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SOME NEW INEQUALITIES FOR DIFFERENTIABLE ARITHMETIC-HARMONICALLY CONVEX FUNCTIONS

MAHIR KADAKAL¹, PRAVEEN AGARWAL^{2,3,4}, AND İMDAT İŞCAN⁵

ABSTRACT. In this study, by using an integral identity together with both the Hölder and the power-mean inequalities for integrals we establish several new inequalities for differentiable arithmetic-harmonically-convex function. Also, we give some applications for special means.

1. PRELIMINARIES AND FUNDAMENTALS

Throughout, we denote any real interval by $I \subseteq \mathbb{R}$ and any functions defined on I by $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$. Let I° denote the interior of I . Also, we denote

$$I_f(a, b) = f(b)b - f(a)a - \int_a^b f(x)dx,$$

for brevity.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of related problems in pure and applied sciences. The following double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

Key words and phrases. Convex function, arithmetic-harmonically convex function Hermite-Hadamard's inequality.

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Theorem 1.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds.

See [2, 4], for the results of the generalization, improvement and extension of the famous integral inequality (1.1).

Definition 1.2 ([1, 5]). A function $f : I \subset \mathbb{R} \rightarrow (0, \infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0, 1]$ the equality

$$(1.2) \quad f(tx + (1-t)y) \leq \frac{f(x)f(y)}{tf(y) + (1-t)f(x)}$$

holds. If the inequality (1.2) is reversed, then the function f is said to be arithmetic-harmonically (AH) concave function.

In order to establish some inequalities of Hermite-Hadamard type integral inequalities for AH-convex functions, we will use the following lemma obtained in the special case of identity given in [3].

Lemma 1.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. We have the identity

$$(1.3) \quad I_f(a, b) = \int_a^b xf'(x)dx.$$

In this study, we use Hölder integral inequality, power-mean integral inequality and the identity (1.3) in order to provide some inequalities for functions whose first derivatives in absolute value at certain power are arithmetic-harmonically convex.

Throughout this paper, we will use the following notations for special means of two nonnegative numbers a, b with $b > a$:

1. the arithmetic mean

$$A := A(a, b) = \frac{a+b}{2}, \quad a, b > 0,$$

2. the geometric mean

$$G := G(a, b) = \sqrt{ab}, \quad a, b \geq 0,$$

3. the logarithmic mean

$$L := L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b, \\ a, & a = b, \end{cases} \quad b > 0,$$

4. the p -logarithmic mean

$$L_p := L_p(a, b) = \begin{cases} \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, & a \neq b, p \in \mathbb{R} \setminus \{-1, 0\}, \\ a, & a = b, \end{cases} \quad a, b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

2. MAIN RESULTS FOR LEMMA

Throughout this section we will denote $K_x = |f'(x)|$ for brevity.

Theorem 2.1. *Let $f : I \subset (0, +\infty) \rightarrow (0, +\infty)$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is an arithmetic-harmonically convex function on the interval $[a, b]$, then the following inequality holds:*

$$(2.1) \quad |I_f(a, b)| \leq \begin{cases} \frac{(b-a)G^2(K_a, K_b)}{K_b - K_a} \left(\frac{bK_b - aK_a}{L(K_a, K_b)} - (b-a) \right), & K_a \neq K_b, \\ (b-a)K_b A(a, b), & K_a = K_b. \end{cases}$$

Proof. Since $|f'|$ is an arithmetic-harmonically convex function on the interval $[a, b]$, we have on setting $t = \frac{b-x}{b-a}$ and $1-t = \frac{x-a}{b-a}$ in (1.2)

$$(2.2) \quad |f'(x)| \leq \frac{(b-a)K_a K_b}{(b-x)K_b + (x-a)K_a},$$

for all $x \in [a, b]$. Substituting (2.2) in

$$(2.3) \quad |I_f(a, b)| \leq \int_a^b x |f'(x)| dx,$$

which follows from (1.3), we have

$$(2.4) \quad |I_f(a, b)| \leq (b-a)K_a K_b \int_a^b \frac{x}{(b-x)K_b + (x-a)K_a} dx.$$

We distinguish two cases. If $K_a = K_b$, then (2.1) follows. Suppose $K_a \neq K_b$. Then, by the change of variable $u = (b-x)K_b + (x-a)K_a$, the integral in (2.4) becomes

$$\begin{aligned} & \frac{(b-a)K_a K_b}{(K_b - K_a)^2} \int_{(b-a)K_a}^{(b-a)K_b} \left(\frac{bK_b - aK_a}{u} - 1 \right) du \\ &= \frac{(b-a)K_a K_b}{K_b - K_a} \left(bK_b - aK_a \frac{\ln K_b - \ln K_a}{K_b - K_a} - (b-a) \right). \end{aligned}$$

Substituting this in (2.4) and using the definition of the logarithmic mean, we conclude (2.1) in this case. This completes the proof. $\square$

Theorem 2.2. *Let $f : I \subset (0, +\infty) \rightarrow (0, +\infty)$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a, b]$, then the following inequality holds:*

$$(2.5) \quad |I_f(a, b)| \leq \begin{cases} \frac{(b-a)L_p(a, b)G^2(K_a, K_b)}{(L(K_a, K_b)L_{q-1}^{q-1}(K_a, K_b))^{\frac{1}{q}}}, & K_a \neq K_b, \\ (b-a)K_b L_p(a, b), & K_a = K_b, \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a, b]$, we have

$$(2.6) \quad |f'(x)|^q \leq \frac{(b-a)(K_a K_b)^q}{(b-x)K_b^q + (x-a)K_a^q},$$

for all $x \in [a, b]$. By using Hölder integral inequality in (2.3), we get

$$(2.7) \quad |I_f(a, b)| \leq \left(\int_a^b x^p dx \right)^{\frac{1}{p}} \left(\int_a^b |f'(x)|^q dx \right)^{\frac{1}{q}}.$$

By combining (2.6) and (2.7) and also using the definitions of the p -logarithmic mean and geometric mean, we obtain

$$(2.8) \quad |I_f(a, b)| \leq (b-a)G^2(K_a, K_b) L_p(a, b) \left(\int_a^b \frac{dx}{(b-x)K_b^q + (x-a)K_a^q} \right)^{\frac{1}{q}}.$$

We distinguish two cases. If $K_a = K_b$, then (2.5) follows. Suppose $K_a \neq K_b$. Then, by the change of variable $u = (b-x)K_b^q + (x-a)K_a^q$, the integral in (2.8) becomes

$$\begin{aligned} & (b-a)G^2(K_a, K_b) L_p(a, b) \left(\int_{(b-a)K_a^q}^{(b-a)K_b^q} \frac{du}{(K_b^q - K_a^q)u} \right)^{\frac{1}{q}} \\ & = (b-a)G^2(K_a, K_b) L_p(a, b) \left(\frac{\ln K_b^q - \ln K_a^q}{K_b^q - K_a^q} \right)^{\frac{1}{q}}. \end{aligned}$$

Substituting this in (2.8) and using the definitions of the logarithmic mean and the p -logarithmic mean, we conclude (2.5) in this case. This completes the proof. $\square$

Theorem 2.3. Let $f : I \subset (0, +\infty) \rightarrow (0, +\infty)$ be a differentiable mapping on I° , and $a, b \in I^\circ$ with $a < b$. If $|f'|^q, q \geq 1$ is an arithmetic-harmonically convex function on the interval $[a, b]$, then the following inequality holds:

$$(2.9) \quad |I_f(a, b)| \leq \begin{cases} \frac{(b-a)A^{1-\frac{1}{q}}(a,b)G^2(K_a, K_b)}{(K_b^q - K_a^q)^{\frac{1}{q}}} \left(\frac{bK_b^q - aK_a^q}{L(K_a, K_b)L_{q-1}^{q-1}(K_a, K_b)} - (b-a) \right)^{\frac{1}{q}}, & K_a \neq K_b, \\ (b-a)K_b A(a, b), & K_a = K_b. \end{cases}$$

Proof. Since $|f'|^q$ is an arithmetic-harmonically convex function on the interval $[a, b]$, we have

$$(2.10) \quad |f'(x)|^q \leq \frac{(b-a)(K_a K_b)^q}{(b-x)K_b^q + (x-a)K_a^q},$$

for all $x \in [a, b]$. By using well known power-mean integral inequality in (2.3), we get

$$(2.11) \quad |I_f(a, b)| \leq \left(\int_a^b x dx \right)^{1-\frac{1}{q}} \left(\int_a^b x |f'(x)|^q dx \right)^{\frac{1}{q}}.$$

By combining (2.10) and (2.11) and also using the definitions of the arithmetic mean and geometric mean, we obtain

$$(2.12) \quad |I_f(a, b)| \leq (b-a) A^{1-\frac{1}{q}}(a, b) G^2(K_a, K_b) \left(\int_a^b \frac{x}{(b-x) K_b^q + (x-a) K_a^q} dx \right)^{\frac{1}{q}}.$$

We distinguish two cases. If $K_a = K_b$, then (2.9) follows. Suppose $K_a \neq K_b$. Then, by the change of variable $u = (b-x) K_b^q + (x-a) K_a^q$, the integral in (2.12) becomes

$$\begin{aligned} & \frac{(b-a) A^{1-\frac{1}{q}}(a, b) G^2(K_a, K_b)}{(K_b^q - K_a^q)^{\frac{2}{q}}} \left(\int_{(b-a) K_a^q}^{(b-a) K_b^q} \frac{b K_b^q - a K_a^q - u}{u} du \right)^{\frac{1}{q}} \\ &= \frac{(b-a) A^{1-\frac{1}{q}}(a, b) G^2(K_a, K_b)}{(K_b^q - K_a^q)^{\frac{1}{q}}} \left(\frac{(b K_b^q - a K_a^q) (\ln K_b^q - \ln K_a^q)}{K_b^q - K_a^q} - (b-a) \right)^{\frac{1}{q}}. \end{aligned}$$

Substituting this in (2.12) and using the definitions of the logarithmic mean and the p -logarithmic mean, we conclude (2.9) in this case. This completes the proof. $\square$

Corollary 2.1. *If we take $q = 1$ in the inequality (2.9), we get the inequality (2.1).*

3. APPLICATIONS FOR SPECIAL MEANS

If $p \in (-1, 0)$, then the function $f(x) = x^p$, $x > 0$, is an arithmetic harmonically-convex [1]. Using this function we obtain following propositions.

Proposition 3.1. *Let $0 < a < b$ and $m \in (-1, 0)$. Then we have the following inequality:*

$$(3.1) \quad L_{m+1}^{m+1}(a, b) \leq \frac{1}{m} \cdot \frac{G^{2m}(a, b)}{L_{m-1}^{m-1}(a, b)} \left((m+1) \frac{L_m^m(a, b)}{L(a^m, b^m)} - 1 \right).$$

Proof. We know that if $m \in (-1, 0)$ then the function $f(x) = \frac{x^{m+1}}{m+1}$, $x > 0$, is an arithmetic harmonically-convex function. Therefore, the assertion follows from the inequality (2.1), for $f : (0, +\infty) \rightarrow \mathbb{R}$, $f(x) = \frac{x^{m+1}}{m+1}$. $\square$

Proposition 3.2. *Let $a, b \in (0, +\infty)$ with $a < b$, $q > 1$ and $m \in (-1, 0)$. Then we have the following inequality:*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a, b) \leq \frac{L_p(a, b) G^{\frac{2m}{q}}(a, b)}{\left(L(a^{m/q}, b^{m/q}) L_{q-1}^{q-1}(a^{m/q}, b^{m/q}) \right)^{\frac{1}{q}}}.$$

Proof. The assertion follows from the inequality (2.5). Let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0, +\infty)$. Then $|f'(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0, +\infty)$ and the result follows directly from Theorem 2.2. $\square$

Proposition 3.3. *Let $a, b \in (0, +\infty)$ with $a < b$, $q > 1$ and $m \in (-1, 0)$. Then, we have the following inequality:*

$$(3.2) \quad L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(a, b) \leq \frac{A^{1-\frac{1}{q}}(a, b) G^{\frac{2m}{q}}(a, b)}{\left(m L_{m-1}^{m-1}(a, b)\right)^{\frac{1}{q}}} \left(\frac{(m+1) L_m^m(a, b)}{L(a^{m/q}, b^{m/q}) L_{q-1}^{q-1}(a^{m/q}, b^{m/q})} - 1 \right)^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (2.9). Let $f(x) = \frac{q}{m+q} x^{\frac{m}{q}+1}$, $x \in (0, +\infty)$. Then $|f'(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0, +\infty)$ and the result follows directly from Theorem 2.3. $\square$

Corollary 3.1. *If we take $q = 1$ in the inequality (3.2), we get the following inequality*

$$(3.3) \quad L_{m+1}^{m+1}(a, b) \leq \frac{G^{2m}(a, b)}{m L_{m-1}^{m-1}(a, b)} \left(\frac{(m+1) L_m^m(a, b)}{L(a^m, b^m)} - 1 \right),$$

which is the same as inequality (3.1).

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A STUDY OF *-PRIME RINGS WITH DERIVATIONS

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AND MUZIBUR RAHMAN MOZUMDER⁴

ABSTRACT. This paper's major goal is to describe the structure of the *-prime ring, with the help of three different derivations α , β and γ such that $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] + [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Further, some more related results have also been discussed. As applications, classical theorems due to Bell-Daif [6] and Herstein [12] are deduced.

1. INTRODUCTION

This research is the extension of the work done by Ali et al. in [3]. If (i) $(s_1 s_2)^* = s_2^* s_1^*$ and (ii) $(s_1^*)^* = s_1$ holds for all $s_1, s_2 \in \chi$, then an additive map $s_1 \mapsto s_1^*$ of χ into itself is said to be an involution. Ring with involution, often known as *-ring or ring with involution. $\mathcal{H}(\chi)$ is the collection of hermitian objects ($s_1^* = s_1$) and $\mathcal{S}(\chi)$ is the collection of skew-hermitian objects ($s_1^* = -s_1$) of χ . If characteristic different from two, then, obviously, $\mathcal{H}(\chi) = \mathcal{S}(\chi)$. Thus, we will consider only *-rings χ with $\text{char}(\chi) \neq 2$. If $\mathcal{Z}(\chi) \subseteq \mathcal{H}(\chi)$, the involution is said to be of the first kind; otherwise, it is of the second kind. In the later case, $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$ (e.g., involution in the case of ring of quaternions). In [11], there's a mention of these rings as well as additional references.

The origins of commuting and centralising maps can be traced back to 1955, when Divinsky [9] proved that “simple Artinian ring is commutative if it has commuting non-trivial automorphisms”. In 1957, Posner [18] found that “existence of nonzero centralizing derivation on a prime ring forces the ring to be commutative”. The study of commuting (centralizing) derivation/additive maps/multiplicative maps and several

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extension of such results begins with the results of Posner [18] along with applications to different areas like Lie theory, matrix theory, operator theory etc. For more details of said work see (see [2, 4, 8–10, 13] and references therein).

In [3], Ali et al. proved that “a prime ring χ must be a commutative integral domain if it admits derivations α and β satisfying any one of the identities: (i) $[\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$, (ii) $\alpha(s_1) \circ \alpha(s_1^*) + \beta([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$, (iii) $\alpha([s_1, s_1^*]) + [\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$, (iv) $\alpha(s_1 \circ s_1^*) + \alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$ ”. Our goal in this work is to continue this line of inquiry and analyse the structure of prime rings with involution satisfying above mentioned $*$ -differential identities which are central. In fact, so many results become corollaries of our results which are in [2, 3, 6, 8, 12, 16, 17] and references therein.

2. THE RESULTS

Herstein [12] proved a classical result “A prime ring χ of $\text{char}(\chi) \neq 2$ with a derivation $\alpha \neq 0$ satisfying the differential identity $[\alpha(s_1), \alpha(s_2)] = 0$ for all $s_1, s_2 \in \chi$, must be commutative”. Further, Daif [7], proved that “Let χ be a 2-torsion free semiprime ring admitting a derivation α such that $[\alpha(s_1), \alpha(s_2)] = 0$ for all $s_1, s_2 \in I$, where I is a nonzero ideal of χ and α is nonzero on I , then χ contains a nonzero central ideal”. Further, this result was extended by second author together with Dar in [8, Theorem 3.1] in case of prime rings involving $*$: $\chi \mapsto \chi$. Indeed, they proved “Let χ be a prime ring with involution $'*$ ' of the second kind such that $\text{char}(\chi) \neq 2$ and satisfying the $*$ -differential identity $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$, then χ must be commutative”. Throughout our discussion $*$ will be of second kind and also as when we consider more than one derivation then it is assume that at least one of them to be nonzero. We begin our investigation with several well-known facts, which lead to the following results repeatedly.

Fact 2.1 ([3, Lemma 2.5]). Let χ be a $*$ -prime ring and α be a derivation and $\alpha(t) = 0$ for all $t \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. Then $\alpha(s_1) = 0$ for all $s_1 \in \mathcal{Z}(\chi)$.

Fact 2.2 ([17, Lemma 2.1]). Let χ be a $*$ -prime ring and χ is normal for all $s_1 \in \chi$. Then χ is commutative.

Fact 2.3 ([17, Lemma 2.2]). Let χ be a $*$ -prime ring and $s_1 \circ s_1^* \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$ if and only if χ is commutative.

Theorem 2.4. *Let χ be a $*$ -prime ring and α, β and γ be derivations of χ satisfying the identity $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. The proof is divided into the following cases.

Case (i) If $\alpha = 0$ and $\beta, \gamma \neq 0$, then we have

$$(2.1) \quad [\beta(s_1), \beta(s_1^*)] \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Taking t for s_1 in (2.1), where $t \in \mathcal{H}(\chi)$, we obtain

$$(2.2) \quad \pm [\gamma(t), t] \in \mathcal{Z}(\chi), \quad \text{for all } t \in \mathcal{H}(\chi).$$

Linearization of (2.2) gives

$$(2.3) \quad \pm [\gamma(t), h_1] \pm [\gamma(h_1), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Replacing h_1 by $h_1 h_0$ in (2.3) and combining (2.3), we get $\pm [h_1, t] \gamma(h_0) \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$ and $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. Applying the primeness of the ring χ , we obtain either $\pm [h_1, t] \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$ or $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. If we consider $\pm [h_1, t] \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$, replacing h_1 by kh_1 , we have $\pm [k, t] h_1 \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$, $k \in \mathcal{S}(\chi)$ and $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$. Since $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$ and χ is prime, implies that $\pm [k, t] \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$ and $k \in \mathcal{S}(\chi)$. This implies that χ is commutative. Now consider $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$, this implies that $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$. Taking kh_1 in place of t in (2.2), we obtain

$$\pm [\gamma(k), k] k_0^2 \in \mathcal{Z}(\chi), \quad \text{for all } k \in \mathcal{S}(\chi) \text{ and } h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi).$$

Since χ is prime and we have $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$, we obtain

$$(2.4) \quad \pm [\gamma(k), k] \in \mathcal{Z}(\chi), \quad \text{for all } k \in \mathcal{S}(\chi) \text{ and } h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi).$$

By linearizing (2.2), we get

$$(2.5) \quad \pm [\gamma(t), h_1] \pm [\gamma(h_1), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Substituting kh_1 for h_1 in (2.5), where $k \in \mathcal{S}(\chi)$ and $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$, we obtain

$$(2.6) \quad \pm [\gamma(t), k] \pm [\gamma(k), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Consider $4[\gamma(s_1), s_2] = [\gamma(2s_1), 2s_2] = [\gamma(t+k), t+k] = [\gamma(t), t] + [\gamma(k), t] + [\gamma(t), k] + [\gamma(k), k]$. Using (2.2), (2.4) and (2.6), we get $4[\gamma(s_1), s_2] \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Since $\text{char}(\chi) \neq 2$, this implies that $[\gamma(s_1), s_2] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Therefore, in view of Posner's result we done.

Case (ii) If $\beta = 0$ and $\alpha, \gamma \neq 0$, then we have $\alpha([s_1, s_1^*]) \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Substituting t for s_1 , we obtain $\pm [\gamma(t), t] \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$, which is same as (2.2), following the line of proof as we did after (2.2), we get χ is commutative.

Case (iii) If $\gamma = 0$ and $\alpha, \beta \neq 0$, then from hypothesis we obtain

$$\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Substituting ss_1 for s_1 in above equation, where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, we get

$$(2.7) \quad [s_1, s_1^*] 2s\alpha(s) + [\alpha(s_1), s_1^*] s\beta(s) + [s_1, \beta(s_1^*)] s\beta(s) + [s_1, s_1^*] (\beta(s))^2 \in \mathcal{Z}(\chi).$$

Linearization of (2.7), gives us

$$(2.8) \quad [s_1, s_2^*] 2s\alpha(s) + [s_2, s_1^*] 2s\alpha(s) + [\alpha(s_1), s_2^*] s\alpha(s) + [\alpha(s_2), s_1^*] s\alpha(s) \\ + [s_1, \beta(s_2^*)] s\beta(s) + [s_2, \beta(s_1^*)] s\beta(s) + [s_1, s_2^*] (\beta(s))^2 + [s_2, s_1^*] (\beta(s))^2 \in \mathcal{Z}(\chi).$$

Now taking s_2s for s_2 in (2.8), where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, and combining it with the obtained result, we find that

$$(2.9) \quad 4[s_2, s_1^*]s^2\alpha(s) + 2[\beta(s_2), s_1^*]s\beta(s)^2 + 2[s_2, s_1^*]s(\beta(s))^2 \\ - [s_1, s_2^*]s(\beta(s))^2 + 2[s_2, \beta(s_1^*)]s^2\beta(s) + [s_2, s_1^*]s(\beta(s))^2 \in \mathcal{Z}(\chi).$$

Substituting s_2s for s_2 in (2.9) and solving with the help of (2.9), we have

$$(2.10) \quad 2[s_1, s_2^*]s^2(\beta(s))^2 + 2[s_2, s_1^*]s(\beta(s))^3 \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Again taking s_2s for s_2 in (2.10), where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, and combining it with (2.10), we get $4[s_2, s_1^*]s^2(\beta(s))^3 \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Replacing s_2 by s_1 , we obtain $4[s_1, s_1^*]s^2(\beta(s))^2 \in \mathcal{Z}(\chi)$, for all $s_1 \in \chi$ and $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation forces that either $[s_1, s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$ or $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. If $[s_1, s_1^*] \in \mathcal{Z}(\chi)$, then by Fact 2.2, χ is commutative. On the other hand, we consider the situation $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Using this in (2.7), we get $2[s_1, s_1^*]s\alpha(s) \in \mathcal{Z}(\chi)$. By the primeness of the ring χ , we conclude that either χ is commutative or $\alpha(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Linearization of $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$, gives us

$$(2.11) \quad \alpha([s_1, s_2^*]) + \alpha([s_2, s_1^*]) + [\beta(s_1), \beta(s_2^*)] + [\beta(s_2), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Replacing s_2 by ss_2 in (2.11) where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$ and using the fact that $\alpha(s) = 0$ and $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, we arrive at

$$2(\alpha([s_2, s_1^*]) + [\beta(s_2), \beta(s_1^*)])s \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation yields

$$\alpha([s_2, s_1^*]) + [\beta(s_2), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

This implies that

$$\alpha([s_2, s_1]) + [\beta(s_2), \beta(s_1)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Replacing s_2 by s_1^2 in the last relation, we get $[\beta(s_1^2), \beta(s_1)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. This further implies that $[(\beta(s_1))^2, s_1] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Thus in view of [14, Theorem 1.1], we get χ is commutative. This proves the theorem.

Case (iv) If $\alpha = 0$, $\beta = 0$ and $\gamma \neq 0$, we have $\pm[\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$, then by [17, Theorem 3.7] χ is commutative.

Case (v) Consider $\beta = 0$, $\gamma = 0$ and $\alpha \neq 0$, then from hypothesis, we have $\alpha([s_1, s_1^*]) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. By [16, Theorem 2.3], we obtain χ is commutative.

Case (vi) Taking $\gamma = 0$, $\alpha = 0$ and $\beta \neq 0$, then by hypothesis we have $[\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Hence, result follows by [17, Theorem 3.1].

Case (vii) Consider the following if $\alpha = 0$, $\beta = 0$ and $\gamma \neq 0$. Substituting t for s_1 in assumption, we obtain $[\gamma(t), t] \in \mathcal{Z}(\chi)$ for all $s_1 \in \mathcal{H}(\chi)$, which is same as (2.2). Therefore χ is commutative by follow the same argument. $\square$

We deduce the following corollaries from Theorem 2.4.

Corollary 2.1 ([8, Theorem 3.1]). *Let χ be a *-prime ring and $\alpha \neq 0$ be a derivation of χ such that $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.2 ([2, Theorem 2.2]). *Let χ be a *-prime ring and $\alpha \neq 0$ be a derivation of χ such that $\alpha([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.3 ([3, Theorem 3.5]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.4. *Let χ be a *-prime ring and α and β be a nonzero derivation of χ satisfying $\alpha(s_1 s_1^*) + \beta(s_1) \beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. By the assumption, we have $\alpha(s_1 s_1^*) + \beta(s_1) \beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Replace s_1 by s_1^* in the last expression to get $\alpha(s_1^* s_1) + \beta(s_1^*) \beta(s_1) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Combining the last two relations, we obtain $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Hence, application of Case (vi) of Theorem 2.4 yields the required result. $\square$

Theorem 2.5. *Let χ be a *-prime ring and α and β be two derivations of χ satisfying the identity $\alpha(s_1 \circ s_1^*) + \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. By the assumption, we have

$$(2.12) \quad \alpha(s_1 \circ s_1^*) + \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Case (i) Assume that $\alpha \neq 0$ and $\beta = 0$. Then it follows from (2.12) that $\alpha(s_1 \circ s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. In view of [16, Theorem 2.5], we get χ is commutative.

Case (ii) Taking $\alpha = 0$ and $\beta \neq 0$. Then (2.12) reduces to

$$(2.13) \quad \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Application of [17, Theorem 3.5] gives the required result.

Case (iii) Assume that both α and β are nonzero. Replacing s_1 by $s_1 + s_2$ in (2.12), we get

$$(2.14) \quad \alpha(s_1 \circ s_2^*) + \alpha(s_2 \circ s_1^*) + \beta(s_1) \circ \beta(s_2^*) + \beta(s_2) \circ \beta(s_1^*) \in \mathcal{Z}(\chi).$$

Substituting $s_2 t$ for s_2 in (2.14), where $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$, we get

$$((s_1 \circ s_2^*) + (s_2 \circ s_1^*))\alpha(t) + (\beta(s_1) \circ s_2^* + s_2 \circ \beta(s_1^*))\beta(t) \in \mathcal{Z}(\chi).$$

Taking $s_2 s_o$ for s_2 where $s_o \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$ and combining it with the obtained relation, we get

$$2((s_2 \circ s_1^*)s_o\alpha(t)) + (s_2 \circ \beta(s_1^*))s_o\beta(t) \in \mathcal{Z}(\chi).$$

Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation yields

$$(2.15) \quad (s_2 \circ s_1^*)\alpha(t) + (s_2 \circ \beta(s_1^*))\beta(t) \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \mathcal{Z}(\chi).$$

This can be further written as

$$[(s_2 \circ s_1^*), r]\alpha(t) + [s_2 \circ \beta(s_1^*), r]\beta(t) = 0, \quad \text{for all } s_1, s_2, r \in \mathcal{Z}(\chi).$$

Replacing χ by $s_2 \circ s_1^*$ we get $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*]\beta(t) = 0$ for all $s_1, s_2 \in \chi$. Then, by the primeness of χ , we get either $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*] = 0$ for all $s_1, s_2 \in \chi$ or $\beta(t) = 0$ for all $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$. If $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*] = 0$ for all $s_1, s_2 \in \chi$, then by substituting z for s_2 in the last relation where $z \in \mathcal{Z}(\chi)$, we obtain $2[\beta(s_1^*), s_1^*]z = 0$ for all $s_1 \in \chi$. Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, this implies that $[\beta(s_1^*), s_1^*] = 0$ for all $s_1 \in \chi$. By the application of Posner's [18] we arrived at conclusion. Now consider the case $\beta(t) = 0$ for all $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$. Then (2.15) reduces to $(s_2 \circ s_1^*)\alpha(t) \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$ and $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$. By the primeness of the ring χ , we get either $s_2 \circ s_1^* \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$ or $\alpha(t) = 0$ for all $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$. If $s_2 \circ s_1^* \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$ by the Fact 2.3 implies that χ is commutative. Finally, we consider the case $\alpha(t) = 0$ for all $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$. Now replacing s_2 by t in (2.14) where $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$, we get

$$(\alpha(s_1) + \alpha(s_1^*))t \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi \text{ and } t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi).$$

Thus in view of the fact $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$ and primeness of the ring χ , we conclude that $\alpha(s_1) + \alpha(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. This can be written as $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$. Hence, χ is commutative by [17, Theorem 3.1]. $\square$

Corollary 2.5 ([2, Theorem 2.3]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.6. *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. From assumption, we have $\alpha(s_1 s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. For any $s_1 \in \chi$, s_1^* also is an element of χ . Substituting s_1^* for s_1 in the given assertion, we obtain $\alpha(s_1^* s_1) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. This implies that $\alpha(s_1 \circ s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Hence, χ is commutative by Corollary 2.5. $\square$

Corollary 2.7 ([8, Theorem 3.2]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.8 ([3, Theorem 3.6]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 \circ s_1^*) + \alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Theorem 2.6. *Let χ be a $*$ -prime ring and α and β be derivations of χ satisfying the identity $[\alpha(s_1), \alpha(s_1^*)] \pm \beta(s_1 \circ s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. We are given that $\alpha, \beta : \chi \rightarrow \chi$ are derivations such that

$$(2.16) \quad [\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Replacing s_1 by s_1^* in the last expression we get

$$(2.17) \quad -[\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Adding the last two relations and using $\text{char}(\chi) \neq 2$ we obtain

$$(2.18) \quad \beta(s_1 \circ s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Hence, the result follows from [13, Theorem 2].

Similarly, we prove the other case with the help of [13, Theorem 2]. $\square$

Corollary 2.9 ([3, Theorem 3.1]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $[\alpha(s_1), \alpha(s_1^*)] \pm \beta(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Theorem 2.7. *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_1^*) \pm \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. First, we consider that

$$\alpha(s_1) \circ \alpha(s_1^*) + \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Replacing s_1 by s_1^* in the last expression we get

$$\alpha(s_1) \circ \alpha(s_1^*) - \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Subtracting the last two relation and using $\text{char}(\chi) \neq 2$ we obtain

$$\beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Hence, the result follow from [13, Theorem 1].

Similarly, we prove the other case with the help of [13, Theorem 1]. $\square$

Corollary 2.10. *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_2) \pm \beta([s_1, s_2]) \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Then χ is commutative.*

Corollary 2.11 ([3, Theorem 3.3]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_1^*) \pm \beta([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

3. SOME EXAMPLES

The first example shows that the restriction of the second kind involution in our theorems is not superfluous.

Example 3.1. Let $\chi = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \mid \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\}$. Of course, χ with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings $*$, α , $\beta : \chi \rightarrow \chi$ such that

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}, \quad \alpha \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}$$

and $\beta \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & \beta_2 \\ -\beta_3 & 0 \end{pmatrix}$. Obviously, $\mathcal{Z}(\chi) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \mid \beta_1 \in \mathbb{Z} \right\}$. Then $s_1^* = s_1$ for all $s_1 \in \mathcal{Z}(\chi)$, and hence $\mathcal{Z}(\chi) \subseteq \mathcal{H}(\chi)$, which shows that the involution

'*' is of the first kind. Moreover, α and β are nonzero derivations of χ and satisfying the identities of the theorems. However, χ is not commutative. Hence, the hypothesis of the second kind involution is crucial in our theorems.

The next example shows that our theorems are not true for semiprime rings.

Example 3.2. Let $S = \chi \times \mathbb{C}$, where χ is same as in Example 3.1 with involution '*' and derivations α and β same as in Example 3.1, $\mathbb{C}$ is the ring of complex numbers with conjugate involution τ . We can easily observe that S is a non commutative semiprime ring with characteristic different from two. Now define an involution α on S , as $(s_1, s_2)^\alpha = (s_1^*, s_2^\tau)$. Clearly, α is an involution of the second kind. Further, we define the mappings α and β from S to S such that $D_1(s_1, s_2) = (\alpha(s_1), 0)$ and $D_2(s_1, s_2) = (\beta(s_1), 0)$ for all $(s_1, s_2) \in S$. It can be easily checked that D_1 and D_2 are derivations on S and satisfying the identities of the Theorem 2.5 and Theorem 2.6 but S is not commutative. Hence, in our theorems, the hypothesis of primeness is essential.

Conclusions. In this paper we have studied some identities involving derivations on prime rings with involution. Purely algebraic methods have been used to describe the structure of rings and we provide the examples, which shows that the assumptions are not superfluous. Applications point of view some well known results are deduced.

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ŁUKASIEWICZ ANTI FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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ABSTRACT. The subalgebra of BCK/BCI-algebra using Łukasiewicz anti fuzzy set introduced by Jun is studied in this article. The concept of Łukasiewicz anti fuzzy subalgebra of a BCK/BCI-algebra is introduced, and several properties are investigated. The relationship between anti fuzzy subalgebra and Łukasiewicz anti fuzzy subalgebra is given, and characterization of a Łukasiewicz anti fuzzy subalgebra is discussed. Conditions are found in which a Łukasiewicz anti fuzzy set is a Łukasiewicz anti fuzzy subalgebra. Finally, conditions under which $\leq$ -subset, Υ -subset, and anti-subset become subalgebra are explored.

1. INTRODUCTION

In [1], Biswas introduced the concept of anti fuzzy subgroups of groups. Modifying Biswas' idea, Hong and Jun [3] applied the idea to BCK-algebras. They introduced the notions of anti fuzzy subalgebras and anti fuzzy ideals of BCK-algebras and investigated several properties. Using anti fuzzy notion and the idea of Łukasiewicz t -conorm, Jun [7] constructed the concept of Łukasiewicz anti fuzzy sets and applied it to BE-algebras. He introduced the notion of Łukasiewicz anti fuzzy BE-ideal and investigated its properties. He discussed the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal and provided conditions for Łukasiewicz anti fuzzy set to be Łukasiewicz anti fuzzy BE-ideal. He also gives three types of subsets so called $\leq$ -subset, Υ -subset, and anti subset, and then he considered the conditions under which they can be BE-ideals.

Key words and phrases. Anti fuzzy subalgebra, Łukasiewicz anti fuzzy set, Łukasiewicz anti fuzzy subalgebra, $\leq$ -subset, Υ -subset, anti subset.

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We would like to study the subalgebra of BCK/BCI-algebra using Łukasiewicz anti fuzzy set introduced by Jun. We introduce Łukasiewicz anti fuzzy subalgebra of a BCK/BCI-algebra and investigate several properties. We give the relationship between anti fuzzy subalgebra and Łukasiewicz anti fuzzy subalgebra. We discuss a characterization of a Łukasiewicz anti fuzzy subalgebra. We find conditions for a Łukasiewicz anti fuzzy set to be a Łukasiewicz anti fuzzy subalgebra. We finally find the condition that $\leq$ -subset, Υ -subset, and anti subset become subalgebra.

2. PRELIMINARIES

This section lists the known default content that will be used later.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5] and [6]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [4, 8] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element “0” and a binary operation “ $*$ ” satisfying the conditions:

$$(I_1) (\forall a, b, c \in X) (((a * b) * (a * c)) * (c * b) = 0);$$

$$(I_2) (\forall a, b \in X) ((a * (a * b)) * b = 0);$$

$$(I_3) (\forall a \in X) (a * a = 0);$$

$$(I_4) (\forall a, b \in X) (a * b = 0, b * a = 0 \Rightarrow a = b),$$

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

$$(K) (\forall a \in X) (0 * a = 0),$$

then X is called a *BCK-algebra*.

The order relation “ $\leq$ ” in a BCK/BCI-algebra X is defined as follows:

$$(2.1) \quad (\forall a, b \in X)(a \leq b \Leftrightarrow a * b = 0).$$

Every BCK/BCI-algebra X satisfies the following conditions (see [4, 8]):

$$(2.2) \quad (\forall a \in X) (a * 0 = a),$$

$$(2.3) \quad (\forall a, b, c \in X) (a \leq b \Rightarrow a * c \leq b * c, c * b \leq c * a),$$

$$(2.4) \quad (\forall a, b, c \in X) ((a * b) * c = (a * c) * b).$$

Every BCI-algebra X satisfies (see [4]):

$$(2.5) \quad (\forall a, b \in X) (a * (a * (a * b)) = a * b),$$

$$(2.6) \quad (\forall a, b \in X) (0 * (a * b) = (0 * a) * (0 * b)).$$

A subset K of a BCK/BCI-algebra X is called a *subalgebra* of X (see [4, 8]) if it satisfies:

$$(2.7) \quad (\forall a, b \in K)(a * b \in K).$$

A fuzzy set g in a set X of the form

$$(2.8) \quad g(b) := \begin{cases} s \in [0, 1), & \text{if } b = a, \\ 1, & \text{if } b \neq a, \end{cases}$$

is called an *anti fuzzy point* with support a and value s , and is denoted by $\frac{a}{s}$. A fuzzy set g in a set X is said to be *non-unit* if there exists $a \in X$ such that $g(a) \neq 1$.

For a fuzzy set g in a set X , we say that an anti fuzzy point $\frac{a}{s}$ is said to

- (i) *beside* in g , denoted by $\frac{a}{s} \triangleleft g$ (see [2]) if $g(a) \leq s$;
- (ii) *be non-quasi coincident* with g , denoted by $\frac{a}{s} \Upsilon g$ (see [2]) if $g(a) + s < 1$.

If $\frac{a}{s} \triangleleft g$ or $\frac{a}{s} \Upsilon g$ (resp., $\frac{a}{s} \triangleleft g$ and $\frac{a}{s} \Upsilon g$), we say that $\frac{a}{s} \triangleleft \vee \Upsilon g$ (resp., $\frac{a}{s} \triangleleft \wedge \Upsilon g$). Given $\beta \in \{\triangleleft, \Upsilon\}$, to indicate $\frac{a}{s} \overline{\beta} g$ means that $\frac{a}{s} \beta g$ is not established.

A fuzzy set f in a BCK/BCI-algebra X is called

- an *anti fuzzy subalgebra* of X (see [3]) if it satisfies:

$$(2.9) \quad (\forall a, b \in X)(f(a * b) \leq \max\{f(a), f(b)\});$$

- an *anti fuzzy ideal* of X (see [3]) if it satisfies:

$$(2.10) \quad (\forall a \in X)(f(0) \leq f(a)),$$

$$(2.11) \quad (\forall a, b \in X)(f(a) \leq \max\{f(a * b), f(b)\}).$$

Let ε be an element of the unit interval $[0, 1]$ and let g be a fuzzy set in a set X . A function $\mathbb{L}_g^\varepsilon : X \rightarrow [0, 1]$, $x \mapsto \min\{1, g(x) + \varepsilon\}$, is called a *Łukasiewicz anti fuzzy set* of g in X (see [7]).

Let $\mathbb{L}_g^\varepsilon$ be a Łukasiewicz anti fuzzy set of a fuzzy set g in X . If $\varepsilon = 0$, then $\mathbb{L}_g^\varepsilon(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x)\} = g(x)$ for all $x \in X$. This shows that if $\varepsilon = 0$, then the Łukasiewicz anti fuzzy set of a fuzzy set g in X is the classical fuzzy set g itself in X . If $\varepsilon = 1$, then $\mathbb{L}_g^\varepsilon(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x) + 1\} = 1$ for all $x \in X$, that is, if $\varepsilon = 1$, then the Łukasiewicz anti fuzzy set is the constant function with value 1. Therefore, in handling the Łukasiewicz anti fuzzy set, the value of ε can always be considered to be in $(0, 1)$.

Let g be a fuzzy set in a set X and $\varepsilon \in (0, 1)$. If $g(x) + \varepsilon \geq 1$ for all $x \in X$, then the Łukasiewicz anti fuzzy set $\mathbb{L}_g^\varepsilon$ of g in X is the constant function with value 1, that is, $\mathbb{L}_g^\varepsilon(x) = 1$ for all $x \in X$. Therefore, for the Łukasiewicz anti fuzzy set to have a meaningful shape, a fuzzy set g in X and $\varepsilon \in (0, 1)$ shall be set to satisfy the condition “ $g(x) + \varepsilon < 1$ for some $x \in X$ ”.

Given a Łukasiewicz anti fuzzy set $\mathbb{L}_g^\varepsilon$ of a fuzzy set g in X and $s \in [0, 1)$, consider the sets:

$$(\mathbb{L}_g^\varepsilon, s)_{\triangleleft} := \{y \in X \mid \frac{y}{s} \triangleleft \mathbb{L}_g^\varepsilon\} \quad \text{and} \quad (\mathbb{L}_g^\varepsilon, s)_{\Upsilon} := \{y \in X \mid \frac{y}{s} \Upsilon \mathbb{L}_g^\varepsilon\},$$

which are called the $\triangleleft$ -subset and Υ -subset of $\mathbb{L}_g^\varepsilon$ in X . Also, we consider the following set

$$\text{Anti}(\mathbb{L}_g^\varepsilon) := \{y \in X \mid \mathbb{L}_g^\varepsilon(y) < 1\}$$

and it is called the *anti subset* of $\mathbb{L}_g^\varepsilon$ in X . It is observed that

$$\text{Anti}(\mathbb{L}_g^\varepsilon) = \{y \in X \mid g(y) + \varepsilon < 1\}.$$

3. ŁUKASIEWICZ ANTI FUZZY SUBALGEBRAS

In this section, let f and γ be a fuzzy set in X and an element of $(0, 1)$, respectively, unless otherwise specified.

Definition 3.1. A Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ in a BCK/BCI-algebra X is called a *Łukasiewicz anti fuzzy subalgebra* of X if it satisfies

$$(3.1) \quad (\forall x, y \in X)(\forall s_a, s_b \in [0, 1)) \left(\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma, \frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma \Rightarrow \frac{x*y}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma \right).$$

Example 3.1. Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation “ $*$ ” given by the Cayley table:

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	b_1	0	b_1	0	0
b_2	b_2	b_2	0	0	0
b_3	b_3	b_3	b_3	0	0
b_4	b_4	b_3	b_4	b_1	0

Then X is a BCK-algebra (see [8]). Define a fuzzy set f in X as follows:

$$f : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.24, & \text{if } x = 0, \\ 0.31, & \text{if } x = b_1, \\ 0.37, & \text{if } x = b_2, \\ 0.43, & \text{if } x = b_3, \\ 0.58, & \text{if } x = b_4. \end{cases}$$

Given $\gamma := 0.58$, the Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ of f in X is given as follows:

$$\mathbb{L}_f^\gamma : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.82, & \text{if } x = 0, \\ 0.89, & \text{if } x = b_1, \\ 0.95, & \text{if } x = b_2, \\ 1.00, & \text{if } x = b_3, \\ 1.00, & \text{if } x = b_4. \end{cases}$$

It is routine to verify that $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X .

Theorem 3.1. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X , then it's Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ in X is a Łukasiewicz anti fuzzy subalgebra of X .*

Proof. Assume that f is an anti fuzzy subalgebra of a BCK/BCI-algebra X . Let $x, y \in X$ and $s_a, s_b \in [0, 1)$ be such that $\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma$ and $\frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma$. Then, $\mathbb{L}_f^\gamma(x) \leq s_a$ and

$\mathbb{L}_f^\gamma(y) \leq s_b$. Hence,

$$\begin{aligned}\mathbb{L}_f^\gamma(x * y) &= \min\{1, f(x * z) + \gamma\} \leq \min\{1, \max\{f(x), f(y)\} + \gamma\} \\ &= \min\{1, \max\{f(x) + \gamma, f(y) + \gamma\}\} \\ &= \max\{\min\{1, f(x) + \gamma\}, \min\{1, f(y) + \gamma\}\} \\ &= \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \leq \max\{s_a, s_b\},\end{aligned}$$

which implies that $\frac{x*y}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma$. Therefore, $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X . $\square$

The following example shows that the converse of Theorem 3.1 may not be true.

Example 3.2. Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation “ $*$ ”

$*$	0	b_1	b_2	b_3	b_4
0	0	0	b_2	b_3	b_4
b_1	b_1	0	b_2	b_3	b_4
b_2	b_2	b_2	0	b_4	b_3
b_3	b_3	b_3	b_4	0	b_2
b_4	b_4	b_4	b_3	b_2	0

Then, X is a BCI-algebra (see [4]). Define a fuzzy set f in X as follows:

$$f : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.28, & \text{if } x = 0, \\ 0.32, & \text{if } x = b_1, \\ 0.39, & \text{if } x = b_2, \\ 0.43, & \text{if } x = b_3, \\ 0.61, & \text{if } x = b_4. \end{cases}$$

Given $\gamma := 0.58$, the γ -Łukasiewicz fuzzy set $\mathbb{L}_f^\gamma$ of f in X is given as follows:

$$\mathbb{L}_f^\gamma : X \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.86, & \text{if } x = 0, \\ 0.90, & \text{if } x = b_1, \\ 0.97, & \text{if } x = b_2, \\ 1.00, & \text{if } x = b_3, \\ 1.00, & \text{if } x = b_4. \end{cases}$$

It is routine to verify that $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X . But f is not an anti fuzzy subalgebra of X because of

$$f(b_2 * b_3) = f(b_4) = 0.61 \not\leq 0.43 = \max\{f(b_2), f(b_3)\}.$$

We explore a characterization of a Łukasiewicz anti fuzzy subalgebra.

Theorem 3.2. *Let f be a fuzzy set in a BCK/BCI-algebra X . Then its Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ in X is a Łukasiewicz anti fuzzy subalgebra of X if and only if it satisfies*

$$(3.2) \quad (\forall x, y \in X)(\mathbb{L}_f^\gamma(x * y) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\}).$$

Proof. Suppose that $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X . Let $x, y \in X$. Since $\frac{x}{\mathbb{L}_f^\gamma(x)} \leq \mathbb{L}_f^\gamma$ and $\frac{y}{\mathbb{L}_f^\gamma(y)} \leq \mathbb{L}_f^\gamma$, it follows from (3.1) that $\frac{x*y}{\max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\}} \leq \mathbb{L}_f^\gamma$. Hence, $\mathbb{L}_f^\gamma(x * y) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\}$.

Conversely, assume that $\mathbb{L}_f^\gamma$ satisfies (3.2). Let $x, y \in X$ and $s_a, s_b \in [0, 1)$ be such that $\frac{x}{s_a} \leq \mathbb{L}_f^\gamma$ and $\frac{y}{s_b} \leq \mathbb{L}_f^\gamma$. Then $\mathbb{L}_f^\gamma(x) \leq s_a$ and $\mathbb{L}_f^\gamma(y) \leq s_b$, and so

$$\mathbb{L}_f^\gamma(x * y) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \leq \max\{s_a, s_b\}.$$

Thus, $\frac{x*y}{\max\{s_a, s_b\}} \leq \mathbb{L}_f^\gamma$, and therefore, $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X . $\square$

Lemma 3.1 ([7]). *If f is a fuzzy set in a set X , then it's Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ satisfies*

$$(3.3) \quad (\forall x, y \in X)(f(x) \geq f(y) \Rightarrow \mathbb{L}_f^\gamma(x) \geq \mathbb{L}_f^\gamma(y)).$$

Lemma 3.2. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X , then it's Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ satisfies*

$$(3.4) \quad (\forall x \in X)(\mathbb{L}_f^\gamma(0) \leq \mathbb{L}_f^\gamma(x)).$$

Proof. If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X , then

$$f(0) = f(x * x) \leq \max\{f(x), f(x)\} = f(x),$$

for all $x \in X$. It follows from (3.3) that $\mathbb{L}_f^\gamma(0) \leq \mathbb{L}_f^\gamma(x)$ for all $x \in X$. $\square$

Proposition 3.1. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X , then it's Łukasiewicz fuzzy set $\mathbb{L}_f^\gamma$ satisfies:*

$$(3.5) \quad (\forall x, y \in X) (\mathbb{L}_f^\gamma(x) = \mathbb{L}_f^\gamma(0) \Leftrightarrow \mathbb{L}_f^\gamma(x * y) \leq \mathbb{L}_f^\gamma(y)).$$

Proof. Let f be an anti fuzzy subalgebra of a BCK/BCI-algebra X . Then $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X (see Theorem 3.1). Assume that $\mathbb{L}_f^\gamma(x) = \mathbb{L}_f^\gamma(0)$ for all $x \in X$. Then,

$$\mathbb{L}_f^\gamma(x * y) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} = \max\{\mathbb{L}_f^\gamma(0), \mathbb{L}_f^\gamma(y)\} = \mathbb{L}_f^\gamma(y),$$

for all $x, y \in X$, by Theorem 3.2 and Lemma 3.2.

Conversely, suppose that $\mathbb{L}_f^\gamma(x * y) \leq \mathbb{L}_f^\gamma(y)$ for all $x, y \in X$. Using (2.2) induces $\mathbb{L}_f^\gamma(x) = \mathbb{L}_f^\gamma(x * 0) \leq \mathbb{L}_f^\gamma(0)$, and so $\mathbb{L}_f^\gamma(x) = \mathbb{L}_f^\gamma(0)$ for all $x \in X$, by Lemma 3.2. $\square$

Proposition 3.2. *If f is an anti fuzzy subalgebra of a BCI-algebra X , then its Łukasiewicz fuzzy set $\mathbb{L}_f^\gamma$ satisfies*

$$(3.6) \quad (\forall x \in X)(\mathbb{L}_f^\gamma(0 * x) \leq \mathbb{L}_f^\gamma(x)).$$

Proof. If f is an anti fuzzy subalgebra of a BCI-algebra X , then

$$f(0 * x) \leq \max\{f(0), f(x)\} = f(x),$$

for all $x \in X$. Hence, $\mathbb{L}_f^\gamma(0 * x) \leq \mathbb{L}_f^\gamma(x)$ for all $x \in X$, by Lemma 3.1. $\square$

Proposition 3.3. *If f is an anti fuzzy subalgebra of a BCI-algebra X , then its Łukasiewicz fuzzy set $\mathbb{L}_f^\gamma$ satisfies*

$$(3.7) \quad (\forall x, y \in X)(\forall s_a, s_b \in [0, 1)) \left(\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma, \frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma \Rightarrow \frac{x*(0*y)}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma \right).$$

Proof. Let $x, y \in X$ and $s_a, s_b \in [0, 1)$ be such that $\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma$ and $\frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma$. Then, $\mathbb{L}_f^\gamma(x) \leq s_a$ and $\mathbb{L}_f^\gamma(y) \leq s_b$, and thus,

$$\begin{aligned} \mathbb{L}_f^\gamma(x * (0 * y)) &= \min\{1, f(x * (0 * y)) + \gamma\} \\ &\leq \min\{1, \max\{f(x), f(0 * y)\} + \gamma\} \\ &\leq \min\{1, \max\{f(x), \max\{f(0), f(y)\}\} + \gamma\} \\ &= \min\{1, \max\{f(x), f(y)\} + \gamma\} \\ &= \min\{1, \max\{f(x) + \gamma, f(y) + \gamma\}\} \\ &= \max\{\min\{1, f(x) + \gamma\}, \min\{1, f(y) + \gamma\}\} \\ &= \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \\ &\leq \max\{s_a, s_b\}. \end{aligned}$$

Hence, $\frac{x*(0*y)}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma$. □

We give conditions for a Łukasiewicz anti fuzzy set to be a Łukasiewicz anti fuzzy subalgebra.

Theorem 3.3. *Let f be a fuzzy set in a BCK/BCI-algebra X . If it's Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ satisfies*

$$(3.8) \quad (\forall x, y \in X)(\forall s_b, s_c \in [0, 1)) \left(z \leq x, \frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma, \frac{z}{s_c} \triangleleft \mathbb{L}_f^\gamma \Rightarrow \frac{x*y}{\max\{s_b, s_c\}} \triangleleft \mathbb{L}_f^\gamma \right),$$

then $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X .

Proof. It is straightforward by (I_3) and (3.8). □

Proposition 3.4. *Let f be a fuzzy set in a BCI-algebra X . Then every Łukasiewicz fuzzy subalgebra $\mathbb{L}_f^\gamma$ of X satisfies*

$$(3.9) \quad (\forall x, y \in X)(\forall s_a, s_b \in [0, 1)) \left(\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma, \frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma \Rightarrow \frac{x*(0*y)}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma \right).$$

Proof. Let $x, y \in X$ and $s_a, s_b \in [0, 1)$ be such that $\frac{x}{s_a} \triangleleft \mathbb{L}_f^\gamma$ and $\frac{y}{s_b} \triangleleft \mathbb{L}_f^\gamma$. Then $\mathbb{L}_f^\gamma(x) \leq s_a$ and $\mathbb{L}_f^\gamma(y) \leq s_b$. Using Theorem 3.2 and Proposition 3.2, we have

$$\mathbb{L}_f^\gamma(x * (0 * y)) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(0 * y)\} \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \leq \max\{s_a, s_b\},$$

and so, $\frac{x*(0*y)}{\max\{s_a, s_b\}} \triangleleft \mathbb{L}_f^\gamma$. □

Corollary 3.1. *If f is an anti fuzzy subalgebra of a BCI-algebra X , then its Łukasiewicz fuzzy set $\mathbb{L}_f^\gamma$ satisfies the condition (3.9).*

Theorem 3.4. Let $\mathbb{L}_f^\gamma$ be a Łukasiewicz anti fuzzy set of a fuzzy set f in a BCK/BCI-algebra X . Then the $\leq$ -set $(\mathbb{L}_f^\gamma, s)_\leq$ of $\mathbb{L}_f^\gamma$ is a subalgebra of X for all $s \in [0, 0.5)$ if and only if the following assertion is valid

$$(3.10) \quad (\forall x, y \in X) \left(\min\{\mathbb{L}_f^\gamma(x * y), 0.5\} \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \right).$$

Proof. Assume that the $\leq$ -set $(\mathbb{L}_f^\gamma, s)_\leq$ of $\mathbb{L}_f^\gamma$ is a subalgebra of X for all $s \in [0, 0.5)$. If the condition (3.10) does not hold, then

$$\max\{\mathbb{L}_f^\gamma(a), \mathbb{L}_f^\gamma(b)\} < \min\{\mathbb{L}_f^\gamma(a * b), 0.5\},$$

for some $a, b \in X$. If we take $s := \max\{\mathbb{L}_f^\gamma(a), \mathbb{L}_f^\gamma(b)\}$, then $s \in [0, 0.5)$, $\frac{a}{s} \leq \mathbb{L}_f^\gamma$ and $\frac{b}{s} \leq \mathbb{L}_f^\gamma$, i.e., $a, b \in (\mathbb{L}_f^\gamma, s)_\leq$. Since $(\mathbb{L}_f^\gamma, s)_\leq$ is a subalgebra of X , we have $a * b \in (\mathbb{L}_f^\gamma, s)_\leq$. But $\frac{a * b}{s} \leq \mathbb{L}_f^\gamma$ implies $a * b \notin (\mathbb{L}_f^\gamma, s)_\leq$, a contradiction. Hence,

$$\max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \geq \min\{\mathbb{L}_f^\gamma(x * y), 0.5\},$$

for all $x, y \in X$.

Conversely, suppose that $\mathbb{L}_f^\gamma$ satisfies (3.10). Let $s \in [0, 0.5)$ and $x, y \in X$ be such that $x \in (\mathbb{L}_f^\gamma, s)_\leq$ and $y \in (\mathbb{L}_f^\gamma, s)_\leq$. Then $\mathbb{L}_f^\gamma(x) \leq s$ and $\mathbb{L}_f^\gamma(y) \leq s$, which imply from (3.10) that

$$0.5 > s \geq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \geq \min\{\mathbb{L}_f^\gamma(x * y), 0.5\}.$$

Hence, $\frac{x * y}{s} \leq \mathbb{L}_f^\gamma$, i.e., $x * y \in (\mathbb{L}_f^\gamma, s)_\leq$. Therefore, $(\mathbb{L}_f^\gamma, s)_\leq$ is a subalgebra of X for $s \in [0, 0.5)$. $\square$

Theorem 3.5. Let $\mathbb{L}_f^\gamma$ be a Łukasiewicz fuzzy set of a fuzzy set f in a BCK/BCI-algebra X . If f is an anti fuzzy subalgebra of X , then the Υ -set $(\mathbb{L}_f^\gamma, s)_\Upsilon$ of $\mathbb{L}_f^\gamma$ is a subalgebra of X for all $s \in [0, 1)$.

Proof. Let $s \in [0, 1)$ and $x, y \in (\mathbb{L}_f^\gamma, s)_\Upsilon$. Then $\frac{x}{s} \Upsilon \mathbb{L}_f^\gamma$ and $\frac{y}{s} \Upsilon \mathbb{L}_f^\gamma$, that is, $\mathbb{L}_f^\gamma(x) + s < 1$ and $\mathbb{L}_f^\gamma(y) + s < 1$. Hence,

$$\mathbb{L}_f^\gamma(x * y) + s \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} + s = \max\{\mathbb{L}_f^\gamma(x) + s, \mathbb{L}_f^\gamma(y) + s\} < 1,$$

by Theorems 3.1 and 3.2. Thus, $\frac{x * y}{s} \Upsilon \mathbb{L}_f^\gamma$, and so, $x * y \in (\mathbb{L}_f^\gamma, s)_\Upsilon$. Therefore, $(\mathbb{L}_f^\gamma, s)_\Upsilon$ is a subalgebra of X . $\square$

Theorem 3.6. Let f be a fuzzy set in a BCK/BCI-algebra X . For a Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ of f in X , if the Υ -set $(\mathbb{L}_f^\gamma, s)_\Upsilon$ is a subalgebra of X , then $\mathbb{L}_f^\gamma$ satisfies

$$(3.11) \quad (\forall x, y \in X)(\forall s_a, s_b \in (0.5, 1]) \left(\frac{x}{s_a} \Upsilon \mathbb{L}_f^\gamma, \frac{y}{s_b} \Upsilon \mathbb{L}_f^\gamma \Rightarrow \frac{x * y}{\min\{s_a, s_b\}} \leq \mathbb{L}_f^\gamma \right).$$

Proof. Let $x, y \in X$ and $s_a, s_b \in (0.5, 1]$ be such that $\frac{x}{s_a} \Upsilon \mathbb{L}_f^\gamma$ and $\frac{y}{s_b} \Upsilon \mathbb{L}_f^\gamma$. Then $x \in (\mathbb{L}_f^\gamma, s_a)_\Upsilon \subseteq (\mathbb{L}_f^\gamma, \min\{s_a, s_b\})_\Upsilon$ and $y \in (\mathbb{L}_f^\gamma, s_b)_\Upsilon \subseteq (\mathbb{L}_f^\gamma, \min\{s_a, s_b\})_\Upsilon$. Hence, $x * y \in (\mathbb{L}_f^\gamma, \min\{s_a, s_b\})_\Upsilon$, and so,

$$\mathbb{L}_f^\gamma(x * y) < 1 - \min\{s_a, s_b\} \leq \min\{s_a, s_b\},$$

since $\min\{s_a, s_b\} > 0.5$. Therefore, $\frac{x * y}{\min\{s_a, s_b\}} \leq \mathbb{L}_f^\gamma$. $\square$

Theorem 3.7. *Let $\mathbb{L}_f^\gamma$ be a Łukasiewicz fuzzy set of a fuzzy set f in a BCK/BCI-algebra X . If f is an anti fuzzy subalgebra of X , then the anti subset $\text{Anti}(\mathbb{L}_f^\gamma)$ of $\mathbb{L}_f^\gamma$ is a subalgebra of X .*

Proof. Let $x, y \in \text{Anti}(\mathbb{L}_f^\gamma)$. Then $f(x) + \gamma < 1$ and $f(y) + \gamma < 1$. If f is an anti fuzzy subalgebra of X , then $\mathbb{L}_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X (see Theorem 3.1). It follows from Theorem 3.2 that

$$\mathbb{L}_f^\gamma(x * y) \leq \max\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} = \max\{f(x) + \gamma, f(y) + \gamma\} < 1.$$

Hence, $x * y \in \text{Anti}(\mathbb{L}_f^\gamma)$, and therefore, $\text{Anti}(\mathbb{L}_f^\gamma)$ is a subalgebra of X . $\square$

Theorem 3.8. *Let f be a fuzzy set in a BCK/BCI-algebra X . If a Łukasiewicz anti fuzzy set $\mathbb{L}_f^\gamma$ of f in X satisfies*

$$(3.12) \quad (\forall x, y \in X)(\forall s_a, s_b \in [0, 1)) \left(\frac{x}{s_a} \leq \mathbb{L}_f^\gamma, \frac{y}{s_b} \leq \mathbb{L}_f^\gamma \Rightarrow \frac{x*y}{\min\{s_a, s_b\}} \Upsilon \mathbb{L}_f^\gamma \right),$$

then the anti subset $\text{Anti}(\mathbb{L}_f^\gamma)$ of $\mathbb{L}_f^\gamma$ is a subalgebra of X .

Proof. Assume that $\mathbb{L}_f^\gamma$ satisfies the condition (3.12) for all $x, y \in X$ and $s_a, s_b \in [0, 1)$. Let $x, y \in \text{Anti}(\mathbb{L}_f^\gamma)$. Then $f(x) + \gamma < 1$ and $f(y) + \gamma < 1$. Since $\frac{x}{\mathbb{L}_f^\gamma(x)} \leq \mathbb{L}_f^\gamma$ and $\frac{y}{\mathbb{L}_f^\gamma(y)} \leq \mathbb{L}_f^\gamma$, it follows from (3.12) that

$$(3.13) \quad \frac{x*y}{\min\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\}} \Upsilon \mathbb{L}_f^\gamma.$$

If $x * y \notin \text{Anti}(\mathbb{L}_f^\gamma)$, then $\mathbb{L}_f^\gamma(x * y) = 1$, and so,

$$\begin{aligned} \mathbb{L}_f^\gamma(x * y) + \min\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} &= 1 + \min\{\mathbb{L}_f^\gamma(x), \mathbb{L}_f^\gamma(y)\} \\ &= 1 + \min\{\min\{1, f(x) + \gamma\}, \min\{1, f(y) + \gamma\}\} \\ &= 1 + \min\{f(x) + \gamma, f(y) + \gamma\} \\ &= 1 + \min\{f(x), f(y)\} + \gamma \\ &\geq 1 + \gamma > 1, \end{aligned}$$

which shows that (3.13) is not valid. This is a contradiction, and thus, $x * y \in \text{Anti}(\mathbb{L}_f^\gamma)$. Hence, $\text{Anti}(\mathbb{L}_f^\gamma)$ is a subalgebra of X . $\square$

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IDEAL RELATIVE UNIFORM CONVERGENCE OF DOUBLE SEQUENCE OF POSITIVE LINEAR FUNCTIONS

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ABSTRACT. In this article, we look into the concept of ideal relative uniform convergence of a double sequence of functions. In addition, we define ideal relative uniform Cauchy and ideal regular relative uniform convergence of a double sequence of positive linear functions defined on a compact domain D with respect to the scale function $\sigma(x)$ defined on D . We also introduced several classes of ideal relative uniform convergent double sequences of functions and investigated their algebraic and topological properties.

1. INTRODUCTION

Kostyrko et al. [21] introduced the concept of $\mathcal{I}$ -convergence of sequences of real numbers, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of natural numbers. $\mathcal{I}$ -convergence is a generalisation and unification of many notions of ordinary convergence. Fast [17] and Steinhaus [29] independently introduced the concept of statistical convergence in 1951 as a generalisation of the concept of ordinary convergence. Furthermore, in 1959, Schoenberg [28] independently investigated some basic properties of statistical convergence. Later, it was studied from a sequence space perspective and linked with summability theory by Fridy [18], Gökhan et al. [19], Tripathy and Sarma [31], and many others. The concept is based on the notion of natural density of $\mathbb{N}$ subsets.

A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k),$$

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exists where χ_E is the characteristics function of E .

A subset E of $\mathbb{N}$ is said to have logarithmic density $d(E)$ if

$$d(E) = \lim_{n \rightarrow +\infty} \frac{1}{s_n} \sum_{k=1}^n \frac{\chi_E(k)}{k},$$

exists, where $s_n = \sum_{k=1}^n \frac{1}{k} = \log n + \gamma + O(\frac{1}{n})$, where γ is the Euler's constant.

The above expression is equivalent to

$$d(E) = \lim_{n \rightarrow +\infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\chi_E(k)}{k}.$$

A double sequence is defined as a double infinite array of numbers (x_{nk}) . Pringsheim [25] introduced the concept of double sequence. Bromwich [2] contains some earlier work on double sequence spaces. Hardy [20] introduced the concept of regular convergence of a double sequence. Basarır and Sonalcan [1, 2], Das et al. [4, 5], Datta and Tripathy [5, 6], and many others have studied the double sequence from various perspectives.

The notion of statistical convergence for double sequences was introduced by Móricz [22], Mursaleen and Edely [24], Tripathy [30] independently. The notion depends on the idea of density of subsets of $\mathbb{N} \times \mathbb{N}$. A subset E of $\mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if

$$\rho(E) = \lim_{p, q \rightarrow +\infty} \frac{1}{pq} \sum_{n=1}^p \sum_{k=1}^q \chi_E(n, k)$$

exists.

Tripathy and Tripathy [39] introduced the notion of logarithmic density for subsets of $\mathbb{N} \times \mathbb{N}$.

A subset $E \subset \mathbb{N} \times \mathbb{N}$ is said to have logarithmic density $\rho^*(E)$ if

$$\rho^*(E) = \lim_{p, q \rightarrow +\infty} \frac{1}{s_p s_q} \sum_{n=1}^p \sum_{k=1}^q \frac{\chi_E(n, k)}{nk}$$

exists.

The above expression is equivalent to the following:

$$\rho^*(E) = \lim_{p, q \rightarrow +\infty} \frac{1}{\log p \log q} \sum_{n=1}^p \sum_{k=1}^q \frac{\chi_E(n, k)}{nk}.$$

A family of sets $\mathcal{I} \subseteq 2^X$, where 2^X is the class of all subsets of non-empty set X , is said to be ideal if and only if $\emptyset \in \mathcal{I}$, for each $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$, and for each $A \in \mathcal{I}$ and each $B \subseteq A$, we have $B \in \mathcal{I}$. If and only if $A \cap B \in \mathcal{F}$ and $B \in \mathcal{F}$ for each instance of $A \in \mathcal{F}$ and $B \supset A$, $\emptyset \notin \mathcal{F}$, $\mathcal{F} \subseteq 2^X$ is said to be a filter on X . If $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$, then an ideal $\mathcal{I}$ is referred to as a non-trivial ideal. If and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = X - A$, then it is evident that $\mathcal{I} \subseteq 2^X$ is a non-trivial ideal: $A \in \mathcal{I}$ is a filter on X . A non-trivial ideal $\mathcal{I} \subseteq 2^X$ is said to be admissible if and only if $\mathcal{I} \subseteq \{\{x\} : x \in X\}$.

Remark 1.1. If we consider subsets A of $\mathbb{N}$ with $\delta(A) = 0, d(A) = 0$ then, these classes of subsets of $\mathbb{N}$ will form an ideal of $\mathbb{N}$. The convergence of sequences will be called as statistical and logarithmic convergence. Similarly, on considering subsets A of $\mathbb{N} \times \mathbb{N}$ with $\rho(A) = 0$ and $\rho^*(A) = 0$, we will get the ideals of $\mathbb{N} \times \mathbb{N}$. The corresponding convergence of sequences are known as Pringsheim's sense statistical and logarithmic convergence of double sequences. Accordingly, the regular convergence can be defined.

For a detail account of $\mathcal{I}$ -convergent sequence, one may refer to [11–16, 27, 32–38].

Moore [23] established the idea of uniform convergence of sequence of functions with respect to a scale function. Chittenden [3] provided the following formulation of Moore's definition.

Definition 1.1. If there are functions g and $\sigma(x)$, defined on D , and for every $\varepsilon > 0$, there is an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$, the inequality

$$|g(x) - f_n(x)| < \varepsilon |\sigma(x)|,$$

holds for every element x of D , then the sequence (f_n) of real, single-valued functions f_n of a real variable x converges relatively uniformly on D . Scale function is the name given to the function $\sigma(x)$. When compared to the scale function, the sequence (f_n) is said to converge relatively uniformly.

The notion was further studied by [7–10, 26] and many others. For the first time, Yıldız [40] introduced the convergence known as ideal relative uniform convergence for double sequences of functions.

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper ${}_2\ell_\infty(ru)$, ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$ denote the classes of relatively uniformly bounded, $\mathcal{I}_2$ -relatively uniformly null, $\mathcal{I}_2$ -relatively uniformly convergent, $\mathcal{I}_2$ -regularly relatively uniformly convergent, $\mathcal{I}_2$ -regularly relatively uniformly null of double sequences of positive linear functions, respectively.

Definition 2.1. A sequence space E is referred to as *solid or normal* if $(x_{nk}) \in E$ implies $(\alpha_{nk}x_{nk}) \in E$, for any (α_{nk}) with $|\alpha_{nk}| \leq 1$, for all $n, k \in \mathbb{N}$.

Definition 2.2. If a sequence space E contains the canonical pre-images of all its step spaces, it is said to be monotone.

Remark 2.1. If a sequence space E is solid, then E is monotone.

Definition 2.3. A sequence space E is said to be *symmetric* if for any $n, k \in \mathbb{N} \times \mathbb{N}$, $(x_{nk}) \in E$ implies $(x_{\pi(n,k)}) \in E$, where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Definition 2.4. For all $n, k \in \mathbb{N}$, a sequence space E is said to be *convergence free* if $(x_{nk}) \in E$ and $x_{nk} = 0$ implies $y_{nk} = 0$ together with $(y_{nk}) \in E$.

Definition 2.5. For all $n, k \in \mathbb{N}$, a sequence space E is said to be a *sequence algebra* if $(x_{nk} \circ y_{nk}) \in E$ whenever (x_{nk}) and (y_{nk}) belongs to E .

Definition 2.6 ([40]). In the class of all subsets of $\mathbb{N} \times \mathbb{N}$, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$. If there are functions $g(x)$ and $\sigma(x)$ defined on D such that for every $\varepsilon > 0$ and for all $x \in D$, then the sequence of functions $(f_{nk}(x))$ of single, real-valued functions $\mathbb{R}$ is said to be $\mathcal{I}_2$ -relatively uniformly convergent on D satisfying the following condition.

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - g(x)| \geq \varepsilon |\sigma(x)|\} \in \mathcal{I}_2.$$

This can also be expressed as for every $\varepsilon > 0$, there exists $M \in \mathcal{I}_2$ such that for any $(n, k) \notin M$,

$$|f_{nk}(x) - f(x)| < \varepsilon |\sigma(x)|, \text{ for all } x \in D.$$

Remark 2.2. We obtain the definition of $\mathcal{I}_2$ -relatively uniformly null of double sequence of positive linear functions if $g = \theta$, the zero function in the previous definition.

Definition 2.7. In the class of all subsets of $\mathbb{N} \times \mathbb{N}$, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$. $\mathcal{I}_2$ -relatively uniformly Cauchy refers to a set of functions $(f_{nk}(x))$ defined on a compact domain D if $s = s(\varepsilon)$, $t = t(\varepsilon)$ and function $\sigma(x)$ are defined on D such that for every $\varepsilon > 0$ and for any $x \in D$

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - f_{st}(x)| \geq \varepsilon |\sigma(x)|\} \in \mathcal{I}_2.$$

Definition 2.8. Considering the class of all subsets of $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, respectively, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$ and $\mathcal{I}$ be an ideal of $2^{\mathbb{N}}$. If there are functions $g(x), f_k(x), f_n(x), \sigma(x), \xi_n(x), \eta_k(x)$ defined on D such that for every $\varepsilon > 0$ and for any $x \in D$, then the sequence of single, real-valued functions $(f_{nk}(x))$ is said to be $\mathcal{I}_2$ -regularly relatively uniformly convergent on D satisfying the following conditions:

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - g(x)| \geq \varepsilon |\sigma(x)|\} \in \mathcal{I}_2, \text{ for any } n, k \in \mathbb{N},$$

$$\{k \in \mathbb{N} : |f_{nk}(x) - f_n(x)| \geq \varepsilon |\xi_n(x)|\} \in \mathcal{I}, \text{ for every } n \in \mathbb{N},$$

$$\{n \in \mathbb{N} : |f_{nk}(x) - f_k(x)| \geq \varepsilon |\eta_k(x)|\} \in \mathcal{I}, \text{ for every } k \in \mathbb{N}.$$

Remark 2.3. We obtain the definition of $\mathcal{I}_2$ -regularly relatively uniformly null of double sequence of positive linear functions if $g = f_k = f_n = \theta$, the zero function in the previous definition.

Remark 2.4. $\mathcal{I}_2 = \mathcal{I}_2(P) \subset 2^{\mathbb{N} \times \mathbb{N}}$ is the class of all subsets of $\mathbb{N} \times \mathbb{N}$ containing terms of sequence of functions $(f_{nk}(x))$ upto n_0 finite term for all n and k w.r.t. the scale function $\sigma(x)$. Then, $\mathcal{I}_2(P)$ is an ideal of $2^{\mathbb{N} \times \mathbb{N}}$ and it corresponds to the double sequence of functions' relative uniform convergence with respect to $\sigma(x)$ on D .

On considering $\mathcal{I}_2(P)$ along with $\mathcal{I}_f$, it corresponds to the double sequence of functions' regular relative uniform convergence with respect to the scale function $\sigma(x)$ on D .

Remark 2.5. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho) \subset 2^{\mathbb{N} \times \mathbb{N}}$, the class of all subsets of $\mathbb{N} \times \mathbb{N}$ of zero natural density w.r.t. the scale function $\sigma(x)$, then, $\mathcal{I}_2(\rho)$ is an ideal of $2^{\mathbb{N} \times \mathbb{N}}$ and $\mathcal{I}_2(\rho)$ corresponds to the statistical relative uniform convergence of double sequence of functions w.r.t. $\sigma(x)$ on D .

On considering $\mathcal{I}_2(\rho)$ along with $\mathcal{I}_\delta$, it corresponds to the statistical regularly relatively uniformly convergent double sequence of functions w.r.t. the scale function $\sigma(x)$ on D .

Remark 2.6. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho^*) \subset 2^{\mathbb{N} \times \mathbb{N}}$, the class of all subsets of $\mathbb{N} \times \mathbb{N}$ of zero logarithmic density w.r.t. the scale function $\sigma(x)$, then, $\mathcal{I}_2(\rho^*)$ is an ideal of $2^{\mathbb{N} \times \mathbb{N}}$ and $\mathcal{I}_2(\rho^*)$ corresponds to the logarithmic relative uniform convergence of double sequence of functions w.r.t. $\sigma(x)$ on D .

On considering $\mathcal{I}_2(\rho^*)$ along with $\mathcal{I}_d$, it corresponds to the logarithmic regularly relatively uniformly convergent double sequence of functions w.r.t. the scale function $\sigma(x)$ on D .

Definition 2.9. Let $(f_{nk}(x))$ and $(g_{nk}(x))$ be two double sequences of real, single-valued functions defined on compact subset D and $\mathcal{I}_2$ be an ideal on $2^{\mathbb{N} \times \mathbb{N}}$. Then, we say that $f_{nk}(x) = g_{nk}(x)$ for almost all n and k relative to $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$ (in short a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$) if for all $x \in D$,

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \neq g_{nk}(x)\} \in \mathcal{I}_2.$$

Definition 2.10. Let $(f_{nk}(x))$ be a sequence of real, single-valued functions defined on compact subset D and $\mathcal{I}_2$ be an ideal on $2^{\mathbb{N} \times \mathbb{N}}$. A subset M of D , is said to contain $f_{nk}(x)$ for a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$ if for all $x \in D$,

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin M\} \in \mathcal{I}_2.$$

We introduce the following sequence spaces:

$${}_2c_0(\mathcal{I}_2, ru) \cap {}_2\ell_\infty(ru) = {}_2c_0^m(\mathcal{I}_2, ru), \quad {}_2c(\mathcal{I}_2, ru) \cap {}_2\ell_\infty(ru) = {}_2c^m(\mathcal{I}_2, ru),$$

$${}_2c_0^R(\mathcal{I}_2, ru) \cap {}_2\ell_\infty(ru) = {}_2c_0^{mR}(\mathcal{I}_2, ru), \quad {}_2c^R(\mathcal{I}_2, ru) \cap {}_2\ell_\infty(ru) = {}_2c^{mR}(\mathcal{I}_2, ru).$$

The double sequence $f = (f_{nk})$ with elements chosen from the space of all real-valued functions on compact domain D is considered. Let $\|f\|_\sigma$ denote the usual sup-norm of f in D with respect to the scale function $\sigma(x)$, which is defined as follows.

$$(2.1) \quad \|f\|_\sigma = \|(f_{nk})\|_\sigma = \sup_{n, k \in \mathbb{N}} \sup_{x \in D} \frac{|f_{nk}(x)|}{|\sigma(x)|}.$$

3. MAIN RESULTS

Theorem 3.1. Let $\mathcal{I}_2$ represent a $2^{\mathbb{N} \times \mathbb{N}}$ ideal. Then, on a compact domain D , a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent if and only if it is $\mathcal{I}_2$ -relatively uniformly Cauchy.

Proof. Consider a compact domain D and a double sequence of functions $(f_{nk}(x))$. In terms of the scale function $\sigma(x)$ defined on D , $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $f(x)$ on D .

Then, for every $\varepsilon > 0$ and for all $x \in D$, there exists $M \in \mathcal{I}_2$ such that

$$(3.1) \quad |f_{nk}(x) - f(x)| \leq \frac{\varepsilon}{2} |\sigma(x)|, \quad \text{for all } (n, k) \notin M.$$

Similarly,

$$(3.2) \quad |f_{st}(x) - f(x)| \leq \frac{\varepsilon}{2} |\sigma(x)|, \quad \text{for all } (s, t) \notin M.$$

Let $n, k, s, t \geq n_0 = n_0(\varepsilon)$. For every $\varepsilon > 0$ and for all $x \in D$, there exists $M \in \mathcal{I}_2$ such that for all $(n, k) \notin M$ and $(s, t) \notin M$, using (3.1) and (3.2) we have

$$\begin{aligned} |f_{nk}(x) - f_{st}(x)| &\leq |f_{nk}(x) - f(x)| + |f_{st}(x) - f(x)| \\ &\leq \frac{\varepsilon}{2} |\sigma(x)| + \frac{\varepsilon}{2} |\sigma(x)| \\ &\leq \varepsilon |\sigma(x)|. \end{aligned}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly Cauchy w.r.t. scale function $\sigma(x)$.

Conversely, let $(f_{nk}(x))$ be $\mathcal{I}_2$ -relatively uniformly Cauchy on D . Then, there exist G, H such that the interval $U = [f_{GH}(x) - 1, f_{GH}(x) + 1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Next, choose G_1, H_1 such that $U' = [f_{G_1, H_1}(x) - 1, f_{G_1, H_1}(x) + 1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let, $U_1 = U \cap U'$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Evidently,

$$\begin{aligned} \{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U \cap U'\} &= \{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U\} \\ &\quad + \{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U'\}. \end{aligned}$$

This implies, $\{(n, k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U \cap U'\} \in \mathcal{I}_2$, for all $x \in D$. Then, for all $x \in D$, U_1 is a closed interval of D with length less than or equal to one that contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$. Next, choose G_2, H_2 such that $U'' = [f_{G_2, H_2}(x) - 1, f_{G_2, H_2}(x) + 1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let $U_2 = U_1 \cap U''$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$, for all $x \in D$. Then, we get, U_2 is a closed interval of D of length less than or equal to $\frac{1}{2}$ that contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Continuing inductively, we get a nested sequence (U_m) of closed intervals of D such that for all $m \in \mathbb{N}$, $U_m \supseteq U_{m+1}$, the length of $U_m \geq 2^{1-m}$, and $(f_{nk}(x)) \in U_m$, a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$. Thus, $\bigcap_{m=1}^{+\infty} U_m$ will contain a function $f(x)$, w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let $\varepsilon > 0$ be given and there exists n_0 such that $\varepsilon > 2^{1-n_0}$. Then, $(f_{nk}(x)) \in U_m$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$. We have

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - f(x)| \geq \varepsilon\} \leq \{f_{nk}(x) \notin U_m\} \in \mathcal{I}_2,$$

for all $x \in D$. Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $f(x)$ w.r.t. the scale function $\sigma(x)$ on D . $\square$

We state the following result without proof, since it can be established using standard technique.

Theorem 3.2. *Let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$. The classes of double sequences of functions ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ are linear spaces.*

Theorem 3.3. *Let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N} \times \mathbb{N}}$. The classes of double sequences of functions ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ are normed linear spaces with respect to the norm defined by (2.1).*

Proof. Let α, β be the scalars and $(f_{nk}(x)), (g_{nk}(x)) \in {}_2c_0^m(\mathcal{I}_2, ru)$. Then, there exist positive real numbers K_1 and K_2 such that

$$\sup_{n,k \in \mathbb{N}} |f_{nk}(x)| < K_1 |\sigma_1(x)| \quad \text{and} \quad \sup_{n,k \in \mathbb{N}} |g_{nk}(x)| < K_2 |\sigma_2(x)|.$$

Hence,

$$\begin{aligned} \sup_{n,k \in \mathbb{N}} |\alpha f_{nk}(x) + \beta g_{nk}(x)| &\leq |\alpha| \sup_{n,k \in \mathbb{N}} |f_{nk}(x)| + |\beta| \sup_{n,k \in \mathbb{N}} |g_{nk}(x)| \\ &\leq |\alpha| K_1 |\sigma_1(x)| + |\beta| K_2 |\sigma_2(x)|. \end{aligned}$$

Without loss of generality we can consider the same scale function, $\sigma(x) = \max\{|\sigma_1(x)|, |\sigma_2(x)|\}$, and we get

$$\sup_{n,k \in \mathbb{N}} |\alpha f_{nk}(x) + \beta g_{nk}(x)| \leq \{|\alpha| K_1 + |\beta| K_2\} \sigma(x).$$

Hence, the space ${}_2c_0^m(\mathcal{I}_2, ru)$ is a linear space. Similarly, we can establish for the rest of the spaces. Now, to verify that the linear space ${}_2c_0^m(\mathcal{I}_2, ru)$ satisfy the norm given in (2.1), the following three conditions must hold true.

Let $(f_{nk}(x)), (g_{nk}(x)) \in {}_2c_0^m(\mathcal{I}_2, ru)$.

- (i) One can easily verify that $\|f\|_\sigma = 0 \Leftrightarrow f(x) = 0$, for all $x \in D$.
- (ii)

$$\begin{aligned} \|(f + g)\|_\sigma &= \sup_{n,k \in \mathbb{N}} \sup_{x \in D} \frac{|f_{nk}(x) + g_{nk}(x)|}{|\sigma(x)|} \\ &\leq \sup_{n,k \in \mathbb{N}} \sup_{x \in D} \frac{|f_{nk}(x)|}{|\sigma(x)|} + \sup_{n,k \in \mathbb{N}} \sup_{x \in D} \frac{|g_{nk}(x)|}{|\sigma(x)|} \\ &\leq \|f\|_\sigma + \|g\|_\sigma. \end{aligned}$$

(iii)

$$\begin{aligned}
\|\lambda f\|_\sigma &= \sup_{n,k \in \mathbb{N}} \sup_{x \in D} \frac{|\lambda f_{nk}(x)|}{|\sigma(x)|} \\
&\leq |\lambda| \sup_{n,k \in \mathbb{N}} \sup_{x \in D} \frac{|f_{nk}(x)|}{|\sigma(x)|} \\
&\leq |\lambda| \|f\|_\sigma.
\end{aligned}$$

Similarly, we can establish for the rest of the sequence spaces. $\square$

Theorem 3.4. *The classes of double sequences of functions ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ are Banach spaces.*

Proof. Let $(f^i(x))$ be a relative uniform Cauchy sequence in ${}_2c^m(\mathcal{I}_2, ru) \subset {}_2\ell_\infty(ru)$, where $f^i(x) = (f_{nk}^i(x))$. Then, $(f^i(x))$ converges relatively uniformly in ${}_2\ell_\infty(ru)$. There exists

$$\lim_{i \rightarrow +\infty} f_{nk}^i(x) = f_{nk}(x), \quad \text{for all } x \in D \text{ and } n, k \in \mathbb{N}.$$

Let $\mathcal{I}_2 - \lim f_{nk}^i(x) = g_i(x)$, for all $x \in D$ and $i \in \mathbb{N}$. Since, $(f^i(x))$ is relatively uniformly Cauchy, for every $\varepsilon > 0$ and for all $x \in D$, there exists $n_0 \in \mathbb{N}$ such that

$$(3.3) \quad |f_{nk}^i(x) - f_{nk}^j(x)| < \frac{\varepsilon}{3} |\sigma(x)|, \quad \text{for all } i, j \geq n_0.$$

Since, $(f_{nk}^i(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $g_i(x)$, there exists $L \in \mathcal{I}_2$ such that for each $(n, k) \notin L$ and for all $x \in D$, we have

$$(3.4) \quad |f_{nk}^i(x) - g_i(x)| \leq \frac{\varepsilon}{3} |\sigma(x)|, \quad \text{for all } i, j \geq n_0.$$

Similarly, $(f_{nk}^j(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $g_j(x)$, there exists $M \in \mathcal{I}_2$ such that for each $(n, k) \notin M$ and for all $x \in D$, we have

$$(3.5) \quad |f_{nk}^j(x) - g_j(x)| \leq \frac{\varepsilon}{3} |\sigma(x)|.$$

Using equations (3.3), (3.4), (3.5), for all $x \in D$, we have

$$\begin{aligned}
|g_i(x) - g_j(x)| &= |f_{nk}^i(x) - g_i(x)| + |f_{nk}^j(x) - g_j(x)| + |f_{nk}^i(x) - f_{nk}^j(x)| \\
&\leq \varepsilon |\sigma(x)|.
\end{aligned}$$

Thus, $(g_i(x))$ is relatively uniformly Cauchy. Then, there exists $\lim_{i \rightarrow +\infty} g_i(x) = g(x)$ (say). We can write, for every $\eta > 0$ and for all $x \in D$, there exists m_0 such that

$$(3.6) \quad |g_i(x) - g(x)| < \frac{\eta}{3} |\sigma(x)|, \quad \text{for all } i \geq m_0.$$

Since, $(f_{nk}^i(x))$ is relatively uniformly Cauchy, for every $\eta > 0$ and for all $x \in D$, there exists m_0 such that

$$(3.7) \quad |f_{nk}^i(x) - f_{nk}(x)| < \frac{\eta}{3} |\sigma(x)|, \quad \text{for all } i \geq m_0.$$

Since, $(f_{nk}^i(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $g_i(x)$, there exists $Q \in \mathcal{I}_2$ such that for all $(n, k) \notin Q$ and for all $x \in D$ we get

$$(3.8) \quad |f_{nk}^i(x) - g_i(x)| < \frac{\eta}{3} |\sigma(x)|.$$

Without loss of generality, for all $(n, k) \notin Q$ and $x \in D$, using equations (3.6), (3.7), (3.8), we get

$$\begin{aligned} |f_{nk}(x) - g(x)| &\leq |f_{nk}(x) - f_{nk}^i(x)| + |f_{nk}^i(x) - g_i(x)| + |g_i(x) - g(x)| \\ &< \eta |\sigma(x)|. \end{aligned}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent to $g(x)$ w.r.t. the scale function $\sigma(x)$. Thus, ${}_2c^m(\mathcal{I}_2, ru)$ is a Banach space.

Similarly, we can prove for the other classes of sequences of functions. □

In view of Theorem 3.4, we state the following theorem without proof.

Theorem 3.5. *The classes of double sequences of functions ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are nowhere dense subsets of ${}_2\ell_\infty(ru)$.*

Theorem 3.6. (a) *The classes of double sequences of functions ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ are solid and hence, are monotone.*

(b) *The classes of double sequences of functions ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are not monotone and hence, are not solid.*

Proof. The proof of the first part follows from the following inclusion relation.

Consider the class of sequences of functions ${}_2c_0(\mathcal{I}_2, ru)$.

Let $(f_{nk}(x)) \in {}_2c_0(\mathcal{I}_2, ru)$ and (α_{nk}) be a sequence of scalars such that

$$|\alpha_{nk}| \leq 1, \quad \text{for all } n, k \in \mathbb{N}.$$

Let $\varepsilon > 0$ be given. Then, for all $x \in D$, we have

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x)| \geq \varepsilon |\sigma(x)|\} \supseteq \{(n, k) \in \mathbb{N} \times \mathbb{N} : |\alpha_{nk} f_{nk}(x)| \geq \varepsilon |\sigma(x)|\}.$$

Hence, $(\alpha_{nk} f_{nk}(x)) \in {}_2c_0(\mathcal{I}_2, ru)$. This implies, ${}_2c_0(\mathcal{I}_2, ru)$ is solid and hence, monotone.

Similarly, we can establish for the rest of the cases. □

The proof of the second part follows from the example below.

Example 3.1. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho^*)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n, k \text{ are prime, } n, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

We get, $(f_{nk}(x))$ is logarithmically relatively uniformly convergent on $[0, 1]$ w.r.t. the scale function $\sigma(x) = 1$. Hence, $(f_{nk}(x)) \in {}_2c(\mathcal{I}_2, ru)$.

Let $(g_{nk}(x))$ be the pre-image of the sequence of functions $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} x, & \text{for } n \text{ is odd, } n, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

One cannot get a scale function for which $(g_{nk}(x))$ is logarithmically relatively uniformly convergent on $[0, 1]$. This implies, $(g_{nk}(x)) \notin {}_2c(\mathcal{I}_2, ru)$. Hence, ${}_2c(\mathcal{I}_2, ru)$ is not monotone and therefore, not solid.

Similarly, we can prove for the other cases.

Result 3.1. The sequence spaces ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are not symmetric.

The result follows from the example below.

Example 3.2. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \rightarrow \mathbb{R}$, defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n = i^2, \text{ for all } i \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

This implies, $(f_{nk}(x)) \in {}_2c(\mathcal{I}_2, ru)$.

Let $(g_{nk}(x))$ be the rearranged sequence of functions of $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} x, & \text{for } n + k \text{ even, } n, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

One cannot get a scale function for which $(g_{nk}(x))$ is statistically relatively uniformly convergent on $[0, 1]$. This implies, $(g_{nk}(x)) \notin {}_2c(\mathcal{I}_2, ru)$. Hence, ${}_2c(\mathcal{I}_2, ru)$ is not symmetric.

Similarly, we can establish for the rest of the classes of double sequences of functions.

Result 3.2. The sequence spaces ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are not convergence free.

The result follows from the example below.

Example 3.3. Let $\mathcal{I}_2 = \mathcal{I}_2(P)$. Consider the double sequences of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_{nk}(x) = \frac{nkx}{1 + n^2k^2x^2}, \quad \text{for each } n, k \in \mathbb{N}.$$

We get, $(f_{nk}(x))$ is relatively uniformly null on $[0, 1]$ w.r.t. the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x \leq 1, \\ 1, & \text{for } x = 0. \end{cases}$$

Hence, $(f_{nk}(x)) \in {}_2c_0(\mathcal{I}_2, ru)$.

Let us consider another class of sequences $(g_{nk}(x))$ of functions $g_{nk} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$g_{nk}(x) = \frac{nk}{nk + x}, \quad \text{for each } n, k \in \mathbb{N}.$$

This implies, $(g_{nk}(x)) \notin {}_2c_0(\mathcal{I}_2, ru)$. Hence, ${}_2c_0(\mathcal{I}_2, ru)$ is not convergence free.

Similarly, we can show for the rest of the cases.

Theorem 3.7. *The sequence spaces ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are sequence algebra.*

Proof. Let the double sequence of functions $(f_{nk}(x))$ and $(g_{nk}(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ belong to the class of sequence of functions ${}_2c(\mathcal{I}_2, ru)$. Then, for every $\varepsilon > 0$, there exists $M \in \mathcal{I}_2$ such that for all $(n, k) \notin M$ and $x \in D$,

$$|f_{nk}(x) - f(x)| < \frac{\varepsilon}{2(|f(x)| + 1)} |\sigma(x)|, \quad \text{for all } n, k \geq n_1.$$

Similarly,

$$|g_{nk}(x) - g(x)| < \frac{\varepsilon}{2(|g(x)| + 1)} |\sigma(x)|, \quad \text{for all } n, k \geq n_2.$$

By applying reverse triangle inequality, there exists n_3 such that for all $n, k \geq n_3$, we have,

$$|f_{nk}(x)| - |f(x)| \leq ||f_{nk}(x)| - |f(x)|| \leq |f_{nk}(x) - f(x)| \leq 1.$$

This implies,

$$|f_{nk}(x)| < |f(x)| + 1, \quad \text{i.e., } \frac{|f_{nk}(x)|}{|f(x)| + 1} < 1.$$

For all $(n, k) \notin M$, there exists n_0 such that for all $n_0 > \max\{n_1, n_2, n_3\}$ and $x \in D$, we have

$$\begin{aligned} |f_{nk}(x)g_{nk}(x) - f(x)g(x)| &= |f_{nk}(x)g_{nk}(x) - f_{nk}(x)g(x) + f_{nk}(x)g(x) - f(x)g(x)| \\ &= |f_{nk}(x)(g_{nk}(x) - g(x)) + g(x)(f_{nk}(x) - f(x))| \\ &\leq |f_{nk}(x)| |g_{nk}(x) - g(x)| + |g(x)| |f_{nk}(x) - f(x)| \\ &\leq |f_{nk}(x)| \frac{\varepsilon}{2(|f(x)| + 1)} |\sigma(x)| + |g(x)| \frac{\varepsilon}{2(|g(x)| + 1)} |\sigma(x)| \\ &\leq \varepsilon |\sigma(x)|. \end{aligned}$$

Hence, $(f_{nk}(x)g_{nk}(x)) \in {}_2c(\mathcal{I}_2, ru)$.

Similarly, we can establish for the rest of the classes of double sequences of functions. $\square$

Result 3.3. On a compact domain D , if a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$ -uniformly convergent, it must also be $\mathcal{I}_2$ -relatively uniformly convergent on D but not vice versa.

The converse of the Result 3.3 is not necessarily true, which is shown in the following example.

Example 3.4. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} \frac{1}{nkx}, & \text{for } 0 < x \leq 1, n, k \in \mathbb{N}, \\ 0, & \text{for } x = 0. \end{cases}$$

We get, $(f_{nk}(x))$ is statistically relatively uniformly convergent w.r.t. the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x \leq 1, \\ 1, & \text{for } x = 0. \end{cases}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent on $[0, 1]$. One can easily see that $(f_{nk}(x))$ is not $\mathcal{I}_2$ -uniformly convergent on $[0, 1]$.

Result 3.4. On a compact domain D , if a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$ -regularly relatively uniformly convergent, it must also be $\mathcal{I}_2$ -relatively uniformly convergent on D but not vice versa.

The converse of the Result 3.4 is not necessarily true, which is shown in the following example.

Example 3.5. Let $\mathcal{I}_2 = \mathcal{I}_2(P)$. We consider the sequence of functions $(f_{nk}(x))$, $f_{nk} : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} -x, & \text{for } n = 1, k \text{ is even, } k = 1, n \text{ is even, } n, k \in \mathbb{N}, \\ x, & \text{otherwise.} \end{cases}$$

Then, $(f_{nk}(x))$ is relatively uniformly convergent on $[0, 1]$ w.r.t. the scale function $\sigma(x) = 1$. Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$ -relatively uniformly convergent on $[0, 1]$.

But the first row and first column of $(f_{nk}(x))$ is not relatively uniformly convergent and hence, $(f_{nk}(x))$ is not $\mathcal{I}_2$ -regularly relatively uniformly convergent.

4. CONCLUSIONS

In this article, we have studied ideal convergence of double sequence of functions from the point of view of relative uniform convergence w.r.t. the scale function $\sigma(x)$ defined on a compact subset $D \subseteq \mathbb{R}$. We introduced the classes of double sequences of functions ${}_2\mathcal{C}(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}_0(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}^R(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}_0^R(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}^m(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}_0^m(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}^{mR}(\mathcal{I}_2, ru)$, ${}_2\mathcal{C}_0^{mR}(\mathcal{I}_2, ru)$ and studied their properties like solid, monotone, symmetric, sequence algebra, convergence free and denseness. We also established the relationship between $\mathcal{I}_2$ -relative uniform convergent and $\mathcal{I}_2$ -relative uniform Cauchy as well as relationship between $\mathcal{I}_2$ -relative uniform convergent and $\mathcal{I}_2$ -regular relative uniform convergent.

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VAGUE WEAK INTERIOR IDEALS OF Γ -SEMIRINGS

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ABSTRACT. The notion of a ((complete-) normal) vague weak interior ideal on a (regular) Γ -semiring is defined. It is proved that the set of all vague weak interior ideals forms a complete lattice. Also, a characterization theorem for a regular Γ -semiring in terms of vague weak interior ideals is derived. Another interesting consequence of the main result is that the cardinal of a non-constant maximal element in the set of all (complete-) normal vague weak interior ideals is 2.

1. INTRODUCTION

In 1934, Vandiver [18] extended the notions of rings and distributive lattices and defined a new algebraic structure as semirings. It is known that semiring theory has many applications to many branches of pure and applied mathematics: functional analysis, combinatorics, graph theory, automata theory, coding and language theory. In 1981, Sen [17] introduced the notion of a Γ -semigroup as a generalization of semigroup. Then Rao [14, 15] generalized a semiring and Γ -ring by introducing Γ -semiring. Ideals play an important role in advance studies and uses of algebraic structures (see, [6, 10]). Hedayati and Shum [9] were considered the congruences and ideals of a Γ -semiring. In 1965, Zadeh [19] introduced the concept of a fuzzy set. Then Gau and Buehrer [8] introduced the concept of vague sets as a generalization of fuzzy sets. Moreover, Ramakrishna [12] studied vague cosets, vague products and several properties related to them. Jun and Park [11] defined the notion of a vague ideal in a subtraction algebra. Rao and Venkateswarlu [16] studied bi-interior ideals of Γ -semirings and get some of its properties. In 2008, Eswarlal [7] introduced the

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concepts of vague ideals and normal vague ideals in semirings. Bhargavi and Eswarlal [1–5] were developed the theory of vague sets on Γ -semirings. In 2019, Rao [13] introduced weak interior ideals and fuzzy weak interior ideals of Γ -semirings. The motivation of this paper, is define the notion of a *((complete)- normal) vague weak interior ideal* of a Γ -semiring. We prove that there is an isomorphism between the set of all vague weak interior ideals and its crisp weak interior ideals. We prove that the set of all vague weak interior ideals forms a complete lattice. Further, we give a characterization theorem for *regular Γ -semiring* in terms of vague weak interior ideals, and a condition to every vague weak interior ideal could be a normal vague weak interior ideal is given.

2. PRELIMINARIES

We recall the basic definitions needed for this paper.

Definition 2.1 ([8]). (a) A vague set ψ is a pair (t_ψ, f_ψ) , where $t_\psi, f_\psi : E \rightarrow [0, 1]$ are mappings s.t. $t_\psi(x) + f_\psi(x) \leq 1$ for all $x \in E$.

(b) The interval $[t_\psi(x), 1 - f_\psi(x)]$ is called the vague value of x in ψ and it is denoted by $V_\psi(x)$, i.e., $V_\psi(x) = [t_\psi(x), 1 - f_\psi(x)]$.

(c) Let $D \subseteq E$, the vague characteristic set of D in $[0, 1]$ is a vague set $\delta_D = (t_{\delta_D}, f_{\delta_D})$ as follows:

$$V_{\delta_D}(f) = \begin{cases} [1, 1], & \text{if } x \in D, \\ [0, 0], & \text{if } x \notin D. \end{cases}$$

i.e.,

$$t_{\delta_D}(x) = \begin{cases} 1, & \text{if } x \in D, \\ 0, & \text{if } x \notin D, \end{cases} \quad \text{and} \quad f_{\delta_D}(x) = \begin{cases} 0, & \text{if } x \in D, \\ 1, & \text{if } x \notin D. \end{cases}$$

(d) Let $\psi = (t_\psi, f_\psi)$ be a vague set. For $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$, the (α, β) -cut or vague cut of ψ is the crisp subset of E is given by:

$$\psi_{(\alpha, \beta)} = \{x \in E : V_\psi(x) \geq [\alpha, \beta]\},$$

i.e.,

$$\psi_{(\alpha, \beta)} = \{x \in E : t_\psi(x) \geq \alpha \text{ and } 1 - f_\psi(x) \geq \beta\}.$$

Denote by $\mathbf{VS}(E)$ the set of all vague sets of E .

Definition 2.2 ([8]). Let $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VS}(E)$. Then, for all $x \in E$:

(a) $\psi^c = (t_{\psi^c}, f_{\psi^c})$, where $t_{\psi^c} = f_\psi, f_{\psi^c} = t_\psi$;

(b) $\psi \subseteq \phi$ if and only if $\psi(x) \leq \phi(x)$;

(c) $\psi \cup \phi := (t_{\psi \cup \phi}, f_{\psi \cup \phi})$, where $t_{\psi \cup \phi}(x) = \max\{t_\psi(x), t_\phi(x)\}$ and $f_{\psi \cup \phi}(x) = \min\{f_\psi(x), f_\phi(x)\}$;

(d) $\psi \cap \phi := (t_{\psi \cap \phi}, f_{\psi \cap \phi})$, where $t_{\psi \cap \phi}(x) = \min\{t_\psi(x), t_\phi(x)\}$ and $f_{\psi \cap \phi}(x) = \max\{f_\psi(x), f_\phi(x)\}$.

Definition 2.3 ([14]). Let $(E, +)$ and $(\Gamma, +)$ be two abelian semigroups. Then E is called a Γ -semiring if there exists a mapping $E \times \Gamma \times E \rightarrow E$ (briefly, images of (a, α, b) will be denoted by $a\alpha b$) satisfying the following axioms:

- (Γ SR₁) $c\alpha(a + b) = c\alpha a + c\alpha b$;
- (Γ SR₂) $(c + a)\alpha b = c\alpha b + a\alpha b$;
- (Γ SR₃) $c(\alpha + \beta)a = c\alpha a + c\beta a$;
- (Γ SR₄) $c\alpha(a\beta b) = (c\alpha a)\beta b$, for all $a, b, c \in E$, $\alpha, \beta \in \Gamma$.

In this paper, E is a Γ -semiring.

Definition 2.4 ([13]). (a) E is called regular if for all $e \in E$, exists $f \in E$, $\alpha, \beta \in \Gamma$ s.t. $e = e\alpha f\beta e$.

(b) A sub- Γ -semiring F of E is called a right (resp. left) weak interior ideal of E if $F\Gamma F\Gamma E \subseteq F$ (resp. $E\Gamma F\Gamma F \subseteq F$). If F is both *right* and *left* weak interior ideal of E , then F is called a *weak interior ideal* of E .

Denote by **RWII**(E) (resp. **LWII**(E)) the set of all right (resp. left) weak interior ideals and **WII**(E) the set of all weak interior ideals of E . One can see that **WII**(E) = **RWII**(E) $\cap$ **LWII**(E).

Definition 2.5 ([4]). Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called a vague Γ -semiring if it satisfies the following axioms:

- (V₁) $V_\psi(a + b) \geq \min\{V_\psi(a), V_\psi(b)\}$;
- (V₂) $V_\psi(a\gamma b) \geq \min\{V_\psi(a), V_\psi(b)\}$ for all $a, b \in E$, $\gamma \in \Gamma$.

Denote by **VT**(E) the set of all vague Γ -semirings of E .

Definition 2.6 ([5]). Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called a right (resp. left) vague ideal of E if it satisfies (V₁) and

- (V₃) $V_\psi(a\gamma b) \geq V_\psi(a)$ (resp. $V_\psi(a\gamma b) \geq V_\psi(b)$), for all $a, b \in E$, $\gamma \in \Gamma$.

If ψ is both *left* and *right* vague ideal of E , then ψ is called a *vague ideal* of E .

Denote by **RVI**(E) (resp. **LVI**(E)) the set of all right (resp. left) vague ideals and **VI**(E) the set of all vague ideals of E . Hence, **VI** = **RVI**(E) $\cap$ **LVI**(E).

Definition 2.7 ([2]). Let $\psi = (t_\psi, f_\psi)$, $\phi = (t_\phi, f_\phi) \in \mathbf{RVI}(E)$ (resp. $\in \mathbf{LVI}(E)$). Then the sum $\psi + \phi$ of ψ and ϕ are defined by:

$$V_{\psi+\phi}(e) = \begin{cases} \sup\{\min\{V_\psi(f), V_\phi(g)\} : e = f + g, \text{ where } f, g \in E\}, \\ [0, 0], \end{cases} \quad \text{otherwise,}$$

i.e.,

$$t_{\psi+\phi}(e) = \begin{cases} \sup\{\min\{t_\psi(f), t_\phi(g)\} : e = f + g, \text{ where } f, g \in E\}, \\ 0, \end{cases} \quad \text{otherwise,}$$

and

$$f_{\psi+\phi}(e) = \begin{cases} \inf\{\max\{f_\psi(f), f_\phi(g)\} : e = f + g, \text{ where } f, g \in E\}, \\ 1, \end{cases} \quad \text{otherwise.}$$

3. VAGUE WEAK INTERIOR IDEALS IN Γ -SEMRINGS

In this section, we define the concept of *vague weak interior ideal* of a Γ -semiring and obtain some of the basic properties. Finally, we give a characterization theorem for a *regular Γ -semiring* in terms of vague weak interior ideals.

From now on, δ stands for vague characteristic set of E unless stated.

Definition 3.1. Let $\psi \in \mathbf{V}\Gamma(E)$. Then ψ is called a *right* (resp. *left*) *vague weak interior ideal* of E if $\psi\Gamma\psi\Gamma\delta \subseteq \psi$ (resp. $\delta\Gamma\psi\Gamma\psi \subseteq \psi$).

If ψ is both right and left vague weak interior ideal, then it is called a *vague weak interior ideal* of E .

Denote by $\mathbf{RVWII}(E)$ (resp. $\mathbf{LVWII}(E)$) the set of all right (resp. left) vague weak interior ideals and $\mathbf{VWII}(E)$ the set of all vague weak interior ideals of E . Hence, $\mathbf{VWII}(E) = \mathbf{RVWII}(E) \cap \mathbf{LVWII}(E)$.

Example 3.1. (i) Let $E := \mathbb{N} \cup \{0\}$ and $\Gamma := \mathbb{N}$. Define the mapping $\cdot : \mathbb{N} \cup \{0\} \times \mathbb{N} \times \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ by $\cdot(a, b, c) = abc$ usual product of a, b, c , for all $a, c \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$. Hence, $\mathbb{N} \cup \{0\}$ is a $\mathbb{N}$ -semiring. Define $t_\psi, f_\psi : \mathbb{N} \cup \{0\} \rightarrow [0, 1]$ as follows:

$$t_\psi(x) = \begin{cases} 0.12, & \text{if } x \in 2\mathbb{N} \text{ or } x = 0, \\ 0.13, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(x) = \begin{cases} 0.18, & \text{if } x \in 2\mathbb{N} \text{ or } x = 0, \\ 0.16, & \text{otherwise.} \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \notin \mathbf{RVWII}(\mathbb{N} \cup \{0\}) \cup \mathbf{LVWII}(\mathbb{N} \cup \{0\})$.

(ii) Let $E = \Gamma := M_{2 \times 2}(\mathbb{N})$. Define the mapping $M_{2 \times 2}(\mathbb{N}) \times M_{2 \times 2}(\mathbb{N}) \times M_{2 \times 2}(\mathbb{N}) \rightarrow M_{2 \times 2}(\mathbb{N})$ by ABC is the matrix multiplication of A, B, C for all $A, B, C \in M_{2 \times 2}(\mathbb{N})$. Hence, $M_{2 \times 2}(\mathbb{N})$ is a $M_{2 \times 2}(\mathbb{N})$ -semiring. Define $t_\psi, f_\psi : M_{2 \times 2}(\mathbb{N}) \rightarrow [0, 1]$ by:

$$t_\psi(M) = \begin{cases} 0.6, & \text{if } M = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}, \text{ where } p, q \in \mathbb{N}; \\ 0.5, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(M) = \begin{cases} 0.1, & \text{if } M = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}, \text{ where } p, q \in \mathbb{N}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(M_{2 \times 2}(\mathbb{N}))$, but $\psi = (t_\psi, f_\psi) \notin \mathbf{LVWII}(M_{2 \times 2}(\mathbb{N}))$.

Now, if define $t_\phi, f_\phi : M_{2 \times 2}(\mathbb{N}) \rightarrow [0, 1]$ by:

$$t_\phi(N) = \begin{cases} 0.72, & \text{if } N = \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix}, \text{ where } m, n \in \mathbb{N}, \\ 0.54, & \text{otherwise,} \end{cases}$$

and

$$f_\phi(N) = \begin{cases} 0.28, & \text{if } N = \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix}, \text{ where } m, n \in \mathbb{N}, \\ 0.37, & \text{otherwise.} \end{cases}$$

Then $\phi = (t_\phi, f_\phi) \in \mathbf{LVWII}(M_{2 \times 2}(\mathbb{N}))$, but $\phi = (t_\phi, f_\phi) \notin \mathbf{RVWII}(M_{2 \times 2}(\mathbb{N}))$.

(iii) Let $E := \{-n : n \in \mathbb{N}\}$ and $\Gamma := \{-2n : n \in \mathbb{N}\}$. Define the mapping $E \times \Gamma \times E \rightarrow E$ by abc usual product of a, b, c for all $a, c \in E; b \in \Gamma$. Hence, E is a Γ -semiring. Define $t_\psi, f_\psi : E \rightarrow [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.53, & \text{if } x = -1, \\ 0.76, & \text{if } x = -2, \\ 0.99, & \text{if } x < -2, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.54, & \text{if } x = -1, \\ 0.28, & \text{if } x = -2, \\ 0.12, & \text{if } x < -2. \end{cases}$$

Therefore, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E) \cap \mathbf{LVWII}(E)$.

Remark 3.1. Consider Example 3.1 (iii), $\psi^c \notin \mathbf{VWII}(E)$.

Theorem 3.1. Let $\psi \in \mathbf{RVI}(E)$ (resp. $\in \mathbf{LVI}(E)$). Then $\psi \in \mathbf{RVWII}(E)$ (resp. $\in \mathbf{LVWII}(E)$).

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{RVI}(E)$. Then $\psi\Gamma\delta \subseteq \psi$. Clearly, $\psi \in \mathbf{VT}(E)$. Now, let $e \in E$. Then

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(e) &= \sup\{\min\{V_\psi(f), V_{\psi\Gamma\delta}(g)\} : e = f\gamma g, \text{ where } f, g \in E; \gamma \in \Gamma\} \\ &\leq \sup\{\min\{V_\psi(f), V_\psi(g)\} : f, g \in E\} \\ &\leq \sup\{V_\psi(e)\} \\ &\leq V_\psi(e). \end{aligned}$$

Thus, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. □

The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.2. Consider Example 3.1 (ii), and define $t_\psi, f_\psi : M_{2 \times 2}(\mathbb{N}) \rightarrow [0, 1]$ by:

$$t_\psi(P) = \begin{cases} 0.8, & \text{if } P = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \text{ where } e, f \in \mathbb{N}, \\ 0.6, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(P) = \begin{cases} 0.1, & \text{if } P = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \text{ where } e, f \in \mathbb{N}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Therefore, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(M_{2 \times 2}(\mathbb{N}))$, but not a right vague ideal of $M_{2 \times 2}(\mathbb{N})$, since $V_\psi(PZQ) < V_\psi(P)$, where $P, Q, Z \in M_{2 \times 2}(\mathbb{N})$.

Proposition 3.1. Let E be regular and $\psi \in \mathbf{VWII}(E)$. Then $\psi \in \mathbf{VI}(E)$.

Proof. Suppose E is regular and $\psi = (t_\psi, f_\psi) \in \mathbf{VWII}(E)$. Now, if $\psi \notin \mathbf{RVI}(E)$, then exists $e \in E$ s.t. $V_{\psi\Gamma\psi\Gamma\delta}(e) > V_\psi(e)$. Since E is regular, $\exists f \in E$ and $\alpha, \beta \in \Gamma$ s.t. $e = e\alpha f\beta e$. Hence,

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(e) &= \sup\{\min\{V_\psi(e\alpha f), V_{\psi\Gamma\delta}(e)\}\} \\ &> \sup\{\min\{V_\psi(e\alpha f), V_\psi(e)\}\} \\ &= V_\psi(e). \end{aligned}$$

This shows that $\psi\Gamma\psi\Gamma\delta \supset \psi$, which is a contraction. Thus, $\psi \in \mathbf{RVI}(E)$. Similarly, we can prove that $\psi \in \mathbf{LVI}(E)$. Therefore, $\psi \in \mathbf{VI}(E)$. $\square$

In the next theorem we show that there is an isomorphism between $\mathbf{RVWII}(E)$ (resp. $\mathbf{LVWII}(E)$) with the set of all vague cuts.

Theorem 3.2. *Let $\psi \in \mathbf{VS}(E)$. Then $\psi \in \mathbf{RVWII}(E)$ (resp. $\psi \in \mathbf{LVWII}(E)$) if and only if $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$ (resp. $\in \mathbf{LWII}(E)$) for all $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$.*

Proof. Suppose $\psi = (t_\psi, f_\psi) \in \mathbf{RVII}(E)$. Using [5, Theorem 3.6], $\psi_{(\alpha,\beta)}$ is a sub- Γ -semiring of E . Given $e \in \psi_{(\alpha,\beta)}\Gamma\psi_{(\alpha,\beta)}\Gamma E$, we get $e = f\gamma g\eta h$ s.t. $f, g \in \psi_{(\alpha,\beta)}$, $h \in E$. Hence, $V_\psi(f) \geq [\alpha, \beta]$ and $V_\psi(g) \geq [\alpha, \beta]$. Now, we have

$$\begin{aligned} V_\psi(e) &\geq V_{\psi\Gamma\psi\Gamma\delta}(e) \\ &= \sup\{\min\{V_\psi(f), V_\psi(g), V_\delta(h)\}\} \\ &\geq [\alpha, \beta]. \end{aligned}$$

Therefore, $e \in \psi_{(\alpha,\beta)}$. This shows that $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$.

Conversely, assume $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$. Using [5, Theorem 3.9], we get $\psi \in \mathbf{VT}(E)$. Now, if $\psi\Gamma\psi\Gamma\delta \not\subseteq \psi$, then exists $s \in E$ s.t. $V_\psi(s) < V_{\psi\Gamma\psi\Gamma\delta}(s)$. Let $[\alpha, \beta] \subseteq [0, 1]$ s.t. $V_\psi(s) < [\alpha, \beta] < V_{\psi\Gamma\psi\Gamma\delta}(s)$. Let $s := f\gamma g\eta h$ s.t. $f, g \notin \psi_{(\alpha,\beta)}$ for all $f, g \in E$, $\gamma, \eta \in \Gamma$. Then $V_\psi(f) < [\alpha, \beta]$, $V_\psi(g) < [\alpha, \beta]$. Now, we have

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(s) &= \sup\{\min\{V_\psi(f), V_\psi(g), V_\delta(h)\}\} \\ &= \sup\{\min\{V_\psi(f), V_\psi(g)\}\} \\ &< [\alpha, \beta]. \end{aligned}$$

This shows that $V_{\psi\Gamma\psi\Gamma\delta}(s) < [\alpha, \beta]$, which is a contraction. Therefore, $\psi \in \mathbf{RVWII}(E)$. Similarly, we can prove that $\psi \in \mathbf{LVWII}(E)$. $\square$

Corollary 3.1. *Let $\psi \in \mathbf{VS}(E)$. Then $\psi \in \mathbf{VWII}(E)$ if and only if $\psi_{(\alpha,\beta)} \in \mathbf{WII}(E)$ for all $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$.*

Theorem 3.3. *Let $\emptyset \neq W \subseteq E$ and δ_W be the vague characteristic set of W . Then $W \in \mathbf{RWII}(E)$ (resp. $\in \mathbf{LWII}(E)$) if and only if $\delta_W \in \mathbf{RVWII}(E)$ (resp. $\in \mathbf{LVWII}(E)$).*

Proof. Suppose $W \in \mathbf{RWII}(E)$. Then $W\Gamma W\Gamma E \subseteq W$. Using [5, Theorem 3.9], we get $\delta_W = (t_{\delta_W}, f_{\delta_W}) \in \mathbf{VT}(E)$. Hence, $\delta_W\Gamma\delta_W\Gamma\delta = \delta_{W\Gamma W\Gamma E} \subseteq \delta_W$. It follows that $\delta_W \in$

RVWII(E). Conversely, assume $\delta_W \in \mathbf{RVWII}(E)$. Then $\delta_W \Gamma \delta_W \Gamma \delta \subseteq \delta_W$. Using [5, Theorem 3.9], W is sub Γ -semiring of E . Thus, $\delta_{W\Gamma W\Gamma E} \subseteq \delta_W$, and so $W\Gamma W\Gamma E \subseteq W$. Therefore, $W \in \mathbf{RWII}(E)$. By a similar argument $W \in \mathbf{LWII}(E)$. $\square$

Corollary 3.2. *Let $\emptyset \neq W \subseteq E$ and δ_W be the vague characteristic set of W . Then $W \in \mathbf{WII}(E)$ if and only if $\delta_W \in \mathbf{VWII}(E)$.*

Theorem 3.4. *Let $\psi, \phi \in \mathbf{VWII}(E)$. Then*

- (i) $\psi \cap \phi \in \mathbf{VWII}(E)$;
- (ii) $\psi + \phi \in \mathbf{VWII}(E)$;
- (iii) $\psi \cap \phi \subseteq \psi, \phi$;
- (iv) $\psi, \phi \subseteq \psi + \phi$.

Proof. (i) Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{RVWII}(E)$. Using [5, Theorem 3.13], we get $\psi \cap \phi \in \mathbf{V}\Gamma(E)$. Given $e \in E$, we have

$$\begin{aligned} V_{(\psi \cap \phi)\Gamma \delta}(e) &= \sup\{\min\{V_{\psi \cap \phi}(f), V_\delta(g)\} : e = f\alpha g; f, g \in E; \alpha \in \Gamma\} \\ &= \sup\{\min\{\min\{V_\psi(f), V_\phi(f)\}, V_\delta(g)\}\} \\ &= \sup\{\min\{\min\{V_\psi(f), V_\delta(g)\}, \min\{V_\phi(f), V_\delta(g)\}\}\} \\ &= \min\{\sup\{\min\{V_\psi(f), V_\delta(g)\}\}, \sup\{\min\{V_\phi(f), V_\delta(g)\}\}\} \\ &= \min\{V_{\psi\Gamma \delta}(e), V_{\phi\Gamma \delta}(e)\} \\ &= V_{(\psi\Gamma \delta) \cap (\phi\Gamma \delta)}(e). \end{aligned}$$

This shows that $(\psi \cap \phi)\Gamma \delta = (\psi\Gamma \delta) \cap (\phi\Gamma \delta)$. Also, we have

$$\begin{aligned} V_{(\psi \cap \phi)\Gamma (\psi \cap \phi)\Gamma \delta}(e) &= \sup\{\min\{V_{\psi \cap \phi}(f), V_{(\psi \cap \phi)\Gamma \delta}(g)\} : e = f\alpha g; f, g \in E; \alpha \in \Gamma\} \\ &= \sup\{\min\{V_{\psi \cap \phi}(f), V_{(\psi\Gamma \delta) \cap (\phi\Gamma \delta)}(g)\}\} \\ &= \sup\{\min\{\min\{V_\psi(f), V_\phi(f)\}, \min\{V_{\psi\Gamma \delta}(g), V_{\phi\Gamma \delta}(g)\}\}\} \\ &= \sup\{\min\{\min\{V_\psi(f), V_{\psi\Gamma \delta}(g)\}, \min\{V_\phi(f), V_{\phi\Gamma \delta}(g)\}\}\} \\ &= \min\{\sup\{\min\{V_\psi(f), V_{\psi\Gamma \delta}(g)\}\}, \sup\{\min\{V_\phi(f), V_{\phi\Gamma \delta}(g)\}\}\} \\ &= \min\{V_{\psi\Gamma \psi\Gamma \delta}(e), V_{\phi\Gamma \phi\Gamma \delta}(e)\} \\ &= V_{(\psi\Gamma \psi\Gamma \delta) \cap (\phi\Gamma \phi\Gamma \delta)}(e). \end{aligned}$$

Therefore, $(\psi \cap \phi)\Gamma (\psi \cap \phi)\Gamma \delta = (\psi\Gamma \psi\Gamma \delta) \cap (\phi\Gamma \phi\Gamma \delta)$. It follows that

$$(\psi \cap \phi)\Gamma (\psi \cap \phi)\Gamma \delta = (\psi \cap \phi)\Gamma [(\psi\Gamma \delta) \cap (\phi\Gamma \delta)] = (\psi\Gamma \psi\Gamma \delta) \cap (\phi\Gamma \phi\Gamma \delta) \subseteq \psi \cap \phi.$$

Thus, $\psi \cap \phi \in \mathbf{RVWII}(E)$. Similarly, we can prove $\psi \cap \phi \in \mathbf{LWII}(E)$. Therefore, $\psi \cap \phi \in \mathbf{VWII}(E)$.

(ii) As similar to the proof of (i), $\psi + \phi \in \mathbf{VWII}(E)$.

(iii) Let $e \in E$. We have $V_{\psi \cap \phi}(e) = \min\{V_\psi(e), V_\phi(e)\} \leq V_\psi(e)$. Therefore, $\psi \cap \phi \subseteq \psi$. Similarly, $\psi \cap \phi \subseteq \phi$.

(iv) Given $x \in E$, we have

$$\begin{aligned} V_{\psi+\phi}(x) &= \sup\{\min\{V_\psi(a), V_\phi(b)\} : x = a + b, \text{ where } a, b \in E\} \\ &\geq \min\{V_\psi(x), V_\phi(0)\} \\ &= V_\psi(x). \end{aligned}$$

It follows that $\psi \subseteq \psi + \phi$. Similarly, $\phi \subseteq \psi + \phi$. □

Corollary 3.3. *If $\psi_i \in \mathbf{VWII}(E)$, where $i \in \Lambda$. Then*

- (i) $\bigcap_{i \in \Lambda} \psi_i \in \mathbf{VWII}(E)$;
- (ii) $\sum_{i \in \Lambda} \psi_i \in \mathbf{VWII}(E)$.

Theorem 3.5. *Let $\psi, \phi \in \mathbf{VWII}(E)$. Then*

- (i) *if $\mathbf{S} := \{\varphi_i : \varphi_i \in \mathbf{VWII}(E), \varphi_i \subseteq \psi, \phi \text{ for all } i \in \Lambda\}$, then $\psi \cap \phi$ is maximal of $\mathbf{S}$;*
- (ii) *if $\mathbf{T} := \{\varrho_i : \varrho_i \in \mathbf{VWII}(E), \psi, \phi \subseteq \varrho_i \text{ for all } i \in \Lambda\}$, then $\psi + \phi$ is minimal of $\mathbf{T}$.*

Proof. Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$.

(i) Using Theorem 3.4 (i) and (iii), we get $\psi \cap \phi \in \mathbf{VWII}(E)$ and $\psi \cap \phi \subseteq \psi, \phi$. Suppose $\varphi \in \mathbf{S}$ s.t. $\varphi \subseteq \psi$ and $\varphi \subseteq \phi$. Now, let $t \in E$. Then

$$V_{\psi \cap \phi}(t) = \min\{V_\psi(t), V_\phi(t)\} \geq \min\{V_\varphi(t), V_\varphi(t)\} = V_\varphi(t).$$

Therefore, $\varphi \subseteq \psi \cap \phi$. Thus, $\psi \cap \phi$ is maximal element in $\mathbf{S}$.

(ii) Applying Theorem 3.4 (ii) and (iv), we get $\psi + \phi \in \mathbf{VWII}(E)$ and $\psi, \phi \subseteq \psi + \phi$. Let $\varphi \in \mathbf{T}$ s.t. $\psi \subseteq \varphi$ and $\phi \subseteq \varphi$. Given $t \in E$, we have

$$\begin{aligned} V_{\psi+\phi}(t) &= \sup\{\min\{V_\psi(r), V_\phi(s)\} : t = r + s, \text{ where } r, s \in E\} \\ &\leq \sup\{\min\{V_\varphi(r), V_\varphi(s)\} : t = r + s, \text{ where } r, s \in E\} \\ &\leq \sup\{V_\varphi(r + s)\} \\ &= V_\varphi(t). \end{aligned}$$

Therefore, $\psi + \phi \subseteq \varphi$ is minimal element of $\mathbf{T}$. □

Theorem 3.6. *The $(\mathbf{VWII}(E), \subseteq)$ is a complete lattice, where $\subseteq$ is defined by:*

$$\psi \subseteq \phi \text{ if and only if } V_\psi(e) \leq V_\phi(e), \text{ for all } e \in E.$$

Proof. It is known that $(\mathbf{VWII}(E), \subseteq)$ is a poset. By Theorem 3.5, every pair of elements in $\mathbf{VWII}(E)$ has a maximal and a minimal element. Hence, $\mathbf{VWII}(E)$ is a lattice. Let $\mathbf{S} := \{\psi_i : \psi_i \in \mathbf{VWII}(E) \text{ for } i \in \Lambda\}$ be a subset of $\mathbf{VWII}(E)$. By Corollary 3.3 (i), $\bigcap_{i \in \Lambda} \psi_i \in \mathbf{VWII}(E)$ and it is the infimum of $\mathbf{S}$. Also, by Corollary 3.3 (ii), $\sum_{i \in \Lambda} \psi_i \in \mathbf{VWII}(E)$ and it is supremum of $\mathbf{S}$. Thus, $(\mathbf{VWII}(E), \subseteq)$ is a complete lattice. □

In the next example we can see that the union of two vague weak interior ideals need not be a vague weak interior ideal.

Example 3.3. Let $E := \mathbb{Z}_8$ and $\Gamma := \{\bar{0}, \bar{2}, \bar{4}\}$. Define $\cdot : \mathbb{Z}_8 \times \Gamma \times \mathbb{Z}_8 \rightarrow \mathbb{Z}_8$ by $\cdot(\bar{x}, \bar{y}, \bar{z}) = \overline{xyz}$ usual product $\bar{x}, \bar{y}, \bar{z}$, for all $\bar{x}, \bar{z} \in \mathbb{Z}_8, \bar{y} \in \{\bar{0}, \bar{2}, \bar{4}\}$. Then $\mathbb{Z}_8$ is a Γ -semiring. Define $t_\psi, f_\psi : \mathbb{Z}_8 \rightarrow [0, 1]$ by:

$$t_\psi(\bar{e}) = \begin{cases} 0.81, & \text{if } \bar{e} = \bar{0}, \\ 0.63, & \text{if } \bar{e} = \bar{1}, \\ 0.45, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_\psi(\bar{e}) = \begin{cases} 0.22, & \text{if } \bar{e} = \bar{0}, \\ 0.31, & \text{if } \bar{e} = \bar{1}, \\ 0.52, & \text{otherwise.} \end{cases}$$

Further, we define $t_\phi, f_\phi : \mathbb{Z}_8 \rightarrow [0, 1]$ by:

$$t_\phi(\bar{e}) = \begin{cases} 0.67, & \text{if } \bar{e} = \bar{0}, \\ 0.54, & \text{if } \bar{e} = \bar{2}, \\ 0.24, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_\phi(\bar{e}) = \begin{cases} 0.32, & \text{if } \bar{e} = \bar{0}, \\ 0.44, & \text{if } \bar{e} = \bar{2}, \\ 0.51, & \text{otherwise.} \end{cases}$$

Therefor, $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(\mathbb{Z}_8)$, but $\psi \cup \phi \notin \mathbf{VWII}(\mathbb{Z}_8)$, since $V_{(\psi \cup \phi)\Gamma(\psi \cup \phi)}(\bar{e}) > V_{\psi \cup \phi}(\bar{e})$ at $\bar{e} = \bar{4}$.

Theorem 3.7. Let $\psi, \phi \in \mathbf{VWII}(E)$. Then $\psi \cup \phi \in \mathbf{VWII}(E)$ if $\psi \subseteq \phi$ or $\phi \subseteq \psi$.

Proof. Assume $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$. Suppose $\psi \subseteq \phi$. Hence, $\psi \cup \phi \in \mathbf{V}\Gamma(E)$. Given $x \in E$, we have

$$\begin{aligned} V_{(\psi \cup \phi)\Gamma(\psi \cup \phi)}(x) &= \sup\{\min\{\sup\{\min\{V_{\psi \cup \phi}(a), V_{\psi \cup \phi}(b)\}\}, V_\delta(c)\} : x = a\alpha b\beta c, \\ &\quad \text{where } a, b, c \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{\sup\{\min\{V_\phi(a), V_\phi(b)\}\}, V_\delta(c)\} : a, b, c \in E\} \\ &= V_{\phi\Gamma\phi\Gamma\delta}(x) \\ &\leq V_\phi(x) \\ &= \max\{V_\psi(x), V_\phi(x)\} \\ &= V_{\psi \cup \phi}(x). \end{aligned}$$

Therefore, $(\psi \cup \phi)\Gamma(\psi \cup \phi)\Gamma\delta \subseteq \psi \cup \phi$. It follows that $\psi \cup \phi \in \mathbf{RVWII}(E)$. Similarly, $\psi \cup \phi \in \mathbf{LVWII}(E)$. Thus, $\psi \cup \phi \in \mathbf{VWII}(E)$. $\square$

Theorem 3.8. E is regular if and only if $\psi = \psi\Gamma\psi\Gamma\delta$, for all $\psi \in \mathbf{RVWII}(E)$.

Proof. Suppose E is regular and let $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. Then $\psi\Gamma\psi\Gamma\delta \subseteq \psi$. Let $x \in E$. Then there exist $a \in E$ and $\alpha, \beta \in \Gamma$ s.t. $x = x\alpha a\beta x$, and so we have:

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(x) &= \sup\{\min\{V_\psi(x), V_{\psi\Gamma\delta}(a\beta x)\}\} \\ &= \sup\{\min\{V_\psi(x), \sup\{\min\{V_\psi(a), V_\delta(x)\}\}\} \\ &= \sup\{\min\{V_\psi(x), V_\psi(a)\}\} \\ &\geq V_\psi(x). \end{aligned}$$

Therefore, $\psi\Gamma\psi\Gamma\delta \supseteq \psi$, and so $\psi\Gamma\psi\Gamma\delta = \psi$.

Conversely, suppose $\psi\Gamma\psi\Gamma\delta = \psi$, and let $W \in \mathbf{RWII}(E)$. Using Theorem 3.3, we get $\delta_W \in \mathbf{RVWII}(E)$. It follows that $\delta_W\Gamma\delta_W\Gamma\delta = \delta_W$, and so $\delta_{W\Gamma W\Gamma E} = \delta_W$. Thus, $W = W\Gamma W\Gamma E$. Using [6, Theorem 4.4], we get E is regular. $\square$

In the following example we show that, for given $\psi \in \mathbf{VWII}(E)$ and $\phi \in \mathbf{VS}(E)$ s.t. $\psi \subseteq \phi$, the extension property is not valid, i.e., maybe $\phi \notin \mathbf{VWII}(E)$.

Example 3.4. Let $E = \Gamma := \mathbb{R}$. Define $\cdot : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $\cdot(a, b, c) = abc$ for all $a, b, c \in \mathbb{R}$. Then $\mathbb{R}$ is a $\mathbb{R}$ -semiring. Define $t_\psi, f_\psi : \mathbb{R} \rightarrow [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.898, & \text{if } x \neq 0, \\ 0.532, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.241, & \text{if } x \neq 0, \\ 0.437, & \text{if } x = 0. \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{VWII}(\mathbb{R})$.

Now, if define $t_\phi, f_\phi : \mathbb{R} \rightarrow [0, 1]$ by:

$$t_\phi(x) = \begin{cases} 0.93, & \text{if } x \in 2\mathbb{Z}, x \neq 0, \\ 0.85, & \text{if } x \in 2\mathbb{Z} + 1, \\ 0.66, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\phi(x) = \begin{cases} 0.13, & \text{if } x \in 2\mathbb{Z}, x \neq 0, \\ 0.25, & \text{if } x \in 2\mathbb{Z} + 1, \\ 0.38, & \text{if } x = 0. \end{cases}$$

We can see that $\psi \subseteq \phi$, but $\phi = (t_\phi, f_\phi) \notin \mathbf{VWII}(\mathbb{R})$.

4. NORMAL VAGUE WEAK INTERIOR IDEALS IN Γ -SEMRINGS

We define the notion of a (*complete*-) *normal vague weak interior ideal*, and show that we can construct it in a Γ -semiring. Additionally, we prove that the cardinal of a maximal element, which is not constant, in the set of all normal vague weak interior ideals of a Γ -semiring is 2.

Definition 4.1. Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called normal, if $V_\psi(0) = [1, 1]$ i.e., $t_\psi(0) = 1$ and $f_\psi(0) = 0$.

Denote by $\mathbf{NVS}(E)$ the set of all normal vague sets of E .

Example 4.1. Consider Example 3.4, and define $t_\psi, f_\psi : \mathbb{R} \rightarrow [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.92, & \text{if } x \in \mathbb{R}^+, \\ 0.75, & \text{if } x \in \mathbb{R}^-, \\ 1, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.13, & \text{if } x \in \mathbb{R}^+, \\ 0.24, & \text{if } x \in \mathbb{R}^-, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{NVS}(\mathbb{R})$.

The following theorem we achieve a necessity condition for a vague set to be normal vague set.

Theorem 4.1. Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$ s.t. $t_\psi(e) + f_\psi(e) \leq t_\psi(0) + f_\psi(0)$ for all $e \in E$. Define $\psi^+ = (t_{\psi^+}, f_{\psi^+})$, where $t_{\psi^+}(e) = t_\psi(e) + 1 - t_\psi(0)$ and $f_{\psi^+}(e) = f_\psi(e) - f_\psi(0)$ for all $e \in E$. Then $\psi^+ \in \mathbf{NVS}(E)$.

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$ and $e \in E$. Then

$$t_{\psi^+}(e) + f_{\psi^+}(e) = t_\psi(e) + 1 - t_\psi(0) + f_\psi(e) - f_\psi(0) \leq 1.$$

Therefore, $\psi^+ \in \mathbf{VS}(E)$. Also, $t_{\psi^+}(0) = 1$ and $f_{\psi^+}(0) = 0$. Thus, $\psi^+ \in \mathbf{NVS}(E)$. $\square$

Proposition 4.1. *Let $\psi, \phi \in \mathbf{VWII}(E)$. Then*

- (i) $\psi^+ \in \mathbf{NVWII}(E)$;
- (ii) $\psi \in \mathbf{NVWII}(E)$ if and only if $\psi^+ = \psi$;
- (iii) $(\psi^+)^+ = \psi$;
- (iv) if exists $\phi \in \mathbf{VWII}(E)$ s.t. $\phi^+ \subseteq \psi$, then $\psi \in \mathbf{NVWII}(E)$;
- (v) if exists $\phi \in \mathbf{VWII}(E)$ s.t. $\phi^+ \subseteq \psi$, then $\psi^+ = \psi$;
- (vi) $(\psi \cap \phi)^+ = \psi^+ \cap \phi^+$;
- (vii) $(\psi \cup \phi)^+ = \psi^+ \cup \phi^+$;
- (viii) $\psi \subseteq \phi$ implies $\psi^+ \subseteq \phi^+$.

Proof. (i) Suppose $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. Given $e, f \in E$, $\gamma \in \Gamma$, we have

$$\begin{aligned} V_{\psi^+}(e + f) &= V_\psi(e + f) + [1, 1] - V_\psi(0) \\ &\geq \min\{V_\psi(e), V_\psi(f)\} + [1, 1] - V_\psi(0) \\ &= \min\{V_\psi(e) + [1, 1] - V_\psi(0), V_\psi(f) + [1, 1] - V_\psi(0)\} \\ &= \min\{V_{\psi^+}(e), V_{\psi^+}(f)\} \end{aligned}$$

and

$$\begin{aligned} V_{\psi^+}(e\gamma f) &= V_\psi(e\gamma f) + [1, 1] - V_\psi(0) \\ &\geq \min\{V_\psi(e), V_\psi(f)\} + [1, 1] - V_\psi(0) \\ &= \min\{V_\psi(e) + [1, 1] - V_\psi(0), V_\psi(f) + [1, 1] - V_\psi(0)\} \\ &= \min\{V_{\psi^+}(e), V_{\psi^+}(f)\}. \end{aligned}$$

Therefore, $\psi^+ \in \mathbf{V}\Gamma(E)$. Also, we have

$$\begin{aligned} V_{\psi^+ \Gamma \psi^+ \Gamma \delta}(e) &= \sup\{\min\{\sup\{\min\{V_{\psi^+}(f), V_{\psi^+}(g)\}\}, V_\delta(h)\} : e = f\alpha g\beta h, \\ &\quad \text{where } f, g, h \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{V_{\psi^+}(f), V_{\psi^+}(g)\}\} \\ &= \sup\{\min\{V_\psi(f) + [1, 1] - V_\psi(0), V_\psi(g) + [1, 1] - V_\psi(0)\}\} \\ &= \sup\{\min\{V_\psi(f), V_\psi(g)\}\} + [1, 1] - V_\psi(0) \\ &= \sup\{\min\{\sup\{\min\{V_\psi(f), V_\psi(g)\}\}, V_\delta(h)\} + [1, 1] - V_\psi(0) \\ &= V_{\psi \Gamma \psi \Gamma \delta}(e) + [1, 1] - V_\psi(0) \\ &\leq V_\psi(e) + [1, 1] - V_\psi(0) \\ &= V_{\psi^+}(e). \end{aligned}$$

Hence, $\psi^+ \in \mathbf{RVWII}(E)$. We can see that $V_{\psi^+}(0) = V_\psi(0) + [1, 1] - V_\psi(0) = [1, 1]$. Therefore, $\psi^+ \in \mathbf{NRVWII}(E)$. Similarly, $\psi^+ \in \mathbf{NLVWII}(E)$. It follows that $\psi^+ \in \mathbf{NVWII}(E)$. Clearly, $\psi \subseteq \psi^+$.

(ii) Assume $\psi = (t_\psi, f_\psi) \in \mathbf{NVWII}(E)$ and $e \in E$. Then

$$V_{\psi^+}(e) = V_\psi(e) + [1, 1] - V_\psi(0) = V_\psi(e) + [1, 1] - [1, 1] = V_\psi(e).$$

Thus, $\psi^+ = \psi$. The converse is obvious.

(iii) Assume $e \in E$. $V_{(\psi^+)^+}(e) = V_{\psi^+}(e) + [1, 1] - V_{\psi^+}(0) = V_{\psi^+}(e)$. Therefore, $(\psi^+)^+ = \psi^+$. Since $\psi \in \mathbf{NVWII}(E)$, using (ii) we get $(\psi^+)^+ = \psi^+ = \psi$.

(iv) Suppose there exists $\phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$, s.t. $\phi^+ \subseteq \psi$. Then, $[1, 1] = V_{\phi^+}(0) \leq V_\psi(0)$. This shows that $V_\psi(0) = [1, 1]$. Thus, $\psi \in \mathbf{NVWII}(E)$.

(v) The proof is clear by using (i) and (iv).

(vi) Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$ and $e \in E$. Then we have

$$\begin{aligned} V_{(\psi \cap \phi)^+}(e) &= V_{\psi \cap \phi}(e) + [1, 1] - V_{\psi \cap \phi}(0) \\ &= \min\{V_\psi(e), V_\phi(e)\} + [1, 1] - \min\{V_\psi(0), V_\phi(0)\} \\ &= \min\{V_\psi(e) + [1, 1] - V_\psi(0), V_\phi(e) + [1, 1] - V_\phi(0)\} \\ &= \min\{V_{\psi^+}(e), V_{\phi^+}(e)\} \\ &= V_{\psi^+ \cap \phi^+}(e). \end{aligned}$$

Hence, $(\psi \cap \phi)^+ = \psi^+ \cap \phi^+$.

(vii) Let $e \in E$. Then we have

$$\begin{aligned} V_{(\psi \cup \phi)^+}(e) &= V_{\psi \cup \phi}(e) + [1, 1] - V_{\psi \cup \phi}(0) \\ &= \max\{V_\psi(e), V_\phi(e)\} + [1, 1] - \max\{V_\psi(0), V_\phi(0)\} \\ &= \max\{V_\psi(e) + [1, 1] - V_\psi(0), V_\phi(e) + [1, 1] - V_\phi(0)\} \\ &= \max\{V_{\psi^+}(e), V_{\phi^+}(e)\} \\ &= V_{\psi^+ \cup \phi^+}(e). \end{aligned}$$

Then $(\psi \cup \phi)^+ = \psi^+ \cup \phi^+$.

(viii) Given $e \in E$, we get

$$V_{\psi^+}(e) = V_\psi(e) + [1, 1] - V_\psi(0) \leq V_\phi(e) + [1, 1] - V_\phi(0) = V_{\phi^+}(e).$$

Therefore, $\psi^+ \subseteq \phi^+$. □

Theorem 4.2. *Let ψ be a maximal element in $\mathbf{NVWII}(E)$, which is not constant. Then $V_\psi(x) \in \{[0, 0], [1, 1]\}$ for all $x \in E$.*

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{NVWII}(E)$. Then $V_\psi(0) = [1, 1]$. Let there exists $s \in E$ s.t. $V_\psi(s) \neq [1, 1]$. It is sufficient to show that $V_\psi(s) = [0, 0]$. Suppose there exists $e_0 \in E$ s.t. $[0, 0] < V_\psi(e_0) < [1, 1]$. Define a vague set $\phi = (t_\phi, f_\phi)$ of E by $t_\phi(e) = \frac{t_\psi(e) + t_\psi(e_0)}{2}$ and $f_\psi(e) = \frac{f_\psi(e) + f_\psi(e_0)}{2}$ for all $e \in E$. Clearly, ϕ is well-defined.

Given $e, f \in E; \gamma \in \Gamma$, we have

$$\begin{aligned} V_\phi(e + f) &= \frac{V_\psi(e + f) + V_\psi(e_0)}{2} \\ &\geq \frac{\min\{V_\psi(e), V_\psi(f)\} + V_\psi(e_0)}{2} \\ &= \min\left\{\frac{V_\psi(e) + V_\psi(e_0)}{2}, \frac{V_\psi(f) + V_\psi(e_0)}{2}\right\} \\ &= \min\{V_\phi(e), V_\phi(f)\} \end{aligned}$$

and

$$\begin{aligned} V_\phi(e\gamma f) &= \frac{V_\psi(e\gamma f) + V_\psi(e_0)}{2} \\ &\geq \frac{\min\{V_\psi(e), V_\psi(f)\} + V_\psi(e_0)}{2} \\ &= \min\left\{\frac{V_\psi(e) + V_\psi(e_0)}{2}, \frac{V_\psi(f) + V_\psi(e_0)}{2}\right\} \\ &= \min\{V_\phi(e), V_\phi(f)\}. \end{aligned}$$

Therefore, $\phi \in \mathbf{V}\Gamma(E)$. Also, we have

$$\begin{aligned} V_{\phi\Gamma\phi\Gamma\delta}(e) &= \sup\{\min\{\sup\{\min\{\phi(f), \phi(g)\}\}, V_\delta(h)\} : e = f\alpha g\beta h, \\ &\quad \text{where } f, g, h \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{V_\phi(f), V_\phi(g)\}\} \\ &= \sup\left\{\min\left\{\frac{V_\psi(f) + V_\psi(e_0)}{2}, \frac{V_\psi(g) + V_\psi(e_0)}{2}\right\}\right\} \\ &= \frac{1}{2} \sup\{\min\{V_\psi(f), V_\psi(g)\}\} + \frac{V_\psi(e_0)}{2} \\ &= \frac{1}{2} \sup\{\min\{\sup\{\min\{V_\psi(f), V_\psi(g)\}\}, V_\delta(h)\}\} + \frac{V_\psi(e_0)}{2} \\ &= \frac{1}{2} V_{\psi\Gamma\psi\Gamma\delta}(e) + \frac{V_\psi(e_0)}{2} \\ &\leq \frac{1}{2} V_\psi(e) + \frac{V_\psi(e_0)}{2} \\ &= V_\phi(e). \end{aligned}$$

Hence $\phi \in \mathbf{RVWII}(E)$. By a similar way we can show that $\phi \in \mathbf{LVWII}(E)$. Thus, $\phi \in \mathbf{VWII}(E)$. Now, we have

$$\begin{aligned} V_{\phi^+}(e) &= V_{\phi}(e) + [1, 1] - V_{\phi}(0) \\ &= \frac{V_{\psi}(e) + V_{\psi}(e_0)}{2} + [1, 1] - \frac{V_{\psi}(0) + V_{\psi}(e_0)}{2} \\ &= \frac{V_{\psi}(e) + [1, 1]}{2}. \end{aligned}$$

That implies $V_{\phi^+}(0) = \frac{V_{\psi}(0) + [1, 1]}{2} = [1, 1]$. Thus, $\phi^+ \in \mathbf{NVWII}(E)$. Now, $V_{\phi^+}(0) = [1, 1] > V_{\psi}(e_0)$. This shows that ϕ^+ is not constant. Further, we have $V_{\phi^+}(e_0) > V_{\psi}(e_0)$, which is a contraction, since ψ is a maximal element. Hence $V_{\psi}(s) = [0, 0]$. Therefore, $V_{\psi}(x) \in \{[0, 0], [1, 1]\}$. $\square$

Corollary 4.1. *If ψ is a maximal element in $\mathbf{NVWII}(E)$, which isn't constant, then $|V_{\psi}(x)| = 2$.*

Definition 4.2. Let $\psi \in \mathbf{NVS}(E)$. Then ψ is called *complete* if there exists $e \in E$ s.t. $V_{\psi}(e) = [0, 0]$.

Denote by $\mathbf{CNVS}(E)$ the set of all normal vague sets of E , resp., $\mathbf{CNVWII}(E)$ the set of all complete normal vague weak interior ideals of E . Then $\mathbf{CNVWII}(E) \subseteq \mathbf{CNVS}(E)$, and so $(\mathbf{CNVWII}(E), \subseteq)$ is a poset.

Example 4.2. Consider Example 3.3, and define $t_{\psi}, f_{\psi} : \mathbb{Z}_8 \rightarrow [0, 1]$ by:

$$t_{\psi}(\bar{x}) = \begin{cases} 1, & \text{if } \bar{x} = \bar{0}, \\ 0.56, & \text{if } \bar{x} = \bar{1}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\psi}(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} = \bar{0}, \\ 0.45, & \text{if } \bar{x} = \bar{1}, \\ 1, & \text{otherwise.} \end{cases}$$

Hence, $\psi = (t_{\psi}, f_{\psi}) \in \mathbf{CNVWII}(E)$.

Theorem 4.3. *If ψ is a maximal element in $(\mathbf{NVWII}(E), \subseteq)$, which is not constant, then it is a maximal element in $(\mathbf{CNVWII}(E), \subseteq)$.*

Proof. Assume $\bar{\psi} = (t_{\bar{\psi}}, f_{\bar{\psi}})$ is a maximal element in $(\mathbf{NVWII}(E), \subseteq)$, which isn't constant. By Theorem 4.2, $V_{\bar{\psi}}(x) \in \{[0, 0], [1, 1]\}$ for all $x \in E$, i.e., $V_{\bar{\psi}}(0) = [1, 1]$ and $V_{\bar{\psi}}(x) = [0, 0]$ for some $x \in E$. It follows that $\bar{\psi} \in \mathbf{CNVWII}(E)$. Suppose $\bar{\phi} = (t_{\bar{\phi}}, f_{\bar{\phi}}) \in \mathbf{CNVWII}(E)$ s.t. $\bar{\psi} \subseteq \bar{\phi}$. Then $\bar{\phi} \in \mathbf{NVWII}(E)$. Since $\bar{\phi}$ is a maximal element in $\mathbf{NVWII}(E)$ and $\bar{\psi} \in \mathbf{NVWII}(E)$ with $\bar{\psi} \subseteq \bar{\phi}$, that gives $\bar{\psi} = \bar{\phi}$. Therefore, $\bar{\psi}$ is a maximal element in $\mathbf{CNVWII}(E)$. $\square$

CONCLUSIONS AND COMMENTS

We defined the notion of a right (resp. left) vague weak interior ideal of a Γ -semiring and the characterization theorem for *regular Γ -semiring* in terms of vague weak interior ideals is derived. In addition, we introduced and studied (*complete*-) *normal* vague weak interior ideals of a Γ -semiring. As a consequence of the results

is that the cardinal of a non-constant maximal element in the set of all (complete-) normal vague weak interior ideals is 2. As a direction of this research will be study on vague (minimal weak interior, bi-interior, quasi-interior) ideals of a Γ -semiring and investigate relations among these notions.

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ULTIMATE BOUNDEDNESS OF SOLUTIONS OF SOME SYSTEM OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents sufficient conditions for the ultimate boundedness of solutions of some system of third-order nonlinear differential equations

$$\ddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

where Ψ, Φ are positive definite symmetric matrices, H, P are n -vectors continuous in their respective arguments, $X \in \mathbb{R}^n$ and $t \in \mathbb{R}^+ = [0, +\infty)$. We do not necessarily require $H(X)$ differentiable to obtain our results. By using the Lyapunov's direct (second) method and constructing a complete Lyapunov function, earlier results are generalized.

1. INTRODUCTION

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and let $\mathbb{R}^n$ denote the real Euclidean n -dimensional space furnished with the usual Euclidean norm denoted by $\|\cdot\|$. Consider the system of third-order nonlinear differential equations

$$(1.1) \quad \ddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

where $t \in \mathbb{R}^+$, $X : \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Psi, \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ are continuous in their respective arguments, H is assumed to be not necessarily differentiable and the dots indicate differentiation with respect to the independent variable t . Thus, for any initial values $X_0, Y_0, Z_0 \in \mathbb{R}^n$, there is a uniquely

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defined solution $X = X(t, X_0, Y_0, Z_0)$ of (1.1), continuous in t, X_0, Y_0, Z_0 satisfying the condition $X(t_0) = X_0, \dot{X}(t_0) = Y_0, \ddot{X}(t_0) = Z_0$ [21]. Equation (1.1) is the vector version for the systems of real third-order nonlinear differential equations of the form

$$\ddot{x}_i + \sum_{k=1}^n \psi_{ik}(\dot{x}_1, \dots, \dot{x}_n) \ddot{x}_k + \sum_{k=1}^n \phi_{ik}(x_1, \dots, x_n) \dot{x}_k + h_i(x_1, \dots, x_n) = p_i(t, x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n, \ddot{x}_1, \dots, \ddot{x}_n),$$

where $i = 1, \dots, n$, in which the functions $\psi_{ik}, \phi_{ik}, h_i, p_i$ are continuous in their respective arguments. In the case $n = 1$, this system reduces to the scalar ordinary differential equations of the form

$$(1.2) \quad \ddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

where ψ, ϕ, h and p are continuous in their respective arguments, see [4–6, 9, 10, 13, 16, 17, 19, 23–26, 28, 29, 34, 35] and the references cited therein. If $\psi(\dot{x}) = a$ and $\phi(x) = b$, (1.2) reduces to

$$\ddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

which has been investigated by Ezeilo [9] for ultimate boundedness and convergence of solutions by assuming

$$(1.3) \quad \frac{h(\xi + \gamma) - h(\gamma)}{\xi} \in I_0, \quad \xi \neq 0,$$

with $I_0 \equiv [\delta, kab] \subset (0, ab)$ the generalized Routh-Hurwitz interval, $\delta > 0$ and $0 < k < 1$. When $\gamma = 0$ in (1.3) we have

$$H_0 = H_0(\xi) \equiv \frac{h(\xi + \gamma) - h(\gamma)}{\xi}$$

and

$$H_0 = \frac{h(\xi)}{\xi}, \quad \text{if } h(0) = 0.$$

On the other hand, if $\Psi(\dot{X}) = A, \Phi(X) = B$ in (1.1), we have

$$(1.4) \quad \ddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

where A, B are real symmetric $n \times n$ matrices. Equation (1.4) has been studied by Afuwape [1] and Meng [12] for the ultimate boundedness and periodicity of solutions for which H is of class $\mathbf{C}(\mathbb{R}^n)$, satisfying

$$(1.5) \quad H(X_2) = H(X_1) + C_h(X_1, X_2)(X_2 - X_1),$$

where $C_h(X_1, X_2)$ is a real $n \times n$ operator for any X_1, X_2 in $\mathbb{R}^n$, and having real eigenvalues $\lambda_i(C_h(X_1, X_2))$, $i = 1, 2, \dots, n$. These eigenvalues satisfy

$$(1.6) \quad 0 < \delta_c \leq \lambda_i(C_h(X_1, X_2)) \leq \Delta_c,$$

with δ_c, Δ_c as fixed constants. Further, the matrices A, B have real positive eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$ respectively, satisfying

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a,$$

$$0 < \delta_b \leq \lambda_i(B) \leq \Delta_b,$$

$i = 1, 2, \dots, n$, and that for some constant $k(< 1)$ the 'generalized' Routh-Hurwitz condition

$$(1.7) \quad \Delta_c \leq k\delta_a\delta_b$$

is satisfied.

In these papers mentioned above, the Lyapunov's direct method was used to obtain results. This entails construction of a quadratic-like function (also known as Lyapunov function) to obtain sufficient conditions which guarantee the properties of solutions, but the construction of this function is difficult since there is no general method to obtaining it ([1]-[35]). Perhaps, reason (1.1) has received no attention in literature.

The present work is concerned with the ultimate boundedness of solutions of (1.1) or its equivalent system form

$$(1.8) \quad \begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -\Psi(Y)Z - \Phi(X)Y - H(X) + P(t, X, Y, Z), \end{aligned}$$

obtained as usual by setting $\dot{X} = Y$, $\dot{Y} = Z$ in (1.1). This problem was left open by Ezeilo and Tejumola [7, page 284]. In this work, by using the Lyapunov's direct method and constructing a suitable complete Lyapunov function, we shall obtain sufficient conditions which guarantee the ultimate boundedness of solutions of (1.1).

2. NOTATION

Our notations are similar to [3]. In this paper, δ 's and Δ 's with or without suffixes represent positive constants whose magnitudes depend on the matrix functions Ψ , Φ , and the vector functions H , P . The δ 's and Δ 's with numerical or alphabetical suffixes shall retain fixed magnitudes while those without suffixes are not necessarily the same at each occurrence. Finally, $\langle X, Y \rangle$ shall represent the scalar product of any vectors $X, Y \in \mathbb{R}^n$, with respective components $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ by $\sum_{i=1}^n x_i y_i$. In particular, $\langle X, X \rangle = \|X\|^2$.

3. STATEMENT OF RESULTS

Our main result in this paper is the following.

Theorem 3.1. *Suppose $H(0) = 0$, and that*

- (i) *there exists $n \times n$ real continuous operator $C_h(X_1, X_2)$ for any vectors X_1, X_2 such that the function H is of class $\mathbf{C}(\mathbb{R}^n)$, satisfy (1.5), with eigenvalues $\lambda_i(C_h(X_1, X_2))$, $i = 1, 2, \dots, n$, satisfying (1.6);*
- (ii) *the matrix functions $\Psi(Y)$, $\Phi(X)$ are continuous in their respective arguments, with eigenvalues $\lambda_i(\Psi(Y))$, $\lambda_i(\Phi(X))$ satisfying*

$$(3.1) \quad 0 < \delta_a \leq \lambda_i(\Psi(Y)) \leq \Delta_a,$$

$$(3.2) \quad 0 < \delta_b \leq \lambda_i(\Phi(X)) \leq \Delta_b,$$

where $i = 1, 2, \dots, n$;

(iii) the matrices Ψ , Φ and the operator C_h are associative and commute pairwise; and

(iv) the vector function $P(t, X, Y, Z)$ satisfies

$$(3.3) \quad \|P(t, X, Y, Z)\| \leq \theta_1(t) + \theta_2(t) \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)^{\frac{\rho}{2}} \\ + \delta_0 \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \right)^{\frac{1}{2}},$$

for any $X, Y, Z \in \mathbb{R}^n$, where $\delta_0 > 0$ is a constant, $\theta_1(t)$, $\theta_2(t)$ are continuous functions in t and $0 \leq \rho < 1$.

Then, there exist constants Δ_1 , Δ_2 , Δ_3 such that every solution $X(t)$ of (1.1) with $X(t_0) = X_0$, $\dot{X}(t_0) = Y_0$, $\ddot{X}(t_0) = Z_0$, and for any constant ν , whatever in the range $\frac{1}{2} \leq \nu \leq 1$, the inequality

$$(3.4) \quad \left(\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \right)^{\nu} \leq \Delta_1 \exp\{-\Delta_2(t - t_0)\} \\ + \Delta_3 \int_{t_0}^t \left(\theta_1^{2\nu}(\tau) + \theta_2^{\frac{2\nu}{1-\rho}}(\tau) \right) \\ \times \exp\{-\Delta_2(t - \tau)\} d\tau$$

holds for all $t \geq t_0$, where $\Delta_1 = \Delta_1(X_0, Y_0, Z_0)$.

A number of quite important results can be deduced from Theorem 3.1. For example, we have the following.

Corollary 3.1. *If $P \equiv 0$ and if all conditions of Theorem 3.1 hold, then every solution $X(t)$ of (1.1) satisfies*

$$(3.5) \quad \|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \rightarrow 0,$$

as $t \rightarrow +\infty$.

Indeed, by setting $\theta_1 = 0 = \theta_2$ in (3.4), we have that, if $\delta_0 \leq \Delta_0$, then

$$\left(\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \right)^{\nu} \leq \Delta_1 \exp\{-\Delta_2(t - t_0)\}, \quad t \geq t_0,$$

from which (3.5) follows on letting $t \rightarrow +\infty$.

Corollary 3.2. *Assume that all conditions of Theorem 3.1 hold and let $\delta_0 \leq \Delta_0$. Suppose also that there are fixed constants η , $1 \leq \eta \leq 2$, and $\mu > 0$ such that*

$$\int_{t_0}^{t+\mu} \left(\theta_1^{\eta}(\tau) + \theta_2^{\frac{\eta}{1-\rho}}(\tau) \right) \exp\{-\Delta_2(t - \tau)\} d\tau \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Then, every solution $X(t)$ of (1.1) satisfies (3.5).

Another interesting result which can be deduced very readily from Theorem 3.1 is the following generalization of the boundedness results in [1, 7] and [12].

Corollary 3.3. *Assume that all the conditions of Theorem 3.1 hold and let $\delta_0 \leq \Delta_0$. Suppose further that each of the functions $\theta_1(t)$, $\theta_2(t)$ satisfies at least one of the following conditions:*

- (i) $\max_{0 \leq t < +\infty} \theta_1(t) < +\infty$, $\max_{0 \leq t < +\infty} \theta_2(t) < +\infty$;
- (ii) $\int_{t_0}^{+\infty} \theta_1^\eta(t) dt < +\infty$, $\int_{t_0}^{+\infty} \theta_2^{\frac{\eta}{1-\rho}}(t) dt < +\infty$,

for some constant η in the range $1 \leq \eta \leq 2$. Then there exists a constant $\Delta_4 > 0$ such that every solution $X(t)$ of (1.1) ultimately satisfies

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \leq \Delta_4.$$

4. PRELIMINARY RESULTS

We need a few important results to prove Theorem 3.1.

Lemma 4.1. *Let D be a real symmetric $n \times n$ positive definite matrix. Then, for any $X \in \mathbb{R}^n$,*

$$(4.1) \quad \delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2,$$

where δ_d , Δ_d are respectively the least and greatest eigenvalues of D .

Proof. See [7]. □

Lemma 4.2. *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then the eigenvalues $\lambda_i(QD)$, $i = 1, 2, \dots, n$, of the product matrix QD are all real and satisfy*

$$\min_{1 \leq j, k \leq n} \lambda_j(Q) \lambda_k(D) \leq \lambda_i(QD) \leq \max_{1 \leq j, k \leq n} \lambda_j(Q) \lambda_k(D),$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of Q and D .

Proof. See [7]. □

The main tool in the proof of Theorem 3.1 is the scalar function $W = W(X, Y, Z)$ defined for arbitrary vectors $X, Y, Z \in \mathbb{R}^n$ by

$$(4.2) \quad 2W = \beta(1-\beta)\delta_b^2 \|X\|^2 + \delta_b(\beta + \alpha\delta_a^{-1}) \|Y\|^2 + \alpha\delta_a^{-1} \|Z\|^2 + \|Z + \delta_a Y + (1-\beta)\delta_b X\|^2,$$

where $\alpha > 0$ and $0 < \beta < 1$.

The following result is immediate from (4.2).

Lemma 4.3. *Assume that all the conditions on $\Psi(Y)$, $\Phi(X)$ and $H(X)$ in Theorem 3.1 are satisfied. Then there are constants $\delta_i > 0$, $i = 1, 2$, such that*

$$(4.3) \quad \delta_1 (\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq W(X, Y, Z) \leq \delta_2 (\|X\|^2 + \|Y\|^2 + \|Z\|^2),$$

for arbitrary $X, Y, Z \in \mathbb{R}^n$.

Proof. The proof of inequalities (4.3) follows if we use Lemma 4.1 (inequalities (4.1)) repeatedly and then choose

$$\delta_1 = \frac{1}{2} \min \left\{ \beta(1 - \beta)\delta_b^2, \delta_b(\beta + \alpha\delta_a^{-1}), \alpha\delta_a^{-1} \right\}$$

and

$$\delta_2 = \frac{1}{2} \max \{ \mu_1, \mu_2, \mu_3 \},$$

where $\mu_1 = \delta_b(1 - \beta)(1 + \delta_a + \delta_b)$, $\mu_2 = \delta_b(\beta + \alpha\delta_a^{-1}) + \delta_a[1 + \delta_b(1 - \beta) + \delta_a]$ and $\mu_3 = 1 + \alpha\delta_a^{-1} + \delta_b(1 - \beta) + \delta_a$. $\square$

5. PROOF OF THEOREM 3.1

To prove Theorem 3.1, it suffices to show that the function W (defined in (4.2)) satisfies for any solution $(X(t), Y(t), Z(t))$ of (1.8) and for any ν in the range $\frac{1}{2} \leq \nu \leq 1$,

$$(5.1) \quad \dot{W} \leq -\delta_3 \psi^2 + \delta_4 \left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t) \right) \psi^{2(1-\nu)},$$

for some constants $\delta_i > 0$, $i = 3, 4$, where $\psi^2 = \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2$. We note that from Lemma 4.3, (5.1) becomes

$$\dot{W} \leq -\delta_5 W + \delta_6 \left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t) \right) W^{(1-\nu)},$$

with $\delta_5 = \delta_1\delta_3$ and $\delta_6 = \delta_2\delta_4$. If we choose $U = W^\nu$, this reduces to

$$\dot{U} \leq -\nu\delta_5 U + \nu\delta_6 \left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t) \right),$$

which when solved for U yields

$$U(t) \leq U(t_0) \exp\{-\nu\delta_5(t - t_0)\} + \Delta_5 \int_{t_0}^t \left(\theta_1^{2\nu}(\tau) + \theta_2^{\frac{2\nu}{1-\rho}}(\tau) \right) \exp\{-\nu\delta_5(t - \tau)\} d\tau,$$

for all $t \geq t_0$.

Rewriting this with $W^\nu = U$ and applying Lemma 4.3, we shall get (3.4) with

$$\Delta_1 = \delta \left(\|X(t_0)\|^2 + \|Y(t_0)\|^2 + \|Z(t_0)\|^2 \right)^\nu, \quad \Delta_2 = \nu\delta_5 \text{ and } \Delta_3 = \delta\Delta_5.$$

It follows that the proof of Theorem 3.1 is complete as soon as inequality (5.1) is proved.

6. DERIVATIVE OF W AND PROOF OF (5.1)

Let $(X(t), Y(t), Z(t))$ be any solution of (1.8). The total derivative of W , with respect to t along the solution path after simplification is

$$(6.1) \quad \dot{W} = -U_1 - U_2 - U_3 - U_4 - U_5 - U_6 - U_7 + U_8,$$

where

$$\begin{aligned}
 U_1 &= \eta_1 \delta_b (1 - \beta) \langle X, H(X) \rangle + \xi_1 \delta_a \langle (\Phi(X) - (1 - \beta) \delta_b I) Y, Y \rangle \\
 &\quad + \gamma_1 \alpha \delta_a^{-1} \langle \Psi(Y) Z, Z \rangle + \langle (\Psi(Y) - \delta_a I) Z, Z \rangle, \\
 U_2 &= \eta_2 \delta_b (1 - \beta) \langle X, H(X) \rangle + \gamma_2 \alpha \delta_a^{-1} \langle \Psi(Y) Z, Z \rangle + (1 + \alpha \delta_a^{-1}) \langle Z, H(X) \rangle, \\
 U_3 &= \eta_3 \delta_b (1 - \beta) \langle X, H(X) \rangle + \xi_2 \delta_a \langle (\Phi(X) - (1 - \beta) \delta_b I) Y, Y \rangle + \delta_a \langle Y, H(X) \rangle, \\
 U_4 &= \eta_4 \delta_b (1 - \beta) \langle X, H(X) \rangle + \gamma_3 \alpha \delta_a^{-1} \langle \Psi(Y) Z, Z \rangle \\
 &\quad + \delta_b (1 - \beta) \langle (\Psi(Y) - \delta_a I) X, Z \rangle, \\
 U_5 &= \eta_5 \delta_b (1 - \beta) \langle X, H(X) \rangle + \xi_3 \delta_a \langle (\Phi(X) - (1 - \beta) \delta_b I) Y, Y \rangle \\
 &\quad + \delta_b (1 - \beta) \langle (\Phi(X) - \delta_b I) X, Y \rangle, \\
 U_6 &= \gamma_4 \alpha \delta_a^{-1} \langle \Psi(Y) Z, Z \rangle + \xi_4 \delta_a \langle (\Phi(X) - (1 - \beta) \delta_b I) Y, Y \rangle \\
 &\quad + (1 + \alpha \delta_a^{-1}) \langle (\Phi(X) - \delta_b I) Y, Z \rangle, \\
 U_7 &= \gamma_5 \alpha \delta_a^{-1} \langle \Psi(Y) Z, Z \rangle + \xi_5 \delta_a \langle (\Phi(X) - (1 - \beta) \delta_b I) Y, Y \rangle \\
 &\quad + \delta_a \langle (\Psi(Y) - \delta_a I) Y, Z \rangle, \\
 U_8 &= \langle \delta_b (1 - \beta) X + \delta_a Y + (1 + \alpha \delta_a^{-1}) Z, P(t, X, Y, Z) \rangle,
 \end{aligned}$$

with η_i, ξ_i, γ_i , $i = 1, 2, 3, 4, 5$, positive constants such that

$$\sum_{i=1}^5 \eta_i = 1, \quad \sum_{i=1}^5 \xi_i = 1 \quad \text{and} \quad \sum_{i=1}^5 \gamma_i = 1.$$

To arrive at (5.1), we prove the following.

Lemma 6.1. *Subject to a conveniently chosen value of k in (1.7), we have*

$$U_j \geq 0, \quad j = 2, 3, 4, 5, 6, 7,$$

for all $X, Y, Z \in \mathbb{R}^n$.

Proof. For some constants $k_i > 0$, $i = 1, 2$, conveniently chosen later, we have

$$\begin{aligned}
 \langle (1 + \alpha \delta_a^{-1}) Z, H(X) \rangle &= \|k_1 (1 + \alpha \delta_a^{-1})^{\frac{1}{2}} Z + 2^{-1} k_1^{-1} (1 + \alpha \delta_a^{-1})^{\frac{1}{2}} H(X)\|^2 \\
 &\quad - \langle k_1^2 (1 + \alpha \delta_a^{-1}) Z, Z \rangle \\
 &\quad - \langle 4^{-1} k_1^{-2} (1 + \alpha \delta_a^{-1}) H(X), H(X) \rangle
 \end{aligned}$$

and

$$\begin{aligned}
 \langle \delta_a Y, H(X) \rangle &= \|k_2 \delta_a^{\frac{1}{2}} Y + 2^{-1} k_2^{-1} \delta_a^{\frac{1}{2}} H(X)\|^2 \\
 &\quad - \langle k_2^2 \delta_a Y, Y \rangle - \langle 4^{-1} k_2^{-2} \delta_a H(X), H(X) \rangle.
 \end{aligned}$$

On using the assumption that $H(0) = 0$ and the hypothesis (1.5), it follows that

$$\begin{aligned} U_2 = & \|k_1(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}Z + 2^{-1}k_1^{-1}(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}C_h(X, 0)X\|^2 \\ & + \langle Z, (\gamma_2\alpha\delta_a^{-1}\Psi(Y) - k_1^2(1 + \alpha\delta_a^{-1}))Z \rangle \\ & + \langle C_h(X, 0)X, (\eta_2\delta_b(1 - \beta) - 4^{-1}k_1^{-1}(1 + \alpha\delta_a^{-1})C_h(X, 0))X \rangle \end{aligned}$$

and

$$\begin{aligned} U_3 = & \|k_2\delta_a^{\frac{1}{2}}Y + 2^{-1}k_2^{-1}\delta_a^{\frac{1}{2}}C_h(X, 0)X\|^2 \\ & + \langle Y, (\xi_2\delta_a[\Phi(X) - (1 - \beta)\delta_bI] - k_2^2\delta_aI)Y \rangle \\ & + \langle C_h(X, 0)X, (\eta_3\delta_b(1 - \beta) - 4^{-1}k_1^{-1}\delta_aC_h(X, 0))X \rangle. \end{aligned}$$

Thus, using (1.6), (3.1), (3.2) and Lemma 4.1 repeatedly, we obtain for all $X, Z \in \mathbb{R}^n$,

$$U_2 \geq 0,$$

if

$$k_1^2 \leq \frac{\gamma_2\alpha\delta_a}{\alpha + \delta_a}, \quad \text{with } \Delta_c \leq \frac{4\alpha(1 - \beta)\eta_2\gamma_2\delta_a^2\delta_b}{(\alpha + \delta_a)^2},$$

and, for all $X, Y \in \mathbb{R}^n$, $U_3 \geq 0$, if

$$k_2^2 \leq \beta\xi_2\delta_b, \quad \text{with } \Delta_c \leq \frac{4\beta(1 - \beta)\eta_2\eta_3\delta_b^2}{\delta_a}.$$

Hence, combining these inequalities (with Δ_c), we have, for all $X, Y, Z \in \mathbb{R}^n$,

$$U_i \geq 0, \quad i = 2, 3, \text{ if } \Delta_c \leq k\delta_a\delta_b,$$

with

$$k = \min \left\{ \frac{4\alpha(1 - \beta)\eta_2\gamma_2\delta_a}{(\alpha + \delta_a)^2}, \frac{4\beta(1 - \beta)\eta_2\eta_3\delta_b}{\delta_a^2} \right\} < 1.$$

To complete the proof of Lemma 6.1, we need to show that

$$U_i \geq 0, \quad i = 4, 5, 6, 7,$$

for all $X, Y, Z \in \mathbb{R}^n$. By (1.5), the assumption that $H(0) = 0$ and for constants $k_j > 0$, $j = 3, 4, 5, 6$, conveniently chosen later, we have

$$\begin{aligned} & \langle \delta_b(1 - \beta)X, (\Psi(Y) - \delta_aI)Z \rangle \\ = & \|k_3\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Psi(Y) - \delta_aI)^{\frac{1}{2}}X + 2^{-1}k_3^{-1}\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Psi(Y) - \delta_aI)^{\frac{1}{2}}Z\|^2 \\ & - \langle k_3^2\delta_b(1 - \beta)(\Psi(Y) - \delta_aI)X, X \rangle - \langle 4^{-1}k_3^{-2}\delta_b(1 - \beta)(\Psi(Y) - \delta_aI)Z, Z \rangle, \end{aligned}$$

$$\begin{aligned}
 & \langle \delta_b(1 - \beta)(\Phi(X) - \delta_b I)X, Y \rangle \\
 &= \|k_4 \delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}X + 2^{-1}k_4^{-1}\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Y\|^2 \\
 &\quad - \langle k_4^2 \delta_b(1 - \beta)(\Phi(X) - \delta_b I)X, X \rangle - \langle 4^{-1}k_4^{-2}\delta_b(1 - \beta)(\Phi(X) - \delta_b I)Y, Y \rangle, \\
 &\quad \langle (1 + \alpha\delta_a^{-1})(\Phi(X) - \delta_b I)Y, Z \rangle \\
 &= \|k_5(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Y + 2^{-1}k_5^{-1}(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Z\|^2 \\
 &\quad - \langle k_5^2(1 + \alpha\delta_a^{-1})(\Phi(X) - \delta_b I)Y, Y \rangle - \langle 4^{-1}k_5^{-2}(1 + \alpha\delta_a^{-1})(\Phi(X) - \delta_b I)Z, Z \rangle, \\
 &\quad \langle \delta_a(\Psi(Y) - \delta_a I)Y, Z \rangle \\
 &= \|k_6\delta_a^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}Y + 2^{-1}k_6^{-1}\delta_a^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}Z\|^2 \\
 &\quad - \langle k_6^2\delta_a(\Psi(Y) - \delta_a I)Y, Y \rangle - \langle 4^{-1}k_6^{-2}\delta_a(\Psi(Y) - \delta_a I)Z, Z \rangle.
 \end{aligned}$$

Then it follows that

$$\begin{aligned}
 U_4 &= \|k_3\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}X + 2^{-1}k_3^{-1}\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}Z\|^2 \\
 &\quad + \left\langle X, \left(\eta_4\delta_b(1 - \beta)C_h(X, 0) - k_3^2\delta_b(1 - \beta)(\Psi(Y) - \delta_a I) \right) X \right\rangle \\
 &\quad + \left\langle Z, \left(\alpha\gamma_3\delta_a^{-1}\Psi(Y) - 4^{-1}k_3^{-2}\delta_b(1 - \beta)(\Psi(Y) - \delta_a I) \right) Z \right\rangle, \\
 U_5 &= \|k_4\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}X + 2^{-1}k_4^{-1}\delta_b^{\frac{1}{2}}(1 - \beta)^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Y\|^2 \\
 &\quad + \left\langle X, \left(\eta_5\delta_b(1 - \beta)C_h(X, 0) - k_4^2\delta_b(1 - \beta)(\Phi(X) - \delta_b I) \right) X \right\rangle \\
 &\quad + \left\langle Y, \left(\xi_3\delta_a[\Phi(X) - (1 - \beta)\delta_b I] - 4^{-1}k_4^{-2}\delta_b(1 - \beta)(\Phi(X) - \delta_b I) \right) Y \right\rangle, \\
 U_6 &= \|k_5(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Y + 2^{-1}k_5^{-1}(1 + \alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X) - \delta_b I)^{\frac{1}{2}}Z\|^2 \\
 &\quad + \left\langle Y, \left(\xi_4\delta_a[\Phi(X) - (1 - \beta)\delta_b I] - k_5^2(1 + \alpha\delta_a^{-1})(\Phi(X) - \delta_b I) \right) Y \right\rangle \\
 &\quad + \left\langle Z, \left(\alpha\gamma_4\delta_a^{-1}\Psi(Y) - 4^{-1}k_5^{-2}(1 + \alpha\delta_a^{-1})(\Phi(X) - \delta_b I) \right) Z \right\rangle
 \end{aligned}$$

and

$$\begin{aligned}
 U_7 &= \|k_6\delta_a^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}Y + 2^{-1}k_6^{-1}\delta_a^{\frac{1}{2}}(\Psi(Y) - \delta_a I)^{\frac{1}{2}}Z\|^2 \\
 &\quad + \left\langle Y, \left(\xi_5\delta_a[\Phi(X) - (1 - \beta)\delta_b I] - k_6^2\delta_a(\Psi(Y) - \delta_a I) \right) Y \right\rangle \\
 &\quad + \left\langle Z, \left(\alpha\gamma_5\delta_a^{-1}\Psi(Y) - 4^{-1}k_6^{-2}\delta_a(\Psi(Y) - \delta_a I) \right) Z \right\rangle.
 \end{aligned}$$

We then obtain the following using the estimates (1.6), (3.1), (3.2) and Lemma 4.1 repeatedly. For all $X, Z \in \mathbb{R}^n$,

$$U_4 \geq 0, \quad \text{if} \quad \frac{(1 - \beta)\delta_b(\Delta_a - \delta_a)}{4\alpha\eta_3} \leq k_3^2 \leq \frac{\eta_4\delta_c}{\Delta_a - \delta_a}.$$

For all $X, Y \in \mathbb{R}^n$,

$$U_5 \geq 0, \quad \text{if} \quad \frac{(1 - \beta)(\Delta_b - \delta_b)}{4\beta\xi_3\delta_a} \leq k_4^2 \leq \frac{\eta_5\delta_c}{\Delta_b - \delta_b}.$$

For all $Y, Z \in \mathbb{R}^n$,

$$U_6 \geq 0, \quad \text{if } \frac{(1 + \alpha\delta_a^{-1})(\Delta_b - \delta_b)}{4\alpha\gamma_4} \leq k_5^2 \leq \frac{\beta\xi_4\delta_a\delta_b}{(1 + \alpha\delta_a^{-1})(\Delta_b - \delta_b)}.$$

For all $Y, Z \in \mathbb{R}^n$,

$$U_7 \geq 0, \quad \text{if } \frac{\delta_a(\Delta_a - \delta_a)}{4\alpha\gamma_5} \leq k_6^2 \leq \frac{\beta\xi_5\delta_a\delta_b}{\delta_a(\Delta_a - \delta_a)}.$$

The proof of Lemma 6.1 is now complete. $\square$

We are now left with the estimates U_1 and U_8 .

From (6.1), we clearly have

$$(6.2) \quad \begin{aligned} U_1 &\geq (1 - \beta)\eta_1\delta_b\delta_c\|X\|^2 + \beta\xi_1\delta_a\delta_b\|Y\|^2 + \alpha\gamma_1\|Z\|^2 \\ &\geq \delta_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \end{aligned}$$

where $\delta_7 = \min\{(1 - \beta)\eta_1\delta_b\delta_c, \beta\xi_1\delta_a\delta_b, \alpha\gamma_1\}$.

For the remaining part of the proof of (5.1), let us for convenience denote $\|X\|^2 + \|Y\|^2 + \|Z\|^2$ by ψ^2 . Since $P(t, X, Y, Z)$ satisfies (3.3), Schwarz's inequality gives U_8 ,

$$(6.3) \quad \begin{aligned} |U_8| &\leq \left((1 - \beta)\delta_b\|X\| + (1 + \alpha\delta_a^{-1})\|Z\| + \delta_a\|Y\| \right) \|P(t, X, Y, Z)\| \\ &\leq \sqrt{3}\delta_8 \left(\delta_0\psi^2 + \theta_2(t)\psi^{1+\rho} + \theta_1(t)\psi \right), \end{aligned}$$

where $\delta_8 = \max\{(1 - \beta)\delta_b, \delta_a, 1 + \alpha\delta_a^{-1}\}$.

Now, combining (6.1) with inequalities (6.2), (6.3), we obtain

$$\dot{W} \leq -(\delta_7 - \sqrt{3}\delta_8\delta_0)\psi^2 + \sqrt{3}\delta_8 \left(\theta_2(t)\psi^{1+\rho} + \theta_1(t)\psi \right).$$

This we can rewrite as

$$(6.4) \quad \dot{W} \leq -\delta_9\psi^2 + \psi_1 + \psi_2,$$

where

$$3\delta_9 = \delta_7 - \sqrt{3}\delta_8\delta_0, \quad \psi_1 = \{\delta_{10}\theta_1(t) - \delta_9\psi\}$$

and

$$\psi_2 = \delta_{10}\theta_2(t)\psi^{1+\rho} - \delta_9\psi^2.$$

If we choose δ_0 small enough such that $\delta_9 > 0$ (following [7, page 306]), with the necessary modification, we obtain

$$\psi_1 \leq \delta_{10}\psi^{2(1-\nu)}\theta_1^{2\nu}(t)$$

and

$$\psi_2 \leq \delta_{11}\psi^{2(1-\nu)}\theta_2^{\frac{2\nu}{1-\rho}}(t),$$

for any constant ν in the range $\frac{1}{2} \leq \nu \leq 1$.

Thus, (6.4) reduces to

$$\dot{W} \leq -\delta_9\psi^2 + \delta_{12} \left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t) \right) \psi^{2(1-\nu)},$$

with $\delta_{12} = \max\{\delta_{10}, \delta_{11}\}$.

This is (5.1) with $\delta_3 = \delta_9$ and $\delta_4 = \delta_{12}$.

This completes the proof of Theorem 3.1.

7. EXAMPLE

Consider (1.1) of the form

$$(7.1) \quad \ddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in \mathbb{R}^2,$$

with

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \Psi(\dot{X}) = \begin{pmatrix} 3 + \frac{1}{1+x_1^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(X) = \begin{pmatrix} 0.00004 + \frac{1}{1+x_1^2} & 0 \\ 0 & 1 \end{pmatrix},$$

$$H(X) = \begin{pmatrix} 0.001 \tan^{-1} x_1 + 0.0001 x_1 \\ 0.0001 x_2 \end{pmatrix}, \quad P(t) = \begin{pmatrix} e^{-t} \\ \sin t \end{pmatrix},$$

where e^{-t} , $\sin t$ are bounded continuous functions on $[0, +\infty)$. A simple calculation (with the earlier notations) gives $\lambda_1(\Psi(\dot{X})) = 1$, $\lambda_2(\Psi(\dot{X})) = 3 + \frac{1}{1+x_1^2}$, $\lambda_1(\Phi(X)) = 1$, $\lambda_2(\Phi(X)) = 0.00004 + \frac{1}{1+x_1^2}$ and $C_h(X, 0) = \begin{pmatrix} 0.0001 + \frac{0.0001}{1+x_1^2} & 0 \\ 0 & 0.0001 \end{pmatrix}$, $\lambda_1(C_h(X, 0)) = 0.0001$, $\lambda_2(C_h(X, 0)) = 0.0001 + \frac{0.0001}{1+x_1^2}$. Following Theorem 3.1, $\delta_a = 1$, $\Delta_a = 3$, $\delta_b = 1$, $\Delta_b = 1.00004$, $\delta_c = 0.0001$, $\Delta_c = 0.0011$. If we choose $\alpha = 3$, $\beta = \frac{1}{2}$, $\gamma_3 = \eta_2 = \eta_3 = \frac{1}{5}$, we obtain $k = \min\{0.015, 0.04\} = 0.015 < 1$. Since $\Delta_c = 0.0011 < 0.015 = k\delta_a\delta_b$, then all the conditions of Theorem 3.1 are satisfied. Thus the solutions of (7.1) are ultimately bounded.

8. CONCLUSION

This paper investigates the ultimate boundedness of solutions of some third-order nonlinear differential equations. By constructing a quadratic-like function (also known as Lyapunov function) and using the Lyapunov second (direct) method, sufficient conditions which guarantee that solutions are ultimately bounded are established. A particular example has been provided to demonstrate results obtained. Results obtained in this paper revise and improve on those in the literature.

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RECONSTRUCTING THE CHARACTERISTIC (PERMANENTAL) POLYNOMIAL OF A DIGRAPH FROM SIMILAR POLYNOMIALS OF ITS ARC-DELETED SUBGRAPHS

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ABSTRACT. Let $D = D(V, E)$ be an arbitrary digraph with the set V of vertices and the set E of arcs ($|V| = n; |E| = m$); loops, if any, are considered reduced arcs with the same head and tail. The *characteristic polynomial* $\phi^-(D; x)$ (*resp. permanental polynomial* $\phi^+(D; x)$) of D is the characteristic (permanental) polynomial of its adjacency matrix A : $\phi(D; x) := \det(xI - A)$ ($\phi^+(D; x) := \text{per}(xI + A)$), where I is an identity matrix. A t -arcs-deleted subgraph D_t of D is the digraph D less exactly t arcs (while all n vertices are preserved). Also, let $\mathcal{D}_t$ and $R_t^-(D; x)$ ($R_t^+(D; x)$) be the collection (multiset) of all t -arc-deleted subgraphs of D and the sum of the characteristic (permanental) polynomials of all subgraphs from $\mathcal{D}_t$, respectively. We consider the reconstruction of the characteristic polynomial $\phi^-(D; x)$ (permanental polynomial $\phi^+(D; x)$) of D from the polynomial sum $R_t^-(D; x)$ ($R_t^+(D; x)$), $t \in \{1, 2, \dots, m - n + n_0\}$, where n_0 is the number of zero roots of $\phi^-(D; x)$ ($\phi^+(D; x)$). Then, we also carry over our reasoning to the case of reconstructing both polynomials of undirected graphs (where edges are deleted).

1. PRELIMINARIES AND THE MAIN PART

The multifaceted topic of reconstructing graph polynomials has long attracted the attention of graphists. It complements the topic of reconstructing the graphs themselves and, probably, in some cases, can help to carry out such a reconstruction. Here, we will not consider the general state of that issue, which we leave for a separate literature review; the interested reader can find quite voluminous information on the topic in [1–9]. We will fully devote ourselves to considering a specific problem,

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referring only to the information directly necessary to solve it. Each square matrix $A = [a_{rs}]_{r,s=1}^n$ is the adjacency matrix of a (weighted) finite (di)graph G , where an entry a_{rs} ($r, s \in \{1, 2, \dots, n\}$) is the weight of an arc $\overrightarrow{rs}$ emanating from vertex r and heading to vertex s . The case $r = s$ corresponds to a loop (a reduced arc whose endpoints coincide); and $a_{rs} = 0$ means that the respective arc does not exist in G . If A is symmetric, with $a_{rs} = a_{sr}$ for all pairs of indices r and s , G can be regarded also as an undirected graph having nonoriented edges $rs = sr$ instead of pairs of opposite arcs $\overrightarrow{rs}$ and $\overleftarrow{sr}$. The case when all nonzero entries of A are equal to 1 corresponds to an unweighted graph G . In general, an entry a_{rs} may be an arbitrary quantity (complex number, algebraic expression, etc.). Herein, we plan to practically consider clearly recognizable cases of graphs or digraphs. In doing so, we will use, without any indication, information that is equally relevant for all graphs in our text.

A vertex-deleted subgraph of a graph G is a subgraph $G - v$ obtained by deleting the vertex v and all edges incident to it from G ; similarly, an edge-deleted (arc-deleted) subgraph of G is a subgraph $G - uv$ ($G - \overrightarrow{uv}$) obtained by deleting the edge uv (arc $\overrightarrow{uv}$), while its end vertices and all other edges (arcs) incident to it are preserved as they are and were in G . This deletion of uv ($\overrightarrow{uv}$) is sometimes called weak deletion of the edge uv (arc $\overrightarrow{uv}$); while the deletion of uv ($\overrightarrow{uv}$) with end vertices and all the edges incident to it is called strong deletion of the edge uv (arc $\overrightarrow{uv}$). In the latter case, the resulting subgraph, denoted by $G - u - v$, is the graph G less the pair u and v of its adjacent vertices ($u \sim v$) and all edges (arcs) incident to them. The collection (in general, a multiset) of all subgraphs $G - v$ ($G - \overrightarrow{uv}$, $G - uv$, $G - u - v$) of the graph G is called a deck and any single element of the deck is called a card. We refer to these four decks as $\mathcal{D}$, $\mathcal{AD}$, $\mathcal{WD}$, and $\mathcal{SD}$, respectively. In our text, we will need two more general decks: $\mathcal{D}_t$ of subgraphs with t ($t = 0, 1, \dots$) vertices deleted and $\mathcal{AD}_t$ subgraphs with t arcs deleted ($\mathcal{D}_0 = \mathcal{D}$; $\mathcal{AD}_0 = \mathcal{AD}$).

The characteristic polynomial $\phi^-(G; x)$ (resp. permanental polynomial $\phi^+(G; x)$) of a (di)graph G is the characteristic polynomial $\phi^-(A; x)$ (permanental polynomial $\phi^+(A; x)$) of its adjacency matrix $A = [a_{u,v}]_{u,v=1}^n$ [10]:

$$(1.1) \quad \begin{aligned} \phi^-(G; x) &:= \phi^-(A; x) = \det(xI - A) = \sum_{k=0}^n c_k^- x^{n-k} = \sum_{k=0}^{n-n_0} c_k^- x^{n-k} \quad (c_0^- = 1), \\ \phi^+(G; x) &:= \phi^+(A; x) = \text{per}(xI + A) = \sum_{k=0}^n c_k^+ x^{n-k} = \sum_{k=0}^{n-n_0} c_k^+ x^{n-k} \quad (c_0^+ = 1), \end{aligned}$$

where I is a diagonal identity matrix of the corresponding dimension; and n_0 is the number of zero roots of $\phi^-(A; x)$ or $\phi^+(A; x)$, respectively.

In what follows, we will use the combined notation $\phi^\pm(D; x)$ (and other $(\pm)$ -notation) wherever it is equally applicable both to the considered characteristic and to permanent polynomials. Hereby we mean a common form of notation, but not equality of results for the corresponding $(-)$ and $(+)$ cases.

In this paper, we demonstrate that the polynomial $\phi^\pm(D; x)$ of an arbitrary digraph D (with $m \geq n - n_0^\pm$, where $n_0^\pm$ is the number of zero roots of $\phi^\pm(D; x)$) is reconstructible from the following polynomial:

$$R_t^\pm(D; x) = \sum_{D_t \in \mathcal{AD}_t(D)} \phi^\pm(D_t; x) = \sum_{k=0}^n r_{t;k}^\pm x^{n-k} \quad (0 \leq t \leq m - n + n_0^\pm),$$

where $D_t := D - \overrightarrow{u_1 v_1} - \overrightarrow{u_2 v_2} - \cdots - \overrightarrow{u_t v_t}$ is an arbitrary subgraph of D obtained by weakly deleting its t arbitrary arcs; and the sum ranges over all deck $\mathcal{AD}_t(D)$ of D .

In order not to leave undirected graphs aside from our reasoning, we will introduce the following terminology. A symmetric digraph $S(G)$ of an undirected graph G (having n vertices and q edges) is obtained by substituting a pair of opposite arcs for every edge in G . We define $B = [b_{\alpha\beta}]_{\alpha, \beta=1}^{2q}$ to be the $2q \times 2q$ matrix with row and columns indexed by the set of arcs of $S(G)$ as follows:

$$(1.2) \quad b_{\alpha\beta} = b(\alpha = (u, v); \beta = (x, y)) = \begin{cases} 1, & v = x \text{ and either } y = u \text{ or } y \neq u, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix B is the adjacency matrix of a derivative digraph $\Gamma(G) = \Gamma[S(G)]$, whose vertex set is the set of all $2q$ arcs of the symmetric digraph $S(G)$, while the adjacency of vertices is defined by (1.2). The digraph $\Gamma(G)$ is called in [11, 12] the line graph of a directed graph $S(G)$ and is called the arc-graph of (undirected) graph G in [13]. In the latter case, the prefix (or adjective) “arc” makes it possible to directly connect this term with the original undirected graph G , without referring to the auxiliary digraph $S(G)$. In what follows, $D = S(G)$ will be automatically considered by us as a special case of an arbitrary digraph D (with $m = 2q$ arcs).

An interesting spectral result concerning the arc-graph Γ of a digraph D is the following theorem [11–13].

Theorem 1.1. *Let $\phi^-(\Gamma; x)$ be the characteristic polynomial of the arc-graph $\Gamma(G)$ of a digraph D . Then,*

$$(1.3) \quad \phi^-(\Gamma; x) = x^{m-n} \phi^-(D; x) = \sum_{k=0}^n c_k^- x^{m-k},$$

where n is the number of vertices, and m is the number of arcs of a digraph D (loops, if any, are also considered reduced arcs).

Remark 1.1. The general version of this theorem (see [11–13]) for the characteristic polynomials remains true for an arbitrary (di)graph H instead of D , possibly with (weighted) loops and (weighted) arcs or edges (having an arbitrary matrix M as its adjacency matrix A in (1.1)). However, Theorem 1.1 cannot be generalized to the case of the permanent polynomials. It is easy to consider the case $D = S(G)$, where G is an undirected graph with $n > 2$ vertices. Then, $m(D) > n(D)$ and $\phi^-[\Gamma(D); x]$ is divisible by x^{m-n} , i.e., has at least $m - n$ zero roots. However, unlike the previous case, $\phi^+[\Gamma(D); x]$ has no zero roots; see Proposition 6 of [13], taking into account that $S(G)$ is an Eulerian digraph.

We also note an important feature of the structure $\Gamma(D)$, which allows us to reconstruct the original digraph D using the adjacency matrix B of $\Gamma(D)$. If we enumerate all the arcs of the digraph D in such a way that the numbers of arcs entering one common vertex of D follow one after another, then we get the matrix B divided into blocks. These blocks are either blocks of all zeros or contain exactly one column of all ones. Further, if we replace each zero block by the number zero, and each block containing ones by the number one, then we get a matrix that is exactly the adjacency matrix of the original digraph graph D ; see [13]. Here, we note that in the case $D = S(G)$, the adjacency matrix of the digraph D coincides with that of an undirected graph G ($A(D) = A(G)$). Thus, this algorithm also reconstructs (the adjacency matrix of) G .

The one-to-one correspondence between each digraph and its arc-graph also allows us to consider the arc-graph $\Gamma(D)$ as the result of the action of the operator Γ on the digraph D , which uniquely maps D to Γ . But we also know the algorithm for converting $\Gamma(D)$ back to D , which we can conventionally denote by Γ^{-1} . Thus, we can summarize what was said like this:

$$D \xrightleftharpoons[\Gamma^{-1}]{\Gamma} \Gamma(D).$$

The above correspondence is valid for an arbitrary digraph D , but we will be especially interested here in its particular case:

$$(1.4) \quad (D - \overrightarrow{u_1 v_1} - \overrightarrow{u_2 v_2} \cdots - \overrightarrow{u_t v_t}) \xrightleftharpoons[\Gamma^{-1}]{\Gamma} \Gamma(D) - \alpha_1 - \alpha_2 - \cdots - \alpha_t,$$

where a vertex α_i removed from the arc-graph $\Gamma(D)$ is an arc $\overrightarrow{u_i v_i}$ of the digraph D ($i \in \{1, 2, \dots, t\}$).

From what has been said, we pass to the following technical lemma.

Lemma 1.1. *Let $\Gamma(D)$ be the arc-graph of a digraph D . Then,*

$$\phi^-(\Gamma - \alpha_1 - \alpha_2 - \cdots - \alpha_t; x) = x^{m-n-t} \phi^-(D - \overrightarrow{u_1 v_1} - \overrightarrow{u_2 v_2} - \cdots - \overrightarrow{u_t v_t}; x),$$

where $\alpha_i = \overrightarrow{u_i v_i}$, $\alpha_i \in V[\Gamma(D)]$ and $\overrightarrow{u_i v_i} \in E(D)$.

Proof. It follows from Theorem 1.1 (see (1.3)) and the correspondence (1.4). □

Lemma 1.1 allows us to calculate the following polynomial sum:

$$(1.5) \quad S_t^-[\Gamma(D); x] = \sum_{[\Gamma(D)]_t \in \mathcal{D}_t[\Gamma(D)]} \phi^- \{[\Gamma(D)]_t; x\} = x^{m-n-t} \sum_{D_t \in \mathcal{AD}_t(D)} \phi^-(D_t; x) \\ = x^{m-n-t} R_t^-(D; x) = \sum_{k=0}^n r_{t;k}^- x^{m-t-k},$$

where $[\Gamma(D)]_t := \Gamma - \alpha_1 - \alpha_2 - \cdots - \alpha_t$ and $D_t := D - \overrightarrow{u_1 v_1} - \overrightarrow{u_2 v_2} - \cdots - \overrightarrow{u_t v_t}$.

Remark 1.2. The fact that $S_t^-[\Gamma(D); x] = x^{m-n-t} R_t^-(D; x)$ in (1.5) prompts us to make some “premature” remark, which will be useful to us when we proceed to consider a

similar method for reconstructing the permanent polynomial. As already indicated in the second part of Remark 1, Theorem 1.1 does not work in the case of the permanent polynomial; therefore, a similar equality for $S_t^+[\Gamma(D); x]$ and $x^{m-n-t}R_t^+(D; x)$ does not hold, although both these polynomials exist separately. Therefore, the calculation of $S_t^+[\Gamma(D); x]$ will be absolutely useless to us, and further we will focus on calculating $R_t^+(D; x)$. But we will use the derived expression $x^{m-n-t}R_t^+(D; x)$.

Here, we recall the known result, whose proof, in particular, can be obtained by multiple application of Clarke's theorem (see Theorem 2.14 of Clarke in [10]) with the addition of the factor $1/t!$, which appears due to the fact that there are $t!$ different sequences of deletion t of vertices from a graph.

Theorem 1.2. *Let G be an arbitrary (di)graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$. And let $G_t := G - v_1 - v_2 - \dots - v_t$ be graph G less its t ($t \in \{1, 2, \dots, n\}$) arbitrary vertices $v_1, v_2, \dots, v_t$ and all edges (arcs, loops) incident to it. Then,*

$$\frac{1}{t!} \cdot \frac{d^t}{dx^t} \phi^-(G; x) = \sum_{G_t \in \mathcal{D}_t(G)} \phi(G_t; x),$$

where the sum ranges over all C_n^t cards of the t -th deck $\mathcal{D}_t$ of G .

Corollary 1.1. *Let $G = \Gamma(D)$. Then,*

$$(1.6) \quad S_t^-[\Gamma(D); x] = \sum_{[\Gamma(D)]_t \in \mathcal{D}_t[\Gamma(D)]} \phi^- \{[\Gamma(D)]_t; x\} = \frac{1}{t!} \cdot \frac{d^t}{dx^t} \phi^-[\Gamma(D); x].$$

The last equation allows us to get $\phi^-[\Gamma(D); x]$ in integral form:

$$(1.7) \quad \phi^-[\Gamma(D); x] = t! \int \dots \int_{t \text{ times}} S_t^-[\Gamma(D); x] dx^t \quad (0 \leq t \leq m - n + n_0^-),$$

where n_0^- is the number of zero roots of $\phi^-[\Gamma(D); x]$ (if n_0^- is not known, use $t \leq m - n$); and the integration at each of the t steps uses the zero integration constant (since the coefficients $r_{t;k}^-$ of x^{m-t-k} in (1.7) must obey their determination in (1.5)).

The equation (1.6) can also be solved for $\phi^-[\Gamma(D); x]$ by comparing the coefficients at the same powers of x in the corresponding polynomials. First, using the last parts (1.5) and (1.6) for an equivalent representation of $S_t[\Gamma(D); x]$, we get

$$\begin{aligned} S_t^-[\Gamma(D); x] &= \sum_{k=0}^n r_{t;k}^- x^{m-t-k} = \frac{1}{t!} \cdot \frac{d^t}{dx^t} \phi^-[\Gamma(G); x] \\ &= \frac{1}{t!} \cdot \frac{d^t}{dx^t} \left[x^{m-n} \sum_{k=0}^n c_k^- x^{n-k} \right] = \frac{1}{t!} \cdot \frac{d^t}{dx^t} \sum_{k=0}^n c_k^- x^{m-k}, \end{aligned}$$

where the coefficients c_k^- belong to the expansion $\phi^-(D; x) = \sum_{k=0}^n c_k^- x^{n-k}$. Hence,

$$\begin{aligned} \sum_{k=0}^n r_{t;k}^- x^{m-t-k} &= \frac{1}{t!} \cdot \frac{d^t}{dx^t} \phi^-[\Gamma(G); x] = \sum_{k=0}^n c_k^- \frac{(m-k)}{t!(m-k-t)!} x^{m-t-k} \\ &= \sum_{k=0}^n c_k^- C_{m-k}^t x^{m-t-k}, \end{aligned}$$

which makes it easy to compare the coefficients of the first and last sums therein:

$$c_k^- = \frac{r_{t;k}^-}{C_{m-k}^t} \quad (k \in \{0, 1, \dots, n\}, 0 \leq t \leq m - n + n_0^-).$$

At this point, we arrive at the following statement (which will later be generalized later to the general case, which also includes the permanental polynomial).

Lemma 1.2. *Let $\phi^-(D; x) = \sum_{k=0}^n c_k^- x^{n-k}$ and $R_t^-(D; x) = \sum_{k=0}^n r_{t;k}^- x^{n-t-k}$ ($0 \leq t \leq m - n + n_0^-$) be the characteristic polynomial of a digraph D and the sum of the characteristic polynomials of all its t -arcs-deleted subgraphs (from $\mathcal{AD}_t$), respectively. Then, the characteristic polynomial $\phi^-(D; x)$ is reconstructible from (the coefficients of) the polynomial $R_t^-(D; x)$ as follows*

$$\phi^-(D; x) = \sum_{k=0}^n \frac{r_{t;k}^-}{C_{m-k}^t} x^{n-k} = \frac{t!}{x^{m-n}} \int \cdots \int_{t \text{ times}} x^{m-n-t} R_t^-(D; x) dx^t \quad (0 \leq t \leq m - n + n_0^-),$$

where the integration at each of the t steps uses the zero integration constant.

Lemma 1.2 can be considered as our final result for the characteristic polynomial $\phi^-(D; x)$ of a directed graph D . Now it remains for us to show that a similar result is also valid for a permanent polynomial $\phi^+(D; x)$. (It is “only” necessary to replace $(-)$ by $(+)$ everywhere in (1.8)).

First, it is important to remember what combinatorial meaning an arbitrary coefficient c_k^+ has in the expansion of a permanent polynomial, $\phi^+(D; x) = \sum_0^n c_k^+ x^{n-k}$. Namely, the coefficient c_k^+ is equal to the number of all coverings by oriented simple p -cycles ($p \in \{1, 2, \dots, k\}$) of exactly k vertices of the digraph D , where 1-cycle and 2-cycle are a loop and a pair of opposite arcs with common endpoints, respectively. See a discussion of the coefficients of the “simple permanent polynomial” of a directed graph, e.g., on page 193 of [14]. But in each directed simple p -cycle, the number of arcs is equal to the number of vertices, p . Therefore, the coefficient c_k^+ is also equal to the number of coverings exactly k of arcs of the digraph D . We need the following lemma.

Lemma 1.3. *Let $R_1^+(D; x) = \sum_{(D-\vec{uv}) \in \mathcal{AD}_1} \phi^+(D-\vec{uv}; x)$ be the sum of the permanent polynomials of all m subgraphs $D-\vec{uv}$ obtained by weakly deleting a single arc $\vec{uv}$*

from D . Then,

$$(1.9) \quad R_1^+(D; x) = \sum_{k=0}^n (m-k)c_k^+ x^{n-k} = x^{n-m+1} \frac{d}{dx} \left[x^{m-n} \phi^+(D; x) \right] \quad (m-n+n_0^+ \geq 1),$$

where n_0^+ is the number of zero roots of $\phi^+(D; x)$ (if n_0^+ is not known, use $m-n \geq 1$).

Proof. Consider an arbitrary covering of k arcs (and vertices) of the digraph D by oriented cycles consisting of arcs $\overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2}, \dots, \overrightarrow{u_k v_k}$, where the head of each arc coincides with the tail of exactly one other arc, which is not specified here. Remove an arbitrary arc $\overrightarrow{uv}$ from the digraph D . Obviously, if this is not one of the arcs belonging to the cover under consideration, then this cover can also be realized in the resulting subgraph $D - \overrightarrow{uv}$, although other covers including $\overrightarrow{uv}$ in D , become impossible. If we consider the complete deck $\mathcal{AD}_1$ of all m one-arc-deleted subgraphs (cards), then among them we will find exactly k subgraphs in which our concrete cover cannot be realized. Since we have considered an arbitrary covering of arbitrary k arcs of the digraph D , we can generalize what has been said to the general case of all such cycle coverings of D . As a result, we can represent the total loss of coverings by all cards in the complete deck $\mathcal{AD}_1$ as the following polynomial, whose coefficients give us the numerical loss of all cycle coverings of the corresponding number of k ($k \in \{1, 2, \dots, n\}$) of arcs of D :

$$\delta^+(D, x) := \sum_{k=0}^n k c_k^+ x^{n-k}.$$

Using (1.9), we get

$$R_1^+(D; x) = \sum_{(D-\overrightarrow{uv}) \in \mathcal{AD}_1(D)} \phi^+(D - \overrightarrow{uv}; x) = m\phi^+(D; x) - \delta^+(D; x) = \sum_{k=0}^n (m-k)c_k^+ x^{n-k}.$$

Thence,

$$\begin{aligned} R_1^+(D; x) &= \sum_{k=0}^n (m-k)c_k^+ x^{n-k} = x^{n-m+1} \frac{d}{dx} \sum_{k=0}^n c_k^+ x^{m-k} \\ &= x^{n-m+1} \frac{d}{dx} \left[x^{m-n} \phi^+(D; x) \right] \quad (m-n+n_0^+ \geq 1), \end{aligned}$$

which completes the proof. $\square$

The following statement plays an essential role in our reasoning.

Lemma 1.4. Let $R_t^+(D; x) = \sum_{D_t \in \mathcal{AD}_t} \phi^+(D_t; x)$ be the sum of the permanent polynomials of all C_m^t subgraphs D_t obtained by weakly deleting t ($t \in \{0, 1, \dots, m-n+n_0^+\}$) arcs from D . Then,

$$(1.10) \quad R_t^+(D; x) = \sum_{k=0}^n C_{m-k}^t c_k^+ x^{n-k} = \frac{x^{n-m+t}}{t!} \cdot \frac{d^t}{dx^t} \left[x^{m-n} \phi^+(D; x) \right] \quad (0 \leq t \leq m-n+n_0^+).$$

Proof. It can be obtained by t -fold application of Lemma 1.3. In this case, as in the case of Theorem 1.2, the multiplier $1/t!$ appears before the differential, since there are $t!$ possibilities of sequential selection of t elements one by one, but we only need one choice. (For a short check, one can consider the first coefficient C_m^t in the expansion of $R_+(D; x)$ in powers of x .) We will mainly focus on the more important part of the proof regarding the first equality in (1.10), while the second equality there is elementarily proved by simple manipulations with the coefficients. By Lemma 1.3, we have $R_1^+(D; x) = \sum_{k=0}^n (m-k)c_k^+ x^{n-k}$, where m is the number of all arcs of the original digraph $D_0 := D$ with 0 deleted arcs; and to unify subsequent entries, we can formally write $R_0^+(D; x) \equiv \phi^+(D; x)$, which is the initial term in the sequence $R_0^+, R_1^+, \dots, R_t^+$. Thus, each stage of sequential calculating of polynomial sums $R_1^+(D; x), R_2^+(D; x), \dots, R_t^+(D; x)$ for decks $\mathcal{AR}_1, \mathcal{AR}_2, \dots, \mathcal{AR}_t$, respectively, means sequential multiplication of the original coefficient c_k^+ ($k \in \{0, 1, \dots, n\}$), of $R_0^+ = \phi^+(D; x)$, first by $(m-k)/1$, then by $(m-k-1)/2$, and so on up to the last multiplier $(m-k-t+1)/t$ in the process, to result in the coefficient

$$r_{t;k} = (t!)^{-1}(m-k)(m-k-1) \cdots (m-k-t+1)c_k^+ = C_{m-k}^t c_k^+$$

of the polynomial $R_t^+(D; x) = \sum_{k=0}^n r_{t;k} x^{n-k}$.

Based on this, we get

$$R_t^+(D; x) = \sum_{k=0}^n C_{m-k}^t c_k^+ x^{n-k},$$

which proves the first equality in (1.10) and, thus, the main part of our statement. It is technically easy to see that the third part of (1.10) is also equal to the same polynomial $R_t^+(D; x)$:

$$\begin{aligned} & \frac{x^{n-m+t}}{t!} \frac{d^t}{dx^t} [x^{m-n} \phi^+(D; x)] = \frac{x^{n-m+t}}{t!} \frac{d^t}{dx^t} \left[\sum_0^n c_k^+ x^{m-k} \right] \\ &= \frac{x^{n-m+t}}{t!} \left[\sum_0^n (m-k)(m-k-1) \cdots (m-k-t+1) c_k^+ x^{m-k} \right] \\ &= \sum_{k=0}^n \frac{(m-k)!}{(t!)(m-k-t)!} c_k^+ x^{n-k} = \sum_{k=0}^n C_{m-k}^t c_k^+ x^{n-k} = R_t^+(D; x) = \sum_{k=0}^n r_{t;k}^+ x^{n-k}, \end{aligned}$$

which completes the proof. □

Now we state a generalizing theorem.

Theorem 1.3. Let $\phi^\pm(D; x) = \sum_{k=0}^n c_k^\pm x^{n-k}$ and $R_t^\pm(D; x) = \sum_{k=0}^n r_{t;k}^\pm x^{n-t-k}$ ($0 \leq t \leq m-n+n_0^\pm$) be the characteristic $(-)$ (permanental $(+)$) polynomial of a digraph D and the sum of the characteristic (permanental) polynomials of all its t -arcs-deleted subgraphs (from $\mathcal{AD}_t$), respectively. Then, the polynomial $\phi^\pm(D; x)$ is reconstructible

from (the coefficients of) the polynomial $R_t^\pm(D; x)$ as follows
(1.11)

$$\phi^\pm(D; x) = \sum_{k=0}^n \frac{r_{t;k}^\pm}{C_{m-k}^t} x^{n-k} = \frac{t!}{x^{m-n}} \int \cdots \int_{t \text{ times}} x^{m-n-t} R_t^\pm(D; x) dx^t \quad (0 \leq t \leq m-n+n_0^\pm),$$

where the integration at each of the t steps uses the zero integration constant.

Proof. The $(-)$ -case has been proven in Lemma 1.2. Now, note that it follows from the last two equalities in (1.11) that

$$(1.12) \quad c_k^+ = \frac{r_{t;k}^+}{C_{m-k}^t} \quad (0 \leq t \leq m-n+n_0^+),$$

which is a $(+)$ -analog of (1.8). Whence we arrive at the overall proof. $\square$

Remark 1.3. All practical applications of Theorem 1.3 (and Lemma 1.2) are related to the values of $t \geq 1$. The last condition can always be satisfied for the case $m-n = -1$, since it corresponds to the oriented tree $\vec{T}$ ($m \geq 2$), whose polynomial $\phi^\pm(\vec{T}; x) \equiv x^n$ has $n_0^\pm = n$ zero roots and allows its formal reconstruction up to the values $t = m-n+n = n-1$. For $m = n$, when an arbitrarily oriented digraph D contains exactly one cycle of length $c < n$, $1 \leq t \leq n-c$; in this case, we can also reconstruct the polynomial $\phi^\pm(D; x)$ (for valid values of t). But in the exceptional case, when D is a consistently oriented cycle, the reconstruction of its polynomial $\phi^\pm(D; x)$ is impossible, since $\phi^\pm(D; x) = x^n \pm 1$, and $m-n+n_0^\pm = m-n+0 = 0 < 1$. For all $m-n \geq 1$, Theorem 1.3 (Lemma 1.2) works for at least $t = 1$. Thus, the polynomial $\phi^\pm(D; x)$ of a consistently oriented cycle remains the only case when its reconstruction using Theorem 1.3 is impossible.

Now we want to move our reasoning to the area of undirected graphs. Earlier, we have already dealt with the problem of recursion of the characteristic $\phi^-(G; x)$ and the permanent $\phi^+(G; x)$ polynomials [15, 16] of the undirected graph G . We use two formulae [15, 16], in which we are now correcting typos made in [16]:

$$(1.13) \quad \begin{aligned} (q-n)\phi^-(G; x) &= \sum_{uv} [\phi^-(G-uv) + \phi^-(G-u-v; x)] - x(d/dx)\phi^-(G; x), \\ (q-n)\phi^+(G; x) &= \sum_{uv} [\phi^+(G-uv) - \phi^+(G-u-v; x)] - x(d/dx)\phi^+(G; x), \end{aligned}$$

where n and q are the numbers of vertices and edges, of G , respectively, and the combined summation ranges over the set of all edges of G and all pairs u and v of adjacent vertices ($u < v; u \sim v$).

We combine these formulae and transform them like what we did before

$$(1.14) \quad (q-n)\phi^\pm(G; x) + x(d/dx)\phi^\pm(G; x) \sum_{uv} [\phi^\pm(G-uv; x) \mp \phi^\pm(G-u-v; x)] \\ = \sum_{k=0}^n g_{1;k}^\pm x^{n-k} = U_1^\pm(G; x),$$

where the coefficients $g_k^\pm$ should be known by recursion. Further, transforming the first side of (1.14), we obtain

$$(1.15) \quad \sum_{k=0}^n [(q-n) + (n-k)] c_k^\pm x^{n-k} = \sum_{k=0}^n (q-k) c_k^\pm x^{n-k} = x^{n-q+1} (d/dx) [x^{q-n} \phi^\pm(G; x)] \\ = \sum_{k=0}^n g_{1;k}^\pm x^{n-k},$$

whence we arrive at the “undirected” generalization of Lemma 1.4.

Theorem 1.4. *Let $U_1^\pm(G; x) = \sum_{k=0}^n g_k^\pm x^{n-k}$ be the sum of the polynomials $\phi^\pm(\cdot; x)$ of all “weak” subgraphs $G-uv$ and all “strong” subgraphs $G-u-v$ ($u < v; u \sim v$) of G . Then,*

$$(1.16) \quad \phi^\pm(G; x) = \sum_{k=0}^n c_k^\pm x^{n-k} = \sum_{k=0}^n \frac{g_{1;k}^\pm}{q-k} x^{n-k} = x^{n-q} \int x^{q-n-1} U_1^\pm(G; x) dx \quad (q > n),$$

where the integration uses the zero integration constant.

Proof. The second equality in (1.16) is related to the comparison of the second and fourth sides of (1.15), while the third equality in (1.16) is a purely technical fact. $\square$

The following corollary allows us to equate two approaches to undirected graphs G – as such and as their symmetric directed equivalents $S(G)$.

Corollary 1.2. *Let $R_1^\pm[S(G); x] = \sum_{k=0}^n r_{1;k}^\pm x^{n-k}$ and $U_1^\pm(G; x) = g_{1;k}^\pm x^{n-k}$ (as above). Then,*

$$(1.17) \quad g_{1;k}^\pm = \frac{q-k}{m-k} r_{1;k}^\pm \quad (k \in \{0, 1, \dots, n\}, m = 2q),$$

whence

$$(1.18) \quad U_1^\pm(G; x) = x^{n-q+1} \frac{d}{dx} \left(\frac{1}{x^q} \int x^{2q-n-1} R_1^\pm[S(G); x] dx \right)$$

and

$$(1.19) \quad R_1^\pm[S(G); x] = x^{n-2q+1} \frac{d}{dx} \left(x^q \int x^{q-n-1} U_1^\pm(G; x) dx \right).$$

Proof. The mutual relation (1.17) of the coefficients follows from (1.12) and the second side (1.15). The former gives, for $t = 1$, $c_k^\pm = r_{1;k}^\pm / C_{m-k}^1 = r_{1;k}^\pm / (m-k)$. Substituting

the obtained expression for $c_k^\pm$ on the second side of (1.15) and equating the result to the last part of (1.15), we obtain:

$$(1.20) \quad \sum_{k=0}^n (q-k)c_k^\pm x^{n-k} = \sum_{k=0}^{\pm} \frac{q-k}{m-k} r_{1;k} x^{n-k} = \sum_{k=0}^n g_{1;k}^\pm x^{n-k},$$

from which, comparing the coefficients of x^{n-k} on the last two sides of (1.20), we arrive at the proof of the first part of our statement, expressed by (1.17).

Integral expressions (1.18) and (1.19), consistent with (1.17), can be obtained using parts of expressions (1.11) and (1.16) used by theorems 1.3 and 1.4, respectively. Prove the first of them, (1.18). First, we equate the last side of (1.16) to the third side of (1.11), assuming that $D = S(G)$ and $t = 1$ in it:

$$(1.21) \quad \begin{aligned} \phi^\pm(G; x) &= \left(x^{n-q} \int x^{q-n-1} U_1^\pm(G; x) dx = \frac{1}{x^{m-n}} \int x^{m-n-1} R_t^\pm[S(G); x] dx \right) \\ &= \phi^\pm[S(G); x]. \end{aligned}$$

Starting from the central equality of (1.21), enclosed in brackets, we will carry out the following sequence of its technical transformations:

$$(1.22) \quad \begin{aligned} \int x^{q-n-1} U_1^\pm(G; x) dx &= \frac{1}{x^{m-q}} \int x^{m-n-1} R_t^\pm[S(G); x] dx, \\ x^{q-n-1} U_1^\pm(G; x) &= \frac{d}{dx} \left(\frac{1}{x^{m-q}} \int x^{m-n-1} R_t^\pm[S(G); x] dx \right), \end{aligned}$$

$$(1.23) \quad U_1^\pm(G; x) = x^{n-q+1} \frac{d}{dx} \left(\frac{1}{x^{m-q}} \int x^{m-n-1} R_t^\pm[S(G); x] dx \right).$$

But due to the fact that the number of arcs m of $D = S(G)$ is equal to $2q$ (where q is the number of edges of G), (1.23) is equivalent to

$$U_1^\pm(G; x) = x^{n-q+1} \frac{d}{dx} \left(\frac{1}{x^q} \int x^{2q-n-1} R_t^\pm[S(G); x] dx \right),$$

which proves (1.18).

The second integral equality (1.19) is proven in a similar way. First, we rewrite equality (1.22) in a different form to obtain

$$\int x^{m-n-1} R_1^\pm[S(G); x] dx = x^{m-q} \int x^{q-n-1} U_1^\pm(G; x) dx.$$

Then, without explanation, we apply a similar sequence of transformations:

$$\begin{aligned} x^{n-q+1} R_1^\pm[S(G); x] &= \frac{d}{dx} \left(x^{m-q} \int x^{q-n-1} U_1^\pm(G; x) dx \right), \\ R_1^\pm[S(G); x] &= x^{n-m+1} \frac{d}{dx} \left(x^{m-q} \int x^{q-n-1} U_1^\pm(G; x) dx \right), \\ R_1^\pm[S(G); x] &= x^{n-2q+1} \frac{d}{dx} \left(x^q \int x^{q-n-1} U_1^\pm(G; x) dx \right), \end{aligned}$$

which proves (1.19) and thus completes the whole proof. $\square$

In addition to Remark 3 (for monocyclic digraphs) and Theorem 1.4 (for undirected graphs), we present the following corollary to formulae (1.13), which is given here with a correction of a typo in [16].

Proposition 1.1. *Let $\overset{q=n}{G}$ be an undirected simple monocyclic graph (whether a cycle or not) with $q = n$. Then,*

$$(1.24) \quad \begin{aligned} \frac{d}{dx} \phi^{\pm}(\overset{q=n}{G}; x) &= x^{-1} \sum_{uv} \left[\phi^{\pm}(\overset{q=n}{G} - uv) \mp \phi^{\pm}(\overset{q=n}{G} - u - v; x) \right] \\ &= \sum_{u \in V(G)} \phi^{\pm}(\overset{q=n}{G} - u; x) \quad (q = n), \end{aligned}$$

where the first combined summation ranges over the set of all edges of $\overset{q=n}{G}$ and all pairs u and v of adjacent vertices ($u < v; u \sim v$).

Thus, the derivative $[\phi^{\pm}(\overset{q=n}{G}; x)]'$ of the polynomial $\phi^{\pm}(\overset{q=n}{G}; x)$ of a monocyclic graph $\overset{q=n}{G}$ is also reconstructible from the first combined sum in (1.24) (due to [16]). Earlier, in the works on the reconstruction of the characteristic polynomial $\phi^{-}(G; x)$ of an arbitrary undirected graph G , the second sum of (1.24) was used; see, e.g., [5–8]. Special attention is paid to $\phi^{-}(\overset{q=n}{G}; x)$ in [8].

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SYMMETRIC N -ADDITIVE MAPPINGS ADMITTING SEMIPRIME RING

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ABSTRACT. Let $\mathcal{R}$ be a ring with centre $Z(\mathcal{R})$. An n -additive map $D : \mathcal{R}^n \rightarrow \mathcal{R}$ is called symmetric n -additive if $D(x_1, \dots, x_n) = D(x_{\pi(1)}, \dots, x_{\pi(n)})$ for all $x_i \in \mathcal{R}$ and for every permutation $(\pi(1), \pi(2), \dots, \pi(n))$. A mapping $\Delta : \mathcal{R} \rightarrow \mathcal{R}$ defined by $\Delta(x) = D(x, x, \dots, x)$ is called the trace of D . In this paper, we prove that a nonzero Lie ideal L of a semiprime ring $\mathcal{R}$ of characteristic different from $(2^n - 2)$ is central, if it satisfies any one of the following properties: (i) $\Delta([x, y]) \mp xy \in Z(\mathcal{R})$; (ii) $\Delta([x, y]) \mp [y, x] \in Z(\mathcal{R})$; (iii) $\Delta(xy) \mp \Delta(x) \mp [x, y] \in Z(\mathcal{R})$; (iv) $\Delta([x, y]) \mp yx \in Z(\mathcal{R})$; (v) $\Delta(xy) \mp \Delta(y) \mp [x, y] \in Z(\mathcal{R})$.

1. INTRODUCTION

Throughout the paper, $\mathcal{R}$ always represents an associative ring, $Z(\mathcal{R})$ is its centre. Let $x, y, z \in \mathcal{R}$. We write the notation $[y, x]$ for the commutator $yx - xy$ and make use of the identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$ semiprime if $a\mathcal{R}a = \{0\}$ implies that $a = 0$. Let $\mathcal{R}$ and $\mathcal{S}$ be abelian groups. A map $q : \mathcal{R} \rightarrow \mathcal{S}$ is called the trace of a biadditive map if there exists a biadditive map $B : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{S}$ such that $q(x) = B(x, x)$ for all $x \in \mathcal{R}$. Assuming further that $\mathcal{R} \subseteq \mathcal{R}'$ are rings, we say that q is commuting if $[q(x), x] = q(x)x - xq(x) = 0$ for all $x \in \mathcal{R}$. An example is a map of the form $q(x) = \lambda x^2 + \mu(x)x + \nu(x)$ where $\lambda \in C$, the centre of $\mathcal{S}$ and $\mu, \nu : \mathcal{R} \rightarrow C$, μ is additive and ν is the trace of a biadditive map. Quite often it turns out that this obvious example is in fact the only possible example of a commuting trace of a biadditive map of $\mathcal{R}$ into $\mathcal{S}$. The basic result of this type states that this is true in the

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case when $\mathcal{R}$ is a prime ring with $\text{char}(\mathcal{R}) \neq 2$ and $\mathcal{S}$ is its central closure provided, however that $\mathcal{R}$ does not satisfy s_4 , the standard polynomial identity of degree 4 ([11], Theorem 1). This theorem has turned out to be the key for solving different problems and to a great extent it initiated the development of the theory of functional identities. We studies about bidervations and the traces of mapping in articles [1, 9, 10, 12] for details. A map $f : \mathcal{R} \rightarrow \mathcal{R}$ is centralizing on $\mathcal{R}$ if $[f(x), x] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. An additive map $D : \mathcal{R} \rightarrow \mathcal{R}$ is called a derivation if it satisfies the Leibnitz rule $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{R}$. Let $n \geq 2$ be a fixed positive integer. A map $D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$ is said to be symmetric (or permuting), if the equation $D(x_1, x_2, \dots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$ for all $x_i \in \mathcal{R}$ and for every permutation $(\pi(1), \pi(2), \dots, \pi(n))$. Let us consider the following maps Let $n \geq 2$ be a fixed positive integer. An n -additive map

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$$

will be called an n -derivation if the relations

$$\begin{aligned} D(x_1 x_1', x_2, \dots, x_n) &= D(x_1, x_2, \dots, x_n) x_1' + x_1 D(x_1', x_2, \dots, x_n), \\ D(x_1, x_2 x_2', \dots, x_n) &= D(x_1, x_2, \dots, x_n) x_2' + x_2 D(x_1, x_2', \dots, x_n), \\ &\vdots \\ D(x_1, x_2, \dots, x_n x_n') &= D(x_1, x_2, \dots, x_n) x_n' + x_n D(x_1, x_2, \dots, x_n'), \end{aligned}$$

are valid for all $x_i, x_i' \in \mathcal{R}$. Of course, an 1-derivation is a derivation and a 2-derivation is called a bi-derivation. If D is symmetric, then the above inequalities are equivalent to each other. Let $n \geq 2$ be a fixed positive integer. If $\mathcal{R}$ is commutative, then a map

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R},$$

defined by

$$(x_1, x_2, \dots, x_n) \rightarrow D(x_1)D(x_2) \cdots D(x_n), \quad \text{for all } x_i \in \mathcal{R}, i = 1, 2, \dots, n,$$

is a symmetric n -derivation, where D is a derivation on $\mathcal{R}$. Let $n \geq 2$ be a fixed positive integer and let a map $\Delta : \mathcal{R} \rightarrow \mathcal{R}$ defined by $\Delta(x) = D(x, x, \dots, x)$ for all $x \in \mathcal{R}$, where

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$$

is a symmetric map, be the trace of D . It is obvious that, in case when

$$D : \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$$

is a symmetric map which is also n -additive, the trace Δ of D satisfies the relation

$$\Delta(x + y) = \Delta(x) + \Delta(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(x, y), \quad \text{for all } x, y \in \mathcal{R},$$

and

$$h_k(x, y) = D(\underbrace{x, x, \dots, x}_{(n-k)\text{-times}}, \underbrace{y, y, \dots, y}_{k\text{-times}}).$$

Gy. Maksa [3] introduced the concept of a symmetric biderivation (see also [2], where an example can be found). It was shown in [3] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [12] and [5]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : \mathcal{R} \rightarrow \mathcal{R}$ gives rise to a biderivation on $\mathcal{R}$. Namely linearizing $[x, f(x)] = 0$ for all $x \in \mathcal{R}$, we get

$$[f(x), y] = [x, f(y)], \quad \text{for all } x \in \mathcal{R},$$

and hence we note that the mapping $(x, y) \rightarrow [f(x), y]$ is a biderivation (moreover, all derivations appearing are inner). Motivated by the aforementioned results we prove that a nonzero Lie ideal L of a semiprime ring $\mathcal{R}$ of characteristic different from $(2^n - 2)$ is central, if it satisfies any one of the following properties: (i) $\Delta([x, y]) \mp xy \in Z(\mathcal{R})$; (ii) $\Delta([x, y]) \mp [y, x] \in Z(\mathcal{R})$; (iii) $\Delta(xy) \mp \Delta(x) \mp [x, y] \in Z(\mathcal{R})$; (iv) $\Delta([x, y]) \mp yx \in Z(\mathcal{R})$; (v) $\Delta(xy) \mp \Delta(y) \mp [x, y] \in Z(\mathcal{R})$.

2. PRELIMINARY RESULTS

We make extensive use of basic commutator identities

$$[xy, z] = [x, z]y + x[y, z], \quad [x, yz] = [x, y]z + y[x, z].$$

Moreover, we shall require the following lemmas.

Lemma 2.1 ([5], Lemma 1.1.5). *If $\mathcal{R}$ is a semiprime ring, then the center of a nonzero one sided ideal is contained in the center of $\mathcal{R}$. As an immediate consequence, any commutative one sided ideal is contained in the center of $\mathcal{R}$.*

Lemma 2.2. *Let $\mathcal{R}$ be a semiprime ring and L be a nonzero Lie ideal of $\mathcal{R}$. If $[L, L] \subseteq Z(\mathcal{R})$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Since $xy \in Z(\mathcal{R})$ for all $x, y \in L$, $xy - yx = [x, y] \in Z(\mathcal{R})$ for all $x, y \in L$. Using Lemma 2.1 we get the required result. $\square$

3. MAIN RESULTS

Theorem 3.1. *Let $\mathcal{R}$ be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \rightarrow \mathcal{R}$ be a symmetric n -additive mapping and Δ be the trace of D . If $\Delta([x, y]) \mp xy \in Z(\mathcal{R})$ for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Let

$$(3.1) \quad \Delta([x, y]) - xy \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Replacing y by $y + z$ in (3.1), we have

$$\Delta([x, y] + [x, z]) - xy - xz \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This implies that

$$\Delta([x, y]) + \Delta([x, z]) + \sum_{k=1}^{n-1} \binom{n}{k} h_k([x, y], [x, z]) - xy - xz \in Z(\mathcal{R}).$$

By using (3.1), we obtain

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k([x, y], [x, z]) \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This gives that

$$(3.2) \quad \binom{n}{1} h_1([x, y], [x, z]) + \binom{n}{2} h_2([x, y], [x, z]) + \binom{n}{3} h_3([x, y], [x, z]) \\ + \cdots + \binom{n}{n-1} h_{n-1}([x, y], [x, z]) \in Z(\mathcal{R}).$$

Substituting y for z in (3.2), we obtain

$$\binom{n}{1} h_1([x, y], [x, y]) + \binom{n}{2} h_2([x, y], [x, y]) + \binom{n}{3} h_3([x, y], [x, y]) \\ + \cdots + \binom{n}{n-1} h_{n-1}([x, y], [x, y]) \in Z(\mathcal{R}).$$

This implies that

$$\binom{n}{1} D(\underbrace{[x, y], [x, y], \dots, [x, y]}_{(n-1)\text{-times}}, \underbrace{[x, y]}_{1\text{-times}}) + \binom{n}{2} D(\underbrace{[x, y], [x, y], \dots, [x, y]}_{(n-2)\text{-times}}, \underbrace{[x, y]}_{2\text{-times}}) \\ + \cdots + \binom{n}{n-1} D(\underbrace{[x, y]}_{1\text{-times}}, \underbrace{[x, y], [x, y], \dots, [x, y]}_{(n-1)\text{-times}}) \in Z(\mathcal{R}).$$

This shows that

$$\left(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} \right) D([x, y], [x, y], \dots, [x, y]) \in Z(\mathcal{R}).$$

We obtain

$$(3.3) \quad (2^n - 2)D([x, y], [x, y], \dots, [x, y]) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Since $\mathcal{R}$ is not of characteristic $(2^n - 2)$, we get

$$D([x, y], [x, y], \dots, [x, y]) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Applying the definition of the trace

$$(3.4) \quad \Delta([x, y]) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Using (3.1), we get $xy \in Z(\mathcal{R})$ for all $x, y \in L$. This implies that $[x, y] \in Z(\mathcal{R})$. By using Lemma 2.2, we get $L \subseteq Z(\mathcal{R})$.

Similarly, we can prove the result if $f([x, y]) + xy \in Z(\mathcal{R})$ for all $x, y \in L$. $\square$

Theorem 3.2. *Let $\mathcal{R}$ be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \rightarrow \mathcal{R}$ be a symmetric n -additive mapping and Δ be the trace of D . If $\Delta([x, y]) \mp [y, x] \in Z(\mathcal{R})$ for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Using the same argument as in Theorem 3.1. $\square$

Theorem 3.3. *Let $\mathcal{R}$ be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \rightarrow \mathcal{R}$ be a symmetric n -additive mapping and Δ be the trace of D . If $\Delta(xy) \mp \Delta(x) \mp [x, y] \in Z(\mathcal{R})$ for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Suppose

$$(3.5) \quad \Delta(xy) - \Delta(x) - [x, y] \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Replacing x by $x + z$ in (3.5), we have

$$\Delta((x + z)y) - \Delta(x + z) - [x + z, y] \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This implies that

$$\Delta(xy + zy) - \Delta(x + z) - [x, y] - [z, y] \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This gives that

$$\begin{aligned} & \Delta(xy) + \Delta(zy) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, zy) - \Delta(x) - \Delta(z) \\ & - \sum_{k=1}^{n-1} \binom{n}{k} h_k(x, z) - [x, y] - [z, y] \in Z(\mathcal{R}). \end{aligned}$$

This implies that

$$\begin{aligned} & \Delta(xy) - \Delta(x) - [x, y] + \Delta(zy) - \Delta(z) - [z, y] \\ & + \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, zy) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(x, z) \in Z(\mathcal{R}). \end{aligned}$$

Using (3.5), we get

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, zy) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(x, z) \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This shows that

$$(3.6) \quad \begin{aligned} & \binom{n}{1} h_1(xy, zy) + \binom{n}{2} h_2(xy, zy) + \cdots + \binom{n}{n-1} h_{n-1}(xy, zy) \\ & - \binom{n}{1} h_1(x, z) - \binom{n}{2} h_2(x, z) - \cdots - \binom{n}{n-1} h_{n-1}(x, z) \in Z(\mathcal{R}). \end{aligned}$$

Substituting x for z in (3.6), we have

$$\begin{aligned} & \binom{n}{1} h_1(xy, xy) + \binom{n}{2} h_2(xy, xy) + \cdots + \binom{n}{n-1} h_{n-1}(xy, xy) \\ & - \binom{n}{1} h_1(x, x) - \binom{n}{2} h_2(x, x) - \cdots - \binom{n}{n-1} h_{n-1}(x, x) \in Z(\mathcal{R}). \end{aligned}$$

We find that

$$\begin{aligned} & \binom{n}{1} D(\underbrace{xy, xy, \dots, xy}_{(n-1)\text{-times}}, \underbrace{xy}_{1\text{-times}}) + \binom{n}{2} D(\underbrace{xy, xy, \dots, xy}_{(n-2)\text{-times}}, \underbrace{xy, xy}_{2\text{-times}}) \\ & + \cdots + \binom{n}{n-1} D(\underbrace{xy}_{1\text{-times}}, \underbrace{xy, xy, \dots, xy}_{(n-1)\text{-times}}) \in Z(\mathcal{R}). \end{aligned}$$

This implies that

$$(2^n - 2) (D(xy, xy, \dots, xy) - D(x, x, \dots, x)) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Since $\mathcal{R}$ is not of characteristic $(2^n - 2)$,

$$D(xy, xy, \dots, xy) - D(x, x, \dots, x) \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

By definition of the trace, we get

$$(3.7) \quad \Delta(xy) - \Delta(x) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Using (3.5), $[x, y] \in Z(\mathcal{R})$ for all $x, y \in L$. Arguing similar manner as in the Theorem 3.1, we get the result. Similarly, we can prove the result if $\Delta(xy) + \Delta(x) + [x, y] \in Z(\mathcal{R})$ for all $x, y \in L$. $\square$

Theorem 3.4. *Let $\mathcal{R}$ be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \rightarrow \mathcal{R}$ be a symmetric n -additive mapping and Δ be the trace of D . If $\Delta([x, y]) \mp yx \in Z(\mathcal{R})$ for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Using the same argument as in Theorem 3.3. $\square$

Theorem 3.5. *Let $\mathcal{R}$ be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero left ideal of $\mathcal{R}$. Let $D : \mathcal{R}^n \rightarrow \mathcal{R}$ be a symmetric n -additive mapping and Δ be the trace of D . If $\Delta(xy) \mp \Delta(y) \mp [x, y] \in Z(\mathcal{R})$ for all $x, y \in L$, then $L \subseteq Z(\mathcal{R})$.*

Proof. Suppose

$$(3.8) \quad \Delta(xy) - \Delta(y) - [x, y] \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Replacing y by $y + z$ in (3.8), we obtain

$$\Delta(x(y + z)) - \Delta(y + z) - [x, y + z] \in Z(\mathcal{R}), \quad \text{for all } x, y, z \in L.$$

This shows that

$$\begin{aligned} & \Delta(xy) + \Delta(xz) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xz) - \Delta(y) \\ & - \Delta(z) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) - [x, y] - [x, z] \in Z(\mathcal{R}). \end{aligned}$$

We find that

$$\begin{aligned} & \Delta(xy) - \Delta(y) - [x, y] + \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xz) + \Delta(xz) \\ & - \Delta(z) - [x, z] - \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathcal{R}). \end{aligned}$$

Using (3.8), we have

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xz) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathcal{R}).$$

On simplifying,

$$(3.9) \quad \begin{aligned} & \binom{n}{1} h_1(xy, xz) + \binom{n}{2} h_2(xy, xz) + \cdots + \binom{n}{n-1} h_{n-1}(xy, xz) \\ & - \binom{n}{1} h_1(y, z) - \binom{n}{2} h_2(y, z) - \cdots - \binom{n}{n-1} h_{n-1}(y, z) \in Z(\mathcal{R}). \end{aligned}$$

Substituting y for z in (3.9), we get

$$\begin{aligned} & \binom{n}{1} h_1(xy, xy) + \binom{n}{2} h_2(xy, xy) + \cdots + \binom{n}{n-1} h_{n-1}(xy, xy) \\ & - \binom{n}{1} h_1(y, y) - \binom{n}{2} h_2(y, y) - \cdots - \binom{n}{n-1} h_{n-1}(y, y) \in Z(\mathcal{R}). \end{aligned}$$

This implies that

$$\begin{aligned} & \binom{n}{1} D(\underbrace{xy, \dots, xy}_{(n-1)\text{-times}}, \underbrace{xy}_{1\text{-times}}) + \binom{n}{2} D(\underbrace{xy, xy, \dots, xy}_{(n-2)\text{-times}}, \underbrace{xy}_{3\text{-times}}) \\ & + \dots + \binom{n}{n-1} D(\underbrace{xy}_{1\text{-times}}, \underbrace{xy, xy, \dots, xy}_{(n-1)\text{-times}}) - \binom{n}{1} D(\underbrace{y, y, \dots, y}_{(n-1)\text{-times}}, \underbrace{y}_{1\text{-times}}) \\ & - \binom{n}{2} D(\underbrace{y, y, \dots, y}_{(n-2)\text{-times}}, \underbrace{y}_{2\text{-times}}) - \dots - \binom{n}{n-1} D(\underbrace{y}_{1\text{-times}}, \underbrace{y, y, \dots, y}_{(n-1)\text{-times}}) \in Z(\mathcal{R}). \end{aligned}$$

Now solving the above equation, we get

$$\begin{aligned} & \left(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} \right) D(xy, xy, \dots, xy) \\ & - \left(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} \right) D(y, y, \dots, y) \in Z(\mathcal{R}). \end{aligned}$$

This gives that

$$(2^n - 2) (D(xy, xy, \dots, xy) - D(y, y, \dots, y)) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Since $\mathcal{R}$ is not characteristic $(2^n - 2)$, we find

$$D(xy, xy, \dots, xy) - D(y, y, \dots, y) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

This shows that

$$(3.10) \quad \Delta(xy) - \Delta(y) \in Z(\mathcal{R}), \quad \text{for all } x, y \in L.$$

Using (3.8) and (3.10), we have $[x, y] \in Z(\mathcal{R})$ for all $x, y \in L$. Arguing in similar manner as in Theorem 3.1, we get the result. Similarly, we can prove the result if $\Delta([x, y]) + \Delta(y) + [x, y] \in Z(\mathcal{R})$ for all $x, y \in L$. $\square$

4. EXAMPLES

The following examples illustrate that $\mathcal{R}$ to be semiprime and characteristic not $(2^n - 2)$ for $n > 1$ is essential in the hypothesis of the above theorem.

Example 4.1. Let $\mathcal{R} = \left\{ \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \mid p, q, r \in \mathbb{Z}, \text{ ring of integers} \right\}$ and the Lie ideal $L = \left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \mid q \in \mathbb{Z} \right\}$. Then $Z(\mathcal{R}) = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \mid p \in \mathbb{Z} \right\}$. Define a map $D : \underbrace{\mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$ by

$$D\left(\begin{pmatrix} p_1 & q_1 \\ 0 & r_1 \end{pmatrix}, \begin{pmatrix} p_2 & q_2 \\ 0 & r_2 \end{pmatrix}, \dots, \begin{pmatrix} p_n & q_n \\ 0 & r_n \end{pmatrix}\right) = \begin{pmatrix} p_1 p_2 p_3 \dots p_n & 0 \\ 0 & 0 \end{pmatrix}.$$

Then D is symmetric n -additive with trace Δ defined by $\Delta : \mathcal{R} \rightarrow \mathcal{R}$ such that $\Delta\left(\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right) = D\left(\begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}, \dots, \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}\right)$ satisfying hypothesis of the above theorems. However, $L \not\subseteq Z(\mathcal{R})$.

Example 4.2. Let $\mathcal{R} = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \mid x, y, z \in \mathbb{Z}, \text{ ring of integers} \right\}$ and the Lie ideal $L = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \mid y \in \mathbb{Z} \right\}$. Then $Z(\mathcal{R}) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z} \right\}$. Define a map $D : \underbrace{\mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}$ by

$$D\left(\begin{pmatrix} x_1 & 0 \\ y_1 & z_1 \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ y_2 & z_2 \end{pmatrix}, \dots, \begin{pmatrix} x_n & 0 \\ y_n & z_n \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & z_1 z_2 z_3 \dots z_n \end{pmatrix}.$$

Then D is symmetric n -additive with trace Δ defined by $\Delta : \mathcal{R} \rightarrow \mathcal{R}$ such that $\Delta\left(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\right) = D\left(\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, \dots, \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}\right)$ satisfying hypothesis of the above theorems. However, $L \not\subseteq Z(\mathcal{R})$.

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ON THE HYPERBOLIC k -MERSENNE AND k -MERSENNE-LUCAS OCTONIONS

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ABSTRACT. In this paper, we introduce the hyperbolic k -Mersenne and k -Mersenne-Lucas octonions and investigate their algebraic properties. We give Binet's formula and present several interrelations and some well-known identities such as Catalan identity, d'Ocagne identity, Vajda identity, generating functions, etc. of these octonions in closed form. Furthermore, we investigate the relations between hyperbolic k -Mersenne octonions and hyperbolic k -Mersenne-Lucas octonions.

1. INTRODUCTION

Number sequences have been studied by researchers for a long time. In particular, the most important and remarkable of these numbers are the Fibonacci numbers. Until today, Fibonacci numbers have been studied and many generalizations have been made. Lucas, Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas, etc. numbers can be given as examples of these generalizations [2, 5, 8, 14, 15, 19–21, 23].

One of these numbers is the Mersenne number. They are named after Marin Mersenne, a French Minim friar, who studied them in the early 17th century. Mersenne numbers have been studied in the literature and various generalizations such as Mersenne-Lucas, k -Mersenne, k -Mersenne-Lucas have been studied [1, 4, 6, 7, 17, 22, 25–27].

Definition 1.1. The Mersenne sequence $\{M_n\}_{n \geq 0}$ is defined recursively as

$$M_{n+2} = 3M_{n+1} - 2M_n, \quad \text{with } M_0 = 0, \quad M_1 = 1, \quad n \geq 0.$$

Key words and phrases. Hyperbolic k -Mersenne octonions, hyperbolic k -Mersenne-Lucas octonions, Binet formula, Catalan identity, generating function.

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Definition 1.2. The Mersenne-Lucas sequence $\{m_n\}_{n \geq 0}$ is defined recursively as

$$m_{n+2} = 3m_{n+1} - 2m_n, \quad \text{with } m_0 = 2, \quad m_1 = 3, \quad n \geq 0.$$

Definition 1.3. The k -Mersenne sequence $\{M_{k,n}\}_{n \geq 0}$ is given recursively as

$$(1.1) \quad M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n}, \quad M_{k,0} = 0, \quad M_{k,1} = 1, \quad n \geq 0.$$

Definition 1.4. The k -Mersenne-Lucas sequence $\{m_{k,n}\}_{n \geq 0}$ is given recursively as

$$(1.2) \quad m_{k,n+2} = 3km_{k,n+1} - 2m_{k,n}, \quad m_{k,0} = 2, \quad m_{k,1} = 3k, \quad n \geq 0.$$

The characteristic equation corresponding to the recurrence relations (1.1) and (1.2) is $\alpha^2 - 3k\alpha + 2 = 0$ and its roots are $\alpha_1 = \frac{3k + \sqrt{9k^2 - 8}}{2}$ and $\alpha_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}$. These characteristics roots hold the following properties

$$(1.3) \quad \alpha_1 + \alpha_2 = 3k, \quad \alpha_1 \alpha_2 = 2, \quad \alpha_1 - \alpha_2 = \sqrt{9k^2 - 8}.$$

The Binet formulas of k -Mersenne and k -Mersenne-Lucas sequences are given, respectively, by

$$(1.4) \quad m_{k,n} = \alpha_1^n + \alpha_2^n \quad \text{and} \quad M_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

The quaternion, an algebraic structure, was first described in 1843 by William Rowan Hamilton [11]. Hamilton defined a quaternion as the quotient of two directed lines in a three-dimensional space, or, equivalently, as the quotient of two vectors. Multiplication of quaternions is noncommutative. A. F. Horadam defined the n th Fibonacci and n th Lucas quaternions and gave their some properties in 1963 [12]. Thus, Macfarlane defined the hyperbolic quaternions and studied their properties [18]. Recently, these numbers have been examined and studies have been carried out [10, 13, 24]. The hyperbolic k -Fibonacci and k -Fibonacci-Lucas, hyperbolic k -Jacobsthal and k -Jacobsthal-Lucas quaternions were defined and given some of their properties [10, 24]. In mathematics, the octonions are a normed division algebra over the real numbers, a kind of hypercomplex number system. Octonions have eight dimensions; twice the number of dimensions of the quaternions, of which they are an extension. They are noncommutative and nonassociative, but satisfy a weaker form of associativity; namely, they are alternative. They are also power associative. Octonions are not as well known as quaternions and complex numbers, which are much more widely studied and used.

A. Cariow and G. Cariow defined the hyperbolic octonions [3]. A hyperbolic octonion O has the form

$$\begin{aligned} O &= h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 i_4 + h_5 i_5 + h_6 i_6 + h_7 i_7 \\ &= (h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7), \end{aligned}$$

where i_1, i_2, i_3 are quaternion imaginary units and $h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7$ are the real components, i_4 ($i_4^2 = 1$) is a counter imaginary unit, and the bases of hyperbolic octonions are defined as in Table 1.

.	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	-1	i_3	$-i_2$	i_5	i_4	$-i_7$	i_6
i_2	$-i_3$	-1	i_1	i_6	i_7	i_4	$-i_5$
i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	i_4
i_4	$-i_5$	$-i_6$	$-i_7$	1	i_1	i_2	i_3
i_5	$-i_4$	$-i_7$	i_6	$-i_1$	1	i_3	$-i_2$
i_6	i_7	$-i_4$	$-i_5$	$-i_2$	$-i_3$	1	i_1
i_7	$-i_6$	i_5	$-i_4$	$-i_3$	i_2	$-i_1$	1

Table 1: Multiplication rule for hyperbolic octonions units.

Godase A. defined the hyperbolic k -Fibonacci and k -Fibonacci-Lucas octonions and gave some of their properties [9]. Özkan E. et al. defined the hyperbolic k -Jacobsthal and k -Jacobsthal-Lucas octonions and gave some of their properties [23]. Kumari M. et al. defined the k -Mersenne, k -Mersenne-Lucas octonions and examined some properties of these numbers [16]. As a new generalization of this study [16], we examine the hyperbolic k -Mersenne and k -Mersenne-Lucas octonions and give their features.

2. HYPERBOLIC k -MERSENNE OCTONIONS

In this section, we introduce the hyperbolic k -Mersenne octonions and establish their Binet formula. Furthermore, we study some well-known identities of them.

Definition 2.1. For $n \geq 0$, the hyperbolic k -Mersenne octonions $\mathcal{H}M_{k,n}$ are defined by

$$\begin{aligned}\mathcal{H}M_{k,n} &= M_{k,n} + M_{k,n+1}i_1 + M_{k,n+2}i_2 + \cdots + M_{k,n+7}i_7 \\ &= (M_{k,n}, M_{k,n+1}, M_{k,n+2}, M_{k,n+3}, M_{k,n+4}, M_{k,n+5}, M_{k,n+6}, M_{k,n+7}).\end{aligned}$$

Definition 2.2. The sequence can be extended to negative indices n , $\mathcal{H}M_{k,-n}$ is defined by

$$\begin{aligned}\mathcal{H}M_{k,-n} &= -\frac{1}{2^n}M_{k,n} - \frac{1}{2^{n-1}}M_{k,n-1}i_1 - \frac{1}{2^{n-2}}M_{k,n-2}i_2 - \frac{1}{2^{n-3}}M_{k,n-3}i_3 \\ &\quad - \frac{1}{2^{n-4}}M_{k,n-4}i_4 - \frac{1}{2^{n-5}}M_{k,n-5}i_5 - \frac{1}{2^{n-6}}M_{k,n-6}i_6 - \frac{1}{2^{n-7}}M_{k,n-7}i_7.\end{aligned}$$

Definition 2.3. Let the scalar and vector parts of the hyperbolic k -Mersenne octonions $\mathcal{H}M_{k,n}$ be denoted by $S(\mathcal{H}M_{k,n})$ and $V(\mathcal{H}M_{k,n})$, respectively, and are defined as follows

$$\begin{aligned}S(\mathcal{H}M_{k,n}) &= M_{k,n}, \\ V(\mathcal{H}M_{k,n}) &= (M_{k,n+1}, M_{k,n+2}, M_{k,n+3}, M_{k,n+4}, M_{k,n+5}, M_{k,n+6}, M_{k,n+7}).\end{aligned}$$

Thus, $\mathcal{H}M_{k,n} = S(\mathcal{H}M_{k,n}) + V(\mathcal{H}M_{k,n})$.

Definition 2.4. For $n \geq 0$, the conjugate of the hyperbolic k -Mersenne octonions $\mathcal{H}M_{k,n}$ is defined by

$$\begin{aligned}\overline{\mathcal{H}M}_{k,n} &= M_{k,n} - M_{k,n+1}i_1 - M_{k,n+2}i_2 - \cdots - M_{k,n+7}i_7 \\ &= (M_{k,n}, -M_{k,n+1}, -M_{k,n+2}, -M_{k,n+3}, -M_{k,n+4}, -M_{k,n+5}, -M_{k,n+6}, -M_{k,n+7}).\end{aligned}$$

Theorem 2.1. *The following equation is provided*

$$\mathcal{H}M_{k,n} + \overline{\mathcal{H}M}_{k,n} = 2S(\mathcal{H}M_{k,n}) = 2M_{k,n}.$$

Proof. From Definition 2.4, we have

$$\begin{aligned}\mathcal{H}M_{k,n} + \overline{\mathcal{H}M}_{k,n} &= S(\mathcal{H}M_{k,n}) + V(\mathcal{H}M_{k,n}) + S(\mathcal{H}M_{k,n}) - V(\mathcal{H}M_{k,n}) \\ &= 2S(\mathcal{H}M_{k,n}) = 2M_{k,n}.\end{aligned}\quad \square$$

Definition 2.5. The norm of the hyperbolic k -Mersenne octonions $\mathcal{H}M_{k,n}$ is defined by

$$N(\mathcal{H}M_{k,n}) = \sqrt{M_{k,n}^2 + \cdots + M_{k,n+3}^2 - M_{k,n+4}^2 - M_{k,n+5}^2 - M_{k,n+6}^2 - M_{k,n+7}^2}.$$

Theorem 2.2. *The recurrence relations of the $\mathcal{H}M_{k,n}$ and $\overline{\mathcal{H}M}_{k,n}$ are as follows:*

- (a) $\mathcal{H}M_{k,n+1} = 3k\mathcal{H}M_{k,n} - 2\mathcal{H}M_{k,n-1}$;
- (b) $\overline{\mathcal{H}M}_{k,n+1} = 3k\overline{\mathcal{H}M}_{k,n} - 2\overline{\mathcal{H}M}_{k,n-1}$.

Proof. (a) We have

$$\begin{aligned}\mathcal{H}M_{k,n+1} &= M_{k,n+1} + M_{k,n+2}i_1 + M_{k,n+3}i_2 + M_{k,n+4}i_3 + M_{k,n+5}i_4 + M_{k,n+6}i_5 \\ &\quad + M_{k,n+7}i_6 + M_{k,n+8}i_7 \\ &= (3kM_{k,n} - 2M_{k,n-1}) + (3kM_{k,n+1} - 2M_{k,n})i_1 + (3kM_{k,n+2} - 2M_{k,n+1})i_2 \\ &\quad + (3kM_{k,n+3} - 2M_{k,n+2})i_3 + (3kM_{k,n+4} - 2M_{k,n+3})i_4 + (3kM_{k,n+5} \\ &\quad - 2M_{k,n+4})i_5 + (3kM_{k,n+6} - 2M_{k,n+5})i_6 + (3kM_{k,n+7} - 2M_{k,n+6})i_7 \\ &= 3k(M_{k,n} + M_{k,n+1}i_1 + M_{k,n+2}i_2 + M_{k,n+3}i_3 + M_{k,n+4}i_4 + M_{k,n+5}i_5 \\ &\quad + M_{k,n+6}i_6 + M_{k,n+7}i_7) - 2(M_{k,n-1} + M_{k,n}i_1 + M_{k,n+1}i_2 + M_{k,n+2}i_3 \\ &\quad + M_{k,n+3}i_4 + M_{k,n+4}i_5 + M_{k,n+5}i_6 + M_{k,n+6}i_7) \\ &= 3k\mathcal{H}M_{k,n} - 2\mathcal{H}M_{k,n-1}.\end{aligned}$$

The proof of (b) is similar to that of (a). $\square$

Theorem 2.3 (Binet Formula). *The Binet formulas for the hyperbolic k -Mersenne octonions are*

$$\begin{aligned}\text{(a)} \quad \mathcal{H}M_{k,n} &= \frac{\alpha^* \alpha_1^n - \beta^* \alpha_2^n}{\alpha_1 - \alpha_2}; \\ \text{(b)} \quad \overline{\mathcal{H}M}_{k,n} &= \frac{\overline{\alpha^*} \alpha_1^n - \overline{\beta^*} \alpha_2^n}{\alpha_1 - \alpha_2}; \\ \text{(c)} \quad \mathcal{H}M_{k,-n} &= \frac{1}{2^n} \left(\frac{\alpha^* \alpha_2^n - \beta^* \alpha_1^n}{\alpha_1 - \alpha_2} \right),\end{aligned}$$

where

$$\begin{aligned}\alpha^* &= 1 + \alpha_1 i_1 + \alpha_1^2 i_2 + \alpha_1^3 i_3 + \alpha_1^4 i_4 + \alpha_1^5 i_5 + \alpha_1^6 i_6 + \alpha_1^7 i_7 \\ &= (1, \alpha_1, \alpha_1^2, \alpha_1^3, \alpha_1^4, \alpha_1^5, \alpha_1^6, \alpha_1^7), \\ \beta^* &= 1 + \alpha_2 i_1 + \alpha_2^2 i_2 + \alpha_2^3 i_3 + \alpha_2^4 i_4 + \alpha_2^5 i_5 + \alpha_2^6 i_6 + \alpha_2^7 i_7 \\ &= (1, \alpha_2, \alpha_2^2, \alpha_2^3, \alpha_2^4, \alpha_2^5, \alpha_2^6, \alpha_2^7), \\ \overline{\alpha^*} &= 1 - \alpha_1 i_1 - \alpha_1^2 i_2 - \alpha_1^3 i_3 - \alpha_1^4 i_4 - \alpha_1^5 i_5 - \alpha_1^6 i_6 - \alpha_1^7 i_7 \\ &= (1, -\alpha_1, -\alpha_1^2, -\alpha_1^3, -\alpha_1^4, -\alpha_1^5, -\alpha_1^6, -\alpha_1^7), \\ \overline{\beta^*} &= 1 - \alpha_2 i_1 - \alpha_2^2 i_2 - \alpha_2^3 i_3 - \alpha_2^4 i_4 - \alpha_2^5 i_5 - \alpha_2^6 i_6 - \alpha_2^7 i_7 \\ &= (1, -\alpha_2, -\alpha_2^2, -\alpha_2^3, -\alpha_2^4, -\alpha_2^5, -\alpha_2^6, -\alpha_2^7).\end{aligned}$$

Proof. (a) Using Definition 2.1 and the Binet formula of $M_{k,n}$, we have

$$\begin{aligned}\mathcal{H}M_{k,n} &= M_{k,n} + M_{k,n+1}i_1 + M_{k,n+2}i_2 + M_{k,n+3}i_3 + M_{k,n+4}i_4 + M_{k,n+5}i_5 \\ &\quad + M_{k,n+6}i_6 + M_{k,n+7}i_7 \\ &= \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) + \left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right)i_1 + \cdots + \left(\frac{\alpha_1^{n+7} - \alpha_2^{n+7}}{\alpha_1 - \alpha_2}\right)i_7 \\ &= \frac{\alpha_1^n}{\alpha_1 - \alpha_2}(1 + \alpha_1 i_1 + \cdots + \alpha_1^7 i_7) - \frac{\alpha_2^n}{\alpha_1 - \alpha_2}(1 + \alpha_2 i_1 + \cdots + \alpha_2^7 i_7) \\ &= \frac{\alpha_1^n \alpha^* - \alpha_2^n \beta^*}{\alpha_1 - \alpha_2}.\end{aligned}$$

The proofs of (b) and (c) are similar to (a), by using Definition 2.4 and Definition 2.2, respectively. $\square$

Theorem 2.4. For $r, s, n \in \mathbb{N}$ such that $s \geq r$, the generating functions for hyperbolic k -Mersenne octonions are given as

$$\begin{aligned}(a) \quad \sum_{n=0}^{\infty} \mathcal{H}M_{k,n}x^n &= \frac{\mathcal{H}M_{k,0} + (\mathcal{H}M_{k,1} - 3k\mathcal{H}M_{k,0})x}{1 - 3kx + 2x^2}; \\ (b) \quad \sum_{n=0}^{\infty} \mathcal{H}M_{k,sn+r}x^n &= \frac{\mathcal{H}M_{k,r} - 2^r \mathcal{H}M_{k,s-r}x}{1 - m_{k,s}x + 2^s x^2}.\end{aligned}$$

The exponential generating functions for hyperbolic k -Mersenne octonions are

$$\begin{aligned}(c) \quad \sum_{n=0}^{\infty} \frac{\mathcal{H}M_{k,sn}x^n}{n!} &= \frac{\alpha^* e^{\alpha_1^s x} - \beta^* e^{\alpha_2^s x}}{\alpha_1 - \alpha_2}; \\ (d) \quad \sum_{n=0}^{\infty} \frac{\mathcal{H}M_{k,sn+r}x^n}{n!} &= \frac{\alpha^* \alpha_1^r e^{\alpha_1^s x} - \beta^* \alpha_2^r e^{\alpha_2^s x}}{\alpha_1 - \alpha_2}.\end{aligned}$$

Proof. (a) Let $G(x) = \sum_{n=0}^{\infty} \mathcal{H}M_{k,n}x^n$. We have

$$G(x) = \sum_{n=0}^{\infty} \mathcal{H}M_{k,n}x^n = \mathcal{H}M_{k,0} + \mathcal{H}M_{k,1}x + \mathcal{H}M_{k,2}x^2 + \mathcal{H}M_{k,3}x^3 + \cdots,$$

$$\begin{aligned} 3kxG(x) &= \mathcal{H}M_{k,0}3kx + \mathcal{H}M_{k,1}3kx^2 + \mathcal{H}M_{k,2}3kx^3 + \mathcal{H}M_{k,3}3kx^4 + \cdots, \\ 2x^2G(x) &= \mathcal{H}M_{k,0}2x^2 + \mathcal{H}M_{k,1}2x^3 + \mathcal{H}M_{k,2}2x^4 + \mathcal{H}M_{k,3}2x^5 + \cdots. \end{aligned}$$

If the necessary mathematical operations are performed, we get the following

$$\begin{aligned} (1 - 3kx + 2x^2)G(x) &= \mathcal{H}M_{k,0} + (\mathcal{H}M_{k,1} - 3k\mathcal{H}M_{k,0})x, \\ G(x) &= \frac{\mathcal{H}M_{k,0} + (\mathcal{H}M_{k,1} - 3k\mathcal{H}M_{k,0})x}{1 - 3kx + 2x^2}. \end{aligned}$$

The proofs of (b), (c) and (d) are similar to that of (a). $\square$

Theorem 2.5. For $k \neq 1$, we have

$$\sum_{i=0}^n \mathcal{H}M_{k,i} = \frac{2\mathcal{H}M_{k,n} - \mathcal{H}M_{k,n+1} + \mathcal{H}M_{k,1} + (1 - 3k)\mathcal{H}M_{k,0}}{3(1 - k)}.$$

Proof. Using Theorem 2.3, we obtain

$$\begin{aligned} \sum_{i=0}^n \mathcal{H}M_{k,i} &= \sum_{i=0}^n \left(\frac{\alpha^* \alpha_1^i - \beta^* \alpha_2^i}{\alpha_1 - \alpha_2} \right) \\ &= \frac{\alpha^*}{\alpha_1 - \alpha_2} \sum_{i=0}^n \alpha_1^i - \frac{\beta^*}{\alpha_1 - \alpha_2} \sum_{i=0}^n \alpha_2^i \\ &= \frac{\alpha^*(1 - \alpha_2 - \alpha_1^{n+1} + \alpha_1^{n+1}\alpha_2) - \beta^*(1 - \alpha_1 - \alpha_2^{n+1} + \alpha_2^{n+1}\alpha_1)}{(\alpha_1 - \alpha_2)(1 - (\alpha_1 + \alpha_2) + \alpha_1\alpha_2)} \\ &= \frac{\alpha^* - \beta^* - \alpha_1\alpha_2(\alpha^* \alpha_1^{-1} - \beta^* \alpha_2^{-1}) - (\alpha^* \alpha_1^{n+1} - \beta^* \alpha_2^{n+1})}{(\alpha_1 - \alpha_2)3(1 - k)} \\ &\quad + \frac{\alpha_1\alpha_2(\alpha^* \alpha_1^n - \beta^* \alpha_2^n)}{(\alpha_1 - \alpha_2)3(1 - k)} \\ &= \frac{\mathcal{H}M_{k,0} - 2\mathcal{H}M_{k,-1} - \mathcal{H}M_{k,n+1} + 2\mathcal{H}M_{k,n}}{3(1 - k)} \quad (\text{from Theorem 2.3}) \\ &= \frac{2\mathcal{H}M_{k,n} - \mathcal{H}M_{k,n+1} + \mathcal{H}M_{k,1} + (1 - 3k)\mathcal{H}M_{k,0}}{3(1 - k)}, \end{aligned}$$

as required. $\square$

Lemma 2.1. We have

- (a) $\alpha^* - \beta^* = \delta\mathcal{H}M_{k,0}$;
- (b) $\alpha^* + \overline{\alpha^*} = m_{k,0} = 2$;
- (c) $\alpha^* + \overline{\beta^*} = 2 + \delta\mathcal{H}M_{k,0}$;
- (d) $\alpha^*\beta^* = (227, -83\alpha_1 + 85\alpha_2, 19\alpha_1^2 - 17\alpha_2^2, -15\alpha_1^3 + 17\alpha_2^3 + 34\alpha_2 - 34\alpha_1, -13\alpha_1^4 + 15\alpha_2^4, 5\alpha_1^5 - 3\alpha_2^5 - 10\alpha_1^3 + 10\alpha_2^3, -\alpha_1^6 + 3\alpha_2^6 + 4\alpha_1^2 - 4\alpha_2^2, \alpha_1^7 + \alpha_2^7 + 2\alpha_1^5 - 2\alpha_2^5 - 4\alpha_1^3 + 4\alpha_2^3 - 8\alpha_1 + 8\alpha_2) = \overline{a_1}$.
- (e) $\beta^*\alpha^* = (227, 85\alpha_1 - 83\alpha_2, -17\alpha_1^2 + 19\alpha_2^2, 17\alpha_1^3 - 15\alpha_2^3 + 34\alpha_1 - 34\alpha_2, 15\alpha_1^4 - 13\alpha_2^4, -3\alpha_1^5 + 5\alpha_2^5 + 10\alpha_1^3 - 10\alpha_2^3, 3\alpha_1^6 - \alpha_2^6 - 4\alpha_1^2 + 4\alpha_2^2, \alpha_1^7 + \alpha_2^7 - 2\alpha_1^5 + 2\alpha_2^5 + 4\alpha_1^3 - 4\alpha_2^3 + 8\alpha_1 - 8\alpha_2) = \overline{a_2}$.

Proof. The proof of the lemma can be easily seen by substituting the values of the roots and performing the necessary operations. $\square$

Theorem 2.6 (Catalan's Identity). *For any positive integers n, s such that $n \geq s$, we have*

$$\mathcal{H}M_{k,n-s}\mathcal{H}M_{k,n+s} - \mathcal{H}M_{k,n}^2 = 2^{n-s}M_{k,s} \frac{\overline{a_1}\alpha_2^s - \overline{a_2}\alpha_1^s}{(\alpha_1 - \alpha_2)}.$$

Proof. By using the Binet formula of the hyperbolic k -Mersenne octonions, we have

$$\begin{aligned} \mathcal{H}M_{k,n-s}\mathcal{H}M_{k,n+s} - \mathcal{H}M_{k,n}^2 &= \frac{1}{(\alpha_1 - \alpha_2)^2} \left[\alpha^* \beta^* \alpha_1^n \alpha_2^n \left(1 - \frac{\alpha_2^s}{\alpha_1^s} \right) + \beta^* \alpha^* \alpha_2^n \alpha_1^n \left(1 - \frac{\alpha_1^s}{\alpha_2^s} \right) \right] \\ &= \frac{(\alpha_1 \alpha_2)^n (\alpha_1^s - \alpha_2^s)}{(\alpha_1 - \alpha_2)^2} \cdot \frac{\alpha^* \beta^* \alpha_2^s - \beta^* \alpha^* \alpha_1^s}{(\alpha_1 \alpha_2)^s} \\ &= (\alpha_1 \alpha_2)^{n-s} M_{k,s} \frac{\overline{a_1}\alpha_2^s - \overline{a_2}\alpha_1^s}{(\alpha_1 - \alpha_2)} \\ &= 2^{n-s} M_{k,s} \frac{\overline{a_1}\alpha_2^s - \overline{a_2}\alpha_1^s}{(\alpha_1 - \alpha_2)}. \end{aligned} \quad \square$$

Theorem 2.7 (Cassini's Identity). *For $n \geq 1$, we have*

$$\mathcal{H}M_{k,n-1}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,n}^2 = 2^{n-1} \frac{\overline{a_1}\alpha_2 - \overline{a_2}\alpha_1}{\sqrt{9k^2 - 8}}.$$

Proof. By substituting $s = 1$ in the Catalan Identity, we obtain the required result. $\square$

Theorem 2.8 (d'Ocagne's Identity). *Let $n, s \geq 0$, then we have*

$$\mathcal{H}M_{k,s}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,s+1}\mathcal{H}M_{k,n} = 2^n \frac{\overline{a_1}\alpha_1^{s-n} - \overline{a_2}\alpha_2^{s-n}}{\sqrt{9k^2 - 8}}.$$

Proof. By using the Binet formula of the hyperbolic k -Mersenne octonions, we have

$$\begin{aligned} \mathcal{H}M_{k,s}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,s+1}\mathcal{H}M_{k,n} &= \frac{\alpha^* \beta^* \alpha_1^s \alpha_2^n (\alpha_1 - \alpha_2) + \beta^* \alpha^* \alpha_2^s \alpha_1^n (\alpha_2 - \alpha_1)}{(\alpha_1 - \alpha_2)^2} \\ &= \frac{(\alpha_1 \alpha_2)^n (\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)^2} (\alpha^* \beta^* \alpha_1^{s-n} - \beta^* \alpha^* \alpha_2^{s-n}) \\ &= 2^n \frac{\overline{a_1}\alpha_1^{s-n} - \overline{a_2}\alpha_2^{s-n}}{\sqrt{9k^2 - 8}}. \end{aligned} \quad \square$$

Theorem 2.9 (Vajda Identity). *For any natural numbers n, i and j , we have*

$$\mathcal{H}M_{k,n+i}\mathcal{H}M_{k,n+j} - \mathcal{H}M_{k,n}\mathcal{H}M_{k,n+i+j} = -2^n M_{k,i} \frac{\overline{a_1}\alpha_2^j - \overline{a_2}\alpha_1^j}{\sqrt{9k^2 - 8}}.$$

Proof. Proof is similar to Theorem 2.8 by using Binet formula of hyperbolic k -Mersenne octonions. $\square$

3. HYPERBOLIC k -MERSENNE-LUCAS OCTONIONS

In this section, we introduce the hyperbolic k -Mersenne-Lucas octonions and establish their Binet formula. Furthermore, we study some well-known identities of them.

Definition 3.1. For $n \geq 0$, the hyperbolic k -Mersenne-Lucas octonions $\mathcal{H}m_{k,n}$ are defined by

$$\begin{aligned}\mathcal{H}m_{k,n} &= m_{k,n} + m_{k,n+1}i_1 + m_{k,n+2}i_2 + \cdots + m_{k,n+7}i_7 \\ &= (m_{k,n}, m_{k,n+1}, m_{k,n+2}, m_{k,n+3}, m_{k,n+4}, m_{k,n+5}, m_{k,n+6}, m_{k,n+7}).\end{aligned}$$

Definition 3.2. For $n \geq 0$, the conjugate of hyperbolic k -Mersenne-Lucas octonions $\mathcal{H}m_{k,n}$ is defined by

$$\begin{aligned}\overline{\mathcal{H}m_{k,n}} &= m_{k,n} - m_{k,n+1}i_1 - m_{k,n+2}i_2 - \cdots - m_{k,n+7}i_7 \\ &= (m_{k,n}, -m_{k,n+1}, -m_{k,n+2}, -m_{k,n+3}, -m_{k,n+4}, -m_{k,n+5}, -m_{k,n+6}, -m_{k,n+7}).\end{aligned}$$

If we use Definition 1.2 in Definition 3.1, then we can define the hyperbolic k -Mersenne-Lucas octonions recursively as

$$\mathcal{H}m_{k,n+2} = 3k\mathcal{H}m_{k,n+1} - 2\mathcal{H}m_{k,n}, \quad n \geq 0.$$

Theorem 3.1 (Binet Formula). *The Binet formulas for the hyperbolic k -Mersenne-Lucas octonions and their conjugate are*

- (a) $\mathcal{H}m_{k,n} = \alpha^* \alpha_1^n + \beta^* \alpha_2^n$;
- (b) $\overline{\mathcal{H}m_{k,n}} = \overline{\alpha^*} \alpha_1^n + \overline{\beta^*} \alpha_2^n$,

where

$$\begin{aligned}\alpha^* &= 1 + \alpha_1 i_1 + \alpha_1^2 i_2 + \alpha_1^3 i_3 + \alpha_1^4 i_4 + \alpha_1^5 i_5 + \alpha_1^6 i_6 + \alpha_1^7 i_7 \\ &= (1, \alpha_1, \alpha_1^2, \alpha_1^3, \alpha_1^4, \alpha_1^5, \alpha_1^6, \alpha_1^7), \\ \beta^* &= 1 + \alpha_2 i_1 + \alpha_2^2 i_2 + \alpha_2^3 i_3 + \alpha_2^4 i_4 + \alpha_2^5 i_5 + \alpha_2^6 i_6 + \alpha_2^7 i_7 \\ &= (1, \alpha_2, \alpha_2^2, \alpha_2^3, \alpha_2^4, \alpha_2^5, \alpha_2^6, \alpha_2^7), \\ \overline{\alpha^*} &= 1 - \alpha_1 i_1 - \alpha_1^2 i_2 - \alpha_1^3 i_3 - \alpha_1^4 i_4 - \alpha_1^5 i_5 - \alpha_1^6 i_6 - \alpha_1^7 i_7 \\ &= (1, -\alpha_1, -\alpha_1^2, -\alpha_1^3, -\alpha_1^4, -\alpha_1^5, -\alpha_1^6, -\alpha_1^7), \\ \overline{\beta^*} &= 1 - \alpha_2 i_1 - \alpha_2^2 i_2 - \alpha_2^3 i_3 - \alpha_2^4 i_4 - \alpha_2^5 i_5 - \alpha_2^6 i_6 - \alpha_2^7 i_7 \\ &= (1, -\alpha_2, -\alpha_2^2, -\alpha_2^3, -\alpha_2^4, -\alpha_2^5, -\alpha_2^6, -\alpha_2^7).\end{aligned}$$

Proof. (a) Using Definition 3.1 and the Binet formula of $m_{k,n}$, we have

$$\begin{aligned}\mathcal{H}m_{k,n} &= m_{k,n} + m_{k,n+1}i_1 + m_{k,n+2}i_2 + \cdots + m_{k,n+7}i_7 \\ &= (\alpha_1^n + \alpha_2^n) + (\alpha_1^{n+1} + \alpha_2^{n+1})i_1 + (\alpha_1^{n+2} + \alpha_2^{n+2})i_2 + (\alpha_1^{n+3} + \alpha_2^{n+3})i_3 \\ &\quad + (\alpha_1^{n+4} + \alpha_2^{n+4})i_4 + (\alpha_1^{n+5} + \alpha_2^{n+5})i_5 + (\alpha_1^{n+6} + \alpha_2^{n+6})i_6 \\ &\quad + (\alpha_1^{n+7} + \alpha_2^{n+7})i_7.\end{aligned}$$

$$\begin{aligned}
& + (\alpha_1^{n+7} + \alpha_2^{n+7}) i_7 \\
& = \alpha_1^n (1 + \alpha_1 i_1 + \alpha_1^2 i_2 + \cdots + \alpha_1^7 i_7) + \alpha_2^n (1 + \alpha_2 i_1 + \alpha_2^2 i_2 + \cdots + \alpha_2^7 i_7) \\
& = \alpha^* \alpha_1^n + \beta^* \alpha_2^n.
\end{aligned}$$

The proof of (b) is similar to (a) by using the Definition 3.2. $\square$

Lemma 3.1. *We have*

- (a) $\alpha^* + \beta^* = \mathcal{H}m_{k,0};$
- (b) $\alpha^* - \beta^* = \mathcal{H}m_{k,0} - 2.$

Proof. The proof of the lemma can be easily seen by substituting the values of α^* and β^* and performing the necessary operations. $\square$

Theorem 3.2 (Catalan's Identity). *For any positive integers n, s such that $n \geq s$, we have*

$$\mathcal{H}m_{k,n-s} \mathcal{H}m_{k,n+s} - \mathcal{H}m_{k,n}^2 = 2^{n-s} M_{k,s} \sqrt{9k^2 - 8} (\beta^* \alpha^* \alpha_1^s - \alpha^* \beta^* \alpha_2^s).$$

Proof. By using the Binet formula of the hyperbolic k -Mersenne-Lucas octonions, we have

$$\begin{aligned}
\mathcal{H}m_{k,n-s} \mathcal{H}m_{k,n+s} - \mathcal{H}m_{k,n}^2 &= (\alpha^* \alpha_1^{n-s} + \beta^* \alpha_2^{n-s}) (\alpha^* \alpha_1^{n+s} + \beta^* \alpha_2^{n+s}) \\
&\quad - (\alpha^* \alpha_1^n + \beta^* \alpha_2^n)^2 \\
&= (\alpha^*)^2 \alpha_1^{2n} + \alpha^* \beta^* \alpha_1^{n-s} \alpha_2^{n+s} + \beta^* \alpha^* \alpha_1^{n+s} \alpha_2^{n-s} + (\beta^*)^2 \alpha_2^{2n} \\
&\quad - ((\alpha^*)^2 \alpha_1^{2n} + \alpha^* \beta^* \alpha_1^n \alpha_2^n + \beta^* \alpha^* \alpha_1^n \alpha_2^n + (\beta^*)^2 \alpha_2^{2n}) \\
&= \alpha^* \beta^* \alpha_1^{n-s} \alpha_2^{n+s} + \beta^* \alpha^* \alpha_1^{n+s} \alpha_2^{n-s} - \alpha^* \beta^* \alpha_1^n \alpha_2^n - \beta^* \alpha^* \alpha_1^n \alpha_2^n \\
&= (\alpha_1 \alpha_2)^n \left[\alpha^* \beta^* \left(\frac{\alpha_2^s}{\alpha_1^s} - 1 \right) + \beta^* \alpha^* \left(\frac{\alpha_1^s}{\alpha_2^s} - 1 \right) \right] \\
&= (\alpha_1 \alpha_2)^n (\alpha_1^s - \alpha_2^s) \left(\frac{\beta^* \alpha^*}{\alpha_2^s} - \frac{\alpha^* \beta^*}{\alpha_1^s} \right) \\
&= (\alpha_1 \alpha_2)^n (\alpha_1^s - \alpha_2^s) \frac{\beta^* \alpha^* \alpha_1^s - \alpha^* \beta^* \alpha_2^s}{\alpha_1^s \alpha_2^s} \\
&= 2^{n-s} M_{k,s} \sqrt{9k^2 - 8} (\beta^* \alpha^* \alpha_1^s - \alpha^* \beta^* \alpha_2^s),
\end{aligned}$$

as required. $\square$

Theorem 3.3 (Cassini's Identity). *For $n \geq 1$, we have*

$$\mathcal{H}m_{k,n-1} \mathcal{H}m_{k,n+1} - \mathcal{H}m_{k,n}^2 = 2^{n-1} \sqrt{9k^2 - 8} (\beta^* \alpha^* \alpha_1 - \alpha^* \beta^* \alpha_2).$$

Proof. By substituting $s = 1$ in the Catalan identity, we obtain the required result. $\square$

Theorem 3.4 (d'Ocagne's Identity). *Let $n, s \geq 0$, then we have*

$$\mathcal{H}m_{k,s} \mathcal{H}m_{k,n+1} - \mathcal{H}m_{k,s+1} \mathcal{H}m_{k,n} = 2^n \sqrt{9k^2 - 8} (\beta^* \alpha^* \alpha_2^{s-n} - \alpha^* \beta^* \alpha_1^{s-n}).$$

Proof. By using the Binet formula of the hyperbolic k -Mersenne-Lucas octonions, we have

$$\begin{aligned}
 & \mathcal{H}m_{k,s}\mathcal{H}m_{k,n+1} - \mathcal{H}m_{k,s+1}\mathcal{H}m_{k,n} \\
 &= (\alpha^*\alpha_1^s + \beta^*\alpha_2^s) (\alpha^*\alpha_1^{n+1} + \beta^*\alpha_2^{n+1}) - (\alpha^*\alpha_1^{s+1} + \beta^*\alpha_2^{s+1}) (\alpha^*\alpha_1^n + \beta^*\alpha_2^n) \\
 &= (\alpha^*)^2\alpha_1^{n+s+1} + \alpha^*\beta^*\alpha_1^s\alpha_2^{n+1} + \beta^*\alpha^*\alpha_1^{n+1}\alpha_2^s + (\beta^*)^2\alpha_2^{n+s+1} \\
 &\quad - \left((\alpha^*)^2\alpha_1^{n+s+1} + \alpha^*\beta^*\alpha_1^{s+1}\alpha_2^n + \beta^*\alpha^*\alpha_1^n\alpha_2^{s+1} + (\beta^*)^2\alpha_2^{n+s+1} \right) \\
 &= \beta^*\alpha^*\alpha_1^n\alpha_2^s(\alpha_1 - \alpha_2) - \alpha^*\beta^*\alpha_1^s\alpha_2^n(\alpha_1 - \alpha_2) \\
 &= (\alpha_1\alpha_2)^n(\alpha_1 - \alpha_2)(\beta^*\alpha^*\alpha_2^{s-n} - \alpha^*\beta^*\alpha_1^{s-n}) \\
 &= 2^n\sqrt{9k^2 - 8}(\beta^*\alpha^*\alpha_2^{s-n} - \alpha^*\beta^*\alpha_1^{s-n}). \quad \square
 \end{aligned}$$

Theorem 3.5 (Vajda Identity). *For any natural numbers n, i and j , we have*

$$\mathcal{H}m_{k,n+i}\mathcal{H}m_{k,n+j} - \mathcal{H}m_{k,n}\mathcal{H}m_{k,n+i+j} = 2^n M_{k,i}\sqrt{9k^2 - 8}(\alpha^*\beta^*\alpha_2^j - \beta^*\alpha^*\alpha_1^j).$$

Proof. Proof is similar to Theorem 3.4 by using Binet formula of hyperbolic k -Mersenne-Lucas octonions. $\square$

Theorem 3.6. *For $r, s, n \in \mathbb{N}$ such that $r \geq s$, the generating functions for hyperbolic k -Mersenne-Lucas octonions are given as*

$$\begin{aligned}
 \text{(a)} \quad & \sum_{n=0}^{\infty} \mathcal{H}m_{k,sn}x^n = \frac{\mathcal{H}m_{k,0} - x(\mathcal{H}m_{k,0}m_{k,s} - \mathcal{H}m_{k,s})}{1 - m_{k,s}x + 2^sx^2}; \\
 \text{(b)} \quad & \sum_{n=0}^{\infty} \mathcal{H}m_{k,sn+r}x^n = \frac{\mathcal{H}m_{k,r} - 2^s\mathcal{H}m_{k,r-s}x}{1 - m_{k,s}x + 2^sx^2}; \\
 \text{(c)} \quad & \text{The exponential generating function is given as}
 \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{H}m_{k,sn}x^n}{n!} = \alpha^*e^{\alpha_1^s x} + \beta^*e^{\alpha_2^s x}.$$

Proof. (a) Using the Theorem 3.1, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \mathcal{H}m_{k,sn}x^n &= \sum_{n=0}^{\infty} (\alpha^*\alpha_1^{sn} + \beta^*\alpha_2^{sn})x^n = \alpha^* \sum_{n=0}^{\infty} (\alpha_1^s x)^n + \beta^* \sum_{n=0}^{\infty} (\alpha_2^s x)^n \\
 &= \alpha^* \left(\frac{1}{1 - \alpha_1^s x} \right) + \beta^* \left(\frac{1}{1 - \alpha_2^s x} \right) \\
 &= \frac{(\alpha^* + \beta^*) - x(\beta^*\alpha_1^s + \alpha^*\alpha_2^s)}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2} \\
 &= \frac{(\alpha^* + \beta^*) - x(\beta^*\alpha_1^s + \beta^*\alpha_2^s - \beta^*\alpha_2^s + \alpha^*\alpha_2^s + \alpha^*\alpha_1^s - \alpha^*\alpha_1^s)}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2} \\
 &= \frac{(\alpha^* + \beta^*) - x[(\alpha^* + \beta^*)(\alpha_1^s + \alpha_2^s) - (\alpha^*\alpha_1^s + \beta^*\alpha_2^s)]}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2} \\
 &= \frac{\mathcal{H}m_{k,0} - x(\mathcal{H}m_{k,0}m_{k,s} - \mathcal{H}m_{k,s})}{1 - m_{k,s}x + 2^sx^2}.
 \end{aligned}$$

The proofs of (b) and (c) are similar to that of (a). $\square$

Theorem 3.7. For $k \neq 1$, we have

$$\sum_{i=0}^n \mathcal{H}m_{k,i} = \frac{2\mathcal{H}m_{k,n} - \mathcal{H}m_{k,n+1} + \mathcal{H}m_{k,1} + \mathcal{H}m_{k,0}(1-3k)}{3(1-k)}.$$

Proof. Using Theorem 3.1, we obtain

$$\begin{aligned} \sum_{i=0}^n \mathcal{H}m_{k,i} &= \sum_{i=0}^n (\alpha^* \alpha_1^i + \beta^* \alpha_2^i) = \alpha^* \sum_{i=0}^n \alpha_1^i + \beta^* \sum_{i=0}^n \alpha_2^i \\ &= \alpha^* \left(\frac{\alpha_1^{n+1} - 1}{\alpha_1 - 1} \right) + \beta^* \left(\frac{\alpha_2^{n+1} - 1}{\alpha_2 - 1} \right) \\ &= \frac{\alpha_1 \alpha_2 (\alpha^* \alpha_1^n + \beta^* \alpha_2^n) - (\alpha^* \alpha_1^{n+1} + \beta^* \alpha_2^{n+1}) - (\alpha^* \alpha_2 + \beta^* \alpha_1) + (\alpha^* + \beta^*)}{\alpha_1 \alpha_2 - (\alpha_1 + \alpha_2) + 1} \\ &= \frac{2\mathcal{H}m_{k,n} - \mathcal{H}m_{k,n+1} + \mathcal{H}m_{k,1} + \mathcal{H}m_{k,0}(1-3k)}{3(1-k)}. \end{aligned} \quad \square$$

4. RELATIONS BETWEEN HYPERBOLIC k -MERSENNE AND k -MERSENNE-LUCAS OCTONIONS

In this section, we have given theorems showing the relations between hyperbolic k -Mersenne octonions and hyperbolic k -Mersenne-Lucas octonions.

Theorem 4.1. For $s, n \in \mathbb{N}$, a generalization of the generating function of hyperbolic k -Mersenne octonions is as follows

$$\sum_{n=0}^{\infty} \mathcal{H}M_{k,sn} x^n = \frac{\mathcal{H}M_{k,0} + (M_{k,s} \mathcal{H}m_{k,0} - \mathcal{H}M_{k,s}) x}{1 - m_{k,s} x + 2^s x^2}.$$

Proof. Using the Theorem 2.3, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{H}M_{k,sn} x^n &= \sum_{n=0}^{\infty} \left(\frac{\alpha^* \alpha_1^{sn} - \beta^* \alpha_2^{sn}}{\alpha_1 - \alpha_2} \right) x^n = \frac{\alpha^*}{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} (\alpha_1^s x)^n - \frac{\beta^*}{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} (\alpha_2^s x)^n \\ &= \frac{\alpha^*}{\alpha_1 - \alpha_2} \cdot \frac{1}{1 - \alpha_1^s x} - \frac{\beta^*}{\alpha_1 - \alpha_2} \cdot \frac{1}{1 - \alpha_2^s x} \\ &= \frac{(\alpha^* - \beta^*) + (\beta^* \alpha_1^s - \alpha^* \alpha_2^s) x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\ &= \frac{(\alpha^* - \beta^*) + (\beta^* \alpha_1^s + \beta^* \alpha_2^s - \beta^* \alpha_2^s - \alpha^* \alpha_1^s + \alpha^* \alpha_1^s - \alpha^* \alpha_2^s) x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\ &= \frac{(\alpha^* - \beta^*) + ((\alpha_1^s - \alpha_2^s)(\alpha^* + \beta^*) - (\alpha^* \alpha_1^s - \beta^* \alpha_2^s)) x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\ &= \frac{\mathcal{H}M_{k,0} + (M_{k,s} \mathcal{H}m_{k,0} - \mathcal{H}M_{k,s}) x}{1 - m_{k,s} x + 2^s x^2}, \end{aligned}$$

as required. $\square$

Theorem 4.2. *For any integer t , we have*

$$\begin{aligned} \text{(a)} \quad \mathcal{H}M_{k,t}^2 + \mathcal{H}m_{k,t}^2 &= \frac{(9k^2 - 7)S_{k,2t}^* + (9k^2 - 9)2^t(\overline{a_1} + \overline{a_2})}{9k^2 - 8}; \\ \text{(b)} \quad \mathcal{H}M_{k,t}^2 - \mathcal{H}m_{k,t}^2 &= \frac{(9 - 9k^2)S_{k,2t}^* - (9k^2 - 7)2^t(\overline{a_1} + \overline{a_2})}{(9k^2 - 8)}, \end{aligned}$$

where $\overline{a_1}$ and $\overline{a_2}$ are given in Lemma 2.1.

Proof. (a) From the Binet formulas of the hyperbolic k -Mersenne and k -Mersenne-Lucas octonions, we write

$$\begin{aligned} & \mathcal{H}M_{k,t}^2 + \mathcal{H}m_{k,t}^2 \\ &= \frac{(\alpha^*)^2(\alpha_1)^{2t} - \alpha^*\beta^*\alpha_1^t\alpha_2^t - \beta^*\alpha^*\alpha_2^t\alpha_1^t + (\beta^*)^2(\alpha_2)^{2t}}{9k^2 - 8} \\ & \quad + \frac{(\alpha^*)^2(\alpha_1)^{2t} + \alpha^*\beta^*\alpha_1^t\alpha_2^t + \beta^*\alpha^*\alpha_2^t\alpha_1^t + (\beta^*)^2(\alpha_2)^{2t}}{(1 + (9k^2 - 8))((\alpha^*)^2(\alpha_1)^{2t} + (\beta^*)^2(\alpha_2)^{2t}) + ((9k^2 - 8) - 1)\alpha_1^t\alpha_2^t(\alpha^*\beta^* + \beta^*\alpha^*)} \\ &= \frac{(9k^2 - 7)S_{k,2t}^* + (9k^2 - 9)2^t(\overline{a_1} + \overline{a_2})}{9k^2 - 8}. \end{aligned}$$

The proof (b) is similar to that of (a). □

Theorem 4.3. *For every integer $r, s \geq t$, there is the following equation*

$$\mathcal{H}M_{k,r+s}\mathcal{H}m_{k,r+t} - \mathcal{H}M_{k,r+t}\mathcal{H}m_{k,r+s} = (\overline{a_1} + \overline{a_2})2^{r-t}M_{k,s-t}.$$

Proof. We write

$$\begin{aligned} & \mathcal{H}M_{k,r+s}\mathcal{H}m_{k,r+t} - \mathcal{H}M_{k,r+t}\mathcal{H}m_{k,r+s} \\ &= \frac{\alpha^*\beta^*\alpha_1^r\alpha_2^r(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s) + \beta^*\alpha^*\alpha_1^r\alpha_2^r(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s)}{\alpha_1 - \alpha_2} \\ &= (\alpha^*\beta^* + \beta^*\alpha^*)(\alpha_1\alpha_2)^{r-t} \frac{\alpha_1^{s-t} - \alpha_2^{s-t}}{\alpha_1 - \alpha_2} \\ &= (\overline{a_1} + \overline{a_2})2^{r-t}M_{k,s-t}. \end{aligned} \quad \square$$

Theorem 4.4. *For any integers s and t , we have*

$$\begin{aligned} \text{(a)} \quad \mathcal{H}M_{k,s}m_{k,t} &= \mathcal{H}M_{k,s+t} + 2^t\mathcal{H}M_{k,s-t}; \\ \text{(b)} \quad \mathcal{H}m_{k,s}M_{k,t} &= \mathcal{H}m_{k,s+t} + 2^t\mathcal{H}m_{k,s-t}. \end{aligned}$$

Proof. (a) We have

$$\begin{aligned} \mathcal{H}M_{k,s}m_{k,t} &= \frac{\alpha^*\alpha_1^{s+t} - \beta^*\alpha_2^{s+t}}{\alpha_1 - \alpha_2} + \frac{(\alpha_1\alpha_2)^t(\alpha^*\alpha_1^{s-t} - \beta^*\alpha_2^{s-t})}{\alpha_1 - \alpha_2} \\ &= \mathcal{H}M_{k,s+t} + 2^t\mathcal{H}M_{k,s-t}. \end{aligned}$$

The proof of (b) is similar to that of (a). □

Theorem 4.5. *For any integer $t \geq s$, the following equations are true.*

$$\begin{aligned}
(a) \quad & \mathcal{H}M_{k,s}\mathcal{H}M_{k,t} - \mathcal{H}M_{k,t}\mathcal{H}M_{k,s} = \frac{2^s(\overline{a_1} - \overline{a_2})M_{k,t-s}}{\sqrt{9k^2 - 8}}; \\
(b) \quad & \mathcal{H}m_{k,s}\mathcal{H}m_{k,t} - \mathcal{H}m_{k,t}\mathcal{H}m_{k,s} = 2^s\sqrt{9k^2 - 8}(\overline{a_1} - \overline{a_2})M_{k,t-s}; \\
(c) \quad & \mathcal{H}M_{k,t}\mathcal{H}m_{k,s} - \mathcal{H}M_{k,s}\mathcal{H}m_{k,t} = 2^s(\overline{a_1} + \overline{a_2})M_{k,t-s}; \\
(d) \quad & \mathcal{H}M_{k,t}\mathcal{H}m_{k,s} - \mathcal{H}m_{k,t}\mathcal{H}M_{k,s} = -2^{s+1}\frac{\overline{a_1}\alpha_2^{t-s} - \overline{a_2}\alpha_1^{t-s}}{\sqrt{9k^2 - 8}}.
\end{aligned}$$

Proof. (a) We have

$$\begin{aligned}
\mathcal{H}M_{k,s}\mathcal{H}M_{k,t} - \mathcal{H}M_{k,t}\mathcal{H}M_{k,s} &= \frac{\alpha^*\beta^*\alpha_1^s\alpha_2^s(\alpha_1^{t-s} - \alpha_2^{t-s}) - \beta^*\alpha^*\alpha_1^s\alpha_2^s(\alpha_1^{t-s} - \alpha_2^{t-s})}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{(\alpha^*\beta^* - \beta^*\alpha^*)(\alpha_1^s\alpha_2^s)(\alpha_1^{t-s} - \alpha_2^{t-s})}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{2^s(\overline{a_1} - \overline{a_2})M_{k,t-s}}{\sqrt{9k^2 - 8}}.
\end{aligned}$$

The other equations are proved similarly to that of (a). $\square$

5. CONCLUSION

In this study, we introduced the hyperbolic k -Mersenne and k -Mersenne-Lucas octonions. We obtained Binet formula, Cassini identity, Catalan identity, d'Ocagne identity, Vajda identity, ordinary and exponential generating function, etc. of these octonions. Also, many properties were obtained and studied the relations between hyperbolic k -Mersenne and k -Mersenne-Lucas octonions. As a consequence, for $k = 1$ results hold for hyperbolic Mersenne and Mersenne-Lucas octonions.

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THE GLOBAL BEHAVIOR OF A SECOND ORDER EXPONENTIAL DIFFERENCE EQUATION

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ABSTRACT. In this paper we present the Julia set and the global behavior of an exponential second order difference equation of the type

$$x_{n+1} = ax_{n-1} + bx_{n-1} \exp(cx_{n-1} + cx_n),$$

where $a \geq 0$, $b > 0$ and $c > 0$ with non-negative initial conditions.

1. INTRODUCTION

In general, difference equations and systems of difference equations in exponential forms have numerous applications in biology, more precisely, they can be used to discuss population model. One of the most simple results on exponential difference equation have been obtained in [8] for the equation of type

$$x_{n+1} = x_n \exp(r(1 - x_n)),$$

known as Ricker's equation, which describes a population with a propensity to simple exponential growth at low densities and tendency to decrease at high densities. In [9] the qualitative behavior of the exponential second order difference equation of the two-dimensional population model

$$x_{n+1} = a + bx_{n-1} \exp(-x_n)$$

is completely investigated and described. In [14] we can find results about boundedness and asymptotic behavior of the positive solution for the difference equation of type

$$x_{n+1} = a + bx_n \exp(-x_{n-1}),$$

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where a and b are positive constants and the initial values x_{-1} , x_0 are nonnegative real numbers. In [11] are given the conditions for the global behavior of the positive solutions for the difference equation

$$x_{n+1} = ax_n + bx_{n-1} \exp(-x_n),$$

where a and b are positive real numbers with positive initial conditions x_{-1} , x_0 . The global stability and bounded nature of the positive solutions of the difference equation

$$x_{n+1} = a + bx_{n-1} + cx_{n-1} \exp(-x_n)$$

are investigated in [10]. In [7] have been obtained results for the local stability of equilibria, parametric conditions for transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation of the following second-order difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-1} \exp(-\sigma x_{n-1}),$$

where the initial conditions satisfy $x_{-1} > 0$, $x_0 > 0$ and α , β and σ are the positive constants. In this paper we will present very unusual results for exponential second order difference equations. Our results are based on the theorems which hold for monotone difference equations. Our principal tool is the theory of monotone maps, and in particular cooperative maps, which guarantee the existence and uniqueness of the stable and unstable invariant manifolds for the fixed points and periodic points (see [5]). Consider the difference equation

$$(1.1) \quad x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots,$$

where f is a continuous and increasing function in both variables. The following result has been obtained in [1].

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of (1.1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following.*

- (i) *Eventually they are both monotonically increasing.*
- (ii) *Eventually they are both monotonically decreasing.*
- (iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

As a consequence of Theorem 1.1 every bounded solution of (1.1) approaches either an equilibrium solution or period-two solution and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of (1.1) is the problem how to determine the basins of attraction of three different types of attractors: the equilibrium solutions, minimal period-two solution(s) and the point(s) at infinity. The following result can be proved by using the techniques of proof of Theorem 11 in [5].

Theorem 1.2. *Consider (1.1) where f is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2)$$

and that E_1 is a local attractor and E_2 is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$, with the neighborhoods where f is strictly increasing. Then the basin of attraction $\mathcal{B}(E_1)$ of E_1 is the region below the global stable manifold $\mathcal{W}^s(E_2)$. More precisely

$$\mathcal{B}(E_1) = \{(x, y) : \text{exists } y_u : y < y_u, (x, y_u) \in \mathcal{W}^s(E_2)\}.$$

The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ is exactly the global stable manifold of E_2 . The global stable manifold extend to the boundary of the domain of (1.1). If there exists a period-two solution, then the end points of the global stable manifold are exactly the period-two solution.

Now, the theorems that are applied in [5] provided the two continuous curves $\mathcal{W}^s(E_2)$ (stable manifold) and $\mathcal{W}^u(E_2)$ (unstable manifold), both passing through the point $E_2(x_2, y_2)$ from Theorem 1.2, such that $\mathcal{W}^s(E_2)$ is a graph of decreasing function and $\mathcal{W}^u(E_2)$ is a graph of an increasing function. The curve $\mathcal{W}^s(E_2)$ splits the first quadrant of initial conditions into two disjoint regions, but we do not know the explicit form of the curve $\mathcal{W}^s(E_2)$. In this paper we investigate the following difference equation

$$(1.2) \quad x_{n+1} = ax_{n-1} + bx_{n-1} \exp(cx_{n-1} + cx_n),$$

where $a \geq 0$, $b > 0$ and $c > 0$ with non-negative initial conditions, that has infinitely many period-two solutions and we expose the explicit form of the curve that separates the first quadrant into two basins of attraction of a locally stable equilibrium point and of the point at infinity. One of the major problems in the dynamics of monotonic maps is determining the basin of attraction of the point at infinity and in particular the boundary of the that basin known as the Julia set. We precisely determined the Julia set of (1.2) and we obtained the global dynamics in the interior of the Julia set, which includes all the points for which solutions are not asymptotic to the point at infinity. It turned out that the Julia set for (1.2) is the union of the stable manifolds of some saddle equilibrium points, nonhyperbolic equilibrium points or period-two points. We first list some results needed for the proofs of our theorems. The main result for studying local stability of equilibria is linearized stability theorem (see Theorem 1.1 in [12]).

Theorem 1.3 (Linearized stability). *Consider the difference equation*

$$(1.3) \quad x_{n+1} = f(x_n, x_{n-1})$$

and let $\bar{x}$ be an equilibrium point of difference equation (1.3). Let $p = \frac{\partial f(\bar{x}, \bar{x})}{\partial u}$ and $q = \frac{\partial f(\bar{x}, \bar{x})}{\partial v}$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\bar{x}$. Let λ_1 and λ_2 roots of the quadratic equation $\lambda^2 - p\lambda - q = 0$.

- a) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the equilibrium $\bar{x}$ is locally asymptotically stable (sink).
- b) If $|\lambda_1| > 1$ or $|\lambda_2| > 1$, then the equilibrium $\bar{x}$ is unstable.
- c) $|\lambda_1| < 1$ and $|\lambda_2| < 1 \Leftrightarrow |p| < 1 - q < 2$. Equilibrium $\bar{x}$ is a sink.
- d) $|\lambda_1| > 1$ and $|\lambda_2| > 1 \Leftrightarrow |q| > 1$ and $|p| < |1 - q|$. Equilibrium $\bar{x}$ is a repeller.
- e) $|\lambda_1| > 1$ and $|\lambda_2| < 1 \Leftrightarrow |p| > |1 - q|$. Equilibrium $\bar{x}$ is a saddle point.
- f) $|\lambda_1| = 1$ or $|\lambda_2| = 1 \Leftrightarrow |p| = |1 - q|$ or $q = -1$ and $|p| \leq 2$. Equilibrium $\bar{x}$ is called a non-hyperbolic point.

The next theorem (Theorem 1.4.1. in [6]) is a very useful tool in establishing bounds for the solutions of nonlinear equations in terms of the solutions of equations with known behaviour.

Theorem 1.4. Let I be an interval of real numbers, let k be a positive integer, and let $F : I^{k+1} \rightarrow I$ be a function which is increasing in all its arguments. Assume that $\{x_n\}_{n=-k}^\infty$, $\{y_n\}_{n=-k}^\infty$ and $\{z_n\}_{n=-k}^\infty$ are sequences of real numbers such that

$$\begin{aligned} x_{n+1} &\leq F(x_n, \dots, x_{n-k}), & n = 0, 1, \dots, \\ y_{n+1} &= F(y_n, \dots, y_{n-k}), & n = 0, 1, \dots, \\ z_{n+1} &\geq F(z_n, \dots, z_{n-k}), & n = 0, 1, \dots \end{aligned}$$

and

$$x_n \leq y_n \leq z_n, \quad \text{for all } -k \leq n \leq 0.$$

Then

$$x_n \leq y_n \leq z_n, \quad \text{for all } n > 0.$$

2. MAIN RESULTS

By using Theorem 1.3, we obtained the following result on local stability of the zero equilibrium of (1.2).

Proposition 2.1. The zero equilibrium of (1.2) is one of the following:

- a) locally asymptotically stable if $a + b < 1$;
- b) non-hyperbolic $a + b = 1$;
- c) unstable if $a + b > 1$.

Set $f(x, y) = ay + by \exp(cy + cx)$ and let $p = \frac{\partial f(\bar{x}, \bar{x})}{\partial x}$ and $q = \frac{\partial f(\bar{x}, \bar{x})}{\partial y}$ denote the partial derivatives of $f(x, y)$ evaluated at the equilibrium $\bar{x}$. The linearized equation

at the positive equilibrium $\bar{x}$ is

$$\begin{aligned} z_{n+1} &= pz_n + qz_{n-1}, \\ p &= bc\bar{x} \exp(2c\bar{x}), \\ q &= a + b(1 + c\bar{x}) \exp(2c\bar{x}). \end{aligned}$$

Now, in view of Theorem 1.3 we obtain the following results on local stability of the positive equilibrium of (1.2).

Proposition 2.2. *The positive equilibrium of (1.2) is one of the following:*

- a) *locally asymptotically stable if $p + q < 1$;*
- b) *non-hyperbolic if $p + q = 1$ or $q - p = 1$;*
- c) *unstable if $p + q > 1$;*
- d) *saddle point if $p > |q - 1|$;*
- e) *repeller if $1 - q < p < q - 1$.*

Theorem 2.1. *If $a \geq 1$ or $b \geq 1$ or $a + b > 1$, then every solution $\{x_n\}$ of (1.2) satisfies $\lim_{n \rightarrow \infty} x_n = \infty$.*

Proof. Let be $a \geq 1$ or $b \geq 1$, then $a + b > 1$. If $\{x_n\}$ is a solution of (1.2), then $\{x_n\}$ satisfies the inequality

$$\begin{aligned} x_{n+1} &= ax_{n-1} + bx_{n-1} \exp(cx_{n-1} + cx_n) \\ &\geq ax_{n-1} + bx_{n-1} = (a + b)x_{n-1}, \quad n = 0, 1, \dots, \end{aligned}$$

which in view of the result on difference inequalities, see Theorem 1.4, implies that $x_n \geq y_n$, $n \geq 1$, where $\{y_n\}$ is a solution of the initial value problem

$$y_{n+1} = (a + b)y_{n-1}, \quad y_{-1} = x_{-1} \text{ and } y_0 = x_0, \quad n = 0, 1, \dots$$

Consequently, if $x_0, x_{-1} > 0$, then $y_0, y_{-1} > 0$, $y_n \geq 0$ for all n , and

$$x_n \geq y_n = \lambda_1 \sqrt{a + b}^n + \lambda_2 \left(-\sqrt{a + b}\right)^n, \quad n = 1, 2, \dots,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $y_n \geq 0$ for all n , which implies $\lim_{n \rightarrow \infty} x_n = \infty$. $\square$

Theorem 2.2. *Consider the difference equation (1.2) in the first quadrant of initial conditions, where $a, b, c > 0$ and $a + b < 1$. Then (1.2) has a zero equilibrium and a unique positive equilibrium $\bar{x}_+ = \frac{1}{2c} \ln \frac{1-a}{b}$. The line $b \exp(cy + cx) = 1 - a$ is the Julia set and separates the first quadrant into two regions: the region below the given line is the basin of attraction of point $E_0(0, 0)$, the region above the line is the basin of attraction of the point at infinity and every point on the line except $E_+(\bar{x}_+, \bar{x}_+)$ is a period-two solution of (1.2).*

Proof. The equilibrium points of (1.2) are the solutions of equation

$$x(a + b \exp(2cx)) = x,$$

that is equivalent to

$$(2.1) \quad x(b \exp(2cx) + a - 1) = 0,$$

which implies that (2.1) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+$. Since $a+b < 1$, then by applying Proposition (2.1) the zero equilibrium is locally asymptotically stable. Denote by $f(x, y) = ay + by \exp(cx + cy)$ and let p and q denote the partial derivatives of function $f(x, y)$ at point E_+ . By straightforward calculation we obtain that the following hold:

$$\begin{aligned} p + q &= a + b(1 + 2c\bar{x}) \exp(2c\bar{x}) \\ &= a + b(1 + 2c\bar{x}) \frac{1-a}{b} = 1 + 2c(1-a)\bar{x} > 1, \\ q - p &= a + b \exp(2c\bar{x}) = a + b \cdot \frac{1-a}{b} = 1. \end{aligned}$$

Hence, by applying Proposition 2.2 the positive equilibrium is an unstable non-hyperbolic point. Period-two solution u, v satisfies the system

$$\begin{aligned} u &= (a + b \exp(cu + cv)) u, \\ v &= (a + b \exp(cu + cv)) v. \end{aligned}$$

Obviously, the point $(0, 0)$ is solution of the system above, but it is not minimal period-two solution. Hence, it has to be $v > 0$ which implies $a + b \exp(cu + cv) = 1$. Therefore, every point of the set $\{(x, y) : a + b \exp(cx + cy) = 1\}$ is a period-two solution of (1.2) except point E_+ . Clearly, the curve $g(x, y) = a + b \exp(cx + cy) = 1$ is a graph of the decreasing function in the first quadrant, more precisely that is line $y = -x + \frac{1}{c} \ln \frac{1-a}{b}$. Let $\{x_n\}$ be a solution of (1.2) for initial condition (x_0, x_{-1}) which lies below the line $g(x, y) = 1$. Then

$$\begin{aligned} g(x_0, x_{-1}) &= a + b \exp(cx_0 + cx_{-1}) < 1, \\ x_{n+1} &= g(x_n, x_{n-1}) x_{n-1} \end{aligned}$$

and

$$\begin{aligned} x_1 &= g(x_0, x_{-1}) x_{-1} < x_{-1}, \\ x_2 &= g(x_1, x_0) x_0 < g(x_{-1}, x_0) x_0 = g(x_0, x_{-1}) x_0 < x_0. \end{aligned}$$

Thus (x_2, x_1) and (x_0, x_{-1}) are two points in North-East ordering $(x_2, x_1) \leq_{ne} (x_0, x_{-1})$ which means that the point (x_2, x_1) is also below the curve $g(x, y) = 1$ and also holds

$$g(x_2, x_1) < 1.$$

Similarly we find

$$\begin{aligned} x_3 &= g(x_2, x_1) x_1 < x_1, \\ x_4 &= g(x_3, x_2) x_2 < g(x_1, x_2) x_2 = g(x_2, x_1) x_2 < x_2. \end{aligned}$$

Continuing on this way we get

$$(0, 0) \leq_{ne} \cdots \leq_{ne} (x_4, x_3) \leq_{ne} (x_2, x_1) \leq_{ne} (x_0, x_{-1}),$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically decreasing and bounded below by 0. Since below the line $g(x, y) = 1$ there are no period-two

solutions it must be $x_{2n} \rightarrow 0$ and $x_{2n+1} \rightarrow 0$. On the other hand, if we consider solution $\{x_n\}$ of (1.2) for initial condition (x_0, x_{-1}) which lies above the line $g(x, y) = 1$ then $g(x_0, x_{-1}) > 1$ and by applying the method shown above we obtain the following condition:

$$(x_{-1}, x_0) \leq_{ne} (x_1, x_2) \leq_{ne} (x_3, x_4) \leq_{ne} \cdots$$

Therefore, both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing, hence $x_{2n} \rightarrow \infty$ and $x_{2n+1} \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

Figure 1 is visual illustration of Theorem 2.2 obtained by using Mathematica 9.0, with the boundaries of the basins of attraction obtained by using the software package Dynamica [6].

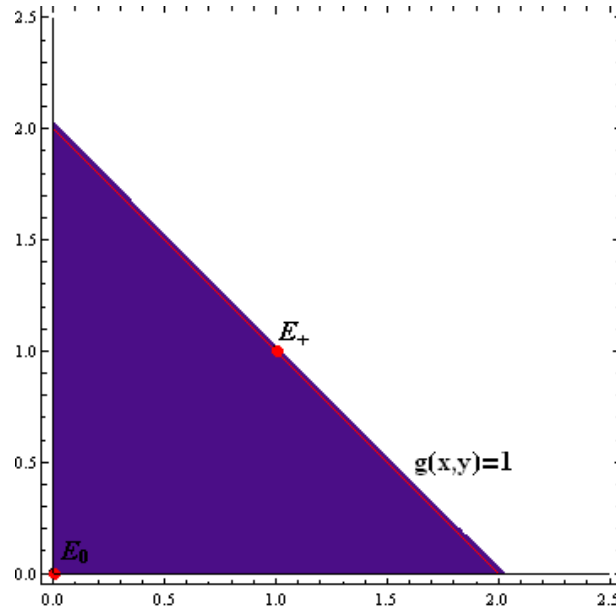


FIGURE 1. Case: $a = 1 - e^{-2}$, $b = e^{-3}$, $c = \frac{1}{2}$

Theorem 2.3. Consider the difference equation (1.2), where $a + b = 1$ and initial conditions $x_{-1}, x_0 \geq 0$ such that $x_{-1}^2 + x_0^2 \neq 0$. Then (1.2) has an unique zero equilibrium and every solution $\{x_n\}$ of (1.2) satisfies $\lim_{n \rightarrow \infty} x_n = \infty$.

Proof. Assume that $a + b = 1$ and $\{x_n\}$ is a solution of (1.2). Since $x_{-1}^2 + x_0^2 \neq 0$, then $\exp(cx_{n-1} + cx_n) > 1$, which implies $\exp(cx_{n-1} + cx_n) = 1 + \alpha_n$, where $\alpha_n > 0$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ satisfies the inequality

$$\begin{aligned} x_{n+1} &= x_{n-1} (a + b \exp(cx_{n-1} + cx_n)) \\ &\geq x_{n-1} (a + b(1 + \alpha_n)) \\ &= x_{n-1} (a + b + b\alpha_n) = x_{n-1} (1 + b\alpha_n) \\ &> x_{n-1}, \end{aligned}$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing. Since there is no positive equilibrium point or period-two solution of (1.2) by applying Theorem 1.1 the both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ approach the point at infinity. $\square$

Now, consider the difference equation of type

$$(2.2) \quad x_{n+1} = Ax_{n-1} + Bx_{n-1} \exp(Cx_{n-1} + Dx_n)$$

in the first quadrant of initial conditions, where the given parameters satisfy conditions $A > 0$, $B > 0$, $C > 0$, $D > 0$ and $A + B < 1$. It is easy to show that (2.2) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+ = \frac{1}{C+D} \ln \frac{1-A}{B}$.

Proposition 2.3. *The zero equilibrium of (2.2) is always locally asymptotically stable. The positive equilibrium $\bar{x}_+ = \frac{1}{C+D} \ln \frac{1-A}{B}$ of (2.2) is one of the following:*

- a) *non-hyperbolic if $C = D$ (or $q - p = 1$);*
- b) *saddle point if $C < D$ (or $p > |q - 1|$);*
- c) *repeller if $C > D$ (or $p < |1 - q|$).*

Proof. Denote by $g(x, y) = Ay + By \exp(Cy + Dx)$ and let p and q denote the partial derivatives of function $g(x, y)$ at equilibrium point $\bar{x}$ of (2.2). By straightforward calculation we obtain that the following hold:

$$\begin{aligned} p(\bar{x}, \bar{x}) &= BD\bar{x} \exp((C + D)\bar{x}), \\ q(\bar{x}, \bar{x}) &= A + B(1 + C\bar{x}) \exp((C + D)\bar{x}). \end{aligned}$$

Hence, if $\bar{x} = 0$, then $p(0, 0) = 0$ and $q(0, 0) = A + B \in (0, 1)$ which implies $|p| < 1 - q < 2$, so by applying Theorem 1.3 the zero equilibrium is locally asymptotically stable. If $\bar{x} = \bar{x}_+$, then $p(\bar{x}_+, \bar{x}_+) = \frac{(1-A)D}{C+D} \ln \frac{1-A}{B} = (1 - A)D\bar{x}_+ > 0$ and

$$q(\bar{x}_+, \bar{x}_+) = 1 + \frac{(1 - A)C}{C + D} \ln \frac{1 - A}{B} = 1 + (1 - A)C\bar{x}_+ > 1.$$

Clearly, $|p| + q = p + q > q > 1$, which implies, by applying Theorem 1.3, the positive equilibrium $\bar{x}_+$ is an unstable. Since $A \in (0, 1)$ and

$$q - p = 1 + (1 - A)(C - D)\bar{x}_+,$$

which yields

$$\begin{aligned} C = D &\Rightarrow q - p = 1 \Leftrightarrow p = q - 1 \Leftrightarrow |p| = |1 - q|, \\ C > D &\Rightarrow q - p > 1 \Leftrightarrow p < q - 1 \Leftrightarrow |p| < |1 - q|, \\ C < D &\Rightarrow q - p < 1 \Leftrightarrow p > q - 1 \Leftrightarrow |p| > |1 - q|. \end{aligned}$$

The rest of proof following from Theorem 1.3. $\square$

Proposition 2.4. (2.2) has prime period-two solution $\left\{P_1\left(0, \frac{1}{C} \ln \frac{1-A}{B}\right), P_2\left(\frac{1}{C} \ln \frac{1-A}{B}, 0\right)\right\}$. If $C > D$, then period-two solution is saddle and if $C < D$, then the period-two solution is repeller.

Proof. Assume that (ϕ, ψ) is a prime period-two solution of (2.2) and $0 \leq \phi < \psi$. Then

$$(2.3) \quad \begin{aligned} \phi &= A\phi + B\psi \exp(C\phi + D\psi), \\ \psi &= A\psi + B\phi \exp(C\psi + D\phi). \end{aligned}$$

If $\phi = 0$, then $\psi = \frac{1}{C} \ln \frac{1-A}{B}$. Let $\phi > 0$. From system (2.3) we find that

$$(C - D)(\phi - \psi) = 0,$$

which implies $C = D$ ($\phi \neq \psi$), this case has already been considered. Set $u_n = x_{n-1}$ and $v_n = x_n$ and write (2.2) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= Au_n + Bu_n \exp(Cu_n + Dv_n). \end{aligned}$$

Let T be the function on $[0, \infty) \times [0, \infty)$ defined by

$$T(u, v) = (v, Au + Bu \exp(Cu + Dv)).$$

Then (ϕ, ψ) is a fixed point of T^2 , the second iterate of T . Furthermore,

$$\begin{aligned} T^2(u, v) &= T(T(u, v)) \\ &= (Au + Bu \exp(Cu + Dv), Av + Bv \exp(Cv + D(Au + Bu \exp(Cu + Dv)))) , \\ T^2(u, v) &= (g(u, v), h(u, v)), \end{aligned}$$

where $g(u, v) = Au + Bu \exp(Cu + Dv)$ and $h(u, v) = g(v, g(u, v))$. Jacobian matrix $J_{T^2}(\phi, \psi)$ evaluated at $(\phi, \psi) = (0, \frac{1}{C} \ln \frac{1-A}{B})$ is given by

$$\begin{aligned} J_{T^2}(\phi, \psi) &= \begin{pmatrix} \frac{\partial g}{\partial u}(\phi, \psi) & \frac{\partial g}{\partial v}(\phi, \psi) \\ \frac{\partial h}{\partial u}(\phi, \psi) & \frac{\partial h}{\partial v}(\phi, \psi) \end{pmatrix} \\ &= \begin{pmatrix} A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} & 0 \\ \frac{(1-A)D}{C} \left(A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} \right) \ln \frac{1-A}{B} & 1 + (1-A) \ln \frac{1-A}{B} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \det(J_T(\phi, \psi)) &= \left(A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} \right) \left(1 + (1-A) \ln \frac{1-A}{B} \right) > 0, \\ \text{tr}(J_T(\phi, \psi)) &= 1 + A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} + (1-A) \ln \frac{1-A}{B} > 1. \end{aligned}$$

If $C < D$, then $-1 + A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} > -1 + A + B \left(\frac{1-A}{B} \right) = 0$ and

$$\text{tr}(J_T(\phi, \psi)) - \det(J_T(\phi, \psi)) = 1 - (1-A) \left(-1 + A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} \right) \ln \frac{1-A}{B} < 1,$$

which yields

$$|\operatorname{tr}(J_T(\phi, \psi))| < |1 + \det(J_T(\phi, \psi))|.$$

Then by applying Theorem 1.3 ($p = \operatorname{tr}(J_T(\phi, \psi))$ and $q = -\det(J_T(\phi, \psi))$), the minimal period-two solution $\{P_1, P_2\}$ is repeller. Similarly, if $C > D$, then

$$-1 + A + B \left(\frac{1-A}{B} \right)^{\frac{D}{C}} < -1 + A + B \left(\frac{1-A}{B} \right) = 0$$

and

$$\operatorname{tr}(J_T(\phi, \psi)) - \det(J_T(\phi, \psi)) > 1,$$

which implies

$$|\operatorname{tr}(J_T(\phi, \psi))| > |1 + \det(J_T(\phi, \psi))|.$$

Now, by applying Theorem 1.3 the minimal period-two solution $\{P_1, P_2\}$ is saddle. $\square$

Proposition 2.5. *Consider the difference equation (2.2) in the first quadrant of initial conditions, where the given parameters satisfy conditions $A > 0$, $B > 0$, $C > 0$, $D > 0$, $C \neq D$ and $A + B < 1$. Set $m = \min\{C, D\}$ and $M = \max\{C, D\}$. Then the global stable manifold of the positive equilibrium is between two lines*

$$(2.4) \quad p_1 : B \exp(mx + my) = 1 - A$$

and

$$(2.5) \quad p_2 : B \exp(Mx + My) = 1 - A.$$

Proof. In a view of Proposition 2.3 the zero equilibrium of (2.2) is always locally asymptotically stable. The theorems applied in [5] provided existence of global stable manifold $\mathcal{W}^s$ through the saddle point. If $C < D$, then by applying Proposition 2.3 the positive equilibrium $\bar{x}_+ = \frac{1}{C+D} \ln \frac{1-A}{B}$ is a saddle point and there exists a global stable manifold which contains point $E_+(\bar{x}_+, \bar{x}_+)$. In this case global behavior of (2.2) is described by Theorem 1.2 where end points of the global stable manifold $\mathcal{W}^s(E_+)$ are exactly the period-two solution $\{P_1, P_2\}$ from Proposition 2.4. If $C > D$, then by applying Proposition 2.3 the positive equilibrium $\bar{x}_+$ is a repeller and in a view of Proposition 2.4 there exists a prime period-two solution $\{P_1, P_2\}$ which is a saddle point. There are two global stable manifolds $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(P_2)$, guaranteed by Theorems 1 and 4 in [13], which contain points $P_1(\phi, \psi)$ and $P_2(\psi, \phi)$. In this case the global behavior of (2.2) is described by Theorem 10 in [2]. Although the Theorems 9 and 10 in [2] have been applied on a polynomial second order difference equation they are special cases of general Theorems in [5] applied on function f , where f is increasing function in its arguments. So, the global dynamics of (2.2) is exactly the same as the global dynamics of equations described by Theorems 9 and 10 in [2]. Furthermore,

$$x_{n+1} = Ax_{n-1} + Bx_{n-1} \exp(Cx_{n-1} + Dx_n) \geq Ax_{n-1} + Bx_{n-1} \exp(mx_{n-1} + mx_n)$$

and

$$x_{n+1} = Ax_{n-1} + Bx_{n-1} \exp(Cx_{n-1} + Dx_n) \leq Ax_{n-1} + Bx_{n-1} \exp(Mx_{n-1} + Mx_n),$$

for all n , by applying Theorem 1.4 for solution $\{x_n\}$ of (2.2) the following inequality holds

$$y_n \leq x_n \leq z_n,$$

for all n , where $\{y_n\}$ is a solution of the difference equation

$$(2.6) \quad y_{n+1} = Ay_{n-1} + By_{n-1} \exp(my_{n-1} + my_n)$$

and $\{z_n\}$ is a solution of the difference equation

$$(2.7) \quad z_{n+1} = Az_{n-1} + Bz_{n-1} \exp(Mz_{n-1} + Mz_n).$$

Since (2.6) and (2.7) satisfy all conditions of Theorem 2.2 this implies that the statement of Proposition 2.5 holds. $\square$

3. CONCLUSION

In this paper we restrict our attention to certain exponential second order difference equation (1.2). It is important to mention that we have accurately determined the Julia set of (1.2) and the basins of attractions for the zero equilibrium and the positive equilibrium point. In general, all theoretical concepts which are very useful in proving the results of global attractivity of equilibrium points and period-two solutions only give us existence of global stable manifold(s) whose computation leads to very uncomfortable calculus (see [3, 4]).

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DIFFERENTIAL INVARIANTS OF COUPLED HIROTA-SATSUMA KDV EQUATIONS

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ABSTRACT. In this paper, we consider a generalized coupled Hirota-Satsuma KdV (CHSK) system of equations. We apply the moving frames method to find a finite generating set of differential invariants for the Lie symmetry group of CHSK equations. Once the generating set of differential invariants is located, we obtain recurrence relations and syzygies among the generating differential invariants. Our approach provides a complete characterization of the structure of algebras of differential invariants of CHSK equations.

1. INTRODUCTION

The equivalence moving frames method was introduced by E. Cartan to solve the equivalence problems on submanifolds under the action of a transformation group. In 1974, P. A. Griffiths has paid to the uniqueness and existence problem on geometric differential equations by using the Cartan method of Lie groups and moving frames [25]. Later on, in the 1990s, Fels and Olver have presented the moving co-frame method as a new formulation of the classical Cartan method for finite-dimensional Lie group actions on manifolds [10, 11]. In the last two decades, the moving frames method has been developed in the general algorithmic and equivariant framework which gives several new powerful tools for finding and classifying the equivalence and symmetry properties of submanifolds, differential invariants, and their syzygies (for instance, see [20–22]).

Key words and phrases. Differential invariants, Symmetry groups, Moving frames, Coupled Hirota-Satsuma KdV equations.

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The equivalence and symmetries of submanifolds are subject to their differential invariants, which have the same symmetry properties and allow us to determine the exact solutions of differential equations [4, 16]. In general, differential invariants are used to solve a broad range of problems appearing in nonlinear theory, mathematical physics, computer science and so on. A significant step for developing these applications is to study the structure of the algebra of differential invariants.

The KdV equations are well-known nonlinear evolution equations (NLEEs) which are a model for many physical phenomena. The simplest form of the KdV equation is $u_t + uu_x + u_{xxx} = 0$, where the uu_x term models nonlinear effects and the u_{xxx} term models dispersive effects of a wave propagation. The KdV equations can give a clear interpretation of both nonlinear effects and dispersive effects of propagation of long waves [3]. In the present paper, we consider a generalized coupled Hirota-Satsuma KdV (CHSK) system as [31]:

$$(1.1) \quad \begin{cases} u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)_x = 0, \\ v_t + v_{xxx} - 3uv_x = 0, \\ w_t + w_{xxx} - 3uw_x = 0, \end{cases}$$

which is the mathematical model of interactions of two long waves with different dispersion relations [6]. Especially, when $v = w$, the system (1.1) gives the well-known coupled Hirota-Satsuma KdV system.

System (1.1) has been studied via several methods such as the classical Lie group method [1], non-local symmetries [5], collocation method with quintic b -spline method [27]. These works mainly focused on obtaining solutions. Recently, a Lie group analysis on the time-fractional (CHSK) system has been done to obtain exact solutions and conservation laws [28]. In analytical works, differential invariants appeared. Indeed, these methods reduce the system with the aid of differential invariants (e.g. [1, 5, 28]). Knowledge of the algebraic structure of the differential invariants enables us to obtain not only the reduced equations but also to construct a class of differential equations which has the same symmetry properties, and it is significant from the physical point of view.

As far as we know, a comprehensive structure of algebras of differential invariants of equations (1.1) is not obtained so far. In this paper, using the moving frames method, we consider the structure of algebras of differential invariants of System (1.1). The advantage of our approach is that we yield the structure of the differential invariants algebra of System (1.1) only by using the infinitesimal determining equations and choosing a proper cross-section. Further, we do not need additional efforts for integration. Moreover, our approach contains straightforward calculations, yet it is more powerful since it yields the relations among the invariants.

The paper has the following organization. In Section 2, first in subsection 2.1, we remember the concept of differential invariants and some results on them. Then, in subsection 2.2, we explain the moving frames method and how one can apply the method to analyze the algebraic structure of groups. In Section 3, we apply the method

to the CHSK system and we analyze the algebraic structure of its Lie symmetry. In fact, using the moving frames method, we locate a finite set of generating differential invariants for the CHSK system and we obtain recurrence relations and the syzygies among the generation of differential invariants.

2. PRELIMINARIES

In this section, we present the preliminary concepts of differential invariants and moving frames method. We assume the reader to be familiar with the concepts of Lie symmetry method which is described in [15] and is used in many papers (e.g. [2, 12, 13, 28, 29]).

First, we remember the concept of jet space. By definition, the *jet space of order n* , $J^n = J^n(M, p)$, is the equivalence classes of p -dimensional submanifolds of a manifold M (of dimension m) under the equivalence relation of n th order contact. For instance, let us consider the local coordinates $z = (x, u)$ on manifold M , such that, the components of $x = (x^1, \dots, x^p)$ are assumed as independent variables and the components of $u = (u^1, \dots, u^q)$ are regarded as dependent ones. So, in these coordinates, a p -submanifold is realized as the graph of a function $u = f(x)$ [17]. Two such submanifolds are equivalent at a point $(x_0, u_0) = (x_0, f(x_0))$ if and only if they have the same n th order Taylor polynomials at x_0 [17]. The induced coordinates on the jet space J^n are denoted by $z^{(n)} = (x, u^{(n)})$, consisting of independent variables x^i , dependent variables u^α , and their derivatives u_J^α , of order $\#J \leq n$.

Now, let G be a local group of transformations acting on M . The induced local action of G on the n th jet space $J^n(M)$ is called the n th prolongation of G denoted by G^n . This prolongation transforms $u = f(x)$ and its derivatives. Studying the infinitesimal generators of prolonged group transformations is much easier than working with the explicit formula for the prolonged group transformations. Therefore, we work with the infinitesimal generators of prolonged group transformations.

If G is assumed to be a connected transformation group, then its infinitesimal generators form the Lie algebra of vector fields as

$$X = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

The n th prolongation of X is

$$X^{(n)} = \sum_{i=1}^p \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_{\#J \leq n} \eta_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha},$$

where

$$\eta_J^\alpha = D_J \left(\eta^\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha,$$

and $D_J = D_{j_1} \cdots D_{j_n}$. Here D_j denotes the total derivative with respect to x^j as

$$(2.1) \quad D_j = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^q \sum_J u_{J,j}^\alpha \frac{\partial}{\partial u_J^\alpha}.$$

2.1. Differential Invariants. A real-valued function $F : J^n \rightarrow \mathbb{R}$ is a *differential invariant* of a group transformation G , if it is remained unchanged by the prolongation of G , i.e. $F(g^{(n)} \cdot z^{(n)}) = F(z^{(n)})$, for all $z^{(n)} \in J^n$ and all $g \in G$ [23].

A well-known theorem ([9, Theorem 42]) of S. Lie and S. Scheffers says that under appropriate assumptions, all the differential invariants can be generated by a finite number of low order invariants and their derivatives. Lie and Scheffers presented the finite-dimensional Lie group case. Then, in 1894, A. Tress extends the theorem to infinite-dimensional pseudo-groups [30]. Indeed, there exists a finite set of generating differential invariants, and p invariant differential operators that preserve the differential invariant algebra, such that any other differential invariant can be locally written as a function of the generating invariants and their invariant derivatives. The order of differentiation is important, since the invariant differential operators need not commute. Furthermore, the differentiated invariants are generally not functionally independent, but are govern by certain functional relations or *syzygies* [24].

To have a complete investigation of the algebra of differential invariants, we need to find a finite set of generating differential invariant, their functional relationships or their syzygies, and the commutation relationships between invariant operators.

2.2. Moving Frames Method. In order to describe the moving frames method, first we recall that a moving frame is an equivariant map $\rho : J^n \rightarrow G$ from the jet space J^n to the group G satisfying $\rho(g^{(n)} \cdot z^{(n)}) = g \cdot \rho(z^{(n)})$ for every $z^{(n)} \in J^n$ and $g \in G$. However, only free actions have moving frames. To prove the necessity of freeness, let g be an arbitrary element in the isotropy subgroup and ρ be a moving frame, then $\rho(z^{(n)}) = \rho(g^{(n)} \cdot z^{(n)}) = g \cdot \rho(z^{(n)})$. Therefore, the isotropy subgroup must be $G_{z^{(n)}} = \{e\}$ for each $z^{(n)} \in J^n$, meaning that the action must be free.

We can make actions free by prolonging the group. A theorem, which was presented by Ovsiannikov and improved by Olver, states that if a group acts (locally) effectively on subsets, then there exists an integer k such that the prolongation of the group action is locally free on an open and dense subset of the k -th order jet space [14, 19]. In cases where G does not act effectively, without loss of any generality, we can replace G with the effectively acting quotient group G/G_M^* , where G_M^* is the global isotropy subgroup [17]. Therefore, in order to make a action free, we prolong the group action to a sufficiently high order jet space. The prolongation makes it possible to apply the moving frames method to any group.

Once the freeness is achieved, we choose a specific local cross-section to the prolonged group orbits. Based on the chosen cross-section, we construct a moving frame. After constructing a moving frame, we use the invariantization process to produce complete

systems of differential invariants and invariant differential operators. So, we start with definition of cross-section.

A *cross-section* is a submanifold $\mathcal{K}^n \subset J^n$, that intersect the prolonged group orbits transversally. The cross-section is called *regular* if $\mathcal{K}^n$ intersects each orbit at most once. The corresponding moving frame associates to each $z^{(n)} \in J^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps $z^{(n)}$ to the cross-section $g \cdot z^n = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathcal{K}$ [22].

For simplicity, we can choose $\mathcal{K} = \{z_1 = c_1, \dots, z_r = c_r\}$ as *coordinate cross-section*, which prescribed by setting the $r = \dim G$ coordinates to proper constants.

Given local coordinates $z^{(n)} = (z, u^{(n)})$ on J^n , let $w(g, z^{(n)}) = g \cdot z^{(n)}$ be the explicit formulae for the group action. The right moving frame $g = \rho^{(n)}(z^{(n)})$ associated with the coordinate cross-section

$$\mathcal{K} = \{z_1 = c_1, \dots, z_r = c_r\},$$

is obtained by solving the *normalization equations*

$$(2.2) \quad w_1(g, z^{(n)}) = c_1, \dots, w_r(g, z^{(n)}) = c_r.$$

Substituting the moving frame formulae for the group parameters into the remaining action rules provides a complete system of functionally independent differential invariants [26].

$$I(z^{(n)}) = w(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}.$$

In particular, the normalization components (2.2) of w will be constant, which are called the *phantom differential invariants*. Other components (2.2) are called *fundamental invariants*.

In particular,

$$H^i(x, u^{(n)}) = \iota(x^i), \quad I_J^\alpha(x, u^{(n)}) = \iota(u_J^\alpha),$$

will denote the normalized differential invariants.

To carry on the moving frames method, we use the concept of invariantization and begin the invariantization process. The invariantization

$$\iota : F(x, u^{(n)}) \rightarrow I(x, u^{(n)}) = F(\rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})),$$

maps the differential function F to the differential invariant $I = \iota(F)$ [24].

Separating the local coordinates (x, u) on M into independent and dependent variables splits the one-forms on J^∞ into *horizontal forms*, which are spanned by $dx^1, \dots, dx^p$, and *vertical forms*, which are spanned by the basic contact one-forms [17]

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha dx^i, \quad \alpha = 1, \dots, q, \#J \geq 0.$$

The pull back of the dual Maurer-Cartan forms $\mu^1, \dots, \mu^r$ on G via the moving frame map: $v^k = \rho^* \mu^k$ produces the *invariantized Maurer-Cartan forms* [8]. We split

the invariantized Maurer-Cartan forms into horizontal forms and vertical forms:

$$v^k = \sum_{i=1}^p R_i^\kappa (\bar{\omega})^i + \sum_{\alpha, J} S_{\alpha}^{\kappa, J} \vartheta_J^\alpha,$$

where R_i^κ and $S_{\alpha}^{\kappa, J}$ are certain differential invariants, $\bar{\omega}^i = \iota(dx^i)$ denote the invariantized horizontal one-forms and the invariantized basis contact forms are denoted by

$$\vartheta_J^\alpha = \iota(\theta_J^\alpha), \quad \alpha = 1, \dots, q, \#J \geq 0.$$

The R_i^κ are called the *Maurer-Cartan invariants* [7]. The Maurer-Cartan invariants will appear in the recurrence formula which will introduce later.

Though invariantization respect all algebraic operators, it does not respect differentiation, i.e., $D[\iota(F)] \neq \iota[D(F)]$. However, there is an explicit formula known as the *recurrence formula* which determines the effect of invariantization on derivatives [11]. Given a differential function $F(x, u^{(n)})$ and $\iota(F)$ its moving frame invariantization. Then the recurrence formula will be

$$(2.3) \quad D_i[\iota(F)] = \iota[D_i(F)] + \sum_{\kappa=1}^r R_i^\kappa \iota[X_\kappa^{(n)}(F)],$$

where R_i^κ are the Maurer-Cartan invariants and $X_\kappa^{(n)}$ are the n th prolongations of the infinitesimal generators X_κ [26]. In our approach, the recurrence formula (2.3) is the key to study the algebra of differential invariants.

The invariant differential operators $\mathcal{D}_i$ map differential invariants to differential invariants. In most cases, they do not commute, but they satisfy in linear commutation relations of the form

$$(2.4) \quad [\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^p Y_{ij}^k \mathcal{D}_k, \quad i, j = 1, \dots, p,$$

where the coefficients Y_{ij}^k are certain differential invariants called the *commutator invariants* [24]. One can obtain the commutator invariants Y_{ij}^k by differentiating the recurrence formula (2.3).

In general, if $K = (k_1, \dots, k_l)$ is an ordered multi-index, then, as a result of (2.4),

$$(2.5) \quad \mathcal{D}_{\pi(K)} = \mathcal{D}_K + \sum_{\#J < \#K} Y_{\pi, K}^J \mathcal{D}_J,$$

for any permutation π of the entries of K . For example,

$$\begin{aligned} \mathcal{D}_i \mathcal{D}_j \mathcal{D}_k &= \mathcal{D}_i \left(\mathcal{D}_k \mathcal{D}_j + \sum_{l=1}^p Y_{jk}^l \mathcal{D}_l \right) \\ &= \mathcal{D}_k \mathcal{D}_i \mathcal{D}_j + \sum_{l=1}^p \left[Y_{ik}^l \mathcal{D}_l \mathcal{D}_j + Y_{jk}^l \mathcal{D}_i \mathcal{D}_l + (\mathcal{D}_i Y_{jk}^l) \mathcal{D}_l \right]. \end{aligned}$$

Using the commutator formulae (2.5), we can construct an infinite number of *commutator syzygies* by applying (2.5) on any one of our generating differential invariants.

3. INVARIANTS OF GENERALIZED COUPLED HIROTA-SATSUMA KdV EQUATIONS

First, we consider the Lie point symmetries of System (1.1). The infinitesimal Lie transformations for equations (1.1) are of the form:

$$\begin{aligned}x &\mapsto x + \lambda \xi^x(x, t, u, v, w), \\t &\mapsto t + \lambda \xi^t(x, t, u, v, w), \\u &\mapsto u + \lambda \eta^u(x, t, u, v, w), \\v &\mapsto v + \lambda \eta^v(x, t, u, v, w), \\w &\mapsto w + \lambda \eta^w(x, t, u, v, w),\end{aligned}$$

with the symmetry generator

$$X = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta^w \frac{\partial}{\partial w}.$$

In [1], using Lie's method, the infinitesimal Lie transformations for equations (1.1) are obtained by solving the following determining system:

$$\begin{aligned}(3.1) \quad \xi_t^x = \xi_u^x = \xi_v^x = \xi_w^x = 0, \quad \xi_x^t = \xi_u^t = \xi_v^t = \xi_w^t = 0, \quad \xi_x^x = \frac{1}{3}\xi_t^t, \\ \eta_x^v = \eta_t^v = \eta_u^v = \eta_w^v = 0, \quad \eta^u = -\frac{2}{3}\xi_t^t u, \quad \eta^w = -\frac{1}{3} \cdot \frac{(4v\xi_t^t + 3\eta^v)w}{v}.\end{aligned}$$

Solving (3.1) yields the following coefficients of the vector field X :

$$\xi^x = a_1 + \frac{1}{3}xa_4, \quad \xi^t = a_1 + ta_4, \quad \eta^u = -\frac{2}{3}ua_4, \quad \eta^v = a_3v, \quad \eta^w = -a_3w - a_4\frac{4}{3}w.$$

where the a_1, a_2, a_3, a_4 are constants. Thus, the Lie algebra of the symmetries is generated by the following four vector fields

$$\begin{aligned}(3.2) \quad X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}, \\ X_4 &= \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u} - \frac{4}{3}w\frac{\partial}{\partial w}.\end{aligned}$$

The action of the symmetry group of equations (1.1) can be yielded by composing the flows of the vector fields (3.2) and is given by

$$(3.3) \quad (X, T, U, V, W) = \exp(\lambda_1 X_1) \circ \exp(\lambda_2 X_2) \circ \exp(\lambda_3 X_3) \circ \exp(\lambda_4 X_4),$$

where $\lambda_1, \dots, \lambda_4$ are the group parameters. Calculating (3.3) leads to

$$X = (x + \lambda_1)e^{1/3\lambda_4}, \quad T = (t + \lambda_2)e^{\lambda_4}, \quad U = ue^{-2/3\lambda_4},$$

$$V = ve^{\lambda_3}, \quad W = we^{-\lambda_3}e^{-\frac{4}{3}\lambda_4}.$$

As noted in Section 2.2, we should choose an appropriate coordinate cross-section. Since the dimension of group action is four, we can choose a coordinate cross-section by setting four arbitrary coordinates equal to constants. Here, we set the coordinate cross-section as $\mathcal{K} = \{x = 1, t = 0, v = 1, w = 1\}$. Our chosen coordinate cross-section defines the following four normalization equations

$$(3.4) \quad X = 1, \quad T = 0, \quad V = 1, \quad W = 1,$$

As a result of our choice of normalization (3.4) the phantom differential invariants are

$$H^1 = \iota(x) = 1, \quad H^2 = \iota(t) = 0, \quad J_{00} = \iota(v) = 1, \quad K_{00} = \iota(w) = 1.$$

Using MAPLE, we found that the normalization equations (3.4) can be solved by the following group parameters:

$$(3.5) \quad \lambda_1 = -\frac{x(vw)^{1/4} - 1}{(vw)^{1/4}}, \quad \lambda_2 = -t, \quad \lambda_3 = -\ln v, \quad \lambda_4 = \frac{3}{4} \ln(vw).$$

The dual implicit differential operators are defined as follow [17]

$$(3.6) \quad D_{X^i} = \sum_{j=1}^p W_j^i D_{x^j}, \quad W_i^j = \left(D_{x^j} X^i \right)^{-1},$$

where D_{x^i} are the total derivative and are (2.1).

From (3.6), we have

$$(3.7) \quad D_X = e^{-\frac{1}{3}\lambda_4} D_x, \quad D_T = e^{-\lambda_4} D_t.$$

Substituting (3.5) into (3.7), the corresponding invariant differential operators are

$$\mathcal{D}_x = (vw)^{-1/4} D_x, \quad \mathcal{D}_t = (vw)^{-3/4} D_t.$$

A complete system of functionally independent normalized differential invariants is yielded by invariantizing the remaining non-phantom differential invariants:

$$\begin{aligned} I_{00} &= \frac{u}{(vw)^{1/2}}, \quad I_{10} = \frac{(vw)^{1/4} u_x}{vw}, \quad I_{01} = \frac{(vw)^{1/4} u_t}{vw}, \\ J_{10} &= \frac{(vw)^{3/4} v_x}{vw}, \quad J_{01} = \frac{(vw)^{1/4} v_t}{vw}, \\ K_{10} &= \frac{(vw)^{3/4} w_x}{vw}, \quad K_{01} = \frac{(vw)^{1/4} w_t}{vw}, \\ I_{20} &= -\frac{1}{4} \cdot \frac{\sqrt{vw} (u_x v_x w + u_x w_x v - 4u_{xx} vw)}{v^2 w^2}, \\ I_{30} &= -\frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(2v_{xx} u_x v w^2 + 2w_{xx} u_x v^2 w - 8u_{xxx} v^2 w^2 - 3u_x v_x^2 w^2 - 2u_x v_x w_x v w \right) \\ &\quad + \frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(-3u_x w_x^2 v^2 + 6u_{xx} v_x v w^2 + 6u_{xx} w_x v^2 w \right), \end{aligned}$$

$$\begin{aligned}
J_{30} &= -\frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(8v_{xx}v_x v w^2 + 6v_{xx}w_x v^2 w + 2w_{xx}v_x v^2 w - 3v_x^3 w^2 \right) \\
&\quad + \frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(2v_x^2 w_x v w + 3v_x w_x^2 v^2 + 8v_{xx}v^2 w^2 \right), \\
K_{30} &= \frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(8v^2 w^2 w_{xxx} - 8v^2 w w_x w_{xx} + 3v^2 w_x^3 - 6v w^2 v_x w_{xx} \right) \\
&\quad - \frac{1}{8} \cdot \frac{(vw)^{1/4}}{v^3 w^3} \left(2v w^2 w_x v_{xx} - 2v w v_x w_x^2 - 3w^2 v_x^2 w_x \right) \\
&\quad \vdots
\end{aligned}$$

where

$$I_{ij} = \iota(u_{i,j}), \quad J_{ij} = \iota(v_{i,j}), \quad K_{ij} = \iota(w_{i,j}).$$

By applying the invariantization process, System (1.1) can be rewritten in terms of the differential invariants as

$$\begin{cases} I_{01} - \frac{1}{2}I_{30} + 3I_{00}I_{10} - 3(J_{00}K_{10} + J_{10}K_{00}) = 0, \\ J_{01} + J_{30} - 3I_{00}J_{10} = 0, \\ K_{01} + K_{30} - 3I_{00}K_{10} = 0. \end{cases}$$

Next, we locate the a finite generating set of differential invariants for Equation (1.1). One can obtain higher order differential invariants by repeatedly applying the invariant differential operators to the lower order differential invariants.

According to (2.3), the recurrence formula for the differential invariants are

$$\begin{aligned}
(3.8) \quad \mathcal{D}_1 H^j &= \delta_1^j + \sum_{\kappa=1}^r \iota(\xi_\kappa^j) R_1^\kappa, & \mathcal{D}_2 H^j &= \delta_2^j + \sum_{\kappa=1}^r \iota(\xi_\kappa^j) R_2^\kappa, \\
\mathcal{D}_1 I_{jk} &= I_{j+1,k} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{u,jk}) R_1^\kappa, & \mathcal{D}_2 I_{jk} &= I_{j,k+1} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{u,jk}) R_2^\kappa, \\
\mathcal{D}_1 J_{jk} &= J_{j+1,k} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{v,jk}) R_1^\kappa, & \mathcal{D}_2 J_{jk} &= J_{j,k+1} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{v,jk}) R_2^\kappa, \\
\mathcal{D}_1 K_{jk} &= K_{j+1,k} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{w,jk}) R_1^\kappa, & \mathcal{D}_2 K_{jk} &= K_{j,k+1} + \sum_{\kappa=1}^r \iota(\eta_\kappa^{w,jk}) R_2^\kappa,
\end{aligned}$$

where R_1^κ and R_2^κ are the Maurer-Cartan invariants and ξ^j , $\eta_\kappa^{u,jk}$, $\eta_\kappa^{v,jk}$ and $\eta_\kappa^{w,jk}$ are the coefficients of $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial u_{jk}}$, $\frac{\partial}{\partial v_{jk}}$ and $\frac{\partial}{\partial w_{jk}}$ in the prolongation of the infinitesimal generator X_κ respectively [26]. Solving the resulting phantom recurrence formula produces the Maurer-Cartan invariants

$$\begin{aligned}
(3.9) \quad R_1^2 &= 0, \quad R_2^2 = -1, \quad R_1^1 = -1 - \frac{1}{3}R_1^4, \quad R_1^3 = -J_{10}, \\
R_2^3 &= -J_{01}, \quad R_1^4 = \frac{3}{4}K_{10} + \frac{3}{4}J_{10}, \quad R_2^4 = \frac{3}{4}K_{01} + \frac{3}{4}J_{01}.
\end{aligned}$$

Substituting the Maurer-Cartan invariants (3.9) back into (3.8) obtain all the non-phantom recurrence formula.

$$\begin{aligned}
 (3.10) \quad \mathcal{D}_1 I_{00} &= I_{10} - \frac{2}{3} I_{00} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 I_{00} &= I_{01} - \frac{2}{3} I_{00} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 I_{10} &= -I_{10} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right) + I_{20}, & \mathcal{D}_2 I_{10} &= -I_{10} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right) + I_{11}, \\
 \mathcal{D}_1 I_{01} &= I_{11} - \frac{5}{3} I_{01} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 I_{01} &= I_{02} - \frac{5}{3} I_{01} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 I_{20} &= I_{30} - \frac{4}{3} I_{20} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 I_{20} &= I_{21} - \frac{4}{3} I_{20} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 I_{02} &= I_{12} - \frac{8}{3} I_{02} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 I_{02} &= I_{03} - \frac{8}{3} I_{02} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 I_{11} &= -2I_{02} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right) + I_{21}, & \mathcal{D}_2 I_{11} &= -2I_{02} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right) + I_{12}, \\
 \mathcal{D}_1 J_{10} &= J_{20} - J_{10}^2 - \frac{1}{3} J_{10} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 J_{10} &= J_{11} - J_{10} J_{01} - \frac{1}{3} J_{10} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 J_{01} &= -J_{10} J_{01} - J_{01} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right) + J_{11}, & \mathcal{D}_2 J_{01} &= -J_{01}^2 - J_{01} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right) + J_{02}, \\
 \mathcal{D}_1 J_{20} &= J_{30} - J_{10} - \frac{2}{3} J_{20} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 J_{20} &= J_{21} - J_{01} - \frac{2}{3} J_{20} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 J_{02} &= -2J_{02} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right) + J_{12} - J_{10}, & \mathcal{D}_2 J_{02} &= -2J_{02} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right) + J_{03} - J_{01}, \\
 \mathcal{D}_1 J_{11} &= J_{21} - J_{10} - \frac{4}{3} J_{20} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 J_{11} &= J_{21} - J_{01} - \frac{4}{3} J_{11} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 K_{10} &= K_{20} + K_{10} J_{10} - \frac{4}{3} K_{10} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 K_{10} &= K_{11} + K_{10} J_{01} - \frac{4}{3} K_{10} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 K_{01} &= K_{11} + K_{01} J_{10} - \frac{7}{3} K_{01} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 K_{01} &= K_{02} + K_{01} J_{01} - \frac{7}{3} K_{01} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 K_{20} &= K_{20} J_{10} - 2K_{20} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right) + K_{30}, & \mathcal{D}_2 K_{20} &= K_{20} J_{01} - 2K_{20} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right) + K_{21}, \\
 \mathcal{D}_1 K_{11} &= K_{21} + K_{11} J_{10} - \frac{8}{3} K_{11} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 K_{11} &= K_{21} + K_{11} J_{01} - \frac{8}{3} K_{11} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right), \\
 \mathcal{D}_1 K_{02} &= K_{12} + K_{20} J_{10} - \frac{10}{3} K_{11} \left(\frac{3}{4} K_{10} + \frac{3}{4} J_{10} \right), & \mathcal{D}_2 K_{02} &= K_{03} + K_{02} J_{01} - \frac{10}{3} K_{02} \left(\frac{3}{4} K_{01} + \frac{3}{4} J_{01} \right).
 \end{aligned}$$

Theorem 3.1. *The entire differential invariant algebra of the CHSK equations (1.1) is generated by the following set:*

$$(3.11) \quad \{I_{00}, J_{10}, J_{01}, K_{10}, K_{01}\}.$$

Proof. From the recurrence formula (3.10), we find that any differential invariants up to third order can be generated by a function composition of $I_{00}, J_{10}, J_{01}, K_{10}, K_{01}$ and their derivatives. By differentiating the differential invariants (3.11), one can find that any higher order differential invariants are also generated by the generating set (3.11). $\square$

Finally, we obtain the commutator invariants which satisfy the commutator relation

$$(3.12) \quad [\mathcal{D}_1, \mathcal{D}_2] = Y_1 \mathcal{D}_1 + Y_2 \mathcal{D}_2,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the invariant differential operators. As a result of general recurrence formulae, [11, 18], we have

$$Y_1 = \sum_{\kappa=1}^r \left[R_2^\kappa \iota(D_x \xi_\kappa^1) - R_1^\kappa \iota(D_t \xi_\kappa^1) \right], \quad Y_2 = \sum_{\kappa=1}^r \left[R_2^\kappa \iota(D_x \xi_\kappa^2) - R_1^\kappa \iota(D_t \xi_\kappa^2) \right],$$

in which ξ_κ^i is the coefficients of ∂_{x^i} , in the infinitesimal generator X_κ .

Substituting our formula (3.9) for the Maurer-Cartan invariants yields

$$(3.13) \quad Y_1 = \frac{1}{4}(K_{01} + J_{01}), \quad Y_2 = -\frac{3}{4}(K_{10} - J_{10}).$$

Thus, from (3.12) and (3.13), we have

$$(3.14) \quad [\mathcal{D}_1, \mathcal{D}_2] = \frac{1}{4}(K_{01} + J_{01}) \mathcal{D}_1 - \frac{3}{4}(K_{10} - J_{10}) \mathcal{D}_2.$$

Indeed, the generating differential invariants $\{I_{00}, J_{10}, J_{01}, K_{10}, K_{01}\}$, the recurrence formulas (3.10), along with the commutation relations (3.14), provide a complete specification of the structure of the differential invariant algebra of CHSK equations (1.1).

4. CONCLUSIONS

In this paper, using the moving frames method, we located a finite generating set of differential invariants and the invariant differential operators for the Lie symmetry group of a generalized coupled Hirota-Satsuma KdV equations (CHSK), and then we obtained the recurrence relations as well as syzygies among the generating differential invariants. In particular, we proved that the differential invariant algebra of CHSK equations can be generated by five differential invariants. The main application of the differential invariants is to construct a class of PDEs, which possess the same symmetry properties, which is important for both mathematics and physical interpretation. Since the CHSK system is the mathematical model of interactions of two long waves with different dispersion relations, our results are applicable to study the invariant properties of interactions of two long waves. In our approach, we also obtained the Maurer-Cartan invariants.

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MATRIX FEJÉR AND LEVIN-STEČKIN INEQUALITIES

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ABSTRACT. Fejér and Levin-Stečkin inequalities treat integrals of the product of convex functions with symmetric functions. The main goal of this article is to present possible matrix versions of these inequalities. In particular, majorization results are shown of Fejér type for both convex and log-convex functions. For the matrix Levin-Stečkin type, we present more rigorous results involving the partial Löwner ordering for Hermitian matrices. Further related results involving synchronous functions are presented, too.

1. INTRODUCTION

The theory of convex functions has played a major role in the study of mathematical inequalities. Related to convex-type inequalities, the Levin-Stečkin's inequality states that if the function $p : [0, 1] \rightarrow \mathbb{R}$ is symmetric about $t = \frac{1}{2}$, namely $p(1 - t) = p(t)$, and non-decreasing on $[0, \frac{1}{2}]$, then for every convex function f on $[0, 1]$, the inequality

$$\int_0^1 p(t) f(t) dt \leq \int_0^1 p(t) dt \int_0^1 f(t) dt$$

holds true [6]. If p is symmetric non-negative (without any knowledge about its monotonicity) and $f : [a, b] \rightarrow \mathbb{R}$ is convex, Fejér inequality states that [4]

$$f\left(\frac{a+b}{2}\right) \int_0^1 p(t) dt \leq \int_0^1 p(t) f((1-t)a + tb) dt \leq \frac{f(a) + f(b)}{2} \int_0^1 p(t) dt.$$

Key words and phrases. Levin-Stečkin inequality, Fejér inequality, positive matrices.

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We notice that Fejér inequality reduces to the Hermite-Hadamard inequality [5] when $p(t) = 1$. In mathematical inequalities, it is of interest to extend known inequalities from the setting of scalars to other objects, such as matrices. In this article, we will be interested in extending both the Levin-Stečkin and Fejér inequalities to the matrices setting.

In the sequel, $\mathcal{M}_n$ will denote the algebra of all $n \times n$ complex matrices. The conjugate transpose (or adjoint) of $A \in \mathcal{M}_n$ is denoted by A^* , and then the matrix A will be called Hermitian if $A^* = A$. When $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$, A is said to be positive semi-definite and is denoted as $A \geq 0$. If $A \geq 0$ and A is invertible, then A is said to be positive (strictly positive or positive definite). When $A, B \in \mathcal{M}_n$ are Hermitian, we say that $A \leq B$ if $B - A \geq 0$. This provides a partial ordering on the class of Hermitian matrices. The eigenvalues of a Hermitian matrix A will be denoted by $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$, repeated according to their multiplicity and arranged decreasingly. That is $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$.

The relation $A \leq B$ implies $\lambda_i(A) \leq \lambda_i(B)$ for any such Hermitian matrices $A, B \in \mathcal{M}_n$. However, the converse is not true. This urges the need to discuss, in some cases, the latter order. For convenience, we will write $\lambda(A) \leq \lambda(B)$ to mean that $\lambda_i(A) \leq \lambda_i(B)$, $i = 1, 2, \dots, n$.

Another weaker ordering among matrices is the so-called weak majorization $\prec_w$, defined for the Hermitian matrices A, B as

$$A \prec_w B \quad \text{if and only if} \quad \sum_{i=1}^k \lambda_i(A) \leq \sum_{i=1}^k \lambda_i(B), \quad k = 1, \dots, n.$$

It is clear that (see [1]) $A \leq B$ implies $\lambda(A) \leq \lambda(B)$, which implies $A \prec_w B$. It is customary to obtain one of these orders when extending a scalar inequality to a matrix inequality. For example, in this article, we obtain

$$\lambda \left(\left(\int_0^1 p(t) dt \right) f \left(\frac{A+B}{2} \right) \right) \leq \lambda \left(\int_0^1 p(t) f((1-t)A + tB) dt \right),$$

as an extension of Fejér inequality, to the Hermitian matrices A, B with spectra in the domain of f .

Further, if f is monotone, then

$$\lambda \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) \leq \lambda \left(\left(\int_0^1 p(t) dt \right) \frac{f(A) + f(B)}{2} \right),$$

as matrix inequalities of the Fejér inequality. We remark that integral inequalities have played a key role in advancing matrix inequalities, as seen in [8, 9, 12], and the references therein.

In the next section, we study the possible matrix versions of Fejér inequality, which implies certain versions of the Hermite-Hadamard matrix inequality [10]. Then log-convex functions will be deployed to obtain new matrix Fejér inequalities for this

type of functions, and we conclude with the discussion of the matrix Levin-Stečkin inequality.

2. FEJÉR MATRIX INEQUALITIES FOR CONVEX FUNCTIONS

We begin with the following weak majorization of Fejér-type inequality.

Theorem 2.1. *Let $f : J \rightarrow \mathbb{R}$ be convex and let $p : [0, 1] \rightarrow [0, +\infty)$ be symmetric about $t = \frac{1}{2}$. If $A, B \in \mathcal{M}_n$ are Hermitian with spectra in the interval J , then*

$$\lambda \left(\left(\int_0^1 p(t) dt \right) f \left(\frac{A+B}{2} \right) \right) \prec_w \lambda \left(\int_0^1 p(t) f((1-t)A + tB) dt \right).$$

Proof. If f is a convex function, then for any $0 \leq t \leq 1$, we have

$$\begin{aligned} f \left(\frac{a+b}{2} \right) &= f \left(\frac{(1-t)a + tb + (1-t)b + ta}{2} \right) \\ &\leq \frac{f((1-t)a + tb) + f((1-t)b + ta)}{2}. \end{aligned}$$

Thus,

$$(2.1) \quad f \left(\frac{a+b}{2} \right) \leq \frac{f((1-t)a + tb) + f((1-t)b + ta)}{2}.$$

If the function p is non-negative, we get from (2.1),

$$p(t) f \left(\frac{a+b}{2} \right) \leq p(t) \left(\frac{f((1-t)a + tb) + f((1-t)b + ta)}{2} \right).$$

Integrating on $t \in [0, 1]$, and using symmetry assumption on p , we get

$$(2.2) \quad \left(\int_0^1 p(t) dt \right) f \left(\frac{a+b}{2} \right) \leq \int_0^1 p(t) f((1-t)a + tb) dt.$$

If we replace a, b by $\langle Ax, x \rangle, \langle Bx, x \rangle$ respectively, in (2.2), we get

$$(2.3) \quad \left(\int_0^1 p(t) dt \right) f \left(\frac{\langle Ax, x \rangle + \langle Bx, x \rangle}{2} \right) \leq \int_0^1 p(t) f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt.$$

On the other hand, it follows from Jensen's inequality [11, Theorem 1.2],

$$f(\langle ((1-t)A + tB)x, x \rangle) \leq \langle f((1-t)A + tB)x, x \rangle.$$

By multiplying both sides by $p(t)$, we get

$$p(t) f(\langle ((1-t)A + tB)x, x \rangle) \leq p(t) \langle f((1-t)A + tB)x, x \rangle.$$

Therefore,

$$(2.4) \quad \int_0^1 p(t) f(\langle ((1-t)A + tB)x, x \rangle) dt \leq \left\langle \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) x, x \right\rangle.$$

Combining inequalities (2.3) with (2.4), we obtain

$$(2.5) \quad \left(\int_0^1 p(t) dt \right) f\left(\frac{\langle Ax, x \rangle + \langle Bx, x \rangle}{2}\right) \leq \left\langle \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) x, x \right\rangle.$$

Suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\frac{A+B}{2}$ with $x_1, \dots, x_n$ as an orthonormal system of corresponding eigenvectors arranged such that $f(\lambda_1) \geq \dots \geq f(\lambda_n)$. We have, for $1 \leq k \leq n$,

$$\begin{aligned} \sum_{j=1}^k \lambda_j \left(\left(\int_0^1 p(t) dt \right) f\left(\frac{A+B}{2}\right) \right) &= \sum_{j=1}^k \left(\int_0^1 p(t) dt \right) f\left(\left\langle \left(\frac{A+B}{2}\right) x_j, x_j \right\rangle\right) \\ &= \sum_{j=1}^k \left(\int_0^1 p(t) dt \right) f\left(\frac{\langle Ax_j, x_j \rangle + \langle Bx_j, x_j \rangle}{2}\right) \\ &\leq \sum_{j=1}^k \left\langle \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) x_j, x_j \right\rangle \\ &\quad \text{(by the inequality (2.5))} \\ &\leq \sum_{j=1}^k \lambda_j \left(\int_0^1 p(t) f((1-t)A + tB) dt \right). \end{aligned}$$

Namely, for $1 \leq k \leq n$,

$$\sum_{j=1}^k \lambda_j \left(\left(\int_0^1 p(t) dt \right) f\left(\frac{A+B}{2}\right) \right) \leq \sum_{j=1}^k \lambda_j \left(\int_0^1 p(t) f((1-t)A + tB) dt \right).$$

Therefore,

$$\lambda \left(\left(\int_0^1 p(t) dt \right) f\left(\frac{A+B}{2}\right) \right) \prec_w \lambda \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) .. \quad \square$$

3. FEJÉR INEQUALITIES VIA LOG-CONVEX FUNCTIONS

In this part of the paper, we show a matrix Fejér inequality for log-convex functions.

Theorem 3.1. *Let $f : (0, +\infty) \rightarrow (0, +\infty)$ be log-convex and $p : [0, 1] \rightarrow (0, +\infty)$ be symmetric and normalized in the sense that $\int_0^1 p(t) dt = 1$. If $A, B \in \mathcal{M}_n$ are positive, then*

$$\lambda \left(\log f\left(\frac{A+B}{2}\right) \right) \prec_w \lambda \left(\log \int_0^1 p(t) f((1-t)A + tB) dt \right).$$

Proof. When f is convex, by (2.3), we have

$$f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \leq \int_0^1 p(t)f(\langle((1-t)A+tB)x, x\rangle) dt,$$

for any unit vector x . Since f is log-convex and A, B are positive, it follows that

$$\log f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \leq \int_0^1 p(t) \log f(\langle((1-t)A+tB)x, x\rangle) dt.$$

Noting that $\log$ is a concave function and that $d\mu(t) := p(t)dt$ is a probability measure, we have

$$\begin{aligned} \log f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) &\leq \int_0^1 p(t) \log f(\langle((1-t)A+tB)x, x\rangle) dt \\ &= \int_0^1 \log f(\langle((1-t)A+tB)x, x\rangle) d\mu(t) \\ &\leq \log \int_0^1 f(\langle((1-t)A+tB)x, x\rangle) d\mu(t) \\ &= \log \int_0^1 p(t)f(\langle((1-t)A+tB)x, x\rangle) dt, \end{aligned}$$

for any unit vector x . Now, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\frac{A+B}{2}$ with orthonormal eigenvectors $x_1, x_2, \dots, x_n$, so that $f(\lambda_1) \geq \dots \geq f(\lambda_n)$. Then, for $1 \leq k \leq n$,

$$\begin{aligned} \sum_{j=1}^k \lambda_j \left(\log f\left(\frac{A+B}{2}\right) \right) &= \sum_{j=1}^k \log f(\lambda_j) \\ &= \sum_{j=1}^k \log f\left(\left\langle \frac{A+B}{2}x_j, x_j \right\rangle\right) \\ &\leq \sum_{j=1}^k \log \int_0^1 p(t)f(\langle((1-t)A+tB)x_j, x_j\rangle) dt \\ &\leq \sum_{j=1}^k \lambda_j \left(\log \int_0^1 p(t)f((1-t)A+tB) dt \right). \end{aligned}$$

This completes the proof. □

As a consequence, we have the following.

Corollary 3.1. *Let $f : (0, +\infty) \rightarrow (0, +\infty)$ be log-convex and $p : [0, 1] \rightarrow (0, +\infty)$ be symmetric and normalized. Then*

$$\prod_{j=1}^k \lambda_j \left(f\left(\frac{A+B}{2}\right) \right) \leq \prod_{j=1}^k \lambda_j \left(\int_0^1 p(t)f((1-t)A+tB) dt \right), \quad k = 1, \dots, n,$$

for any positive matrices $A, B \in \mathcal{M}_n$.

Proof. From Theorem 3.1, we have

$$\sum_{j=1}^k \lambda_j \left(\log f \left(\frac{A+B}{2} \right) \right) \leq \sum_{j=1}^k \lambda_j \left(\log \int_0^1 p(t) f((1-t)A + tB) dt \right),$$

which is equivalent to

$$\sum_{j=1}^k \log \lambda_j \left(f \left(\frac{A+B}{2} \right) \right) \leq \sum_{j=1}^k \log \lambda_j \left(\int_0^1 p(t) f((1-t)A + tB) dt \right).$$

Consequently,

$$\log \prod_{j=1}^k \lambda_j \left(f \left(\frac{A+B}{2} \right) \right) \leq \log \prod_{j=1}^k \lambda_j \left(\int_0^1 p(t) f((1-t)A + tB) dt \right),$$

which implies the desired inequality. $\square$

4. LEVIN-STEČKIN MATRIX INEQUALITIES

We present a new inequality of Levin-Stečkin type. The significance of this inequality is its validity for any positive function p without imposing any conditions on its symmetry or monotony.

Theorem 4.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be convex differentiable and let $p : [0, 1] \rightarrow [0, +\infty)$ be continuous. Then*

$$\int_0^1 f(t) dt \int_0^1 p(t) dt + \left(\int_0^1 f'(t) dt \int_0^1 tp(t) dt - \int_0^1 tf'(t) dt \int_0^1 p(t) dt \right) \leq \int_0^1 f(t)p(t) dt.$$

Further,

$$\int_0^1 p(t) f(t) dt + \frac{1}{2} \int_0^1 p(t) f'(t) dt - \int_0^1 p(t) tf'(t) dt \leq \int_0^1 p(t) dt \int_0^1 f(t) dt.$$

Proof. For the convex differentiable function f and $s, t \in [0, 1]$ we have

$$(4.1) \quad f(s) + f'(s)(t-s) \leq f(t).$$

Since $p(t) \geq 0$, it follows that

$$p(t)f(s) + p(t)f'(s)(t-s) \leq p(t)f(t), \quad s, t \in [0, 1].$$

Integrating this inequality over $t \in [0, 1]$ then over $s \in [0, 1]$ implies

$$\int_0^1 f(s) ds \int_0^1 p(t) dt + \left(\int_0^1 f'(s) ds \int_0^1 tp(t) dt - \int_0^1 sf'(s) ds \int_0^1 p(t) dt \right) \leq \int_0^1 f(t)p(t) dt,$$

which is equivalent to the first desired inequality.

For the second inequality, integrating (4.1) over $t \in [0, 1]$, we obtain

$$f(s) + f'(s) \left(\frac{1}{2} - s \right) \leq \int_0^1 f(t) dt.$$

If we put $s = t$, we have

$$f(t) + f'(t) \left(\frac{1}{2} - t \right) \leq \int_0^1 f(t) dt.$$

Multiplying both sides by $p(t)$, we get

$$p(t) f(t) + p(t) f'(t) \left(\frac{1}{2} - t \right) \leq p(t) \int_0^1 f(t) dt.$$

Again, if we take integral over $t \in [0, 1]$, we infer that

$$\int_0^1 p(t) f(t) dt + \frac{1}{2} \int_0^1 p(t) f'(t) dt - \int_0^1 p(t) t f'(t) dt \leq \int_0^1 p(t) dt \int_0^1 f(t) dt.$$

This completes the proof. $\square$

Corollary 4.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be convex differentiable and let $p : [0, 1] \rightarrow [0, +\infty)$ be symmetric about $\frac{1}{2}$ and non-decreasing on $[0, \frac{1}{2}]$. Then*

$$\int_0^1 f'(t) dt \int_0^1 t p(t) dt \leq \int_0^1 t f'(t) dt \int_0^1 p(t) dt.$$

Proof. This follows from the first inequality in Theorem 4.1 because when p is symmetric about $\frac{1}{2}$ and non-decreasing on $[0, \frac{1}{2}]$, we have

$$\int_0^1 f(t) p(t) dt \leq \int_0^1 f(t) dt \int_0^1 p(t) dt. \quad \square$$

Assume that τ_t and σ_t are two arbitrary weighted symmetric operator means with $0 \leq t \leq 1$. A real-valued continuous function $f : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is operator τ_t - σ_t -convex if

$$f(A\tau_t B) \leq f(A)\sigma_t f(B),$$

for Hermitian $A, B \in \mathcal{M}_n$ whose spectra are contained in J . For $t = \frac{1}{2}$, we say f is operator τ - σ , and we write

$$(4.2) \quad f(A\tau B) \leq f(A)\sigma f(B).$$

An important example of operator mean is the arithmetic mean, which is denoted by ∇_t , as the weighted version, for $0 \leq t \leq 1$.

To prove the next lemma, we need the following important property of the weighted operator means:

$$(4.3) \quad (A\tau_\alpha B)\tau_\gamma(A\tau_\beta B) = A\tau_{(1-\gamma)\alpha+\gamma\beta} B, \quad \alpha, \beta, \gamma \in [0, 1].$$

Lemma 4.1. *Let $f : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an operator τ_t - σ_t -convex and let $x \in \mathbb{C}^n$. Then*

$$F(t) = \langle f(A\tau_t B)x, x \rangle$$

is convex on $0 \leq t \leq 1$.

Proof. Indeed,

$$\begin{aligned}
 F\left(\frac{t+s}{2}\right) &= \left\langle f\left(A\tau_{\frac{t+s}{2}}B\right)x, x\right\rangle \\
 &= \left\langle f\left((A\tau_tB)\tau(A\tau_sB)\right)x, x\right\rangle \quad (\text{by (4.3)}) \\
 &\leq \left\langle f\left(A\tau_tB\right)\sigma f\left(A\tau_sB\right)x, x\right\rangle \quad (\text{by (4.2)}) \\
 &\leq \left\langle f\left(A\tau_tB\right)\nabla f\left(A\tau_sB\right)x, x\right\rangle \\
 &\quad (\text{since arithmetic mean is the biggest one among symmetric means}) \\
 &= \frac{\left\langle f\left(A\tau_tB\right)x, x\right\rangle + \left\langle f\left(A\tau_sB\right)x, x\right\rangle}{2} \\
 &= \frac{F(t) + F(s)}{2}.
 \end{aligned}$$

This completes the proof. $\square$

Theorem 4.2. Let $A, B \in \mathcal{M}_n$ be two Hermitian matrices with spectra contained in J , let f be operator τ_t - σ_t -convex and $p : [0, 1] \rightarrow [0, +\infty)$ be symmetric about $t = \frac{1}{2}$ and non-decreasing on $[0, \frac{1}{2}]$. Then

$$\int_0^1 p(t) f(A\tau_tB) dt \leq \int_0^1 p(t) dt \int_0^1 f(A\tau_tB) dt.$$

Proof. Let $x \in \mathcal{M}_n$ be a unit vector. Then

$$\begin{aligned}
 \left\langle \left(\int_0^1 p(t) f(A\tau_tB) dt \right) x, x \right\rangle &= \int_0^1 p(t) \langle f(A\tau_tB)x, x \rangle dt \\
 &\leq \int_0^1 p(t) dt \int_0^1 \langle f(A\tau_tB)x, x \rangle dt \\
 &= \left\langle \left(\int_0^1 p(t) dt \int_0^1 f(A\tau_tB) dt \right) x, x \right\rangle
 \end{aligned}$$

where we have employed Lemma 4.1. This completes the proof. $\square$

The case $\tau_t = \sigma_t = \nabla_t$, in Theorem 4.2, reduces to

$$(4.4) \quad \int_0^1 p(t) f((1-t)A + tB) dt \leq \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt.$$

The following theorem gives a reverse for the inequality (4.4) by employing the Mond-Pečarić method [11].

Theorem 4.3. Let $f : [m, M] \rightarrow \mathbb{R}$ be convex and let $p : [0, 1] \rightarrow [0, +\infty)$ be symmetric about $t = \frac{1}{2}$. If $A, B \in \mathcal{M}_n$ are Hermitian with spectra in the interval $[m, M]$, then for

any $\alpha \geq 0$

$$\int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \leq \beta \int_0^1 p(t) dt I + \alpha \int_0^1 p(t) f((1-t)A + tB) dt,$$

where $\beta = \max_{m \leq x \leq M} \{a_f x + b_f - \alpha f(x)\}$, $a_f = \frac{f(M) - f(m)}{M - m}$ and $b_f = \frac{Mf(m) - mf(M)}{M - m}$.

Proof. Since $f(x) \leq a_f x + b_f$, we get by the functional calculus

$$f((1-t)A + tB) \leq a_f((1-t)A + tB) + b_f I.$$

By taking integral over $0 \leq t \leq 1$, we reach

$$\int_0^1 f((1-t)A + tB) dt \leq a_f \left(\frac{A+B}{2} \right) + b_f I.$$

This implies

$$\int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \leq a_f \left(\int_0^1 p(t) dt \right) \frac{A+B}{2} + b_f \left(\int_0^1 p(t) dt \right) I.$$

Hence for any vector y ,

$$\left\langle \left(\int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \right) y, y \right\rangle \leq a_f \left(\int_0^1 p(t) dt \right) \left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle + \int_0^1 p(t) dt b_f.$$

Now, by (2.3), we can write

$$\begin{aligned} & \left\langle \left(\int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \right) y, y \right\rangle - \alpha \int_0^1 p(t) f(\langle (1-t)A + tB y, y \rangle) dt \\ & \leq a_f \left(\int_0^1 p(t) dt \right) \left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle + b_f \int_0^1 p(t) dt - \alpha \int_0^1 p(t) f(\langle (1-t)A + tB y, y \rangle) dt \\ & \leq a_f \left(\int_0^1 p(t) dt \right) \left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle + b_f \int_0^1 p(t) dt - \alpha \left(\int_0^1 p(t) dt \right) f \left(\left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle \right) \\ & = \left(\int_0^1 p(t) dt \right) \left(a_f \left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle + b_f - \alpha f \left(\left\langle \left(\frac{A+B}{2} \right) y, y \right\rangle \right) \right) \\ & \leq \left(\int_0^1 p(t) dt \right) \max_{m \leq x \leq M} \{a_f x + b_f - \alpha f(x)\}. \end{aligned}$$

Thus,

$$\begin{aligned}
 & \left\langle \left(\int_0^1 p(t) dt \int_0^1 f((1-t)A + tB) dt \right) y, y \right\rangle \\
 & \leq \beta \int_0^1 p(t) dt + \alpha \int_0^1 p(t) f(\langle ((1-t)A + tB)y, y \rangle) dt \\
 & \leq \beta \int_0^1 p(t) dt + \alpha \int_0^1 p(t) \langle f((1-t)A + tB)y, y \rangle dt \\
 & \quad (\text{by [11, Theorem 1.2]}) \\
 & = \left\langle \left(\beta \left(\int_0^1 p(t) dt \right) I + \alpha \int_0^1 p(t) f((1-t)A + tB) dt \right) y, y \right\rangle
 \end{aligned}$$

as desired. $\square$

5. FURTHER INEQUALITIES VIA SYNCHRONOUS FUNCTIONS

We say that the functions $f, g : J \rightarrow \mathbb{R}$ are synchronous (asynchronous) on the interval J if they satisfy the following condition, for all $s, t \in J$,

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0.$$

It is obvious that if f, g are monotonic and have the same monotonicity on the interval J , then they are synchronous on J while if they have opposite monotonicity, they are asynchronous.

Related to the Levin-Stečkin inequality, the celebrated Čebyšev inequality [2] states that if f and g are two functions having the same monotonicity on $[0, 1]$, then

$$\int_0^1 f(t) dt \int_0^1 g(t) dt \leq \int_0^1 f(t) g(t) dt.$$

For some Čebyšev type inequalities for Hilbert space operators, see [7].

The following result provides a refinement and a reverse of this inequality via synchronous functions.

Theorem 5.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be synchronous functions on the interval $[a, b]$. Then*

$$\begin{aligned}
 & \min \left\{ \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2, \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right\} \\
 & \leq \frac{1}{b-a} \int_a^b f(t) g(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right) \left(\frac{1}{b-a} \int_a^b g(t) dt \right)
 \end{aligned}$$

$$\leq \max \left\{ \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2, \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right\}.$$

If f and g have opposite monotonicity then

$$\begin{aligned} & \min \left\{ \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2, \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right\} \\ & \leq \left(\frac{1}{b-a} \int_a^b f(t) dt \right) \left(\frac{1}{b-a} \int_a^b g(t) dt \right) - \frac{1}{b-a} \int_a^b f(t) g(t) dt \\ & \leq \max \left\{ \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2, \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right\}. \end{aligned}$$

Proof. We prove the first inequality. The second inequality goes likewise, and we omit the details. We have

$$\begin{aligned} & f(t)g(t) + f(s)g(s) - (f(t)g(s) + f(s)g(t)) \\ & = (f(t) - f(s))(g(t) - g(s)) \\ & = |(f(t) - f(s))(g(t) - g(s))| \\ & = |f(t) - f(s)| |g(t) - g(s)| \\ & \geq \min \{ (f(t) - f(s))^2, (g(t) - g(s))^2 \} \\ & = \min \{ f^2(t) + f^2(s) - 2f(t)f(s), g^2(t) + g^2(s) - 2g(t)g(s) \}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \min \{ f^2(s) + f^2(t) - 2f(s)f(t), g^2(t) + g^2(s) - 2g(t)g(s) \} \\ & \leq f(t)g(t) + f(s)g(s) - (f(t)g(s) + f(s)g(t)). \end{aligned}$$

Consequently,

$$\begin{aligned} & \min \left\{ (b-a)f^2(s) + \int_a^b f^2(t) dt - 2f(s) \int_a^b f(t) dt, \int_a^b g^2(t) dt + (b-a)g^2(s) - 2g(s) \int_a^b g(t) dt \right\} \\ & \leq \int_a^b f(t)g(t) dt + (b-a)f(s)g(s) - g(s) \int_a^b f(t) dt - f(s) \int_a^b g(t) dt. \end{aligned}$$

Upon integration, this implies

$$\begin{aligned} & \min \left\{ 2(b-a) \int_a^b f^2(t) dt - 2 \left(\int_a^b f(t) dt \right)^2, 2(b-a) \int_a^b g^2(t) dt - 2 \left(\int_a^b g(t) dt \right)^2 \right\} \\ & \leq 2(b-a) \int_a^b f(t)g(t) dt - 2 \int_a^b f(t) dt \int_a^b g(t) dt. \end{aligned}$$

Multiplying both sides by $\frac{1}{2(b-a)^2}$, we obtain,

$$\begin{aligned} & \min \left\{ \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2, \frac{1}{b-a} \int_a^b g^2(t) dt - \left(\frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right\} \\ & \leq \frac{1}{b-a} \int_a^b f(t) g(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right) \left(\frac{1}{b-a} \int_a^b g(t) dt \right). \end{aligned}$$

The second inequality is obtained from the same arguments and the following relation

$$\begin{aligned} & \max \{ f^2(s) + f^2(t) - 2f(s)f(t), g^2(t) + g^2(s) - 2g(t)g(s) \} \\ & \geq f(t)g(t) + f(s)g(s) - (f(t)g(s) + f(s)g(t)). \end{aligned} \quad \square$$

In the following result, we establish a refinement and a reverse for the Levin-Stečkin inequality.

Theorem 5.2. *Let $p : [0, 1] \rightarrow \mathbb{R}$ be a symmetric about $t = \frac{1}{2}$, namely $p(1-t) = p(t)$, and non-decreasing on $[0, \frac{1}{2}]$, then for every convex function f on $[0, 1]$,*

$$\begin{aligned} & \int_0^1 p(t) f(t) dt \leq \int_0^1 p(t) dt \int_0^1 f(t) dt \\ & - \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(\int_0^1 p(t) dt \right)^2, \frac{1}{2} \int_0^{1/2} (f(t) + f(1-t))^2 dt - \left(\frac{1}{2} \int_0^1 (f(t) + f(1-t)) dt \right)^2 \right\}. \end{aligned}$$

A similar but reversed inequality holds if we replace min with max.

Proof. If f is symmetric and convex, by Theorem 5.1, we have

$$\begin{aligned} & \int_0^1 p(t) dt \int_0^1 f(t) dt \\ & = \left(\int_0^{1/2} p(t) dt + \int_{1/2}^1 p(t) dt \right) \left(\int_0^{1/2} f(t) dt + \int_{1/2}^1 f(t) dt \right) = 4 \int_0^{1/2} p(t) dt \int_0^{1/2} f(t) dt \\ & \geq 2 \int_0^{1/2} p(t) f(t) dt + \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(2 \int_0^{1/2} p(t) dt \right)^2, 2 \int_0^{1/2} f^2(t) dt - \left(2 \int_0^{1/2} f(t) dt \right)^2 \right\} \\ & = \int_0^1 p(t) f(t) dt + \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(\int_0^1 p(t) dt \right)^2, 2 \int_0^{1/2} f^2(t) dt - \left(\int_0^1 f(t) dt \right)^2 \right\}. \end{aligned}$$

Namely,

$$\begin{aligned} & \int_0^1 p(t) f(t) dt + \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(\int_0^1 p(t) dt \right)^2, 2 \int_0^{1/2} f^2(t) dt - \left(\int_0^1 f(t) dt \right)^2 \right\} \\ & \leq \int_0^1 p(t) dt \int_0^1 f(t) dt. \end{aligned}$$

We shall now consider an arbitrary f . For convex f , the function $\frac{f(x)+f(1-x)}{2}$ is convex and symmetric so that we can use the above inequality. Hence,

$$\begin{aligned} & \int_0^1 p(t) f(t) dt \\ &= \frac{\int_0^1 p(t) f(t) dt + \int_0^1 p(1-t) f(1-t) dt}{2} = \int_0^1 p(t) \frac{f(t) + f(1-t)}{2} dt \\ &\leq \int_0^1 p(t) dt \int_0^1 \frac{f(t) + f(1-t)}{2} dt \\ &\quad - \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(\int_0^1 p(t) dt \right)^2, \frac{1}{2} \int_0^{1/2} (f(t) + f(1-t))^2 dt - \left(\frac{1}{2} \int_0^1 (f(t) + f(1-t)) dt \right)^2 \right\} \\ &= \int_0^1 p(t) dt \int_0^1 f(t) dt \\ &\quad - \min \left\{ 2 \int_0^{1/2} p^2(t) dt - \left(\int_0^1 p(t) dt \right)^2, \frac{1}{2} \int_0^{1/2} (f(t) + f(1-t))^2 dt - \left(\frac{1}{2} \int_0^1 (f(t) + f(1-t)) dt \right)^2 \right\}, \end{aligned}$$

which yields the desired inequality. $\square$

We can improve the second inequality in Theorem 5.1 in the following way.

Theorem 5.3. *Let $f, g : J \rightarrow \mathbb{R}$ be synchronous functions on the interval $[0, 1]$. Then*

$$\begin{aligned} & \int_0^1 f(t) g(t) dt - \int_0^1 f(t) dt \int_0^1 g(t) dt \\ & \leq \frac{1}{2} \left(\int_0^1 f^2(t) dt - \left(\int_0^1 f(t) dt \right)^2 + \int_0^1 g^2(t) dt - \left(\int_0^1 g(t) dt \right)^2 \right). \end{aligned}$$

Proof. We have

$$\begin{aligned}
 & f(t)g(t) + f(s)g(s) - (f(t)g(s) + f(s)g(t)) \\
 &= (f(t) - f(s))(g(t) - g(s)) \\
 &= |(f(t) - f(s))(g(t) - g(s))| \\
 &= |f(t) - f(s)| |g(t) - g(s)| \\
 &\leq \frac{1}{2} \left((f(t) - f(s))^2 + (g(t) - g(s))^2 \right) \\
 &= \frac{1}{2} \left(f^2(t) + f^2(s) + g^2(t) + g^2(s) - 2(g(t)g(s) + f(t)f(s)) \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & f(t)g(t) + f(s)g(s) - (f(t)g(s) + f(s)g(t)) \\
 &\leq \frac{1}{2} \left(f^2(t) + f^2(s) + g^2(t) + g^2(s) - 2(g(t)g(s) + f(t)f(s)) \right).
 \end{aligned}$$

The remaining part of the proof is similar to the proof of Theorem 5.1, so we omit the details. $\square$

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