

A STUDY OF *-PRIME RINGS WITH DERIVATIONS

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ABSTRACT. This paper's major goal is to describe the structure of the *-prime ring, with the help of three different derivations α , β and γ such that $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] + [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Further, some more related results have also been discussed. As applications, classical theorems due to Bell-Daif [6] and Herstein [12] are deduced.

1. INTRODUCTION

This research is the extension of the work done by Ali et al. in [3]. If (i) $(s_1 s_2)^* = s_2^* s_1^*$ and (ii) $(s_1^*)^* = s_1$ holds for all $s_1, s_2 \in \chi$, then an additive map $s_1 \mapsto s_1^*$ of χ into itself is said to be an involution. Ring with involution, often known as *-ring or ring with involution. $\mathcal{H}(\chi)$ is the collection of hermitian objects ($s_1^* = s_1$) and $\mathcal{S}(\chi)$ is the collection of skew-hermitian objects ($s_1^* = -s_1$) of χ . If characteristic different from two, then, obviously, $\mathcal{H}(\chi) = \mathcal{S}(\chi)$. Thus, we will consider only *-rings χ with $\text{char}(\chi) \neq 2$. If $\mathcal{Z}(\chi) \subseteq \mathcal{H}(\chi)$, the involution is said to be of the first kind; otherwise, it is of the second kind. In the later case, $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$ (e.g., involution in the case of ring of quaternions). In [11], there's a mention of these rings as well as additional references.

The origins of commuting and centralising maps can be traced back to 1955, when Divinsky [9] proved that "simple Artinian ring is commutative if it has commuting non-trivial automorphisms". In 1957, Posner [18] found that "existence of nonzero centralizing derivation on a prime ring forces the ring to be commutative". The study of commuting (centralizing) derivation/additive maps/multiplicative maps and several

Key words and phrases. Prime ring, involution, derivation, central identities.

2020 *Mathematics Subject Classification.* Primary: 16N60. Secondary: 16W10, 16W25.

DOI 10.46793/KgJMat2505.677A

Received: June 02, 2022.

Accepted: November 10, 2022.

extension of such results begins with the results of Posner [18] along with applications to different areas like Lie theory, matrix theory, operator theory etc. For more details of said work see (see [2, 4, 8–10, 13] and references therein).

In [3], Ali et al. proved that “a prime ring χ must be a commutative integral domain if it admits derivations α and β satisfying any one of the identities: (i) $[\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$, (ii) $\alpha(s_1) \circ \alpha(s_1^*) + \beta([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$, (iii) $\alpha([s_1, s_1^*]) + [\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$, (iv) $\alpha(s_1 \circ s_1^*) + \alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$ ”. Our goal in this work is to continue this line of inquiry and analyse the structure of prime rings with involution satisfying above mentioned $*$ -differential identities which are central. In fact, so many results become corollaries of our results which are in [2, 3, 6, 8, 12, 16, 17] and references therein.

2. THE RESULTS

Herstein [12] proved a classical result “A prime ring χ of $\text{char}(\chi) \neq 2$ with a derivation $\alpha \neq 0$ satisfying the differential identity $[\alpha(s_1), \alpha(s_2)] = 0$ for all $s_1, s_2 \in \chi$, must be commutative”. Further, Daif [7], proved that “Let χ be a 2-torsion free semiprime ring admitting a derivation α such that $[\alpha(s_1), \alpha(s_2)] = 0$ for all $s_1, s_2 \in I$, where I is a nonzero ideal of χ and α is nonzero on I , then χ contains a nonzero central ideal”. Further, this result was extended by second author together with Dar in [8, Theorem 3.1] in case of prime rings involving $*$: $\chi \mapsto \chi$. Indeed, they proved “Let χ be a prime ring with involution $'*$ ' of the second kind such that $\text{char}(\chi) \neq 2$ and satisfying the $*$ -differential identity $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$, then χ must be commutative”. Throughout our discussion $*$ will be of second kind and also as when we consider more than one derivation then it is assume that at least one of them to be nonzero. We begin our investigation with several well-known facts, which lead to the following results repeatedly.

Fact 2.1 ([3, Lemma 2.5]). Let χ be a $*$ -prime ring and α be a derivation and $\alpha(t) = 0$ for all $t \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. Then $\alpha(s_1) = 0$ for all $s_1 \in \mathcal{Z}(\chi)$.

Fact 2.2 ([17, Lemma 2.1]). Let χ be a $*$ -prime ring and χ is normal for all $s_1 \in \chi$. Then χ is commutative.

Fact 2.3 ([17, Lemma 2.2]). Let χ be a $*$ -prime ring and $s_1 \circ s_1^* \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$ if and only if χ is commutative.

Theorem 2.4. *Let χ be a $*$ -prime ring and α, β and γ be derivations of χ satisfying the identity $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. The proof is divided into the following cases.

Case (i) If $\alpha = 0$ and $\beta, \gamma \neq 0$, then we have

$$(2.1) \quad [\beta(s_1), \beta(s_1^*)] \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Taking t for s_1 in (2.1), where $t \in \mathcal{H}(\chi)$, we obtain

$$(2.2) \quad \pm [\gamma(t), t] \in \mathcal{Z}(\chi), \quad \text{for all } t \in \mathcal{H}(\chi).$$

Linearization of (2.2) gives

$$(2.3) \quad \pm [\gamma(t), h_1] \pm [\gamma(h_1), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Replacing h_1 by $h_1 h_0$ in (2.3) and combining (2.3), we get $\pm [h_1, t] \gamma(h_0) \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$ and $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. Applying the primeness of the ring χ , we obtain either $\pm [h_1, t] \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$ or $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$. If we consider $\pm [h_1, t] \in \mathcal{Z}(\chi)$ for all $h_1, t \in \mathcal{H}(\chi)$, replacing h_1 by kh_1 , we have $\pm [k, t] h_1 \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$, $k \in \mathcal{S}(\chi)$ and $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$. Since $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$ and χ is prime, implies that $\pm [k, t] \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$ and $k \in \mathcal{S}(\chi)$. This implies that χ is commutative. Now consider $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{H}(\chi) \cap \mathcal{Z}(\chi)$, this implies that $\gamma(h_1) = 0$ for all $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$. Taking kh_1 in place of t in (2.2), we obtain

$$\pm [\gamma(k), k] h_1^2 \in \mathcal{Z}(\chi), \quad \text{for all } k \in \mathcal{S}(\chi) \text{ and } h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi).$$

Since χ is prime and we have $\mathcal{S}(\chi) \cap \mathcal{Z}(\chi) \neq (0)$, we obtain

$$(2.4) \quad \pm [\gamma(k), k] \in \mathcal{Z}(\chi), \quad \text{for all } k \in \mathcal{S}(\chi) \text{ and } h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi).$$

By linearizing (2.2), we get

$$(2.5) \quad \pm [\gamma(t), h_1] \pm [\gamma(h_1), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Substituting kh_1 for h_1 in (2.5), where $k \in \mathcal{S}(\chi)$ and $h_1 \in \mathcal{S}(\chi) \cap \mathcal{Z}(\chi)$, we obtain

$$(2.6) \quad \pm [\gamma(t), k] \pm [\gamma(k), t] \in \mathcal{Z}(\chi), \quad \text{for all } t, h_1 \in \mathcal{H}(\chi).$$

Consider $4[\gamma(s_1), s_2] = [\gamma(2s_1), 2s_2] = [\gamma(t+k), t+k] = [\gamma(t), t] + [\gamma(k), t] + [\gamma(t), k] + [\gamma(k), k]$. Using (2.2), (2.4) and (2.6), we get $4[\gamma(s_1), s_2] \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Since $\text{char}(\chi) \neq 2$, this implies that $[\gamma(s_1), s_2] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Therefore, in view of Posner's result we done.

Case (ii) If $\beta = 0$ and $\alpha, \gamma \neq 0$, then we have $\alpha([s_1, s_1^*]) \pm [\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Substituting t for s_1 , we obtain $\pm [\gamma(t), t] \in \mathcal{Z}(\chi)$ for all $t \in \mathcal{H}(\chi)$, which is same as (2.2), following the line of proof as we did after (2.2), we get χ is commutative.

Case (iii) If $\gamma = 0$ and $\alpha, \beta \neq 0$, then from hypothesis we obtain

$$\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Substituting ss_1 for s_1 in above equation, where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, we get

$$(2.7) \quad [s_1, s_1^*] 2s\alpha(s) + [\alpha(s_1), s_1^*] s\beta(s) + [s_1, \beta(s_1^*)] s\beta(s) + [s_1, s_1^*] (\beta(s))^2 \in \mathcal{Z}(\chi).$$

Linearization of (2.7), gives us

$$(2.8) \quad [s_1, s_2^*] 2s\alpha(s) + [s_2, s_1^*] 2s\alpha(s) + [\alpha(s_1), s_2^*] s\alpha(s) + [\alpha(s_2), s_1^*] s\alpha(s) \\ + [s_1, \beta(s_2^*)] s\beta(s) + [s_2, \beta(s_1^*)] s\beta(s) + [s_1, s_2^*] (\beta(s))^2 + [s_2, s_1^*] (\beta(s))^2 \in \mathcal{Z}(\chi).$$

Now taking s_2s for s_2 in (2.8), where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, and combining it with the obtained result, we find that

$$(2.9) \quad 4[s_2, s_1^*]s^2\alpha(s) + 2[\beta(s_2), s_1^*]s\beta(s)^2 + 2[s_2, s_1^*]s(\beta(s))^2 - [s_1, s_2^*]s(\beta(s))^2 + 2[s_2, \beta(s_1^*)]s^2\beta(s) + [s_2, s_1^*]s(\beta(s))^2 \in \mathcal{Z}(\chi).$$

Substituting s_2s for s_2 in (2.9) and solving with the help of (2.9), we have

$$(2.10) \quad 2[s_1, s_2^*]s^2(\beta(s))^2 + 2[s_2, s_1^*]s(\beta(s))^3 \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Again taking s_2s for s_2 in (2.10), where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, and combining it with (2.10), we get $4[s_2, s_1^*]s^2(\beta(s))^3 \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Replacing s_2 by s_1 , we obtain $4[s_1, s_1^*]s^2(\beta(s))^2 \in \mathcal{Z}(\chi)$, for all $s_1 \in \chi$ and $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation forces that either $[s_1, s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$ or $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. If $[s_1, s_1^*] \in \mathcal{Z}(\chi)$, then by Fact 2.2, χ is commutative. On the other hand, we consider the situation $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Using this in (2.7), we get $2[s_1, s_1^*]s\alpha(s) \in \mathcal{Z}(\chi)$. By the primeness of the ring χ , we conclude that either χ is commutative or $\alpha(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$. Linearization of $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$, gives us

$$(2.11) \quad \alpha([s_1, s_2^*]) + \alpha([s_2, s_1^*]) + [\beta(s_1), \beta(s_2^*)] + [\beta(s_2), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Replacing s_2 by ss_2 in (2.11) where $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$ and using the fact that $\alpha(s) = 0$ and $\beta(s) = 0$ for all $s \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$, we arrive at

$$2(\alpha([s_2, s_1^*]) + [\beta(s_2), \beta(s_1^*)])s \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation yields

$$\alpha([s_2, s_1^*]) + [\beta(s_2), \beta(s_1^*)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

This implies that

$$\alpha([s_2, s_1]) + [\beta(s_2), \beta(s_1)] \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \chi.$$

Replacing s_2 by s_1^2 in the last relation, we get $[\beta(s_1^2), \beta(s_1)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. This further implies that $[(\beta(s_1))^2, s_1] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Thus in view of [14, Theorem 1.1], we get χ is commutative. This proves the theorem.

Case (iv) If $\alpha = 0, \beta = 0$ and $\gamma \neq 0$, we have $\pm[\gamma(s_1), s_1^*] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$, then by [17, Theorem 3.7] χ is commutative.

Case (v) Consider $\beta = 0, \gamma = 0$ and $\alpha \neq 0$, then from hypothesis, we have $\alpha([s_1, s_1^*]) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. By [16, Theorem 2.3], we obtain χ is commutative.

Case (vi) Taking $\gamma = 0, \alpha = 0$ and $\beta \neq 0$, then by hypothesis we have $[\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Hence, result follows by [17, Theorem 3.1].

Case (vii) Consider the following if $\alpha = 0, \beta = 0$ and $\gamma \neq 0$. Substituting t for s_1 in assumption, we obtain $[\gamma(t), t] \in \mathcal{Z}(\chi)$ for all $s_1 \in \mathcal{H}(\chi)$, which is same as (2.2). Therefore χ is commutative by follow the same argument. \square

We deduce the following corollaries from Theorem 2.4.

Corollary 2.1 ([8, Theorem 3.1]). *Let χ be a *-prime ring and $\alpha \neq 0$ be a derivation of χ such that $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.2 ([2, Theorem 2.2]). *Let χ be a *-prime ring and $\alpha \neq 0$ be a derivation of χ such that $\alpha([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.3 ([3, Theorem 3.5]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.4. *Let χ be a *-prime ring and α and β be a nonzero derivation of χ satisfying $\alpha(s_1s_1^*) + \beta(s_1)\beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. By the assumption, we have $\alpha(s_1s_1^*) + \beta(s_1)\beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Replace s_1 by s_1^* in the last expression to get $\alpha(s_1^*s_1) + \beta(s_1^*)\beta(s_1) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Combining the last two relations, we obtain $\alpha([s_1, s_1^*]) + [\beta(s_1), \beta(s_1^*)] \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Hence, application of Case (vi) of Theorem 2.4 yields the required result. □

Theorem 2.5. *Let χ be a *-prime ring and α and β be two derivations of χ satisfying the identity $\alpha(s_1 \circ s_1^*) + \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. By the assumption, we have

$$(2.12) \quad \alpha(s_1 \circ s_1^*) + \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Case (i) Assume that $\alpha \neq 0$ and $\beta = 0$. Then it follows from (2.12) that $\alpha(s_1 \circ s_1^*) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. In view of [16, Theorem 2.5], we get χ is commutative.

Case (ii) Taking $\alpha = 0$ and $\beta \neq 0$. Then (2.12) reduces to

$$(2.13) \quad \beta(s_1) \circ \beta(s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Application of [17, Theorem 3.5] gives the required result.

Case (iii) Assume that both α and β are nonzero. Replacing s_1 by $s_1 + s_2$ in (2.12), we get

$$(2.14) \quad \alpha(s_1 \circ s_2^*) + \alpha(s_2 \circ s_1^*) + \beta(s_1) \circ \beta(s_2^*) + \beta(s_2) \circ \beta(s_1^*) \in \mathcal{Z}(\chi).$$

Substituting s_2t for s_2 in (2.14), where $t \in \mathcal{Z}(\chi) \cap \mathcal{H}(\chi)$, we get

$$((s_1 \circ s_2^*) + (s_2 \circ s_1^*))\alpha(t) + (\beta(s_1) \circ s_2^* + s_2 \circ \beta(s_1^*))\beta(t) \in \mathcal{Z}(\chi).$$

Taking s_2s_o for s_2 where $s_o \in \mathcal{Z}(\chi) \cap \mathcal{S}(\chi)$ and combining it with the obtained relation, we get

$$2((s_2 \circ s_1^*)s_o\alpha(t)) + (s_2 \circ \beta(s_1^*))s_o\beta(t) \in \mathcal{Z}(\chi).$$

Since $\text{char}(\chi) \neq 2$ and $\mathcal{Z}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, the above relation yields

$$(2.15) \quad (s_2 \circ s_1^*)\alpha(t) + (s_2 \circ \beta(s_1^*))\beta(t) \in \mathcal{Z}(\chi), \quad \text{for all } s_1, s_2 \in \mathcal{Z}(\chi).$$

This can be further written as

$$[(s_2 \circ s_1^*), r]\alpha(t) + [s_2 \circ \beta(s_1^*), r]\beta(t) = 0, \quad \text{for all } s_1, s_2, r \in \mathcal{L}(\chi).$$

Replacing χ by $s_2 \circ s_1^*$ we get $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*]\beta(t) = 0$ for all $s_1, s_2 \in \chi$. Then, by the primeness of χ , we get either $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*] = 0$ for all $s_1, s_2 \in \chi$ or $\beta(t) = 0$ for all $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$. If $[s_2 \circ \beta(s_1^*), s_2 \circ s_1^*] = 0$ for all $s_1, s_2 \in \chi$, then by substituting z for s_2 in the last relation where $z \in \mathcal{L}(\chi)$, we obtain $2[\beta(s_1^*), s_1^*]z = 0$ for all $s_1 \in \chi$. Since $\text{char}(\chi) \neq 2$ and $\mathcal{L}(\chi) \cap \mathcal{S}(\chi) \neq (0)$, this implies that $[\beta(s_1^*), s_1^*] = 0$ for all $s_1 \in \chi$. By the application of Posner's [18] we arrived at conclusion. Now consider the case $\beta(t) = 0$ for all $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$. Then (2.15) reduces to $(s_2 \circ s_1^*)\alpha(t) \in \mathcal{L}(\chi)$ for all $s_1, s_2 \in \chi$ and $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$. By the primeness of the ring χ , we get either $s_2 \circ s_1^* \in \mathcal{L}(\chi)$ for all $s_1, s_2 \in \chi$ or $\alpha(t) = 0$ for all $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$. If $s_2 \circ s_1^* \in \mathcal{L}(\chi)$ for all $s_1, s_2 \in \chi$ by the Fact 2.3 implies that χ is commutative. Finally, we consider the case $\alpha(t) = 0$ for all $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$. Now replacing s_2 by t in (2.14) where $t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi)$, we get

$$(\alpha(s_1) + \alpha(s_1^*))t \in \mathcal{L}(\chi), \quad \text{for all } s_1 \in \chi \text{ and } t \in \mathcal{L}(\chi) \cap \mathcal{H}(\chi).$$

Thus in view of the fact $\mathcal{L}(\chi) \cap \mathcal{S}(\chi) \neq (0)$ and primeness of the ring χ , we conclude that $\alpha(s_1) + \alpha(s_1^*) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. This can be written as $[\alpha(s_1), \alpha(s_1^*)] = 0$ for all $s_1 \in \chi$. Hence, χ is commutative by [17, Theorem 3.1]. \square

Corollary 2.5 ([2, Theorem 2.3]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.6. *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 s_1^*) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. From assumption, we have $\alpha(s_1 s_1^*) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. For any $s_1 \in \chi$, s_1^* also is an element of χ . Substituting s_1^* for s_1 in the given assertion, we obtain $\alpha(s_1^* s_1) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. This implies that $\alpha(s_1 \circ s_1^*) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. Hence, χ is commutative by Corollary 2.5. \square

Corollary 2.7 ([8, Theorem 3.2]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Corollary 2.8 ([3, Theorem 3.6]). *Let χ be a $*$ -prime ring and $\alpha \neq 0$ be a derivation of χ satisfying $\alpha(s_1 \circ s_1^*) + \alpha(s_1) \circ \alpha(s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Theorem 2.6. *Let χ be a $*$ -prime ring and α and β be derivations of χ satisfying the identity $[\alpha(s_1), \alpha(s_1^*)] \pm \beta(s_1 \circ s_1^*) \in \mathcal{L}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. We are given that $\alpha, \beta : \chi \rightarrow \chi$ are derivations such that

$$(2.16) \quad [\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) \in \mathcal{L}(\chi), \quad \text{for all } s_1 \in \chi.$$

Replacing s_1 by s_1^* in the last expression we get

$$(2.17) \quad -[\alpha(s_1), \alpha(s_1^*)] + \beta(s_1 \circ s_1^*) \in \mathcal{L}(\chi), \quad \text{for all } s_1 \in \chi.$$

Adding the last two relations and using $\text{char}(\chi) \neq 2$ we obtain

$$(2.18) \quad \beta(s_1 \circ s_1^*) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Hence, the result follows from [13, Theorem 2].

Similarly, we prove the other case with the help of [13, Theorem 2]. □

Corollary 2.9 ([3, Theorem 3.1]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $[\alpha(s_1), \alpha(s_1^*)] \pm \beta(s_1 \circ s_1^*) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

Theorem 2.7. *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_1^*) \pm \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi)$ for all $s_1 \in \chi$. Then χ is commutative.*

Proof. First, we consider that

$$\alpha(s_1) \circ \alpha(s_1^*) + \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Replacing s_1 by s_1^* in the last expression we get

$$\alpha(s_1) \circ \alpha(s_1^*) - \beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Subtracting the last two relation and using $\text{char}(\chi) \neq 2$ we obtain

$$\beta([s_1, s_1^*]) \in \mathcal{Z}(\chi), \quad \text{for all } s_1 \in \chi.$$

Hence, the result follow from [13, Theorem 1].

Similarly, we prove the other case with the help of [13, Theorem 1]. □

Corollary 2.10. *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_2) \pm \beta([s_1, s_2]) \in \mathcal{Z}(\chi)$ for all $s_1, s_2 \in \chi$. Then χ is commutative.*

Corollary 2.11 ([3, Theorem 3.3]). *Let χ be a *-prime ring and α and β be derivations of χ satisfying the identity $\alpha(s_1) \circ \alpha(s_1^*) \pm \beta([s_1, s_1^*]) = 0$ for all $s_1 \in \chi$. Then χ is commutative.*

3. SOME EXAMPLES

The first example shows that the restriction of the second kind involution in our theorems is not superfluous.

Example 3.1. Let $\chi = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \mid \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\}$. Of course, χ with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings $*$, α , $\beta : \chi \rightarrow \chi$ such that

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}, \quad \alpha \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}$$

and $\beta \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & \beta_2 \\ -\beta_3 & 0 \end{pmatrix}$. Obviously, $\mathcal{Z}(\chi) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \mid \beta_1 \in \mathbb{Z} \right\}$. Then $s_1^* = s_1$ for all $s_1 \in \mathcal{Z}(\chi)$, and hence $\mathcal{Z}(\chi) \subseteq \mathcal{H}(\chi)$, which shows that the involution

'*' is of the first kind. Moreover, α and β are nonzero derivations of χ and satisfying the identities of the theorems. However, χ is not commutative. Hence, the hypothesis of the second kind involution is crucial in our theorems.

The next example shows that our theorems are not true for semiprime rings.

Example 3.2. Let $S = \chi \times \mathbb{C}$, where χ is same as in Example 3.1 with involution '*' and derivations α and β same as in Example 3.1, \mathbb{C} is the ring of complex numbers with conjugate involution τ . We can easily observe that S is a non commutative semiprime ring with characteristic different from two. Now define an involution α on S , as $(s_1, s_2)^\alpha = (s_1^*, s_2^\tau)$. Clearly, α is an involution of the second kind. Further, we define the mappings α and β from S to S such that $D_1(s_1, s_2) = (\alpha(s_1), 0)$ and $D_2(s_1, s_2) = (\beta(s_1), 0)$ for all $(s_1, s_2) \in S$. It can be easily checked that D_1 and D_2 are derivations on S and satisfying the identities of the Theorem 2.5 and Theorem 2.6 but S is not commutative. Hence, in our theorems, the hypothesis of primeness is essential.

Conclusions. In this paper we have studied some identities involving derivations on prime rings with involution. Purely algebraic methods have been used to describe the structure of rings and we provide the examples, which shows that the assumptions are not superfluous. Applications point of view some well known results are deduced.

Acknowledgements. The authors are indebted to the referee for his/her useful suggestions.

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