Kragujevac Journal of Mathematics
Volume 49(4) (2025), Pages 555–566.

EXISTENCE RESULTS OF IMPULSIVE HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL AND BOUNDARY HYBRID CONDITIONS

MOHAMED HANNABOU[1], MOHAMED BOUAOUID[1], AND KHALID HILAL[1]

Abstract. In this paper, we establish sufficient conditions for the existence and uniqueness of solution of impulsive hybrid fractional differential equations with initial and boundary hybrid conditions. The proof of the main result is based on the classical fixed point theorems such as Banach fixed point theorem and Leray-Schauder alternative fixed point theorem. Two examples are included to show the applicability of our results.

1. Introduction

Fractional calculus refers to integration or differentiation of any order. The field has a history as old as calculus itself, which did not attract enough attention for a long time. In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences. For a nice introduction, we refer the reader to [9, 10] and references cited therein.

Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process [8]. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The governing equations of such phenomena may be modeled as impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations as these

Key words and phrases. Hybrid systems of ordinary differential equations, fractional derivatives and integrals, fixed-point theorems.
2020 *Mathematics Subject Classification.* Primary: 34A38. Secondary: 26A33, 47H10.
DOI 10.46793/KgJMat2504.555H
Received: March 16, 2022.
Accepted: July 30, 2022.

equations provide a natural frame work for mathematical modelling of many real world phenomena, namely in the control theory, physics, chemistry, population dynamics, biotechnology, economics and medical fields.

In [11], Surang Sitho, Sotiris K. Ntouyas and Jessada Tariboon, discussed the existence results for the following hybrid fractional integro-differential equation:

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J=[0,T],\\ x(0)=0, \end{cases}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0<\alpha\leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\dots,\beta_m\}$, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g\in C(J\times\mathbb{R},\mathbb{R})$, with $h_i\in C(J\times\mathbb{R},\mathbb{R})$ and $h_i(0,0)=0$, $i=1,2,\dots,m$.

In [4], K. Hilal and A. Kajouni, considered boundary value problems for hybrid differential equations with fractional order (BVPHDEF of short) involving Caputo differential operator of order $0<\alpha<1$:

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J=[0,T],\\ a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c, \end{cases}$$

where $f\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $g\in C(J\times\mathbb{R},\mathbb{R})$ and a,b,c are real constants with $a+b\neq 0$.

Dhage and Lakshmikantham [2], discussed the following first order hybrid differential equation:

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J=[0,T],\\ x(t_0)=x_0\in\mathbb{R}, \end{cases}$$

where $f\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $g\in C(J\times\mathbb{R},\mathbb{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and comparison results.

Zhao, Sun, Han and Li [13], are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operator:

$$\begin{cases} D^{q}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J=[0,T],\\ x(0)=0, \end{cases}$$

where $f\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $g\in C(J\times\mathbb{R},\mathbb{R})$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.

Benchohra et al. [1] discussed the following boundary value problems for differential equations with fractional order:

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad t \in J = [0,T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a, b, c are real constants with $a + b \neq 0$.

Motivated by some recent studies related to the boundary value problem of a class of impulsive hybrid fractional differential equations and by the nice works [12, 14], we consider the following Cauchy problem of hybrid fractional differential equations:

$$\begin{cases} D^\alpha\Big(\frac{u(t)}{f(t,u(t))}\Big) = g(t, u(t)), \quad t \in [0,1], t \neq t_i, i = 1, 2, \ldots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), \quad t_i \in (0,1), i = 1, 2, \ldots, n, \\ \frac{u(0)}{f(0,u(0))} = \phi(u), \end{cases} \tag{1.1}$$

D^α stands for Caputo fractional derivative of order α, $f \in C([0,1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $\phi : C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions such that $\phi(u) = \sum_{i=1}^n \lambda_i u(\xi_i)$, where $\xi_i \in (0,1)$ for $i = 1, 2, \ldots, n$, and $I_k : \mathbb{R} \to \mathbb{R}$ with $u(t_k^+) = \lim_{\epsilon\to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\epsilon\to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = i$.

In the sequel of this work, we assume that $\sum_{i=1}^n \lambda_i u(\xi_i)^{\alpha-1} < 1$.

This paper is arranged as follows. In Section 2, we recall some tools related to the fractional calculus as well as some needed results. In Section 3, we present the main results. Section 4 is devoted to examples of application of the main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Throughout this paper, let $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{n-1} = (t_{n-1}, t_n], J_n = (t_n, 1]$, $n \in \mathbb{N}$, $n > 1$.

For $t_i \in (0,1)$ such that $t_1 < t_2 < \cdots < t_n$ we define the following spaces:

$$\begin{aligned} I' =& I \setminus \{t_1, t_2, \ldots, t_n\}, \\ X =& \{u \in C([0,1], \mathbb{R}) : u \in C(I') \text{ and left } u(t_i^+) \text{ and right limit } u(t_i^-) \\ & \quad \text{exist and } u(t_i^-) = u(t_i), 1 \leq i \leq n\}. \end{aligned}$$

Then, clearly $(X, \|\cdot\|)$ is a Banach space under the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Definition 2.1 ([6])**.** The fractional integral of the function $h \in L^1([a,b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function.

Definition 2.2 ([6])**.** For a function h defined on the interval $[a,b]$, the Riemann-Liouville fractional-order derivative of h, is defined by

$$({}^cD^{\alpha}_{a^+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([6])**.** For a function h defined on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD^{\alpha}_{a^+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.1 ([10])**.** *Let $\alpha, \beta \geq 0$, then the following relations hold:*

1. $I^{\alpha}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}$;
2. ${}^cD^{\alpha}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}$.

Lemma 2.2 ([10])**.** *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha}\, {}^cD^{\alpha} f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}.$$

3. Main Results

In this section, we prove the existence of a solution for Cauchy problem (1.1).

To do so, we will need the following assumptions.

(H_1) The function $u \mapsto \frac{u}{f(t,u)}$ is increasing in $\mathbb{R}$ for every $t \in [0,1]$.

(H_2) The function f is continuous and bounded, that is, there exists a positive number $L > 0$ such that $|f(t,u)| \leq L$ for all $(t,u) \in [0,1] \times \mathbb{R}$.

(H_3) There exists a positive number $M_g > 0$, such that

$$|g(t,u) - g(t,\bar{u})| \leq M_g|u-\bar{u}|, \quad \text{for all } u, \bar{u} \in \mathbb{R} \text{ and } t \in [0,1].$$

(H_4) There exists a constant $A > 0$, such that

$$|I_i(u) - I_i(\bar{u})| \leq A|u-\bar{u}|, \quad i = 1,2,\ldots,n, \text{ for all } u, \bar{u}, \in \mathbb{R}.$$

(H_5) There exists a constant $K_{\phi} > 0$, such that

$$|\phi(u) - \phi(v)| \leq K_{\phi}\|u-v\|, \quad \text{for all } u, v \in C([0,1],\mathbb{R}).$$

(H_6) There exist constants $M_{\phi} > 0$ and $N_I > 0$, such that

$$|\phi(u)| \leq M_{\phi}\|u\|, \quad |I_i(v)| \leq N_I|v|, \quad i = 1,2,\ldots,n,$$

for all $u \in C([0,1],\mathbb{R})$ and $v \in \mathbb{R}$.

(H_7) There exists a constant $C > 0$, such that

$$|I_i(u)| \leq C, \quad i = 1, 2, \dots, n, \text{ for all } u \in \mathbb{R}.$$

(H_8) There exists a constant $\rho > 0$, such that

$$|\phi(u)| \leq \rho, \quad \text{for all } u \in X.$$

(H_9) There exist constants $\rho_0, \rho_1 > 0$, such that

$$|g(t, u(t))| \leq \rho_0 + \rho_1 \|u\|, \quad \text{for all } u \in X \text{ and } t \in [0, 1].$$

For brevity, let us set

$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\right). \tag{3.1}$$

Lemma 3.1. *Let $\alpha \in (0, 1)$ and $h : [0, T] \to \mathbb{R}$ be continuous. A function $u \in C([0, T], \mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = u_0 - \int_0^a \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

if and only if u is a solution of the following fractional Cauchy problem:

$$\begin{cases} D^\alpha u(t) = h(t), & t \in [0, T], \\ u(a) = u_0, & a > 0. \end{cases}$$

Lemma 3.2. *Assume that hypotheses (H_1) and (H_2) hold. Let $\alpha \in (0, 1)$ and $h : [0, 1] \to \mathbb{R}$ be continuous. A function u is a solution of the fractional integral equation* (3.2)

$$u(t) = f(t, u(t))\left[\phi(u) + \theta(t)\sum_{i=1}^n \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right], \quad t \in [t_i, t_{i+1}],$$

where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \notin [t_0, t_1[, \end{cases}$$

if and only if u is a solution of the following impulsive problem:

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) = h(t), & t \in [0, 1], t \neq t_i, i = 1, 2, \dots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), & t_i \in (0, 1), i = 1, 2, \dots, n, \\ \frac{u(0)}{f(0,u(0))} = \phi(u). \end{cases} \tag{3.3}$$

Proof. Assume that u satisfies (3.3). If $t \in [t_0, t_1[$, then

$$D^\alpha\left(\frac{u(t)}{f(t, u(t))}\right) = h(t), \quad t \in [t_0, t_1[, \tag{3.4}$$

$$\frac{u(0)}{f(0, u(0))} = \phi(u). \tag{3.5}$$

Applying I^α on both sides of (3.4), we obtain

$$\frac{u(t)}{f(t,u(t))} = \frac{u(0)}{f(0,u(0))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds = \phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.$$

Then we get

$$u(t) = f(t,u(t))\left(\phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

If $t \in [t_1, t_2[$, then

$$D^\alpha\Big(\frac{u(t)}{f(t,u(t))}\Big) = h(t), \quad t \in [t_1, t_2[, \tag{3.6}$$

$$u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)). \tag{3.7}$$

According to Lemma 3.1 and the continuity of $t \mapsto f(t,u(t))$, we have

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} &= \frac{u(t_1^+)}{f(t_1,u(t_1))} - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\ &= \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f(t_1,u(t_1))} - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.\end{aligned}$$

Since

$$u(t_1^-) = f(t_1,u(t_1))\left(\phi(u) + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right),$$

then we get

$$\begin{aligned}\frac{u(t)}{f(t,u(t)))} =& \left(\phi(u) + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))}\\ &- \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\ =&\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.\end{aligned}$$

So, one has

$$u(t) = f(t,u(t))\left(\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

For $t \in [t_2, t_3[$, we have

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} &= \frac{u(t_2^+)}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\ &= \frac{(u(t_2^-) + I_2(u(t_2^-)))}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\end{aligned}$$

and

$$u(t_2^-) = f(t_2,u(t_2))\left(\phi(u) + \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f(t_1,u(t_1))} + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right).$$

Therefore, we obtain

$$\begin{aligned}\frac{u(t)}{f(t,u(t))} =&\phi(u) + \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f(t_1,u(t_1))} + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &+ \frac{I_2(u(t_2^-))}{f(t_2,u(t_2))} - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ =&\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \frac{I_2(u(t_2^-))}{f(t_2,u(t_2))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.\end{aligned}$$

Consequently, we get

$$u(t) = f(t,u(t))\left(\phi(u) + \sum_{i=1}^{2} \frac{I_i(u(t_i^-))}{f(t_i,u(t_i))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

By using the same method, for $t \in [t_i, t_{i+1}[$, $i = 3, 4, \dots, n$, one has

$$u(t) = f(t,u(t))\left(\phi(u) + \sum_{i=1}^{k} \frac{I_i(u(t_i^-))}{f(t_i,u(t_i))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

Conversely, assume that u satisfies (3.2). Then for $t \in [t_0, t_1[$, we have

$$u(t) = f(t,u(t))\left(\phi(u) + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right). \tag{3.8}$$

Then, dividing by $f(t,u(t))$ and applying D^α on both sides of (3.8), we get equation (3.4).

Again, substituting $t = 0$ in (3.8), we obtain $\frac{u(0)}{f(0,u(0))} = \phi(u)$. Since $u \mapsto \frac{u}{f(t,u)}$ is increasing in $\mathbb{R}$ for $t \in [t_0, t_1[$, the map $u \mapsto \frac{u}{f(t,u)}$ is injective in $\mathbb{R}$. Then we get (3.5).

If $t \in [t_1, t_2[$, then we have

$$u(t) = f(t,u(t))\left(\phi(u) + \frac{I_1(u(t_1^-))}{f(t_1,u(t_1))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right). \tag{3.9}$$

Then, dividing by $f(t,u(t))$ and applying D^α on both sides of (3.9), we get equation (3.6). Again by (H_3), substituting $t = t_1$ in (3.8) and taking the limit in (3.9), then (3.9) minus (3.8) gives (3.7).

Similarly, for $t \in [t_i, t_{i+1}[$, $i = 2, 3, \dots, n$, we get

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) = h(t), \quad t \in [t_k, t_{k+1}[, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)). \end{cases} \tag{3.10}$$

This completes the proof. □

Lemma 3.3. *Let g be continuous, then $u \in X$ is a solution of Cauchy problem* (1.1) *if and only if u is a solution of the integral equation*

(3.11)

$$u(t) = f(t,u(t))\left(\phi(u) + \theta(t)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(t,u(t))ds\right), \quad t \in [t_i, t_{i+1}],$$

where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \notin [t_0, t_1[. \end{cases}$$

Now we are in a position to present our first result which deals with the existence and uniqueness of solution for Cauchy problem (1.1). This result is based on Banach's fixed point theorem. To do so, we define the operator $\Psi : X \to X$ by

$$\Psi(u)(t) = f(t,u(t)) \left(\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds \right). \tag{3.12}$$

Theorem 3.1. *Assume that conditions* (H_1)*-*(H_6) *hold and the function* $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous. Then Cauchy problem* (1.1) *has an unique solution provided that* $\Delta < 1$*, where* Δ *is the constant given in equation* (3.1)*.*

Proof. Let us set $\sup_{t\in[0,1]} g(t,0) = \kappa < \infty$, and define a closed ball $\bar{B}$ as follows

$$\bar{B} = \{u \in X : \|u\| \le r\},$$

where

$$r \ge \frac{L\kappa}{1 - L\left(M_\phi + nN_I + \frac{1}{\Gamma(\alpha+1)}M_g\right)}. \tag{3.13}$$

We show that $\Psi(\bar{B}) \subset \bar{B}$. For $u \in \bar{B}$, we obtain

$$\begin{aligned}
|\Psi(u)(t)| &\le L\left|\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds\right| \\
&\le L\left[M_\phi\|u\| + nN_I\|u\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|g(s,u(s)) - g(s,0)| + |g(s,0)|)ds\right] \\
&\le L\left[M_\phi\|u\| + nN_I\|u\| + \frac{1}{\Gamma(\alpha+1)}\left(M_g\|u\| + \kappa\right)\right] \\
&\le L\left[(M_\phi + nN_I)r + \frac{1}{\Gamma(\alpha+1)}(M_g r + \kappa)\right].
\end{aligned}$$

Hence, we get

$$\|\Psi(u)\| \le L\left((M_\phi + nN_I)r + \frac{1}{\Gamma(\alpha+1)}\left(M_g r + \kappa_1\right)\right). \tag{3.14}$$

From (3.14), it follows that $\|\Psi(u)\| \le r$.

Next, for $(u, \bar{u}) \in X^2$ and for any $t \in [0,1]$, we have

$$\begin{aligned}|\Psi(u)(t) - \Psi(\bar{u})(t)| =& \Bigg| f(t,u(t)) \left[\phi(u) + \theta(t) \sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t,u(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))ds \right] \\ & - f(t,\bar{u}(t)) \Bigg[\phi(\bar{u}) + \theta(t) \sum_{i=1}^{n} \frac{I_i(\bar{u}(t_i^-))}{f(t,\bar{u}(t_i))} \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,\bar{u}(s))ds \Bigg] \Bigg| \\ \leq & L \left(K_\phi |u-\bar{u}| + nA|u-\bar{u}| + \frac{M_g}{\Gamma(\alpha+1)} |u-\bar{u}| \right),\end{aligned}$$

which implies that

$$\|\Psi(u) - \Psi(\bar{u})\| \leq L\Big(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)} \Big) \|u-\bar{u}\| = \Delta \|u-\bar{u}\|. \tag{3.15}$$

In view of condition $\Delta < 1$, it follows that Ψ is a contraction operator. So Banach's fixed point theorem applies and hence the operator Ψ has an unique fixed point, which is an unique solution of Cauchy problem (1.1). This completes the proof. $\square$

In our second result, we discuss the existence of solutions for Cauchy problem (1.1) by means of Leray-Schauder alternative.

For brevity, let us set

$$\mu_1 = \frac{L}{\Gamma(\alpha+1)}, \tag{3.16}$$

$$\mu_0 = 1 - \mu_1 \rho_1. \tag{3.17}$$

Lemma 3.4 (Leray-Schauder alternative see [3])**.** *Let $\mathcal{F} : G \to G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $P(\mathcal{F}) = \{u \in G : u = \lambda \mathcal{F} u$ for some $0 < \lambda < 1\}$. Then either the set $P(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fixed point.*

Theorem 3.2. *Assume that conditions (H_1)-(H_3) and (H_7)-(H_9) hold. Furthermore, it is assumed that $\mu_1 \rho_1 < 1$, where μ_1 is given by (3.16). Then Cauchy problem (1.1) has at least one solution.*

Proof. We will show that the operator $\Psi : X \to X$ satisfies all the assumptions of Lemma 3.4.

Step 1. We prove that the operator Ψ is completely continuous.

Clearly, it follows from the continuity of functions f and g that the operator Ψ is continuous.

Let $S \subset X$ be bounded. Then we can find a positive constant H such that $|g(t, u(t))| \leq H$, $u \in S$. Thus, for any $u \in S$, we can get

$$\begin{aligned}|\Psi(u)(t)| &\leq L\left(\rho + \sum_{i=1}^{n} C + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} H ds\right)\\ &\leq L\left(\rho + nC + \frac{H}{\Gamma(\alpha+1)}\right),\end{aligned}$$

which yields

$$\|\Psi(u)\| \leq L\left(\rho + nC + \frac{H}{\Gamma(\alpha+1)}\right). \tag{3.18}$$

From the inequality (3.18), we deduce that the operator Ψ is uniformly bounded.

Step 2. Now we show that the operator Ψ is equicontinuous.

For $\tau_1, \tau_2 \in [0,1]$ with $\tau_1 < \tau_2$, we obtain

$$\begin{aligned}&|\Psi(u(\tau_2)) - \Psi(u(\tau_1))|\\ \leq& L\Bigg|\left(\phi(u) + \theta(\tau_2)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + H\int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\right)\\ &- \left(\phi(u) + \theta(\tau_1)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))} + H\int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\right)\Bigg|\\ \leq& L\left(\left|(\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))}\right| + H\left|\int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds - \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\right|\right)\\ \leq& L\Bigg(\left|(\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f(t, u(t_i))}\right| + H\Bigg|\int_0^{\tau_1} \frac{(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\\ &+ \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds\Bigg|\Bigg),\end{aligned}$$

which tends to 0 independently of u. This implies that the operator $\Psi(u)$ is equicontinuous. Thus, by the above findings, the operator $\Psi(u)$ is completely continuous.

In the next step, it will be established that the set $P = \{u \in X : u = \lambda\Psi(u), 0 < \lambda < 1\}$ is bounded.

For $u \in P$, we have $u = \lambda\Psi(u)$. Thus, for any $t \in [0,1]$, we can write $u(t) = \lambda\Psi(u)(t)$. Then we obtain

$$\begin{aligned}\|u\| &\leq L\left(\rho + nC + \frac{1}{\Gamma(\alpha+1)}(\rho_0 + \rho_1\|u\|)\right)\\ &\leq L(\rho + nC) + \mu_1(\rho_0 + \rho_1\|u\|).\end{aligned}$$

Hence, we get

$$\|u\| \leq \frac{L(\rho + nC) + \mu_1\rho_0}{\mu_0}.$$

This shows that the set P is bounded. In consequence, all the conditions of Lemma 3.4 are satisfied. Finally, the operator Ψ has at least one fixed point, which is a solution of Cauchy problem (1.1). This completes the proof. ☐

4. Examples

Example 4.1. Consider the hybrid fractional differential equation:

$$\begin{cases} {}^cD^{\frac{1}{2}}\left(\frac{u(t)}{\frac{e^{-1}+t+\sqrt{u(t)}}{40+t^2}}\right) = \frac{e^{-t}+|\sin u(t)|}{20}, \quad t \in [0,1] \setminus \{t_1\}, \\ u(t_1^+) = u(t_1^-) + (-2u(t_1^-)), \quad t_1 \neq 0, 1, \\ \frac{u(0)}{f(0,u(0)} = \sum_{i=1}^{n} \lambda_i u(t_i). \end{cases} \tag{4.1}$$

Here, we have

$$f(t,u(t)) = \frac{e^{-1}+t+\sqrt{u(t)}}{40+t^2},$$

$$g(t,u(t)) = \frac{e^{-t}+|\sin u(t)|}{20},$$

$$|g(t,u_1) - g(t,u_2)| \leq \frac{1}{40}|u_2 - u_1|, \quad t \in [0,1] \text{ and } u_1, u_2 \in \mathbb{R},$$

$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha+1)}\right) \simeq 0.0012345687 < 1.$$

Then all the assumptions of Theorem 3.2 are satisfied, thus our results can be applied to Cauchy problem (4.1).

Example 4.2. Consider another example for hybrid fractional differential equations of the following form

$$\begin{cases} {}^cD^{\frac{1}{2}}\left(\frac{v(t)}{\frac{e^{-1}+t^2+\sqrt{v(t)}}{32+t}}\right) = \frac{e^{-2t}+\cos^2(v(t))}{20}, \quad t \in [0,1] \setminus \{t_1\}, \\ v(t_1^+) = v(t_1^-) + (-2v(t_1^-)), \quad t_1 \neq 0, 1, \\ \frac{v(0)}{f(0,v(0))} = \sum_{j=1}^{n} \lambda_j v(t_j)). \end{cases} \tag{4.2}$$

Here, we have

$$f(t,v(t)) = \frac{e^{-1}+t^2+\sqrt{v(t)}}{32+t},$$

$$g(t,v(t)) = \frac{e^{-2t}+\cos^2(v(t))}{20},$$

$$|g(t, v_1) - g(t, v_2)| \leq \frac{1}{20}|v_2 - v_1|, \quad t \in [0, 1] \text{ and } v_1, v_2 \in \mathbb{R},$$
$$\Delta = L\left(K_\phi + nA + \frac{M_g}{\Gamma(\alpha + 1)}\right) \simeq 0.3354687 < 1.$$

Then all the assumptions of Theorem 3.2 are satisfied, thus our results can be applied to Cauchy problem (4.2).

References

[1] M. Benchohra, S. Hamani and S. K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl. **2008** (2008), 1–12. `http://www.utgjiu.ro/math/sma`

[2] B. C. Dhage, and V. Lakshmikantham, *Basic results on hybrid differential equations*, Nonlinear Analysis: Hybrid Systems, **4** (2010), 414–424. `https://doi.org/10.1016/j.nahs.2009.10.005`

[3] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, NY, USA, 2003. `https://doi.org/10.1007/978-0-387-21593-8`

[4] K. Hilal, and A. Kajouni, *Boundary value problems for hybrid differential equations with fractional order*, Advances in Difference Equations **2015** (2015) `https://doi10.1186/s13662-015-0530-7`

[5] M. Hannabou and K. Hilal, *Existence results for a system of coupled hybrid differential equations with fractional order*, Int. J. Differ. Equ. **2020** (2020), Article ID 3038427. `https://doi:10.1155/2020/3038427`

[6] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science B.V, Amsterdam, 2006.

[7] V. Lakshmikantham, S. Leela and J. D. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009. `https://SBN13:9781904868644`

[8] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989. `http://dx.doi.org/10.1142/0906`

[9] V. Lakshmikantham, S. Leela and J. D. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009. `https://SBN13:9781904868644`

[10] I. Podlubny, *Fractional Differential Equations, Academic Press*, New York, 1993.

[11] S. Sitho, S. K. Ntouyas and J. Tariboon, *Existence results for hybrid fractional integro-differential equations*, Boundary Value Problems **2015** (2015). `https://10.1186/s13661-015-0376-7`

[12] L. Zhang and G. Wang, *Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions*, Electronic Journal of Qualitative Theory of Differential Equations **7** (2011), 1–11.

[13] Y. Zhao, S. Suna, Z. Han and Q. Li, *Theory of fractional hybrid differential equations*, Computers and Mathematics with Application **62** (2011), 1312–1324. `https://doi10.1016/j.camwa.2011.03.041`

[14] G. Zhenghui, L. Yang and G. Liu, *Existence and uniqueness of solutions to impulsive fractional integro-differential equations with nonlocal*, Applied Mathematics **4** (2013), 859–863. `http://dx.doi.org/10.4236/am.2013.46118`

[1]Laboratoire de Mathématiques Appliquées & Calcul Scientifique,
Université Sultan Moulay Slimane, BP 523, 23000 Beni Mellal, Morocco
Email address: `hnnabou@gmail.com`
Email address: `Bouaouidfst@gmail.com`
Email address: `hilalkhalid2005@yahoo.fr`