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ULTIMATE BOUNDEDNESS OF SOLUTIONS OF SOME SYSTEM OF THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper presents sufficient conditions for the ultimate boundedness of solutions of some system of third-order nonlinear differential equations

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

where Ψ, Φ are positive definite symmetric matrices, H, P are $n-$vectors continuous in their respective arguments, $X \in \mathbb{R}^n$ and $t \in \mathbb{R}^+ = [0, +\infty)$. We do not necessarily require $H(X)$ differentiable to obtain our results. By using the Lyapunov's direct (second) method and constructing a complete Lyapunov function, earlier results are generalized.

1. Introduction

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and let $\mathbb{R}^n$ denote the real Euclidean n-dimensional space furnished with the usual Euclidean norm denoted by $\|\cdot\|$. Consider the system of third-order nonlinear differential equations

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

where $t \in \mathbb{R}^+,\ X : \mathbb{R}^+ \to \mathbb{R}^n,\ H : \mathbb{R}^n \to \mathbb{R}^n,\ P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ \Psi, \Phi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are continuous in their respective arguments, H is assumed to be not necessarily differentiable and the dots indicate differentiation with respect to the independent variable t. Thus, for any initial values $X_0, Y_0, Z_0 \in \mathbb{R}^n$, there is a uniquely

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defined solution $X = X(t, X_0, Y_0, Z_0)$ of (1.1), continuous in t, X_0, Y_0, Z_0 satisfying the condition $X(t_0) = X_0$, $\dot{X}(t_0) = Y_0$, $\ddot{X}(t_0) = Z_0$ [21]. Equation (1.1) is the vector version for the systems of real third-order nonlinear differential equations of the form

$$\begin{aligned}&\dddot{x}_i + \sum_{k=1}^{n} \psi_{ik}(\dot{x}_1, \ldots, \dot{x}_n)\ddot{x}_k + \sum_{k=1}^{n} \phi_{ik}(x_1, \ldots, x_n)\dot{x}_k + h_i(x_1, \ldots, x_n)\\ =&p_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ddot{x}_1, \ldots, \ddot{x}_n),\end{aligned}$$

where $i = 1, \ldots, n$, in which the functions ψ_{ik}, ϕ_{ik}, h_i, p_i are continuous in their respective arguments. In the case $n = 1$, this system reduces to the scalar ordinary differential equations of the form

$$\dddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.2}$$

where ψ, ϕ, h and p are continuous in their respective arguments, see [4–6, 9, 10, 13, 16, 17, 19, 23–26, 28, 29, 34, 35] and the references cited therein. If $\psi(\dot{x}) = a$ and $\phi(x) = b$, (1.2) reduces to

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}),$$

which has been investigated by Ezeilo [9] for ultimate boundedness and convergence of solutions by assuming

$$\frac{h(\xi + \gamma) - h(\gamma)}{\xi} \in I_0, \quad \xi \neq 0, \tag{1.3}$$

with $I_0 \equiv [\delta, kab] \subset (0, ab)$ the generalized Routh-Hurwitz interval, $\delta > 0$ and $0 < k < 1$. When $\gamma = 0$ in (1.3) we have

$$H_0 = H_0(\xi) \equiv \frac{h(\xi + \gamma) - h(\gamma)}{\xi}$$

and

$$H_0 = \frac{h(\xi)}{\xi}, \quad \text{if} \;\; h(0) = 0.$$

On the other hand, if $\Psi(\dot{X}) = A, \Phi(X) = B$ in (1.1), we have

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.4}$$

where A, B are real symmetric $n \times n$ matrices. Equation (1.4) has been studied by Afuwape [1] and Meng [12] for the ultimate boundedness and periodicity of solutions for which H is of class $\mathbf{C}(\mathbb{R}^n)$, satisfying

$$H(X_2) = H(X_1) + C_h(X_1, X_2)(X_2 - X_1), \tag{1.5}$$

where $C_h(X_1, X_2)$ is a real $n \times n$ operator for any X_1, X_2 in $\mathbb{R}^n$, and having real eigenvalues $\lambda_i(C_h(X_1, X_2)),\ i = 1, 2, \ldots, n$. These eigenvalues satisfy

$$0 < \delta_c \leq \lambda_i(C_h(X_1, X_2)) \leq \Delta_c, \tag{1.6}$$

with δ_c, Δ_c as fixed constants. Further, the matrices A, B have real positive eigenvalues $\lambda_i(A)$ and $\lambda_i(B)$ respectively, satisfying

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a,$$

$$0 < \delta_b \leq \lambda_i(B) \leq \Delta_b,$$

$i = 1, 2, \ldots, n$, and that for some constant $k(< 1)$ the 'generalized' Routh-Hurwitz condition

$$\Delta_c \leq k\delta_a\delta_b \tag{1.7}$$

is satisfied.

In these papers mentioned above, the Lyapunov's direct method was used to obtain results. This entails construction of a quadratic-like function (also known as Lyapunov function) to obtain sufficient conditions which guarantee the properties of solutions, but the construction of this function is difficult since there is no general method to obtaining it ([1]-[35]). Perhaps, reason (1.1) has received no attention in literature.

The present work is concerned with the ultimate boundedness of solutions of (1.1) or its equivalent system form

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -\Psi(Y)Z - \Phi(X)Y - H(X) + P(t, X, Y, Z), \end{aligned} \tag{1.8}$$

obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$ in (1.1). This problem was left open by Ezeilo and Tejumola [7, page 284]. In this work, by using the Lyapunov's direct method and constructing a suitable complete Lyapunov function, we shall obtain sufficient conditions which guarantee the ultimate boundedness of solutions of (1.1).

2. Notation

Our notations are similar to [3]. In this paper, $\delta's$ and $\Delta's$ with or without suffixes represent positive constants whose magnitudes depend on the matrix functions Ψ, Φ, and the vector functions H, P. The $\delta's$ and $\Delta's$ with numerical or alphabetical suffixes shall retain fixed magnitudes while those without suffixes are not necessarily the same at each occurrence. Finally, $\langle X, Y\rangle$ shall represent the scalar product of any vectors $X, Y \in \mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X\rangle = \|X\|^2$.

3. Statement of Results

Our main result in this paper is the following.

Theorem 3.1. *Suppose* $H(0) = 0$, *and that*

(i) *there exists* $n \times n$ *real continuous operator* $C_h(X_1, X_2)$ *for any vectors* X_1, X_2 *such that the function* H *is of class* $\mathbf{C}(\mathbb{R}^n)$, *satisfy* (1.5), *with eigenvalues* $\lambda_i(C_h(X_1, X_2))$, $i = 1, 2, \ldots, n$, *satisfying* (1.6)*;*

(ii) *the matrix functions* $\Psi(Y)$, $\Phi(X)$ *are continuous in their respective arguments, with eigenvalues* $\lambda_i(\Psi(Y))$, $\lambda_i(\Phi(X))$ *satisfying*

$$0 < \delta_a \leq \lambda_i(\Psi(Y)) \leq \Delta_a, \tag{3.1}$$

$$0 < \delta_b \leq \lambda_i(\Phi(X)) \leq \Delta_b, \tag{3.2}$$

where $i = 1, 2, \ldots, n$;

(iii) *the matrices* Ψ, Φ *and the operator* C_h *are associative and commute pairwise; and*

(iv) *the vector function* $P(t, X, Y, Z)$ *satisfies*

$$\begin{aligned} \|P(t,X,Y,Z)\| \leq & \theta_1(t) + \theta_2(t)\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{\rho}{2}} \\ & + \delta_0\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

for any $X, Y, Z \in \mathbb{R}^n$, *where* $\delta_0 > 0$ *is a constant,* $\theta_1(t)$, $\theta_2(t)$ *are continuous functions in* t *and* $0 \leq \rho < 1$.

Then, there exist constants Δ_1, Δ_2, Δ_3 *such that every solution* $X(t)$ *of* (1.1) *with* $X(t_0) = X_0$, $\dot{X}(t_0) = Y_0$, $\ddot{X}(t_0) = Z_0$, *and for any constant* ν, *whatever in the range* $\frac{1}{2} \leq \nu \leq 1$, *the inequality*

$$\begin{aligned} \left(\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\right)^{\nu} \leq & \Delta_1 \exp\{-\Delta_2(t - t_0)\} \\ & + \Delta_3 \int_{t_0}^{t} \left(\theta_1^{2\nu}(\tau) + \theta_2^{\frac{2\nu}{1-\rho}}(\tau)\right) \\ & \times \exp\{-\Delta_2(t - \tau)\} d\tau \end{aligned} \tag{3.4}$$

holds for all $t \geq t_0$, *where* $\Delta_1 = \Delta_1(X_0, Y_0, Z_0)$.

A number of quite important results can be deduced from Theorem 3.1. For example, we have the following.

Corollary 3.1. *If* $P \equiv 0$ *and if all conditions of Theorem* 3.1 *hold, then every solution* $X(t)$ *of* (1.1) *satisfies*

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \to 0, \tag{3.5}$$

as $t \to +\infty$.

Indeed, by setting $\theta_1 = 0 = \theta_2$ in (3.4), we have that, if $\delta_0 \leq \Delta_0$, then

$$\left(\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\right)^{\nu} \leq \Delta_1 \exp\{-\Delta_2(t - t_0)\}, \quad t \geq t_0,$$

from which (3.5) follows on letting $t \to +\infty$.

Corollary 3.2. *Assume that all conditions of Theorem* 3.1 *hold and let* $\delta_0 \leq \Delta_0$. *Suppose also that there are fixed constants* η, $1 \leq \eta \leq 2$, *and* $\mu > 0$ *such that*

$$\int_{t_0}^{t+\mu} \left(\theta_1^{\eta}(\tau) + \theta_2^{\frac{\eta}{1-\rho}}(\tau)\right) \exp\{-\Delta_2(t - \tau)\} d\tau \to 0, \quad \text{as } t \to +\infty.$$

Then, every solution $X(t)$ *of* (1.1) *satisfies* (3.5).

Another interesting result which can be deduced very readily from Theorem 3.1 is the following generalization of the boundedness results in [1, 7] and [12].

Corollary 3.3. *Assume that all the conditions of Theorem* 3.1 *hold and let* $\delta_0 \leq \Delta_0$. *Suppose further that each of the functions* $\theta_1(t)$, $\theta_2(t)$ *satisfies at least one of the following conditions:*

(i) $\max_{0\leq t<+\infty} \theta_1(t) < +\infty$, $\max_{0\leq t<+\infty} \theta_2(t) < +\infty$;

(ii) $\int_{t_0}^{+\infty} \theta_1^{\eta}(t)dt < +\infty$, $\int_{t_0}^{+\infty} \theta_2^{\frac{\eta}{1-\rho}}(t)dt < +\infty$,

for some constant η *in the range* $1 \leq \eta \leq 2$. *Then there exists a constant* $\Delta_4 > 0$ *such that every solution* $X(t)$ *of* (1.1) *ultimately satisfies*

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \leq \Delta_4.$$

4. Preliminary Results

We need a few important results to prove Theorem 3.1.

Lemma 4.1. *Let* D *be a real symmetric* $n \times n$ *positive definite matrix. Then, for any* $X \in \mathbb{R}^n$,

$$\delta_d\|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d\|X\|^2, \tag{4.1}$$

where δ_d, Δ_d *are respectively the least and greatest eigenvalues of* D.

Proof. See [7]. □

Lemma 4.2. *Let* Q, D *be any two real* $n \times n$ *commuting symmetric matrices. Then the eigenvalues* $\lambda_i(QD)$, $i = 1, 2, \ldots, n$, *of the product matrix* QD *are all real and satisfy*

$$\min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D),$$

where $\lambda_j(Q)$ *and* $\lambda_k(D)$ *are respectively the eigenvalues of* Q *and* D.

Proof. See [7]. □

The main tool in the proof of Theorem 3.1 is the scalar function $W = W(X, Y, Z)$ defined for arbitrary vectors $X, Y, Z \in \mathbb{R}^n$ by

$$2W = \beta(1-\beta)\delta_b^2\|X\|^2+\delta_b(\beta+\alpha\delta_a^{-1})\|Y\|^2+\alpha\delta_a^{-1}\|Z\|^2+\|Z+\delta_aY+(1-\beta)\delta_bX\|^2, \tag{4.2}$$

where $\alpha > 0$ and $0 < \beta < 1$.

The following result is immediate from (4.2).

Lemma 4.3. *Assume that all the conditions on* $\Psi(Y)$, $\Phi(X)$ *and* $H(X)$ *in Theorem* 3.1 *are satisfied. Then there are constants* $\delta_i > 0$, $i = 1, 2$, *such that*

$$\delta_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \leq W(X, Y, Z) \leq \delta_2\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right), \tag{4.3}$$

for arbitrary $X, Y, Z \in \mathbb{R}^n$.

Proof. The proof of inequalities (4.3) follows if we use Lemma 4.1 (inequalities (4.1)) repeatedly and then choose

$$\delta_1 = \frac{1}{2}\min\left\{\beta(1-\beta)\delta_b^2, \delta_b(\beta+\alpha\delta_a^{-1}), \alpha\delta_a^{-1}\right\}$$

and

$$\delta_2 = \frac{1}{2}\max\left\{\mu_1, \mu_2, \mu_3\right\},$$

where $\mu_1 = \delta_b(1-\beta)(1+\delta_a+\delta_b)$, $\mu_2 = \delta_b(\beta+\alpha\delta_a^{-1}) + \delta_a[1+\delta_b(1-\beta)+\delta_a]$ and $\mu_3 = 1+\alpha\delta_a^{-1}+\delta_b(1-\beta)+\delta_a$. □

5. Proof of Theorem 3.1

To prove Theorem 3.1, it suffices to show that the function W (defined in (4.2)) satisfies for any solution $(X(t), Y(t), Z(t))$ of (1.8) and for any ν in the range $\frac{1}{2} \leq \nu \leq 1$,

$$\dot{W} \leq -\delta_3\psi^2 + \delta_4\left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t)\right)\psi^{2(1-\nu)}, \tag{5.1}$$

for some constants $\delta_i > 0,\ i = 3, 4$, where $\psi^2 = \|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2$. We note that from Lemma 4.3, (5.1) becomes

$$\dot{W} \leq -\delta_5 W + \delta_6\left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t)\right)W^{(1-\nu)},$$

with $\delta_5 = \delta_1\delta_3$ and $\delta_6 = \delta_2\delta_4$. If we choose $U = W^\nu$, this reduces to

$$\dot{U} \leq -\nu\delta_5 U + \nu\delta_6\left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t)\right),$$

which when solved for U yields

$$U(t) \leq U(t_0)\exp\{-\nu\delta_5(t-t_0)\} + \Delta_5\int_{t_0}^{t}\left(\theta_1^{2\nu}(\tau) + \theta_2^{\frac{2\nu}{1-\rho}}(\tau)\right)\exp\{-\nu\delta_5(t-\tau)\}d\tau,$$

for all $t \geq t_0$.

Rewriting this with $W^\nu = U$ and applying Lemma 4.3, we shall get (3.4) with

$$\Delta_1 = \delta\left(\|X(t_0)\|^2 + \|Y(t_0)\|^2 + \|Z(t_0)\|^2\right)^\nu, \quad \Delta_2 = \nu\delta_5 \text{ and } \Delta_3 = \delta\Delta_5.$$

It follows that the proof of Theorem 3.1 is complete as soon as inequality (5.1) is proved.

6. Derivative of W and Proof of (5.1)

Let $(X(t), Y(t), Z(t))$ be any solution of (1.8). The total derivative of W, with respect to t along the solution path after simplification is

$$\dot{W} = -U_1 - U_2 - U_3 - U_4 - U_5 - U_6 - U_7 + U_8, \tag{6.1}$$

where

$$\begin{aligned}
U_1 =&\eta_1\delta_b(1-\beta)\langle X, H(X)\rangle + \xi_1\delta_a\langle(\Phi(X)-(1-\beta)\delta_b I)Y, Y\rangle\\
&+\gamma_1\alpha\delta_a^{-1}\langle\Psi(Y)Z, Z\rangle + \langle(\Psi(Y)-\delta_a I)Z, Z\rangle,\\
U_2 =&\eta_2\delta_b(1-\beta)\langle X, H(X)\rangle + \gamma_2\alpha\delta_a^{-1}\langle\Psi(Y)Z, Z\rangle + (1+\alpha\delta_a^{-1})\langle Z, H(X)\rangle,\\
U_3 =&\eta_3\delta_b(1-\beta)\langle X, H(X)\rangle + \xi_2\delta_a\langle(\Phi(X)-(1-\beta)\delta_b I)Y, Y\rangle + \delta_a\langle Y, H(X)\rangle,\\
U_4 =&\eta_4\delta_b(1-\beta)\langle X, H(X)\rangle + \gamma_3\alpha\delta_a^{-1}\langle\Psi(Y)Z, Z\rangle\\
&+\delta_b(1-\beta)\langle(\Psi(Y)-\delta_a I)X, Z\rangle,\\
U_5 =&\eta_5\delta_b(1-\beta)\langle X, H(X)\rangle + \xi_3\delta_a\langle(\Phi(X)-(1-\beta)\delta_b I)Y, Y\rangle\\
&+\delta_b(1-\beta)\langle(\Phi(X)-\delta_b I)X, Y\rangle,\\
U_6 =&\gamma_4\alpha\delta_a^{-1}\langle\Psi(Y)Z, Z\rangle + \xi_4\delta_a\langle(\Phi(X)-(1-\beta)\delta_b I)Y, Y\rangle\\
&+(1+\alpha\delta_a^{-1})\langle(\Phi(X)-\delta_b I)Y, Z\rangle,\\
U_7 =&\gamma_5\alpha\delta_a^{-1}\langle\Psi(Y)Z, Z\rangle + \xi_5\delta_a\langle(\Phi(X)-(1-\beta)\delta_b I)Y, Y\rangle\\
&+\delta_a\langle(\Psi(Y)-\delta_a I)Y, Z\rangle,\\
U_8 =&\langle\delta_b(1-\beta)X + \delta_a Y + (1+\alpha\delta_a^{-1})Z, P(t, X, Y, Z)\rangle,
\end{aligned}$$

with $\eta_i, \xi_i, \gamma_i,\ i = 1, 2, 3, 4, 5,$ positive constants such that

$$\sum_{i=1}^{5}\eta_i = 1, \quad \sum_{i=1}^{5}\xi_i = 1 \quad \text{and} \quad \sum_{i=1}^{5}\gamma_i = 1.$$

To arrive at (5.1), we prove the following.

Lemma 6.1. *Subject to a conveniently chosen value of k in* (1.7)*, we have*

$$U_j \geq 0, \quad j = 2, 3, 4, 5, 6, 7,$$

for all $X, Y, Z \in \mathbb{R}^n$.

Proof. For some constants $k_i > 0,\ i = 1, 2,$ conveniently chosen later, we have

$$\begin{aligned}
\langle(1+\alpha\delta_a^{-1})Z, H(X)\rangle =&\|k_1(1+\alpha\delta_a^{-1})^{\frac{1}{2}}Z + 2^{-1}k_1^{-1}(1+\alpha\delta_a^{-1})^{\frac{1}{2}}H(X)\|^2\\
&-\langle k_1^2(1+\alpha\delta_a^{-1})Z, Z\rangle\\
&-\langle 4^{-1}k_1^{-2}(1+\alpha\delta_a^{-1})H(X), H(X)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle\delta_a Y, H(X)\rangle =&\|k_2\delta_a^{\frac{1}{2}}Y + 2^{-1}k_2^{-1}\delta_a^{\frac{1}{2}}H(X)\|^2\\
&-\langle k_2^2\delta_a Y, Y\rangle - \langle 4^{-1}k_2^{-2}\delta_a H(X), H(X)\rangle.
\end{aligned}$$

On using the assumption that $H(0) = 0$ and the hypothesis (1.5), it follows that

$$\begin{aligned} U_2 =& \|k_1(1+\alpha\delta_a^{-1})^{\frac{1}{2}}Z + 2^{-1}k_1^{-1}(1+\alpha\delta_a^{-1})^{\frac{1}{2}}C_h(X,0)X\|^2 \\ &+ \langle Z, (\gamma_2\alpha\delta_a^{-1}\Psi(Y) - k_1^2(1+\alpha\delta_a^{-1}))Z\rangle \\ &+ \langle C_h(X,0)X, (\eta_2\delta_b(1-\beta) - 4^{-1}k_1^{-1}(1+\alpha\delta_a^{-1})C_h(X,0))X\rangle \end{aligned}$$

and

$$\begin{aligned} U_3 =& \|k_2\delta_a^{\frac{1}{2}}Y + 2^{-1}k_2^{-1}\delta_a^{\frac{1}{2}}C_h(X,0)X\|^2 \\ &+ \langle Y, (\xi_2\delta_a[\Phi(X) - (1-\beta)\delta_b I] - k_2^2\delta_a I)Y\rangle \\ &+ \langle C_h(X,0)X, (\eta_3\delta_b(1-\beta) - 4^{-1}k_1^{-1}\delta_a C_h(X,0))X\rangle. \end{aligned}$$

Thus, using (1.6), (3.1), (3.2) and Lemma 4.1 repeatedly, we obtain for all $X, Z \in \mathbb{R}^n$,

$$U_2 \geq 0,$$

if

$$k_1^2 \leq \frac{\gamma_2\alpha\delta_a}{\alpha+\delta_a}, \quad \text{with } \Delta_c \leq \frac{4\alpha(1-\beta)\eta_2\gamma_2\delta_a^2\delta_b}{(\alpha+\delta_a)^2},$$

and, for all $X, Y \in \mathbb{R}^n$, $U_3 \geq 0$, if

$$k_2^2 \leq \beta\xi_2\delta_b, \quad \text{with } \Delta_c \leq \frac{4\beta(1-\beta)\eta_2\eta_3\delta_b^2}{\delta_a}.$$

Hence, combining these inequalities (with Δ_c), we have, for all $X, Y, Z \in \mathbb{R}^n$,

$$U_i \geq 0, \quad i = 2, 3, \text{ if } \Delta_c \leq k\delta_a\delta_b,$$

with

$$k = \min\left\{\frac{4\alpha(1-\beta)\eta_2\gamma_2\delta_a}{(\alpha+\delta_a)^2}, \frac{4\beta(1-\beta)\eta_2\eta_3\delta_b}{\delta_a^2}\right\} < 1.$$

To complete the proof of Lemma 6.1, we need to show that

$$U_i \geq 0, \quad i = 4, 5, 6, 7,$$

for all $X, Y, Z \in \mathbb{R}^n$. By (1.5), the assumption that $H(0) = 0$ and for constants $k_j > 0$, $j = 3, 4, 5, 6$, conveniently chosen later, we have

$$\begin{aligned} &\langle \delta_b(1-\beta)X, (\Psi(Y) - \delta_a I)Z\rangle \\ =& \|k_3\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}X + 2^{-1}k_3^{-1}\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Z\|^2 \\ &- \langle k_3^2\delta_b(1-\beta)(\Psi(Y)-\delta_a I)X, X\rangle - \langle 4^{-1}k_3^{-2}\delta_b(1-\beta)(\Psi(Y)-\delta_a I)Z, Z\rangle, \end{aligned}$$

$$
\begin{aligned}
&\langle \delta_b(1-\beta)(\Phi(X)-\delta_b I)X, Y\rangle \\
=&\|k_4\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}X + 2^{-1}k_4^{-1}\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Y\|^2 \\
&-\langle k_4^2\delta_b(1-\beta)(\Phi(X)-\delta_b I)X, X\rangle - \langle 4^{-1}k_4^{-2}\delta_b(1-\beta)(\Phi(X)-\delta_b I)Y, Y\rangle, \\
&\langle (1+\alpha\delta_a^{-1})(\Phi(X)-\delta_b I)Y, Z\rangle \\
=&\|k_5(1+\alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Y + 2^{-1}k_5^{-1}(1+\alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Z\|^2 \\
&-\langle k_5^2(1+\alpha\delta_a^{-1})(\Phi(X)-\delta_b I)Y, Y\rangle - \langle 4^{-1}k_5^{-2}(1+\alpha\delta_a^{-1})(\Phi(X)-\delta_b I)Z, Z\rangle, \\
&\langle \delta_a(\Psi(Y)-\delta_a I)Y, Z\rangle \\
=&\|k_6\delta_a^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Y + 2^{-1}k_6^{-1}\delta_a^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Z\|^2 \\
&-\langle k_6^2\delta_a(\Psi(Y)-\delta_a I)Y, Y\rangle - \langle 4^{-1}k_6^{-2}\delta_a(\Psi(Y)-\delta_a I)Z, Z\rangle.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
U_4 =&\|k_3\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}X + 2^{-1}k_3^{-1}\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Z\|^2 \\
&+\left\langle X, \left(\eta_4\delta_b(1-\beta)C_h(X,0) - k_3^2\delta_b(1-\beta)(\Psi(Y)-\delta_a I)\right)X\right\rangle \\
&+\left\langle Z, \left(\alpha\gamma_3\delta_a^{-1}\Psi(Y) - 4^{-1}k_3^{-2}\delta_b(1-\beta)(\Psi(Y)-\delta_a I)\right)Z\right\rangle, \\
U_5 =&\|k_4\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}X + 2^{-1}k_4^{-1}\delta_b^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Y\|^2 \\
&+\left\langle X, \left(\eta_5\delta_b(1-\beta)C_h(X,0) - k_4^2\delta_b(1-\beta)(\Phi(Y)-\delta_b I)\right)X\right\rangle \\
&+\left\langle Y, \left(\xi_3\delta_a[\Phi(X)-(1-\beta)\delta_b I] - 4^{-1}k_4^{-2}\delta_b(1-\beta)(\Phi(X)-\delta_b I)\right)Y\right\rangle, \\
U_6 =&\|k_5(1+\alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Y + 2^{-1}k_5^{-1}(1+\alpha\delta_a^{-1})^{\frac{1}{2}}(\Phi(X)-\delta_b I)^{\frac{1}{2}}Z\|^2 \\
&+\left\langle Y, \left(\xi_4\delta_a[\Phi(X)-(1-\beta)\delta_b I] - k_5^2(1+\alpha\delta_a^{-1})(\Phi(X)-\delta_b I)\right)Y\right\rangle \\
&+\left\langle Z, \left(\alpha\gamma_4\delta_a^{-1}\Psi(Y) - 4^{-1}k_5^{-2}(1+\alpha\delta_a^{-1})(\Phi(X)-\delta_b I)\right)Z\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
U_7 =&\|k_6\delta_a^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Y + 2^{-1}k_6^{-1}\delta_a^{\frac{1}{2}}(\Psi(Y)-\delta_a I)^{\frac{1}{2}}Z\|^2 \\
&+\left\langle Y, \left(\xi_5\delta_a[\Phi(X)-(1-\beta)\delta_b I] - k_6^2\delta_a(\Psi(Y)-\delta_a I)\right)Y\right\rangle \\
&+\left\langle Z, \left(\alpha\gamma_5\delta_a^{-1}\Psi(Y) - 4^{-1}k_6^{-2}\delta_a(\Psi(Y)-\delta_a I)\right)Z\right\rangle.
\end{aligned}
$$

We then obtain the following using the estimates (1.6), (3.1), (3.2) and Lemma 4.1 repeatedly. For all $X, Z \in \mathbb{R}^n$,

$$U_4 \geq 0, \quad \text{if} \quad \frac{(1-\beta)\delta_b(\Delta_a-\delta_a)}{4\alpha\eta_3} \leq k_3^2 \leq \frac{\eta_4\delta_c}{\Delta_a-\delta_a}.$$

For all $X, Y \in \mathbb{R}^n$,

$$U_5 \geq 0, \quad \text{if} \quad \frac{(1-\beta)(\Delta_b-\delta_b)}{4\beta\xi_3\delta_a} \leq k_4^2 \leq \frac{\eta_5\delta_c}{\Delta_b-\delta_b}.$$

For all $Y, Z \in \mathbb{R}^n$,

$$U_6 \geq 0, \quad \text{if} \quad \frac{(1+\alpha\delta_a^{-1})(\Delta_b - \delta_b)}{4\alpha\gamma_4} \leq k_5^2 \leq \frac{\beta\xi_4\delta_a\delta_b}{(1+\alpha\delta_a^{-1})(\Delta_b - \delta_b)}.$$

For all $Y, Z \in \mathbb{R}^n$,

$$U_7 \geq 0, \quad \text{if} \quad \frac{\delta_a(\Delta_a - \delta_a)}{4\alpha\gamma_5} \leq k_6^2 \leq \frac{\beta\xi_5\delta_a\delta_b}{\delta_a(\Delta_a - \delta_a)}.$$

The proof of Lemma 6.1 is now complete. □

We are now left with the estimates U_1 and U_8.

From (6.1), we clearly have

$$\begin{aligned} U_1 &\geq (1-\beta)\eta_1\delta_b\delta_c\|X\|^2 + \beta\xi_1\delta_a\delta_b\|Y\|^2 + \alpha\gamma_1\|Z\|^2 \\ &\geq \delta_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \end{aligned} \tag{6.2}$$

where $\delta_7 = \min\{(1-\beta)\eta_1\delta_b\delta_c, \beta\xi_1\delta_a\delta_b, \alpha\gamma_1\}$.

For the remaining part of the proof of (5.1), let us for convenience denote $\|X\|^2 + \|Y\|^2 + \|Z\|^2$ by ψ^2. Since $P(t, X, Y, Z)$ satisfies (3.3), Schwarz's inequality gives U_8,

$$\begin{aligned} |U_8| &\leq \left((1-\beta)\delta_b\|X\| + (1+\alpha\delta_a^{-1})\|Z\| + \delta_a\|Y\|\right)\|P(t,X,Y,Z)\| \\ &\leq \sqrt{3}\delta_8\left(\delta_0\psi^2 + \theta_2(t)\psi^{1+\rho} + \theta_1(t)\psi\right), \end{aligned} \tag{6.3}$$

where $\delta_8 = \max\{(1-\beta)\delta_b, \delta_a, 1+\alpha\delta_a^{-1}\}$.

Now, combining (6.1) with inequalities (6.2), (6.3), we obtain

$$\dot{W} \leq -(\delta_7 - \sqrt{3}\delta_8\delta_0)\psi^2 + \sqrt{3}\delta_8\left(\theta_2(t)\psi^{1+\rho} + \theta_1(t)\psi\right).$$

This we can rewrite as

$$\dot{W} \leq -\delta_9\psi^2 + \psi_1 + \psi_2, \tag{6.4}$$

where

$$3\delta_9 = \delta_7 - \sqrt{3}\delta_8\delta_0, \quad \psi_1 = \{\delta_{10}\theta_1(t) - \delta_9\psi\}\psi$$

and

$$\psi_2 = \delta_{10}\theta_2(t)\psi^{1+\rho} - \delta_9\psi^2.$$

If we choose δ_0 small enough such that $\delta_9 > 0$ (following [7, page 306]), with the necessary modification, we obtain

$$\psi_1 \leq \delta_{10}\psi^{2(1-\nu)}\theta_1^{2\nu}(t)$$

and

$$\psi_2 \leq \delta_{11}\psi^{2(1-\nu)}\theta_2^{\frac{2\nu}{1-\rho}}(t),$$

for any constant ν in the range $\frac{1}{2} \leq \nu \leq 1$.

Thus, (6.4) reduces to

$$\dot{W} \leq -\delta_9\psi^2 + \delta_{12}\left(\theta_1^{2\nu}(t) + \theta_2^{\frac{2\nu}{1-\rho}}(t)\right)\psi^{2(1-\nu)},$$

with $\delta_{12} = \max\{\delta_{10}, \delta_{11}\}$.

This is (5.1) with $\delta_3 = \delta_9$ and $\delta_4 = \delta_{12}$.
This completes the proof of Theorem 3.1.

7. Example

Consider (1.1) of the form

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + \Phi(X)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in \mathbb{R}^2, \tag{7.1}$$

with

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \Psi(\dot{X}) = \begin{pmatrix} 3 + \frac{1}{1+\dot{x_1}^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi(X) = \begin{pmatrix} 0.00004 + \frac{1}{1+x_1^2} & 0 \\ 0 & 1 \end{pmatrix},$$

$$H(X) = \begin{pmatrix} 0.001 \tan^{-1} x_1 + 0.0001x_1 \\ 0.0001x_2 \end{pmatrix}, \quad P(t) = \begin{pmatrix} e^{-t} \\ \sin t \end{pmatrix},$$

where e^{-t}, $\sin t$ are bounded continuous functions on $[0, +\infty)$. A simple calculation (with the earlier notations) gives $\lambda_1(\Psi(\dot{X})) = 1$, $\lambda_2(\Psi(\dot{X})) = 3 + \frac{1}{1+\dot{x_1}^2}$, $\lambda_1(\Phi(X)) = 1$, $\lambda_2(\Phi(X)) = 0.00004 + \frac{1}{1+x_1^2}$ and $C_h(X, 0) = \begin{pmatrix} 0.0001 + \frac{0.0001}{1+x_1^2} & 0 \\ 0 & 0.0001 \end{pmatrix}$, $\lambda_1(C_h(X, 0)) = 0.0001$, $\lambda_2(C_h(X, 0)) = 0.0001 + \frac{0.0001}{1+x_1^2}$. Following Theorem 3.1, $\delta_a = 1$, $\Delta_a = 3$, $\delta_b = 1$, $\Delta_b = 1.00004$, $\delta_c = 0.0001$, $\Delta_c = 0.0011$. If we choose $\alpha = 3$, $\beta = \frac{1}{2}$, $\gamma_3 = \eta_2 = \eta_3 = \frac{1}{5}$, we obtain $k = \min\{0.015, 0.04\} = 0.015 < 1$. Since $\Delta_c = 0.0011 < 0.015 = k\delta_a\delta_b$, then all the conditions of Theorem 3.1 are satisfied. Thus the solutions of (7.1) are ultimately bounded.

8. Conclusion

This paper investigates the ultimate boundedness of solutions of some third-order nonlinear differential equations. By constructing a quadratic-like function (also known as Lyapunov function) and using the Lyapunov second (direct) method, sufficient conditions which guarantee that solutions are ultimately bounded are established. A particular example has been provided to demonstrate results obtained. Results obtained in this paper revise and improve on those in the literature.

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