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ON THE ZEROS OF APOLAR POLYNOMIALS

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Abstract. The classical notion of apolarity is defined for two complex polynomials of same degree. The main property of two apolar polynomials, $f(z)$ and $g(z)$ was given by Grace's theorem which states that "every circular domain containing all the zeros of $f(z)$ contains at least one zero of $g(z)$ and vice-versa". A. Aziz [1] dropped the condition that $f(z)$ and $g(z)$ are of the same degree in case the circular domain is a disk. In this paper, we extend the result of A. Aziz to every kind of circular domain and hence an extension of Grace's theorem for two arbitrary polynomials is proved. This also allows us to generalise the results of Walsh, Szego, Takagi, Aziz and several other results about apolar polynomials.

1. Introduction

Two polynomials

$$f(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu z^\nu \quad \text{and} \quad g(z)=\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu z^\nu$$

are called *apolar* if

$$\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}a_{n-\nu}b_\nu=0.$$

Circular domain is (open or closed) interior or exterior of any circle, or (open or closed) half plane. As to the relative location of the zeros of two apolar polynomials $f(z)$ and $g(z)$, we have the following fundamental result known as Grace's Apolarity theorem [3].

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Theorem 1.1. *If $f(z)$ and $g(z)$ are apolar polynomials, then any circular domain C which contains all zeros of one of the polynomials $f(z)$ or $g(z)$ contains at least one zero of the other.*

In case C is a disk or complement of a disk, A. Aziz [2] proved the following generalization of Theorem 1.1.

Theorem 1.2. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, where $m \leq n$. Assume*

$$\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} a_{n-\nu} b_\nu = 0.$$

(a) *If $g(z)$ has all zeros in the disc $|z| \geq r$, then $f(z)$ has at least one zero there.*
(b) *If $f(z)$ has all zeros in the region $|z| \leq r$, then $g(z)$ has at least one zero there.*

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu = \prod_{\nu=1}^{n} (z - z_\nu)$ be two monic polynomials (leading coefficient unity). Let ζ be a zero of f. Then

$$\prod_{\nu=1}^{n} (\zeta - z_\nu) = g(\zeta) - f(\zeta) = \sum_{\nu=0}^{n-1} (b_\nu - a_\nu)\zeta^\nu$$

and so,

$$\min_{1 \leq \nu \leq n} |\zeta - z_\nu| \leq \left(\sum_{\nu=0}^{n-1} |b_\nu - a_\nu| \cdot |\zeta|^\nu \right)^{\frac{1}{n}}.$$

Hence, for each zero ζ of f and any $\epsilon > 0$, there exists a $\delta > 0$ such that every monic polynomial $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ satisfying $|b_\nu - a_\nu| < \delta$, for $\nu = 0, \dots, n-1$, has a zero w with $|\zeta - w| < \epsilon$. Thus, each zero depends continously on the coefficients. This property can be stated in the following theorem.

Theorem 1.3 (Continuity Theorem)**.** *Let*

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = \prod_{\nu=1}^{k} (z - z_\nu)^{m_\nu}, \quad m_1 + m_2 + \cdots + m_k = n,$$

be a monic polynomial of degree n with distinct zeros $z_1, z_2, \dots, z_k$ of multiplicities $m_1, m_2, \dots, m_k$. Then given a positive $\epsilon < \min_{1 \leq i < j \leq k} \frac{|z_i - z_j|}{2}$, there exists $\delta > 0$ so that any monic polynomial $g(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ whose coefficients satisfy $|b_\nu - a_\nu| < \delta$, for $\nu = 1, 2, \dots, n-1$, has exactly m_j zeros in the disc

$$|z - z_j| < \epsilon, \quad j = 1, 2, \dots, k.$$

Remark 1.1. If f is not monic, then its zeros depend continously on $\frac{a_\nu}{a_n}$ for $\nu = 0, 1, 2, \dots, n-1$. If $a_n \to 0$ while other coefficients remain fixed, then at least one zero tends to infinity.

2. Main Results

We first prove the following result, which extends Theorem 1.1 (Grace's Theorem) and Theorem 1.2 to polynomials of arbitrary degree and to every circular domain.

Theorem 2.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, *where* $m \leq n$, *such that*

$$\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} a_{n-\nu} b_\nu = 0. \tag{2.1}$$

Then every circular domain C *which contains all zeros of one of the polynomials* $f(z)$ *or* $g(z)$ *contains at least one zero of the other.*

Proof. If $m = n$, then the result reduces to Theorem 1.1 (Grace's Theorem). So, assume that $m < n$, where n is degree of $f(z)$ and m is degree of $g(z)$.

For any $\epsilon > 0$, consider the polynomial

$$g_\epsilon(z) = \epsilon z^n + \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu(\epsilon) z^\nu = c \prod_{\nu=1}^{n} (z - \zeta_\nu(\epsilon)).$$

It is possible to choose the coefficients $\{b_\nu(\epsilon) : \nu = 0, 1, \ldots, m\}$, so that $b_\nu(\epsilon)$ approaches b_ν, as ϵ approaches to 0. In that case, the polynomials $g_\epsilon(z)$ and $f(z)$ are apolar.

As $\epsilon \to 0$, m of the zeros $\{\zeta_1(\epsilon), \zeta_2(\epsilon), \ldots, \zeta_n(\epsilon)\}$ approach the finite zeros of $g(z)$ and $(n - m)$ zeros tend to ∞. The classical Grace's Theorem is valid for polynomials $g_\epsilon(z)$ and $f(z)$. Hence, by continuity theorem, it is also valid for polynomials $g(z)$ and $f(z)$.

This completes the proof. □

Remark 2.1. Theorem 2.1 says that zeros of two polynomials having different degrees and satisfying condition (2.1) cannot be separated by the boundary of a circular domain which is either a circle or a straight line.

The following special case of Theorem 2.1 is a generalisation of the result due to Takagi [8].

Corollary 2.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, *where* $m \leq n$, *satisfying condition* (2.1), *then any convex region* C_1 *enclosing all the zeros of* $f(z)$ *must have at least one point in common with any convex region* C_2 *enclosing all the zeros of* $g(z)$.

Proof. Assume that two convex regions C_1 and C_2 have no point in common. Then one can separate them by means of the boundary of a circle or straight line. This would contradict Theorem 2.1. Hence, C_1 must have at least one point in common with C_2. □

From Corollary 2.1, we also deduce the following result for polynomials having only real zeros.

Corollary 2.2. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ *and* $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$*, where* $m \leq n$*, satisfying condition* (2.1)*, having only real zeros, then any interval* I_1 *containing the zeros of* $f(z)$ *must have at least one point in common with any interval* I_2 *containing the zeros of* $g(z)$*.*

3. Applications of Theorem 2.1

As an application, we prove the following result, which is generalisation of Szego's Convolution Theorem [5, p. 108].

Theorem 3.1. *Let* $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$*, be a polynomial of degree* n*, satisfying the following relation*

$$\sum_{\nu=0}^{m} \binom{m}{\nu} a_{n-\nu} l_\nu = 0, \quad m \leq n,$$

then every circular domain that contains all the zeros of

$$g(z) = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} l_\nu z^\nu$$

contains at least one zero of $f(z)$*.*

Proof. Under the given hypothesis, the polynomials $f(z)$ and $g(z)$ satisfy the condition (2.1), by Theorem 2.1, $f(z)$ has at least one zero in every circular domain that contains all the zeros of $g(z)$. This completes the proof. □

Next, we obtain the following coincidence theorem, which is in fact, a generalisation of Walsh's Coincidence Theorem [9] and Aziz's result [2, Theorem 2] involving convex circular domain.

Theorem 3.2. *Let* $\phi(z_1, z_2, \dots, z_n)$ *be a symmetric* n*-linear form of total degree* m*,* $m \leq n$ *in* $z_1, z_2, \dots, z_n$ *and let* C *be the convex circular domain containing* n *points* $w_1, w_2, \dots, w_n$*. Then in* C *there exists at least one point* w *such that*

$$\phi(w, w, \dots, w) = \phi(w_1, w_2, \dots, w_n).$$

Proof. We write

$$f(z) = a_n \prod_{\nu=1}^{n} (z - z_\nu) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu,$$

so that

$$\binom{n}{\nu} a_{n-\nu} = (-1)^\nu S(n, \nu) a_n, \tag{3.1}$$

where $S(n, \nu)$ are the symmetric functions consisting of the sum of all possible products of $z_1, z_2, \dots, z_n$ taken ν at a time.

Let $\phi(w, w, \dots, w) = \phi_0$, then the difference $\phi(z_1, z_2, \dots, z_n) - \phi_0$ is linear, symmetric and of total degree $m \leq n$ in the variables $z_1, z_2, \dots, z_n$, by the well-known theorem of algebra, any function linear and symmetric in the variables $z_1, z_2, \ldots, z_n$ may be expressed as a linear combination of the elementary symmetric functions $S(n,\nu)$, $\nu = 0, 1, \ldots, m$, that is, we may find constants b_ν so that

$$\begin{aligned}\phi(z_1, z_2, \ldots, z_n) - \phi_0 &= b_0 + S(n,1)b_1 + S(n,2)b_2 + \cdots + S(n,m)b_m \\ &= \frac{1}{a_n}\left(b_0 a_n - \binom{n}{1} b_1 a_{n-1} + \cdots + (-1)^m \binom{n}{m} a_{n-m} b_m\right),\end{aligned}$$

by using (3.1).

We define the polynomial $g(z)$ by

$$g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} \frac{\binom{n}{\nu}}{\binom{m}{\nu}} b_\nu z^\nu = \phi(z_1, z_2, \ldots, z_n) - \phi_0.$$

Then the relation

$$\phi(w_1, w_2, \ldots, w_n) - \phi_0 = 0$$

shows that the polynomials $f(z)$ and $g(z)$ satisfy the condition of Theorem 2.1. Since all the zeros of $f(z)$ lie in C, at least one zero of $g(z)$ lies in C, i.e, there exists one point w in C such that

$$\phi(w, w, \ldots, w) = \phi(w_1, w_2, \ldots, w_n).$$

This completes the proof. □

Szego [7] used Grace's Theorem to obtain following interesting result about the zeros of the polynomial $h(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu$ (obtained by certain composition of two given polynomials $f(z)$ and $g(z)$) of degree n.

Theorem 3.3. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$. Let C be the circular domain containing all the zeros of $g(z)$.Then each zero γ of*

$$h(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu$$

is of the form $\gamma = -\alpha\beta$, $\alpha \in C$, $f(\beta) = 0$.

By applying Theorem 2.1, we have obtained following generalisation of Theorem 3.3 and Aziz's result [1, Theorem 2].

Theorem 3.4. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} b_\nu z^\nu$, where $m \le n$ and let C be the circular domain containing all the zeros of $g(z)$. Then each zero γ of*

$$h(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu b_\nu z^\nu$$

is of the form $\gamma = -\alpha\beta$, $\alpha \in C$, $f(\beta) = 0$.

Proof. Let w be any zero of $h(z)$, then

$$h(w) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu b_\nu w^\nu = 0. \tag{3.2}$$

Equation (3.2) shows that the polynomials

$$\begin{aligned} z^n f\left(\frac{-w}{z}\right) = &\binom{n}{0}(-1)^n a_n w^n + \cdots + \binom{n}{n-2} a_2 w^2 z^{n-2} \\ &- \binom{n}{n-1} a_1 w z^{n-1} + \binom{n}{n} a_0 z^n \end{aligned}$$

and

$$g(z) = \binom{m}{0} b_0 + \binom{m}{1} b_1 z + \cdots + \binom{m}{m-1} b_{m-1} z^{m-1} + \binom{m}{m} b_m z^m$$

satisfy conditions of Theorem 2.1. Since all the zeros of $g(z)$ lie in C, then at least one zero of $z^n f\left(\frac{-w}{z}\right)$ lie in C. If $\beta_1, \beta_2, \ldots, \beta_n$ are the zeros of $f(z)$, then the zeros of $z^n f\left(\frac{-w}{z}\right)$ are $\frac{-w}{\beta_1}, \frac{-w}{\beta_2}, \ldots, \frac{-w}{\beta_n}$. One of these zeros must be α, where α is suitably chosen point in C, that is, $w = -\alpha\beta_\nu$, for some ν. This completes the proof. □

Following application of Theorem 2.1 is generalisation of of Aziz's result [2, Theorem 5].

Theorem 3.5. *From the two given polynomials*

$$f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - \alpha_\nu)$$

and

$$g(z) = \sum_{\nu=0}^{m} \binom{m}{\nu} a_\nu z^\nu = b_n \prod_{\nu=1}^{m} (z - \beta_\nu)$$

of degree n and m, $m \le n$, form the third polynomial

$$h(z) = \sum_{\nu=0}^{m} (n-\nu)! a_{n-\nu} g^{(\nu)}(z).$$

If all the zeros of $f(z)$ lie in circular domain C, then every zero of $h(z)$ has the form $w = \alpha + \beta$, where α is suitably point in C and β is zero of $g(z)$.

Proof. Let w be any zero of $h(z)$. Then

$$h(w) = \sum_{\nu=0}^{m} (n-\nu)! a_{n-\nu} g^{(\nu)}(w) = 0. \tag{3.3}$$

Equation (3.3) shows that the polynomials

$$f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \quad \text{and} \quad g(w-z) = \sum_{\nu=0}^{m} (-1)^\nu \frac{g^{(\nu)}(w)}{\nu!} z^\nu$$

of degree n and m respectively, $m \leq n$, satisfy all the conditions of Theorem 2.1. Since all the zeros of $f(z)$ lie in C, $g(w-z)$ has at least one zero in C. But the zeros of $g(w-z)$ are of the form $w-\beta_1, w-\beta_2, \ldots, w-\beta_m$. One of these zeros must be some α, where $\alpha \in C$, that is, we must have $w=\alpha+\beta_\nu$, for some ν. This completes the proof. $\square$

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