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VAGUE WEAK INTERIOR IDEALS OF Γ-SEMIRINGS

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ABSTRACT. The notion of a ((*complete-*) *normal*) *vague weak interior ideal* on a (*regular*) Γ-semiring is defined. It is proved that the set of all vague weak interior ideals forms a complete lattice. Also, a characterization theorem for a regular Γ-semiring in terms of vague weak interior ideals is derived. Another interesting consequence of the main result is that the cardinal of a non-constant maximal element in the set of all (complete-) normal vague weak interior ideals is 2.

1. INTRODUCTION

In 1934, Vandiver [18] extended the notions of rings and distributive lattices and defined a new algebraic structure as semirings. It is known that semiring theory has many applications to many branches of pure and applied mathematics: functional analysis, combinatorics, graph theory, automata theory, coding and language theory. In 1981, Sen [17] introduced the notion of a Γ-semigroup as a generalization of semigroup. Then Rao [14, 15] generalized a semiring and Γ-ring by introducing Γ-semiring. Ideals play an important role in advance studies and uses of algebraic structures (see, [6, 10]). Hedayati and Shum [9] were considered the congruences and ideals of a Γ-semiring. In 1965, Zadeh [19] introduced the concept of a fuzzy set. Then Gau and Buehrer [8] introduced the concept of vague sets as a generalization of fuzzy sets. Moreover, Ramakrishna [12] studied vague cosets, vague products and several properties related to them. Jun and Park [11] defined the notion of a vague ideal in a subtraction algebra. Rao and Venkateswarlu [16] studied bi-interior ideals of Γ-semirings and get some of its properties. In 2008, Eswarlal [7] introduced the

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concepts of vague ideals and normal vague ideals in semirings. Bhargavi and Eswarlal [1–5] were developed the theory of vague sets on Γ-semirings. In 2019, Rao [13] introduced weak interior ideals and fuzzy weak interior ideals of Γ-semirings. The motivation of this paper, is define the notion of a ((*complete*)- *normal*) *vague weak interior ideal* of a Γ-semiring. We prove that there is an isomorphism between the set of all vague weak interior ideals and its crisp weak interior ideals. We prove that the set of all vague weak interior ideals forms a complete lattice. Further, we give a characterization theorem for *regular* Γ-*semiring* in terms of vague weak interior ideals, and a condition to every vague weak interior ideal could be a normal vague weak interior ideal is given.

2. Preliminaries

We recall the basic definitions needed for this paper.

Definition 2.1 ([8]). (a) A vague set ψ is a pair (t_ψ, f_ψ), where $t_\psi, f_\psi : E \to [0,1]$ are mappings s.t. $t_\psi(x) + f_\psi(x) \leq 1$ for all $x \in E$.

(b) The interval $[t_\psi(x), 1-f_\psi(x)]$ is called the vague value of x in ψ and it is denoted by $V_\psi(x)$, i.e., $V_\psi(x) = [t_\psi(x), 1 - f_\psi(x)]$.

(c) Let $D \subseteq E$, the vague characteristic set of D in $[0,1]$ is a vague set $\delta_D = (t_{\delta_D}, f_{\delta_D})$ as follows:

$$V_{\delta_D}(f) = \begin{cases} [1,\ 1]\ , & \text{if } x \in D, \\ [0,\ 0], & \text{if } x \notin D. \end{cases}$$

i.e.,

$$t_{\delta_D}(x) = \begin{cases} 1, & \text{if } x \in D, \\ 0, & \text{if } x \notin D, \end{cases} \quad \text{and} \quad f_{\delta_D}(x) = \begin{cases} 0, & \text{if } x \in D, \\ 1, & \text{if } x \notin D. \end{cases}$$

(d) Let $\psi = (t_\psi, f_\psi)$ be a vague set. For $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α,β)-cut or vague cut of ψ is the crisp subset of E is given by:

$$\psi_{(\alpha,\beta)} = \{x \in E : V_\psi(x) \geq [\alpha, \beta]\},$$

i.e.,

$$\psi_{(\alpha,\beta)} = \{x \in E : t_\psi(x) \geq \alpha \text{ and } 1 - f_\psi(x) \geq \beta\}.$$

Denote by $\mathbf{VS}(E)$ the set of all vague sets of E.

Definition 2.2 ([8]). Let $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VS}(E)$. Then, for all $x \in E$:

(a) $\psi^c = (t_{\psi^c}, f_{\psi^c})$, where $t_{\psi^c} = f_\psi,\ f_{\psi^c} = t_\psi$;

(b) $\psi \subseteq \phi$ if and only if $\psi(x) \leq \phi(x)$;

(c) $\psi \cup \phi := (t_{\psi\cup\phi}, f_{\psi\cup\phi})$, where $t_{\psi\cup\phi}(x) = \max\{t_\psi(x), t_\phi(x)\}$ and $f_{\psi\cup\phi}(x) = \min\{f_\psi(x), f_\phi(x)\}$;

(d) $\psi \cap \phi := (t_{\psi\cap\phi}, f_{\psi\cap\phi})$, where $t_{\psi\cap\phi}(x) = \min\{t_\psi(x), t_\phi(x)\}$ and $f_{\psi\cap\phi}(x) = \max\{f_\psi(x), f_\phi(x)\}$.

Definition 2.3 ([14])**.** Let $(E, +)$ and $(\Gamma, +)$ be two abelian semigroups. Then E is called a Γ-semiring if there exists a mapping $E \times \Gamma \times E \to E$ (briefly, images of (a, α, b) will be denoted by $a\alpha b$) satisfying the following axioms:

(ΓSR_1) $c\alpha(a + b) = c\alpha a + c\alpha b$;
(ΓSR_2) $(c + a)\alpha b = c\alpha b + a\alpha b$;
(ΓSR_3) $c(\alpha + \beta)a = c\alpha a + c\beta c$;
(ΓSR_4) $c\alpha(a\beta b) = (c\alpha a)\beta b$, for all $a, b, c \in E$, $\alpha, \beta \in \Gamma$.

In this paper, E is a Γ-semiring.

Definition 2.4 ([13])**.** (a) E is called regular if for all $e \in E$, exists $f \in E$, α, $\beta \in \Gamma$ s.t. $e = e\alpha f\beta e$.

(b) A sub-Γ-semiring F of E is called a right (resp. left) weak interior ideal of E if $F\Gamma F\Gamma E \subseteq F$ (resp. $E\Gamma F\Gamma F \subseteq F$). If F is both *right* and *left* weak interior ideal of E, then F is called a *weak interior ideal* of E.

Denote by $\mathbf{RWII}(E)$ (resp. $\mathbf{LWII}(E)$) the set of all right (resp. left) weak interior ideals and $\mathbf{WII}(E)$ the set of all weak interior ideals of E. One can see that $\mathbf{WII}(E) = \mathbf{RWII}(E) \cap \mathbf{LWII}(E)$.

Definition 2.5 ([4])**.** Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called a vague Γ-semiring if it satisfies the following axioms:

(V_1) $V_\psi(a + b) \geq \min\{V_\psi(a), V_\psi(b)\}$;
(V_2) $V_\psi(a\gamma b) \geq \min\{V_\psi(a), V_\psi(b)\}$ fora ll $a, b \in E$, $\gamma \in \Gamma$.

Denote by $\mathbf{V\Gamma}(E)$ the set of all vague Γ-semirings of E.

Definition 2.6 ([5])**.** Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called a right (resp. left) vague ideal of E if it satisfies (V_1) and

(V_3) $V_\psi(a\gamma b) \geq V_\psi(a)$ (resp. $V_\psi(a\gamma b) \geq V_\psi(b)$), for all $a, b \in E$, $\gamma \in \Gamma$.

If ψ is both *left* and *right* vague ideal of E, then ψ is called a *vague ideal* of E.

Denote by $\mathbf{RVI}(E)$ (resp. $\mathbf{LVI}(E)$) the set of all right (resp. left) vague ideals and $\mathbf{VI}(E)$ the set of all vague ideals of E. Hence, $\mathbf{VI} = \mathbf{RVI}(E) \cap \mathbf{LVI}(E)$.

Definition 2.7 ([2])**.** Let $\psi = (t_\psi, f_\psi)$, $\phi = (t_\phi, f_\phi) \in \mathbf{RVI}(E)$ (resp. $\in \mathbf{LVI}(E)$). Then the sum $\psi + \phi$ of ψ and ϕ are defined by:

$$V_{\psi+\phi}(e) = \begin{cases} \sup\{\min\{V_\psi(f), V_\phi(g)\} : e = f + g, \text{ where } f, g \in E\}, & \\ [0,0], & \text{otherwise,} \end{cases}$$

i.e.,

$$t_{\psi+\phi}(e) = \begin{cases} \sup\{\min\{t_\psi(f), t_\phi(g)\} : e = f + g, \text{ where } f, g \in E\}, & \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_{\psi+\phi}(e) = \begin{cases} \inf\{\max\{f_\psi(f), f_\phi(g)\} : e = f + g, \text{where } f, g \in E\}, & \\ 1, & \text{otherwise.} \end{cases}$$

3. Vague Weak Interior Ideals in Γ-semirings

In this section, we define the concept of *vague weak interior ideal* of a Γ-semiring and obtain some of the basic properties. Finally, we give a characterization theorem for a *regular* Γ*-semiring* in terms of vague weak interior ideals.

From now on, δ stands for vague characteristic set of E unless stated.

Definition 3.1. Let $\psi \in \mathbf{V\Gamma}(E)$. Then ψ is called a *right* (resp. *left*) *vague weak interior ideal* of E if $\psi\Gamma\psi\Gamma\delta \subseteq \psi$ (resp. $\delta\Gamma\psi\Gamma\psi \subseteq \psi$).

If ψ is both right and left vague weak interior ideal, then it is called a *vague weak interior ideal* of E.

Denote by $\mathbf{RVWII}(E)$ (resp. $\mathbf{LVWII}(E)$) the set of all right (resp. left) vague weak interior ideals and $\mathbf{VWII}(E)$ the set of all vague weak interior ideals of E. Hence, $\mathbf{VWII}(E) = \mathbf{RVWII}(E) \cap \mathbf{LVWII}(E)$.

Example 3.1. (i) Let $E := \mathbb{N} \cup \{0\}$ and $\Gamma := \mathbb{N}$. Define the mapping $\cdot : \mathbb{N} \cup \{0\} \times \mathbb{N} \times \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ by $\cdot(a, b, c) = abc$ usual product of a, b, c, for all $a, c \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$. Hence, $\mathbb{N} \cup \{0\}$ is a $\mathbb{N}$-semiring. Define $t_\psi, f_\psi : \mathbb{N} \cup \{0\} \to [0, 1]$ as follows:

$$t_\psi(x) = \begin{cases} 0.12, & \text{if } x \in 2\mathbb{N} \text{ or } x = 0, \\ 0.13, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(x) = \begin{cases} 0.18, & \text{if } x \in 2\mathbb{N} \text{ or } x = 0, \\ 0.16, & \text{otherwise.} \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \notin \mathbf{RVWII}(\mathbb{N} \cup \{0\}) \cup \mathbf{LVWII}(\mathbb{N} \cup \{0\})$.

(ii) Let $E = \Gamma := M_{2\times2}(\mathbb{N})$. Define the mapping $M_{2\times2}(\mathbb{N}) \times M_{2\times2}(\mathbb{N}) \times M_{2\times2}(\mathbb{N}) \to M_{2\times2}(\mathbb{N})$ by ABC is the matrix multiplication of A, B, C for all $A, B, C \in M_{2\times2}(\mathbb{N})$. Hence, $M_{2\times2}(\mathbb{N})$ is a $M_{2\times2}(\mathbb{N})$-semiring. Define $t_\psi, f_\psi : M_{2\times2}(\mathbb{N}) \to [0, 1]$ by:

$$t_\psi(M) = \begin{cases} 0.6, & \text{if } M = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}, \text{ where } p, q \in \mathbb{N}; \\ 0.5, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(M) = \begin{cases} 0.1, & \text{if } M = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}, \text{ where } p, q \in \mathbb{N}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(M_{2\times2}(\mathbb{N}))$, but $\psi = (t_\psi, f_\psi) \notin \mathbf{LVWII}(M_{2\times2}(\mathbb{N}))$.

Now, if define $t_\phi, f_\phi : M_{2\times2}(\mathbb{N}) \to [0, 1]$ by:

$$t_\phi(N) = \begin{cases} 0.72, & \text{if } N = \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix}, \text{ where } m, n \in \mathbb{N}, \\ 0.54, & \text{otherwise,} \end{cases}$$

and

$$f_\phi(N) = \begin{cases} 0.28, & \text{if } N = \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix}, \text{ where } m, n \in \mathbb{N}, \\ 0.37, & \text{otherwise.} \end{cases}$$

Then $\phi = (t_\phi, f_\phi) \in \mathbf{LVWII}(M_{2\times 2}(\mathbb{N}))$, but $\phi = (t_\phi, f_\phi) \notin \mathbf{RVWII}(M_{2\times 2}(\mathbb{N}))$.

(iii) Let $E := \{-n : n \in \mathbb{N}\}$ and $\Gamma := \{-2n : n \in \mathbb{N}\}$. Define the mapping $E \times \Gamma \times E \to E$ by abc usual product of a, b, c for all $a, c \in E; b \in \Gamma$. Hence, E is a Γ-semiring. Define $t_\psi, f_\psi : E \to [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.53, & \text{if } x = -1, \\ 0.76, & \text{if } x = -2, \\ 0.99, & \text{if } x < -2, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.54, & \text{if } x = -1, \\ 0.28, & \text{if } x = -2, \\ 0.12, & \text{if } x < -2. \end{cases}$$

Therefore, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E) \cap \mathbf{LVWII}(E)$.

Remark 3.1. Consider Example 3.1 (iii), $\psi^c \notin \mathbf{VWII}(E)$.

Theorem 3.1. *Let $\psi \in \mathbf{RVI}(E)$ (resp. $\in \mathbf{LVI}(E)$). Then $\psi \in \mathbf{RVWII}(E)$ (resp. $\in \mathbf{LVWII}(E)$).*

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{RVI}(E)$. Then $\psi\Gamma\delta \subseteq \psi$. Clearly, $\psi \in \mathbf{V\Gamma}(E)$. Now, let $e \in E$. Then

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(e) &= \sup\{\min\{V_\psi(f), V_{\psi\Gamma\delta}(g)\} : e = f\gamma g, \text{ where } f, g \in E;\ \gamma \in \Gamma\} \\ &\leq \sup\{\min\{V_\psi(f), V_\psi(g)\} : f, g \in E\} \\ &\leq \sup\{V_\psi(e)\} \\ &\leq V_\psi(e). \end{aligned}$$

Thus, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. □

The following example shows that the converse of Theorem 3.1 need not be true.

Example 3.2. Consider Example 3.1 (ii), and define $t_\psi, f_\psi : M_{2\times 2}(\mathbb{N}) \to [0, 1]$ by:

$$t_\psi(P) = \begin{cases} 0.8, & \text{if } P = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \text{ where } e, f \in \mathbb{N}, \\ 0.6, & \text{otherwise,} \end{cases}$$

and

$$f_\psi(P) = \begin{cases} 0.1, & \text{if } P = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \text{ where } e, f \in \mathbb{N}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Therefore, $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(M_{2\times 2}(\mathbb{N}))$, but not a right vague ideal of $M_{2\times 2}(\mathbb{N})$, since $V_\psi(PZQ) < V_\psi(P)$, where $P, Q, Z \in M_{2\times 2}(\mathbb{N})$.

Proposition 3.1. *Let E be regular and $\psi \in \mathbf{VWII}(E)$. Then $\psi \in \mathbf{VI}(E)$.*

Proof. Suppose E is regular and $\psi = (t_\psi, f_\psi) \in \mathbf{VWII}(E)$. Now, if $\psi \notin \mathbf{RVI}(E)$, then exists $e \in E$ s.t. $V_{\psi\Gamma\psi\Gamma\delta}(e) > V_\psi(e)$. Since E is regular, $\exists f \in E$ and $\alpha, \beta \in \Gamma$ s.t. $e = e\alpha f\beta e$. Hence,

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(e) &= \sup\{\min\{V_\psi(e\alpha f), V_{\psi\Gamma\delta}(e)\}\} \\ &> \sup\{\min\{V_\psi(e\alpha f), V_\psi(e)\}\} \\ &= V_\psi(e). \end{aligned}$$

This shows that $\psi\Gamma\psi\Gamma\delta \supset \psi$, which is a contraction. Thus, $\psi \in \mathbf{RVI}(E)$. Similarly, we can prove that $\psi \in \mathbf{LVI}(E)$. Therefore, $\psi \in \mathbf{VI}(E)$. □

In the next theorem we show that there is an isomorphism between $\mathbf{RVWII}(E)$ (resp. $\mathbf{LVWII}(E)$) with the set of all vague cuts.

Theorem 3.2. *Let $\psi \in \mathbf{VS}(E)$. Then $\psi \in \mathbf{RVWII}(E)$ (resp. $\psi \in \mathbf{LVWII}(E)$) if and only if $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$ (resp. $\in \mathbf{LWII}(E)$) for all $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$.*

Proof. Suppose $\psi = (t_\psi, f_\psi) \in \mathbf{RVII}(E)$. Using [5, Theorem 3.6], $\psi_{(\alpha,\beta)}$ is a sub-Γ-semiring of E. Given $e \in \psi_{(\alpha,\beta)}\Gamma\psi_{(\alpha,\beta)}\Gamma E$, we get $e = f\gamma g\eta h$ s.t. $f, g \in \psi_{(\alpha,\beta)}$, $h \in E$. Hence, $V_\psi(f) \geq [\alpha, \beta]$ and $V_\psi(g) \geq [\alpha, \beta]$. Now, we have

$$\begin{aligned} V_\psi(e) &\geq V_{\psi\Gamma\psi\Gamma\delta}(e) \\ &= \sup\{\min\{V_\psi(f), V_\psi(g), V_\delta(h)\} \\ &\geq [\alpha, \beta]. \end{aligned}$$

Therefore, $e \in \psi_{(\alpha,\beta)}$. This shows that $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$.

Conversely, assume $\psi_{(\alpha,\beta)} \in \mathbf{RWII}(E)$. Using [5, Theorem 3.9], we get $\psi \in \mathbf{V\Gamma}(E)$. Now, if $\psi\Gamma\psi\Gamma\delta \nsubseteq \psi$, then exists $s \in E$ s.t. $V_\psi(s) < V_{\psi\Gamma\psi\Gamma\delta}(s)$. Let $[\alpha, \beta] \subseteq [0,1]$ s.t. $V_\psi(s) < [\alpha, \beta] < V_{\psi\Gamma\psi\Gamma\delta}(s)$. Let $s := f\gamma g\eta h$ s.t. $f, g \notin \psi_{(\alpha,\beta)}$ for all $f, g \in E$, $\gamma, \eta \in \Gamma$. Then $V_\psi(f) < [\alpha, \beta]$, $V_\psi(g) < [\alpha, \beta]$. Now, we have

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(s) &= \sup\{\min\{V_\psi(f), V_\psi(g), V_\delta(h)\}\} \\ &= \sup\{\min\{V_\psi(f), V_\psi(g)\}\} \\ &< [\alpha, \beta]. \end{aligned}$$

This shows that $V_{\psi\Gamma\psi\Gamma\delta}(s) < [\alpha, \beta]$, which is a contraction. Therefore, $\psi \in \mathbf{RVWII}(E)$. Similarly, we can prove that $\psi \in \mathbf{LVWII}(E)$. □

Corollary 3.1. *Let $\psi \in \mathbf{VS}(E)$. Then $\psi \in \mathbf{VWII}(E)$ if and only if $\psi_{(\alpha,\beta)} \in \mathbf{WII}(E)$ for all $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$.*

Theorem 3.3. *Let $\emptyset \neq W \subseteq E$ and δ_W be the vague characteristic set of W. Then $W \in \mathbf{RWII}(E)$ (resp. $\in \mathbf{LWII}(E)$) if and only if $\delta_W \in \mathbf{RVWII}(E)$ (resp. $\in \mathbf{LVWII}(E)$).*

Proof. Suppose $W \in \mathbf{RWII}(E)$. Then $W\Gamma W\Gamma E \subseteq W$. Using [5, Theorem 3.9], we get $\delta_W = (t_{\delta_W}, f_{\delta_W}) \in \mathbf{V\Gamma}(E)$. Hence, $\delta_W\Gamma\delta_W\Gamma\delta = \delta_{W\Gamma W\Gamma E} \subseteq \delta_W$. It follows that $\delta_W \in$

RVWII(E). Conversely, assume $\delta_W \in$ **RVWII**(E). Then $\delta_W \Gamma \delta_W \Gamma \delta \subseteq \delta_W$. Using [5, Theprem 3.9], W is sub Γ-semiring of E. Thus, $\delta_{W\Gamma W\Gamma E} \subseteq \delta_W$, and so $W\Gamma W\Gamma E \subseteq W$. Therefore, $W \in$ **RWII**(E). By a similar argument $W \in$ **LWII**(E). □

Corollary 3.2. *Let $\emptyset \neq W \subseteq E$ and δ_W be the vague characteristic set of W. Then $W \in$ **WII**(E) if and only if $\delta_W \in$ **VWII**(E).*

Theorem 3.4. *Let $\psi, \phi \in$ **VWII**(E). Then*

(i) $\psi \cap \phi \in$ **VWII**(E);
(ii) $\psi + \phi \in$ **VWII**(E);
(iii) $\psi \cap \phi \subseteq \psi, \phi$;
(iv) $\psi,\ \phi \subseteq \psi + \phi$.

Proof. (i) Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in$ **RVWII**(E). Using [5, Theorem 3.13], we get $\psi \cap \phi \in \mathbf{V\Gamma}(E)$. Given $e \in E$, we have

$$\begin{aligned}
V_{(\psi\cap\phi)\Gamma\delta}(e) &= \sup\{\min\{V_{\psi\cap\phi}(f), V_\delta(g)\} : e = f\alpha g;\ f, g \in E; \alpha \in \Gamma\} \\
&= \sup\{\min\{\min\{V_\psi(f), V_\phi(f)\}, V_\delta(g)\}\} \\
&= \sup\{\min\{\min\{V_\psi(f), V_\delta(g),\}, \min\{V_\phi(f), V_\delta(g)\}\}\} \\
&= \min\{\sup\{\min\{V_\psi(f), V_\delta(g)\}\}, \sup\{\min\{V_\phi(f), V_\delta(g)\}\}\} \\
&= \min\{V_{\psi\Gamma\delta}(e), V_{\phi\Gamma\delta}(e)\} \\
&= V_{(\psi\Gamma\delta)\cap(\phi\Gamma\delta)}(e).
\end{aligned}$$

This shows that $(\psi \cap \phi)\Gamma\delta = (\psi\Gamma\delta) \cap (\phi\Gamma\delta)$. Also, we have

$$\begin{aligned}
V_{(\psi\cap\phi)\Gamma(\psi\cap\phi)\Gamma\delta}(e) &= \sup\{\min\{V_{\psi\cap\phi}(f), V_{(\psi\cap\phi)\Gamma\delta}(g)\} : e = f\alpha g;\ f, g \in E; \alpha \in \Gamma\} \\
&= \sup\{\min\{V_{\psi\cap\phi}(f), V_{(\psi\Gamma\delta)\cap(\phi\Gamma\delta)}(g)\}\} \\
&= \sup\{\min\{\min\{V_\psi(f), V_\phi(f)\}, \min\{V_{\psi\Gamma\delta}(g), V_{\phi\Gamma\delta}(g)\}\}\} \\
&= \sup\{\min\{\min\{V_\psi(f), V_{\psi\Gamma\delta}(g)\}, \min\{V_\phi(f), V_{\phi\Gamma\delta}(g)\}\}\} \\
&= \min\{\sup\{\min\{V_\psi(f), V_{\psi\Gamma\delta}(g)\}, \sup\{\min\{V_\phi(f), V_{\phi\Gamma\delta}(g)\}\}\} \\
&= \min\{V_{\psi\Gamma\psi\Gamma\delta}(e), V_{\phi\Gamma\phi\Gamma\delta}(e)\} \\
&= V_{(\psi\Gamma\psi\Gamma\delta)\cap(\phi\Gamma\phi\Gamma\delta)}(e).
\end{aligned}$$

Therefore, $(\psi \cap \phi)\Gamma(\psi \cap \phi)\Gamma\delta = (\psi\Gamma\psi\Gamma\delta) \cap (\phi\Gamma\phi\Gamma\delta)$. It follows that

$$(\psi \cap \phi)\Gamma(\psi \cap \phi)\Gamma\delta = (\psi \cap \phi)\Gamma[(\psi\Gamma\delta) \cap (\phi\Gamma\delta)] = (\psi\Gamma\psi\Gamma\delta) \cap (\phi\Gamma\phi\Gamma\delta) \subseteq \psi \cap \phi.$$

Thus, $\psi \cap \phi \in$ **RVWII**(E). Similarly, we can prove $\psi \cap \phi \in$ **LVWII**(E). Therefore, $\psi \cap \phi \in$ **VWII**(E).

(ii) As similar to the proof of (i), $\psi + \phi \in$ **VWII**(E).

(iii) Let $e \in E$. We have $V_{\psi\cap\phi}(e) = \min\{V_\psi(e), V_\phi(e)\} \leq V_\psi(e)$. Therefore, $\psi \cap \phi \subseteq \psi$. Similarly, $\psi \cap \phi \subseteq \phi$.

(iv) Given $x \in E$, we have

$$\begin{aligned} V_{\psi+\phi}(x) &= \sup\{\min\{V_\psi(a), V_\phi(b)\} : x = a + b, \text{ where } a, b \in E\} \\ &\geq \min\{V_\psi(x), V_\phi(0)\} \\ &= V_\psi(x). \end{aligned}$$

It follows that $\psi \subseteq \psi + \phi$. Similarly, $\phi \subseteq \psi + \phi$. □

Corollary 3.3. *If* $\psi_i \in \mathbf{VWII}(E)$*, where* $i \in \Lambda$*. Then*

(i) $\bigcap_{i\in\Lambda} \psi_i \in \mathbf{VWII}(E)$;

(ii) $\sum_{i\in\Lambda} \psi_i \in \mathbf{VWII}(E)$.

Theorem 3.5. *Let* $\psi, \phi \in \mathbf{VWII}(E)$*. Then*

(i) *if* $\mathbf{S} := \{\varphi_i : \varphi_i \in \mathbf{VWII}(E), \varphi_i \subseteq \psi, \phi \text{ for all } i \in \Lambda\}$*, then* $\psi \cap \phi$ *is maximal of* $\mathbf{S}$*;*

(ii) *if* $\mathbf{T} := \{\varrho_i : \varrho_i \in \mathbf{VWII}(E),\ \psi, \phi \subseteq \varrho_i \text{ for all } i \in \Lambda\}$*, then* $\psi + \phi$ *is minimal of* $\mathbf{T}$*.*

Proof. Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$.

(i) Using Theorem 3.4 (i) and (iii), we get $\psi \cap \phi \in \mathbf{VWII}(E)$ and $\psi \cap \phi \subseteq \psi, \phi$. Suppose $\varphi \in \mathbf{S}$ s.t. $\varphi \subseteq \psi$ and $\varphi \subseteq \phi$. Now, let $t \in E$. Then

$$V_{\psi\cap\phi}(t) = \min\{V_\psi(t), V_\phi(t)\} \geq \min\{V_\varphi(t), V_\varphi(t)\} = V_\varphi(t).$$

Therefore, $\varphi \subseteq \psi \cap \phi$. Thus, $\psi \cap \phi$ is maximal element in $\mathbf{S}$.

(ii) Applying Theorem 3.4 (ii) and (iv), we get $\psi + \phi \in \mathbf{VWII}(E)$ and $\psi, \phi \subseteq \psi + \phi$. Let $\varphi \in \mathbf{T}$ s.t. $\psi \subseteq \varphi$ and $\phi \subseteq \varphi$. Given $t \in E$, we have

$$\begin{aligned} V_{\psi+\phi}(t) &= \sup\{\min\{V_\psi(r), V_\phi(s)\} : t = r + s, \text{ where } r, s \in E\} \\ &\leq \sup\{\min\{V_\varphi(r), V_\varphi(s)\} : t = r + s, \text{ where } r, s \in E\} \\ &\leq \sup\{V_\varphi(r + s)\} \\ &= V_\varphi(t). \end{aligned}$$

Therefore, $\psi + \phi \subseteq \varphi$ is minimal element of $\mathbf{T}$. □

Theorem 3.6. *The* $(\mathbf{VWII}(E), \subseteq)$ *is a complete lattice, where* $\subseteq$ *is defined by:*

$$\psi \subseteq \phi \text{ if and only if } V_\psi(e) \leq V_\phi(e), \quad \text{for all } e \in E.$$

Proof. It is known that $(\mathbf{VWII}(E), \subseteq)$ is a poset. By Theorem 3.5, every pair of elements in $\mathbf{VWII}(E)$ has a maximal and a minimal element. Hence, $\mathbf{VWII}(E)$ is a lattice. Let $\mathbf{S} := \{\psi_i : \psi \in \mathbf{VWII}(S) \text{ for } i \in \Lambda\}$ be a subset of $\mathbf{VWII}(E)$. By Corollary 3.3 (i), $\bigcap_{i\in\Lambda} \psi_i \in \mathbf{VWII}(E)$ and it is the infimum of $\mathbf{S}$. Also, by Corollary 3.3 (ii), $\sum_{i\in\Lambda} \psi_i \in \mathbf{VWII}(E)$ and it is supremum of $\mathbf{S}$. Thus, $(\mathbf{VWII}(E), \subseteq)$ is a complete lattice. □

In the next example we can see that the union of two vague weak interior ideals need not be a vague weak interior ideal.

Example 3.3. Let $E := \mathbb{Z}_8$ and $\Gamma := \{\bar{0}, \bar{2}, \bar{4}\}$. Define $\cdot : \mathbb{Z}_8 \times \Gamma \times \mathbb{Z}_8 \to \mathbb{Z}_8$ by $\cdot(\bar{x}, \bar{y}, \bar{z}) = \overline{xyz}$ usual product $\bar{x}, \bar{y}, \bar{z}$, for all $\bar{x}, \bar{z} \in \mathbb{Z}_8$, $\bar{y} \in \{\bar{0}, \bar{2}, \bar{4}\}$. Then $\mathbb{Z}_8$ is a Γ-semiring. Define $t_\psi, f_\psi : \mathbb{Z}_8 \to [0,1]$ by:

$$t_\psi(\bar{e}) = \begin{cases} 0.81, & \text{if } \bar{e} = \bar{0}, \\ 0.63, & \text{if } \bar{e} = \bar{1}, \\ 0.45, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_\psi(\bar{e}) = \begin{cases} 0.22, & \text{if } \bar{e} = \bar{0}, \\ 0.31, & \text{if } \bar{e} = \bar{1}, \\ 0.52, & \text{otherwise.} \end{cases}$$

Further, we define $t_\phi, f_\phi : \mathbb{Z}_8 \to [0,1]$ by:

$$t_\psi(\bar{e}) = \begin{cases} 0.67, & \text{if } \bar{e} = \bar{0}, \\ 0.54, & \text{if } \bar{e} = \bar{2}, \\ 0.24, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_\phi(\bar{e}) = \begin{cases} 0.32, & \text{if } \bar{e} = \bar{0}, \\ 0.44, & \text{if } \bar{e} = \bar{2}, \\ 0.51, & \text{otherwise.} \end{cases}$$

Therefor, $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(\mathbb{Z}_8)$, but $\psi \cup \phi \notin \mathbf{VWII}(\mathbb{Z}_8)$, since $V_{(\psi\cup\phi)\Gamma(\psi\cup\phi)}(\bar{e}) > V_{\psi\cup\phi}(\bar{e})$ at $\bar{e} = \bar{4}$.

Theorem 3.7. *Let $\psi, \phi \in \mathbf{VWII}(E)$. Then $\psi \cup \phi \in \mathbf{VWII}(E)$ if $\psi \subseteq \phi$ or $\phi \subseteq \psi$.*

Proof. Assume $\psi = (t_\psi, f_\psi)$, $\phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$. Suppose $\psi \subseteq \phi$. Hence, $\psi \cup \phi \in \mathbf{V\Gamma}(E)$. Given $x \in E$, we have

$$\begin{aligned} V_{(\psi\cup\phi)\Gamma(\psi\Gamma\phi)}(x) &= \sup\{\min\{\sup\{\min\{V_{\psi\cup\ \phi}(a), V_{\psi\cup\phi}(b)\}\}, V_\delta(c)\} : x = a\alpha b\beta c, \\ &\quad \text{where } a, b, c \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{\sup\{\min\{V_\phi(a), V_\phi(b)\}\}, V_\delta(c)\} : a, b, c \in E\} \\ &= V_{\phi\Gamma\phi\Gamma\delta}(x) \\ &\leq V_\phi(x) \\ &= \max\{V_\psi(x), V_\phi(x)\} \\ &= V_{\psi\cup\phi}(x). \end{aligned}$$

Therefore, $(\psi \cup \phi)\Gamma(\psi \cup \phi)\Gamma\delta \subseteq \psi \cup \phi$. It follows that $\psi \cup \phi \in \mathbf{RVWII}(E)$. Similarly, $\psi \cup \phi \in \mathbf{LVWII}(E)$. Thus, $\psi \cup \phi \in \mathbf{VWII}(E)$. □

Theorem 3.8. *E is regular if and only if $\psi = \psi\Gamma\psi\Gamma\delta$, for all $\psi \in \mathbf{RVWII}(E)$.*

Proof. Suppose E is regular and let $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. Then $\psi\Gamma\psi\Gamma\delta \subseteq \psi$. Let $x \in E$. Then there exist $a \in E$ and $\alpha, \beta \in \Gamma$ s.t. $x = x\alpha a\beta x$, and so we have:

$$\begin{aligned} V_{\psi\Gamma\psi\Gamma\delta}(x) &= \sup\{\min\{V_\psi(x), V_{\psi\Gamma\delta}(a\beta x)\}\} \\ &= \sup\{\min\{V_\psi(x), \sup\{\min\{V_\psi(a), V_\delta(x)\}\} \\ &= \sup\{\min\{V_\psi(x), V_\psi(a)\}\} \\ &\geq V_\psi(x). \end{aligned}$$

Therefore, $\psi\Gamma\psi\Gamma\delta \supseteq \psi$, and so $\psi\Gamma\psi\Gamma\delta = \psi$.

Conversely, suppose $\psi\Gamma\psi\Gamma\delta = \psi$, and let $W \in \mathbf{RWII}(E)$. Using Theorem 3.3, we get $\delta_W \in \mathbf{RVWII}(E)$. It follows that $\delta_W\Gamma\delta_W\Gamma\delta = \delta_W$, and so $\delta_{W\Gamma W\Gamma E} = \delta_W$. Thus, $W = W\Gamma W\Gamma E$. Using [6, Theorem 4.4], we get E is regular. □

In the following example we show that, for given $\psi \in \mathbf{VWII}(E)$ and $\phi \in \mathbf{VS}(E)$ s.t. $\psi \subseteq \phi$, the extension property is not valid, i.e., maybe $\phi \notin \mathbf{VWII}(E)$.

Example 3.4. Let $E = \Gamma := \mathbb{R}$. Define $\cdot : \mathbb{R}^3 \to \mathbb{R}$ by $\cdot(a, b, c) = abc$ for all $a, b, c \in \mathbb{R}$. Then $\mathbb{R}$ is a $\mathbb{R}$-semiring. Define $t_\psi, f_\psi : \mathbb{R} \to [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.898, & \text{if } x \neq 0, \\ 0.532, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.241, & \text{if } x \neq 0, \\ 0.437, & \text{if } x = 0. \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{VWII}(\mathbb{R})$.

Now, if define $t_\phi, f_\phi : \mathbb{R} \to [0, 1]$ by:

$$t_\phi(x) = \begin{cases} 0.93, & \text{if } x \in 2\mathbb{Z},\ x \neq 0, \\ 0.85, & \text{if } x \in 2\mathbb{Z} + 1, \\ 0.66, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\phi(e) = \begin{cases} 0.13, & \text{if } x \in 2\mathbb{Z},\ x \neq 0, \\ 0.25, & \text{if } x \in 2\mathbb{Z} + 1, \\ 0.38, & \text{if } x = 0. \end{cases}$$

We can see that $\psi \subseteq \phi$, but $\phi = (t_\phi, f_\phi) \notin \mathbf{VWII}(\mathbb{R})$.

4. Normal Vague Weak Interior Ideals in Γ-Semirings

We define the notion of a (*complete-*) *normal vague weak interior ideal*, and show that we can construct it in a Γ-semiring. Additionally, we prove that the cardinal of a maximal element, which is not constant, in the set of all normal vague weak interior ideals of a Γ-semiring is 2.

Definition 4.1. Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$. Then ψ is called normal, if $V_\psi(0) = [1, 1]$ i.e., $t_\psi(0) = 1$ and $f_\psi(0) = 0$.

Denote by $\mathbf{NVS}(E)$ the set of all normal vague sets of E.

Example 4.1. Consider Example 3.4, and define $t_\psi, f_\psi : \mathbb{R} \to [0, 1]$ by:

$$t_\psi(x) = \begin{cases} 0.92, & \text{if } x \in \mathbb{R}^+, \\ 0.75, & \text{if } x \in \mathbb{R}^-, \\ 1, & \text{if } x = 0, \end{cases} \quad \text{and} \quad f_\psi(x) = \begin{cases} 0.13, & \text{if } x \in \mathbb{R}^+, \\ 0.24, & \text{if } x \in \mathbb{R}^-, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\psi = (t_\psi, f_\psi) \in \mathbf{NVS}(\mathbb{R})$.

The following theorem we achieve a necessity condition for a vague set to be normal vague set.

Theorem 4.1. *Let $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$ s.t. $t_\psi(e) + f_\psi(e) \leq t_\psi(0) + f_\psi(0)$ for all $e \in E$. Define $\psi^+ = (t_{\psi^+}, f_{\psi^+})$, where $t_{\psi^+}(e) = t_\psi(e) + 1 - t_\psi(0)$ and $f_{\psi^+}(e) = f_\psi(e) - f_\psi(0)$ for all $e \in E$. Then $\psi^+ \in \mathbf{NVS}(E)$.*

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{VS}(E)$ and $e \in E$. Then

$$t_{\psi^+}(e) + f_{\psi^+}(e) = t_\psi(e) + 1 - t_\psi(0) + f_\psi(e) - f_\psi(0) \leq 1.$$

Therefore, $\psi^+ \in \mathbf{VS}(E)$. Also, $t_{\psi^+}(0) = 1$ and $f_{\psi^+}(0) = 0$. Thus, $\psi^+ \in \mathbf{NVS}(E)$. □

Proposition 4.1. *Let $\psi, \phi \in \mathbf{VWII}(E)$. Then*

(i) $\psi^+ \in \mathbf{NVWII}(E)$;
(ii) $\psi \in \mathbf{NVWII}(E)$ *if and only if* $\psi^+ = \psi$;
(iii) $(\psi^+)^+ = \psi$;
(iv) *if exists* $\phi \in \mathbf{VWII}(E)$ *s.t.* $\phi^+ \subseteq \psi$, *then* $\psi \in \mathbf{NVWII}(E)$;
(v) *if exists* $\phi \in \mathbf{VWII}(E)$ *s.t.* $\phi^+ \subseteq \psi$, *then* $\psi^+ = \psi$;
(vi) $(\psi \cap \phi)^+ = \psi^+ \cap \phi^+$;
(vii) $(\psi \cup \phi)^+ = \psi^+ \cup \phi^+$;
(viii) $\psi \subseteq \phi$ *implies* $\psi^+ \subseteq \phi^+$.

Proof. (i) Suppose $\psi = (t_\psi, f_\psi) \in \mathbf{RVWII}(E)$. Given $e, f \in E$, $\gamma \in \Gamma$, we have

$$\begin{aligned} V_{\psi^+}(e+f) &= V_\psi(e+f) + [1,1] - V_\psi(0) \\ &\geq \min\{V_\psi(e), V_\psi(f)\} + [1,1] - V_\psi(0) \\ &= \min\{V_\psi(e) + [1,1] - V_\psi(0), V_\psi(f) + [1,1] - V_\psi(0)\} \\ &= \min\{V_{\psi^+}(e), V_{\psi^+}(f)\} \end{aligned}$$

and

$$\begin{aligned} V_{\psi^+}(e\gamma f) &= V_\psi(e\gamma f) + [1,1] - V_\psi(0) \\ &\geq \min\{V_\psi(e), V_\psi(f)\} + [1,1] - V_\psi(0) \\ &= \min\{V_\psi(e) + [1,1] - V_\psi(0), V_\psi(f) + [1,1] - V_\psi(0)\} \\ &= \min\{V_{\psi^+}(e), V_{\psi^+}(f)\}. \end{aligned}$$

Therefore, $\psi^+ \in \mathbf{V\Gamma}(E)$. Also, we have

$$\begin{aligned} V_{\psi^+\Gamma\psi^+\Gamma\delta}(e) &= \sup\{\min\{\sup\{\min\{V_{\psi^+}(f), V_{\psi^+}(g)\}\}, V_\delta(h)\} : e = f\alpha g\beta h, \\ &\quad \text{where } f, g, h \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{V_{\psi^+}(f), V_{\psi^+}(g)\}\} \\ &= \sup\{\min\{V_\psi(f) + [1,1] - V_\psi(0), V_\psi(g) + [1,1] - V_\psi(0)\}\} \\ &= \sup\{\min\{V_\psi(f), V_\psi(g)\}\} + [1,1] - V_\psi(0) \\ &= \sup\{\min\{\sup\{\min\{V_\psi(f), V_\psi(g)\}\}, V_\delta(h)\} + [1,1] - V_\psi(0) \\ &= V_{\psi\Gamma\psi\Gamma\delta}(e) + [1,1] - V_\psi(0) \\ &\leq V_\psi(e) + [1,1] - V_\psi(0) \\ &= V_{\psi^+}(e). \end{aligned}$$

Hence, $\psi^+ \in \mathbf{RVWII}(E)$. We can see that $V_{\psi^+}(0) = V_\psi(0) + [1,1] - V_\psi(0) = [1,1]$. Therefore, $\psi^+ \in \mathbf{NRVWII}(E)$. Similarly, $\psi^+ \in \mathbf{NLVWII}(E)$. It follows that $\psi^+ \in \mathbf{NVWII}(E)$. Clearly, $\psi \subseteq \psi^+$.

(ii) Assume $\psi = (t_\psi, f_\psi) \in \mathbf{NVWII}(E)$ and $e \in E$. Then

$$V_{\psi^+}(e) = V_\psi(e) + [1,1] - V_\psi(0) = V_\psi(e) + [1,1] - [1,1] = V_\psi(e).$$

Thus, $\psi^+ = \psi$. The converse is obvious.

(iii) Assume $e \in E$. $V_{(\psi^+)^+}(e) = V_{\psi^+}(e) + [1,1] - V_{\psi^+}(0) = V_{\psi^+}(e)$. Therefore, $(\psi^+)^+ = \psi^+$. Since $\psi \in \mathbf{NVWII}(E)$, using (ii) we get $(\psi^+)^+ = \psi^+ = \psi$.

(iv) Suppose there exists $\phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$, s.t. $\phi^+ \subseteq \psi$. Then, $[1,1] = V_{\phi^+}(0) \leq V_\psi(0)$. This shows that $V_\psi(0) = [1,1]$. Thus, $\psi \in \mathbf{NVWII}(E)$.

(v) The proof is clear by using (i) and (iv).

(vi) Suppose $\psi = (t_\psi, f_\psi), \phi = (t_\phi, f_\phi) \in \mathbf{VWII}(E)$ and $e \in E$. Then we have

$$\begin{aligned}
V_{(\psi\cap\phi)^+}(e) &= V_{\psi\cap\phi}(e) + [1,1] - V_{\psi\cap\phi}(0) \\
&= \min\{V_\psi(e), V_\phi(e)\} + [1,1] - \min\{V_\psi(0), V_\phi(0)\} \\
&= \min\{V_\psi(e) + [1,1] - V_\psi(0), V_\phi(e) + [1,1] - V_\phi(0)\} \\
&= \min\{V_{\psi^+}(e), V_{\phi^+}(e)\} \\
&= V_{\psi^+\cap\phi^+}(e).
\end{aligned}$$

Hence,$(\psi \cap \phi)^+ = \psi^+ \cap \phi^+$.

(vii) Let $e \in E$. Then we have

$$\begin{aligned}
V_{(\psi\cup\phi)^+}(e) &= V_{\psi\cup\phi}(e) + [1,1] - V_{\psi\cup\phi}(0) \\
&= \max\{V_\psi(e), V_\phi(e)\} + [1,1] - \max\{V_\psi(0), V_\phi(0)\} \\
&= \max\{V_\psi(e) + [1,1] - V_\psi(0), V_\phi(e) + [1,1] - V_\phi(0)\} \\
&= \max\{V_{\psi^+}(e), V_{\phi^+}(e)\} \\
&= V_{\psi^+\cup\phi^+}(e).
\end{aligned}$$

Then $(\psi \cup \phi)^+ = \psi^+ \cup \phi^+$.

(viii) Given $e \in E$, we get

$$V_{\psi^+}(e) = V_\psi(e) + [1,1] - V_\psi(0) \leq V_\phi(e) + [1,1] - V_\phi(0) = V_{\phi^+}(e).$$

Therefore, $\psi^+ \subseteq \phi^+$. □

Theorem 4.2. *Let ψ be a maximal element in $\mathbf{NVWII}(E)$, which is not constant. Then $V_\psi(x) \in \{[0,0],[1,1]\}$ for all $x \in E$.*

Proof. Assume $\psi = (t_\psi, f_\psi) \in \mathbf{NVWII}(E)$. Then $V_\psi(0) = [1,1]$. Let there exists $s \in E$ s.t. $V_\psi(s) \neq [1,1]$. It is sufficient to show that $V_\psi(s) = [0,0]$. Suppose there exists $e_0 \in E$ s.t. $[0,0] < V_\psi(e_0) < [1,1]$. Define a vague set $\phi = (t_\phi, f_\phi)$ of E by $t_\phi(e) = \frac{t_\psi(e)+t_\psi(e_0)}{2}$ and $f_\psi(e) = \frac{f_\psi(e)+f_\psi(e_0)}{2}$ for all $e \in E$. Clearly, ϕ is well-defined.

Given $e, f \in E; \gamma \in \Gamma$, we have

$$\begin{aligned} V_\phi(e+f) &= \frac{V_\psi(e+f) + V_\psi(e_0)}{2} \\ &\geq \frac{\min\{V_\psi(e), V_\psi(f)\} + V_\psi(e_0)}{2} \\ &= \min\left\{\frac{V_\psi(e) + V_\psi(e_0)}{2}, \frac{V_\psi(f) + V_\psi(e_0)}{2}\right\} \\ &= \min\{V_\phi(e), V_\phi(f)\} \end{aligned}$$

and

$$\begin{aligned} V_\phi(e\gamma f) &= \frac{V_\psi(e\gamma f) + V_\psi(e_0)}{2} \\ &\geq \frac{\min\{V_\psi(e), V_\psi(f)\} + V_\psi(e_0)}{2} \\ &= \min\left\{\frac{V_\psi(e) + V_\psi(e_0)}{2}, \frac{V_\psi(f) + V_\psi(e_0)}{2}\right\} \\ &= \min\{V_\phi(e), V_\phi(f)\}. \end{aligned}$$

Therefore, $\phi \in \mathbf{V\Gamma}(E)$. Also, we have

$$\begin{aligned} V_{\phi\Gamma\phi\Gamma\delta}(e) &= \sup\{\min\{\sup\{\min\{\phi(f), \phi(g)\}\}, V_\delta(h)\} : e = f\alpha g\beta h, \\ &\quad \text{where } f, g, h \in E, \alpha, \beta \in \Gamma\} \\ &= \sup\{\min\{V_\phi(f), V_\phi(g)\}\} \\ &= \sup\left\{\min\left\{\frac{V_\psi(f) + V_\psi(e_0)}{2}, \frac{V_\psi(g) + V_\psi(e_0)}{2}\right\}\right\} \\ &= \frac{1}{2}\sup\{\min\{V_\psi(f), V_\psi(g)\}\} + \frac{V_\psi(e_0)}{2} \\ &= \frac{1}{2}\sup\{\min\{\sup\{\min\{V_\psi(f), V\psi(g)\}\}, V_\delta(h)\}\} + \frac{V_\psi(e_0)}{2} \\ &= \frac{1}{2}V_{\psi\Gamma\psi\Gamma\delta}(e) + \frac{V_\psi(e_0)}{2} \\ &\leq \frac{1}{2}V_\psi(e) + \frac{V_\psi(e_0)}{2} \\ &= V_\phi(e). \end{aligned}$$

Hence $\phi \in \mathbf{RVWII}(E)$. By a similar way we can show that $\phi \in \mathbf{LVWII}(E)$. Thus, $\phi \in \mathbf{VWII}(E)$. Now, we have

$$\begin{aligned} V_{\phi^+}(e) &= V_\phi(e) + [1,1] - V_\phi(0) \\ &= \frac{V_\psi(e) + V_\psi(e_0)}{2} + [1,1] - \frac{V_\psi(0) + V_\psi(e_0)}{2} \\ &= \frac{V_\psi(e) + [1,1]}{2}. \end{aligned}$$

That implies $V_{\phi^+}(0) = \frac{V_\psi(0)+[1,1]}{2} = [1,1]$. Thus, $\phi^+ \in \mathbf{NVWII}(E)$. Now, $V_{\phi^+}(0) = [1,1] > V_\psi(e_0)$. This shows that ϕ^+ is not constant. Further, we have $V_{\phi^+}(e_0) > V_\psi(e_0)$, which is a contraction, since ψ is a maximal element. Hence $V_\psi(s) = [0,0]$. Therefore, $V_\psi(x) \in \{[0,0],[1,1]\}$. □

Corollary 4.1. *If ψ is a maximal element in $\mathbf{NVWII}(E)$, which isn't constant, then $|V_\psi(x)| = 2$.*

Definition 4.2. Let $\psi \in \mathbf{NVS}(E)$. Then ψ is called *complete* if there exists $e \in E$ s.t. $V_\psi(e) = [0,0]$.

Denote by $\mathbf{CNVS}(E)$ the set of all normal vague sets of E, resp., $\mathbf{CNVWII}(E)$ the set of all complete normal vague weak interior ideals of E. Then $\mathbf{CNVWII}(E) \subseteq \mathbf{CNVS}(E)$, and so $(\mathbf{CNVWII}(E), \subseteq)$ is a poset.

Example 4.2. Consider Example 3.3, and define $t_\psi, f_\psi : \mathbb{Z}_8 \to [0,1]$ by:

$$t_\psi(\bar{x}) = \begin{cases} 1, & \text{if } \bar{x} = \bar{0}, \\ 0.56, & \text{if } \bar{x} = \bar{1}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad f_\psi(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} = \bar{0}, \\ 0.45, & \text{if } \bar{x} = \bar{1}, \\ 1, & \text{otherwise}. \end{cases}$$

Hence, $\psi = (t_\psi, f_\psi) \in \mathbf{CNVWII}(E)$.

Theorem 4.3. *If ψ is a maximal element in $(\mathbf{NVWII}(E), \subseteq)$, which is not constant, then it is a maximal element in $(\mathbf{CNVWII}(E), \subseteq)$.*

Proof. Assume $\bar{\psi} = (t_{\bar{\psi}}, f_{\bar{\psi}})$ is a maximal element in $(\mathbf{NVWII}(E), \subseteq)$, which isn't constant. By Theorem 4.2, $V_{\bar{\psi}}(x) \in \{[0,0],[1,1]\}$ for all $x \in E$, i.e., $V_{\bar{\psi}}(0) = [1,1]$ and $V_{\bar{\psi}}(x) = [0,0]$ for some $x \in E$. It follows that $\bar{\psi} \in \mathbf{CNVWII}(E)$. Suppose $\bar{\phi} = (t_{\bar{\phi}}, f_{\bar{\phi}}) \in \mathbf{CNVWII}(E)$ s.t. $\bar{\psi} \subseteq \bar{\phi}$. Then $\bar{\phi} \in \mathbf{NVWII}(E)$. Since $\bar{\phi}$ is a maximal element in $\mathbf{NVWII}(E)$ and $\bar{\phi} \in \mathbf{NVWII}(E)$ with $\bar{\psi} \subseteq \bar{\phi}$, that gives $\bar{\psi} = \bar{\phi}$. Therefor, $\bar{\psi}$ is a maximal element in $\mathbf{CNVWII}(E)$. □

Conclusions and Comments

We defined the notion of a right (resp. left) vague weak interior ideal of a Γ-semiring and the characterization theorem for *regular* Γ*-semiring* in terms of vague weak interior ideals is derived. In addition, we introduced and studied (*complete-*) *normal* vague weak interior ideals of a Γ-semiring. As a consequence of the results is that the cardinal of a non-constant maximal element in the set of all (complete-) normal vague weak interior ideals is 2. As a direction of this research will be study on vague (minimal weak interior, bi-interior, quasi-interior) ideals of a Γ-semiring and investigate relations among these notions.

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