*

Proof. Given $(n+2r)$-board. If it ends with a square, then there are $bq_n^{(r)}$ ways to tile the $(n + 2r - 1)$-board for n even and $aq_n^{(r)}$ for n odd. If it ends with a domino, then there are $q_{n+1}^{(r-1)}$ ways to tile the $(n + 2(r - 1))$-board. When $n = 0$, there is one way to tile a $2r$-board with at least r dominoes and there are q_{n+1} ways to tile a n-board with at least 0 dominoes. There is no way to tile an $(n + 2r)$-board with at least r dominoes for $n < 0$. □

Let $f(n, k)$ be the number of weighted tilings having n tiles and exactly k dominoes. Then

$$f(n, k) = a^{\xi(n+k)} b^{\xi(n+k+1)} f(n - 1, k) + f(n - 1, k - 1).$$

In fact, if the $(n+k)$-board ends in a square there are $a^{\xi(n+k)} b^{\xi(n+k+1)} f(n-1, k)$ ways to tile the board. If it ends with a domino, then there are $f(n - 1, k - 1)$ ways.

Lemma 2.1. *The number of weighted tilings having n tiles and exactly k dominoes is*

$$a^{\xi(n+k)} \binom{n}{k} (ab)^{\lfloor (n-k)/2 \rfloor}.$$

Proof. Let $g(n,k)=a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor}$. Then

$$a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor}=a^{\xi(n+k)}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)(ab)^{\lfloor (n-k)/2\rfloor}.$$

Using $\lfloor (n-k)/2\rfloor=\lfloor (n-k-1)/2\rfloor+\xi(n+k+1)$, we get

$$\begin{aligned}a^{\xi(n+k)}\binom{n}{k}(ab)^{\lfloor (n-k)/2\rfloor}&=a^{\xi(n+k)}(ab)^{\xi(n+k+1)}\binom{n-1}{k}(ab)^{\lfloor (n-k-1)/2\rfloor}\\&\quad+a^{\xi(n+k)}\binom{n-1}{k-1}(ab)^{\lfloor (n-k)/2\rfloor}\\&=a^{\xi(n+k)}b^{\xi(n+k+1)}g(n-1,k)+g(n-1,k-1).\end{aligned}$$

Since $g(n,k)$ satisfies the same recurrence of $f(n,k)$ and the same initial conditions, we get result. □

In the following theorems, we establish an explicit formula for the bi-periodic hyper-Fibonacci sequence.

Theorem 2.2. *For $n,r\geq 0$, we have*

$$q_{n+1}^{(r)}=a^{\xi(n)}\sum_{k=r}^{\lfloor n/2\rfloor+r}\binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor+r-k}. \tag{2.1}$$

Proof. From Theorem 2.1, $q_{n+1}^{(r)}$ counts the number of ways to tile an $(n+2r)$-board with at least r dominoes. On the other hand, using Lemma 2.1, the possible tilings with exactly k dominoes contains $n+2r-2k$ squares and $n+2r-k$ tiles, have cardinality $a^{\xi(n)}\binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor+r-k}$. Since it contains at least r dominoes, the sum over $k\geq r$ gives the identity. □

Now, we establish a double-summation formula for even-numbered bi-periodic hyper-Fibonacci numbers $q_{2n+2}^{(r)}$.

Theorem 2.3. *For $n,r\geq 0$, we have*

$$q_{2n+2}^{(r)}=a\sum_{k=r}^{n+r}\sum_{j=0}^{k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor}. \tag{2.2}$$

Proof. Consider an $(n+2r+1)$-board. Since the length of the board is odd, there are an odd number of squares such that we have at least one in each tiling. Suppose there are i dominoes to the left of its median square and j dominoes to its right, whose total is at least r dominoes, i.e., $i+j\geq r$. The median square contributes an $a^{\xi(n+r-i-j+1)}b^{\xi(n+r-i-j)}$ to the weight (according to the position of the median square). Such tiling contains $2n+2r-2i-2j+1$ squares, so there are $n+r-i-j$ squares on each side of the median square. The left side gives $n+r-j$ tiles with i dominos. Hence, there are $a^{\xi(n+r-i-j)}\binom{n+r-j}{i}(ab)^{\lfloor (n+r-i-j)/2\rfloor}$ different ways. Similarly, we have $a^{\xi(n+r-i-j)}\binom{n+r-i}{j}(ab)^{\lfloor (n+r-i-j)/2\rfloor}$ different ways to tile the right side. Thus, the possible tilings have cardinality $a(ab)^{\xi(n+r-i-j)}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-i-j)/2\rfloor}$. Summing over $i+j\geq r$, we get

$$\begin{aligned} & a\sum_{r\leq i+j\leq n+r}(ab)^{\xi(n+r-i-j)}\binom{n+r-j}{i}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-i-j)/2\rfloor} \\ =& a\sum_{k=r}^{n+r}\sum_{i+j=k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{i}\binom{n+r-i}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor} \\ =& a\sum_{k=r}^{n+r}\sum_{j=0}^{k}(ab)^{\xi(n+r-k)}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}(ab)^{2\lfloor (n+r-k)/2\rfloor}. \end{aligned}$$ □

For $a=b=1$, we get the following identity.

Corollary 2.1. *For $n,r\geq 0$, the following identity holds*

$$F_{2n+2}^{(r)}=\sum_{k=r}^{n+r}\sum_{j=0}^{k}\binom{n+r-j}{k-j}\binom{n+r-k+j}{j}. \tag{2.3}$$

From the explicit formulas (1.8) and (2.1), we state the bi-periodic hyper-Fibonacci sequence in terms of the bi-periodic Fibonacci sequence and binomial sum.

Theorem 2.4. *Let $n\geq 0$ and $r\geq 1$ be integers, then we have*

$$q_{n+1}^{(r)}=q_{n+1+2r}-a^{\xi(n)}\sum_{k=0}^{r-1}\binom{n+2r-k}{k}(ab)^{\lfloor n/2\rfloor+r-k}. \tag{2.4}$$

Note that, if we take $a=b=1$, we get the following identity, see [3],

$$F_{n+1}^{(r)}=F_{n+1+2r}-\sum_{k=0}^{r-1}\binom{n+2r-k}{k}.$$

Theorem 2.5. *For $n,r\geq 1$, we have*

$$q_{n+1}^{(r)}=q_{n-1}+\sum_{k=0}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}. \tag{2.5}$$

Proof. There exists $q_{n+1}^{(r)}$ ways to tile a board of length $n+2r$ containing at least r dominoes. Consider the number of dominoes at the end of each tiling. If tiling ends in at least r dominoes, then the final r dominoes cover cells $n+1$ through $n+2r$, while the remaining tilings can be done in q_{n+1} ways. On the other hand, if tilings ends in exactly $r-k$ dominoes for some $1\leq k\leq r$, preceded by a square at position $n+2k$ and contribute $a^{\xi(n)}b^{\xi(n+1)}$ to the weight, then the remaining $(n-1+2k)$-board can be tiled with at least k dominoes in $q_n^{(k)}$ ways. The result follows from the sum of over k, i.e.,

$$q_{n+1}^{(r)}=q_{n+1}+\sum_{k=1}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}=q_{n-1}+\sum_{k=0}^{r}a^{\xi(n)}b^{\xi(n+1)}q_n^{(k)}.$$

□

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity.

Corollary 2.2. *For $n, r \geq 1$, we have*

$$F_{n+1}^{(r)}(x) = F_{n-1}(x) + \sum_{k=0}^{r} xF_n^{(k)}(x). \tag{2.6}$$

For $a = b = 1$, we obtain the following identity, see [2],

$$F_{n+1}^{(r)} = F_{n-1} + \sum_{k=0}^{r} F_n^{(k)}.$$

In the following theorem, we give the recurrence relation of the bi-periodic hyper-Fibonacci sequence.

Theorem 2.6. *For $n \geq 0$ and $r \geq 2$, we have*

$$q_{n+2}^{(r)} = abq_n^{(r)} + 2q_{n+2}^{(r-1)} - q_{n+2}^{(r-2)}. \tag{2.7}$$

Proof. We will construct a 3-to-1 correspondence between the following two sets.

- The set of all tiled $(n + 2r - 1)$-boards with at least r dominoes. There are $q_n^{(r)}$ ways.
- The set of all tiled $(n+2r+1)$-boards with at least r dominoes and $(n+2r-3)$-boards with at least $r - 1$ dominoes. There are $q_{n+2}^{(r)} + q_n^{(r-1)}$ ways.

Consider an arbitrary tiling T of length $n + 2r - 1$, we can do the following.

1. Add two squares at the end of T to get an $(n + 2r + 1)$-board ending in a square. Then there are $abq_n^{(r)}$ ways.
2. Add a domino at the end of T to get an $(n+2r+1)$-board ending in a domino. Then there are $q_{n+2}^{(r-1)}$ ways.
3. Condition on whether T ends in a square or a domino.
   i. Suppose T ends in a square, then insert a domino immediately to the left of the square to creates $(n + 2r + 1)$-board ending in a square. Then there are $a^{\xi(n+1)}b^{\xi(n)}q_{n+1}^{(r-1)}$ ways to do it.
   ii. Suppose T ends in a domino, we remove the domino to get an $(n+2r-2)$-board. Then there are $q_n^{(r-1)}$ ways.

So, we conclude that

$$\begin{aligned} q_{n+2}^{(r)} + q_n^{(r-1)} &= abq_n^{(r)} + q_{n+2}^{(r-1)} + a^{\xi(n+1)}b^{\xi(n)}q_{n+1}^{(r-1)} + q_n^{(r-1)} \\ &= abq_n^{(r)} + 2q_{n+2}^{(r-1)} + q_n^{(r-1)} - q_{n+2}^{(r-2)}. \end{aligned}$$

Therefore

$$q_{n+2}^{(r)} = abq_n^{(r)} + 2q_{n+2}^{(r-1)} - q_{n+2}^{(r-2)}.$$ □

Note that, if we take $a = b = 1$, we get the following hyper-Fibonacci identity.

Corollary 2.3. *For $n \geq 0$ and $r \geq 2$, we have*

$$F_{n+2}^{(r)} = F_n^{(r)} + 2F_{n+2}^{(r-1)} - F_{n+2}^{(r-2)}. \tag{2.8}$$

The following theorem gives the nonhomogeneous recurrence relation for the bi-periodic hyper-Fibonacci sequence.

Theorem 2.7. *For $n, r \geq 1$, we have*

$$q_{n+1}^{(r)} = a^{\xi(n)} b^{\xi(n+1)} q_n^{(r)} + q_{n-1}^{(r)} + a^{\xi(n)} (ab)^{\lfloor n/2 \rfloor} \binom{n+r-1}{r-1}. \tag{2.9}$$

Proof. There are $q_{n+1}^{(r)}$ ways to tile a $(n+2r)$-board with at least r dominoes. We consider the last tile in a tiling, which can be either a square or a domino. If the board ends in a square, then there are $bq_n^{(r)}$ ways to tile $(n+2r-1)$-boards with at least r dominoes for n even and $aq_n^{(r)}$ ways to do it for n odd. If the board ends in a domino, we separate the tilings into two disjoint sets A and B. The set A with exactly r dominoes and the set B whose contain tilings with at least $r+1$ dominoes. Having in mind that one domino is fixed, the tilings in the set A has $n+r-1$ tiles with exactly $r-1$ dominoes, then by Lemma 2.1, we have $|A| = a^{\xi(n)}(ab)^{\lfloor n/2 \rfloor}\binom{n+r-1}{r-1}$. The tilings in the set B are equivalent to the tilings of an $(n+2r-2)$-boards with at least r dominoes, i.e., $|B| = q_{n-1}^{(r)}$. Therefore,

$$q_{n+1}^{(r)} = a^{\xi(n)} b^{\xi(n+1)} q_n^{(r)} + |A| + |B|.$$

□

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity, see [4],

$$F_{n+1}^{(r)}(x) = xF_n^{(r)}(x) + F_{n-1}^{(r)}(x) + x^n \binom{n+r-1}{r-1}.$$

Theorem 2.8. *For $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq r$, we have*

$$q_{n+m}^{(r)} = \sum_{k=0}^{m} a^{\xi(n+m+1)\xi(n+k)} b^{\xi(n+m)\xi(n+k+1)} \binom{m}{k} (ab)^{\lfloor (m-k)/2 \rfloor} q_{n+k}^{(r-k)}. \tag{2.10}$$

Proof. There exists $q_{n+m}^{(r)}$ ways to tile a board of length $(n+m+2r-1)$ containing at least r dominoes. Consider the number of dominoes among the first m tiles. The k dominoes can be placed among the first m tiles in $\binom{m}{k}$ ways and the remaining tiles which consisting of squares, contribute $a^{\xi(n+m+1)\xi(n+k)} b^{\xi(n+m)\xi(n+k+1)} (ab)^{\lfloor (m-k)/2 \rfloor}$ to the weight. The remaining right board has a length of $n-1+2r-k$, with at least $r-k$ dominos that can be tiled in $q_{n+k}^{(r-k)}$ ways. Summing over all possible k completes the proof. □

Note that, if we take $a = b = x$ and $m = r$, we get the following hyper-Fibonacci identity, see [4],

$$F_{n+r}^{(r)} = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} F_{n+k}^{(r-k)}.$$

The bi-periodic hyper-Fibonacci sequence can be expressed in terms of the combinatorial sum of bi-periodic Fibonacci sequence.

Theorem 2.9. *For $n, r \geq 1$, we have*

$$q_n^{(r)} = \sum_{k=1}^{n} a^{\xi(n+1)\xi(k)} b^{\xi(n)\xi(k+1)} \binom{n+r-k-1}{r-1} (ab)^{\lfloor (n-k)/2 \rfloor} q_k. \tag{2.11}$$

Proof. The left-hand side of this equality counts the number of ways to tile a board of length $n + 2r - 1$ containing at least r dominoes.

The right-hand side is obtained by conditioning on the location of the rth domino. Suppose that the rth domino occupies cell k and $k+1$ $(1 \leq k \leq n)$ (from the right). The left part is a tiling of some section of length $k-1$ which can be done in q_k ways. The rigth part is a tiling of the remaining portion of length $n + 2r - 2 - k$ (i.e., cells $k+2$ through $n + 2r - 1$) with exactly $r - 1$ dominos, which can be done in a $a^{\xi(n+1)\xi(k)} b^{\xi(n)\xi(k+1)} \binom{n+r-k-1}{r-1} (ab)^{\lfloor (n-k)/2 \rfloor}$ ways (according to the parity of the numbers n and k). The result follows from considering the sum of all possible locations of the r^{th} domino. □

Note that, if we take $a = b = x$, we get the following hyper-Fibonacci identity, see [4],

$$F_n^{(r)}(x) = \sum_{k=1}^{n} x^{n-k} \binom{n+r-k-1}{r-1} F_k(x).$$

In the following theorem, we give the alternating binomial sum of the bi-periodic hyper-Fibonacci numbers.

Theorem 2.10. *For $r, m, n \in \mathbb{N} \cup \{0\}$ with $m \leq r$,we have*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} q_{n+m}^{(r-j)} = a^{\xi(n)\xi(m)} b^{\xi(n+1)\xi(m)} (ab)^{\lfloor m/2 \rfloor} q_n^{(r)}. \tag{2.12}$$

Proof. We proceed by induction on $m \leq r$. For $m = 1$ and $m = 2$, we get (1.10) and Theorem 2.6, respectively. Suppose that the result holds for all $i \leq m$. Then we can prove it for $m + 1$

$$\begin{aligned}
\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} q_{n+m+1}^{(r-j)} &= \sum_{j=0}^{m+1} (-1)^j \left(\binom{m}{j} + \binom{m}{j-1} \right) q_{n+m+1}^{(r-j)} \\
&= \sum_{j \geq 0} (-1)^j \binom{m}{j} q_{n+m+1}^{(r-j)} - \sum_{j \geq 0} (-1)^j \binom{m}{j} q_{n+m+1}^{(r-j-1)}.
\end{aligned}$$

From (1.10), we obtain

$$\begin{aligned}
\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} q_{n+m+1}^{(r-j)} &= \sum_{j \geq 0} (-1)^j \binom{m}{j} a^{\xi(n+m)} b^{\xi(n+m+1)} q_{n+m}^{(r-j)} \\
&= a^{\xi(n+m)} b^{\xi(n+m+1)} a^{\xi(n)\xi(m)} b^{\xi(n+1)\xi(m)} (ab)^{\lfloor m/2 \rfloor} q_n^{(r)}.
\end{aligned}$$

Using $\xi(n+m)=\xi(n)+\xi(m)-2\xi(n)\xi(m)$ and $\lfloor m/2\rfloor=\lfloor (m+1)/2\rfloor-\xi(m)$, we get

$$\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}q_{n+m+1}^{(r-j)}=a^{\xi(n)\xi(m+1)}b^{\xi(n+1)\xi(m+1)}(ab)^{\lfloor (m+1)/2\rfloor}q_n^{(r)}.$$

Therefore, the identity is valid for all $m\le r$. □

Note that, for $a=b=x$, we get the following result.

Corollary 2.4. *The following equality holds for any nonnegative integers $r\ge m$*

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}F_{n+m}^{(r-j)}=x^mF_n^{(r)}. \tag{2.13}$$

The bi-periodic Fibonacci sequence can be expressed in terms of the bi-periodic hyper-Fibonacci sequence.

Theorem 2.11. *For $r,m\in\mathbb{N}\cup\{0\}$, we have*

$$q_{m+1}=\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m)}b^{\xi(k)\xi(m+1)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}. \tag{2.14}$$

Proof. We proceed by induction on m. This is true for $m=0$. Suppose that the result holds for all $i\le m$. Then we can prove it for $m+1$. From (1.10), we get

$$\begin{aligned}q_{m+2}&=a^{\xi(m+1)}b^{\xi(m)}q_{m+1}+q_m\\&=a^{\xi(m+1)}b^{\xi(m)}\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m)}b^{\xi(k)\xi(m+1)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}\\&\quad+\sum_{k=0}^{m-1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m-k}^{(r)}.\end{aligned}$$

Using $\xi(m+1)=\xi(m-k+1)+\xi(k)\xi(m+1)-\xi(k)\xi(m)$ and $\xi(m)=\xi(m-k)+\xi(k)\xi(m)-\xi(k)\xi(m+1)$ we get $\xi(k)\xi(m)+\xi(m+1)=\xi(k)\xi(m+1)+\xi(m-k+1)$ and $\xi(k)\xi(m+1)+\xi(m)=\xi(k)\xi(m)+\xi(m-k)$. Therefore, we have

$$\begin{aligned}q_{m+2}&=\sum_{k=0}^{m}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)+\xi(m-k+1)}b^{\xi(k)\xi(m)+\xi(m-k)}(ab)^{\lfloor k/2\rfloor}q_{m+1-k}^{(r)}\\&\quad+\sum_{k=0}^{m-1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m-k}^{(r)}\\&=\sum_{k\ge0}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}\left(a^{\xi(m-k+1)}b^{\xi(m-k)}q_{m+1-k}^{(r)}+q_{m-k}^{(r)}\right)\\&=\sum_{k=0}^{m+1}\binom{r}{k}(-1)^ka^{\xi(k)\xi(m+1)}b^{\xi(k)\xi(m)}(ab)^{\lfloor k/2\rfloor}q_{m+2-k}^{(r)}.\end{aligned}$$ □

Note that, for $a=b=x$, we get the following result.

Corollary 2.5. *The following equality holds for any integers $r, m \geq 0$*

$$F_{m+1}(x) = \sum_{k=0}^{m} \binom{r}{k} (-1)^k x^k F_{m+1-k}^{(r)}(x). \tag{2.15}$$

3. Generating Function

We start by establishing the relationship between the bi-periodic hyper-Fibonacci sequence and the hyper-Fibonacci polynomials.

Lemma 3.1. *For $n, r \geq 0$, we have*

$$q_n^{(r)} = \frac{1}{2}\left(\left(1+\sqrt{\frac{a}{b}}\right) - (-1)^n \left(1-\sqrt{\frac{a}{b}}\right)\right) F_n^{(r)}\left(\sqrt{ab}\right). \tag{3.1}$$

Proof. Using (1.4), (2.1) and $\lfloor n/2 \rfloor = (n - \xi(n))/2$, we have

$$\begin{aligned}
q_n^{(r)} &= a^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2 \rfloor + r} \binom{n-1+2r-k}{k} (ab)^{(n-1-\xi(n-1))/2+r-k} \\
&= \left(\frac{a}{\sqrt{ab}}\right)^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2 \rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k} \\
&= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n-1)} \sum_{k=r}^{\lfloor (n-1)/2 \rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k} \\
&= \frac{\left(1+\sqrt{\frac{a}{b}}\right) - (-1)^n \left(1-\sqrt{\frac{a}{b}}\right)}{2} \sum_{k=r}^{\lfloor (n-1)/2 \rfloor + r} \binom{n-1+2r-k}{k} \left(\sqrt{ab}\right)^{n-1+2r-2k}. \square
\end{aligned}$$

Theorem 3.1. *The generating function of the bi-periodic hyper-Fibonacci sequence is given by*

$$\sum_{n\geq 0} q_n^{(r)} z^n = z \frac{\left(1+\sqrt{\frac{a}{b}}\right)\left(1+\sqrt{ab}z - z^2\right)\left(1+\sqrt{ab}z\right)^r + \left(1-\sqrt{\frac{a}{b}}\right)\left(1-\sqrt{ab}z - z^2\right)\left(1-\sqrt{ab}z\right)^r}{2\left(1-(ab+2)z^2+z^4\right)\left(1-abz^2\right)^r}.$$

Proof. Using Lemma 3.1 and (1.5), we get

$$\begin{aligned}
\sum_{n\geq 0} q_n^{(r)} z^n &= \frac{1}{2}\left(1+\sqrt{\frac{a}{b}}\right) \sum_{n\geq 0} F_n^{(r)}\left(\sqrt{ab}\right) z^n - \frac{1}{2}\left(1-\sqrt{\frac{a}{b}}\right) \sum_{n\geq 0} F_n^{(r)}\left(\sqrt{ab}\right) (-z)^n \\
&= \frac{1}{2}\left(1+\sqrt{\frac{a}{b}}\right) \frac{z}{\left(1-\sqrt{ab}z - z^2\right)\left(1-\sqrt{ab}z\right)^r} \\
&\quad - \frac{1}{2}\left(1-\sqrt{\frac{a}{b}}\right) \frac{-z}{\left(1+\sqrt{ab}z - z^2\right)\left(1+\sqrt{ab}z\right)^r},
\end{aligned}$$

which gives the desired result. $\square$

Note that, if we take $r = 0$, we obtain the generating function of the bi-periodic Fibonacci sequence (1.7). If we take $a = b = x$, we obtain the generating function of hyper-Fibonacci polynomials (1.5) with $y = 1$.

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NUMERICAL TREATMENT OF VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AND ITS CONVERGENCE ANALYSIS

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Abstract. This work deals with semi-analytical and numerical methods to solve a class of fractional order Volterra-Fredholm integro-differential equations. First, a semi-analytical method is proposed using the Chebyshev and Bernstein polynomials in the Adomian decomposition method. The uniqueness of the solution and convergence of the method are proved. Further, we solve the model using a numerical scheme comparing the L1 scheme for the fractional order derivative in combination with appropriate quadrature rules for the integral parts. Numerical experiments are done by the proposed methods to show their efficiency through a few tabular data and plots. Some comparisons with the existing results show that the proposed methods are highly productive and reliable.

1. Introduction

The considerable interest in integro-differential equations (IDEs) has mainly arisen due to its major applications in the theory of mechanical engineering, elasticity [29] and several others. The well-known mathematician Niels Henrik Abel obtained the famous integral equation of the first kind with kernel function $\mathcal{K}(x,t) = (x-t)^{-\mu}$, for $\mu = 1/2$ by solving the mechanical problem of *Tautochrone* as described in [23], which he then generalized it for $0 < \mu < 1$. The theory given by Abel in [23] further paved the way for researchers to look deep into the idea of fractional order integro-differential equations (FracIDEs). The wide application of FracIDEs for electromagnetic waves in dielectric media and unsteady aerodynamics have generated great interest in exploring more in

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this field. Several analytical and numerical techniques have been introduced to obtain the solutions of FracIDEs with singular and nonsingular kernels [12, 17]. Aghajani et al., established the existence of solutions for FracIDEs in [2]. The operational tau approximation method based on orthogonal polynomials was implemented on a class of FracIDEs by Vanani and Aminataei in [30]. Heydari and Hooshmandasl [14] used the Chebyshev wavelet method to solve the nonlinear FracIDEs on a large interval by converting the fractional differential and integral parts of the FracIDE to some operational matrices. Then, they obtained the solutions by solving a set of algebraic equations. Also, based on the Haar wavelet collocation method, Marasi and Derakhshan in [20] focused on finding a numerical method for solving the variable-order Caputo-Prabhakar FracIDEs. Higher order FracIDEs, such as the fourth-order FracIDEs, were solved by Amer et al. [5] using the Adomian decomposition method (ADM) and variational iteration method (VIM), where the solution was given by an infinite convergent series. Also, quite a few approximated techniques described in [9, 24] have been discussed in the past to solve the linear and nonlinear FracIDEs.

But all the model problems solved have considered the source term as a polynomial function which is comparatively easier to approximate. Thus, we propose a new modification of ADM for obtaining the solution of a class of FracIDE where the source function is not a polynomial one. The general way of ADM was first introduced by G. Adomian [1] to solve linear and nonlinear problems. Gradually, ADM was improvised using the Chebyshev [15], Legendre [19] and Bernstein polynomials [25]. These modified techniques are used to solve a class of ordinary and partial differential equations where the source function is exponential, trigonometric, or hyperbolic functions rather than the polynomial one. The approximation of functions by polynomials is extremely important as different scientific experiments rely on them, such as the study of statistics in population dynamics [28], temperatures, and also in the approximation theory [7]. Moreover, polynomials are the best mathematical techniques to approximate as they can be characterized, figured, separated, and incorporated effortlessly. Orthogonal polynomials such as the Chebyshev polynomials have been widely used in approximating functions in a wide variety of problems. These are the eigen functions of singular Sturm-Liouville problems. It is well known that these eigen functions allow the approximation of functions in $C^\infty[a, b]$, where the truncation error approaches zero faster than any function used in the approximation as described in [8]. Gottlieb in [10] described this effect as the "spectral accuracy". For more information, one may refer [6]. In this article, we are using the first kind orthogonal Chebyshev polynomials $\{\mathcal{T}_k\}_{k=0}^{\infty}$ given as

$$\sqrt{1-x^2}\mathcal{T}'(x) + \frac{k^2}{\sqrt{1-x^2}}\mathcal{T}_k(x) = 0.$$

Also, we have used the Bernstein polynomials for the modification of ADM. These polynomials approximate the function with a few terms in comparison to the approximations done using the Taylor series. They are utilized in the fields of connected arithmetic and material science as well as computer-assisted geometric outlines. They are also used in conjunction with other techniques like the Galerkin and collocation methods to solve some differential and integral problems.

Though researchers have widely studied the semi-analytical approaches for solving the mixed FracIDE, a few numerical solutions to such model problems have been studied in the past. Certain works are done, such as Ali et al. [3] used the hybrid orthonormal Bernstein and block-pulse functions wavelet method, Alkan and Hatipoglu [4] introduced the sinc-collocation method for solving the mixed FracIDE. One may also refer to the work done in [16, 21, 26]. Keeping this literature gap in mind, this article also proposes an efficient numerical scheme for finding the numerical solution of a class of Volterra-Fredholm FracIDE. The novel L1 scheme is applied for the fractional derivatives and the quadrature rule for the integral parts. The Composite trapezoidal scheme approximates the Volterra integral whereas the Fredholm integral is solved using the rectangular rule. The error analysis is briefly carried out. Computational data in the numerical section prove the robustness of the proposed numerical technique.

The paper is structured as follows. Section 2 outlines some of the definitions and properties, while the model problem is defined in Section 3. Section 4 describes the semi-analytical approximations along with the convergence analysis. The numerical approximation of the solution and the error analysis are described in Section 5. Some test examples are considered in Section 6, satisfying the theoretical findings and finally, Section 7 draws the concluding remarks.

2. Some Definitions and Properties

Definition 2.1. The Bernstein basis polynomials of degree m over the interval $[0,1]$ are defined as:

$$\mathcal{B}_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m.$$

Definition 2.2. The Riemann-Liouville fractional integral of order $\mu > 0$ for a function f is defined as:

$$\mathbb{I}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\tau)^{\mu-1} f(\tau) d\tau.$$

Definition 2.3. The Caputo derivative of order $\mu \in \mathbb{R}^+$ for a function f is defined as:

$$\mathbb{D}^\mu f(x) = \begin{cases} \frac{1}{\Gamma(n-\mu)} \int_0^x (x-\tau)^{n-\mu-1} f(\tau) d\tau, & n-1 < \mu < n, \\ \frac{d^n f(x)}{dx^n}, & \mu = n, n \in \mathbb{N}. \end{cases}$$

Some of the important properties of fractional derivatives and integrals are discussed as follows.

- $\mathbb{D}^\mu \mathbb{I}^\mu f(x) = f(x)$ and $\mathbb{I}^\mu \mathbb{D}^\mu f(x) = f(x) - f(0+)$, $0 < \mu \leq 1$, where $f(0+) = \lim_{h\to 0} f(0+h)$.

- Linearity property is sustained while defining the derivative in the Caputo sense, given as:
$$\mathbb{D}^{\mu}(\psi_1 m(x) + \psi_2 n(x)) = \psi_1 \mathbb{D}^{\mu} m(x) + \psi_2 \mathbb{D}^{\mu} n(x).$$
- For $0 < \mu \leq 1$ and $\vartheta \in \mathbb{R}$, $\mathbb{I}^{\mu} x^{\vartheta} = \frac{\Gamma(\vartheta+1)}{\Gamma(\mu+\vartheta+1)}$, $\mu > 0$, $\vartheta > -1$, $x > 0$.

3. Model Problem

Consider the Volterra-Fredholm FracIDE of order μ described as:

(3.1)
$$\begin{cases} \mathbb{D}^{\mu} z(x) + a(x) z(x) = f(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds, \\ z(0) = z_0, \end{cases}$$

where $0 < \mu \leq 1$ and $x \in [0,1]$. The fractional order derivative $\mathbb{D}^{\mu}$ is defined in the Caputo sense which is assumed to be invertible. The source function $f(x)$, $a(x)$, and the nonlinear operators denoted by $\mathcal{N}_1$ and $\mathcal{N}_2$ are continuous functions on $[0,1]$. $\mathcal{K}_1(x,s)$ and $\mathcal{K}_2(x,s)$ are smooth kernel functions defined on $[0,1] \times [0,1]$. The given initial condition is symbolized as z_0. Throughout this article, for a function $f(x)$, defined on $\Omega = [0,1]$, we define $||f(x)||_{\infty} = \max_{x \in \Omega} |f(x)|$ and C is defined as a generic constant, independent of μ.

4. Semi-Analytical Approximations

4.1. **Adomian decomposition method (ADM).** A brief description of the modified ADM is discussed in this section. Consider the FracIDE (3.1). The nonlinear operator is approximated using the Adomian polynomials $\mathcal{A}_n$. One may refer to [1,24] for the formula of $\mathcal{A}_n$. The solution z is represented as a series solution given by $z = \sum_{n=0}^{\infty} z_n$. Operating the inverse operator $\mathbb{I}^{\mu}$ on both sides of (3.1), we get

$$\begin{aligned} z(x) = & z(0^+) + \mathbb{I}^{\mu} \left[f(x) - a(x) z(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds \right. \\ & \left. + \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds \right]. \end{aligned} \tag{4.1}$$

Following the classical ADM, the recurrence relation for the solution of (4.1) is obtained as:

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(f(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(-a(x) z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(-a(x) z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right), \\ \vdots \end{cases}$$

Finally, we calculate the solution as $z = \sum_{n=0}^{\infty} z_n$, if $\sum_{n=0}^{\infty} z_n$ converges.

4.2. **ADM based on Chebyshev polynomials (ADM-CP).** In the usual algorithm of ADM, the approximation of f is made using Taylor's series expansion as $f(x) = \sum_{i=0}^{n} \frac{f^n(0)}{n!} x^n$ for an arbitrary $\mathbb{N}$. Hosseini [15] modified the ADM by expanding f using the Chebyshev polynomial approximation

$$f_C(x) \approx \sum_{i=0}^{n} C_n \mathcal{T}_n(x), \tag{4.2}$$

where $\mathcal{T}_n(x)$ is the first kind of orthogonal Chebyshev polynomial. Some of the Chebyshev polynomials are noted below:

$$\begin{cases} \mathcal{T}_0(x) = 1, \\ \mathcal{T}_1(x) = x, \\ \mathcal{T}_2(x) = 2x^2 - 1, \\ \mathcal{T}_3(x) = 4x^3 - 3x, \\ \vdots \\ \mathcal{T}_{n+1}(x) = 2x\mathcal{T}_n - \mathcal{T}_{n-1}, \quad n \geq 1. \end{cases} \tag{4.3}$$

Using (4.2) and (4.3), the following approximations for the solution of (3.1) are obtained as

(4.4)
$$\begin{cases} z_0 = \mathbb{I}^{\mu}(C_0\mathcal{T}_0(x) + C_1\mathcal{T}_1(x) + C_2\mathcal{T}_2(x) + \cdots + C_n\mathcal{T}_n(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(-a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(-a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right), \\ \vdots \end{cases}$$

This work will prove that the approximated solution obtained by (4.4) is more reliable than any other existing methods. In addition, one may also approximate using the following algorithm as described in [15]

$$\begin{cases} z_n = \mathbb{I}^{\mu}(C_n\mathcal{T}_n(x)) + z(0^+), \quad n = 0, \\ z_{n+1} = \mathbb{I}^{\mu}\left(C_{n+1}\mathcal{T}_{n+1}(x) - a(x)z_n(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_n(s))ds\right) \\ \qquad + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_n(s))ds\right), \quad n \geq 1. \end{cases} \tag{4.5}$$

Now, (4.2) can also be written in the standard form as $f(x) \approx p_0 + p_1x + p_2x^2 + \cdots + p_rx^r$,

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & -3 & 0 & 5 & \cdots \\ 0 & 0 & 2 & 0 & -8 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & -20 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix}$$

and

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(p_0) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(p_1 - a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds\right) + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(p_2 - a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds\right) + \mathbb{I}^{\mu}\left(\int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right). \end{cases}$$

Finally, using (4.5), the series solution $z(x)$ is obtained as follows:

$$z(x) = z_0(x) + z_1(x) + z_2(x) + \cdots. \tag{4.6}$$

4.3. **ADM based on Bernstein polynomials (ADM-BP).** In this segment, in order to improve the accuracy and reliability of ADM, the source function is expressed in the form of Bernstein polynomial approximation

$$f_B(x) = \sum_{i=0}^{n} D_i \mathcal{B}_i(x), \tag{4.7}$$

where $\mathcal{B}_i(x)$ are the Bernstein polynomials. Using (4.1) and (4.7), the approximated solution for FracIDE (3.1) is obtained as follows:

(4.8)

$$\begin{cases} z_0 = \mathbb{I}^{\mu}(D_0\mathcal{B}_0(x) + D_1\mathcal{B}_1(x) + D_2\mathcal{B}_2(x) + \cdots + D_n\mathcal{B}_n(x)) + z(0^+), \\ z_1 = \mathbb{I}^{\mu}\left(-a(x)z_0(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_0(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_0(s))ds\right), \\ z_2 = \mathbb{I}^{\mu}\left(-a(x)z_1(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{A}_1(z_1(s))ds + \int_0^1 \mathcal{K}_2(x,s)\mathcal{A}_2(z_1(s))ds\right), \\ \vdots \end{cases}$$

The Bernstein polynomials of degree m are obtained as $\mathcal{B}_m f(x) = \sum_{i=0}^{m} \binom{m}{i} x^i$ $(1-x)^{(m-i)} f\left(\frac{i}{m}\right)$. For each function $f : [0,1] \to$ R, we have $\lim_{m\to+\infty} \mathcal{B}_m f(x) = f(x)$. Finally using (4.8), the solution is obtained as

$$z(x) = z_0(x) + z_1(x) + z_2(x) + \cdots. \tag{4.9}$$

4.4. **Convergence analysis.**

4.4.1. *Existence and uniqueness of the solution.* In this segment, some of the hypothesies are stated, which will be further used in the analysis.

- (**H1**) Consider two Lipschitz constants $C_1, C_2 > 0$ such that $\mathcal{N}_1(z(x))$ and $\mathcal{N}_2(z(x))$ satisfy the Lipschitz conditions given as

$$\begin{cases} ||\mathcal{N}_1(z_1(x)) - \mathcal{N}_1(z_2(x))|| \le C_1||z_1 - z_2||, \\ ||\mathcal{N}_2(z_1(x)) - \mathcal{N}_2(z_2(x))|| \le C_2||z_1 - z_2||. \end{cases}$$

- (**H2**) Consider $\mathcal{Q} = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \le t \le x \le 1\}$ and $\mathcal{K}_1^*,\ \mathcal{K}_2^* \in C(\mathcal{Q}, \mathbb{R}^+)$, such that

$$\mathcal{K}_1^* = \sup_{x\in[0,1]} \int_0^x |\mathcal{K}_1(x,s)|dt < +\infty, \quad \mathcal{K}_2^* = \sup_{x\in[0,1]} \int_0^x |\mathcal{K}_2(x,s)|dt < +\infty.$$

Theorem 4.1. *Assuming that* (**H1**) *and* (**H2**) *hold, if* $\frac{\|a\|_\infty+(\mathcal{K}_1^*C_1+\mathcal{K}_2^*C_2)}{\Gamma(\mu+1)} < 1$, *then there exists a unique solution* $z(x) \in C[0,1]$ *for* (3.1).

Proof. The proof of the above theorem is well explained in Theorem 7 of [13]. Here we provide the outline of the proof in very few lines. Applying $\mathbb{I}^\mu$ on both sides of (3.1) we get, $z(x) = Tz(x)$, where

$$\begin{aligned}(Tz)(x) =& z_0 + \mathbb{I}^\mu \left[-a(x)z(x) + f(x) + \int_0^x \mathcal{K}_1(x,s)\mathcal{N}_1(z(s))ds \right.\\ &\left. - \int_0^1 \mathcal{K}_2(x,s)\mathcal{N}_2(z(s))ds)\right].\end{aligned}$$

Since, we know $z_1(x), z_2(x) \in C[0,1]$, so

$$\begin{aligned}|(Tz_1)(x) - (Tz_2)(x)| \leq& \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}|a(s)||z_1(s)-z_2(s)|ds\\ &+ \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\left[\int_0^t |\mathcal{K}_1(t,s)|\cdot|\mathcal{N}_1(z_1(s)) - \mathcal{N}_1(z_2(s))|ds\right.\\ &\left. + \int_0^1 |\mathcal{K}_2(t,s)|\cdot|\mathcal{N}_2(z_1(s)) - \mathcal{N}_2(z_2(s))|ds\right]dt\\ \leq& \frac{\|a\|_\infty}{\Gamma(\mu+1)}|z_1-z_2| + \frac{\mathcal{K}_1^*}{\Gamma(\mu+1)}\int_0^x (x-s)^{\mu-1}\\ &\times\left[\int_0^t |z_1(s)-z_2(s)|ds\right]dt + \frac{\mathcal{K}_2^*}{\Gamma(\mu+1)}\int_0^x (x-s)^{\mu-1}\\ &\times\left[\int_0^1 |z_1(s)-z_2(s)|ds\right]dt\\ \leq& \frac{\|a\|_\infty}{\Gamma(\mu+1)}|z_1-z_2| + \frac{\mathcal{K}_1^*C_1+\mathcal{K}_2^*C_2}{\Gamma(\mu+1)}|z_1-z_2|.\end{aligned}$$

As $\frac{\|a\|_\infty+(\mathcal{K}_1^*C_1+\mathcal{K}_2^*C_2)}{\Gamma(\mu+1)} < 1$, we have, $||T(z_1(x)) - T(z_2(x))|| \leq ||z_1 - z_2||$. This proves that T is a contraction mapping in Banach space $C([0,1], ||\cdot||)$. So, we can conclude that (3.1) has a unique solution in $C[0,1]$ using the Banach contraction principle. □

Theorem 4.2. *Suppose* $C([0,1], ||\cdot||)$ *is the Banach space of all continuous functions on* Ω. *Then* $z = \sum_{i=0}^\infty z_i(x)$ *uniformly converges to the exact solution on* $[0,1]$.

Proof. As proved in [12], consider $|z_1(x)| < +\infty$ for all $x \in [0,1]$. The sequence of the partial sum of the series is denoted as s_p. Let s_p and s_q be arbitrary partial sums with $p \geq q$. We need to prove that $s_p = \sum_{i=0}^p z_i(x)$ is a Cauchy sequence in $C([0,1], ||\cdot||)$. We have

$$\begin{aligned}\|s_p - s_q\|_\infty &= \max_{x\in[0,1]} |s_p - s_q|\\ &= \max_{x\in[0,1]}\left|\sum_{i=0}^p z_i(x) - \sum_{i=0}^q z_i(x)\right|\end{aligned}$$

$$
\begin{aligned}
&= \max_{x\in[0,1]} \left| \sum_{i=q+1}^{p} z_i(x) \right| \\
&= \max_{x\in[0,1]} \left| \sum_{i=q+1}^{p} \left(\frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t) z_i(t) + \int_0^t \mathcal{K}_1(t,s) \mathcal{A}_{1i}(s) ds \right.\right.\right. \\
&\quad \left.\left.\left. + \int_0^1 \mathcal{K}_2(t,s) \mathcal{A}_{2i}(s) ds \right] \right) dt \right| \\
&= \max_{x\in[0,1]} \left| \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t) \sum_{i=q}^{p-1} z_i(t) + \int_0^t \mathcal{K}_1(t,s) \sum_{i=q}^{p-1} \mathcal{A}_{1i}(s) ds \right.\right. \\
&\quad \left.\left. + \int_0^1 \mathcal{K}_2(t,s) \sum_{i=q}^{p-1} \mathcal{A}_{2i}(s) ds \right] dt \right|.
\end{aligned}
$$

Since, we know $\sum_{i=q}^{p-1} \mathcal{A}_{1i} = \mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1})$, $\sum_{i=q}^{p-1} \mathcal{A}_{2i} = \mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1})$ and $\sum\limits_{i=q}^{p-1} z_i = z(s_{p-1}) - z(s_{q-1})$. So, we reach at

$$
\begin{aligned}
||s_p - s_q||_\infty &= \max_{x\in[0,1]} \left(\left| \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \left[a(t)(z(s_{p-1}) - z(s_{q-1})) \right.\right.\right. \\
&\quad + \int_0^t \mathcal{K}_1(t,s) \Big(\mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1}) \Big) ds \\
&\quad \left.\left.\left. + \int_0^1 \mathcal{K}_2(t,s) \Big(\mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1}) \Big) ds \right] dt \right| \right) \\
&\le \max_{x\in[0,1]} \left(\frac{1}{\Gamma(\mu)} \int_0^x |x-t|^{\mu-1} \left[|a(t)| \, |z(s_{p-1}) - z(s_{q-1})| \right.\right. \\
&\quad + \int_0^t |\mathcal{K}_1(t,s)| \cdot |\mathcal{N}_1(s_{p-1}) - \mathcal{N}_1(s_{q-1})| \, ds \\
&\quad \left.\left. + \int_0^1 |\mathcal{K}_2(t,s)| \cdot |\mathcal{N}_2(s_{p-1}) - \mathcal{N}_2(s_{q-1})| \, ds \right] dt \right) \\
&\le \frac{1}{\Gamma(\mu+1)} \Big[\|a(t)\|_\infty \|s_{p-1} - s_{q-1}\|_\infty + \mathcal{K}_1^* C_1 ||s_{p-1} - s_{q-1}||_\infty \\
&\quad + \mathcal{K}_2^* C_2 ||s_{p-1} - s_{q-1}||_\infty \Big] \\
&= \left(\frac{\|a\|_\infty + \mathcal{K}_1^* C_1 + \mathcal{K}_2^* C_2}{\Gamma(\mu+1)} \right) ||s_{p-1} - s_{q-1}||_\infty = \gamma_1 ||s_{p-1} - s_{q-1}||_\infty,
\end{aligned}
$$

where

$$
\gamma_1 = \left(\frac{\|a\|_\infty + \mathcal{K}_1^* C_1 + \mathcal{K}_2^* C_2}{\Gamma(\mu+1)} \right). \tag{4.10}
$$

Also, for $p = q+1$,

$$
\|s_p - s_q\|_\infty \le \gamma_1 \|s_q - s_{q-1}\|_\infty \le \gamma_1^2 \|s_{q-1} - s_{q-2}\|_\infty \le \gamma_1^3 \|s_{q-2} - s_{q-3}\|_\infty
$$

$$\leq \cdots \leq \gamma_1^q \|s_1 - s_0\|_\infty.$$

So, we can write

$$\begin{aligned}\|s_p - s_q\|_\infty &\leq \|s_{q+1} - s_q\|_\infty + |s_{q+2} - s_{q+1}\|_\infty + \cdots + |s_p - s_{p-1}\|_\infty \\ &\leq \left[\gamma_1^q + \gamma_1^{q+1} + \cdots + \gamma_1^{p-1}\right] ||s_1 - s_0||_\infty \\ &\leq \gamma_1^q \left(\frac{1-\gamma_1^{p-q}}{1-\gamma_1}\right) ||z_1||_\infty.\end{aligned}$$

Since $0 < \gamma_1 < 1$, we have $(1 - \gamma_1^{p-q}) < 1$, then $||s_p - s_q||_\infty \leq \frac{\gamma_1^q}{1-\gamma_1}||z_1||_\infty$. As $||z_1(x)|| < \infty$ and $m \to \infty$, we get $||s_p - s_q||_\infty \to 0$. Hence, it can be concluded that s_p is a Cauchy sequence in $C[0,1]$ and $z = \lim_{n\to\infty} z_n$. Thus, the series is proved to be convergent by Weierstrass M-test. □

4.4.2. *Error bound.* The exact solution for (3.1) is given by $z(x) = \lim_{N\to\infty} z_N$ and the numerical solution can be obtained by truncating the series (4.6) and (4.9) up to a finite number of terms. If z_N gives the N terms approximated solution then, the absolute pointwise error bound depends on the partial sum $\sum_{n=0}^{N-1} z_n(x)$ which is bounded by $\frac{\mathcal{M}\gamma_1^N}{1-\gamma_1}$. γ_1 is defined in (4.10) which satisfies $0 < \gamma_1 < 1$ and $z_0 \leq \mathcal{M}$.

5. Numerical Approximation

In this section, we propose the numerical solution for (3.1). The approximation of fractional derivative $\mathbb{D}^\mu$ is made using the L1 scheme in [22]. The composite trapezoidal rule is used for approximating the Volterra integral and the rectangular rule for the Fredholm integral.

Now, to construct the mesh points, consider $\mathcal{N}$ to be any positive integer and $h = 1/\mathcal{N}$. Then, the mesh can be obtained as $\{x_n = nh : n = 0, 1, \ldots, \mathcal{N}\}$. The Caputo fractional order derivative is defined as

$$\mathbb{D}^\mu z(x_n) = \frac{1}{\Gamma(1-\mu)} \sum_{i=0}^{n-1} \int_{p=x_i}^{x_{i+1}} \frac{z'(p)}{(x_n - p)^\mu} dp. \tag{5.1}$$

Approximating $\mathbb{D}^\mu$ in (5.1) using the L1 approach at each x_n for $1 \leq n \leq \mathcal{N}$, we reach at

$$\mathbb{D}^\mu z(x_n) \approx \mathbb{D}^\mu_{\mathcal{N}} z_n := \frac{1}{\Gamma(1-\mu)} \sum_{i=0}^{n-1} \frac{z(x_{i+1}) - z(x_i)}{h} \int_{p=x_i}^{x_{i+1}} \frac{dp}{(x_n - p)^\mu}$$

$$= \frac{1}{h^\mu \Gamma(2-\mu)} \sum_{i=0}^{n-1} (z(x_{i+1}) - z(x_i)) c_{n-i} + \mathcal{R}_n^{(1)}, \tag{5.2}$$

where $c_k = k^{1-\mu} - (c-1)^{1-\mu}$, $k \geq 1$. Approximating an integral part using the composite trapezoidal rule for the Volterra integral and rectangular rule for the

Fredholm integral for $1 \leq n \leq N$, we have

$$\int_0^{x_n} \mathcal{K}_1(x_n, s)z(s)ds = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \mathcal{K}_1(x_n, s)z(s)ds,$$

$$= \frac{h}{2}\sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z(x_{i+1}) + \mathcal{K}_1(x_n, x_i)z(x_i)] + \mathcal{R}_n^{(2)}, \tag{5.3}$$

$$\int_0^{l} \mathcal{K}_2(x_n, s)z(s)ds = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \mathcal{K}_2(x_n, s)z(s)ds$$

$$= h\sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z(x_i)] + \mathcal{R}_n^{(3)}, \tag{5.4}$$

where the remainder terms $\mathcal{R}_n^{(i)}$ for $i = 1, 2, 3$ are given by

$$\mathcal{R}_n^{(1)} = (\mathbb{D}^\mu - \mathbb{D}_{\mathcal{N}}^\mu)z(x_n) = \left[\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1} \frac{x_{i+1} + x_i - 2p}{(x_n - p)^\mu} + O(h^2)\right], \tag{5.5}$$

$$\mathcal{R}_n^{(2)} = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p)\frac{\partial}{\partial p}\left[\mathcal{K}_1(x_n, p)z(p)\right] dp + O(h^2), \tag{5.6}$$

$$\mathcal{R}_n^{(3)} = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (x_{i-1} - p)\frac{\partial}{\partial p}\left[\mathcal{K}_2(x_n, p)z(p)\right] dp + O(h). \tag{5.7}$$

Finally, using (5.2), (5.3) and (5.4), we construct the difference scheme as

$$\begin{aligned}
&\mathbb{D}_{\mathcal{N}}^\mu z(x_n) + a(x_n)z(x_n) + \frac{h}{2}\sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z(x_{i+1}) + \mathcal{K}_1(x_n, x_i)z(x_i)] \\
&+ h\sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z(x_i)] = f(x_n) + \mathcal{R}_n^{(i)}, \quad \text{for } n = 1, 2, \ldots \mathcal{N}, \\
&z(0) = z_0,
\end{aligned}$$

where $\mathcal{R}_n^{(i)} = \mathcal{R}_n^{(1)} + \mathcal{R}_n^{(2)} + \mathcal{R}_n^{(3)}$ described as in (5.5), (5.6) and (5.7). Neglecting the remainder terms for $n = 1, 2, \ldots, \mathcal{N}$, we get the fully discrete scheme as

$$\begin{aligned}
&\mathbb{D}_{\mathcal{N}}^\mu z_n + a_n z_n + \frac{h}{2}\sum_{i=0}^{n-1} [\mathcal{K}_1(x_n, x_{i+1})z_{i+1} + \mathcal{K}_1(x_n, x_i)z_i] + h\sum_{i=1}^{n} [\mathcal{K}_2(x_n, x_i)z_i] = f_n, \\
&z(0) = z_0^N.
\end{aligned} \tag{5.8}$$

5.1. **Convergence analysis.** In this section, we find the error estmates for approximating (3.1) using the numerical scheme (5.8).

Lemma 5.1. *For all* $\mu \in [0, 1]$ *and* $n \geq 1$. *If*

$$B(n) = n^{1-\mu} + 2\Big((n-1)^{1-\mu} + (n-2)^{1-\mu} + (n-3)^{1-\mu} + \cdots + 1^{1-\mu}\Big) - \frac{2}{2-\mu}n^{2-\mu},$$

then $|B(n)| \leq C$, *where* C *is independent of* n *and* μ.

Proof. The detailed proof of this lemma is discussed in [18]. □

Theorem 5.1. *For a constant C and fractional order derivative $\mu \in (0,1)$, the following inequality follows:*

$$|\mathcal{R}_n^{(1)}| \leq Ch^{2-\mu}. \tag{5.9}$$

Proof. Solving the L.H.S of (5.9)

$$\begin{aligned}
&\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{x_{i+1}+x_i-2p}{(x_n-p)^\mu}dp\\
=&\frac{-1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{1}{1-\mu}(2i+1)h^{2-\mu}\left[(n-i-1)^{1-\mu}-(n-i)^{1-\mu}\right]\\
&+\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{2}{1-\mu}h^{2-\mu}\Big[(i+1)(n-i-1)^{1-\mu}-i(n-i)^{1-\mu}\\
&+\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{2}{(2-\mu)(1-\mu)}h^{2-\mu}[(n-i-1)^{2-\mu}-(n-i)^{2-\mu}]\Big]\\
=&\frac{h^{2-\mu}}{\Gamma(2-\mu)}\left[n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu})+\cdots+1^{1-\mu}\right]-\frac{2h^{2-\mu}}{\Gamma(3-\mu)}n^{2-\mu}\\
=&\frac{h^{2-\mu}}{\Gamma(2-\mu)}\left[n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu}+\cdots+1^{1-\mu})-\frac{2}{2-\mu}n^{2-\mu}\right].
\end{aligned}$$

Let $B(n)=n^{1-\mu}+2((n-1)^{1-\mu}+(n-2)^{1-\mu}+(n-3)^{1-\mu}+\cdots+1^{1-\mu})-\frac{2}{2-\mu}n^{2-\mu}$. From Lemma 5.1, $|B(n)|$ is bounded for all $\mu\in[0,1]$ and all $n\geq 1$. So, taking into fact that $\frac{1}{\Gamma(2-\mu)}\leq 2$ for all $\mu\in[0,1]$, we get

$$\left|\frac{1}{\Gamma(1-\mu)}\sum_{i=0}^{n-1}\frac{x_{i+1}+x_i-2p}{(x_n-p)^\mu}dp\right|\leq 2h^{2-\mu}. \tag{5.10}$$

As a result, from (5.10), we obtain $\mathcal{R}_{nr}^{(1)}\leq C\mathcal{N}^{-(2-\mu)}$. □

The above theorem proves that the solution obtained using the L1 scheme on a uniform mesh is $O(\mathcal{N}^{-(2-\mu)})$ accurate. But, when the solutions have a mild singularity at the initial mesh point $x=0$, then the order of accuracy will be $O(\mathcal{N}^{-\mu})$ and $O(\mathcal{N}^{-1})$ on any sub-domain that is bounded away from $x=0$. For the analysis of such cases, one may refer to [11, 22]. We have considered $\mathcal{R}_n^{(1)}=\mathcal{R}_{ns}^{(1)}+\mathcal{R}_{nr}^{(1)}$, where $\mathcal{R}_{ns}^{(1)}$ is the remainder term for the case, where there is a mild singularity at $x=0$ and $\mathcal{R}_{nr}^{(1)}$ is the remainder term for the case where the solution is regular. The following lemma gives the truncation error for the Caputo order derivative due to the presence of weak singularity at the initial mesh point.

Lemma 5.2. *For each mesh point x_n, $n=1,2,\ldots,\mathcal{N}$, we have the following estimate while there is a mild singularity at the initial mesh point $x=0$*

$$|\mathcal{R}_{ns}^{(1)}|\leq Cn^{-(\mu+1)},\quad \textit{for all}\, n=1,2\ldots,\mathcal{N}.$$

Proof. One may refer to [22, 27] for the detailed proof of the lemma. □

Lemma 5.3. *The remainder term $\mathcal{R}_n^{(2)}$, $n = 1, 2, \ldots, \mathcal{N}$, satisfies the following estimate:*

$$|\mathcal{R}_n^{(2)}| \leq C\mathcal{N}^{-1}.$$

Proof. From (5.6), we get

$$\begin{aligned}
|\mathcal{R}_n^{(2)}| &= \left|\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \frac{d}{dp}[\mathcal{K}(x_n, p) z(p)] dp\right| \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \left|\frac{d}{dp}[\mathcal{K}(x_n, p) z(p)]\right| dp \\
&\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_{i+1/2} - p) \left|\frac{\partial}{\partial p}[\mathcal{K}(x_n, p) z(p)] + \frac{\partial}{\partial z}[\mathcal{K}(x_n, p) z(p)] z'(p)\right| dp \\
&\leq Ch \int_0^{x_n} (1 + z'(p)) dp \leq Ch \leq C\mathcal{N}^{-1}, \quad \text{for all } n = 1, 2, \ldots, \mathcal{N},
\end{aligned}$$

which is the desired bound. □

Lemma 5.4. *Assuming that $\mathcal{K}_2$ is a continuous bounded function on $[0, 1]$. The remainder term $\mathcal{R}_n^{(3)}$, $n = 1, 2, \ldots, \mathcal{N}$, satisfies the following estimate:*

$$|\mathcal{R}_n^{(3)}| \leq C\mathcal{N}^{-1}.$$

Proof. From (5.7), we get

$$\begin{aligned}
|\mathcal{R}_n^{(3)}| &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (x_i - p) \left|\frac{\partial}{\partial p}[\mathcal{K}_2(x_n, p) z(p)] dp\right| \\
&\leq h \int_0^1 \left|\frac{\partial}{\partial p}[\mathcal{K}_2(x_n, p) z(p)] dp\right| \\
&\leq h \int_0^1 \left\{\left|\frac{\partial \mathcal{K}_2(x_n, p)}{\partial p}\right| |z(p)| + |\mathcal{K}_2(x_n, p)||z'(p)|\right\} dp \\
&\leq Ch \leq C\mathcal{N}^{-1}, \quad \text{for all } n = 1, 2, \ldots, \mathcal{N}.
\end{aligned}$$

This proves the required estimate. □

Consider e_n to be the error function. $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ be the exact solution of the continuous problem (3.1) and $\{z_n\}_{n=1}^{\mathcal{N}}$ be the numerical solution of (5.8), then the error function is defined as:

$$\mathrm{e}_n = |z(x_n) - z_n|, \quad \mathrm{e}_0 = 0, \quad \text{for } n = 1, 2, \ldots, \mathcal{N}. \tag{5.11}$$

Theorem 5.2. *If $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ is the exact solution to the continuous problem (3.1) and $\{z_n\}_{n=1}^{\mathcal{N}}$ is the numerical solution of (5.8), then the error bound when there exists a weak singularity at the initial mesh point is given by*

$$|\mathrm{e}_n| \leq |z(x_n) - z_n| \leq h^\mu \Gamma(2 - \mu) \sum_{i=1}^{n} |\mathcal{R}_n^{(i)}| \leq C\left[h x_n^{\mu-1} + h^\mu \mathcal{N}^{-(1-\mu)}\right]. \tag{5.12}$$

Proof. From (5.11) and using Lemma 3 of [27], we have

$$\begin{aligned}|\mathrm{e}_n| &\le |z(x_n)-z_n| \le h^{\mu}\Gamma(2-\mu)\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(i)}| \\ &\le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_{ns}^{(1)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(2)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(3)}|. \end{aligned} \tag{5.13}$$

Applying Lemma 5.2, Lemma 5.3 and Lemma 5.4, (5.13) reduces to

$$|\mathrm{e}_n| \le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}i^{-(1+\mu)} + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}\mathcal{N}^{-1}.$$

Finally employing Lemma 3 of [11] and Lemma 4.3 of [22] to the above inequality, the desired result (5.12) is obtained. □

Theorem 5.3. *If* $\{z(x_n)\}_{n=1}^{\mathcal{N}}$ *is the exact solution to the continuous problem* (3.1) *and* $\{z_n\}_{n=1}^{\mathcal{N}}$ *is the numerical solution of* (5.8)*, then the error bound is given by*

$$|\mathrm{e}_n| \le |z(x_n)-z_n| \le CX^{\mu}h, \quad n = 1,2,\ldots,\mathcal{N}.$$

Proof. We have

$$\begin{aligned}|z(x_n)-z_n| &\le h^{\mu}\Gamma(2-\mu)\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(i)}| \\ &\le Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(1)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(2)}| + Ch^{\mu}\sum_{i=1}^{n}\gamma_{n-i}|\mathcal{R}_n^{(3)}|.\end{aligned}$$

Combining Theorem 5.1, Lemma 5.3 and Lemma 5.4, we get

$$|z(x_n)-z_n| \le C\mathcal{N}^{-1} + C\gamma_{n-1}^{-1}h^2.$$

By the definition of γ_n, we have $n^{-\mu}\gamma_{n-1}^{-1} \le \frac{1}{1-\mu}$, $n = 1,2,\ldots,\mathcal{N}$. Consequently, for all n such that $nh \le X$, we have

$$\begin{aligned}|z(x_n)-z_n| &\le C\mathcal{N}^{-1} + C\gamma_{n-1}^{-1}h^2 \\ &= C\mathcal{N}^{-1} + Cn^{-\mu}\gamma_{n-1}^{-1}h^2 = C\mathcal{N}^{-1} + Cn^{-\mu}n^{-\mu}\gamma_{n-1}^{-1}n^{\mu}h^2 \\ &= C\mathcal{N}^{-1} + C\left(\frac{1}{1-\mu}\right)(nh)^{\mu}h^{2-\mu} \le C\mathcal{N}^{-1} + CX^{\mu}h^{2-\mu},\end{aligned}$$

which gives the desired result. □

$\mathcal{E}_{\mu}^{\mathcal{N}} = \max_{0\le n\le \mathcal{N}}|z(x_n)-z_n|$ denotes the pointwise error while using the numerical scheme, while $\mathcal{P}_{\mu}^{\mathcal{N}} = \frac{\mathcal{E}_{\mu}^{\mathcal{N}}/\mathcal{E}_{\mu}^{2\mathcal{N}}}{\ln 2}$ denotes the order of convergence.

6. Numerical Experiments

This section consists of two numerical examples which clearly depict the efficiency of the proposed techniques.

Example 6.1. Consider the following model

$$\mathbb{D}^{\mu}z(x)+a(x)z(x)=f(x)+\int_0^x \mathcal{K}_1(x,s)z(s)ds+\int_0^1 \mathcal{K}_2(x,s)z(s)ds,$$

with the initial condition $z(0)=1$. Here $f(x)=\exp(x)-1+x^{1-\mu}E_{1,2-\mu}(x),\ E_{1,2-\mu}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+2-\mu)}$, $a(x)=0$, $\mathcal{K}_1(x,s)=1$ and $\mathcal{K}_2(x,s)=2s-1$. The exact solution is $z(x)=\exp(x)$.

First approximating $f(x)$ using the Chebyshev polynomials, $f_C(x)=\sum_{i=0}^{5}C_i\mathcal{T}_i(2x-1),\ x\in[0,1]$. Here,

$$\begin{aligned}C_0&=\frac{1}{\pi}\int_{-1}^{1}\frac{f(0.5x+0.5)\mathcal{T}_0(x)}{\sqrt{1-x^2}}dx,\\ C_i&=\frac{2}{\pi}\int_{-1}^{1}\frac{f(0.5x+0.5)\mathcal{T}_i(x)}{\sqrt{1-x^2}}dx,\quad i=0,1,\dots,5.\end{aligned}$$

So, we get

$$\begin{aligned}f_C(x)\approx&\frac{x^{1-\mu}}{\Gamma(2-\mu)}+\frac{x^{2-\mu}}{\Gamma(3-\mu)}+\frac{x^{3-\mu}}{\Gamma(4-\mu)}+\frac{x^{4-\mu}}{\Gamma(5-\mu)}+\frac{x^{5-\mu}}{\Gamma(6-\mu)}+\frac{x^{6-\mu}}{\Gamma(7-\mu)}\\&-1.0002x-0.499197x^2-0.166489x^3-0.0437939x^4-0.00868682x^5\\&-0.00004.\end{aligned}\tag{6.1}$$

Substituting (6.1) and applying (4.4), we obtain the two term approximated solution as follows:

$$\begin{aligned}z(x)=&z_0(x)+z_1(x)\\=&1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}-1.0002\frac{x^{1+\mu}}{\Gamma(2+\mu)}-0.499197\frac{(x^{2+\mu})}{\Gamma(3+\mu)}\\&-0.166489\frac{x^{3+\mu}}{\Gamma(4+\mu)}-0.0437939\frac{x^{4+\mu}}{\Gamma(5+\mu)}-0.0086862\frac{x^{5+\mu}}{\Gamma(6+\mu)}\\&-4.00e-05\frac{x^{\mu}}{\Gamma(1+\mu)}+0.000198413\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)}+0.00138889\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)}\\&+0.00833333\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)}+0.0416667\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)}+0.166667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)}\\&+0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)}+\frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)}-1.0002\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)}\\&-0.499197\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)}-0.166489\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)}\end{aligned}$$

$$- 0.0437939\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)} - 0.0086862\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}$$
$$- 0.00004\frac{x^{1+2\mu}\Gamma(2+\mu)}{\Gamma(2+2\mu)(1+\mu)}.$$

Simplify, the problem using the Bernstein polynomials $\sum_{i=0}^{n} \mathcal{D}_i \mathcal{B}_i(x)$ with $i = 5$ gives the approximation for $f(x)$ as

$$f_B(x) \approx 1 + \frac{x^{1-\mu}}{\Gamma(2-\mu)} + \frac{x^{2-\mu}}{\Gamma(3-\mu)} + \frac{x^{3-\mu}}{\Gamma(4-\mu)} + \frac{x^{4-\mu}}{\Gamma(5-\mu)} + \frac{x^{5-\mu}}{\Gamma(6-\mu)} + \frac{x^{6-\mu}}{\Gamma(7-\mu)}$$
$$- 1.10701379x - 0.490191813x^2 - 0.108529819x^3 - 0.0120144007x^4$$
$$- 0.00053200429x^5.$$

Using (4.8), the series solution using ADM-BP is obtained. We get the two term approximated solution as follows:

$$z(x) = z_0(x) + z_1(x)$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} - 1.10701\frac{x^{1+\mu}}{\Gamma(2+\mu)} - 0.49019\frac{x^{2+\mu}}{\Gamma(3+\mu)}$$
$$- 0.10853\frac{x^{3+\mu}}{\Gamma(4+\mu)} - 0.012014\frac{x^{4+\mu}}{\Gamma(5+\mu)} - 0.00053\frac{x^{5+\mu}}{\Gamma(6+\mu)}$$
$$+ 0.00019\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)} + 0.001389\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)} + 0.00833\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)}$$
$$+ 0.04167\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)} + 0.16667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)} + 0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)} + \frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)}$$
$$- 1.10701\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)} - 0.49019\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)}$$
$$- 0.10853\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)} - 0.01201\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)}$$
$$- 0.00053\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}.$$

Finally, using the classical ADM and approximating $f(x)$ using Taylor's polynomial denoted as $f_T(x)$, we get

$$f_T(x) \approx 1 + \frac{x^{1-\mu}}{\Gamma(2-\mu)} + \frac{x^{2-\mu}}{\Gamma(3-\mu)} + \frac{x^{3-\mu}}{\Gamma(4-\mu)} + \frac{x^{4-\mu}}{\Gamma(5-\mu)} + \frac{x^{5-\mu}}{\Gamma(6-\mu)}$$
$$+ \frac{x^{6-\mu}}{\Gamma(7-\mu)} - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{x^5}{5!}.$$

Recursively, using the scheme for ADM, the solution is obtained as

$$z(x) = z_0(x) + z_1(x)$$

$$
\begin{aligned}
=&1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^{1+\mu}}{\Gamma(2+\mu)}-\frac{x^{2+\mu}}{\Gamma(3+\mu)}-\frac{x^{3+\mu}}{\Gamma(4+\mu)}\\
&-\frac{x^{4+\mu}}{\Gamma(5+\mu)}-\frac{x^{5+\mu}}{\Gamma(6+\mu)}+0.00019\frac{x^{7+\mu}\Gamma(8)}{\Gamma(8+\mu)}+0.00139\frac{x^{6+\mu}\Gamma(7)}{\Gamma(7+\mu)}\\
&+0.008333\frac{x^{5+\mu}\Gamma(6)}{\Gamma(6+\mu)}+0.041667\frac{x^{4+\mu}\Gamma(5)}{\Gamma(5+\mu)}+0.16667\frac{x^{3+\mu}\Gamma(4)}{\Gamma(4+\mu)}\\
&+0.5\frac{x^{2+\mu}\Gamma(3)}{\Gamma(3+\mu)}+\frac{x^{1+\mu}\Gamma(2)}{\Gamma(2+\mu)}-\frac{x^{2+2\mu}\Gamma(3+\mu)}{\Gamma(3+2\mu)(2+\mu)}-\frac{x^{3+2\mu}\Gamma(4+\mu)}{\Gamma(4+2\mu)(3+\mu)}\\
&-\frac{x^{4+2\mu}\Gamma(5+\mu)}{\Gamma(5+2\mu)(4+\mu)}-\frac{x^{5+2\mu}\Gamma(6+\mu)}{\Gamma(6+2\mu)(5+\mu)}-\frac{x^{6+2\mu}\Gamma(7+\mu)}{\Gamma(7+2\mu)(6+\mu)}.
\end{aligned}
$$

For the semi-analytical methods, the error is calculated using $\mathbb{E}_n^\infty = |z(x) - \sum_{i=0}^{n} z_i(x)|$. Figure 1(a) shows the error plot at $\mu = 0.25$ using the two term expansion of the modified ADM and the classical ADM. One can observe the robustness of ADM-BP and ADM-CP over the classical ADM, as the decrement in error is more in the case of our proposed techniques as compared to the classical technique. Similarly, Figure 1(b) depicts the comparison of $\mathbb{E}_2^\infty$ between all the three techniques. The error in the case of ADM-CP and ADM-BP is minimal compared to the classical ADM which makes it efficient for use when the source term in the model problem is any function rather than a polynomial function. The solution plots are graphically shown in Figure 2 for the proposed techniques and the classical ADM. The accuracy of the semi-analytical methods can be seen. Table 1 shows the error computed with one term and two term solutions. The data depicts that the error decreases gradually with the increase in number of iterations. Tables 2 and 3 give the pointwise error for $x \in [0, 1]$ at $\mu = 0.01$ and $\mu = 0.95$, respectively. At some points close to zero, the error in classical ADM seems less than our proposed methods. But at rest all of the node points, the proposed methods prove to be more accurate and efficient which clearly shows their reliability.

Example 6.2. Consider the following Volterra-Fredholm FracIDE

$$\mathbb{D}^{0.75}z(x) + \frac{x^2e^x}{5}z(x) = \frac{6}{\Gamma(2.25)}x^{2.25} + \int_0^x e^x s z(s)ds + \int_0^1 (4 - s^{-3})z(s)ds,$$

with the initial condition $z(0) = 0$. The exact solution is $z(x) = x^3$.

Here, the source function is in the form of a polynomial function. We first approximate $f(x)$ using the Chebyshev polynomials, and then apply the recursive algorithm to obtain the series solution.

$$f_C(x) = \sum_{i=0}^{4} C_i\mathcal{T}_i(2x-1), \quad 0 \le x \le 1,$$

where $C_0 = \frac{1}{\pi}\int_{-1}^1 \frac{f(0.5x+0.5)\mathcal{T}_0(x)}{\sqrt{1-x^2}}dx$ and $C_i = \frac{2}{\pi}\int_{-1}^1 \frac{f(0.5x+0.5)\mathcal{T}_i(x)}{\sqrt{1-x^2}}dx$, $i = 1, 2, \dots, 6$. It implies that $f_C(x) \approx -0.2293888x^4 + 0.9765696x^3 + 1.666256x^2 - 0.0610748x +$

0.0010169. Applying (4.4), we obtain the approximated solution as

$$z(x) = -0.2293888x^{4.75}\frac{\Gamma(5)}{\Gamma(5.75)} + 0.9765696x^{3.75}\frac{\Gamma(4)}{\Gamma(4.75)} + 1.666256x^{2.75}\frac{\Gamma(3)}{\Gamma(3.75)} - 0.0610748x^{1.75}\frac{\Gamma(2)}{\Gamma(2.75)} + 0.0010169x^{0.75}\frac{\Gamma(1)}{\Gamma(1.75)}.$$

$f(x)$ is approximated using the Bernstein polynomials $f_B(x) = \sum_{i=0}^{n} \mathcal{D}_i\mathcal{B}_i(x)$ with $i = 10$. Then applying the recursive algorithm for ADM to obtain the series solution, we get

$$f_B(x) \approx -0.00056944194 93x^{10} + 0.006482451697x^9 - 0.03408x^8 + 0.11017x^7 - 0.247127x^6 + 0.416952685x^5 - 0.584450992x^4 + 0.9119425917x^3 + 1.641949867x^2 + 0.132354172x.$$

Substituting $f_B(x)$ in (4.8), we obtain the approximated solution, which converges to the exact solution as shown in Figure 3(a). Also, the pointwise errors of the proposed techniques are shown using Figure 3(b). Hamoud and Ghadle in [12] solved this example using the classical ADM and obtained the exact solution in the first iteration. Since, the source term is already a polynomial function (in Taylor's series expansion), the proposed techniques (ADM-BP and ADM-CP) do not contribute much to decreasing the error in comparison to the solutions obtained in [12]. Table 4 shows the pointwise error obtained after the first term series solution using ADM-BP and ADM-CP. Though the error is less, the proposed methods are still ineffective for such model problems. Hence, one can conclude that the proposed techniques are suitable for the model problems where the source term is any other function except the polynomials.

We have also solved this example using the proposed numerical scheme (5.8). The solution is regular in its considered domain. The computed results are recorded in Table 5. One can clearly observe that the order of accuracy is almost first order accurate over the entire domain which satisfies the theoretical estimates. Figure 4(a) shows the solution plot for both the approximated and the exact solution at $\mu = 0.75$.

Example 6.3. Consider the following numerical experiment:

$$\mathbb{D}^{\mu}z(x) + a(x)z(x) = f(x) + \int_0^x sz(s)ds + \int_0^1 (x-s)z(s)ds,$$

where $a(x) = 0$ and the exact solution is $z(x) = x^{\mu} + x$.

The problem is solved using the proposed numerical scheme (5.8). Table 6 shows the error and rate of convergence for Example 6.3. Due to the presence of weak singularity, the order of accuracy is $O(\mathcal{N}^{-\mu})$ over the entire domain. A sharp singularity is present at the initial mesh point $x = 0$ which is evident from Figure 4(b) at $\mu = 0.1$.

Example 6.4. Consider a nonlinear model of Volterra-Fredholm FracIDE:

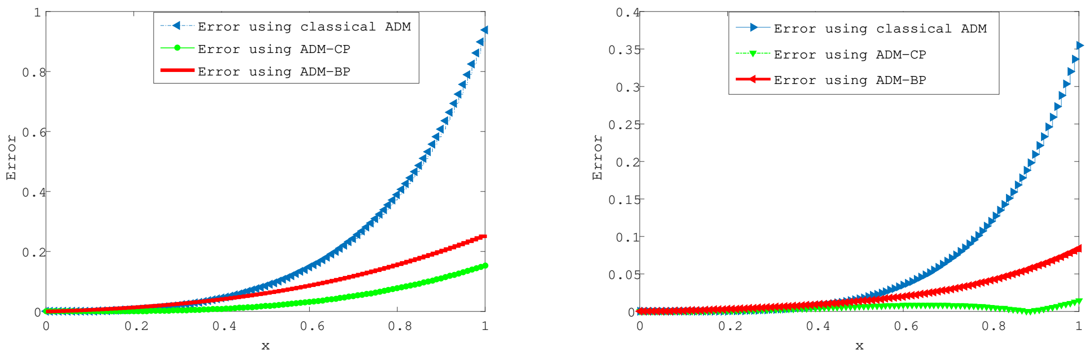
$$\mathbb{D}^{\mu}z(x)+a(x)z(x)=f(x)+\int_0^x z^4(s)ds-\int_0^1 f(x+s)z(s)ds,$$

where $a(x)=0$, $f(x)=\frac{t^4\Gamma(5+\mu)}{24}+t^{5+\mu}-\frac{t^{17+4\mu}}{17+4\mu}+\frac{t}{5+\mu}+\frac{1}{6+\mu}$ and the exact solution is $z(x)=x^{\mu+4}$.

Table 7 shows the computed values of maximum pointwise error and order of convergence for arbitrary order fractional derivatives. The tabular data proves that the proposed numerical scheme also works well for a class of nonlinear Volterra-Fredholm FracIDEs.

7. Conclusion

This article intends to solve the fractional order Volterra-Fredholm integro-differential equations using semi-analytical and numerical methods. At first, we used the modified Adomian decomposition technique for the model problem where the source term is generalized as any kind of function (other than the polynomial function). The uniqueness and existence of the solutions are properly established and convergence of the method is carried out. Secondly, we have developed a fully discrete scheme for obtaining the numerical solution. Error analysis is done and it is validated with the help of a few numerical experiments. Finally, a comparison with some existing methods shows that the proposed methods are more efficient and robust.



(a) $\mu=0.25$ (b) $\mu=0.75$

Figure 1. Error plots using semi-analytical methods for Example 6.1.

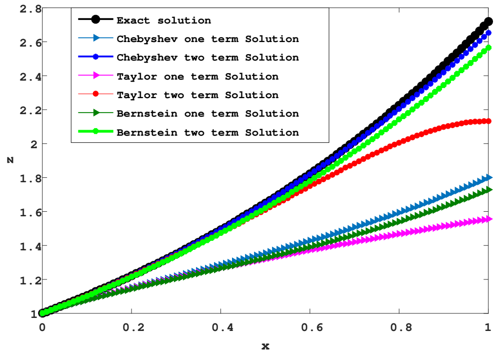
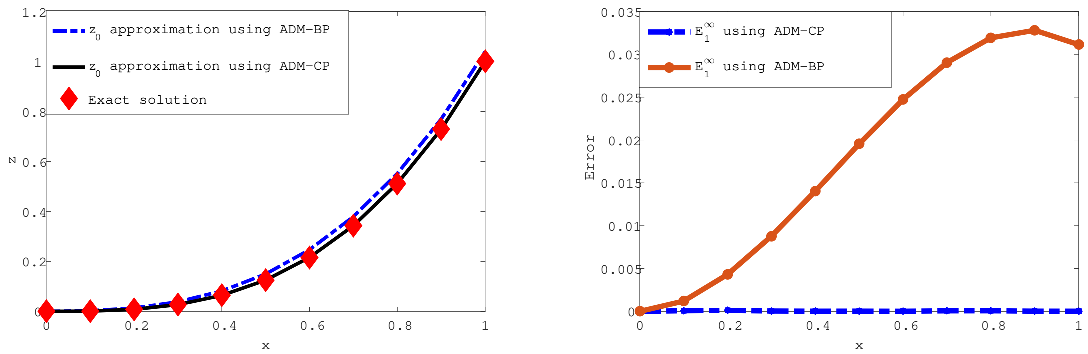


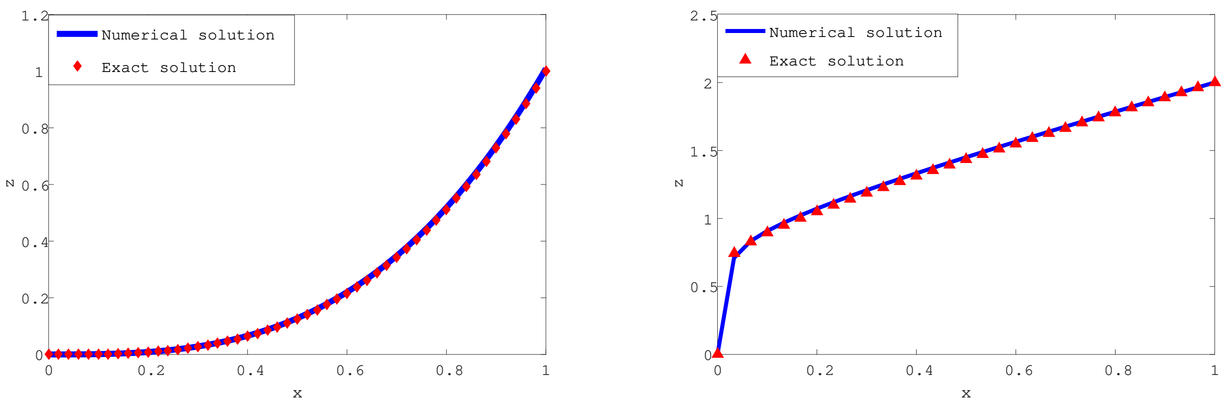
FIGURE 2. Solution plots using semi-analytical methods at $\mu = 0.5$ for Example 6.1.



(a) Solution plots

(b) Error plots

FIGURE 3. Plots using semi-analytical methods for Example 6.2.



(a) Example 6.2.

(b) Example 6.3.

FIGURE 4. Solution plots using the numerical method.

TABLE 1. Absolute pointwise errors using semi-analytical methods with $\mu = 0.5$ for Example 6.1.

x	ADM-BP		ADM-CP	
	$\mathbb{E}_1^\infty$	$\mathbb{E}_2^\infty$	$\mathbb{E}_1^\infty$	$\mathbb{E}_2^\infty$
0.2	7.7156e-2	6.2421e-3	7.0056e-2	1.0425e-3
0.4	2.2598e-1	1.8957e-2	2.0617e-1	2.2681e-3
0.6	4.2975e-1	4.2329e-2	3.9409e-1	2.2862e-3
0.8	6.8471e-1	8.3729e-2	6.3132e-1	2.1288e-2
1.0	9.9004e-1	1.5230e-1	9.1806e-1	6.6316e-2

TABLE 2. Absolute pointwise errors using semi-analytical methods with $\mu = 0.01$ for Example 6.1.

x	$\mathbb{E}_2^\infty$ using ADM-BP	$\mathbb{E}_2^\infty$ using ADM-CP	$\mathbb{E}_2^\infty$ using classical ADM
0.2	3.2232e-2	9.6072e-3	2.1724e-2
0.4	8.8292e-2	3.9878e-2	1.0156e-1
0.6	1.6813e-1	9.2759e-2	2.7822e-1
0.8	2.7053e-1	1.7013e-1	6.4637e-1
1.0	3.9279e-1	2.7399e-1	1.4248

TABLE 3. Absolute pointwise errors using semi-analytical methods with $\mu = 0.95$ for Example 6.1.

x	$\mathbb{E}_2^\infty$ using ADM-BP	$\mathbb{E}_2^\infty$ using ADM-CP	$\mathbb{E}_2^\infty$ using classical ADM
0.2	1.7740e-3	6.4018e-4	1.9533e-4
0.4	5.3420e-3	4.0840e-3	3.3104e-3
0.6	1.0931e-2	1.0136e-2	1.8948e-2
0.8	2.2620e-2	1.4696e-2	7.2710e-2
1.0	4.9022e-2	8.5299e-3	2.3222e-1

TABLE 5. Absolute pointwise errors using numerical approximation with $\mu = 0.75$ for Example 6.2.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mathcal{E}_\mu^\mathcal{N}$	3.8450e-3	1.6347e-3	6.9200e-4	2.9214e-4	1.2312e-4	5.1839e-5
$\mathcal{P}_\mu^\mathcal{N}$	1.234	1.240	1.244	1.246	1.248	

TABLE 4. Absolute pointwise errors using semi-analytical methods with $\mu = 0.75$ for Example 6.2.

x	$\mathbb{E}_1^\infty$ using ADM-BP	$\mathbb{E}_1^\infty$ using ADM-CP
0.1	1.5167e-3	7.7086e-5
0.2	5.4081e-3	1.1525e-4
0.3	1.0989e-2	4.4583e-5
0.4	1.7565e-2	2.3456e-5
0.5	2.4440e-2	2.9868e-5
0.6	3.0919e-2	1.6814e-5
0.7	3.6305e-2	6.7225e-5
0.8	3.9902e-2	7.1810e-5
0.9	4.1014e-2	2.6809e-5
1	3.8945e-2	2.2190e-5

TABLE 6. Absolute pointwise errors using numerical approximation for Example 6.3.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mu = 0.2$	5.3434e-2 0.144	4.8349e-2 0.173	4.2892e-2 0.186	3.7690e-2 0.193	3.2964e-2 0.197	2.8763e-2
$\mu = 0.4$	3.1270e-2 0.364	2.4291e-2 0.382	1.8635e-2 0.391	1.4209e-2 0.396	1.0801e-2 0.398	8.1980e-3
$\mu = 0.6$	1.2472e-2 0.566	8.4267e-3 0.583	5.6251e-3 0.592	3.7329e-3 0.596	2.4700e-3 0.598	1.6319e-3
$\mu = 0.8$	3.5118e-3 0.709	2.1476e-3 0.744	1.2819e-3 0.770	7.5155e-4 0.785	4.3619e-4 0.791	2.5213e-4

TABLE 7. Absolute pointwise errors using numerical approximation for Example 6.4.

$\mathcal{N}$	100	200	400	800	1600	3200
$\mu = 0.5$	1.9248e-2 0.975	9.7900e-3 0.518	6.8373e-3 0.502	4.8276e-3 0.501	3.4111e-3 0.501	2.4111e-3
$\mu = 0.7$	2.3163e-2 1.015	1.1464e-2 1.023	5.6412e-3 1.024	2.7745e-3 1.022	1.3666e-3 1.019	6.7455e-4
$\mu = 0.9$	3.2869e-2 1.009	1.6331e-2 1.020	8.0508e-3 1.026	3.9539e-3 1.028	1.9390e-3 1.028	9.5055e-4

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ALL EVEN (UNITARY) PERFECT POLYNOMIALS OVER $\mathbb{F}_2$ WITH ONLY MERSENNE PRIMES AS ODD DIVISORS

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Abstract. We address an arithmetic problem in the ring $\mathbb{F}_2[x]$. We prove that the only (unitary) perfect polynomials over $\mathbb{F}_2$ that are products of x, $x+1$ and of Mersenne primes are precisely the nine (resp. nine "classes") known ones. This follows from a new result about the factorization of $M^{2h+1}+1$, for a Mersenne prime M and for a positive integer h.

1. Introduction

Let $A \in \mathbb{F}_2[x]$ be a nonzero binary polynomial. Let $\sigma(A)$ denote the sum of all divisors of A (including 1 and A). If $\sigma(A) = A$, then one says that A is a *one-ring* [5] or in other words, A is *perfect* [4]. In addition to polynomials of the form $(x^2+x)^{2^n-1}$, with some positive integer n, E. F. Canaday [5] discovered eleven non-splitting perfect polynomials (see **Notation**): $T_1, \ldots, T_9$ and C_1, C_2. The T_j's are divisible only by x, $x+1$ and by irreducible polynomials of the form $U_{a,b} := x^a(x+1)^b + 1$, for some positive integers a, b. The last two C_1 and C_2 are divisible by x^4+x+1 which is not of the form $U_{a,b}$. The parallel with the integer case is then natural to be considered. We know that all perfect numbers are of the form $2^m(2^m-1)$, where m is a prime number and 2^m-1 is a Mersenne prime number. So, we may consider the following notions. We say that a binary polynomial is *even* if it has a linear factor [6]. It is *odd*, otherwise. We also define a *Mersenne prime* (polynomial) over $\mathbb{F}_2$ as an irreducible polynomial of the above form $U_{a,b}$ [9]. The name comes as an analogue of the integral Mersenne primes, taking $x^a(x+1)^b$ as an analogue of the prime power 2^{a+b}.

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Note that the notion of Mersenne prime polynomial is only useful over $\mathbb{F}_2$, whereas one may consider the "parity" of a polynomial over any finite field.

Unitary perfect polynomials are defined and studied in several directions by J. T. B. Beard Jr. et al. [1, 2, 4]. As over the integers, for $A \in \mathbb{F}_2[x]$, a divisor D of A is *unitary* if $\gcd(D, A/D) = 1$. Let $\sigma^*(A)$ denote the sum of all unitary divisors of A (including 1 and A). If $\sigma^*(A) = A$, then A is *unitary perfect*.

We say that a (unitary) perfect polynomial is *indecomposable* if it is not a product of two coprime nonconstant (unitary) perfect polynomials.

Any unitary perfect polynomial is even (Lemma 3.4). The known ones, which are only divisible by Mersenne primes (as odd factors), belong to the equivalence classes (see Lemma 3.5) of $B_1, \ldots, B_9$ (see **Notation**). The other ones (which are divisible by non-Mersenne primes) belong to several different (perhaps, infinitely many) classes (see [2] and [11]).

Since a few moments, we would like to continue this investigation (with more or less success). In particular, we want to find all non-splitting (unitary) perfect binary polynomials which are only divisible by x, $x+1$ and by Mersenne primes. Some results are obtained [7, Theorems 1.1 and 1.3] but they are not complete. The main obstacle is the fact that we cannot understand how $M^{2h+1} + 1 = (M+1)\sigma(M^{2h})$ factors over $\mathbb{F}_2$, for a Mersenne prime M and a positive integer h. We have formulated [9] a conjecture about that (Conjecture 4.1). The further we make progress on that conjecture, the better we reach our goal. Conjecture 4.1 is already proved under some conditions on M and h [9, Theorem 1.4]. In this paper, we continue working toward its proof with some new conditions on M and h, where the sets $\mathcal{M}$ and Δ defined below intersect. We get Proposition 1.1 which in turn, allows us to obtain Theorems 1.1 and 1.2.

The study of Mersenne primes have some interest. For example, we have established [9, Theorem 1.3] that if $\gcd(a, b) = 1$, then $U_{a,b} = x^a(x+1)^b + 1$ has exactly the same number of irreducible divisors as the trinomial $x^{a+b} + x^b + 1$. In particular, they are both irreducible or both not irreducible. So, they would be useful in the domain of error-correcting codes.

It is convenient to fix some notation.

Notation.

- The set of integers (resp. of nonnegative integers, of positive integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{N}$, $\mathbb{N}^*$).
- For $S, T \in \mathbb{F}_2[x]$ and for $m \in \mathbb{N}^*$, $S^m \mid T$ (resp. $S^m \| T$) means that S divides T (resp. $S^m \mid T$ but $S^{m+1} \nmid T$). We also denote by $\overline{S}$ the polynomial defined as $\overline{S}(x) = S(x+1)$ and by $val_x(S)$ (resp. $val_{x+1}(S)$) the valuation of S, at x (resp. at $x+1$).
- We put

$$M_1 = 1 + x(x+1), \quad M_2 = 1 + x(x+1)^2, \quad M_3 = 1 + x(x+1)^3,$$
$$T_1 = x^2(x+1)M_1, \quad T_3 = x^4(x+1)^3 M_3, \quad T_2 = \overline{T_1}, \quad T_4 = \overline{T_3},$$

$$\begin{aligned}
T_5 =& x^4(x+1)^4 M_3\overline{M_3} = \overline{T_5}, \quad T_6 = x^6(x+1)^3 M_2\overline{M_2}, \quad T_7 = \overline{T_6},\\
T_8 =& x^4(x+1)^6 M_2\overline{M_2}M_3, \quad T_9 = \overline{T_8},\\
C_1 =& x^2(x+1)M_1^2(x^4+x+1), \quad C_2 = \overline{C_1},\\
B_1 =& x^3(x+1)^3 M_1^2, \quad B_2 = x^3(x+1)^2 M_1, \quad B_3 = x^5(x+1)^4 M_3,\\
B_4 =& x^7(x+1)^4 M_2\overline{M_2}, \quad B_5 = x^5(x+1)^6 M_1^2 M_3, \quad B_6 = x^5(x+1)^5 M_3\overline{M_3},\\
B_7 =& x^7(x+1)^7 M_2^2\overline{M_2}^2, \quad B_8 = x^7(x+1)^6 M_1^2 M_2\overline{M_2},\\
B_9 =& x^7(x+1)^5 M_2\overline{M_2 M_3}.
\end{aligned}$$

- The following sets play important roles:

$$\begin{aligned}
\mathcal{M} =& \{M_1, M_2, \overline{M_2}, M_3, \overline{M_3}\},\\
\mathcal{P} =& \{T_1, \ldots, T_9\}, \quad \mathcal{P}_u = \{B_1, \ldots, B_9\},\\
\Delta_1 =& \{p \in \mathbb{N}^* : p \text{ is a Mersenne prime}\},\\
\Delta_2 =& \{p \in \mathbb{N}^* : p \text{ is prime and } ord_p(2) \equiv 0 \bmod 8\},\\
\Delta =& \Delta_1 \cup \Delta_2,
\end{aligned}$$

where $ord_p(2)$ denotes the order of 2 in $\mathbb{F}_p \setminus \{0\}$. In particular, Δ contains all Fermat primes greater than 5.

Throughout this paper, we always suppose that any (unitary) perfect polynomial is indecomposable. We have often used Maple software for computations. Our main results are the following.

Proposition 1.1. *Let $h \in \mathbb{N}^*$ and let $M \in \mathbb{F}_2[x]$ be a Mersenne prime. Then in the following cases, $\sigma(M^{2h})$ is divisible by a non-Mersenne prime:*
(i) *$M \in \{M_1, M_3, \overline{M_3}\}$ or ($M \in \{M_2, \overline{M_2}\}$ and $h \geq 2$);*
(ii) *$M \notin \mathcal{M}$ and $2h+1$ is divisible by a prime number p lying in $\Delta \setminus \{7\}$.*

Theorem 1.1. *Let $A = x^a(x+1)^b \prod_{i \in I} P_i^{h_i} \in \mathbb{F}_2[x]$ be such that each P_i is a Mersenne prime and $a, b, h_i \in \mathbb{N}^*$. Then, A is perfect if and only if $A \in \mathcal{P}$.*

Theorem 1.2. *Let $A = x^a(x+1)^b \prod_{i \in I} P_i^{h_i} \in \mathbb{F}_2[x]$ be such that each P_i is a Mersenne prime and $a, b, h_i \in \mathbb{N}^*$. Then, A is unitary perfect if and only if $A = B^{2^n}$, for some $n \in \mathbb{N}$ and $B \in \mathcal{P}_u$.*

We first prove the two theorems before the proposition.

2. Proof of Theorem 1.1

Sufficiencies are obtained by direct computations. For the necessities, we shall apply Lemma 2.3 and Proposition 2.1. We fix: $A = x^a(x+1)^b \prod_{i \in I} P_i^{h_i} = A_1 A_2$, where $a, b, h_i \in \mathbb{N}$, P_i is a Mersenne prime, $A_1 = x^a(x+1)^b \prod_{P_i \in \mathcal{M}} P_i^{h_i}$ and $A_2 = \prod_{P_j \notin \mathcal{M}} P_j^{h_j}$.

Lemma 2.1. *If A is perfect, then $\sigma(x^a)$, $\sigma((x+1)^b)$ and each $\sigma({P_i}^{h_i})$, with $i \in I$, are only divisible by x, $x+1$ or by Mersenne primes.*

Proof. Since σ is multiplicative, $\sigma(A) = \sigma(x^a)\sigma((x+1)^b)\prod_{i\in I}\sigma({P_i}^{h_i})$. Any divisor of $\sigma(x^a)$, $\sigma((x+1)^b)$ and $\sigma({P_i}^{h_i})$ divides $\sigma(A) = A$. □

Lemma 2.2 ([4], Lemma 2)**.** *A polynomial S is perfect if and only if for any irreducible polynomial P and for any $m_1, m_2 \in \mathbb{N}^*$, we have*

$$(P^{m_1}\|S, P^{m_2}\|\sigma(S)) \Rightarrow m_1 = m_2.$$

The following example will be useful for Proposition 2.1.

Example 2.1. $S_1 = x^{13}(x+1)^2 M_1^3 {M_2}^2 \overline{M_2}^{\,2} M_3\overline{M_3}$ is not perfect because $x^{13}\|S_1$ and $x^7\|\sigma(S_1)$.

Lemma 2.3 ([7], Theorem 1.1)**.** *If $h_i = 2^{n_i} - 1$ for any $i \in I$, then $A \in \mathcal{P}$.*

We get from Theorem 8 in [5] and from Proposition 1.1.

Lemma 2.4. (i) *If $h \in \mathbb{N}^*$ and if $\sigma(x^{2h})$ is only divisible by Mersenne primes, then $2h \in \{2,4,6\}$ and all its divisors lie in $\mathcal{M}$. More precisely, $\sigma(x^2) = M_1 = \sigma((x+1)^2)$, $\sigma(x^4) = M_3$, $\sigma((x+1)^4) = \overline{M_3}$ and $\sigma(x^6) = M_2\overline{M_2} = \sigma((x+1)^6)$.*

(ii) *Let $M \in \mathcal{M}$ and $h \in \mathbb{N}^*$ be such that $\sigma(M^{2h})$ is only divisible by Mersenne primes, then $2h = 2$, $M \in \{M_2, \overline{M_2}\}$ and $\sigma(M^2) \in \{M_1M_3, M_1\overline{M_3}\}$.*

We dress from Lemma 2.4, the following tables of all the forms of a, b, P_i and h_i which satisfy Lemma 2.1, if $P_i \in \mathcal{M}$ and if $h_i \neq 2^{n_i} - 1$.

TABLE 1. Some $\sigma(x^a)$ and $\sigma((x+1)^b)$

a	$\sigma(x^a)$	b	$\sigma((x+1)^b)$
$3\cdot 2^n - 1$	$(x+1)^{2^n-1}{M_1}^{2^n}$	$3\cdot 2^m - 1$	$x^{2^m-1}{M_1}^{2^m}$
$5\cdot 2^n - 1$	$(x+1)^{2^n-1}{M_3}^{2^n}$	$5\cdot 2^m - 1$	$x^{2^m-1}\overline{M_3}^{\,2^m}$
$7\cdot 2^n - 1$	$(x+1)^{2^n-1}{M_2}^{2^n}\ \overline{M_2}^{\,2^n}$	$7\cdot 2^m - 1$	$x^{2^m-1}{M_2}^{2^m}\ \overline{M_2}^{\,2^m}$

TABLE 2. Some $\sigma({P_i}^{h_i})$

P_i	h_i	$\sigma({P_i}^{h_i})$
M_2	$3\cdot 2^{n_i} - 1$	$(1+M_2)^{2^{n_i}-1}{M_1}^{2^{n_i}}\ \overline{M_3}^{\,2^{n_i}}$
$\overline{M_2}$	$3\cdot 2^{n_i} - 1$	$(1+\overline{M_2})^{2^{n_i}-1}{M_1}^{2^{n_i}}\ {M_3}^{2^{n_i}}$

Corollary 2.1. *Suppose that A_1 is perfect. Then, neither M_2 nor $\overline{M_2}$ divides $\sigma({P_i}^{h_i})$ if $P_i \in \mathcal{M}$. Moreover, $\overline{M_2}$ divides A_1 whenever M_2 divides A_1 and their exponents (in A_1) are equal.*

Proof. The first statement follows from Lemma 2.4 (ii). Now, if M_2 divides $A_1 = \sigma(A_1)$, then M_2 divides $\sigma(x^a)\,\sigma((x+1)^b)\prod_{P_i \in \mathcal{M}} \sigma(P_i^{h_i})$. Hence, M_2 divides $\sigma(x^a)\sigma((x+1)^b)$. Table 1 shows that a or b is of the form $7 \cdot 2^n - 1$, where $n \in \mathbb{N}$. So, $\overline{M_2}$ divides $\sigma(A_1) = A_1$. It suffices to consider two cases. If $a = 7 \cdot 2^n - 1$ and $b = 7 \cdot 2^m - 1$, then ${M_2}^{\ell} \| A_1$ and $\overline{M_2}^{\ \ell} \| A_1$, with $\ell = 2^n + 2^m$. If $a = 7 \cdot 2^n - 1$ and $b = 3 \cdot 2^m - 1$ or $b = 5 \cdot 2^m - 1$, then ${M_2}^{\ell} \| A_1$ and $\overline{M_2}^{\ \ell} \| A_1$, with $\ell = 2^n$. ☐

Lemma 2.5. *If P is a Mersenne prime divisor of $\sigma(A_1)$, then $P, \overline{P} \in \{M_1, M_2, M_3\}$.*

Proof. One has $\sigma(A_1) = \sigma(x^a)\sigma((x+1)^b)\prod_{P_i \in \mathcal{M}} \sigma(P_i^{h_i})$. If P divides $\sigma(x^a)\sigma((x+1)^b)$, then $P \in \mathcal{M}$, by Lemma 2.4 (i). If P divides $\sigma(P_i^{h_i})$ with $P_i \in \mathcal{M}$, then $P_i \in \{M_2, \overline{M_2}\}$, $h_i = 2$ or h_i is of the form $3 \cdot 2^{n_i} - 1$ and $P, \overline{P} \in \{M_1, M_3\}$ (see Table 2). ☐

Lemma 2.6. *If A is perfect, then $A = A_1$.*

Proof. We claim that $A_2 = 1$. Let $P_j \notin \mathcal{M}$ and $Q_i \in \mathcal{M}$. Then, P_j divides neither $\sigma(x^a)$, $\sigma((x+1)^b)$ nor $\sigma(Q_i^{h_i})$. Thus $\gcd(P_j^{h_j}, \sigma(A_1)) = 1$. Observe that $P_j^{h_j}$ divides $\sigma(A_2)$ because $P_j^{h_j}$ divides $A = \sigma(A) = \sigma(A_1)\sigma(A_2)$. Hence, A_2 divides $\sigma(A_2)$. So, A_2 is perfect and it is equal to 1, A being indecomposable. ☐

Proposition 2.1. *If A_1 is perfect, then $h_j = 2^{n_j} - 1$ for any $P_j \in \mathcal{M}$.*

Proof. We refer to Table 2.

(i) Suppose that $P_j \notin \{M_2, \overline{M_2}\}$. If h_j is even, then $\sigma(P_j^{h_j})$ is divisible by a non-Mersenne prime. It contradicts Lemma 2.1. If $hj = 2^{n_j} u_j - 1$ with $u_j \geq 3$ odd, then $\sigma(P_j^{h_j}) = (1+P_j)^{2^{n_j}-1}(1+P_j+\cdots+P_j^{u_j-1})^{2^{n_j}}$. Since $1+P_j+\cdots+P_j^{u_j-1} = \sigma({P_j}^{u_j-1})$ is divisible by a non-Mersenne prime, we also get a contradiction to Lemma 2.1.

(ii) If $P_j \in \{M_2, \overline{M_2}\}$ and h_j is even or it is of the form $2^{n_j} u_j - 1$, with $u_j \geq 3$ odd and $n_j \geq 1$, then Corollary 2.1 implies that there exists $\ell \in \mathbb{N}^*$ such that ${M_2}^{\ell} \| A_1$ and $\overline{M_2}^{\ \ell} \| A_1$. Recall that $\sigma({M_2}^2) = M_1\overline{M_3}$ and $\sigma(\overline{M_2}^{\ 2}) = M_1 M_3$. We proceed as in the proof of Corollary 2.1. It suffices to distinguish four cases which give contradictions.

• Case 1: $a = 7 \cdot 2^n - 1$ and $b = 7 \cdot 2^m - 1$. One has $\ell = 2^n + 2^m$ and neither M_1 nor M_3 divides $\sigma(x^a)\ \sigma((x+1)^b)$.

If h_j is even, then $h_j = 2 = \ell$. So, $n = m = 0$, ${M_1}^2 \| \sigma(A_1) = A_1$. It contradicts the part (i) of our proof.

If $h_j = 2^{n_j} u_j - 1$ with $u_j \geq 3$ odd and $n_j \geq 1$, then $u_j = 3$ and ${M_1}^{2 \cdot 2^{n_j}} \| A_1$.

• Case 2: $a = 7 \cdot 2^n - 1$ and $b = 5 \cdot 2^m - 1$.

One has $\ell = 2^n$ and $M_1 \nmid \sigma(x^a)\sigma((x+1)^b)$. If h_j is even, then $2^n = \ell = h_j = 2$. So, $n = 1$ and ${M_1}^2 \| A_1$. If $h_j = 2^{n_j} u_j - 1$, with $u_j \geq 3$ odd and $n_j \geq 1$, then $u_j = 3$ and $2^n = \ell = h_j = 3 \cdot 2^{n_j} - 1$. It is impossible.

• Case 3: $a = 7 \cdot 2^n - 1$, $b = 3 \cdot 2^m - 1$ and h_j is even.

As above, $2^n = \ell = h_j = 2$, ${M_1}^{2^m}$ divides $\sigma((x+1)^b)$ and ${M_1}^{2^n + 2^m}$ divides $\sigma(A_1) = A_1$. So, $n = 1$ and ${M_1}^{2^m + 2} \| A_1$. Thus, the part (i) implies that $m = 0$.

Hence, $A_1 = S_1 = x^{13}(x+1)^2 M_1^3 {M_2}^2 \overline{M_2}^{\,2} M_3 \overline{M_3}$ which is not perfect (see Example 2.1).

• Case 4: $a = 7 \cdot 2^n - 1$, $b = 3 \cdot 2^m - 1$, $h_j = 2^{n_j} u_j - 1$, $u_j \geq 3$ odd, $n_j \geq 1$.
One has $u_j = 3$ and $2^n = \ell = h_j = 3 \cdot 2^{n_j} - 1$. It is impossible. ☐

Lemma 2.6, Proposition 2.1 and Lemma 2.3 imply the following result.

Corollary 2.2. *If A is perfect, then $A = A_1 \in \mathcal{P}$.*

3. Proof of Theorem 1.2

As in Section 2, we fix $A = x^a(x+1)^b \prod_{i \in I} P_i^{h_i} = A_1 A_2$, where $a, b, h_i \in \mathbb{N}$, P_i is a Mersenne prime, $A_1 = x^a(x+1)^b \prod_{P_i \in \mathcal{M}} P_i^{h_i}$ and $A_2 = \prod_{P_j \notin \mathcal{M}} P_j^{h_j}$.

Sufficiencies are obtained by direct computations. For the necessities, we shall apply Lemma 3.6 and Proposition 3.1. The latter is proved (by similar arguments) as Lemma 2.6 and Proposition 2.1.

Lemma 3.1. *If A is unitary perfect, then $\sigma^*(x^a)$, $\sigma^*((x+1)^b)$, $\sigma^*({P_i}^{h_i})$, for any $i \in I$, are only divisible by x, $x+1$ or by Mersenne primes.*

Proof. Since σ^* is multiplicative, $\sigma^*(A) = \sigma^*(x^a)\sigma^*((x+1)^b) \prod_{i \in I} \sigma^*({P_i}^{h_i})$, any divisor of $\sigma^*(x^a)$, $\sigma^*((x+1)^b)$, $\sigma^*({P_i}^{h_i})$ divides $\sigma^*(A) = A$. ☐

Lemma 3.2 ([4], Lemma 2)**.** *A polynomial S is unitary perfect if and only if for any irreducible polynomial P and for any $m_1, m_2 \in \mathbb{N}^*$, we have*

$$P^{m_1} \| S, P^{m_2} \| \sigma^*(S) \Rightarrow m_1 = m_2.$$

We shall need the example below to prove Proposition 3.1.

Example 3.1. Since $x^{14} \| S_2$ and $x^{10} \| \sigma^*(S_2)$, the polynomial

$$S_2 = x^{14}(x+1)^7 {M_1}^2 {M_2}^3 \overline{M_2}^{\,3} M_3 \overline{M_3}$$

is not unitary perfect.

Lemma 3.3. *Let $S \in \mathbb{F}_2[x]$ be an irreducible polynomial. Then, for any $n, u \in \mathbb{N}$ with u odd, $\sigma^*(S^{2^n u}) = (1+S)^{2^n} (\sigma(S^{u-1}))^{2^n}$.*

Lemma 3.4. *Let $C \in \mathbb{F}_2[x] \setminus \{0, 1\}$ be u.p. Then C is even, $\overline{C}$ and C^{2^r} are also u.p, for any $r \in \mathbb{N}$.*

Proof. If D is a divisor of C, then $\overline{D}$ divides $\overline{C}$ and D^{2^r} divides C^{2^r}. Thus, $\sigma^*(\overline{C}) = \overline{\sigma^*(C)} = \overline{C}$ and $\sigma^*(C^{2^r}) = (\sigma^*(C))^{2^r} = C^{2^r}$. It remains to prove that C is even. Consider an irreducible divisor P of C and $k \in \mathbb{N}^*$ such that $P^k \| C$. The polynomial $1 + P$ is even and divides $1 + P^k = \sigma^*(P^k)$. So, $1 + P$ divides $\sigma^*(C) = C$. ☐

Definition 3.1. We denote by $\sim$ the relation on $\mathbb{F}_2[x]$ defined as: $S \sim T$ if there exists $\ell \in \mathbb{Z}$ such that $S = T^{2^\ell}$.

Lemma 3.5 ([3], Section 2)**.** *The relation $\sim$ is an equivalence relation on $\mathbb{F}_2[x]$. Each equivalence class contains a unique polynomial B which is not a square, with $val_x(B) \leq val_{x+1}(B)$.*

Lemma 3.6 ([7], Theorem 1.3)**.** *If $h_i = 2^{n_i}$ for any $i \in I$, then A (or $\overline{A}$) is of the form B^{2^n}, where $B \in \mathcal{P}_u$.*

The following tables, obtained from Lemmas 2.1, 2.4 and 3.3, are useful to prove Proposition 3.1.

TABLE 3. Some $\sigma^*(x^a)$ and $\sigma^*((x+1)^b)$

a	$\sigma^*(x^a)$	b	$\sigma^*((x+1)^b)$
$3 \cdot 2^n$	$(x+1)^{2^n} {M_1}^{2^n}$	$3 \cdot 2^m$	$x^{2^m} {M_1}^{2^m}$
$5 \cdot 2^n$	$(x+1)^{2^n} {M_3}^{2^n}$	$5 \cdot 2^m$	$x^{2^m} \overline{M_3}^{\,2^m}$
$7 \cdot 2^n$	$(x+1)^{2^n} {M_2}^{2^n}\, \overline{M_2}^{\,2^n}$	$7 \cdot 2^m$	$x^{2^m} {M_2}^{2^m}\, \overline{M_2}^{\,2^m}$

TABLE 4. Some $\sigma^*({P_i}^{h_i})$

P_i	h_i	$\sigma^*({P_i}^{h_i})$
M_2	$3 \cdot 2^{n_i}$	$(1+M_2)^{2^{n_i}} {M_1}^{2^{n_i}}\, \overline{M_3}^{\,2^{n_i}}$
$\overline{M_2}$	$3 \cdot 2^{n_i}$	$(1+\overline{M_2})^{2^{n_i}} {M_1}^{2^{n_i}}\, {M_3}^{2^{n_i}}$

Proposition 3.1. (i) *If A is u.p, then $A = A_1$.*
(ii) *If A_1 is u.p, then $h_j = 2^{n_j}$ for any $P_j \in \mathcal{M}$.*
(iii) *If A is u.p, then A or $\overline{A}$ is of the form B^{2^n}, where $B \in \mathcal{P}_u$.*

Proof. The proof of (i) is analogous to that of Lemma 2.6. The statement (iii) follows from (i), (ii) and Lemma 3.6. We only sketch the proof of (ii). Set $h_j = 2^{n_j} u_j$, where u_j is odd and $n_j \geq 0$. Suppose that $P_j \notin \{M_2, \overline{M_2}\}$. If $u_j \geq 3$, then $\sigma(P_j^{u_j-1})$ and thus $\sigma^*(P_j^{h_j})$ are divisible by a non-Mersenne prime. It contradicts Lemma 2.1. Now, if $P_j \in \{M_2, \overline{M_2}\}$ and if $u_j \geq 3$, then $u_j = 3$ and (a or b is of the form $7 \cdot 2^n$). Recall that $\sigma^*({M_2}^3) = (1+M_2)M_1\overline{M_3}$ and $\sigma^*(\overline{M_2}^{\,3}) = (1+\overline{M_2})M_1M_3$. We consider two cases. The first gives non unitary perfect polynomials whereas the second leads to a contradiction.

• Case 1: $a = 7 \cdot 2^n$ and $b = 7 \cdot 2^m$, with $n, m \geq 0$.

One has ${M_2}^{\ell} \| A_1$ and $\overline{M_2}^{\,\ell} \| A_1$, with $\ell = 2^n + 2^m$. Neither M_1 nor M_3 divides $\sigma(x^a)\ \sigma((x+1)^b)$. Thus, $3 \cdot 2^{n_j} = h_j = \ell = 2^n + 2^m$. So, $n = m+1$ and $n_j = m$ or $m = n+1$ and $n_j = n$. Therefore, $({M_1}^2)^{2^{n_j}}$, ${M_3}^{2^{n_j}}$ and $\overline{M_3}^{\,2^{n_j}}$ divide $\sigma^*(M_2^{h_j})\sigma^*(\overline{M_2}^{\,h_j})$ and they divide $\sigma^*(A_1) = A_1$. Thus, $A_1 = {S_2}^{2^m}$ or $A_1 = \overline{S_2}^{\,2^n}$ where $S_2 = x^{14}(x+1)^7{M_1}^2{M_2}^3\overline{M_2}^{\,3}M_3\overline{M_3}$. In both cases, A_1 is not unitary perfect because S_2 is not u.p (Example 3.1).

• Case 2: $a = 7 \cdot 2^n$ and $b = 5 \cdot 2^m$ or $b = 3 \cdot 2^m$, with $n, m \geq 0$.
One has $\ell = 2^n$. So, we get the contradiction $3 \cdot 2^{n_j} = h_j = \ell = 2^n$. □

4. Proof of Proposition 1.1

That proposition partially solves [9, Conjecture 1.1], which we recall here.

Conjecture 4.1 ([9], Conjecture 1.1). Let $h \in \mathbb{N}^*$ and let $M \in \mathbb{F}_2[x]$ be a Mersenne prime. Then, $\sigma(M^{2h})$ is always divisible by a non-Mersenne prime, except for $M \in \{M_2, M_3\}$ and $h = 1$.

We mainly prove it by contradiction (to Corollary 4.1). Lemma 4.1 states that $\sigma(M^{2h})$ is square-free, for any $h \in \mathbb{N}^*$. Recall that we set $M = x^a(x+1)^b + 1$, $U_{2h} = \sigma(\sigma(M^{2h}))$ and

$$\sigma(M^{2h}) = \prod_{j \in J} P_j, \quad P_j = 1 + x^{a_j}(x+1)^{b_j} \text{ irreducible}, P_i \neq P_j \text{ if } i \neq j. \tag{4.1}$$

By Lemma 4.3, if there exists a prime divisor p of $2h+1$ such that $\sigma(M^{p-1})$ is divisible by a non-Mersenne prime, then $\sigma(M^{2h})$ is also divisible by a non-Mersenne. Therefore, it suffices to consider that $2h + 1 = p$ is a prime number, except for $p = 3$ with $M \in \{M_2, \overline{M_2}\}$ (see Section 4.1).

4.1. **Useful facts.** For $S \in \mathbb{F}_2[x] \backslash \{0, 1\}$, of degree s, we denote by $\alpha_l(S)$ the coefficient of x^{s-l} in S, $0 \leq l \leq s$. One has $\alpha_0(S) = 1$.

Lemma 4.1 ([9], Lemmas 4.6 and 4.8). *The polynomial $\sigma(M^{2h})$ is square-free and $M \neq M_1$.*

Lemma 4.2 ([9], Theorem 1.4). *Let $h \in \mathbb{N}^*$ be such that $p = 2h+1$ is prime and let M be a Mersenne prime such that $M \notin \{M_2, \overline{M_2}\}$ and $\omega(\sigma(M^{2h})) = 2$. Then, $\sigma(M^{2h})$ is divisible by a non-Mersenne prime.*

The lemma below generalizes Lemma 4.10 in [9] (with an analogous proof).

Lemma 4.3. *If k is a divisor (prime or not) of $2h+1$, then $\sigma(M^{k-1})$ divides $\sigma(M^{2h})$.*

We sometimes apply Lemmas 4.4 and 4.5 without explicit mentions.

Lemma 4.4. *Let $S \in \mathbb{F}_2[x]$ be such that $s = \deg(S) \geq 1$ and $l, t, r, r_1, \ldots, r_k \in \mathbb{N}$ be such that $r_1 > \cdots > r_k$, $t \leq k, r_1 - r_t \leq l \leq r \leq s$. Then*

(i) $\alpha_l[(x^{r_1} + \cdots + x^{r_k})S] = \alpha_l(S) + \alpha_{l-(r_1-r_2)}(S) + \cdots + \alpha_{l-(r_1-r_t)}(S)$;

(ii) $\alpha_l(\sigma(S)) = \alpha_l(S)$ *if any divisor of S has degree at least $r+1$.*

Proof. The equality in (i) (resp. in (ii)) follows from the definition of α_l (resp. from the fact $\sigma(S) = S + T$, where $\deg(T) \leq \deg(S) - r - 1$). □

Corollary 4.1. (i) *The integers $u = \sum_{j \in J} a_j$ and $v = \sum_{j \in J} b_j$ are both even.*

(ii) *The polynomial U_{2h} splits (over $\mathbb{F}_2$) and it is a square.*

(iii) *The polynomial $\sigma(M^{2h})$ is reducible.*

Proof. (i) See [9, Corollary 4.9]. For (ii), one has $U_{2h} = \sigma(\sigma(M^{2h})) = \sigma(\prod_{j \in J} P_j)$, from Assumption (4.1). Hence, $U_{2h} = \prod_{j \in J} x^{a_j}(x+1)^{b_j} = x^u(x+1)^v$, where u and v are both even.

(iii) If $\sigma(M^{2h}) = Q$ is irreducible, then $U_{2h} = 1 + Q$ is not a square. □

Lemma 4.5. *One has* $\alpha_l(\sigma(M^{2h})) = \alpha_l(M^{2h})$ *if* $l \leq a+b-1$ *and* $\alpha_l(\sigma(M^{2h})) = \alpha_l(M^{2h} + M^{2h-1})$ *if* $a+b \leq l \leq 2(a+b)-1$.

Proof. Since $\sigma(M^{2h}) = M^{2h} + M^{2h-1} + T$, with $\deg(T) \leq (a+b)(2h-2) = 2h(a+b) - 2(a+b)$, Lemma 4.4 (ii) implies that $\alpha_l(\sigma(M^{2h})) = \alpha_l(M^{2h})$ if $l \leq a+b-1$ and $\alpha_l(\sigma(M^{2h})) = \alpha_l(M^{2h} + M^{2h-1})$ if $a+b \leq l \leq 2(a+b)-1$. □

Lemma 4.6. *Denote by* $N_2(m)$ *the number of irreducible polynomials over* $\mathbb{F}_2$*, of degree* $m \geq 1$*. Then*

(i) $N_2(m) \geq \frac{2^m - 2(2^{m/2}-1)}{m}$;
(ii) $\varphi(m) < N_2(m)$ *if* $m \geq 4$*, where* φ *is the Euler totient function;*
(iii) *for each* $m \geq 4$*, there exists an irreducible polynomial of degree* m*, which is not a Mersenne prime.*

Proof. (i) See [10, Exercise 3.27, page 142].

(ii) If $m \in \{4, 5\}$, then direct computations give $\varphi(4) = 2$, $N_2(4) = 3$ and $\varphi(5) = 4$, $N_2(5) = 6$. Now, suppose that $m \geq 6$. Consider the function $f(x) = 2^x - 2(2^{x/2} - 1) - x^2$, for $x \geq 6$. The derivative of f is a positive function. So, $f(x) \geq f(6) > 0$ and $x < \frac{2^x - 2(2^{x/2}-1)}{x}$. Thus, $\varphi(m) \leq m < \frac{2^m - 2(2^{m/2}-1)}{m} \leq N_2(m)$.

(iii) We remark that if $1 + x^c(x+1)^d$ is a Mersenne prime, then $\gcd(c, d) = 1$. So, $\gcd(c, c+d) = 1$. Therefore, the set $\mathcal{M}_m$ of Mersenne primes of degree m is a subset of $\{x^c(x+1)^{m-c} + 1 : 1 \leq c \leq m,\ \gcd(c, m) = 1\}$. Thus,

$$\#\mathcal{M}_m \leq \#\{c : 1 \leq c \leq m,\ \gcd(c, m) = 1\} = \varphi(m).$$

Hence, there exist at least $N_2(m) - \varphi(m)$ irreducible non-Mersenne polynomials, with $N_2(m) - \varphi(m) \geq 1$, by (ii). □

Lemma 4.7. *For any* $j \in J$*,* $ord_p(2)$ *divides* $a_j + b_j = \deg(P_j)$.

Proof. Set $d = \gcd_{i \in J}(a_i + b_i)$. By Lemma 4.13 in [9], p divides $2^d - 1$. Thus, $ord_p(2)$ divides d. □

Lemma 4.8 ([10], Chapter 2 and 3)**.** *Let* $q = 2^r - 1$ *be a Mersenne prime number. Then, any irreducible polynomial* P *of degree* r *is primitive. In particular, each root* β *of* P *is a primitive element of the field* $\mathbb{F}_{2^r}$*, so that* β *is of order* q *in* $\mathbb{F}_{2^r} \setminus \{0\}$.

Lemma 4.9. *Let* $P_i = 1 + x^{a_i}(x+1)^{b_i}$ *be a prime divisor of* $\sigma(M^{p-1})$*, where* $2^{a_i+b_i} - 1 = p_i$ *is a prime number. Then,* $p_i = p$ *and* $\sigma(M^{p-1})$ *is divisible by any irreducible polynomial of degree* $a_i + b_i$*. Furthermore, at least one of those divisors is not a Mersenne prime if* $a_i + b_i \geq 4$.

Proof. The polynomial P_i is primitive. If α is a root of P_i, then $(M^p+1)(\alpha) = 0$ and $M(\alpha) = \alpha^r$ for some $1 \leq r \leq p_i - 1$. Thus, $1 = M(\alpha)^p = \alpha^{rp}$, with $ord(\alpha) = p_i$. So, p_i divides rp and $p_i = p$. Any irreducible polynomial S of degree $a_i + b_i$ is primitive. Let β be a root of S. One has $ord(\beta) = p_i = p$, $S(\beta) = 0$ and $M(\beta) = \beta^s$, for some $1 \leq s \leq p_i - 1$. Thus, $M(\beta)^p = \beta^{ps} = 1$ and S divides $M^p + 1 = x^a(x+1)^b\sigma(M^{p-1})$. The third statement follows from Lemma 4.6 (iii). ☐

Corollary 4.2. *For any $i \in J$, $a_i + b_i \leq 3$ or $2^{a_i+b_i} - 1$ is not prime.*

Lemma 4.10. *Let $P, Q \in \mathbb{F}_2[x]$ be such that $\deg(P) = r$, $2^r - 1$ is prime, $P \nmid Q(Q+1)$ but $P \mid Q^p + 1$. Then $2^r - 1 = p$.*

Proof. The polynomial P is primitive. If β is a root of P, then $ord(\beta) = 2^r - 1$. Moreover, $Q(\beta) \notin \{0, 1\}$ because $P \nmid Q(Q+1)$. Thus, $Q(\beta) = \beta^t$ for some $1 \leq t \leq 2^r - 2$. Hence, $1 = Q(\beta)^p = \beta^{tp}$. So, $2^r - 1$ divides tp and $2^r - 1 = p$. ☐

Corollary 4.3. *Let $r \in \mathbb{N}^*$ be such that $2^r - 1$ is a prime distinct from p. Then, no irreducible polynomial of degree r divides $\sigma(M^{p-1})$.*

Proof. If P is a prime divisor of $\sigma(M^{p-1})$ with $\deg(P) = r$, then P divides $M^p + 1$ and by taking $Q = M$ in the above lemma, we get a contradiction. ☐

In the following lemma and two corollaries, we suppose that p is a Mersenne prime of the form $2^m - 1$ (with m prime).

Lemma 4.11. *Let $P, Q \in \mathbb{F}_2[x]$ be such that P is irreducible of degree m and $P \nmid Q(Q+1)$. Then, P divides $Q^p + 1$.*

Proof. The polynomial P is primitive. If β is a root of P, then $ord(\beta) = 2^m - 1 = p$, $Q(\beta) \notin \{0, 1\}$ because $P \nmid Q(Q+1)$. Thus, $Q(\beta) = \beta^t$ for some $1 \leq t \leq p - 1$. Hence, $Q(\beta)^p = \beta^{tp} = 1$. So, P divides $Q^p + 1$. ☐

Corollary 4.4. *Any irreducible polynomial $P \neq M$ (Mersenne or not), of degree m, divides $\sigma(M^{p-1})$.*

Proof. We may apply Lemma 4.11, with $Q = M$, because P does not divide $x^a(x+1)^bM = M(M+1) = Q(Q+1)$. So, P is odd and it divides $M^p + 1 = (M+1)\ \sigma(M^{p-1}) = x^a(x+1)^b\ \sigma(M^{p-1})$. ☐

Corollary 4.5. *The polynomial M_1 (resp. M_2, $\overline{M_2}$) divides $\sigma(M^{p-1})$ if and only if ($M \neq M_1$ and $p = 3$) (resp. $M \neq M_2$ and $p = 7$, $M \neq \overline{M_2}$ and $p = 7$).*

Proof. Apply Corollary 4.4 with $m \in \{2, 3\}$. ☐

In order to carry on the proof (of Proposition 1.1), we distinguish three cases.
Case I: $M \in \{M_1, M_3, \overline{M_3}\}$.

Lemma 4.1 implies that $M \neq M_1$. It suffices to suppose that $M = M_3$. We refer to Section 5.2 in [8]. Put $D = M_1M_2\overline{M_2}$. By [8, Lemma 5.4], we have to consider four situations:

(i) $\gcd(\sigma(M^{2h}), D) = 1$;
(ii) $\sigma(M^{2h}) = M_1 B$, with $\gcd(B, D) = 1$;
(iii) $\sigma(M^{2h}) = M_2 \overline{M_2} B$, with $\gcd(B, D) = 1$;
(iv) $\sigma(M^{2h}) = DB$, with $\gcd(B, D) = 1$, where any irreducible divisor of B has degree exceeding 5.

The following lemma contradicts the fact that U_{2h} is a square.

Lemma 4.12. *One has $\alpha_3(U_{2h}) = 1$ or $\alpha_5(U_{2h}) = 1$.*

Proof. For (i), (iii) and (iv), use [8, Lemmas 5.9, 5.10, 5.15 and 5.17].

(ii) Since $\sigma(M^{2h}) = (x^2+x+1)B$ and $U_{2h} = (x^2+x)\sigma(B)$, we obtain (by Lemmas 4.4 and 4.5): $0 = \alpha_1(M^{2h}) = \alpha_1(\sigma(M^{2h})) = \alpha_1(B)+1$, $\alpha_3(U_{2h}) = \alpha_3(\sigma(B))+\alpha_2(\sigma(B)) = \alpha_3(B) + \alpha_2(B)$, $0 = \alpha_3(M^{2h}) = \alpha_3(\sigma(M^{2h})) = \alpha_3(B) + \alpha_2(B) + \alpha_1(B)$.

Thus, $\alpha_3(U_{2h}) = \alpha_3(B) + \alpha_2(B) = \alpha_1(B) = 1$. ☐

Case II: $M \in \{M_2, \overline{M_2}\}$ and $h \geq 2$.

It suffices to consider that $M = M_2$.

Lemma 4.13. (i) *If $h \geq 4$, then M_1 divides $\sigma(M^{2h})$ if and only if 3 divides $2h + 1$.*

(ii) *If $h \geq 4$, then M_2 divides $\sigma(M^{2h})$ if and only if 7 divides $2h + 1$.*

(iii) *If $h \geq 4$ and if $2h + 1$ is divisible by a prime $p \notin \{3, 7\}$, then any irreducible divisor of $\sigma(M^{2h})$ is of degree at least 4.*

Proof. The assertion (iii) follows from (i) and (ii) which in turn, are obtained from Corollaries 4.3 and 4.4. ☐

We consider three possibilities since $\sigma(M^{p-1}) = \sigma({M_2}^2) = M_1 \overline{M_3}$ (product of two Mersenne primes), if $p = 3$.

Case II-1: $2h + 1$ is (divisible by) a prime $p \in \{5, 7\}$.

Lemma 4.14. *For $p \in \{5, 7\}$, some non-Mersenne prime divides $\sigma(M^{p-1})$.*

Proof. Here, $h \in \{2, 3\}$. By direct computations, $U_4 = x^3(x + 1)^6(x^3 + x + 1)$ and $U_6 = x^8(x + 1)^4(x^3 + x + 1)^2$ which do not split (despite that U_6 is a square). ☐

Case II-2: $2h + 1 = 3^w$, for some $w \geq 2$.

In this case, 9 divides $2h + 1$ and $\sigma(M^8)$ divides $\sigma(M^{2h})$ (by Lemma 4.3). But, $\sigma(M^8) = (x^2 + x + 1)(x^4 + x^3 + 1)(x^6 + x + 1)(x^{12} + x^8 + x^7 + x^4 + 1)$, where $x^6 + x + 1 = 1 + x(x + 1)M_3$ is not a Mersenne prime.

Case II-3: $2h + 1$ is (divisible by) a prime $p \notin \{3, 5, 7\}$. We may write $p = 2h + 1$ with $h \geq 4$.

Lemma 4.15. (i) *If $l \in \{1, 2, 3\}$, then $\alpha_l(U_{2h}) = \alpha_l(\sigma(M^{2h}))$.*

(ii) *If $l \in \{1, 2\}$, then $\alpha_l(\sigma(M^{2h})) = \alpha_l(M^{2h})$.*

(iii) *The coefficients $\alpha_3(\sigma(M^{2h}))$ and $\alpha_3(M^{2h} + M^{2h-1})$ are equal.*

Proof. (i) It follows from Lemma 4.13. For $l \leq 2$, $6h - l = \deg(\sigma(M^{2h})) - l = \deg((M^{2h}) - l > 3(2h - 1) = \deg(M^{2h-1})$ and for $3 \leq l \leq 5$, $6h - l > 3(2h - 2) = \deg(M^{2h-2})$. Hence, we get (ii) and (iii). ☐

Corollary 4.6. *The coefficient* $\alpha_3(U_{2h})$ *equals* 1.

Proof. The previous lemma implies that $\alpha_3(U_{2h}) = \alpha_3(M^{2h} + M^{2h-1}) = \alpha_3[(x^3 + x)M^{2h-1}] = \alpha_3(M^{2h-1}) + \alpha_1(M^{2h-1})$. But, $M^{2h-1} = (x^3+x+1)^{2h-1} = (x^3+x)^{2h-1} + (x^3+x)^{2h-2} + \cdots$. The coefficient of x^{6h-6} (resp. of x^{6h-4}) in M^{2h-1} is exactly $\alpha_3(M^{2h-1})$ (resp. $\alpha_1(M^{2h-1})$). So, $\alpha_3(M^{2h-1}) = 1$ and $\alpha_1(M^{2h-1}) = 0$. ☐

Case III: $M \notin \mathcal{M}$.

Here, we have two possibilities.

III-1: the prime p is such that $ord_p(2) \equiv 0 \mod 8$. Lemmas 4.16 and 4.7 imply Corollary 4.7.

Lemma 4.16. *There exists no Mersenne prime of degree multiple of* 8.

Proof. If $Q = 1 + x^{c_1}(x+1)^{c_2}$ with $c_1 + c_2 = 8k$, then $\omega(Q)$ is even by [9, Corollary 3.3]. So, Q is reducible. ☐

Corollary 4.7. *If* $ord_p(2) \equiv 0 \mod 8$, *then* $\sigma(M^{2h})$ *is divisible by a non-Mersenne prime.*

Proof. Suppose that $\sigma(M^{2h}) = \prod_{j \in J} P_j$, where each P_j is a Mersenne prime. Then, Lemma 4.7 implies that $ord_p(2)$ divides $\deg(P_j)$, for any $j \in J$. So, 8 divides $\deg(P_j)$. It contradicts Lemma 4.16. ☐

III-2: p is a Mersenne prime number with $p \neq 7$.

Set $p = 2^m - 1$, with m and p are both prime. Note that there are (at present) 51 known Mersenne prime numbers (OEIS Sequences A000043 and A000668). The first five of them are: $3, 7, 31, 127$ and 8191.

Lemma 4.17. *If* $p \geq 31$ *is a Mersenne prime number, then* $\sigma(M^{p-1})$ *is divisible by a non-Mersenne prime.*

Proof. Here, $a + b = \deg(M) \geq 5$ since $M \notin \mathcal{M}$. We get our result from Corollary 4.4 and Lemma 4.6 (iii). ☐

It remains then the case $p = 3$ (since $p \neq 7$, in this section). Lemma 4.2 has already treated the case where $\omega(\sigma(M^2)) = 2$. So, we suppose that $\omega(\sigma(M^2)) \geq 3$. Put $\sigma(M^2) = M_1 \cdots M_r,\ r \geq 3$ and $U_2 = \sigma(\sigma(M^2))$. We shall prove that $\alpha_3(U_2) = 1$ (Corollary 4.9), a contradiction to the fact that U_2 is a square. Corollary 4.5 gives the following lemma.

Lemma 4.18. (i) *The trinomial* $1 + x + x^2$ *divides* $\sigma(M^2)$.

(ii) *No irreducible polynomial of degree* $r \geq 3$ *such that* $2^r - 1$ *is prime, divides* $\sigma(M^2)$.

Corollary 4.8. *The polynomial* $\sigma(M^2)$ *is of the form* $(1 + x + x^2)B$, *where* $\gcd(1 + x + x^2, B) = 1$ *and any prime divisor of* B *has degree at least* 4.

Lemma 4.19. *If* $\sigma(M^2) = (1 + x + x^2)B$ *with* $\gcd(1 + x + x^2, B) = 1$, *then*

(i) $\alpha_1(\sigma(M^2)) = \alpha_1(B) + 1,\ \alpha_2(\sigma(M^2)) = \alpha_2(B) + \alpha_1(B) + 1$;
(ii) $\alpha_3(\sigma(M^2)) = \alpha_3(B) + \alpha_2(B) + \alpha_1(B)$;
(iii) $\alpha_3(\sigma(M^2)) = 0$.

Proof. We directly get (i) and (ii). For (iii), $\sigma(M^2) = 1 + M + M^2 = x^{2a}(x+1)^{2b} + x^a(x+1)^b + 1$. Moreover, $2a + 2b - 3 > a + b$ because $a + b \geq 4$ and $x^{2a}(x+1)^{2b}$ is a square. So, $\alpha_3(\sigma(M^2)) = \alpha_3(x^{2a}(x+1)^{2b}) = 0$. □

Lemma 4.20. *Some coefficients of* U_2 *and* B *satisfy:*

$$\alpha_1(U_2) = \alpha_1(B) + 1, \quad \alpha_2(U_2) = \alpha_2(B) + \alpha_1(B), \quad \alpha_3(U_2) = \alpha_3(B) + \alpha_2(B).$$

Proof. Corollary 4.8 implies that $U_2 = \sigma(\sigma(M^2)) = \sigma((1 + x + x^2)B) = \sigma(1 + x + x^2)\sigma(B) = (x^2 + x)\sigma(B)$. Any irreducible divisor of B has degree more than 3. Hence, $\alpha_l(\sigma(B)) = \alpha_l(B)$, for $1 \leq l \leq 3$. One gets

$$\begin{aligned}\alpha_1(U_2) =& \alpha_1(\sigma(B)) + 1 = \alpha_1(B) + 1,\\ \alpha_2(U_2) =& \alpha_2(\sigma(B)) + \alpha_1(\sigma(B)) = \alpha_2(B) + \alpha_1(B),\\ \alpha_3(U_2) =& \alpha_3(\sigma(B)) + \alpha_2(\sigma(B)) = \alpha_3(B) + \alpha_2(B).\end{aligned}$$ □

Corollary 4.9. *The coefficient* $\alpha_3(U_2)$ *equals* 1.

Proof. The polynomial U_2 is a square, so $0 = \alpha_1(U_2) = \alpha_1(B) + 1$ and thus $\alpha_1(B) = 1$. Lemma 4.19 (iii) implies that $0 = \alpha_3(\sigma(M^2)) = \alpha_3(B) + \alpha_2(B) + \alpha_1(B)$. Therefore, $\alpha_3(U_2) = \alpha_3(B) + \alpha_2(B) = \alpha_1(B) = 1$. □

Remark 4.1. Our method fails for $p = 7$. Indeed, for many M, one has $\alpha_3(U_6) = \alpha_5(U_6) = 0$. So, we do not reach a contradiction. We should find a large enough odd integer l such that, $\alpha_l(U_6) = 0$. But, this does not appear always possible.

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ON THE ZEROS OF APOLAR POLYNOMIALS

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ABSTRACT. The classical notion of apolarity is defined for two complex polynomials of same degree. The main property of two apolar polynomials, $f(z)$ and $g(z)$ was given by Grace's theorem which states that "every circular domain containing all the zeros of $f(z)$ contains at least one zero of $g(z)$ and vice-versa". A. Aziz [1] dropped the condition that $f(z)$ and $g(z)$ are of the same degree in case the circular domain is a disk. In this paper, we extend the result of A. Aziz to every kind of circular domain and hence an extension of Grace's theorem for two arbitrary polynomials is proved. This also allows us to generalise the results of Walsh, Szego, Takagi, Aziz and several other results about apolar polynomials.

1. INTRODUCTION

Two polynomials

$$f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu \quad \text{and} \quad g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$$

are called *apolar* if

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} a_{n-\nu} b_\nu = 0.$$

Circular domain is (open or closed) interior or exterior of any circle, or (open or closed) half plane. As to the relative location of the zeros of two apolar polynomials $f(z)$ and $g(z)$, we have the following fundamental result known as Grace's Apolarity theorem [3].

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Theorem 1.1. *If $f(z)$ and $g(z)$ are apolar polynomials, then any circular domain C which contains all zeros of one of the polynomials $f(z)$ or $g(z)$ contains at least one zero of the other.*

In case C is a disk or complement of a disk, A. Aziz [2] proved the following generalization of Theorem 1.1.

Theorem 1.2. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, where $m \leq n$. Assume*

$$\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} a_{n-\nu} b_\nu = 0.$$

(a) *If $g(z)$ has all zeros in the disc $|z| \geq r$, then $f(z)$ has at least one zero there.*
(b) *If $f(z)$ has all zeros in the region $|z| \leq r$, then $g(z)$ has at least one zero there.*

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu = \prod_{\nu=1}^{n}(z - z_\nu)$ be two monic polynomials (leading coefficient unity). Let ζ be a zero of f. Then

$$\prod_{\nu=1}^{n} (\zeta - z_\nu) = g(\zeta) - f(\zeta) = \sum_{\nu=0}^{n-1} (b_\nu - a_\nu)\zeta^\nu$$

and so,

$$\min_{1\leq\nu\leq n} |\zeta - z_\nu| \leq \left(\sum_{\nu=0}^{n-1} |b_\nu - a_\nu| \cdot |\zeta|^\nu \right)^{\frac{1}{n}} .$$

Hence, for each zero ζ of f and any $\epsilon > 0$, there exists a $\delta > 0$ such that every monic polynomial $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ satisfying $|b_\nu - a_\nu| < \delta$, for $\nu = 0, \dots, n-1$, has a zero w with $|\zeta - w| < \epsilon$. Thus, each zero depends continously on the coefficients. This property can be stated in the following theorem.

Theorem 1.3 (Continuity Theorem). *Let*

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = \prod_{\nu=1}^{k} (z - z_\nu)^{m_\nu}, \quad m_1 + m_2 + \cdots + m_k = n,$$

be a monic polynomial of degree n with distinct zeros $z_1, z_2, \dots, z_k$ of multiplicities $m_1, m_2, \dots, m_k$. Then given a positive $\epsilon < \min_{1\leq i<j\leq k} \frac{|z_i - z_j|}{2}$, there exists $\delta > 0$ so that any monic polynomial $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ whose coefficients satisfy $|b_\nu - a_\nu| < \delta$, for $\nu = 1, 2, \dots, n-1$, has exactly m_j zeros in the disc

$$|z - z_j| < \epsilon, \quad j = 1, 2, \dots, k.$$

Remark 1.1. If f is not monic, then its zeros depend continuosly on $\frac{a_\nu}{a_n}$ for $\nu = 0, 1, 2, \dots, n-1$. If $a_n \to 0$ while other coefficients remain fixed, then at least one zero tends to infinity.

2. Main Results

We first prove the following result, which extends Theorem 1.1 (Grace's Theorem) and Theorem 1.2 to polynomials of arbitrary degree and to every circular domain.

Theorem 2.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$*, where* $m \leq n$*, such that*

$$\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} a_{n-\nu} b_\nu = 0. \tag{2.1}$$

Then every circular domain C *which contains all zeros of one of the polynomials* $f(z)$ *or* $g(z)$ *contains at least one zero of the other.*

Proof. If $m = n$, then the result reduces to Theorem 1.1 (Grace's Theorem). So, assume that $m < n$, where n is degree of $f(z)$ and m is degree of $g(z)$.

For any $\epsilon > 0$, consider the polynomial

$$g_\epsilon(z) = \epsilon z^n + \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu(\epsilon) z^\nu = c \prod_{\nu=1}^{n} (z - \zeta_\nu(\epsilon)).$$

It is possible to choose the coefficients $\{b_\nu(\epsilon) : \nu = 0, 1, \dots, m\}$, so that $b_\nu(\epsilon)$ approaches b_ν, as ϵ approaches to 0. In that case, the polynomials $g_\epsilon(z)$ and $f(z)$ are apolar.

As $\epsilon \to 0$, m of the zeros $\{\zeta_1(\epsilon), \zeta_2(\epsilon), \dots, \zeta_n(\epsilon)\}$ approach the finite zeros of $g(z)$ and $(n - m)$ zeros tend to ∞. The classical Grace's Theorem is valid for polynomials $g_\epsilon(z)$ and $f(z)$. Hence, by continuity theorem, it is also valid for polynomials $g(z)$ and $f(z)$.

This completes the proof. □

Remark 2.1. Theorem 2.1 says that zeros of two polynomials having different degrees and satisfying condition (2.1) cannot be separated by the boundary of a circular domain which is either a circle or a straight line.

The following special case of Theorem 2.1 is a generalisation of the result due to Takagi [8].

Corollary 2.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$*, where* $m \leq n$*, satisfying condition* (2.1)*, then any convex region* C_1 *enclosing all the zeros of* $f(z)$ *must have at least one point in common with any convex region* C_2 *enclosing all the zeros of* $g(z)$*.*

Proof. Assume that two convex regions C_1 and C_2 have no point in common. Then one can separate them by means of the boundary of a circle or straight line. This would contradict Theorem 2.1. Hence, C_1 must have at least one point in common with C_2. □

From Corollary 2.1, we also deduce the following result for polynomials having only real zeros.

Corollary 2.2. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$*, where* $m \leq n$*, satisfying condition* (2.1)*, having only real zeros, then any interval* I_1 *containing the zeros of* $f(z)$ *must have at least one point in common with any interval* I_2 *containing the zeros of* $g(z)$*.*

3. Applications of Theorem 2.1

As an application, we prove the following result, which is generalisation of Szego's Convolution Theorem [5, p. 108].

Theorem 3.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$*, be a polynomial of degree* n*, satisfying the following relation*

$$\sum_{\nu=0}^{m} \binom{m}{\nu} a_{n-\nu} l_\nu = 0, \quad m \leq n,$$

then every circular domain that contains all the zeros of

$$g(z) = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} l_\nu z^\nu$$

contains at least one zero of $f(z)$*.*

Proof. Under the given hypothesis, the polynomials $f(z)$ and $g(z)$ satisfy the condition (2.1), by Theorem 2.1, $f(z)$ has at least one zero in every circular domain that contains all the zeros of $g(z)$. This completes the proof. □

Next, we obtain the following coincidence theorem, which is in fact, a generalisation of Walsh's Coincidence Theorem [9] and Aziz's result [2, Theorem 2] involving convex circular domain.

Theorem 3.2. *Let* $\phi(z_1, z_2, \ldots, z_n)$ *be a symmetric* n*-linear form of total degree* m*,* $m \leq n$ *in* $z_1, z_2, \ldots, z_n$ *and let* C *be the convex circular domain containing* n *points* $w_1, w_2, \ldots, w_n$*. Then in* C *there exists at least one point* w *such that*

$$\phi(w, w, \ldots, w) = \phi(w_1, w_2, \ldots, w_n).$$

Proof. We write

$$f(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu,$$

so that

$$\binom{n}{\nu} a_{n-\nu} = (-1)^\nu S(n, \nu) a_n, \tag{3.1}$$

where $S(n, \nu)$ are the symmetric functions consisting of the sum of all possible products of $z_1, z_2, \ldots, z_n$ taken ν at a time.

Let $\phi(w, w, \ldots, w) = \phi_0$, then the difference $\phi(z_1, z_2, \ldots, z_n) - \phi_0$ is linear, symmetric and of total degree $m \leq n$ in the variables $z_1, z_2, \ldots, z_n$, by the well-known theorem of algebra, any function linear and symmetric in the variables $z_1, z_2, \ldots, z_n$ may be expressed as a linear combination of the elementary symmetric functions $S(n,\nu)$, $\nu = 0, 1, \ldots, m$, that is, we may find constants b_ν so that

$$\begin{aligned}\phi(z_1, z_2, \ldots, z_n) - \phi_0 &= b_0 + S(n,1)b_1 + S(n,2)b_2 + \cdots + S(n,m)b_m \\ &= \frac{1}{a_n}\left(b_0 a_n - \binom{n}{1} b_1 a_{n-1} + \cdots + (-1)^m \binom{n}{m} a_{n-m} b_m\right),\end{aligned}$$

by using (3.1).

We define the polynomial $g(z)$ by

$$g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} \frac{\binom{n}{\nu}}{\binom{m}{\nu}} b_\nu z^\nu = \phi(z_1, z_2, \ldots, z_n) - \phi_0.$$

Then the relation

$$\phi(w_1, w_2, \ldots, w_n) - \phi_0 = 0$$

shows that the polynomials $f(z)$ and $g(z)$ satisfy the condition of Theorem 2.1. Since all the zeros of $f(z)$ lie in C, at least one zero of $g(z)$ lies in C, i.e, there exists one point w in C such that

$$\phi(w, w, \ldots, w) = \phi(w_1, w_2, \ldots, w_n).$$

This completes the proof. □

Szego [7] used Grace's Theorem to obtain following interesting result about the zeros of the polynomial $h(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu$ (obtained by certain composition of two given polynomials $f(z)$ and $g(z)$) of degree n.

Theorem 3.3. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$. Let C be the circular domain containing all the zeros of $g(z)$. Then each zero γ of*

$$h(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu$$

is of the form $\gamma = -\alpha\beta$, $\alpha \in C$, $f(\beta) = 0$.

By applying Theorem 2.1, we have obtained following generalisation of Theorem 3.3 and Aziz's result [1, Theorem 2].

Theorem 3.4. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, where $m \leq n$ and let C be the circular domain containing all the zeros of $g(z)$. Then each zero γ of*

$$h(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu b_\nu z^\nu$$

is of the form $\gamma = -\alpha\beta$, $\alpha \in C$, $f(\beta) = 0$.

Proof. Let w be any zero of $h(z)$, then

$$h(w) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu b_\nu w^\nu = 0. \tag{3.2}$$

Equation (3.2) shows that the polynomials

$$z^n f\left(\frac{-w}{z}\right) = \binom{n}{0}(-1)^n a_n w^n + \cdots + \binom{n}{n-2} a_2 w^2 z^{n-2} - \binom{n}{n-1} a_1 w z^{n-1} + \binom{n}{n} a_0 z^n$$

and

$$g(z) = \binom{m}{0} b_0 + \binom{m}{1} b_1 z + \cdots + \binom{m}{m-1} b_{m-1} z^{m-1} + \binom{m}{m} b_m z^m$$

satisfy conditions of Theorem 2.1. Since all the zeros of $g(z)$ lie in C, then at least one zero of $z^n f\left(\frac{-w}{z}\right)$ lie in C. If $\beta_1, \beta_2, \dots, \beta_n$ are the zeros of $f(z)$, then the zeros of $z^n f\left(\frac{-w}{z}\right)$ are $\frac{-w}{\beta_1}, \frac{-w}{\beta_2}, \dots, \frac{-w}{\beta_n}$. One of these zeros must be α, where α is suitably chosen point in C, that is, $w = -\alpha\beta_\nu$, for some ν. This completes the proof. □

Following application of Theorem 2.1 is generalisation of of Aziz's result [2, Theorem 5].

Theorem 3.5. *From the two given polynomials*

$$f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - \alpha_\nu)$$

and

$$g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu z^\nu = b_n \prod_{\nu=1}^{m} (z - \beta_\nu)$$

of degree n and m, $m \le n$, form the third polynomial

$$h(z) = \sum_{\nu=0}^{m} (n-\nu)! a_{n-\nu} g^{(\nu)}(z).$$

If all the zeros of $f(z)$ lie in circular domain C, then every zero of $h(z)$ has the form $w = \alpha + \beta$, where α is suitably point in C and β is zero of $g(z)$.

Proof. Let w be any zero of $h(z)$. Then

$$h(w) = \sum_{\nu=0}^{m} (n-\nu)! a_{n-\nu} g^{(\nu)}(w) = 0. \tag{3.3}$$

Equation (3.3) shows that the polynomials

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \quad \text{and} \quad g(w-z) = \sum_{\nu=0}^{m} (-1)^\nu \frac{g^{(\nu)}(w)}{\nu!} z^\nu$$

of degree n and m respectively, $m \leq n$, satisfy all the conditions of Theorem 2.1. Since all the zeros of $f(z)$ lie in C, $g(w-z)$ has at least one zero in C. But the zeros of $g(w-z)$ are of the form $w-\beta_1, w-\beta_2, \ldots, w-\beta_m$. One of these zeros must be some α, where $\alpha \in C$, that is, we must have $w = \alpha + \beta_\nu$, for some ν. This completes the proof. □

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