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IDEAL RELATIVE UNIFORM CONVERGENCE OF DOUBLE SEQUENCE OF POSITIVE LINEAR FUNCTIONS

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Abstract. In this article, we look into the concept of ideal relative uniform convergence of a double sequence of functions. In addition, we define ideal relative uniform Cauchy and ideal regular relative uniform convergence of a double sequence of positive linear functions defined on a compact domain D with respect to the scale function $\sigma(x)$ defined on D. We also introduced several classes of ideal relative uniform convergent double sequences of functions and investigated their algebraic and topological properties.

1. Introduction

Kostyrko et al. [21] introduced the concept of $\mathcal{I}$-convergence of sequences of real numbers, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of natural numbers. $\mathcal{I}$-convergence is a generalisation and unification of many notions of ordinary convergence. Fast [17] and Steinhaus [29] independently introduced the concept of statistical convergence in 1951 as a generalisation of the concept of ordinary convergence. Furthermore, in 1959, Schoenberg [28] independently investigated some basic properties of statistical convergence. Later, it was studied from a sequence space perspective and linked with summability theory by Fridy [18], Gökhan et al. [19], Tripathy and Sarma [31], and many others. The concept is based on the notion of natural density of $\mathbb{N}$ subsets.

A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

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exists where χ_E is the characteristics function of E.

A subset E of $\mathbb{N}$ is said to have logarithmic density $d(E)$ if

$$d(E) = \lim_{n\to+\infty} \frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_E(k)}{k},$$

exists, where $s_n = \sum\limits_{k=1}^{n}\frac{1}{k} = \log n + \gamma + O(\frac{1}{n})$, where γ is the Euler's constant.

The above expression is equivalent to

$$d(E) = \lim_{n\to+\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{\chi_E(k)}{k}.$$

A double sequence is defined as a double infinite array of numbers (x_{nk}). Pringsheim [25] introduced the concept of double sequence. Bromwich [2] contains some earlier work on double sequence spaces. Hardy [20] introduced the concept of regular convergence of a double sequence. Basarır and Sonalcan [1, 2], Das et al. [4, 5], Datta and Tripathy [5, 6], and many others have studied the double sequence from various perspectives.

The notion of statistical convergence for double sequences was introduced by Móricz [22], Mursaleen and Edely [24], Tripathy [30] independently. The notion depends on the idea of density of subsets of $\mathbb{N}\times\mathbb{N}$. A subset E of $\mathbb{N}\times\mathbb{N}$ is said to have density $\rho(E)$ if

$$\rho(E) = \lim_{p,q\to+\infty} \frac{1}{pq}\sum_{n=1}^{p}\sum_{k=1}^{q}\chi_E(n,k)$$

exists.

Tripathy and Tripathy [39] introduced the notion of logarithmic density for subsets of $\mathbb{N}\times\mathbb{N}$.

A subset $E\subset\mathbb{N}\times\mathbb{N}$ is said to have logarithmic density $\rho^*(E)$ if

$$\rho^*(E) = \lim_{p,q\to+\infty} \frac{1}{s_p s_q}\sum_{n=1}^{p}\sum_{k=1}^{q}\frac{\chi_E(n,k)}{nk}$$

exists.

The above expression is equivalent to the following:

$$\rho^*(E) = \lim_{p,q\to+\infty} \frac{1}{\log p\log q}\sum_{n=1}^{p}\sum_{k=1}^{q}\frac{\chi_E(n,k)}{nk}.$$

A family of sets $I\subseteq 2^X$, where 2^X is the class of all subsets of non-empty set X, is said to be ideal if and only if $\emptyset\in\mathcal{I}$, for each $A,B\in\mathcal{I}$, we have $A\cup B\in\mathcal{I}$, and for each $A\in\mathcal{I}$ and each $B\subseteq A$, we have $B\in\mathcal{I}$. If and only if $A\cap B\in\mathcal{F}$ and $B\in\mathcal{F}$ for each instance of $A\in\mathcal{F}$ and $B\supset A$, $\emptyset\notin\mathcal{F}$, $\mathcal{F}\subseteq 2^X$ is said to be a filter on X. If $\mathcal{I}\neq\{\emptyset\}$ and $X\notin\mathcal{I}$, then an ideal $\mathcal{I}$ is referred to as a non-trivial ideal. If and only if $\mathcal{F}=\mathcal{F}(\mathcal{I})=X-A$, then it is evident that $\mathcal{I}\subseteq 2^X$ is a non-trivial ideal: $A\in\mathcal{I}$ is a filter on X. A non-trivial ideal $\mathcal{I}\subseteq 2^X$ is said to be admissible if and only if $\mathcal{I}\supset\{\{x\}:x\in X\}$.

Remark 1.1. If we consider subsets A of $\mathbb{N}$ with $\delta(A) = 0, d(A) = 0$ then, these classes of subsets of $\mathbb{N}$ will form an ideal of $\mathbb{N}$. The convergence of sequences will be called as statistical and logarithmic convergence. Similarly, on considering subsets A of $\mathbb{N} \times \mathbb{N}$ with $\rho(A) = 0$ and $\rho^*(A) = 0$, we will get the ideals of $\mathbb{N} \times \mathbb{N}$. The corresponding convergence of sequences are known as Pringsheim's sense statistical and logarithmic convergence of double sequences. Accordingly, the regular convergence can be defined.

For a detail account of $\mathcal{I}$-convergent sequence, one may refer to [11–16, 27, 32–38].

Moore [23] established the idea of uniform convergence of sequence of functions with respect to a scale function. Chittenden [3] provided the following formulation of Moore's definition.

Definition 1.1. If there are functions g and $\sigma(x)$, defined on D, and for every $\varepsilon > 0$, there is an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$, the inequality

$$|g(x) - f_n(x)| < \varepsilon|\sigma(x)|,$$

holds for every element x of D, then the sequence (f_n) of real, single-valued functions f_n of a real variable x converges relatively uniformly on D. Scale function is the name given to the function $\sigma(x)$. When compared to the scale function, the sequence (f_n) is said to converge relatively uniformly.

The notion was further studied by [7–10, 26] and many others. For the first time, Yıldız [40] introduced the convergence known as ideal relative uniform convergence for double sequences of functions.

2. Definitions and Preliminaries

Throughout the paper ${}_2\ell_\infty(ru)$, ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$ denote the classes of relatively uniformly bounded, $\mathcal{I}_2$-relatively uniformly null, $\mathcal{I}_2$-relatively uniformly convergent, $\mathcal{I}_2$- regularly relatively uniformly convergent, $\mathcal{I}_2$-regularly relatively uniformly null of double sequences of positive linear functions, respectively.

Definition 2.1. A sequence space E is referred to as *solid or normal* if $(x_{nk}) \in E$ implies $(\alpha_{nk}x_{nk}) \in E$, for any (α_{nk}) with $|\alpha_{nk}| \leq 1$, for all $n, k \in \mathbb{N}$.

Definition 2.2. If a sequence space E contains the canonical pre-images of all its step spaces, it is said to be monotone.

Remark 2.1. If a sequence space E is solid, then E is monotone.

Definition 2.3. A sequence space E is said to be *symmetric* if for any $n, k \in \mathbb{N} \times \mathbb{N}$, $(x_{nk}) \in E$ implies $(x_{\pi(n,k)}) \in E$, where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Definition 2.4. For all $n, k \in \mathbb{N}$, a sequence space E is said to be *convergence free* if $(x_{nk}) \in E$ and $x_{nk} = 0$ implies $y_{nk} = 0$ together with $(y_{nk}) \in E$.

Definition 2.5. For all $n, k \in \mathbb{N}$, a sequence space E is said to be a *sequence algebra* if $(x_{nk} \circ y_{nk}) \in E$ whenever (x_{nk}) and (y_{nk}) belongs to E.

Definition 2.6 ([40])**.** In the class of all subsets of $\mathbb{N} \times \mathbb{N}$, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$. If there are functions $g(x)$ and $\sigma(x)$ defined on D such that for every $\varepsilon > 0$ and for all $x \in D$, then the sequence of functions $(f_{nk}(x))$ of single, real-valued functions $\mathbb{R}$ is said to be $\mathcal{I}_2$-relatively uniformly convergent on D satisfying the following condition.

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - g(x)| \geq \varepsilon|\sigma(x)|\} \in \mathcal{I}_2.$$

This can also be expressed as for every $\varepsilon > 0$, there exists $M \in \mathcal{I}_2$ such that for any $(n, k) \notin M$,

$$|f_{nk}(x) - f(x)| < \varepsilon|\sigma(x)|, \quad \text{for all } x \in D.$$

Remark 2.2. We obtain the definition of $\mathcal{I}_2$-relatively uniformly null of double sequence of positive linear functions if $g = \theta$, the zero function in the previous definition.

Definition 2.7. In the class of all subsets of $\mathbb{N} \times \mathbb{N}$, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$. $\mathcal{I}_2$-relatively uniformly Cauchy refers to a set of functions $(f_{nk}(x))$ defined on a compact domain D if $s = s(\varepsilon)$, $t = t(\varepsilon)$ and function $\sigma(x)$ are defined on D such that for every $\varepsilon > 0$ and for any $x \in D$

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - f_{st}(x)| \geq \varepsilon|\sigma(x)|\} \in \mathcal{I}_2.$$

Definition 2.8. Considering the class of all subsets of $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, respectively, let $\mathcal{I}_2$ be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$ and $\mathcal{I}$ be an ideal of $2^{\mathbb{N}}$. If there are functions $g(x), f_k(x), f_n(x)$, $\sigma(x), \xi_n(x), \eta_k(x)$ defined on D such that for every $\varepsilon > 0$ and for any $x \in D$, then the sequence of single, real-valued functions $(f_{nk}(x))$ is said to be $\mathcal{I}_2$-regularly relatively uniformly convergent on D satisfying the following conditions:

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - g(x)| \geq \varepsilon|\sigma(x)|\} \in \mathcal{I}_2, \text{ for any } n, k \in \mathbb{N},$$

$$\{k \in \mathbb{N} : |f_{nk}(x) - f_n(x)| \geq \varepsilon|\xi_n(x)|\} \in \mathcal{I}, \text{ for every } n \in \mathbb{N},$$

$$\{n \in \mathbb{N} : |f_{nk}(x) - f_k(x)| \geq \varepsilon|\eta_k(x)|\} \in \mathcal{I}, \text{ for every } k \in \mathbb{N}.$$

Remark 2.3. We obtain the definition of $\mathcal{I}_2$-regularly relatively uniformly null of double sequence of positive linear functions if $g = f_k = f_n = \theta$, the zero function in the previous definition.

Remark 2.4. $\mathcal{I}_2 = \mathcal{I}_2(P) \subset 2^{\mathbb{N}\times\mathbb{N}}$ is the class of all subsets of $\mathbb{N} \times \mathbb{N}$ containing terms of sequence of functions $(f_{nk}(x))$ upto n_0 finite term for all n and k w.r.t. the scale function $\sigma(x)$. Then, $\mathcal{I}_2(P)$ is an ideal of $2^{\mathbb{N}\times\mathbb{N}}$ and and it corresponds to the double sequence of functions' relative uniform convergence with respect to $\sigma(x)$ on D.

On considering $\mathcal{I}_2(P)$ along with $\mathcal{I}_f$, it corresponds to the double sequence of functions' regular relative uniform convergence with respect to the scale function $\sigma(x)$ on D.

Remark 2.5. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho) \subset 2^{\mathbb{N}\times\mathbb{N}}$, the class of all subsets of $\mathbb{N}\times\mathbb{N}$ of zero natural density w.r.t. the scale function $\sigma(x)$, then, $\mathcal{I}_2(\rho)$ is an ideal of $2^{\mathbb{N}\times\mathbb{N}}$ and $\mathcal{I}_2(\rho)$ corresponds to the statistical relative uniform convergence of double sequence of functions w.r.t. $\sigma(x)$ on D.

On considering $\mathcal{I}_2(\rho)$ along with $\mathcal{I}_\delta$, it corresponds to the statistical regularly relatively uniformly convergent double sequence of functions w.r.t. the scale function $\sigma(x)$ on D.

Remark 2.6. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho^*) \subset 2^{\mathbb{N}\times\mathbb{N}}$, the class of all subsets of $\mathbb{N}\times\mathbb{N}$ of zero logarithmic density w.r.t. the scale function $\sigma(x)$, then, $\mathcal{I}_2(\rho^*)$ is an ideal of $2^{\mathbb{N}\times\mathbb{N}}$ and $\mathcal{I}_2(\rho^*)$ corresponds to the logarithmic relative uniform convergence of double sequence of functions w.r.t. $\sigma(x)$ on D.

On considering $\mathcal{I}_2(\rho^*)$ along with $\mathcal{I}_d$, it corresponds to the logarithmic regularly relatively uniformly convergent double sequence of functions w.r.t. the scale function $\sigma(x)$ on D.

Definition 2.9. Let $(f_{nk}(x))$ and $(g_{nk}(x))$ be two double sequences of real, single-valued functions defined on compact subset D and $\mathcal{I}_2$ be an ideal on $2^{\mathbb{N}\times\mathbb{N}}$. Then, we say that $f_{nk}(x) = g_{nk}(x)$ *for almost all n and k relative to $\mathcal{I}_2$* w.r.t. the scale function $\sigma(x)$ (in short a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$) if for all $x \in D$,

$$\{(n,k) \in \mathbb{N}\times\mathbb{N} : f_{nk}(x) \neq g_{nk}(x)\} \in \mathcal{I}_2.$$

Definition 2.10. Let $(f_{nk}(x))$ be a sequence of real, single-valued functions defined on compact subset D and $\mathcal{I}_2$ be an ideal on $2^{\mathbb{N}\times\mathbb{N}}$. A subset M of D, is said to contain $f_{nk}(x)$ for a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$ if for all $x \in D$,

$$\{(n,k) \in \mathbb{N}\times\mathbb{N} : f_{nk}(x) \notin M\} \in \mathcal{I}_2.$$

We introduce the following sequence spaces:

$${}_2c_0(\mathcal{I}_2, ru) \cap_2 \ell_\infty(ru) =_2 c_0^m(\mathcal{I}_2, ru), \quad {}_2c(\mathcal{I}_2, ru) \cap_2 \ell_\infty(ru) =_2 c^m(\mathcal{I}_2, ru),$$

$${}_2c_0^R(\mathcal{I}_2, ru) \cap_2 \ell_\infty(ru) =_2 c_0^{mR}(\mathcal{I}_2, ru), \quad {}_2c^R(\mathcal{I}_2, ru) \cap_2 \ell_\infty(ru) =_2 c^{mR}(\mathcal{I}_2, ru).$$

The double sequence $f = (f_{nk})$ with elements chosen from the space of all real-valued functions on compact domain D is considered. Let $||f||_\sigma$ denote the usual sup-norm of f in D with respect to the scale function $\sigma(x)$, which is defined as follows.

$$||f||_\sigma = ||(f_{nk})||_\sigma = \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|f_{nk}(x)|}{|\sigma(x)|}. \tag{2.1}$$

3. Main Results

Theorem 3.1. *Let $\mathcal{I}_2$ represent a $2^{\mathbb{N}\times\mathbb{N}}$ ideal. Then, on a compact domain D, a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent if and only if it is $\mathcal{I}_2$-relatively uniformly Cauchy.*

Proof. Consider a compact domain D and a double sequence of functions $(f_{nk}(x))$. In terms of the scale function $\sigma(x)$ defined on D, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent to $f(x)$ on D.

Then, for every $\varepsilon > 0$ and for all $x \in D$, there exists $M \in \mathcal{I}_2$ such that

$$|f_{nk}(x) - f(x)| \leq \frac{\varepsilon}{2}|\sigma(x)|, \quad \text{for all } (n,k) \notin M. \tag{3.1}$$

Similarly,

$$|f_{st}(x) - f(x)| \leq \frac{\varepsilon}{2}|\sigma(x)|, \quad \text{for all } (s,t) \notin M. \tag{3.2}$$

Let $n, k, s, t \geq n_0 = n_0(\varepsilon)$. For every $\varepsilon > 0$ and for all $x \in D$, there exists $M \in \mathcal{I}_2$ such that for all $(n,k) \notin M$ and $(s,t) \notin M$, using (3.1) and (3.2) we have

$$\begin{aligned} |f_{nk}(x) - f_{st}(x)| &\leq |f_{nk}(x) - f(x)| + |f_{st}(x) - f(x)| \\ &\leq \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| \\ &\leq \varepsilon|\sigma(x)|. \end{aligned}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly Cauchy w.r.t. scale function $\sigma(x)$.

Conversely, let $(f_{nk}(x))$ be $\mathcal{I}_2$-relatively uniformly Cauchy on D. Then, there exist G, H such that the interval $U = [f_{GH}(x)-1, f_{GH}(x)+1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Next, choose G_1, H_1 such that $U' = [f_{G_1,H_1}(x) - 1, f_{G_1,H_1}(x) + 1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let, $U_1 = U \cap U'$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Evidently,

$$\begin{aligned} \{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U \cap U'\} =& \{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U\} \\ &+ \{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U'\}. \end{aligned}$$

This implies, $\{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk}(x) \notin U \cap U'\} \in \mathcal{I}_2$, for all $x \in D$. Then, for all $x \in D$, U_1 is a closed interval of D with length less than or equal to one that contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$. Next, choose G_2, H_2 such that $U'' = [f_{G_2H_2}(x) - 1, f_{G_2H_2}(x) + 1]$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let $U_2 = U_1 \cap U''$ contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$, for all $x \in D$. Then, we get, U_2 is a closed interval of D of length less than or equal to $\frac{1}{2}$ that contains $f_{nk}(x)$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Continuing inductively, we get a nested sequence (U_m) of closed intervals of D such that for all $m \in \mathbb{N}, U_m \supseteq U_{m+1}$, the length of $U_m \geq 2^{1-m}$, and $(f_{nk}(x)) \in U_m$, a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$. Thus, $\bigcap_{m=1}^{+\infty} U_m$ will contain a function $f(x)$, w.r.t. the scale function $\sigma(x)$, for all $x \in D$.

Let $\varepsilon > 0$ be given and there exists n_0 such that $\varepsilon > 2^{1-n_0}$. Then, $(f_{nk}(x)) \in U_m$ a.a.n&k.r. $\mathcal{I}_2$ w.r.t. the scale function $\sigma(x)$, for all $x \in D$. We have

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : |f_{nk}(x) - f(x)| \geq \varepsilon\} \leq \{f_{nk}(x) \notin U_m\} \in \mathcal{I}_2,$$

for all $x \in D$. Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent to $f(x)$ w.r.t. the scale function $\sigma(x)$ on D. □

We state the following result without proof, since it can be established using standard technique.

Theorem 3.2. *Let* $\mathcal{I}_2$ *be an ideal of* $2^{\mathbb{N}\times\mathbb{N}}$*. The classes of double sequences of functions* ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru), {}_2c^m(\mathcal{I}_2, ru), {}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ *are linear spaces.*

Theorem 3.3. *Let* $\mathcal{I}_2$ *be an ideal of* $2^{\mathbb{N}\times\mathbb{N}}$*. The classes of double sequences of functions* ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ *are normed linear spaces with respect to the norm defined by* (2.1).

Proof. Let α, β be the scalars and $(f_{nk}(x)), (g_{nk}(x)) \in_2 c_0^m(I_2, ru)$. Then, there exist positive real numbers K_1 and K_2 such that

$$\sup_{n,k\in\mathbb{N}} |f_{nk}(x)| < K_1|\sigma_1(x)| \quad \text{and} \quad \sup_{n,k\in\mathbb{N}} |g_{nk}(x)| < K_2|\sigma_2(x)|.$$

Hence,

$$\begin{aligned}\sup_{n,k\in\mathbb{N}} |\alpha f_{nk}(x) + \beta g_{nk}(x)| &\leq |\alpha| \sup_{n,k\in\mathbb{N}} |f_{nk}(x)| + |\beta| \sup_{n,k\in\mathbb{N}} |g_{nk}(x)| \\ &\leq |\alpha| K_1 |\sigma_1(x)| + |\beta| K_2 |\sigma_2(x)|.\end{aligned}$$

Without loss of generality we can consider the same scale function, $\sigma(x) = \max\{|\sigma_1(x)|, |\sigma_2(x)|\}$, and we get

$$\sup_{n,k\in\mathbb{N}} |\alpha f_{nk}(x) + \beta g_{nk}(x)| \leq \{|\alpha| K_1 + |\beta| K_2\}\sigma(x).$$

Hence, the space ${}_2c_0^m(I_2, ru)$ is a linear space. Similarly, we can establish for the rest of the spaces. Now, to verify that the linear space ${}_2c_0^m(I_2, ru)$ satisfy the norm given in (2.1), the following three conditions must hold true.

Let $(f_{nk}(x)), (g_{nk}(x)) \in_2 c_0^m(\mathcal{I}_2, ru)$.

(i) One can easily verify that $||f||_\sigma = 0 \Leftrightarrow f(x) = 0$, for all $x \in D$.

(ii)

$$\begin{aligned}||(f+g)||_\sigma &= \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|f_{nk}(x) + g_{nk}(x)|}{|\sigma(x)|} \\ &\leq \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|f_{nk}(x)|}{|\sigma(x)|} + \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|g_{nk}(x)|}{|\sigma(x)|} \\ &\leq ||f||_\sigma + ||g||_\sigma.\end{aligned}$$

(iii)

$$\begin{aligned}||\lambda f||_\sigma &= \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|\lambda f_{nk}(x)|}{|\sigma(x)|} \\ &\le |\lambda| \sup_{n,k\in\mathbb{N}} \sup_{x\in D} \frac{|f_{nk}(x)|}{|\sigma(x)|} \\ &\le |\lambda|\ ||f||_\sigma.\end{aligned}$$

Similarly, we can establish for the rest of the sequence spaces. □

Theorem 3.4. *The classes of double sequences of functions* ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ *are Banach spaces.*

Proof. Let $(f^i(x))$ be a relative uniform Cauchy sequence in ${}_2c^m(\mathcal{I}_2, ru) \subset_2 \ell_\infty(ru)$, where $f^i(x) = (f^i_{nk}(x))$. Then, $(f^i(x))$ converges relatively uniformly in ${}_2\ell_\infty(ru)$. There exists

$$\lim_{i\to+\infty} f^i_{nk}(x) = f_{nk}(x), \quad \text{for all } x \in D \text{ and } n, k \in \mathbb{N}.$$

Let $\mathcal{I}_2 - \lim f^i_{nk}(x) = g_i(x)$, for all $x \in D$ and $i \in \mathbb{N}$. Since, $(f^i(x))$ is relatively uniformly Cauchy, for every $\varepsilon > 0$ and for all $x \in D$, there exists $n_0 \in \mathbb{N}$ such that

$$|f^i_{nk}(x) - f^j_{nk}(x)| < \frac{\varepsilon}{3}|\sigma(x)|, \quad \text{for all } i, j \ge n_0. \tag{3.3}$$

Since, $(f^i_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent to $g_i(x)$, there exists $L \in \mathcal{I}_2$ such that for each $(n, k) \notin L$ and for all $x \in D$, we have

$$|f^i_{nk}(x) - g_i(x)| \le \frac{\varepsilon}{3}|\sigma(x)|, \quad \text{for all } i, j \ge n_0. \tag{3.4}$$

Similarly, $(f^j_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent to $g_j(x)$, there exists $M \in \mathcal{I}_2$ such that for each $(n, k) \notin M$ and for all $x \in D$, we have

$$|f^j_{nk}(x) - g_j(x)| \le \frac{\varepsilon}{3}|\sigma(x)|. \tag{3.5}$$

Using equations (3.3), (3.4), (3.5), for all $x \in D$, we have

$$\begin{aligned}|g_i(x) - g_j(x)| &= |f^i_{nk}(x) - g_i(x)| + |f^j_{nk}(x) - g_j(x)| + |f^i_{nk}(x) - f^j_{nk}(x)| \\ &\le \varepsilon|\sigma(x)|.\end{aligned}$$

Thus, $(g_i(x))$ is relatively uniformly Cauchy. Then, there exists $\lim_{i\to+\infty} g_i(x) = g(x)$ (say). We can write, for every $\eta > 0$ and for all $x \in D$, there exists m_0 such that

$$|g_i(x) - g(x)| < \frac{\eta}{3}|\sigma(x)|, \quad \text{for all } i \ge m_0. \tag{3.6}$$

Since, $(f^i_{nk}(x))$ is relatively uniformly Cauchy, for every $\eta > 0$ and for all $x \in D$, there exists m_0 such that

$$|f^i_{nk}(x) - f_{nk}(x)| < \frac{\eta}{3}|\sigma(x)|, \quad \text{for all } i \ge m_0. \tag{3.7}$$

Since, $(f^i_{nk}(x))$ is $\mathcal{I}_2-$relatively uniformly convergent to $g_i(x)$, there exists $Q \in \mathcal{I}_2$ such that for all $(n,k) \notin Q$ and for all $x \in D$ we get

$$|f^i_{nk}(x) - g_i(x)| < \frac{\eta}{3}|\sigma(x)|. \tag{3.8}$$

Without loss of generality, for all $(n,k) \notin Q$ and $x \in D$, using equations (3.6), (3.7), (3.8), we get

$$\begin{aligned}|f_{nk}(x) - g(x)| &\le |f_{nk}(x) - f^i_{nk}(x)| + |f^i_{nk}(x) - g_i(x)| + |g_i(x) - g(x)| \\ &< \eta|\sigma(x)|.\end{aligned}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent to $g(x)$ w.r.t. the scale function $\sigma(x)$. Thus, ${}_2c^m(\mathcal{I}_2, ru)$ is a Banach space.

Similarly, we can prove for the other classes of sequences of functions. □

In view of Theorem 3.4, we state the following theorem without proof.

Theorem 3.5. *The classes of double sequences of functions* ${}_2c^m_0(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}_0(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ *are nowhere dense subsets of* ${}_2\ell_\infty(ru)$.

Theorem 3.6. (a) *The classes of double sequences of functions* ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c^R_0(\mathcal{I}_2, ru)$, ${}_2c^m_0(\mathcal{I}_2, ru)$, ${}_2c^{mR}_0(\mathcal{I}_2, ru)$ *are solid and hence, are monotone.*

(b) *The classes of double sequences of functions* ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ *are not monotone and hence, are not solid.*

Proof. The proof of the first part follows from the following inclusion relation.

Consider the class of sequences of functions ${}_2c_0(\mathcal{I}_2, ru)$.

Let $(f_{nk}(x)) \in_2 c_0(\mathcal{I}_2, ru)$ and (α_{nk}) be a sequence of scalars such that

$$|\alpha_{nk}| \le 1, \quad \text{for all } n,k \in \mathbb{N}.$$

Let $\varepsilon > 0$ be given. Then, for all $x \in D$, we have

$$\{(n,k) \in \mathbb{N}\times\mathbb{N} : |f_{nk}(x)| \ge \varepsilon|\sigma(x)|\} \supseteq \{(n,k) \in \mathbb{N}\times\mathbb{N} : |\alpha_{nk} f_{nk}(x)| \ge \varepsilon|\sigma(x)|\}.$$

Hence, $(\alpha_{nk} f_{nk}(x)) \in_2 c_0(\mathcal{I}_2, ru)$. This implies, ${}_2c_0(\mathcal{I}_2, ru)$ is solid and hence, monotone.

Similarly, we can establish for the rest of the cases. □

The proof of the second part follows from the example below.

Example 3.1. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho^*)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0,1] \to \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n,k \text{ are prime}, n,k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

We get, $(f_{nk}(x))$ is logarithmically relatively uniformly convergent on $[0,1]$ w.r.t. the scale function $\sigma(x) = 1$. Hence, $(f_{nk}(x)) \in_2 c(\mathcal{I}_2, ru)$.

Let $(g_{nk}(x))$ be the pre-image of the sequence of functions $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} x, & \text{for } n \text{ is odd, } n,k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

One cannot get a scale function for which $(g_{nk}(x))$ is logarithmically relatively uniformly convergent on $[0,1]$. This implies, $(g_{nk}(x)) \notin_2 c(\mathcal{I}_2, ru)$. Hence, ${}_2c(\mathcal{I}_2, ru)$ is not monotone and therefore, not solid.

Similarly, we can prove for the other cases.

Result 3.1. The sequence spaces ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are not symmetric.

The result follows from the example below.

Example 3.2. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0,1] \to \mathbb{R}$, defined by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n = i^2, \text{ for all } i \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

This implies, $(f_{nk}(x)) \in_2 c(\mathcal{I}_2, ru)$.

Let $(g_{nk}(x))$ be the rearranged sequence of functions of $(f_{nk}(x))$ defined by

$$g_{nk}(x) = \begin{cases} x, & \text{for } n+k \text{ even , } n,k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

One cannot get a scale function for which $(g_{nk}(x))$ is statistically relatively uniformly convergent on $[0,1]$. This implies, $(g_{nk}(x)) \notin_2 c(\mathcal{I}_2, ru)$. Hence, ${}_2c(\mathcal{I}_2, ru)$ is not symmetric.

Similarly, we can establish for the rest of the classes of double sequences of functions.

Result 3.2. The sequence spaces ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ are not convergence free.

The result follows from the example below.

Example 3.3. Let $\mathcal{I}_2 = \mathcal{I}_2(P)$. Consider the double sequences of functions $(f_{nk}(x))$, $f_{nk} : [0,1] \to \mathbb{R}$ defined by

$$f_{nk}(x) = \frac{nkx}{1 + n^2k^2x^2}, \quad \text{for each } n,k \in \mathbb{N}.$$

We get, $(f_{nk}(x))$ is relatively uniformly null on $[0,1]$ w.r.t. the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x \leq 1, \\ 1, & \text{for } x = 0. \end{cases}$$

Hence, $(f_{nk}(x)) \in_2 c_0(\mathcal{I}_2, ru)$.

Let us consider another class of sequences $(g_{nk}(x))$ of functions $g_{nk} : [0,1] \to \mathbb{R}$ defined by

$$g_{nk}(x) = \frac{nk}{nk+x}, \quad \text{for each } n,k \in \mathbb{N}.$$

This implies, $(g_{nk}(x)) \notin_2 c_0(\mathcal{I}_2, ru)$. Hence, ${}_2c_0(\mathcal{I}_2, ru)$ is not convergence free.

Similarly, we can show for the rest of the cases.

Theorem 3.7. *The sequence spaces* ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$, ${}_2c(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$ *are sequence algebra.*

Proof. Let the double sequence of functions $(f_{nk}(x))$ and $(g_{nk}(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ belong to the class of sequence of functions ${}_2c(\mathcal{I}_2, ru)$. Then, for every $\varepsilon > 0$, there exists $M \in \mathcal{I}_2$ such that for all $(n,k) \notin M$ and $x \in D$,

$$|f_{nk}(x) - f(x)| < \frac{\varepsilon}{2(|f(x)|+1)}|\sigma(x)|, \quad \text{for all } n,k \geq n_1.$$

Similarly,

$$|g_{nk}(x) - g(x)| < \frac{\varepsilon}{2(|g(x)|+1)}|\sigma(x)|, \quad \text{for all } n,k \geq n_2.$$

By applying reverse triangle inequality, there exists n_3 such that for all $n,k \geq n_3$, we have,

$$|f_{nk}(x)| - |f(x)| \leq ||f_{nk}(x)| - |f(x)|| \leq |f_{nk}(x) - f(x)| \leq 1.$$

This implies,

$$f_{nk}(x)| < |f(x)| + 1, \quad \text{i.e.,} \ \frac{|f_{nk}(x)|}{|f(x)|+1} < 1.$$

For all $(n,k) \notin M$, there exists n_0 such that for all $n_0 > \max\{n_1, n_2, n_3\}$ and $x \in D$, we have

$$\begin{aligned}
|f_{nk}(x)g_{nk}(x) - f(x)g(x)| &= |f_{nk}(x)g_{nk}(x) - f_{nk}(x)g(x) + f_{nk}(x)g(x) - f(x)g(x)| \\
&= |f_{nk}(x)(g_{nk}(x) - g(x)) + g(x)(f_{nk}(x) - f(x))| \\
&\leq |f_{nk}(x)| \ |g_{nk}(x) - g(x)| + |g(x)| \ |f_{nk}(x) - f(x)| \\
&\leq |f_{nk}(x)| \ \frac{\varepsilon}{2(|(f(x)|+1)}|\sigma(x)| + |g(x)| \ \frac{\varepsilon}{2(|g(x)|+1)}|\sigma(x)| \\
&\leq \varepsilon|\sigma(x)|.
\end{aligned}$$

Hence, $(f_{nk}(x)g_{nk}(x)) \in_2 c(\mathcal{I}_2, ru)$.

Similarly, we can establish for the rest of the classes of double sequences of functions.

□

Result 3.3. On a compact domain D, if a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$-uniformly convergent, it must also be $\mathcal{I}_2$-relatively uniformly convergent on D but not vice versa.

The converse of the Result 3.3 is not necessarily true, which is shown in the following example.

Example 3.4. Let $\mathcal{I}_2 = \mathcal{I}_2(\rho)$, consider the double sequence of functions $(f_{nk}(x))$, $f_{nk} : [0,1] \to \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} \frac{1}{nkx}, & \text{for } 0 < x \leq 1, n, k \in \mathbb{N}, \\ 0, & \text{for } x = 0. \end{cases}$$

We get, $(f_{nk}(x))$ is statistically relatively uniformly convergent w.r.t. the scale function

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x \leq 1, \\ 1, & \text{for } x = 0. \end{cases}$$

Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent on $[0,1]$. One can easily see that $(f_{nk}(x))$ is not $\mathcal{I}_2$-uniformly convergent on $[0,1]$.

Result 3.4. On a compact domain D, if a double sequence of functions $(f_{nk}(x))$ is $\mathcal{I}_2$-regularly relatively uniformly convergent, it must also be $\mathcal{I}_2$-relatively uniformly convergent on D but not vice versa.

The converse of the Result 3.4 is not necessarily true, which is shown in the following example.

Example 3.5. Let $\mathcal{I}_2 = \mathcal{I}_2(P)$. We consider the sequence of functions $(f_{nk}(x))$, $f_{nk} : [0,1] \to \mathbb{R}$ defined by

$$f_{nk}(x) = \begin{cases} -x, & \text{for } n = 1, k \text{ is even, } \ k = 1, n \text{ is even, } \ n, k \in \mathbb{N}, \\ x, & \text{otherwise.} \end{cases}$$

Then, $(f_{nk}(x))$ is relatively uniformly convergent on $[0,1]$ w.r.t. the scale function $\sigma(x) = 1$. Hence, $(f_{nk}(x))$ is $\mathcal{I}_2$-relatively uniformly convergent on $[0,1]$.

But the first row and first column of $(f_{nk}(x))$ is not relatively uniformly convergent and hence, $(f_{nk}(x))$ is not $\mathcal{I}_2$-regularly relatively uniformly convergent.

4. Conclusions

In this article, we have studied ideal convergence of double sequence of functions from the point of view of relative uniform convergence w.r.t. the scale function $\sigma(x)$ defined on a compact subset $D \subseteq \mathbb{R}$. We introduced the classes of double sequences of functions ${}_2c(\mathcal{I}_2, ru)$, ${}_2c_0(\mathcal{I}_2, ru)$, ${}_2c^R(\mathcal{I}_2, ru)$, ${}_2c_0^R(\mathcal{I}_2, ru)$, ${}_2c^m(\mathcal{I}_2, ru)$, ${}_2c_0^m(\mathcal{I}_2, ru)$, ${}_2c^{mR}(\mathcal{I}_2, ru)$, ${}_2c_0^{mR}(\mathcal{I}_2, ru)$ and studied their properties like solid, monotone, symmetric, sequence algebra, convergence free and denseness. We also established the relationship between $\mathcal{I}_2$-relative uniform convergent and $\mathcal{I}_2$- relative uniform Cauchy as well as relationship between $\mathcal{I}_2$-relative uniform convergent and $\mathcal{I}_2$-regular relative uniform convergent.

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