Kragujevac Journal of Mathematics
Volume 49(5) (2025), Pages 781–792.

THE GLOBAL BEHAVIOR OF A SECOND ORDER EXPONENTIAL DIFFERENCE EQUATION

VAHIDIN HADŽIABDIĆ[1], JASMIN BEKTEŠEVIĆ[1], AND MIDHAT MEHULJIĆ[1]

Abstract. In this paper we present the Julia set and the global behavior of an exponential second order difference equation of the type

$$x_{n+1} = ax_{n-1} + bx_{n-1} \exp\left(cx_{n-1} + cx_n\right),$$

where $a \geq 0$, $b > 0$ and $c > 0$ with non-negative initial conditions.

1. Introduction

In general, difference equations and systems of difference equations in exponential forms have numerous applications in biology, more precisely, they can be used to discuss population model. One of the most simple results on exponential difference equation have been obtained in [8] for the equation of type

$$x_{n+1} = x_n \exp\left(r\left(1 - x_n\right)\right),$$

known as Ricker's equation, which describes a population with a propensity to simple exponential growth at low densities and tendency to decrease at high densities. In [9] the qualitative behavior of the exponential second order difference equation of the two-dimensional population model

$$x_{n+1} = a + bx_{n-1} \exp\left(-x_n\right)$$

is completely investigated and described. In [14] we can find results about boundedness and asymptotic behavior of the positive solution for the difference equation of type

$$x_{n+1} = a + bx_n \exp\left(-x_{n-1}\right),$$

Key words and phrases. Basin of attraction, period-two solutions, Julia set, difference equation.
2020 *Mathematics Subject Classification.* Primary: 39A10. Secondary: 39A23.
DOI 10.46793/KgJMat2505.781H
Received: March 31, 2022.
Accepted: January 18, 2023.

where a and b are positive constants and the initial values x_{-1}, x_0 are nonnegative real numbers. In [11] are given the conditions for the global behavior of the positive solutions for the difference equation

$$x_{n+1} = ax_n + bx_{n-1} \exp(-x_n),$$

where a and b are positive real numbers with positive initial conditions x_{-1}, x_0. The global stability and bounded nature of the positive solutions of the difference equation

$$x_{n+1} = a + bx_{n-1} + cx_{n-1} \exp(-x_n)$$

are investigated in [10]. In [7] have been obtained results for the local stability of equilibria, parametric conditions for transcritical bifurcation, period-doubling bifurcation and Neimark-Sacker bifurcation of the following second-order difference equation

$$x_{n+1} = \alpha x_n + \beta x_{n-1} \exp(-\sigma x_{n-1}),$$

where the initial conditions satisfy $x_{-1} > 0$, $x_0 > 0$ and α, β and σ are the positive constants. In this paper we will present very unusual results for exponential second order difference equations. Our results are based on the theorems which hold for monotone difference equations. Our principal tool is the theory of monotone maps, and in particular cooperative maps, which guarantee the existence and uniqueness of the stable and unstable invariant manifolds for the fixed points and periodic points (see [5]). Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{1.1}$$

where f is a continuous and increasing function in both variables. The following result has been obtained in [1].

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of (1.1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following.*

- (i) *Eventually they are both monotonically increasing.*
- (ii) *Eventually they are both monotonically decreasing.*
- (iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

As a consequence of Theorem 1.1 every bounded solution of (1.1) approaches either an equilibrium solution or period-two solution and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of (1.1) is the problem how to determine the basins of attraction of three different types of attractors: the equilibrium solutions, minimal period-two solution(s) and the point(s) at infinity. The following result can be proved by using the techniques of proof of Theorem 11 in [5].

Theorem 1.2. *Consider* (1.1) *where f is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2)$$

and that E_1 is a local attractor and E_2 is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$, with the neighborhoods where f is strictly increasing. Then the basin of attraction $\mathcal{B}(E_1)$ of E_1 is the region below the global stable manifold $\mathcal{W}^s(E_2)$. More precisely

$$\mathcal{B}(E_1) = \{(x, y) : \text{exists } y_u : \ y < y_u, (x, y_u) \in \ \mathcal{W}^s(E_2)\}.$$

The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ is exactly the global stable manifold of E_2. The global stable manifold extend to the boundary of the domain of (1.1)*. If there exists a period-two solution, then the end points of the global stable manifold are exactly the period-two solution.*

Now, the theorems that are applied in [5] provided the two continuous curves $\mathcal{W}^s(E_2)$ (stable manifold) and $\mathcal{W}^u(E_2)$ (unstable manifold), both passing through the point $E_2(x_2, y_2)$ from Theorem 1.2, such that $\mathcal{W}^s(E_2)$ is a graph of decreasing function and $\mathcal{W}^u(E_2)$ is a graph of an increasing function. The curve $\mathcal{W}^s(E_2)$ splits the first quadrant of initial conditions into two disjoint regions, but we do not know the explicit form of the curve $\mathcal{W}^s(E_2)$. In this paper we investigate the following difference equation

$$x_{n+1} = ax_{n-1} + bx_{n-1}\exp(cx_{n-1} + cx_n), \tag{1.2}$$

where $a \geq 0$, $b > 0$ and $c > 0$ with non-negative initial conditions, that has infinitely many period-two solutions and we expose the explicit form of the curve that separates the first quadrant into two basins of attraction of a locally stable equilibrium point and of the point at infinity. One of the major problems in the dynamics of monotonic maps is determining the basin of attraction of the point at infinity and in particular the boundary of the that basin known as the Julia set. We precisely determined the Julia set of (1.2) and we obtained the global dynamics in the interior of the Julia set, which includes all the points for which solutions are not asymptotic to the point at infinity. It turned out that the Julia set for (1.2) is the union of the stable manifolds of some saddle equilibrium points, nonhyperbolic equilibrium points or period-two points. We first list some results needed for the proofs of our theorems. The main result for studying local stability of equilibria is linearized stability theorem (see Theorem 1.1 in [12]).

Theorem 1.3 (Linearized stability)**.** *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \tag{1.3}$$

and let $\bar{x}$ be an equilibrium point of difference equation (1.3) *. Let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial u}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial v}$ denote the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\bar{x}$. Let λ_1 and λ_2 roots of the quadratic equation $\lambda^2 - p\lambda - q = 0$.*

a) *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the equilibrium $\bar{x}$ is locally asymptotically stable* (*sink*)*.*
b) *If $|\lambda_1| > 1$ or $|\lambda_2| > 1$, then the equilibrium $\bar{x}$ is unstable.*
c) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| < 1 - q < 2$*. Equilibrium $\bar{x}$ is a sink.*
d) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1 \Leftrightarrow |q| > 1$ *and* $|p| < |1 - q|$*. Equilibrium $\bar{x}$ is a repeller.*
e) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| > |1 - q|$*. Equilibrium $\bar{x}$ is a saddle point.*
f) $|\lambda_1| = 1$ *or* $|\lambda_2| = 1 \Leftrightarrow |p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$*. Equilibrium $\bar{x}$ is called a non-hyperbolic point.*

The next theorem (Theorem 1.4.1. in [6]) is a very useful tool in establishing bounds for the solutions of nonlinear equations in terms of the solutions of equations with known behaviour.

Theorem 1.4. *Let I be an interval of real numbers, let k be a positive integer, and let $F : I^{k+1} \to I$ be a function which is increasing in all its arguments. Assume that $\{x_n\}_{n=-k}^{\infty}, \{y_n\}_{n=-k}^{\infty}$ and $\{z_n\}_{n=-k}^{\infty}$ are sequences of real numbers such that*

$$\begin{aligned} x_{n+1} \leq & F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \\ y_{n+1} = & F(y_n, \ldots, y_{n-k}), \quad n = 0, 1, \ldots, \\ z_{n+1} \geq & F(z_n, \ldots, z_{n-k}), \quad n = 0, 1, \ldots \end{aligned}$$

and

$$x_n \leq y_n \leq z_n, \quad \textit{for all } -k \leq n \leq 0.$$

Then

$$x_n \leq y_n \leq z_n, \quad \textit{for all } n > 0.$$

2. Main Results

By using Theorem 1.3, we obtained the following result on local stability of the zero equilibrium of (1.2).

Proposition 2.1. *The zero equilibrium of* (1.2) *is one of the following:*

a) *locally asymptotically stable if $a + b < 1$;*
b) *non-hyperbolic $a + b = 1$;*
c) *unstable if $a + b > 1$.*

Set $f(x, y) = ay + by \exp(cy + cx)$ and let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial x}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial y}$ denote the partial derivatives of $f(x, y)$ evaluated at the equilibrium $\bar{x}$. The linearized equation at the positive equilibrium $\bar{x}$ is

$$\begin{aligned} z_{n+1} =& p z_n + q z_{n-1}, \\ p =& bc\bar{x}\exp(2c\bar{x}), \\ q =& a + b(1+c\bar{x})\exp(2c\bar{x}). \end{aligned}$$

Now, in view of Theorem 1.3 we obtain the following results on local stability of the positive equilibrium of (1.2).

Proposition 2.2. *The positive equilibrium of* (1.2) *is one of the following:*

a) *locally asymptotically stable if* $p+q<1$*;*
b) *non-hyperbolic if* $p+q=1$ *or* $q-p=1$*;*
c) *unstable if* $p+q>1$*;*
d) *saddle point if* $p>|q-1|$*;*
e) *repeller if* $1-q<p<q-1$.

Theorem 2.1. *If* $a \geq 1$ *or* $b \geq 1$ *or* $a+b>1$*, then every solution* $\{x_n\}$ *of* (1.2) *satisfies* $\lim\limits_{n\to\infty} x_n = \infty$.

Proof. Let be $a \geq 1$ or $b \geq 1$, then $a+b>1$. If $\{x_n\}$ is a solution of (1.2), then $\{x_n\}$ satisfies the inequality

$$\begin{aligned} x_{n+1} =& a x_{n-1} + b x_{n-1}\exp(c x_{n-1} + c x_n) \\ \geq& a x_{n-1} + b x_{n-1} = (a+b) x_{n-1}, \quad n = 0,1,\ldots, \end{aligned}$$

which in view of the result on difference inequalities, see Theorem 1.4, implies that $x_n \geq y_n$, $n \geq 1$, where $\{y_n\}$ is a solution of the initial value problem

$$y_{n+1} = (a+b)\, y_{n-1}, \quad y_{-1} = x_{-1} \text{ and } y_0 = x_0, \quad n = 0,1,\ldots$$

Consequently, if $x_0, x_{-1} > 0$, then $y_0, y_{-1} > 0$, $y_n \geq 0$ for all n, and

$$x_n \geq y_n = \lambda_1 \sqrt{a+b}^{\,n} + \lambda_2 \left(-\sqrt{a+b}\right)^n, \quad n = 1,2,\ldots,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $y_n \geq 0$ for all n, which implies $\lim\limits_{n\to\infty} x_n = \infty$. □

Theorem 2.2. *Consider the difference equation* (1.2) *in the first quadrant of initial conditions, where* $a,b,c>0$ *and* $a+b<1$*. Then* (1.2) *has a zero equilibrium and a unique positive equilibrium* $\bar{x}_+ = \frac{1}{2c}\ln\frac{1-a}{b}$*. The line* $b\exp(cy+cx) = 1-a$ *is the Julia set and separates the first quadrant into two regions: the region below the given line is the basin of attraction of point* $E_0(0,0)$*, the region above the line is the basin of attraction of the point at infinity and every point on the line except* $E_+(\bar{x}_+, \bar{x}_+)$ *is a period-two solution of* (1.2).

Proof. The equilibrium points of (1.2) are the solutions of equation

$$x\left(a + b\exp(2cx)\right) = x,$$

that is equivalent to

$$x\left(b\exp(2cx) + a - 1\right) = 0, \tag{2.1}$$

which implies that (2.1) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+$. Since $a+b<1$, then by applying Proposition (2.1) the zero equilibrium is locally asymptotically stable. Denote by $f(x,y)=ay+by\exp(cx+cy)$ and let p and q denote the partial derivatives of function $f(x,y)$ at point E_+. By straightforward calculation we obtain that the following hold:

$$\begin{aligned} p+q &= a+b(1+2c\bar{x})\exp(2c\bar{x}) \\ &= a+b(1+2c\bar{x})\frac{1-a}{b} = 1+2c(1-a)\bar{x}>1, \\ q-p &= a+b\exp(2c\bar{x}) = a+b\cdot\frac{1-a}{b}=1. \end{aligned}$$

Hence, by applying Proposition 2.2 the positive equilibrium is an unstable non-hyperbolic point. Period-two solution u,v satisfies the system

$$\begin{aligned} u &= (a+b\exp(cu+cv))\,u, \\ v &= (a+b\exp(cu+cv))\,v. \end{aligned}$$

Obviously, the point $(0,0)$ is solution of the system above, but it is not minimal period-two solution. Hence, it has to be $v>0$ which implies $a+b\exp(cu+cv)=1$. Therefore, every point of the set $\{(x,y): a+b\exp(cx+cy)=1\}$ is a period-two solution of (1.2) except point E_+. Clearly, the curve $g(x,y)=a+b\exp(cx+cy)=1$ is a graph of the decreasing function in the first quadrant, more precisely that is line $y=-x+\frac{1}{c}\ln\frac{1-a}{b}$. Let $\{x_n\}$ be a solution of (1.2) for initial condition (x_0,x_{-1}) which lies below the line $g(x,y)=1$. Then

$$\begin{aligned} g(x_0,x_{-1}) &= a+b\exp(cx_0+cx_{-1})<1, \\ x_{n+1} &= g(x_n,x_{n-1})\,x_{n-1} \end{aligned}$$

and

$$\begin{aligned} x_1 =& g(x_0,x_{-1})\,x_{-1}<x_{-1}, \\ x_2 =& g(x_1,x_0)\,x_0<g(x_{-1},x_0)\,x_0=g(x_0,x_{-1})\,x_0<x_0. \end{aligned}$$

Thus (x_2,x_1) and (x_0,x_{-1}) are two points in North-East ordering $(x_2,x_1)\le_{ne}(x_0,x_{-1})$ which means that the point (x_2,x_1) is also below the curve $g(x,y)=1$ and also holds

$$g(x_2,x_1)<1.$$

Similarly we find

$$\begin{aligned} x_3 =& g(x_2,x_1)\,x_1<x_1, \\ x_4 =& g(x_3,x_2)\,x_2<g(x_1,x_2)\,x_2=g(x_2,x_1)\,x_2<x_2. \end{aligned}$$

Continuing on this way we get

$$(0,0)\le_{ne}\cdots\le_{ne}(x_4,x_3)\le_{ne}(x_2,x_1)\le_{ne}(x_0,x_{-1}),$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically decreasing and bounded below by 0. Since below the line $g(x,y)=1$ there are no period-two solutions it must be $x_{2n} \to 0$ and $x_{2n+1} \to 0$. On the other hand, if we consider solution $\{x_n\}$ of (1.2) for initial condition (x_0, x_{-1}) which lies above the line $g(x,y) = 1$ then $g(x_0, x_{-1}) > 1$ and by applying the method shown above we obtain the following condition:

$$(x_{-1}, x_0) \leq_{ne} (x_1, x_2) \leq_{ne} (x_3, x_4) \leq_{ne} \cdots$$

Therefore, both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing, hence $x_{2n} \to \infty$ and $x_{2n+1} \to \infty$ as $n \to \infty$. □

Figure 1 is visual illustration of Theorem 2.2 obtained by using Mathematica 9.0, with the boundaries of the basins of attraction obtained by using the software package Dynamica [6].

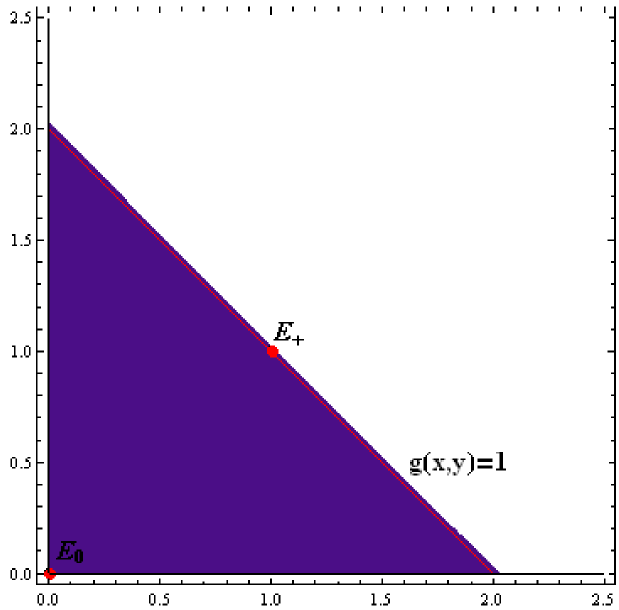

FIGURE 1. Case: $a = 1 - e^{-2}$, $b = e^{-3}$, $c = \frac{1}{2}$

Theorem 2.3. *Consider the difference equation* (1.2)*, where* $a + b = 1$ *and initial conditions* $x_{-1}, x_0 \geq 0$ *such that* $x_{-1}^2 + x_0^2 \neq 0$*. Then* (1.2) *has an unique zero equilibrium and every solution* $\{x_n\}$ *of* (1.2) *satisfies* $\lim_{n\to\infty} x_n = \infty$*.*

Proof. Assume that $a + b = 1$ and $\{x_n\}$ is a solution of (1.2). Since $x_{-1}^2 + x_0^2 \neq 0$, then $\exp(cx_{n-1} + cx_n) > 1$, which implies $\exp(cx_{n-1} + cx_n) = 1 + \alpha_n$, where $\alpha_n > 0$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ satisfies the inequality

$$\begin{aligned} x_{n+1} &= x_{n-1}(a + b\exp(cx_{n-1} + cx_n)) \\ &\geq x_{n-1}(a + b(1 + \alpha_n)) \\ &= x_{n-1}(a + b + b\alpha_n) = x_{n-1}(1 + b\alpha_n) \\ &> x_{n-1}, \end{aligned}$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing. Since there is no positive equilibrium point or period-two solution of (1.2) by applying Theorem 1.1 the both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ approache the point at infinity. $\square$

Now, consider the difference equation of type

$$x_{n+1} = Ax_{n-1} + Bx_{n-1}\exp\left(Cx_{n-1} + Dx_n\right) \tag{2.2}$$

in the first quadrant of initial conditions, where the given parameters satisfy conditions $A > 0$, $B > 0$, $C > 0$, $D > 0$ and $A + B < 1$. It is easy to show that (2.2) has two equilibria: zero equilibrium and unique positive equilibrium $\bar{x}_+ = \frac{1}{C+D}\ln\frac{1-A}{B}$.

Proposition 2.3. *The zero equilibrium of* (2.2) *is always locally asymptotically stable. The positive equilibrium* $\bar{x}_+ = \frac{1}{C+D}\ln\frac{1-A}{B}$ *of* (2.2) *is one of the following:*

a) *non-hyperbolic if* $C = D$ *(or* $q - p = 1$*);*
b) *saddle point if* $C < D$ *(or* $p > |q-1|$*);*
c) *repeller if* $C > D$ *(or* $p < |1-q|$*).*

Proof. Denote by $g(x,y) = Ay + By\exp(Cy + Dx)$ and let p and q denote the partial derivatives of function $g(x,y)$ at equilibrium point $\overline{x}$ of (2.2). By straightforward calculation we obtain that the following hold:

$$p(\overline{x},\overline{x}) = BD\overline{x}\exp\left((C+D)\overline{x}\right),$$
$$q(\overline{x},\overline{x}) = A + B(1 + C\overline{x})\exp\left((C+D)\overline{x}\right).$$

Hence, if $\overline{x} = 0$, then $p(0,0) = 0$ and $q(0,0) = A + B \in (0,1)$ which implies $|p| < 1 - q < 2$, so by applying Theorem 1.3 the zero equilibrium is locally asymptotically stable. If $\overline{x} = \bar{x}_+$, then $p(\bar{x}_+,\bar{x}_+) = \frac{(1-A)D}{C+D}\ln\frac{1-A}{B} = (1-A)D\bar{x}_+ > 0$ and

$$q(\bar{x}_+,\bar{x}_+) = 1 + \frac{(1-A)C}{C+D}\ln\frac{1-A}{B} = 1 + (1-A)C\bar{x}_+ > 1.$$

Clearly, $|p| + q = p + q > q > 1$, which implies, by applying Theorem 1.3, the positive equilibrium $\bar{x}_+$ is an unstable. Since $A \in (0,1)$ and

$$q - p = 1 + (1-A)(C-D)\bar{x}_+,$$

which yields

$$C = D \Rightarrow q - p = 1 \Leftrightarrow p = q - 1 \Leftrightarrow |p| = |1-q|,$$
$$C > D \Rightarrow q - p > 1 \Leftrightarrow p < q - 1 \Leftrightarrow |p| < |1-q|,$$
$$C < D \Rightarrow q - p < 1 \Leftrightarrow p > q - 1 \Leftrightarrow |p| > |1-q|.$$

The rest of proof following from Theorem 1.3. $\square$

Proposition 2.4. (2.2) *has prime period-two solution* $\left\{P_1\left(0, \frac{1}{C}\ln\frac{1-A}{B}\right), P_2\left(\frac{1}{C}\ln\frac{1-A}{B}, 0\right)\right\}$. *If* $C > D$, *then period-two solution is saddle and if* $C < D$, *then the period-two solution is repeller.*

Proof. Assume that (ϕ, ψ) is a prime period-two solution of (2.2) and $0 \le \phi < \psi$. Then

$$\begin{aligned} \phi &= A\phi + B\phi \exp\left(C\phi + D\psi\right), \\ \psi &= A\psi + B\psi \exp\left(C\psi + D\phi\right). \end{aligned} \tag{2.3}$$

If $\phi = 0$, then $\psi = \frac{1}{C} \ln \frac{1-A}{B}$. Let $\phi > 0$. From system (2.3) we find that

$$(C - D)(\phi - \psi) = 0,$$

which implies $C = D$ $(\phi \neq \psi)$, this case has already been considered. Set $u_n = x_{n-1}$ and $v_n = x_n$ and write (2.2) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= Au_n + Bu_n \exp\left(Cu_n + Dv_n\right). \end{aligned}$$

Let T be the function on $[0, \infty) \times [0, \infty)$ defined by

$$T(u, v) = \left(v, Au + Bu \exp\left(Cu + Dv\right)\right).$$

Then (ϕ, ψ) is a fixed point of T^2, the second iterate of T. Furthermore,

$$\begin{aligned} &T^2(u, v) = T(T(u, v)) \\ &= \left(Au + Bu \exp\left(Cu + Dv\right), Av + Bv \exp\left(Cv + D\left(Au + Bu \exp\left(Cu + Dv\right)\right)\right)\right), \\ &T^2(u, v) = \left(g(u, v), h(u, v)\right), \end{aligned}$$

where $g(u, v) = Au + Bu \exp\left(Cu + Dv\right)$ and $h(u, v) = g(v, g(u, v))$. Jacobian matrix $J_{T^2}(\phi, \psi)$ evaluated at $(\phi, \psi) = \left(0, \frac{1}{C} \ln \frac{1-A}{B}\right)$ is given by

$$\begin{aligned} J_{T^2}(\phi, \psi) &= \begin{pmatrix} \frac{\partial g}{\partial u}(\phi, \psi) & \frac{\partial g}{\partial v}(\phi, \psi) \\ \frac{\partial h}{\partial u}(\phi, \psi) & \frac{\partial h}{\partial v}(\phi, \psi) \end{pmatrix} \\ &= \begin{pmatrix} A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}} & 0 \\ \frac{(1-A)D}{C}\left(A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}}\right) \ln \frac{1-A}{B} & 1 + (1-A) \ln \frac{1-A}{B} \end{pmatrix} \end{aligned}$$

and

$$\det\left(J_T(\phi, \psi)\right) = \left(A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}}\right)\left(1 + (1-A) \ln \frac{1-A}{B}\right) > 0,$$

$$\operatorname{tr}\left(J_T(\phi, \psi)\right) = 1 + A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}} + (1-A) \ln \frac{1-A}{B} > 1.$$

If $C < D$, then $-1 + A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}} > -1 + A + B\left(\frac{1-A}{B}\right) = 0$ and

$$\operatorname{tr}\left(J_T(\phi, \psi)\right) - \det\left(J_T(\phi, \psi)\right) = 1 - (1-A)\left(-1 + A + B\left(\frac{1-A}{B}\right)^{\frac{D}{C}}\right) \ln \frac{1-A}{B} < 1,$$

which yields

$$|\mathrm{tr}\left(J_T\left(\phi,\psi\right)\right)| < |1+\det\left(J_T\left(\phi,\psi\right)\right)|.$$

Then by applying Theorem 1.3 ($p = tr\left(J_T\left(\phi,\psi\right)\right)$ and $q = -\det\left(J_T\left(\phi,\psi\right)\right)$), the minimal period-two solution $\{P_1, P_2\}$ is repeller. Similarly, if $C > D$, then

$$-1+A+B\left(\frac{1-A}{B}\right)^{\frac{D}{C}} < -1+A+B\left(\frac{1-A}{B}\right) = 0$$

and

$$\mathrm{tr}\left(J_T\left(\phi,\psi\right)\right) - \det\left(J_T\left(\phi,\psi\right)\right) > 1,$$

which implies

$$|\mathrm{tr}\left(J_T\left(\phi,\psi\right)\right)| > |1+\det\left(J_T\left(\phi,\psi\right)\right)|.$$

Now, by applying Theorem 1.3 the minimal period-two solution $\{P_1, P_2\}$ is saddle. □

Proposition 2.5. *Consider the difference equation* (2.2) *in the first quadrant of initial conditions, where the given parameters satisfy conditions $A > 0$, $B > 0$, $C > 0$, $D > 0$, $C \neq D$ and $A + B < 1$. Set $m = \min\{C, D\}$ and $M = \max\{C, D\}$. Then the global stable manifold of the positive equilibrium is between two lines*

$$p_1 : B\exp\left(mx+my\right) = 1-A \tag{2.4}$$

and

$$p_2 : B\exp\left(Mx+My\right) = 1-A. \tag{2.5}$$

Proof. In a view of Proposition 2.3 the zero equilibrium of (2.2) is always locally asymptotically stable. The theorems applied in [5] provided existence of global stable manifold $\mathcal{W}^s$ through the saddle point. If $C < D$, then by applying Proposition 2.3 the positive equilibrium $\bar{x}_+ = \frac{1}{C+D}\ln\frac{1-A}{B}$ is a saddle point and there exists a global stable manifold which contains point $E_+(\bar{x}_+, \bar{x}_+)$. In this case global behavior of (2.2) is described by Theorem 1.2 where end points of the global stable manifold $\mathcal{W}^s\left(E_+\right)$ are exactly the period-two solution $\{P_1, P_2\}$ from Proposition 2.4. If $C > D$, then by applying Proposition 2.3 the positive equilibrium $\bar{x}_+$ is a repeller and in a view of Proposition 2.4 there exists a prime period-two solution $\{P_1, P_2\}$ which is a saddle point. There are two global stable manifolds $\mathcal{W}^s\left(P_1\right)$ and $\mathcal{W}^s\left(P_2\right)$, guaranteed by Theorems 1 and 4 in [13], which contain points $P_1(\phi,\psi)$ and $P_2(\psi,\phi)$. In this case the global behavior of (2.2) is described by Theorem 10 in [2]. Although the Theorems 9 and 10 in [2] have been applied on a polynomial second order difference equation they are special cases of general Theorems in [5] applied on function f, where f is increasing function in its arguments. So, the global dynamics of (2.2) is exactly the same as the global dynamics of equations decribed by Theorems 9 and 10 in [2]. Furthermore,

$$x_{n+1} = Ax_{n-1} + Bx_{n-1}\exp\left(Cx_{n-1}+Dx_n\right) \geq Ax_{n-1} + Bx_{n-1}\exp\left(mx_{n-1}+mx_n\right)$$

and

$$x_{n+1} = Ax_{n-1} + Bx_{n-1}\exp\left(Cx_{n-1}+Dx_n\right) \leq Ax_{n-1} + Bx_{n-1}\exp\left(Mx_{n-1}+Mx_n\right),$$

for all n, by applying Theorem 1.4 for solution $\{x_n\}$ of (2.2) the following inequality holds

$$y_n \leq x_n \leq z_n,$$

for all n, where $\{y_n\}$ is a solution of the difference equation

$$y_{n+1} = Ay_{n-1} + By_{n-1} \exp\left(my_{n-1} + my_n\right) \tag{2.6}$$

and $\{z_n\}$ is a solution of the difference equation

$$z_{n+1} = Az_{n-1} + Bz_{n-1} \exp\left(Mz_{n-1} + Mz_n\right). \tag{2.7}$$

Since (2.6) and (2.7) satisfy all conditions of Theorem 2.2 this implies that the statement of Proposition 2.5 holds. □

3. Conclusion

In this paper we restrict our attention to certain exponential second order difference equation (1.2). It is important to mention that we have accurately determined the Julia set of (1.2) and the basins of attractions for the zero equilibrium and the positive equilibrium point. In general, all theoretical concepts which are very useful in proving the results of global attractivity of equilibrium points and period-two solutions only give us existence of global stable manifold(s) whose computation leads to very uncomfortable calculus (see [3, 4]).

Acknowledgements. The authors are very grateful to anonymous reviewers who read the entire paper with great care and whose corrections and suggestions have improved this paper.

References

[1] A. M. Amleh, E. Camouzis and G. Ladas, *On the dynamics of rational difference equation*, Part I, Int. J. Differ. Equ. **3**(1) (2008), 1–35.

[2] J. Bektešević, M. R. S. Kulenović and E. Pilav, *Global dynamics of quadratic second order difference equation in the first quadrant*, Appl. Math. Comput. **227** (2014), 50–65. https://doi.org/10.1016/j.amc.2013.10.048

[3] J. Bektešević, M. R. S. Kulenović and E. Pilav, *Asymptotic approximations of the stable and unstable manifolds of fixed points of a two-dimensional quadratic map*, J. Comput. Anal. Appl. **21**(1) (2016), 35–51.

[4] J. Bektešević, M. R. S. Kulenović and E. Pilav, *Asymptotic approximations of the stable and unstable manifolds of fixed points of a two-dimensional cubic map*, Int. J. Differ. Equ. **10**(1) (2015), 39–58.

[5] A. Brett and M. R. S. Kulenović, *Basins of attraction of equilibrium points of monotone difference equations*, Sarajevo J. Math. **5**(18) (2009), 211–233.

[6] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, London, 2008.

[7] Q. Din, E. M. Elabbasy, A. A. Elsadany and S. Ibrahim, *Bifurcation analysis and chaos control of a second-order exponential difference equation*, Filomat **33**(15) (2019), 5003–5022. https://doi.org/10.2298/FIL1915003D

[8] S. Elaydi, *An Introduction to Difference Equations*, 3rd Edition, Springer - Verlag, New York, 2005.

[9] E. El-Metwally, E. A. Grove, G. Ladas, R. Levins and M. Radin, *On the difference equation* $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, Nonlinear Anal. **47**(7) (2001), 4623–4634. `https://doi.org/10.1016/S0362-546X(01)00575-2`

[10] H. Feng, H. Ma and W. Ding, *Global asymptotic behavior of positive solutions for exponential form difference equation with three parameters*, J. Appl. Anal. Comput. **6**(3) (2016), 600–606. `https://doi.org/10.11948/2016041`

[11] E. A. Grove, G. Ladas, N. R. Prokup and R. Levis, *On the global behavior of solutions of a biological model*, Commun. Appl. Nonlinear Anal. **7**(2) (2000), 33–46.

[12] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, London, 2001.

[13] M. R. S. Kulenović and O. Merino, *Invariant manifolds for competitive discrete systems in the plane*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **20**(8) (2010), 2471–2486. `https://doi.org/10.1142/S0218127410027118`

[14] W. Wang and H. Feng, *On the dynamics of positive solutions for the difference equation in a new population model*, J. Nonlinear Sci. Appl. **9**(4) (2016), 1748–1754. `http://dx.doi.org/10.22436/jnsa.009.04.30`

[1]Faculty of Mechanical Engineering,
University of Sarajevo,
Vilsonovo šetalište 9, Sarajevo, Bosnia and Herzegovina
Email address: `hadziabdic@mef.unsa.ba`
Email address: `bektesevic@mef.unsa.ba`
Email address: `mehuljic@mef.unsa.ba`