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ON THE HYPERBOLIC k-MERSENNE AND k-MERSENNE-LUCAS OCTONIONS

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Abstract. In this paper, we introduce the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions and investigate their algebraic properties. We give Binet's formula and present several interrelations and some well-known identities such as Catalan identity, d'Ocagne identity, Vajda identity, generating functions, etc. of these octonions in closed form. Furthermore, we investigate the relations between hyperbolic k-Mersenne octonions and hyperbolic k-Mersenne-Lucas octonions.

1. Introduction

Number sequences have been studied by researchers for a long time. In particular, the most important and remarkable of these numbers are the Fibonacci numbers. Until today, Fibonacci numbers have been studied and many generalizations have been made. Lucas, Jacobsthal, Jacobsthal-Lucas, Pell, Pell-Lucas, etc. numbers can be given as examples of these generalizations [2, 5, 8, 14, 15, 19–21, 23].

One of these numbers is the Mersenne number. They are named after Marin Mersenne, a French Minim friar, who studied them in the early 17th century. Mersenne numbers have been studied in the literature and various generalizations such as Mersenne-Lucas, k-Mersenne, k-Mersenne-Lucas have been studied [1, 4, 6, 7, 17, 22, 25–27].

Definition 1.1. The Mersenne sequence $\{M_n\}_{n\geq 0}$ is defined recursively as

$$M_{n+2} = 3M_{n+1} - 2M_n, \quad \text{with} \quad M_0 = 0, \quad M_1 = 1, \quad n \geq 0.$$

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Definition 1.2. The Mersenne-Lucas sequence $\{m_n\}_{n\geq 0}$ is defined recursively as

$$m_{n+2} = 3m_{n+1} - 2m_n, \quad \text{with} \quad m_0 = 2, \quad m_1 = 3, \quad n \geq 0.$$

Definition 1.3. The k-Mersenne sequence $\{M_{k,n}\}_{n\geq 0}$ is given recursively as

$$M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n}, \quad M_{k,0} = 0, \quad M_{k,1} = 1, \quad n \geq 0. \tag{1.1}$$

Definition 1.4. The k-Mersenne-Lucas sequence $\{m_{k,n}\}_{n\geq 0}$ is given recursively as

$$m_{k,n+2} = 3km_{k,n+1} - 2m_{k,n}, \quad m_{k,0} = 2, \quad m_{k,1} = 3k, \quad n \geq 0. \tag{1.2}$$

The characteristic equation corresponding to the recurrence relations (1.1) and (1.2) is $\alpha^2 - 3k\alpha + 2 = 0$ and its roots are $\alpha_1 = \frac{3k+\sqrt{9k^2-8}}{2}$ and $\alpha_2 = \frac{3k-\sqrt{9k^2-8}}{2}$. These characteristics roots hold the following properties

$$\alpha_1 + \alpha_2 = 3k, \quad \alpha_1\alpha_2 = 2, \quad \alpha_1 - \alpha_2 = \sqrt{9k^2 - 8}. \tag{1.3}$$

The Binet formulas of k-Mersenne and k-Mersenne-Lucas sequences are given, respectively, by

$$m_{k,n} = \alpha_1^n + \alpha_2^n \quad \text{and} \quad M_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}. \tag{1.4}$$

The quaternion, an algebraic structure, was first described in 1843 by William Rowan Hamilton [11]. Hamilton defined a quaternion as the quotient of two directed lines in a three-dimensional space, or, equivalently, as the quotient of two vectors. Multiplication of quaternions is noncommutative. A. F. Horadam defined the nth Fibonacci and nth Lucas quaternions and gave their some properties in 1963 [12]. Thus, Macfarlane defined the hyperbolic quaternions and studied their properties [18]. Recently, these numbers have been examined and studies have been carried out [10, 13, 24]. The hyperbolic k-Fibonacci and k-Fibonacci-Lucas, hyperbolic k-Jacobsthal and k-Jacobsthal-Lucas quaternions were defined and given some of their properties [10, 24]. In mathematics, the octonions are a normed division algebra over the real numbers, a kind of hypercomplex number system. Octonions have eight dimensions; twice the number of dimensions of the quaternions, of which they are an extension. They are noncommutative and nonassociative, but satisfy a weaker form of associativity; namely, they are alternative. They are also power associative. Octonions are not as well known as quaternions and complex numbers, which are much more widely studied and used.

A. Cariow and G. Cariow defined the hyperbolic octonions [3]. A hyperbolic octonion O has the form

$$\begin{aligned} O &= h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 i_4 + h_5 i_5 + h_6 i_6 + h_7 i_7 \\ &= (h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7), \end{aligned}$$

where i_1, i_2, i_3 are quaternion imaginary units and $h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7$ are the real components, i_4 $({i_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions are defined as in Table 1.

$\cdot$	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	-1	i_3	$-i_2$	i_5	i_4	$-i_7$	i_6
i_2	$-i_3$	-1	i_1	i_6	i_7	i_4	$-i_5$
i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	i_4
i_4	$-i_5$	$-i_6$	$-i_7$	1	i_1	i_2	i_3
i_5	$-i_4$	$-i_7$	i_6	$-i_1$	1	i_3	$-i_2$
i_6	i_7	$-i_4$	$-i_5$	$-i_2$	$-i_3$	1	i_1
i_7	$-i_6$	i_5	$-i_4$	$-i_3$	i_2	$-i_1$	1

Table 1: Multiplication rule for hyperbolic octonions units.

Godase A. defined the hyperbolic k-Fibonacci and k-Fibonacci-Lucas octonions and gave some of their properties [9]. Özkan E. et al. defined the hyperbolic k-Jacobsthal and k-Jacobsthal-Lucas octonions and gave some of their properties [23]. Kumari M. et al. defined the k-Mersenne, k-Mersenne-Lucas octonions and examined some properties of these numbers [16]. As a new generalization of this study [16], we examine the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions and give their features.

2. Hyperbolic k-Mersenne Octonions

In this section, we introduce the hyperbolic k-Mersenne octonions and establish their Binet formula. Furthermore, we study some well-known identities of them.

Definition 2.1. For $n \geq 0$, the hyperbolic k-Mersenne octonions $\mathcal{H}M_{k,n}$ are defined by

$$\begin{aligned}\mathcal{H}M_{k,n} &= M_{k,n} + M_{k,n+1}i_1 + M_{k,n+2}i_2 + \cdots + M_{k,n+7}i_7 \\ &= (M_{k,n}, M_{k,n+1}, M_{k,n+2}, M_{k,n+3}, M_{k,n+4}, M_{k,n+5}, M_{k,n+6}, M_{k,n+7}).\end{aligned}$$

Definition 2.2. The sequence can be extended to negative indices n, $\mathcal{H}M_{k,-n}$ is defined by

$$\begin{aligned}\mathcal{H}M_{k,-n} = &-\frac{1}{2^n}M_{k,n} - \frac{1}{2^{n-1}}M_{k,n-1}i_1 - \frac{1}{2^{n-2}}M_{k,n-2}i_2 - \frac{1}{2^{n-3}}M_{k,n-3}i_3 \\ &- \frac{1}{2^{n-4}}M_{k,n-4}i_4 - \frac{1}{2^{n-5}}M_{k,n-5}i_5 - \frac{1}{2^{n-6}}M_{k,n-6}i_6 - \frac{1}{2^{n-7}}M_{k,n-7}i_7.\end{aligned}$$

Definition 2.3. Let the scalar and vector parts of the hyperbolic k-Mersenne octonions $\mathcal{H}M_{k,n}$ be denoted by $S(\mathcal{H}M_{k,n})$ and $V(\mathcal{H}M_{k,n})$, respectively, and are defined as follows

$$\begin{aligned}S(\mathcal{H}M_{k,n}) &= M_{k,n}, \\ V(\mathcal{H}M_{k,n}) &= (M_{k,n+1}, M_{k,n+2}, M_{k,n+3}, M_{k,n+4}, M_{k,n+5}, M_{k,n+6}, M_{k,n+7}).\end{aligned}$$

Thus, $\mathcal{H}M_{k,n} = S(\mathcal{H}M_{k,n}) + V(\mathcal{H}M_{k,n})$.

Definition 2.4. For $n \geq 0$, the conjugate of the hyperbolic k-Mersenne octonions $\mathcal{H}M_{k,n}$ is defined by

$$\begin{aligned}\overline{\mathcal{H}M}_{k,n} &= M_{k,n} - M_{k,n+1}i_1 - M_{k,n+2}i_2 - \cdots - M_{k,n+7}i_7 \\ &= (M_{k,n}, -M_{k,n+1}, -M_{k,n+2}, -M_{k,n+3}, -M_{k,n+4}, -M_{k,n+5}, -M_{k,n+6}, -M_{k,n+7}).\end{aligned}$$

Theorem 2.1. *The following equation is provided*

$$\mathcal{H}M_{k,n} + \overline{\mathcal{H}M}_{k,n} = 2S(\mathcal{H}M_{k,n}) = 2M_{k,n}.$$

Proof. From Definition 2.4, we have

$$\begin{aligned}\mathcal{H}M_{k,n} + \overline{\mathcal{H}M}_{k,n} &= S(\mathcal{H}M_{k,n}) + V(\mathcal{H}M_{k,n}) + S(\mathcal{H}M_{k,n}) - V(\mathcal{H}M_{k,n}) \\ &= 2S(\mathcal{H}M_{k,n}) = 2M_{k,n}.\end{aligned}$$

□

Definition 2.5. The norm of the hyperbolic k-Mersenne octonions $\mathcal{H}M_{k,n}$ is defined by

$$N(\mathcal{H}M_{k,n}) = \sqrt{M_{k,n}^2 + \cdots + M_{k,n+3}^2 - M_{k,n+4}^2 - M_{k,n+5}^2 - M_{k,n+6}^2 - M_{k,n+7}^2}.$$

Theorem 2.2. *The recurrence relations of the $\mathcal{H}M_{k,n}$ and $\overline{\mathcal{H}M}_{k,n}$ are as follows:*

(a) $\mathcal{H}M_{k,n+1} = 3k\mathcal{H}M_{k,n} - 2\mathcal{H}M_{k,n-1}$;
(b) $\overline{\mathcal{H}M}_{k,n+1} = 3k\overline{\mathcal{H}M}_{k,n} - 2\overline{\mathcal{H}M}_{k,n-1}$.

Proof. (a) We have

$$\begin{aligned}\mathcal{H}M_{k,n+1} &= M_{k,n+1} + M_{k,n+2}i_1 + M_{k,n+3}i_2 + M_{k,n+4}i_3 + M_{k,n+5}i_4 + M_{k,n+6}i_5 \\ &\quad + M_{k,n+7}i_6 + M_{k,n+8}i_7 \\ &= (3kM_{k,n} - 2M_{k,n-1}) + (3kM_{k,n+1} - 2M_{k,n})i_1 + (3kM_{k,n+2} - 2M_{k,n+1})i_2 \\ &\quad + (3kM_{k,n+3} - 2M_{k,n+2})i_3 + (3kM_{k,n+4} - 2M_{k,n+3})i_4 + (3kM_{k,n+5} \\ &\quad - 2M_{k,n+4})i_5 + (3kM_{k,n+6} - 2M_{k,n+5})i_6 + (3kM_{k,n+7} - 2M_{k,n+6})i_7 \\ &= 3k(M_{k,n} + M_{k,n+1}i_1 + M_{k,n+2}i_2 + M_{k,n+3}i_3 + M_{k,n+4}i_4 + M_{k,n+5}i_5 \\ &\quad + M_{k,n+6}i_6 + M_{k,n+7}i_7) - 2(M_{k,n-1} + M_{k,n}i_1 + M_{k,n+1}i_2 + M_{k,n+2}i_3 \\ &\quad + M_{k,n+3}i_4 + M_{k,n+4}i_5 \quad + M_{k,n+5}i_6 + M_{k,n+6}i_7) \\ &= 3k\mathcal{H}M_{k,n} - 2\mathcal{H}M_{k,n-1}.\end{aligned}$$

The proof of (b) is similar to that of (a). □

Theorem 2.3 (Binet Formula)**.** *The Binet formulas for the hyperbolic k-Mersenne octonions are*

(a) $\mathcal{H}M_{k,n} = \dfrac{\alpha^*\alpha_1^n - \beta^*\alpha_2^n}{\alpha_1 - \alpha_2}$;
(b) $\overline{\mathcal{H}M}_{k,n} = \dfrac{\overline{\alpha^*}\alpha_1^n - \overline{\beta^*}\alpha_2^n}{\alpha_1 - \alpha_2}$;
(c) $\mathcal{H}M_{k,-n} = \dfrac{1}{2^n}\left(\dfrac{\alpha^*\alpha_2^n - \beta^*\alpha_1^n}{\alpha_1 - \alpha_2}\right)$,

where

$$\begin{aligned}
\alpha^* &= 1+\alpha_1 i_1+\alpha_1^2 i_2+\alpha_1^3 i_3+\alpha_1^4 i_4+\alpha_1^5 i_5+\alpha_1^6 i_6+\alpha_1^7 i_7\\
&= (1,\alpha_1,\alpha_1^2,\alpha_1^3,\alpha_1^4,\alpha_1^5,\alpha_1^6,\alpha_1^7),\\
\beta^* &= 1+\alpha_2 i_1+\alpha_2^2 i_2+\alpha_2^3 i_3+\alpha_2^4 i_4+\alpha_2^5 i_5+\alpha_2^6 i_6+\alpha_2^7 i_7\\
&= (1,\alpha_2,\alpha_2^2,\alpha_2^3,\alpha_2^4,\alpha_2^5,\alpha_2^6,\alpha_2^7),\\
\overline{\alpha^*} &= 1-\alpha_1 i_1-\alpha_1^2 i_2-\alpha_1^3 i_3-\alpha_1^4 i_4-\alpha_1^5 i_5-\alpha_1^6 i_6-\alpha_1^7 i_7\\
&= (1,-\alpha_1,-\alpha_1^2,-\alpha_1^3,-\alpha_1^4,-\alpha_1^5,-\alpha_1^6,-\alpha_1^7),\\
\overline{\beta^*} &= 1-\alpha_2 i_1-\alpha_2^2 i_2-\alpha_2^3 i_3-\alpha_2^4 i_4-\alpha_2^5 i_5-\alpha_2^6 i_6-\alpha_2^7 i_7\\
&= (1,-\alpha_2,-\alpha_2^2,-\alpha_2^3,-\alpha_2^4,-\alpha_2^5,-\alpha_2^6,-\alpha_2^7).
\end{aligned}$$

Proof. (a) Using Definition 2.1 and the Binet formula of $M_{k,n}$, we have

$$\begin{aligned}
\mathcal{H}M_{k,n} =& M_{k,n}+M_{k,n+1}i_1+M_{k,n+2}i_2+M_{k,n+3}i_3+M_{k,n+4}i_4+M_{k,n+5}i_5\\
&+M_{k,n+6}i_6+M_{k,n+7}i_7\\
=&\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)+\left(\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right)i_1+\cdots+\left(\frac{\alpha_1^{n+7}-\alpha_2^{n+7}}{\alpha_1-\alpha_2}\right)i_7\\
=&\frac{\alpha_1^n}{\alpha_1-\alpha_2}(1+\alpha_1 i_1+\cdots+\alpha_1^7 i_7)-\frac{\alpha_2^n}{\alpha_1-\alpha_2}(1+\alpha_2 i_1+\cdots+\alpha_2^7 i_7)\\
=&\frac{\alpha_1^n\alpha^*-\alpha_2^n\beta^*}{\alpha_1-\alpha_2}.
\end{aligned}$$

The proofs of (b) and (c) are similar to (a), by using Definition 2.4 and Definition 2.2, respectively. ☐

Theorem 2.4. *For $r,s,n\in\mathbb{N}$ such that $s\geq r$, the generating functions for hyperbolic k-Mersenne octonions are given as*

(a) $$\sum_{n=0}^{\infty}\mathcal{H}M_{k,n}x^n=\frac{\mathcal{H}M_{k,0}+(\mathcal{H}M_{k,1}-3k\mathcal{H}M_{k,0})\,x}{1-3kx+2x^2};$$

(b) $$\sum_{n=0}^{\infty}\mathcal{H}M_{k,sn+r}x^n=\frac{\mathcal{H}M_{k,r}-2^r\mathcal{H}M_{k,s-r}x}{1-m_{k,s}x+2^s x^2}.$$

The exponential generating functions for hyperbolic k-Mersenne octonions are

(c) $$\sum_{n=0}^{\infty}\frac{\mathcal{H}M_{k,sn}x^n}{n!}=\frac{\alpha^* e^{\alpha_1^s x}-\beta^* e^{\alpha_2^s x}}{\alpha_1-\alpha_2};$$

(d) $$\sum_{n=0}^{\infty}\frac{\mathcal{H}M_{k,sn+r}x^n}{n!}=\frac{\alpha^*\alpha_1^r e^{\alpha_1^s x}-\beta^*\alpha_2^r e^{\alpha_2^s x}}{\alpha_1-\alpha_2}.$$

Proof. (a) Let $G(x)=\sum_{n=0}^{\infty}\mathcal{H}M_{k,n}x^n$. We have

$$G(x)=\sum_{n=0}^{\infty}\mathcal{H}M_{k,n}x^n=\mathcal{H}M_{k,0}+\mathcal{H}M_{k,1}x+\mathcal{H}M_{k,2}x^2+\mathcal{H}M_{k,3}x^3+\cdots,$$

$$3kxG(x) = \mathcal{H}M_{k,0}3kx + \mathcal{H}M_{k,1}3kx^2 + \mathcal{H}M_{k,2}3kx^3 + \mathcal{H}M_{k,3}3kx^4 + \cdots,$$
$$2x^2G(x) = \mathcal{H}M_{k,0}2x^2 + \mathcal{H}M_{k,1}2x^3 + \mathcal{H}M_{k,2}2x^4 + \mathcal{H}M_{k,3}2x^5 + \cdots.$$

If the necessary mathematical operations are performed, we get the following

$$(1 - 3kx + 2x^2)G(x) = \mathcal{H}M_{k,0} + (\mathcal{H}M_{k,1} - 3k\mathcal{H}M_{k,0})\,x,$$
$$G(x) = \frac{\mathcal{H}M_{k,0} + (\mathcal{H}M_{k,1} - 3k\mathcal{H}M_{k,0})\,x}{1 - 3kx + 2x^2}.$$

The proofs of (b), (c) and (d) are similar to that of (a). □

Theorem 2.5. *For $k \neq 1$, we have*

$$\sum_{i=0}^{n} \mathcal{H}M_{k,i} = \frac{2\mathcal{H}M_{k,n} - \mathcal{H}M_{k,n+1} + \mathcal{H}M_{k,1} + (1-3k)\mathcal{H}M_{k,0}}{3(1-k)}.$$

Proof. Using Theorem 2.3, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} \mathcal{H}M_{k,i} &= \sum_{i=0}^{n} \left(\frac{\alpha^*\alpha_1^i - \beta^*\alpha_2^i}{\alpha_1 - \alpha_2} \right) \\
&= \frac{\alpha^*}{\alpha_1 - \alpha_2} \sum_{i=0}^{n} \alpha_1^i - \frac{\beta^*}{\alpha_1 - \alpha_2} \sum_{i=0}^{n} \alpha_2^i \\
&= \frac{\alpha^*(1 - \alpha_2 - \alpha_1^{n+1} + \alpha_1^{n+1}\alpha_2) - \beta^*(1 - \alpha_1 - \alpha_2^{n+1} + \alpha_2^{n+1}\alpha_1)}{(\alpha_1 - \alpha_2)(1 - (\alpha_1 + \alpha_2) + \alpha_1\alpha_2)} \\
&= \frac{\alpha^* - \beta^* - \alpha_1\alpha_2(\alpha^*\alpha_1^{-1} - \beta^*\alpha_2^{-1}) - (\alpha^*\alpha_1^{n+1} - \beta^*\alpha_2^{n+1})}{(\alpha_1 - \alpha_2)3(1-k)} \\
&\quad + \frac{\alpha_1\alpha_2(\alpha^*\alpha_1^n - \beta^*\alpha_2^n)}{(\alpha_1 - \alpha_2)3(1-k)} \\
&= \frac{\mathcal{H}M_{k,0} - 2\mathcal{H}M_{k,-1} - \mathcal{H}M_{k,n+1} + 2\mathcal{H}M_{k,n}}{3(1-k)} \qquad \text{(from Theorem 2.3)} \\
&= \frac{2\mathcal{H}M_{k,n} - \mathcal{H}M_{k,n+1} + \mathcal{H}M_{k,1} + (1-3k)\mathcal{H}M_{k,0}}{3(1-k)},
\end{aligned}$$

as required. □

Lemma 2.1. *We have*

(a) $\alpha^* - \beta^* = \delta\mathcal{H}M_{k,0}$;
(b) $\alpha^* + \overline{\alpha^*} = m_{k,0} = 2$;
(c) $\alpha^* + \overline{\beta^*} = 2 + \delta\mathcal{H}M_{k,0}$;
(d) $\alpha^*\beta^* = (227, -83\alpha_1 + 85\alpha_2, 19\alpha_1^2 - 17\alpha_2^2, -15\alpha_1^3 + 17\alpha_2^3 + 34\alpha_2 - 34\alpha_1,$
$-13\alpha_1^4 + 15\alpha_2^4, 5\alpha_1^5 - 3\alpha_2^5 - 10\alpha_1^3 + 10\alpha_2^3, -\alpha_1^6 + 3\alpha_2^6 + 4\alpha_1^2 - 4\alpha_2^2,$
$\alpha_1^7 + \alpha_2^7 + 2\alpha_1^5 - 2\alpha_2^5 - 4\alpha_1^3 + 4\alpha_2^3 - 8\alpha_1 + 8\alpha_2) = \overline{a_1}$.
(e) $\beta^*\alpha^* = (227, 85\alpha_1 - 83\alpha_2, -17\alpha_1^2 + 19\alpha_2^2, 17\alpha_1^3 - 15\alpha_2^3 + 34\alpha_1 - 34\alpha_2,$
$15\alpha_1^4 - 13\alpha_2^4, -3\alpha_1^5 + 5\alpha_2^5 + 10\alpha_1^3 - 10\alpha_2^3, 3\alpha_1^6 - \alpha_2^6 - 4\alpha_1^2 + 4\alpha_2^2,$
$\alpha_1^7 + \alpha_2^7 - 2\alpha_1^5 + 2\alpha_2^5 + 4\alpha_1^3 - 4\alpha_2^3 + 8\alpha_1 - 8\alpha_2) = \overline{a_2}$.

Proof. The proof of the lemma can be easily seen by substituting the values of the roots and performing the necessary operations. □

Theorem 2.6 (Catalan's Identity)**.** *For any positive integers n, s such that $n \geq s$, we have*

$$\mathcal{H}M_{k,n-s}\mathcal{H}M_{k,n+s} - \mathcal{H}M_{k,n}^2 = 2^{n-s}M_{k,s}\frac{\overline{a_1}\alpha_2^s - \overline{a_2}\alpha_1^s}{(\alpha_1 - \alpha_2)}.$$

Proof. By using the Binet formula of the hyperbolic k-Mersenne octonions, we have

$$\begin{aligned}\mathcal{H}M_{k,n-s}\mathcal{H}M_{k,n+s}-\mathcal{H}M_{k,n}^2 &= \frac{1}{(\alpha_1-\alpha_2)^2}\left[\alpha^*\beta^*\alpha_1^n\alpha_2^n\left(1-\frac{\alpha_2^s}{\alpha_1^s}\right)+\beta^*\alpha^*\alpha_2^n\alpha_1^n\left(1-\frac{\alpha_1^s}{\alpha_2^s}\right)\right]\\ &= \frac{(\alpha_1\alpha_2)^n(\alpha_1^s-\alpha_2^s)}{(\alpha_1-\alpha_2)^2}\cdot\frac{\alpha^*\beta^*\alpha_2^s-\beta^*\alpha^*\alpha_1^s}{(\alpha_1\alpha_2)^s}\\ &= (\alpha_1\alpha_2)^{n-s}M_{k,s}\frac{\overline{a_1}\alpha_2^s-\overline{a_2}\alpha_1^s}{(\alpha_1-\alpha_2)}\\ &= 2^{n-s}M_{k,s}\frac{\overline{a_1}\alpha_2^s-\overline{a_2}\alpha_1^s}{(\alpha_1-\alpha_2)}.\end{aligned}$$ □

Theorem 2.7 (Cassini's Identity)**.** *For $n \geq 1$, we have*

$$\mathcal{H}M_{k,n-1}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,n}^2 = 2^{n-1}\frac{\overline{a_1}\alpha_2 - \overline{a_2}\alpha_1}{\sqrt{9k^2-8}}.$$

Proof. By substituting $s = 1$ in the Catalan Identity, we obtain the required result. □

Theorem 2.8 (d'Ocagne's Identity)**.** *Let $n, s \geq 0$, then we have*

$$\mathcal{H}M_{k,s}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,s+1}\mathcal{H}M_{k,n} = 2^n\frac{\overline{a_1}\alpha_1^{s-n} - \overline{a_2}\alpha_2^{s-n}}{\sqrt{9k^2-8}}.$$

Proof. By using the Binet formula of the hyperbolic k-Mersenne octonions, we have

$$\begin{aligned}\mathcal{H}M_{k,s}\mathcal{H}M_{k,n+1} - \mathcal{H}M_{k,s+1}\mathcal{H}M_{k,n} &= \frac{\alpha^*\beta^*\alpha_1^s\alpha_2^n(\alpha_1-\alpha_2)+\beta^*\alpha^*\alpha_2^s\alpha_1^n(\alpha_2-\alpha_1)}{(\alpha_1-\alpha_2)^2}\\ &= \frac{(\alpha_1\alpha_2)^n(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2}\left(\alpha^*\beta^*\alpha_1^{s-n}-\beta^*\alpha^*\alpha_2^{s-n}\right)\\ &= 2^n\frac{\overline{a_1}\alpha_1^{s-n}-\overline{a_2}\alpha_2^{s-n}}{\sqrt{9k^2-8}}.\end{aligned}$$ □

Theorem 2.9 (Vajda Identity)**.** *For any natural numbers n, i and j, we have*

$$\mathcal{H}M_{k,n+i}\mathcal{H}M_{k,n+j} - \mathcal{H}M_{k,n}\mathcal{H}M_{k,n+i+j} = -2^nM_{k,i}\frac{\overline{a_1}\alpha_2^j - \overline{a_2}\alpha_1^j}{\sqrt{9k^2-8}}.$$

Proof. Proof is similar to Theorem 2.8 by using Binet formula of hyperbolic k-Mersenne octonions. □

3. Hyperbolic k-Mersenne-Lucas Octonions

In this section, we introduce the hyperbolic k-Mersenne-Lucas octonions and establish their Binet formula. Furthermore, we study some well-known identities of them.

Definition 3.1. For $n \geq 0$, the hyperbolic k-Mersenne-Lucas octonions $\mathcal{H}m_{k,n}$ are defined by

$$\begin{aligned}\mathcal{H}m_{k,n} &= m_{k,n} + m_{k,n+1}i_1 + m_{k,n+2}i_2 + \cdots + m_{k,n+7}i_7\\ &= (m_{k,n}, m_{k,n+1}, m_{k,n+2}, m_{k,n+3}, m_{k,n+4}, m_{k,n+5}, m_{k,n+6}, m_{k,n+7}).\end{aligned}$$

Definition 3.2. For $n \geq 0$, the conjugate of hyperbolic k-Mersenne-Lucas octonions $\mathcal{H}m_{k,n}$ is defined by

$$\begin{aligned}\overline{\mathcal{H}m}_{k,n} &= m_{k,n} - m_{k,n+1}i_1 - m_{k,n+2}i_2 - \cdots - m_{k,n+7}i_7\\ &= (m_{k,n}, -m_{k,n+1}, -m_{k,n+2}, -m_{k,n+3}, -m_{k,n+4}, -m_{k,n+5}, -m_{k,n+6}, -m_{k,n+7}).\end{aligned}$$

If we use Definition 1.2 in Definition 3.1, then we can define the hyperbolic k-Mersenne-Lucas octonions recursively as

$$\mathcal{H}m_{k,n+2} = 3k\mathcal{H}m_{k,n+1} - 2\mathcal{H}m_{k,n}, \quad n \geq 0.$$

Theorem 3.1 (Binet Formula)**.** *The Binet formulas for the hyperbolic k-Mersenne-Lucas octonions and their conjugate are*

(a) $\mathcal{H}m_{k,n} = \alpha^*\alpha_1^n + \beta^*\alpha_2^n$;
(b) $\overline{\mathcal{H}m}_{k,n} = \overline{\alpha^*}\alpha_1^n + \overline{\beta^*}\alpha_2^n$,

where

$$\begin{aligned}\alpha^* &= 1 + \alpha_1 i_1 + \alpha_1^2 i_2 + \alpha_1^3 i_3 + \alpha_1^4 i_4 + \alpha_1^5 i_5 + \alpha_1^6 i_6 + \alpha_1^7 i_7\\ &= (1, \alpha_1, \alpha_1^2, \alpha_1^3, \alpha_1^4, \alpha_1^5, \alpha_1^6, \alpha_1^7),\\ \beta^* &= 1 + \alpha_2 i_1 + \alpha_2^2 i_2 + \alpha_2^3 i_3 + \alpha_2^4 i_4 + \alpha_2^5 i_5 + \alpha_2^6 i_6 + \alpha_2^7 i_7\\ &= (1, \alpha_2, \alpha_2^2, \alpha_2^3, \alpha_2^4, \alpha_2^5, \alpha_2^6, \alpha_2^7),\\ \overline{\alpha^*} &= 1 - \alpha_1 i_1 - \alpha_1^2 i_2 - \alpha_1^3 i_3 - \alpha_1^4 i_4 - \alpha_1^5 i_5 - \alpha_1^6 i_6 - \alpha_1^7 i_7\\ &= (1, -\alpha_1, -\alpha_1^2, -\alpha_1^3, -\alpha_1^4, -\alpha_1^5, -\alpha_1^6, -\alpha_1^7),\\ \overline{\beta^*} &= 1 - \alpha_2 i_1 - \alpha_2^2 i_2 - \alpha_2^3 i_3 - \alpha_2^4 i_4 - \alpha_2^5 i_5 - \alpha_2^6 i_6 - \alpha_2^7 i_7\\ &= (1, -\alpha_2, -\alpha_2^2, -\alpha_2^3, -\alpha_2^4, -\alpha_2^5, -\alpha_2^6, -\alpha_2^7).\end{aligned}$$

Proof. (a) Using Definition 3.1 and the Binet formula of $m_{k,n}$, we have

$$\begin{aligned}\mathcal{H}m_{k,n} =& m_{k,n} + m_{k,n+1}i_1 + m_{k,n+2}i_2 + \cdots + m_{k,n+7}i_7\\ =& (\alpha_1^n + \alpha_2^n) + \left(\alpha_1^{n+1} + \alpha_2^{n+1}\right) i_1 + \left(\alpha_1^{n+2} + \alpha_2^{n+2}\right) i_2 + \left(\alpha_1^{n+3} + \alpha_2^{n+3}\right) i_3\\ &+ \left(\alpha_1^{n+4} + \alpha_2^{n+4}\right) i_4 + \left(\alpha_1^{n+5} + \alpha_2^{n+5}\right) i_5 + \left(\alpha_1^{n+6} + \alpha_2^{n+6}\right) i_6\end{aligned}$$

$$
\begin{aligned}
&+\left(\alpha_1^{n+7}+\alpha_2^{n+7}\right)i_7\\
=&\alpha_1^n(1+\alpha_1 i_1+\alpha_1^2 i_2+\cdots+\alpha_1^7 i_7)+\alpha_2^n(1+\alpha_2 i_1+\alpha_2^2 i_2+\cdots+\alpha_2^7 i_7)\\
=&\alpha^*\alpha_1^n+\beta^*\alpha_2^n.
\end{aligned}
$$

The proof of (b) is similar to (a) by using the Definition 3.2. □

Lemma 3.1. *We have*

(a) $\alpha^*+\beta^*=\mathcal{H}m_{k,0}$;
(b) $\alpha^*-\overline{\beta^*}=\mathcal{H}m_{k,0}-2$.

Proof. The proof of the lemma can be easily seen by substituting the values of α^* and β^* and performing the necessary operations. □

Theorem 3.2 (Catalan's Identity)**.** *For any positive integers n, s such that $n \geq s$, we have*

$$\mathcal{H}m_{k,n-s}\mathcal{H}m_{k,n+s}-\mathcal{H}m_{k,n}^2=2^{n-s}M_{k,s}\sqrt{9k^2-8}(\beta^*\alpha^*\alpha_1^s-\alpha^*\beta^*\alpha_2^s).$$

Proof. By using the Binet formula of the hyperbolic k-Mersenne-Lucas octonions, we have

$$
\begin{aligned}
\mathcal{H}m_{k,n-s}\mathcal{H}m_{k,n+s}-\mathcal{H}m_{k,n}^2=&\left(\alpha^*\alpha_1^{n-s}+\beta^*\alpha_2^{n-s}\right)\left(\alpha^*\alpha_1^{n+s}+\beta^*\alpha_2^{n+s}\right)\\
&-(\alpha^*\alpha_1^n+\beta^*\alpha_2^n)^2\\
=&(\alpha^*)^2\alpha_1^{2n}+\alpha^*\beta^*\alpha_1^{n-s}\alpha_2^{n+s}+\beta^*\alpha^*\alpha_1^{n+s}\alpha_2^{n-s}+(\beta^*)^2\alpha_2^{2n}\\
&-\left((\alpha^*)^2\alpha_1^{2n}+\alpha^*\beta^*\alpha_1^n\alpha_2^n+\beta^*\alpha^*\alpha_1^n\alpha_2^n+(\beta^*)^2\alpha_2^{2n}\right)\\
=&\alpha^*\beta^*\alpha_1^{n-s}\alpha_2^{n+s}+\beta^*\alpha^*\alpha_1^{n+s}\alpha_2^{n-s}-\alpha^*\beta^*\alpha_1^n\alpha_2^n-\beta^*\alpha^*\alpha_1^n\alpha_2^n\\
=&(\alpha_1\alpha_2)^n\left[\alpha^*\beta^*\left(\frac{\alpha_2^s}{\alpha_1^s}-1\right)+\beta^*\alpha^*\left(\frac{\alpha_1^s}{\alpha_2^s}-1\right)\right]\\
=&(\alpha_1\alpha_2)^n(\alpha_1^s-\alpha_2^s)\left(\frac{\beta^*\alpha^*}{\alpha_2^s}-\frac{\alpha^*\beta^*}{\alpha_1^s}\right)\\
=&(\alpha_1\alpha_2)^n(\alpha_1^s-\alpha_2^s)\frac{\beta^*\alpha^*\alpha_1^s-\alpha^*\beta^*\alpha_2^s}{\alpha_1^s\alpha_2^s}\\
=&2^{n-s}M_{k,s}\sqrt{9k^2-8}(\beta^*\alpha^*\alpha_1^s-\alpha^*\beta^*\alpha_2^s),
\end{aligned}
$$

as required. □

Theorem 3.3 (Cassini's Identity)**.** *For $n \geq 1$, we have*

$$\mathcal{H}m_{k,n-1}\mathcal{H}m_{k,n+1}-\mathcal{H}m_{k,n}^2=2^{n-1}\sqrt{9k^2-8}\left(\beta^*\alpha^*\alpha_1-\alpha^*\beta^*\alpha_2\right).$$

Proof. By substituting $s=1$ in the Catalan identity, we obtain the required result. □

Theorem 3.4 (d'Ocagne's Identity)**.** *Let $n, s \geq 0$, then we have*

$$\mathcal{H}m_{k,s}\mathcal{H}m_{k,n+1}-\mathcal{H}m_{k,s+1}\mathcal{H}m_{k,n}=2^n\sqrt{9k^2-8}(\beta^*\alpha^*\alpha_2^{s-n}-\alpha^*\beta^*\alpha_1^{s-n}).$$

Proof. By using the Binet formula of the hyperbolic k-Mersenne-Lucas octonions, we have

$$\begin{aligned}
&\mathcal{H}m_{k,s}\mathcal{H}m_{k,n+1} - \mathcal{H}m_{k,s+1}\mathcal{H}m_{k,n}\\
&= (\alpha^*\alpha_1^s + \beta^*\alpha_2^s)\left(\alpha^*\alpha_1^{n+1} + \beta^*\alpha_2^{n+1}\right) - \left(\alpha^*\alpha_1^{s+1} + \beta^*\alpha_2^{s+1}\right)(\alpha^*\alpha_1^n + \beta^*\alpha_2^n)\\
&=(\alpha^*)^2\alpha_1^{n+s+1} + \alpha^*\beta^*\alpha_1^s\alpha_2^{n+1} + \beta^*\alpha^*\alpha_1^{n+1}\alpha_2^s + (\beta^*)^2\alpha_2^{n+s+1}\\
&\quad - \left((\alpha^*)^2\alpha_1^{n+s+1} + \alpha^*\beta^*\alpha_1^{s+1}\alpha_2^n + \beta^*\alpha^*\alpha_1^n\alpha_2^{s+1} + (\beta^*)^2\alpha_2^{n+s+1}\right)\\
&=\beta^*\alpha^*\alpha_1^n\alpha_2^s(\alpha_1 - \alpha_2) - \alpha^*\beta^*\alpha_1^s\alpha_2^n(\alpha_1 - \alpha_2)\\
&=(\alpha_1\alpha_2)^n(\alpha_1 - \alpha_2)(\beta^*\alpha^*\alpha_2^{s-n} - \alpha^*\beta^*\alpha_1^{s-n})\\
&=2^n\sqrt{9k^2 - 8}(\beta^*\alpha^*\alpha_2^{s-n} - \alpha^*\beta^*\alpha_1^{s-n}).
\end{aligned}$$

□

Theorem 3.5 (Vajda Identity)**.** *For any natural numbers n, i and j, we have*

$$\mathcal{H}m_{k,n+i}\mathcal{H}m_{k,n+j} - \mathcal{H}m_{k,n}\mathcal{H}m_{k,n+i+j} = 2^n M_{k,i}\sqrt{9k^2 - 8}(\alpha^*\beta^*\alpha_2^j - \beta^*\alpha^*\alpha_1^j).$$

Proof. Proof is similar to Theorem 3.4 by using Binet formula of hyperbolic k-Mersenne-Lucas octonions. □

Theorem 3.6. *For $r, s, n \in \mathbb{N}$ such that $r \geq s$, the generating functions for hyperbolic k-Mersenne-Lucas octonions are given as*

(a) $\sum_{n=0}^{\infty} \mathcal{H}m_{k,sn}x^n = \dfrac{\mathcal{H}m_{k,0} - x\left(\mathcal{H}m_{k,0}m_{k,s} - \mathcal{H}m_{k,s}\right)}{1 - m_{k,s}x + 2^s x^2}$;

(b) $\sum_{n=0}^{\infty} \mathcal{H}m_{k,sn+r}x^n = \dfrac{\mathcal{H}m_{k,r} - 2^s\mathcal{H}m_{k,r-s}x}{1 - m_{k,s}x + 2^s x^2}$;

(c) *The exponential generating function is given as*

$$\sum_{n=0}^{\infty} \frac{\mathcal{H}m_{k,sn}x^n}{n!} = \alpha^* e^{\alpha_1^s x} + \beta^* e^{\alpha_2^s x}.$$

Proof. (a) Using the Theorem 3.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}m_{k,sn}x^n &= \sum_{n=0}^{\infty} \left(\alpha^*\alpha_1^{sn} + \beta^*\alpha_2^{sn}\right)x^n = \alpha^* \sum_{n=0}^{\infty}(\alpha_1^s x)^n + \beta^* \sum_{n=0}^{\infty}(\alpha_2^s x)^n\\
&= \alpha^*\left(\frac{1}{1 - \alpha_1^s x}\right) + \beta^*\left(\frac{1}{1 - \alpha_2^s x}\right)\\
&= \frac{(\alpha^* + \beta^*) - x\left(\beta^*\alpha_1^s + \alpha^*\alpha_2^s\right)}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2}\\
&= \frac{(\alpha^* + \beta^*) - x\left(\beta^*\alpha_1^s + \beta^*\alpha_2^s - \beta^*\alpha_2^s + \alpha^*\alpha_2^s + \alpha^*\alpha_1^s - \alpha^*\alpha_1^s\right)}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2}\\
&= \frac{(\alpha^* + \beta^*) - x\left[(\alpha^* + \beta^*)(\alpha_1^s + \alpha_2^s) - (\alpha^*\alpha_1^s + \beta^*\alpha_2^s)\right]}{1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1\alpha_2)^s x^2}\\
&= \frac{\mathcal{H}m_{k,0} - x\left(\mathcal{H}m_{k,0}m_{k,s} - \mathcal{H}m_{k,s}\right)}{1 - m_{k,s}x + 2^s x^2}.
\end{aligned}$$

The proofs of (b) and (c) are similar to that of (a). □

Theorem 3.7. *For $k \neq 1$, we have*

$$\sum_{i=0}^{n} \mathcal{H}m_{k,i} = \frac{2\mathcal{H}m_{k,n} - \mathcal{H}m_{k,n+1} + \mathcal{H}m_{k,1} + \mathcal{H}m_{k,0}(1-3k)}{3(1-k)}.$$

Proof. Using Theorem 3.1, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} \mathcal{H}m_{k,i} &= \sum_{i=0}^{n} \left(\alpha^* \alpha_1^i + \beta^* \alpha_2^i\right) = \alpha^* \sum_{i=0}^{n} \alpha_1^i + \beta^* \sum_{i=0}^{n} \alpha_2^i \\
&= \alpha^* \left(\frac{\alpha_1^{n+1} - 1}{\alpha_1 - 1}\right) + \beta^* \left(\frac{\alpha_2^{n+1} - 1}{\alpha_2 - 1}\right) \\
&= \frac{\alpha_1 \alpha_2 (\alpha^* \alpha_1^n + \beta^* \alpha_2^n) - (\alpha^* \alpha_1^{n+1} + \beta^* \alpha_2^{n+1}) - (\alpha^* \alpha_2 + \beta^* \alpha_1) + (\alpha^* + \beta^*)}{\alpha_1 \alpha_2 - (\alpha_1 + \alpha_2) + 1} \\
&= \frac{2\mathcal{H}m_{k,n} - \mathcal{H}m_{k,n+1} + \mathcal{H}m_{k,1} + \mathcal{H}m_{k,0}(1-3k)}{3(1-k)}.
\end{aligned}$$

□

4. Relations Between Hyperbolic k-Mersenne and k-Mersenne-Lucas Octonions

In this section, we have given theorems showing the relations between hyperbolic k-Mersenne octonions and hyperbolic k-Mersenne-Lucas octonions.

Theorem 4.1. *For $s, n \in \mathbb{N}$, a generalization of the generating function of hyperbolic k-Mersenne octonions is as follows*

$$\sum_{n=0}^{\infty} \mathcal{H}M_{k,sn} x^n = \frac{\mathcal{H}M_{k,0} + (M_{k,s}\mathcal{H}m_{k,0} - \mathcal{H}M_{k,s})\, x}{1 - m_{k,s}x + 2^s x^2}.$$

Proof. Using the Theorem 2.3, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{H}M_{k,sn} x^n &= \sum_{n=0}^{\infty} \left(\frac{\alpha^* \alpha_1^{sn} - \beta^* \alpha_2^{sn}}{\alpha_1 - \alpha_2}\right) x^n = \frac{\alpha^*}{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} (\alpha_1^s x)^n - \frac{\beta^*}{\alpha_1 - \alpha_2} \sum_{n=0}^{\infty} (\alpha_2^s x)^n \\
&= \frac{\alpha^*}{\alpha_1 - \alpha_2} \cdot \frac{1}{1 - \alpha_1^s x} - \frac{\beta^*}{\alpha_1 - \alpha_2} \cdot \frac{1}{1 - \alpha_2^s x} \\
&= \frac{(\alpha^* - \beta^*) + (\beta^* \alpha_1^s - \alpha^* \alpha_2^s)\, x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\
&= \frac{(\alpha^* - \beta^*) + (\beta^* \alpha_1^s + \beta^* \alpha_2^s - \beta^* \alpha_2^s - \alpha^* \alpha_1^s + \alpha^* \alpha_1^s - \alpha^* \alpha_2^s)\, x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\
&= \frac{(\alpha^* - \beta^*) + ((\alpha_1^s - \alpha_2^s)(\alpha^* + \beta^*) - (\alpha^* \alpha_1^s - \beta^* \alpha_2^s))\, x}{(\alpha_1 - \alpha_2)(1 - (\alpha_1^s + \alpha_2^s)x + (\alpha_1 \alpha_2)^s x^2)} \\
&= \frac{\mathcal{H}M_{k,0} + (M_{k,s}\mathcal{H}m_{k,0} - \mathcal{H}M_{k,s})\, x}{1 - m_{k,s}x + 2^s x^2},
\end{aligned}$$

as required. □

Theorem 4.2. *For any integer t, we have*

(a) $\mathcal{H}M_{k,t}^2 + \mathcal{H}m_{k,t}^2 = \dfrac{(9k^2-7)S_{k,2t}^* + (9k^2-9)2^t(\overline{a_1}+\overline{a_2})}{9k^2-8}$;

(b) $\mathcal{H}M_{k,t}^2 - \mathcal{H}m_{k,t}^2 = \dfrac{(9-9k^2)S_{k,2t}^* - (9k^2-7)2^t(\overline{a_1}+\overline{a_2})}{(9k^2-8)}$,

where $\overline{a_1}$ and $\overline{a_2}$ are given in Lemma 2.1.

Proof. (a) From the Binet formulas of the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions, we write

$$\begin{aligned}
&\mathcal{H}M_{k,t}^2 + \mathcal{H}m_{k,t}^2\\
=&\frac{(\alpha^*)^2(\alpha_1)^{2t} - \alpha^*\beta^*\alpha_1^t\alpha_2^t - \beta^*\alpha^*\alpha_2^t\alpha_1^t + (\beta^*)^2(\alpha_2)^{2t}}{9k^2-8}\\
&+ (\alpha^*)^2(\alpha_1)^{2t} + \alpha^*\beta^*\alpha_1^t\alpha_2^t + \beta^*\alpha^*\alpha_2^t\alpha_1^t + (\beta^*)^2(\alpha_2)^{2t}\\
=&\frac{(1+(9k^2-8))((\alpha^*)^2(\alpha_1)^{2t} + (\beta^*)^2(\alpha_2)^{2t}) + ((9k^2-8)-1)\alpha_1^t\alpha_2^t(\alpha^*\beta^*+\beta^*\alpha^*)}{9k^2-8}\\
=&\frac{(9k^2-7)S_{k,2t}^* + (9k^2-9)2^t(\overline{a_1}+\overline{a_2})}{9k^2-8}.
\end{aligned}$$

The proof (b) is similar to that of (a). ☐

Theorem 4.3. *For every integer $r, s \geq t$, there is the following equation*

$$\mathcal{H}M_{k,r+s}\mathcal{H}m_{k,r+t} - \mathcal{H}M_{k,r+t}\mathcal{H}m_{k,r+s} = (\overline{a_1}+\overline{a_2})2^{r-t}M_{k,s-t}.$$

Proof. We write

$$\begin{aligned}
&\mathcal{H}M_{k,r+s}\mathcal{H}m_{k,r+t} - \mathcal{H}M_{k,r+t}\mathcal{H}m_{k,r+s}\\
=&\frac{\alpha^*\beta^*\alpha_1^r\alpha_2^r(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s) + \beta^*\alpha^*\alpha_1^r\alpha_2^r(\alpha_1^s\alpha_2^t - \alpha_1^t\alpha_2^s)}{\alpha_1-\alpha_2}\\
=&(\alpha^*\beta^*+\beta^*\alpha^*)(\alpha_1\alpha_2)^{r-t}\frac{\alpha_1^{s-t}-\alpha_2^{s-t}}{\alpha_1-\alpha_2}\\
=&(\overline{a_1}+\overline{a_2})2^{r-t}M_{k,s-t}.
\end{aligned}$$

☐

Theorem 4.4. *For any integers s and t, we have*

(a) $\mathcal{H}M_{k,s}m_{k,t} = \mathcal{H}M_{k,s+t} + 2^t\mathcal{H}M_{k,s-t}$;

(b) $\mathcal{H}m_{k,s}M_{k,t} = \mathcal{H}m_{k,s+t} + 2^t\mathcal{H}m_{k,s-t}$.

Proof. (a) We have

$$\begin{aligned}
\mathcal{H}M_{k,s}m_{k,t} &= \frac{\alpha^*\alpha_1^{s+t} - \beta^*\alpha_2^{s+t}}{\alpha_1-\alpha_2} + \frac{(\alpha_1\alpha_2)^t(\alpha^*\alpha_1^{s-t} - \beta^*\alpha_2^{s-t})}{\alpha_1-\alpha_2}\\
&= \mathcal{H}M_{k,s+t} + 2^t\mathcal{H}M_{k,s-t}.
\end{aligned}$$

The proof of (b) is similar to that of (a). ☐

Theorem 4.5. *For any integer $t \geq s$, the following equations are true.*

(a) $\mathcal{H}M_{k,s}\mathcal{H}M_{k,t} - \mathcal{H}M_{k,t}\mathcal{H}M_{k,s} = \dfrac{2^s(\overline{a_1} - \overline{a_2})M_{k,t-s}}{\sqrt{9k^2-8}}$;

(b) $\mathcal{H}m_{k,s}\mathcal{H}m_{k,t} - \mathcal{H}m_{k,t}\mathcal{H}m_{k,s} = 2^s\sqrt{9k^2-8}(\overline{a_1} - \overline{a_2})M_{k,t-s}$;

(c) $\mathcal{H}M_{k,t}\mathcal{H}m_{k,s} - \mathcal{H}M_{k,s}\mathcal{H}m_{k,t} = 2^s(\overline{a_1} + \overline{a_2})M_{k,t-s}$;

(d) $\mathcal{H}M_{k,t}\mathcal{H}m_{k,s} - \mathcal{H}m_{k,t}\mathcal{H}M_{k,s} = -2^{s+1}\dfrac{\overline{a_1}\alpha_2^{t-s} - \overline{a_2}\alpha_1^{t-s}}{\sqrt{9k^2-8}}$.

Proof. (a) We have

$$\begin{aligned}
\mathcal{H}M_{k,s}\mathcal{H}M_{k,t} - \mathcal{H}M_{k,t}\mathcal{H}M_{k,s} &= \frac{\alpha^*\beta^*\alpha_1^s\alpha_2^s(\alpha_1^{t-s} - \alpha_2^{t-s}) - \beta^*\alpha^*\alpha_1^s\alpha_2^s(\alpha_1^{t-s} - \alpha_2^{t-s})}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{(\alpha^*\beta^* - \beta^*\alpha^*)(\alpha_1^s\alpha_2^s)(\alpha_1^{t-s} - \alpha_2^{t-s})}{(\alpha_1 - \alpha_2)^2} \\
&= \frac{2^s(\overline{a_1} - \overline{a_2})M_{k,t-s}}{\sqrt{9k^2-8}}.
\end{aligned}$$

The other equations are proved similarly to that of (a). □

5. Conclusion

In this study, we introduced the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions. We obtained Binet formula, Cassini identity, Catalan identity, d'Ocagne identity, Vajda identity, ordinary and exponential generating function, etc. of these octonions. Also, many properties were obtained and studied the relations between hyperbolic k-Mersenne and k-Mersenne-Lucas octonions. As a consequence, for $k = 1$ results hold for hyperbolic Mersenne and Mersenne-Lucas octonions.

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