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RECONSTRUCTING THE CHARACTERISTIC (PERMANENTAL) POLYNOMIAL OF A DIGRAPH FROM SIMILAR POLYNOMIALS OF ITS ARC-DELETED SUBGRAPHS

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Abstract. Let $D = D(V, E)$ be an arbitrary digraph with the set V of vertices and the set E of arcs ($|V| = n; |E| = m$); loops, if any, are considered reduced arcs with the same head and tail. The *characteristic polynomial* $\phi^-(D; x)$ (*resp. permanental polynomial* (ϕ^+)) *of* D is the characteristic (permanental) polynomial of its adjacency matrix A: $\phi(D; x) := \det(xI - A)$ ($\phi^+(D; x) := \mathrm{per}(xI + A)$), where I is an identity matrix. A t-arcs-deleted subgraph D_t of D is the digraph D less exactly t arcs (while all n vertices are preserved). Also, let $\mathcal{D}_t$ and $R_t^-(D; x)$ $\left(R_t^+(D; x)\right)$ be the collection (multiset) of all t-arc-deleted subgraphs of D and the sum of the characteristic (permanental) polynomials of all subgraphs from $\mathcal{D}_t$, respectively. We consider the reconstruction of the characteristic polynomial $\phi^-(D; x)$ (permanental polynomial $\phi^+(D; x)$) of D from the polynomial sum $R_t^-(D; x)$ $\left(R_t^+(D; x)\right)$, $t \in \{1, 2, \ldots, m - n + n_0\}$, where n_0 is the number of zero roots of $\phi^-(D; x)$ ($\phi^+(D; x)$). Then, we also carry over our reasoning to the case of reconstructing both polynomials of undirected graphs (where edges are deleted).

1. Preliminaries and the Main Part

The multifaceted topic of reconstructing graph polynomials has long attracted the attention of graphists. It complements the topic of reconstructing the graphs themselves and, probably, in some cases, can help to carry out such a reconstruction. Here, we will not consider the general state of that issue, which we leave for a separate literature review; the interested reader can find quite voluminous information on the topic in [1–9]. We will fully devote ourselves to considering a specific problem,

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referring only to the information directly necessary to solve it. Each square matrix $A = [a_{rs}]_{r,s=1}^{n}$ is the adjacency matrix of a (weighted) finite (di)graph G, where an entry a_{rs} $(r, s \in \{1, 2, \ldots, n\})$ is the weight of an arc $\overrightarrow{rs}$ emanating from vertex r and heading to vertex s. The case $r = s$ corresponds to a loop (a reduced arc whose endpoints coincide); and $a_{rs} = 0$ means that the respective arc does not exist in G. If A is symmetric, with $a_{rs} = a_{sr}$ for all pairs of indices r and s, G can be regarded also as an undirected graph having nonoriented edges $rs = sr$ instead of pairs of opposite arcs $\overrightarrow{rs}$ and $\overleftarrow{sr}$. The case when all nonzero entries of A are equal to 1 corresponds to an unweighted graph G. In general, an entry a_{rs} may be an arbitrary quantity (complex number, algebraic expression, etc.). Herein, we plan to practically consider clearly recognizable cases of graphs or digraphs. In doing so, we will use, without any indication, information that is equally relevant for all graphs in our text.

A vertex-deleted subgraph of a graph G is a subgraph $G-v$ obtained by deleting the vertex v and all edges incident to it from G; similarly, an edge-deleted (arc-deleted) subgraph of G is a subgraph $G - uv$ $(G - \overrightarrow{uv})$ obtained by deleting the edge uv (arc $\overrightarrow{uv}$), while its end vertices and all other edges (arcs) incident to it are preserved as they are and were in G. This deletion of uv $(\overrightarrow{uv})$ is sometimes called weak deletion of the edge uv (arc $\overrightarrow{uv}$); while the deletion of uv $(\overrightarrow{uv})$ with end vertices and all the edges incident to it is called strong deletion of the edge uv (arc $\overrightarrow{uv}$). In the latter case, the resulting subgraph, denoted by $G - u - v$, is the graph G less the pair u and v of its adjacent vertices $(u \sim v)$ and all edges (arcs) incident to them. The collection (in general, a multiset) of all subgraphs $G - v$ $(G - \overrightarrow{uv}, G - uv, G - u - v)$ of the graph G is called a deck and any single element of the deck is called a card. We refer to these four decks as $\mathcal{D}$, $\mathcal{AD}$, $\mathcal{WD}$, and $\mathcal{SD}$, respectively. In our text, we will need two more general decks: $\mathcal{D}_t$ of subgraphs with t $(t = 0, 1, \ldots)$ vertices deleted and $\mathcal{AD}_t$ subgraphs with t arcs deleted $(\mathcal{D}_0 = \mathcal{D}; \mathcal{AD}_0 = \mathcal{AD})$.

The characteristic polynomial $\phi^-(G; x)$ (resp. permanental polynomial $\phi^+(G; x)$) of a (di)graph G is the characteristic polynomial $\phi^-(A; x)$ (permanental polynomial $\phi^+(G; x)$) of its adjacency matrix $A = [a_{u,v}]_{u,v=1}^{n}$ [10]:

(1.1)

$$\phi^-(G; x) := \phi^-(A; x) = \det(xI - A) = \sum_{k=0}^{n} c_k^- x^{n-k} = \sum_{k=0}^{n-n_0} c_k^- x^{n-k} \quad (c_0^- = 1),$$

$$\phi^+(G; x) := \phi^+(A; x) = \mathrm{per}(xI + A) = \sum_{k=0}^{n} c_k^+ x^{n-k} = \sum_{k=0}^{n-n_0} c_k^+ x^{n-k} \quad (c_0^+ = 1),$$

where I is a diagonal identity matrix of the corresponding dimension; and n_0 is the number of zero roots of $\phi^-(A; x)$ or $\phi^+(A; x)$, respectively.

In what follows, we will use the combined notation $\phi^{\pm}(D; x)$ (and other $(\pm)$-notation) wherever it is equally applicable both to the considered characteristic and to permanent polynomials. Hereby we mean a common form of notation, but not equality of results for the corresponding $(-)$ and $(+)$ cases.

In this paper, we demonstrate that the polynomial $\phi^{\pm}(D;x)$ of an arbitrary digraph D (with $m \geq n - n_0^{\pm}$, where $n_0^{\pm}$ is the number of zero roots of $\phi^{\pm}(D;x)$) is reconstructible from the following polynomial:

$$R_t^{\pm}(D;x) = \sum_{D_t \in \mathcal{AD}_t(D)} \phi^{\pm}(D_t;x) = \sum_{k=0}^{n} r_{t;k}^{\pm} x^{n-k} \quad (0 \leq t \leq m - n + n_0^{\pm}),$$

where $D_t := D - \overrightarrow{u_1v_1} - \overrightarrow{u_2v_2} - \cdots - \overrightarrow{u_tv_t}$ is an arbitrary subgraph of D obtained by weakly deleting its t arbitrary arcs; and the sum ranges over all deck $\mathcal{AD}_t(D)$ of D.

In order not to leave undirected graphs aside from our reasoning, we will introduce the following terminology. A symmetric digraph $S(G)$ of an undirected graph G (having n vertices and q edges) is obtained by substituting a pair of opposite arcs for every edge in G. We define $B = [b_{\alpha\beta}]_{\alpha,\beta=1}^{2q}$ to be the $2q \times 2q$ matrix with row and columns indexed by the set of arcs of $S(G)$ as follows:

$$b_{\alpha\beta} = b\ (\alpha = (u,v); \beta = (x,y)) = \begin{cases} 1, & v = x \text{ and either } y = u \text{ or } y \neq u, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The matrix B is the adjacency matrix of a derivative digraph $\Gamma(G) = \Gamma[S(G)]$, whose vertex set is the set of all $2q$ arcs of the symmetric digraph $S(G)$, while the adjacency of vertices is defined by (1.2). The digraph $\Gamma(G)$ is called in [11, 12] the line graph of a directed graph $S(G)$ and is called the arc-graph of (undirected) graph G in [13]. In the latter case, the prefix (or adjective) "arc" makes it possible to directly connect this term with the original undirected graph G, without referring to the auxiliary digraph $S(G)$. In what follows, $D = S(G)$ will be automatically considered by us as a special case of an arbitrary digraph D (with $m = 2q$ arcs).

An interesting spectral result concerning the arc-graph Γ of a digraph D is the following theorem [11–13].

Theorem 1.1. *Let $\phi^{-}(\Gamma;x)$ be the characteristic polynomial of the arc-graph $\Gamma(G)$ of a digraph D. Then,*

$$\phi^{-}(\Gamma;x) = x^{m-n}\phi^{-}(D;x) = \sum_{k=0}^{n} c_k^{-} x^{m-k}, \tag{1.3}$$

where n is the number of vertices, and m is the number of arcs of a digraph D (loops, if any, are also considered reduced arcs).

Remark 1.1. The general version of this theorem (see [11–13]) for the characteristic polynomials remains true for an arbitrary (di)graph H instead of D, possibly with (weighted) loops and (weighted) arcs or edges (having an arbitrary matrix M as its adjacency matrix A in (1.1)). However, Theorem 1.1 cannot be generalized to the case of the permanental polynomials. It is easy to consider the case $D = S(G)$, where G is an undirected graph with $n > 2$ vertices. Then, $m(D) > n(D)$ and $\phi^{-}[\Gamma(D);x]$ is divisible by x^{m-n}, i.e., has at least $m - n$ zero roots. However, unlike the previous case, $\phi^{+}[\Gamma(D);x]$ has no zero roots; see Proposition 6 of [13], taking into account that $S(G)$ is an Eulerian digraph.

We also note an important feature of the structure $\mathit{\Gamma}(D)$, which allows us to reconstruct the original digraph D using the adjacency matrix B of $\mathit{\Gamma}(D)$. If we enumerate all the arcs of the digraph D in such a way that the numbers of arcs entering one common vertex of D follow one after another, then we get the matrix B divided into blocks. These blocks are either blocks of all zeros or contain exactly one column of all ones. Further, if we replace each zero block by the number zero, and each block containing ones by the number one, then we get a matrix that is exactly the adjacency matrix of the original digraph graph D; see [13]. Here, we note that in the case $D = \mathcal{S}(G)$, the adjacency matrix of the digraph D coincides with that of an undirected graph G $(A(D) = A(G))$. Thus, this algorithm also reconstructs (the adjacency matrix of) G.

The one-to-one correspondence between each digraph and its arc-graph also allows us to consider the arc-graph $\mathit{\Gamma}(D)$ as the result of the action of the operator $\mathit{\Gamma}$ on the digraph D, which uniquely maps D to $\mathit{\Gamma}$. But we also know the algorithm for converting $\mathit{\Gamma}(D)$ back to D, which we can conventionally denote by $\mathit{\Gamma}^{-1}$. Thus, we can summarize what was said like this:

$$D \underset{\mathit{\Gamma}^{-1}}{\overset{\mathit{\Gamma}}{\rightleftarrows}} \mathit{\Gamma}(D).$$

The above correspondence is valid for an arbitrary digraph D, but we will be especially interested here in its particular case:

$$(D - \overrightarrow{u_1v_1} - \overrightarrow{u_2v_2} \cdots - \overrightarrow{u_tv_t}) \underset{\mathit{\Gamma}^{-1}}{\overset{\mathit{\Gamma}}{\rightleftarrows}} \mathit{\Gamma}(D) - \alpha_1 - \alpha_2 - \cdots - \alpha_t, \tag{1.4}$$

where a vertex α_i removed from the arc-graph $\mathit{\Gamma}(D)$ is an arc $\overrightarrow{u_iv_i}$ of the digraph D $(i \in \{1, 2, \ldots, t\})$.

From what has been said, we pass to the following technical lemma.

Lemma 1.1. *Let $\mathit{\Gamma}(D)$ be the arc-graph of a digraph D. Then,*

$$\phi^-(\mathit{\Gamma} - \alpha_1 - \alpha_2 - \cdots - \alpha_t; x) = x^{m-n-t}\phi^-(D - \overrightarrow{u_1v_1} - \overrightarrow{u_2v_2} - \cdots - \overrightarrow{u_tv_t}; x),$$

where $\alpha_i = \overrightarrow{u_iv_i}$, $\alpha_i \in V[\mathit{\Gamma}(D)]$ and $\overrightarrow{u_iv_i} \in E(D)$.

Proof. It follows from Theorem 1.1 (see (1.3)) and the correspondence (1.4). $\square$

Lemma 1.1 allows us to calculate the following polynomial sum:

$$\begin{aligned} S_t^-[\mathit{\Gamma}(D); x] &= \sum_{[\mathit{\Gamma}(D)]_{t\cdot} \in \mathcal{D}_t[\mathit{\Gamma}(D)]} \phi^- \{[\mathit{\Gamma}(D)]_{t\cdot}; x\} = x^{m-n-t} \sum_{D_t \in \mathcal{AD}_t(D)}^{n} \phi^-(D_t; x) \\ &= x^{m-n-t} R_t^-(D; x) = \sum_{k=0}^{n} r_{t;k}^- x^{m-t-k}, \end{aligned} \tag{1.5}$$

where $[\mathit{\Gamma}(D)]_{t\cdot} := \mathit{\Gamma} - \alpha_1 - \alpha_2 - \cdots - \alpha_t$ and $D_t := D - \overrightarrow{u_1v_1} - \overrightarrow{u_2v_2} - \cdots \overrightarrow{u_tv_t}$.

Remark 1.2. The fact that $S_t^-[\mathit{\Gamma}(D); x] = x^{m-n-t} R_t^-(D; x)$ in (1.5) prompts us to make some "premature" remark, which will be useful to us when we proceed to consider a similar method for reconstructing the permanent polynomial. As already indicated in the second part of Remark 1, Theorem 1.1 does not work in the case of the permanent polynomial; therefore, a similar equality for $S_t^+[\Gamma(D);x]$ and $x^{m-n-t}R_t^+(D;x)$ does not hold, although both these polynomials exist separately. Therefore, the calculation of $S_t^+[\Gamma(D);x]$ will be absolutely useless to us, and further we will focus on calculating $R_t^+(D;x)$. But we will use the derived expression $x^{m-n-t}R_t^+(D;x)$.

Here, we recall the known result, whose proof, in particular, can be obtained by multiple application of Clarke's theorem (see Theorem 2.14 of Clarke in [10]) with the addition of the factor $1/t!$, which appears due to the fact that there are $t!$ different sequences of deletion t of vertices from a graph.

Theorem 1.2. *Let G be an arbitrary (di)graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$. And let $G_{t\cdot} := G - v_1 - v_2 - \cdots - v_t$ be graph G less its t ($t \in \{1, 2, \dots, n\}$) arbitrary vertices $v_1, v_2, \dots, v_t$ and all edges (arcs, loops) incident to it. Then,*

$$\frac{1}{t!} \cdot \frac{\mathrm{d}^t}{\mathrm{d}x^t}\phi^-(G;x) = \sum_{G_{t\cdot} \in \mathcal{D}_t(G)} \phi(G_{t\cdot};x),$$

where the sum ranges over all C_n^t cards of the t-th deck $\mathcal{D}_t$ of G.

Corollary 1.1. *Let $G = \Gamma(D)$. Then,*

$$S_t^-[\Gamma(D);x] = \sum_{[\Gamma(D)]_{t\cdot} \in \mathcal{D}_t[\Gamma(D)]} \phi^-\{[\Gamma(D)]_{t\cdot};x\} = \frac{1}{t!} \cdot \frac{\mathrm{d}^t}{\mathrm{d}x^t}\phi^-[\Gamma(D);x]. \tag{1.6}$$

The last equation allows us to get $\phi^-[\Gamma(D);x]$ in integral form:

$$\phi^-[\Gamma(D);x] = t!\underbrace{\int \cdots \int}_{t \text{ times}} S_t^-[\Gamma(D);x]\,\mathrm{d}x^t \quad (0 \le t \le m - n + n_0^-), \tag{1.7}$$

where n_0^- is the number of zero roots of $\phi^-[\Gamma(D);x]$ (if n_0^- is not known, use $t \le m-n$); and the integration at each of the t steps uses the zero integration constant (since the coefficients $r_{t;k}^-$ of x^{m-t-k} in (1.7) must obey their determination in (1.5)).

The equation (1.6) can also be solved for $\phi^-[\Gamma(D);x]$ by comparing the coefficients at the same powers of x in the corresponding polynomials. First, using the last parts (1.5) and (1.6) for an equivalent representation of $S_t[\Gamma(D);x]$, we get

$$\begin{aligned} S_t^-[\Gamma(D);x] &= \sum_{k=0}^{n} r_{t;k}^- x^{m-t-k} = \frac{1}{t!} \cdot \frac{\mathrm{d}^t}{\mathrm{d}x^t}\phi^-[\Gamma(G);x] \\ &= \frac{1}{t!} \cdot \frac{\mathrm{d}^t}{\mathrm{d}x^t}\left[x^{m-n}\sum_{k=0}^{n} c_k^- x^{n-k}\right] = \frac{1}{t!} \cdot \frac{\mathrm{d}^t}{\mathrm{d}x^t}\sum_{k=0}^{n} c_k^- x^{m-k}, \end{aligned}$$

where the coefficients c_k^- belong to the expansion $\phi^-(D;x)=\sum_{k=0}^{n} c_k^- x^{n-k}$. Hence,

$$\begin{aligned}\sum_{k=0}^{n} r_{t;k}^- x^{m-t-k} &= \frac{1}{t!}\cdot\frac{\mathrm{d}^t}{\mathrm{d}x^t}\phi^-[\Gamma(G);x] = \sum_{k=0}^{n} c_k^- \frac{(m-k)}{t!(m-k-t)!}x^{m-t-k}\\ &= \sum_{k=0}^{n} c_k^- C_{m-k}^t x^{m-t-k},\end{aligned}$$

which makes it easy to compare the coefficients of the first and last sums therein:

$$c_k^- = \frac{r_{t;k}^-}{C_{m-k}^t} \quad (k\in\{0,1,\dots,n\}, 0\le t\le m-n+n_0^-).$$

At this point, we arrive at the following statement (which will later be generalized later to the general case, which also includes the permanental polynomial).

Lemma 1.2. *Let $\phi^-(D;x)=\sum_{k=0}^{n} c_k^- x^{n-k}$ and $R_t^-(D;x)=\sum_{k=0}^{n} r_{t;k}^- x^{n-t-k}$ $(0\le t\le m-n+n_0^-)$ be the characteristic polynomial of a digraph D and the sum of the characteristic polynomials of all its t-arcs-deleted subgraphs (from $\mathcal{AD}_t$), respectively. Then, the characteristic polynomial $\phi^-(D;x)$ is reconstructible from (the coefficients of) the polynomial $R_t^-(D;x)$ as follows*
(1.8)
$$\phi^-(D;x)=\sum_{k=0}^{n}\frac{r_{t;k}^-}{C_{m-k}^t}x^{n-k}=\frac{t!}{x^{m-n}}\underbrace{\int\cdots\int}_{t\ times} x^{m-n-t}R_t^-(D;x)\,\mathrm{d}x^t \quad (0\le t\le m-n+n_0^-),$$

where the integration at each of the t steps uses the zero integration constant.

Lemma 1.2 can be considered as our final result for the characteristic polynomial $\phi^-(D;x)$ of a directed graph D. Now it remains for us to show that a similar result is also valid for a permanent polynomial $\phi^+(D;x)$. (It is "only" necessary to replace $(-)$ by $(+)$ everywhere in (1.8)).

First, it is important to remember what combinatorial meaning an arbitrary coefficient c_k^+ has in the expansion of a permanent polynomial, $\phi^+(D;x)=\sum_0^n c_k^+ x^{n-k}$. Namely, the coefficient c_k^+ is equal to the number of all coverings by oriented simple p-cycles $(p\in\{1,2,\dots,k\})$ of exactly k vertices of the digraph D, where 1-cycle and 2-cycle are a loop and a pair of opposite arcs with common endpoints, respectively. See a discussion of the coefficients of the "simple permanent polynomial" of a directed graph, e.g., on page 193 of [14]. But in each directed simple p-cycle, the number of arcs is equal to the number of vertices, p. Therefore, the coefficient c_k^+ is also equal to the number of coverings exactly k of arcs of the digraph D. We need the following lemma.

Lemma 1.3. *Let $R_1^+(D;x)=\sum_{(D-\overrightarrow{uv})\in\mathcal{AD}_1}\phi^+(D-\overrightarrow{uv};x)$ be the sum of the permanent polynomials of all m subgraphs $D-\overrightarrow{uv}$ obtained by weakly deleting a single arc $\overrightarrow{uv}$*

from D. Then,

(1.9)

$$R_1^+(D;x) = \sum_{k=0}^{n}(m-k)c_k^+x^{n-k} = x^{n-m+1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{m-n}\phi^+(D;x)\right] \quad (m-n+n_0^+ \geq 1),$$

where n_0^+ is the number of zero roots of $\phi^+(D;x)$ (if n_0^+ is not known, use $m-n \geq 1$).

Proof. Consider an arbitrary covering of k arcs (and vertices) of the digraph D by oriented cycles consisting of arcs $\overrightarrow{u_1v_1}, \overrightarrow{u_2v_2}, \ldots, \overrightarrow{u_kv_k}$, where the head of each arc coincides with the tail of exactly one other arc, which is not specified here. Remove an arbitrary arc $\overrightarrow{uv}$ from the digraph D. Obviously, if this is not one of the arcs belonging to the cover under consideration, then this cover can also be realized in the resulting subgraph $D - \overrightarrow{uv}$, although other covers including $\overrightarrow{uv}$ in D, become impossible. If we consider the complete deck $\mathcal{AD}_1$ of all m one-arc-deleted subgraphs (cards), then among them we will find exactly k subgraphs in which our concrete cover cannot be realized. Since we have considered an arbitrary covering of arbitrary k arcs of the digraph D, we can generalize what has been said to the general case of all such cycle coverings of D. As a result, we can represent the total loss of coverings by all cards in the complete deck $\mathcal{AD}_1$ as the following polynomial, whose coefficients give us the numerical loss of all cycle coverings of the corresponding number of k ($k \in \{1, 2, \ldots, n\}$) of arcs of D:

$$\delta^+(D,x) := \sum_{k=0}^{n} kc_k^+x^{n-k}.$$

Using (1.9), we get

$$R_1^+(D;x) = \sum_{(D-\overrightarrow{uv})\in\mathcal{AD}_1(D)} \phi^+\left(D-\overrightarrow{uv};x\right) = m\phi^+(D;x) - \delta^+(D;x) = \sum_{k=0}^{n}(m-k)c_k^+x^{n-k}.$$

Thence,

$$\begin{aligned} R_1^+(D;x) =& \sum_{k=0}^{n}(m-k)c_k^+x^{n-k} = x^{n-m+1}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n}c_k^+x^{m-k} \\ &= x^{n-m+1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{m-n}\phi^+(D;x)\right] \quad (m-n+n_0^+ \geq 1), \end{aligned}$$

which completes the proof. □

The following statement plays an essential role in our reasoning.

Lemma 1.4. *Let $R_t^+(D;x) = \sum_{D_t\in\mathcal{AD}_t}\phi^+(D_t;x)$ be the sum of the permanent polynomials of all C_m^t subgraphs D_t obtained by weakly deleting t ($t \in \{0, 1, \ldots, m-n+n_0^+\}$) arcs from D. Then,*

(1.10)

$$R_t^+(D;x) = \sum_{k=0}^{n}C_{m-k}^t c_k^+x^{n-k} = \frac{x^{n-m+t}}{t!}\cdot\frac{\mathrm{d}^t}{\mathrm{d}x^t}\left[x^{m-n}\phi^+(D;x)\right] \quad \left(0 \leq t \leq m-n+n_0^+\right).$$

Proof. It can be obtained by t-fold application of Lemma 1.3. In this case, as in the case of Theorem 1.2, the multiplier $1/t!$ appears before the differential, since there are $t!$ possibilities of sequential selection of t elements one by one, but we only need one choice. (For a short check, one can consider the first coefficient C_m^t in the expansion of $R_+(D;x)$ in powers of x.) We will mainly focus on the more important part of the proof regarding the first equality in (1.10), while the second equality there is elementarily proved by simple manipulations with the coefficients. By Lemma 1.3, we have $R_1^+(D;x) = \sum_{k=0}^{n}(m-k)c_k^+ x^{n-k}$, where m is the number of all arcs of the original digraph $D_0 := D$ with 0 deleted arcs; and to unify subsequent entries, we can formally write $R_0^+(D;x) \equiv \phi^+(D;x)$, which is the initial term in the sequence $R_0^+, R_1^+, \ldots, R_t^+$. Thus, each stage of sequential calculating of polynomial sums $R_1^+(D;x)$, $R_2^+(D;x), \ldots, R_t^+(D;x)$ for decks $\mathcal{AR}_1, \mathcal{AR}_2, \ldots, \mathcal{AR}_t$, respectively, means sequential multiplication of the original coefficient c_k^+ ($k \in \{0,1,\ldots,n\}$), of $R_0^+ = \phi^+(D;x)$, first by $(m-k)/1$, then by $(m-k-1)/2$, and so on up to the last multiplier $(m-k-t+1)/t$ in the process, to result in the coefficient

$$r_{t;k} = (t!)^{-1}(m-k)(m-k-1)\cdots(m-k-t+1)c_k^+ = C_{m-k}^t c_k^+$$

of the polynomial $R_t^+(D;x) = \sum_{k=0}^{n} r_{t;k}x^{n-k}$.

Based on this, we get

$$R_t^+(D;x) = \sum_{k=0}^{n} C_{m-k}^t c_k^+ x^{n-k},$$

which proves the first equality in (1.10) and, thus, the main part of our statement. It is technically easy to see that the third part of (1.10) is also equal to the same polynomial $R_t^+(D;x)$:

$$\begin{aligned}
&\frac{x^{n-m+t}}{t!}\frac{\mathrm{d}^t}{\mathrm{d}x^t}\left[x^{m-n}\phi^+(D;x)\right] = \frac{x^{n-m+t}}{t!}\frac{\mathrm{d}^t}{\mathrm{d}x^t}\left[\sum_0^n c_k^+ x^{m-k}\right]\\
&= \frac{x^{n-m+t}}{t!}\left[\sum_0^n (m-k)(m-k-1)\cdots(m-k-t+1)c_k^+ x^{m-k}\right]\\
&= \sum_{k=0}^{n}\frac{(m-k)!}{(t!)(m-k-t)!}c_k^+ x^{n-k} = \sum_{k=0}^{n} C_{m-k}^t c_k^+ x^{n-k} = R_t^+(D;x) = \sum_{k=0}^{n} r_{t;k}^+ x^{n-k},
\end{aligned}$$

which completes the proof. ☐

Now we state a generalizing theorem.

Theorem 1.3. *Let* $\phi^\pm(D;x) = \sum_{k=0}^{n} c_k^\pm x^{n-k}$ *and* $R_t^\pm(D;x) = \sum_{k=0}^{n} r_{t;k}^\pm x^{n-t-k}$ ($0 \le t \le m-n+n_0^\pm$) *be the characteristic* ($-$) (*permanental* ($+$)) *polynomial of a digraph* D *and the sum of the characteristic* (*permanental*) *polynomials of all its* t*-arcs-deleted subgraphs* (*from* $\mathcal{AD}_t$), *respectively. Then, the polynomial* $\phi^\pm(D;x)$ *is reconstructible* *from (the coefficients of) the polynomial $R_t^{\pm}(D;x)$ as follows*

$$\phi^{\pm}(D;x)=\sum_{k=0}^{n}\frac{r_{t;k}^{\pm}}{C_{m-k}^{t}}x^{n-k}=\frac{t!}{x^{m-n}}\underbrace{\int\cdots\int}_{t\ times}x^{m-n-t}R_t^{\pm}(D;x)\,\mathrm{d}x^t\quad(0\le t\le m-n+n_0^{\pm}),\tag{1.11}$$

where the integration at each of the t steps uses the zero integration constant.

Proof. The $(-)$-case has been proven in Lemma 1.2. Now, note that it follows from the last two equalities in (1.11) that

$$c_k^{+}=\frac{r_{t;k}^{+}}{C_{m-k}^{t}}\quad(0\le t\le m-n+n_0^{+}),\tag{1.12}$$

which is a $(+)$-analog of (1.8). Whence we arrive at the overall proof. □

Remark 1.3. All practical applications of Theorem 1.3 (and Lemma 1.2) are related to the values of $t\ge 1$. The last condition can always be satisfied for the case $m-n=-1$, since it corresponds to the oriented tree $\overrightarrow{T}$ $(m\ge 2)$, whose polynomial $\phi^{\pm}\left(\overrightarrow{T};x\right)\equiv x^n$ has $n_0^{\pm}=n$ zero roots and allows its formal reconstruction up to the values $t=m-n+n=n-1$. For $m=n$, when an arbitrarily oriented digraph D contains exactly one cycle of length $c<n$, $1\le t\le n-c$; in this case, we can also reconstruct the polynomial $\phi^{\pm}(D;x)$ (for valid values of t). But in the exceptional case, when D is a consistently oriented cycle, the reconstruction of its polynomial $\phi^{\pm}(D;x)$ is impossible, since $\phi^{\pm}(D;x)=x^n\pm 1$, and $m-n+n_0^{\pm}=m-n+0=0<1$. For all $m-n\ge 1$, Thereom 1.3 (Lemma 1.2) works for at least $t=1$. Thus, the polynomial $\phi^{\pm}(D;x)$ of a consistently oriented cycle remains the only case when its reconstruction using Theorem 1.3 is impossible.

Now we want to move our reasoning to the area of undirected graphs. Earlier, we have already dealt with the problem of recursion of the characteristic $\phi^{-}(G;x)$ and the permanent $\phi^{+}(G;x)$ polynomials [15, 16] of the undirected graph G. We use two formulae [15, 16], in which we are now correcting typos made in [16]:

$$\begin{aligned}(q-n)\phi^{-}(G;x)&=\sum_{uv}[\phi^{-}(G-uv)+\phi^{-}(G-u-v;x)]-x(\mathrm{d}/\mathrm{d}x)\phi^{-}(G;x),\\(q-n)\phi^{+}(G;x)&=\sum_{uv}[\phi^{+}(G-uv)-\phi^{+}(G-u-v;x)]-x(\mathrm{d}/\mathrm{d}x)\phi^{+}(G;x),\end{aligned}\tag{1.13}$$

where n and q are the numbers of vertices and edges, of G, respectively, and the combined summation ranges over the set of all edges of G and all pairs u and v of adjacent vertices $(u<v;u\sim v)$.

We combine these formulae and transform them like what we did before

$$(q-n)\phi^{\pm}(G;x)+x(\mathrm{d}/\mathrm{d}x)\phi^{\pm}(G;x)\sum_{uv}[\phi^{\pm}(G-uv;x)\mp\phi^{\pm}(G-u-v;x)] \tag{1.14}$$
$$=\sum_{k=0}^{n}g^{\pm}_{1;k}x^{n-k}=U^{\pm}_{1}(G;x),$$

where the coefficients $g^{\pm}_{k}$ should be known by recursion. Further, transforming the first side of (1.14), we obtain

$$\sum_{k=0}^{n}[(q-n)+(n-k)]c^{\pm}_{k}x^{n-k}=\sum_{k=0}^{n}(q-k)c^{\pm}_{k}x^{n-k}=x^{n-q+1}(\mathrm{d}/\mathrm{d}x)[x^{q-n}\phi^{\pm}(G;x)] \tag{1.15}$$
$$=\sum_{k=0}^{n}g^{\pm}_{1;k}x^{n-k},$$

whence we arrive at the "undirected" generalization of Lemma 1.4.

Theorem 1.4. *Let $U^{\pm}_{1}(G;x)=\sum_{k=0}^{n}g^{\pm}_{k}x^{n-k}$ be the sum of the polynomials $\phi^{\pm}[(\cdot);x]$ of all "weak" subgraphs $G-uv$ and all "strong" subgraphs $G-u-v$ $(u<v;u\sim v)$ of G. Then,*

$$\phi^{\pm}(G;x)=\sum_{k=0}^{n}c^{\pm}_{k}x^{n-k}=\sum_{k=0}^{n}\frac{g^{\pm}_{1;k}}{q-k}x^{n-k}=x^{n-q}\int x^{q-n-1}U^{\pm}_{1}(G;x)\,\mathrm{d}x\quad(q>n), \tag{1.16}$$

where the integration uses the zero integration constant.

Proof. The second equality in (1.16) is related to the comparison of the second and fourth sides of (1.15), while the third equality in (1.16) is a purely technical fact. □

The following corollary allows us to equate two approaches to undirected graphs G – as such and as their symmetric directed equivalents $S(G)$.

Corollary 1.2. *Let $R^{\pm}_{1}[S(G);x]=\sum_{k=0}^{n}r^{\pm}_{1;k}x^{n-k}$ and $U^{\pm}_{1}(G;x)=g^{\pm}_{1;k}x^{n-k}$ (as above). Then,*

$$g^{\pm}_{1;k}=\frac{q-k}{m-k}r^{\pm}_{1;k}\quad(k\in\{0,1,\dots,n\},m=2q)\,, \tag{1.17}$$

whence

$$U^{\pm}_{1}(G;x)=x^{n-q+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^{q}}\int x^{2q-n-1}R^{\pm}_{1}[S(G);x]\,\mathrm{d}x\right) \tag{1.18}$$

and

$$R^{\pm}_{1}[S(G);x]=x^{n-2q+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{q}\int x^{q-n-1}U^{\pm}_{1}(G;x)\,\mathrm{d}x\right). \tag{1.19}$$

Proof. The mutual relation (1.17) of the coefficients follows from (1.12) and the second side (1.15). The former gives, for $t=1$, $c^{\pm}_{k}=r^{\pm}_{1,k}/C^{1}_{m-k}=r^{\pm}_{1;k}/(m-k)$. Substituting the obtained expression for $c_k^{\pm}$ on the second side of (1.15) and equating the result to the last part of (1.15), we obtain:

$$\sum_{k=0}^{n}(q-k)c_k^{\pm}x^{n-k} = \sum_{k=0}^{\pm}\frac{q-k}{m-k}r_{1;k}x^{n-k} = \sum_{k=0}^{n}g_{1;k}^{\pm}x^{n-k}, \tag{1.20}$$

from which, comparing the coefficients of x^{n-k} on the last two sides of (1.20), we arrive at the proof of the first part of our statement, expressed by (1.17).

Integral expressions (1.18) and (1.19), consistent with (1.17), can be obtained using parts of expressions (1.11) and (1.16) used by theorems 1.3 and 1.4, respectively. Prove the first of them, (1.18). First, we equate the last side of (1.16) to the third side of (1.11), assuming that $D = S(G)$ and $t = 1$ in it:

$$\begin{aligned}\phi^{\pm}(G;x) &= \left(x^{n-q}\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x = \frac{1}{x^{m-n}}\int x^{m-n-1}R_t^{\pm}[S(G);x]\,\mathrm{d}x\right)\\ &= \phi^{\pm}[S(G);x].\end{aligned} \tag{1.21}$$

Starting from the central equality of (1.21), enclosed in brackets, we will carry out the following sequence of its technical transformations:

$$\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x =\frac{1}{x^{m-q}}\int x^{m-n-1}R_t^{\pm}[S(G);x]\,\mathrm{d}x, \tag{1.22}$$

$$x^{q-n-1}U_1^{\pm}(G;x) =\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^{m-q}}\int x^{m-n-1}R_t^{\pm}[S(G);x]\,\mathrm{d}x\right),$$

$$U_1^{\pm}(G;x) =x^{n-q+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^{m-q}}\int x^{m-n-1}R_t^{\pm}[S(G);x]\,\mathrm{d}x\right). \tag{1.23}$$

But due to the fact that the number of arcs m of $D = S(G)$ is equal to $2q$ (where q is the number of edges of G), (1.23) is equivalent to

$$U_1^{\pm}(G;x) = x^{n-q+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^{q}}\int x^{2q-n-1}R_t^{\pm}[S(G);x]\,\mathrm{d}x\right),$$

which proves (1.18).

The second integral equality (1.19) is proven in a similar way. First, we rewrite equality (1.22) in a different form to obtain

$$\int x^{m-n-1}R_1^{\pm}[S(G);x]\,\mathrm{d}x = x^{m-q}\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x.$$

Then, without explanation, we apply a similar sequence of transformations:

$$x^{n-q+1}R_1^{\pm}[S(G);x] =\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{m-q}\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x\right),$$

$$R_1^{\pm}[S(G);x] =x^{n-m+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{m-q}\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x\right),$$

$$R_1^{\pm}[S(G);x] =x^{n-2q+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{q}\int x^{q-n-1}U_1^{\pm}(G;x)\,\mathrm{d}x\right),$$

which proves (1.19) and thus completes the whole proof. □

In addition to Remark 3 (for monocyclic digraphs) and Theorem 1.4 (for undirected graphs), we present the following corollary to formulae (1.13), which is given here with a correction of a typo in [16].

Proposition 1.1. *Let* $\overset{q=n}{G}$ *be an undirected simple monocyclic graph (whether a cycle or not) with* $q = n$*. Then,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\phi^{\pm}(\overset{q=n}{G};x) &= x^{-1}\sum_{uv}\Big[\phi^{\pm}\Big(\overset{q=n}{G}-uv\Big) \mp \phi^{\pm}(\overset{q=n}{G}-u-v;x)\Big] \\ &= \sum_{u\in V(G)} \phi^{\pm}(\overset{q=n}{G}-u;x) \quad (q=n), \end{aligned} \tag{1.24}$$

where the first combined summation ranges over the set of all edges of $\overset{q=n}{G}$ *and all pairs* u *and* v *of adjacent vertices* $(u < v; u \sim v)$*.*

Thus, the derivative $[\phi^{\pm}(\overset{q=n}{G};x)]'$ of the polynomial $\phi^{\pm}(\overset{q=n}{G};x)$ of a monocyclic graph $\overset{q=n}{G}$ is also reconstructible from the first combined sum in (1.24) (due to [16]). Earlier, in the works on the reconstruction of the characteristic polynomial $\phi^{-}(G;x)$ of an arbitrary undirected graph G, the second sum of (1.24) was used; see, e.g., [5–8]. Special attention is paid to $\phi^{-}(\overset{q=n}{G};x)$ in [8].

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