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ŁUKASIEWICZ ANTI FUZZY SUBALGEBRAS OF BCK/BCI-ALGEBRAS

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Abstract. The subalgebra of BCK/BCI-algebra using Łukasiewicz anti fuzzy set introduced by Jun is studied in this article. The concept of Łukasiewicz anti fuzzy subalgebra of a BCK/BCI-algebra is introduced, and several properties are investigated. The relationship between anti fuzzy subalgebra and Łukasiewicz anti fuzzy subalgebra is given, and characterization of a Łukasiewicz anti fuzzy subalgebra is discussed. Conditions are found in which a Lukasiewicz anti fuzzy set is a Lukasiewicz anti fuzzy subalgebra Finally, conditions under which $\lessdot$-subset, Υ-subset, and anti-subset become subalgebra are explored.

1. Introduction

In [1], Biswas introduced the concept of anti fuzzy subgroups of groups. Modifying Biswas' idea, Hong and Jun [3] applied the idea to BCK-algebras. They introduced the notions of anti fuzzy subalgebras and anti fuzzy ideals of BCK-algebras and investigated several properties. Using anti fuzzy notion and the idea of Łukasiewicz t-conorm, Jun [7] constructed the concept of Łukasiewicz anti fuzzy sets and applied it to BE-algebras. He introduced the notion of Łukasiewicz anti fuzzy BE-ideal and investigated its properties. He discussed the relationship between anti fuzzy BE-ideal and Łukasiewicz anti fuzzy BE-ideal and provided conditions for Łukasiewicz anti fuzzy set to be Łukasiewicz anti fuzzy BE-ideal. He also gives three types of subsets so called $\lessdot$-subset, Υ-subset, and anti subset, and then he considered the conditions under which they can be BE-ideals.

Key words and phrases. Anti fuzzy subalgebra, Łukasiewicz anti fuzzy set, Łukasiewicz anti fuzzy subalgebra, $\lessdot$-subset, Υ-subset, anti subset.
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We would like to study the subalgebra of BCK/BCI-algebra using Łukasiewicz anti fuzzy set introduced by Jun. We introduce Łukasiewicz anti fuzzy subalgebra of a BCK/BCI-algebra and investigate several properties. We give the relationship between anti fuzzy subalgebra and Łukasiewicz anti fuzzy subalgebra. We discuss a characterization of a Łukasiewicz anti fuzzy subalgebra. We find conditions for a Lukasiewicz anti fuzzy set to be a Lukasiewicz anti fuzzy subalgebra. We finally find the condition that $\lessdot$-subset, Υ-subset, and anti subset become subalgebra.

2. Preliminaries

This section lists the known default content that will be used later.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5] and [6]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [4, 8] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element "0" and a binary operation " $*$ " satisfying the conditions:

(I_1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$;
(I_2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$;
(I_3) $(\forall a \in X)\ (a * a = 0)$;
(I_4) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b\ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions (see [4, 8]):

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2.2}$$

$$(\forall a, b, c \in X)\,(a \leq b\ \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a)\,, \tag{2.3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{2.4}$$

Every BCI-algebra X satisfies (see [4]):

$$(\forall a, b \in X)\,(a * (a * (a * b)) = a * b)\,, \tag{2.5}$$

$$(\forall a, b \in X)\,(0 * (a * b) = (0 * a) * (0 * b))\,. \tag{2.6}$$

A subset K of a BCK/BCI-algebra X is called a *subalgebra* of X (see [4, 8]) if it satisfies:

$$(\forall a, b \in K)(a * b \in K). \tag{2.7}$$

A fuzzy set g in a set X of the form

$$g(b) := \begin{cases} s \in [0,1), & \text{if } b = a, \\ 1, & \text{if } b \neq a, \end{cases} \tag{2.8}$$

is called an *anti fuzzy point* with support a and value s, and is denoted by $\frac{a}{s}$. A fuzzy set g in a set X is said to be *non-unit* if there exists $a \in X$ such that $g(a) \neq 1$.

For a fuzzy set g in a set X, we say that an anti fuzzy point $\frac{a}{s}$ is said to

(i) *beside* in g, denoted by $\frac{a}{s} \lessdot g$ (see [2]) if $g(a) \leq s$;

(ii) *be non-quasi coincident* with g, denoted by $\frac{a}{s} \Upsilon g$ (see [2]) if $g(a) + s < 1$.

If $\frac{a}{s} \lessdot g$ or $\frac{a}{s} \Upsilon g$ (resp., $\frac{a}{s} \lessdot g$ and $\frac{a}{s} \Upsilon g$), we say that $\frac{a}{s} \lessdot\vee\Upsilon\, g$ (resp., $\frac{a}{s} \lessdot\wedge\Upsilon\, g$). Given $\beta \in \{\lessdot, \Upsilon\}$, to indicate $\frac{a}{s}\,\overline{\beta}\, g$ means that $\frac{a}{s}\,\beta\, g$ is not established.

A fuzzy set f in a BCK/BCI-algebra X is called

- an *anti fuzzy subalgebra* of X (see [3]) if it satisfies:

$$(\forall a, b \in X)(f(a * b) \leq \max\{f(a), f(b)\}); \tag{2.9}$$

- an *anti fuzzy ideal* of X (see [3]) if it satisfies:

$$(\forall a \in X)(f(0) \leq f(a)), \tag{2.10}$$

$$(\forall a, b \in X)(f(a) \leq \max\{f(a * b), f(b)\}). \tag{2.11}$$

Let ε be an element of the unit interval $[0,1]$ and let g be a fuzzy set in a set X. A function $Ł_g^\varepsilon : X \to [0,1]$, $x \mapsto \min\{1, g(x) + \varepsilon\}$, is called a *Łukasiewicz anti fuzzy set* of g in X (see [7]).

Let $Ł_g^\varepsilon$ be a Łukasiewicz anti fuzzy set of a fuzzy set g in X. If $\varepsilon = 0$, then $Ł_g^\varepsilon(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x)\} = g(x)$ for all $x \in X$. This shows that if $\varepsilon = 0$, then the Łukasiewicz anti fuzzy set of a fuzzy set g in X is the classical fuzzy set g itself in X. If $\varepsilon = 1$, then $Ł_g^\varepsilon(x) = \min\{1, g(x) + \varepsilon\} = \min\{1, g(x) + 1\} = 1$ for all $x \in X$, that is, if $\varepsilon = 1$, then the Łukasiewicz anti fuzzy set is the constant function with value 1. Therefore, in handling the Łukasiewicz anti fuzzy set, the value of ε can always be considered to be in $(0,1)$.

Let g be a fuzzy set in a set X and $\varepsilon \in (0,1)$. If $g(x) + \varepsilon \geq 1$ for all $x \in X$, then the Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ of g in X is the constant function with value 1, that is, $Ł_g^\varepsilon(x) = 1$ for all $x \in X$. Therefore, for the Łukasiewicz anti fuzzy set to have a meaningful shape, a fuzzy set g in X and $\varepsilon \in (0,1)$ shall be set to satisfy the condition "$g(x) + \varepsilon < 1$ for some $x \in X$".

Given a Łukasiewicz anti fuzzy set $Ł_g^\varepsilon$ of a fuzzy set g in X and $s \in [0,1)$, consider the sets:

$$(Ł_g^\varepsilon, s)_{\lessdot} := \{y \in X \mid \tfrac{y}{s} \lessdot Ł_g^\varepsilon\} \quad \text{and} \quad (Ł_g^\varepsilon, s)_{\Upsilon} := \{y \in X \mid \tfrac{y}{s}\, \Upsilon\, Ł_g^\varepsilon\},$$

which are called the $\lessdot$-*subset* and Υ-*subset* of $Ł_g^\varepsilon$ in X. Also, we consider the following set

$$\mathrm{Anti}\,(Ł_g^\varepsilon) := \{y \in X \mid Ł_g^\varepsilon(y) < 1\}$$

and it is called the *anti subset* of $Ł_g^\varepsilon$ in X. It is observed that

$$\text{Anti}\,(Ł_g^\varepsilon) = \{y \in X \mid g(y) + \varepsilon < 1\}.$$

3. Łukasiewicz Anti Fuzzy Subalgebras

In this section, let f and γ be a fuzzy set in X and an element of $(0,1)$, respectively, unless otherwise specified.

Definition 3.1. A Łukasiewicz anti fuzzy set $Ł_f^\gamma$ in a BCK/BCI-algebra X is called a *Łukasiewicz anti fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)(\forall s_a, s_b \in [0,1)) \left(\tfrac{x}{s_a} \lessdot Ł_f^\gamma,\ \tfrac{y}{s_b} \lessdot Ł_f^\gamma \ \Rightarrow\ \tfrac{x*y}{\max\{s_a, s_b\}} \lessdot Ł_f^\gamma \right). \tag{3.1}$$

Example 3.1. Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation "$*$" given by the Cayley table:

$*$	0	b_1	b_2	b_3	b_4
0	0	0	0	0	0
b_1	b_1	0	b_1	0	0
b_2	b_2	b_2	0	0	0
b_3	b_3	b_3	b_3	0	0
b_4	b_4	b_3	b_4	b_1	0

Then X is a BCK-algebra (see [8]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1], \quad x \mapsto \begin{cases} 0.24, & \text{if } x = 0, \\ 0.31, & \text{if } x = b_1, \\ 0.37, & \text{if } x = b_2, \\ 0.43, & \text{if } x = b_3, \\ 0.58, & \text{if } x = b_4. \end{cases}$$

Given $\gamma := 0.58$, the Łukasiewicz anti fuzzy set $Ł_f^\gamma$ of f in X is given as follows:

$$Ł_f^\gamma : X \to [0,1], \quad x \mapsto \begin{cases} 0.82, & \text{if } x = 0, \\ 0.89, & \text{if } x = b_1, \\ 0.95, & \text{if } x = b_2, \\ 1.00, & \text{if } x = b_3, \\ 1.00, & \text{if } x = b_4. \end{cases}$$

It is routine to verify that $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X.

Theorem 3.1. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X, then it's Łukasiewicz anti fuzzy set $Ł_f^\gamma$ in X is a Łukasiewicz anti fuzzy subalgebra of X.*

Proof. Assume that f is an anti fuzzy subalgebra of a BCK/BCI-algebra X. Let $x, y \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_f^\gamma$ and $\frac{y}{s_b} \lessdot Ł_f^\gamma$. Then, $Ł_f^\gamma(x) \leq s_a$ and

$Ł_f^\gamma(y) \leq s_b$. Hence,

$$\begin{aligned} Ł_f^\gamma(x * y) &= \min\{1, f(x * z) + \gamma\} \leq \min\{1, \max\{f(x), f(y)\} + \gamma\} \\ &= \min\{1, \max\{f(x) + \gamma, f(y) + \gamma\}\} \\ &= \max\{\min\{1, f(x) + \gamma\}, \min\{1, f(y) + \gamma\}\} \\ &= \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \leq \max\{s_a, s_b\}, \end{aligned}$$

which implies that $\frac{x*y}{\max\{s_a, s_b\}} \lessdot Ł_f^\gamma$. Therefore, $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.1 may not be true.

Example 3.2. Let $X = \{0, b_1, b_2, b_3, b_4\}$ be a set with a binary operation "$*$"

$*$	0	b_1	b_2	b_3	b_4
0	0	0	b_2	b_3	b_4
b_1	b_1	0	b_2	b_3	b_4
b_2	b_2	b_2	0	b_4	b_3
b_3	b_3	b_3	b_4	0	b_2
b_4	b_4	b_4	b_3	b_2	0

Then, X is a BCI-algebra (see [4]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1], \quad x \mapsto \begin{cases} 0.28, & \text{if } x = 0, \\ 0.32, & \text{if } x = b_1, \\ 0.39, & \text{if } x = b_2, \\ 0.43, & \text{if } x = b_3, \\ 0.61, & \text{if } x = b_4. \end{cases}$$

Given $\gamma := 0.58$, the γ-Łukasiewicz fuzzy set $Ł_f^\gamma$ of f in X is given as follows:

$$Ł_f^\gamma : X \to [0,1], \quad x \mapsto \begin{cases} 0.86, & \text{if } x = 0, \\ 0.90, & \text{if } x = b_1, \\ 0.97, & \text{if } x = b_2, \\ 1.00, & \text{if } x = b_3, \\ 1.00, & \text{if } x = b_4. \end{cases}$$

It is routine to verify that $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X. But f is not an anti fuzzy subalgebra of X because of

$$f(b_2 * b_3) = f(b_4) = 0.61 \nleq 0.43 = \max\{f(b_2), f(b_3)\}.$$

We explore a characterization of a Łukasiewicz anti fuzzy subalgebra.

Theorem 3.2. *Let f be a fuzzy set in a BCK/BCI-algebra X. Then its Łukasiewicz anti fuzzy set $Ł_f^\gamma$ in X is a Łukasiewicz anti fuzzy subalgebra of X if and only if it satisfies*

$$(\forall x, y \in X)(Ł_f^\gamma(x * y) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\}). \tag{3.2}$$

Proof. Suppose that $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X. Let $x, y \in X$. Since $\frac{x}{Ł_f^\gamma(x)} \lessdot Ł_f^\gamma$ and $\frac{y}{Ł_f^\gamma(y)} \lessdot Ł_f^\gamma$, it follows from (3.1) that $\frac{x*y}{\max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\}} \lessdot Ł_f^\gamma$. Hence, $Ł_f^\gamma(x*y) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\}$.

Conversely, assume that $Ł_f^\gamma$ satisfies (3.2). Let $x, y \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_f^\gamma$ and $\frac{y}{s_b} \lessdot Ł_f^\gamma$. Then $Ł_f^\gamma(x) \leq s_a$ and $Ł_f^\gamma(y) \leq s_b$, and so

$$Ł_f^\gamma(x*y) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \leq \max\{s_a, s_b\}.$$

Thus, $\frac{x*y}{\max\{s_a, s_b\}} \lessdot Ł_f^\gamma$, and therefore, $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X. □

Lemma 3.1 ([7]). *If f is a fuzzy set in a set X, then it's Łukasiewicz anti fuzzy set $Ł_f^\gamma$ satisfies*

$$(\forall x, y \in X)(f(x) \geq f(y) \;\Rightarrow\; Ł_g^\gamma(x) \geq Ł_g^\gamma(y)). \tag{3.3}$$

Lemma 3.2. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X, then it's Łukasiewicz anti fuzzy set $Ł_f^\gamma$ satisfies*

$$(\forall x \in X)(Ł_f^\gamma(0) \leq Ł_f^\gamma(x)). \tag{3.4}$$

Proof. If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X, then

$$f(0) = f(x*x) \leq \max\{f(x), f(x)\} = f(x),$$

for all $x \in X$. It follows from (3.3) that $Ł_f^\gamma(0) \leq Ł_f^\gamma(x)$ for all $x \in X$. □

Proposition 3.1. *If f is an anti fuzzy subalgebra of a BCK/BCI-algebra X, then it's Łukasiewicz fuzzy set $Ł_f^\gamma$ satisfies:*

$$(\forall x, y \in X)\left(Ł_f^\gamma(x) = Ł_f^\gamma(0) \;\Leftrightarrow\; Ł_f^\gamma(x*y) \leq Ł_f^\gamma(y)\right). \tag{3.5}$$

Proof. Let f be an anti fuzzy subalgebra of a BCK/BCI-algebra X. Then $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X (see Theorem 3.1). Assume that $Ł_f^\gamma(x) = Ł_f^\gamma(0)$ for all $x \in X$. Then,

$$Ł_f^\gamma(x*y) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} = \max\{Ł_f^\gamma(0), Ł_f^\gamma(y)\} = Ł_f^\gamma(y),$$

for all $x, y \in X$, by Theorem 3.2 and Lemma 3.2.

Conversely, suppose that $Ł_f^\gamma(x*y) \leq Ł_f^\gamma(y)$ for all $x, y \in X$. Using (2.2) induces $Ł_f^\gamma(x) = Ł_f^\gamma(x*0) \leq Ł_f^\gamma(0)$, and so $Ł_f^\gamma(x) = Ł_f^\gamma(0)$ for all $x \in X$, by Lemma 3.2. □

Proposition 3.2. *If f is an anti fuzzy subalgebra of a BCI-algebra X, then its Łukasiewicz fuzzy set $Ł_f^\gamma$ satisfies*

$$(\forall x \in X)(Ł_f^\gamma(0*x) \leq Ł_f^\gamma(x)). \tag{3.6}$$

Proof. If f is an anti fuzzy subalgebra of a BCI-algebra X, then

$$f(0*x) \leq \max\{f(0), f(x)\} = f(x),$$

for all $x \in X$. Hence, $Ł_f^\gamma(0*x) \leq Ł_f^\gamma(x)$ for all $x \in X$, by Lemma 3.1. □

Proposition 3.3. *If f is an anti fuzzy subalgebra of a BCI-algebra X, then its Łukasiewicz fuzzy set $Ł_f^\gamma$ satisfies*

$$(\forall x, y \in X)(\forall s_a, s_b \in [0,1)) \left(\tfrac{x}{s_a} \lessdot Ł_f^\gamma,\ \tfrac{y}{s_b} \lessdot Ł_f^\gamma \ \Rightarrow \tfrac{x*(0*y)}{\max\{s_a,s_b\}}] \lessdot Ł_f^\gamma \right). \tag{3.7}$$

Proof. Let $x, y \in X$ and $s_a, s_b \in [0.1)$ be such that $\frac{x}{s_a} \lessdot Ł_f^\gamma$ and $\frac{y}{s_b} \lessdot Ł_f^\gamma$. Then, $Ł_f^\gamma(x) \leq s_a$ and $Ł_f^\gamma(y) \leq s_b$, and thus,

$$\begin{aligned}
Ł_f^\gamma(x*(0*y)) &= \min\{1, f(x*(0*y)) + \gamma\} \\
&\leq \min\{1, \max\{f(x), f(0*y)\} + \gamma\} \\
&\leq \min\{1, \max\{f(x), \max\{f(0), f(y)\}\} + \gamma\} \\
&= \min\{1, \max\{f(x), f(y)\} + \gamma\} \\
&= \min\{1, \max\{f(x) + \gamma, f(y) + \gamma\}\} \\
&= \max\{\min\{1, f(x) + \gamma\}, \min\{1, f(y) + \gamma\}\} \\
&= \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \\
&\leq \max\{s_a, s_b\}.
\end{aligned}$$

Hence, $\frac{x*(0*y)}{\max\{s_a,s_b\}}] \lessdot Ł_f^\gamma$. □

We give conditions for a Lukasiewicz anti fuzzy set to be a Lukasiewicz anti fuzzy subalgebra.

Theorem 3.3. *Let f be a fuzzy set in a BCK/BCI-algebra X. If it's Łukasiewicz anti fuzzy set $Ł_f^\gamma$ satisfies*

$$(\forall x, y \in X)(\forall s_b, s_c \in [0,1)) \left(z \leq x,\ \tfrac{y}{s_b} \lessdot Ł_f^\gamma,\ \tfrac{z}{s_c} \lessdot Ł_f^\gamma \ \Rightarrow \ \tfrac{x*y}{\max\{s_b,s_c\}} \lessdot Ł_f^\gamma \right), \tag{3.8}$$

then $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X.

Proof. It is straightforward by (I_3) and (3.8). □

Proposition 3.4. *Let f be a fuzzy set in a BCI-algebra X. Then every Łukasiewicz fuzzy subalgebra $Ł_f^\gamma$ of X satisfies*

$$(\forall x, y \in X)(\forall s_a, s_b \in [0,1)) \left(\tfrac{x}{s_a} \lessdot Ł_f^\gamma,\ \tfrac{y}{s_b} \lessdot Ł_f^\gamma \ \Rightarrow \tfrac{x*(0*y)}{\max\{s_a,s_b\}} \lessdot Ł_f^\gamma \right). \tag{3.9}$$

Proof. Let $x, y \in X$ and $s_a, s_b \in [0,1)$ be such that $\frac{x}{s_a} \lessdot Ł_f^\gamma$ and $\frac{y}{s_b} \lessdot Ł_f^\gamma$. Then $Ł_f^\gamma(x) \leq s_a$ and $Ł_f^\gamma(y) \leq s_b$. Using Theorem 3.2 and Proposition 3.2, we have

$$Ł_f^\gamma(x*(0*y)) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(0*y)\} \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \leq \max\{s_a, s_b\},$$

and so, $\frac{x*(0*y)}{\max\{s_a,s_b\}} \lessdot Ł_f^\gamma$. □

Corollary 3.1. *If f is an anti fuzzy subalgebra of a BCI-algebra X, then its Łukasiewicz fuzzy set $Ł_f^\gamma$ satisfies the condition (3.9).*

Theorem 3.4. *Let $Ł_f^\gamma$ be a Łukasiewicz anti fuzzy set of a fuzzy set f in a BCK/BCI-algebra X. Then the $\lessdot$-set $(Ł_f^\gamma, s)_\lessdot$ of $Ł_f^\gamma$ is a subalgebra of X for all $s \in [0, 0.5)$ if and only if the following assertion is valid*

$$(\forall x, y \in X)\left(\min\{Ł_f^\gamma(x * y), 0.5\} \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\}\right). \tag{3.10}$$

Proof. Assume that the $\lessdot$-set $(Ł_f^\gamma, s)_\lessdot$ of $Ł_f^\gamma$ is a subalgebra of X for all $s \in [0, 0.5)$. If the condition (3.10) does not hold, then

$$\max\{Ł_f^\gamma(a), Ł_f^\gamma(b)\} < \min\{Ł_f^\gamma(a * b), 0.5\},$$

for some $a, b \in X$. If we take $s := \max\{Ł_f^\gamma(a), Ł_f^\gamma(b)\}$, then $s \in [0, 0.5)$, $\frac{a}{s} \lessdot Ł_f^\gamma$ and $\frac{b}{s} \lessdot Ł_f^\gamma$, i.e., a, $b \in (Ł_f^\gamma, s)_\lessdot$. Since $(Ł_f^\gamma, s)_\lessdot$ is a subalgebra of X, we have $a*b \in (Ł_f^\gamma, s)_\lessdot$. But $\frac{a*b}{s} \overline{\lessdot}\, Ł_f^\gamma$ implies $a * b \notin (Ł_f^\gamma, s)_\lessdot$, a contradiction. Hence,

$$\max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \geq \min\{Ł_f^\gamma(x * y), 0.5\},$$

for all $x, y \in X$.

Conversely, suppose that $Ł_f^\gamma$ satisfies (3.10). Let $s \in [0, 0.5)$ and $x, y \in X$ be such that $x \in (Ł_f^\gamma, s)_\lessdot$ and $y \in (Ł_f^\gamma, s)_\lessdot$. Then $Ł_f^\gamma(x) \leq s$ and $Ł_f^\gamma(y) \leq s$, which imply from (3.10) that

$$0.5 > s \geq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \geq \min\{Ł_f^\gamma(x * y), 0.5\}.$$

Hence, $\frac{x*y}{s} \lessdot Ł_f^\gamma$, i.e., $x * y \in (Ł_f^\gamma, s)_\lessdot$. Therefore, $(Ł_f^\gamma, s)_\lessdot$ is a subalgebra of X for $s \in [0, 0.5)$. □

Theorem 3.5. *Let $Ł_f^\gamma$ be a Łukasiewicz fuzzy set of a fuzzy set f in a BCK/BCI-algebra X. If f is an anti fuzzy subalgebra of X, then the Υ-set $(Ł_f^\gamma, s)_\Upsilon$ of $Ł_f^\gamma$ is a subalgebra of X for all $s \in [0, 1)$.*

Proof. Let $s \in [0, 1)$ and $x, y \in (Ł_f^\gamma, s)_\Upsilon$. Then $\frac{x}{s} \,\Upsilon\, Ł_f^\gamma$ and $\frac{y}{s} \,\Upsilon\, Ł_f^\gamma$, that is, $Ł_f^\gamma(x)+s < 1$ and $Ł_f^\gamma(y) + s < 1$. Hence,

$$Ł_f^\gamma(x * y) + s \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} + s = \max\{Ł_f^\gamma(x) + s, Ł_f^\gamma(y) + s\} < 1,$$

by Theorems 3.1 and 3.2. Thus, $\frac{x*y}{s} \,\Upsilon\, Ł_f^\gamma$, and so, $x * y \in (Ł_f^\gamma, s)_\Upsilon$. Therefore, $(Ł_f^\gamma, s)_\Upsilon$ is a subalgebra of X. □

Theorem 3.6. *Let f be a fuzzy set in a BCK/BCI-algebra X. For a Łukasiewicz anti fuzzy set $Ł_f^\gamma$ of f in X, if the Υ-set $(Ł_f^\gamma, s)_\Upsilon$ is a subalgebra of X, then $Ł_f^\gamma$ satisfies*

$$(\forall x, y \in X)(\forall s_a, s_b \in (0.5, 1])\left(\tfrac{x}{s_a} \,\Upsilon\, Ł_f^\gamma,\ \tfrac{y}{s_b} \,\Upsilon\, Ł_f^\gamma \;\Rightarrow \tfrac{x*y}{\min\{s_a,s_b\}} \lessdot Ł_f^\gamma\right). \tag{3.11}$$

Proof. Let $x, y \in X$ and $s_a, s_b \in (0.5, 1]$ be such that $\frac{x}{s_a} \,\Upsilon\, Ł_f^\gamma$ and $\frac{y}{s_b} \,\Upsilon\, Ł_f^\gamma$. Then $x \in (Ł_f^\gamma, s_a)_\Upsilon \subseteq (Ł_f^\gamma, \min\{s_a, s_b\})_\Upsilon$ and $y \in (Ł_f^\gamma, s_b)_\Upsilon \subseteq (Ł_f^\gamma, \min\{s_a, s_b\})_\Upsilon$. Hence, $x * y \in (Ł_f^\gamma, \min\{s_a, s_b\})_\Upsilon$, and so,

$$Ł_f^\gamma(x * y) < 1 - \min\{s_a, s_b\} \leq \min\{s_a, s_b\},$$

since $\min\{s_a, s_b\} > 0.5$. Therefore, $\frac{x*y}{\min\{s_a,s_b\}} \lessdot Ł_f^\gamma$. □

Theorem 3.7. *Let* $Ł_f^\gamma$ *be a Łukasiewicz fuzzy set of a fuzzy set* f *in a BCK/BCI-algebra* X. *If* f *is an anti fuzzy subalgebra of* X, *then the anti subset* $\mathrm{Anti}\,(Ł_f^\gamma)$ *of* $Ł_f^\gamma$ *is a subalgebra of* X.

Proof. Let $x, y \in \mathrm{Anti}\,(Ł_f^\gamma)$. Then $f(x)+\gamma < 1$ and $f(y)+\gamma < 1$. If f is an anti fuzzy subalgebra of X, then $Ł_f^\gamma$ is a Łukasiewicz anti fuzzy subalgebra of X (see Theorem 3.1). It follows from Theorem 3.2 that

$$Ł_f^\gamma(x * y) \leq \max\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} = \max\{f(x)+\gamma, f(y)+\gamma\} < 1.$$

Hence, $x * y \in \mathrm{Anti}\,(Ł_f^\gamma)$, and therefore, $\mathrm{Anti}\,(Ł_f^\gamma)$ is a subalgebra of X. □

Theorem 3.8. *Let* f *be a fuzzy set in a BCK/BCI-algebra* X. *If a Łukasiewicz anti fuzzy set* $Ł_f^\gamma$ *of* f *in* X *satisfies*

$$(\forall x, y \in X)(\forall s_a, s_b \in [0,1)) \left(\tfrac{x}{s_a} \lessdot Ł_f^\gamma,\ \tfrac{y}{s_b} \lessdot Ł_f^\gamma \ \Rightarrow \tfrac{x*y}{\min\{s_a, s_b\}} \Upsilon\, Ł_f^\gamma \right), \tag{3.12}$$

then the anti subset $\mathrm{Anti}\,(Ł_f^\gamma)$ *of* $Ł_f^\gamma$ *is a subalgebra of* X.

Proof. Assume that $Ł_f^\gamma$ satisfies the condition (3.12) for all $x, y \in X$ and $s_a, s_b \in [0,1)$. Let $x, y \in \mathrm{Anti}\,(Ł_f^\gamma)$. Then $f(x)+\gamma < 1$ and $f(y)+\gamma < 1$. Since $\frac{x}{Ł_f^\gamma(x)} \lessdot Ł_f^\gamma$ and $\frac{y}{Ł_f^\gamma(y)} \lessdot Ł_f^\gamma$, it follows from (3.12) that

$$\tfrac{x*y}{\min\{Ł_f^\gamma(x), Ł_f^\gamma(y)\}} \Upsilon\, Ł_f^\gamma. \tag{3.13}$$

If $x * y \notin \mathrm{Anti}\,(Ł_f^\gamma)$, then $Ł_f^\gamma(x * y) = 1$, and so,

$$\begin{aligned} Ł_f^\gamma(x * y) + \min\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} &= 1 + \min\{Ł_f^\gamma(x), Ł_f^\gamma(y)\} \\ &= 1 + \min\{\min\{1, f(x)+\gamma\}, \min\{1, f(y)+\gamma\}\} \\ &= 1 + \min\{f(x)+\gamma, f(y)+\gamma\} \\ &= 1 + \min\{f(x), f(y)\} + \gamma \\ &\geq 1 + \gamma > 1, \end{aligned}$$

which shows that (3.13) is not valid. This is a contradiction, and thus, $x*y \in \mathrm{Anti}\,(Ł_f^\gamma)$. Hence, $\mathrm{Anti}\,(Ł_f^\gamma)$ is a subalgebra of X. □

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