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Contents

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ON THE q-BESSEL TRANSFORM OF LIPSCHITZ AND DINI-LIPSCHITZ FUNCTIONS ON WEIGHTED SPACE $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$

OTHMAN TYR[1] AND RADOUAN DAHER[1]

Abstract. E. C. Titchmarsh proved some theorems (Theorems 84 and 85) on the classical Fourier transform of functions satisfying conditions related to the Cauchy-Lipschitz conditions in the one-dimensional case. In this paper, we obtain a generalization of those theorems for the q-Bessel transform of a set of functions satisfying the q-Bessel-Lipschitz condition of certain order in suitable weighted spaces $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, where $1 < p \leq 2$. In addition, we introduce the q-Bessel-Dini-Lipschitz condition and we obtain analogous of Titchmarsh's theorems in this occurrence.

1. Introduction

By definition, a function $f = f(t)$ on $\mathbb{R}$ belong to the Lipschitz class $\operatorname{Lip}(\alpha, p; \mathbb{R})$, $0 < \alpha \leq 1$, $p \in [1, +\infty)$, if $f \in L^p(\mathbb{R})$ and

$$\left(\int_{\mathbb{R}} |f(t+h) - f(t)|^p dt\right)^{1/p} = \mathcal{O}(h^\alpha), \tag{1.1}$$

as $h \to 0$. It was first considered by Lipschitz in 1864 while studying the convergence of the Fourier series of a periodic function f. He proved that the inequality (1.1) is sufficient to have that the Fourier series of f converges everywhere to the value of f. A strengthening criterion was introduced by Dini in 1872 whose conclusion states that the convergence is in addition uniform.

A first classical result of Titchmarsh [19, Theorem 84] says that if $0 < \alpha \leq 1$, $1 < p \leq 2$, and

$$f \in \operatorname{Lip}(\alpha, p; \mathbb{R}),$$

Key words and phrases. q-Bessel operator, q-Bessel transform, generalized q-Bessel translation, Lipschitz class, Dini-Lipschitz class, Titchmarsh's theorems.

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then the classical Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{p}{p+\alpha p-1}<\beta\leq\overline{p}=\frac{p}{p-1}.$$

On the other hand, Younis in [21] studied the same phenomena for the wider Dini-Lipschitz class as well as for some other allied classes of functions. More precisely, he proved that if $f\in L^p(\mathbb{R})$ with $1<p\leq 2$, is such that

$$\left(\int_{\mathbb{R}}|f(t+h)-f(t)|^p dt\right)^{1/p}=\mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right),$$

as $h\to 0$, where $0<\alpha\leq 1$, then its Fourier transform $\widehat{f}$ belongs to $L^\beta(\mathbb{R})$, for

$$\frac{p}{p+\alpha p-1}<\beta\leq\overline{p}=\frac{p}{p-1}\quad\text{and}\quad\frac{1}{\beta}<\gamma.$$

A second result of Titchmarch [19, Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform. He proved that if $\alpha\in(0,1)$, then the following statement

$$f\in\operatorname{Lip}(\alpha,2;\mathbb{R}),$$

is equivalent to the statement

$$\int_{|\lambda|\geq N}|\widehat{f}(\lambda)|^2 d\lambda=\mathcal{O}(N^{-2\alpha})\quad\text{as}\quad N\to+\infty.$$

An extension of these theorems to functions of several variables on $\mathbb{R}^n$ and on the torus group $\mathbb{T}^n$ was studied by Younis [20,21], and has also been generalized to general compact Lie groups [22]. Recently, it has also been extended to the case of compact groups [3]. Later, analogous results were given, where considering generalized Fourier transforms: Bessel, Dunkl, q-Dunkl, Jacobi, ... One can cite [2,4–7,11,14].

One may naturally ask what are the analogous results for the q-Bessel transform of Titchmarsh theorems? As far as we know, this question has not been answered yet. In this paper, we try to explore the validity of those theorems in case of the q-Bessel transform in the space $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q,t^{2\nu+1}d_qt)$, where $1<p\leq 2$. For this generalization, we use a generalized q-Bessel translation operator.

This paper is arranged as follows. In Section 2, we state some basic notions and results from q-harmonic analysis related to the q-Bessel transform $\mathcal{F}_{q,\nu}$ that will be needed throughout this paper. Section 3 is devoted to proving Titchmarsh's theorem [19, Theorem 84] for the q-Bessel transform for functions satisfying the q-Bessel-Lipschitz condition in the weighted space $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, where $1<p\leq 2$, and we extend this theorem to functions satisfying the q-Bessel-Dini-Lipschitz condition. In the last section, we obtain a generalization of Titchmarsh's theorem [19, Theorem 85] on the image under the q-Bessel transform of a class functions satisfying a generalized Lipschitz and Dini-Lipschitz condition in the Sobolev space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$.

2. Harmonic Analysis Associated with the q-Bessel Operator

Throughout this paper we consider $0 < q < 1$ and $\nu > -1/2$. We refer to [15] and [17] for the definitions, notations and properties of the q-shifted factorials, the q-hypergeometric functions, the Jackson's q-derivative and the Jackson's q-integrals. The references [8–10, 13, 18] are devoted to the q-Bessel Fourier analysis.

We introduce the following set

$$\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}.$$

We notice that in q-calculus all functions are assumed to have $\mathbb{R}_q^+$ as a domain of definition.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{l=0}^{n-1}(1-aq^l), \quad n = 1, 2, \dots, \tag{2.1}$$

$$(a;q)_\infty = \lim_{n\to+\infty}(a;q)_n = \prod_{l=0}^{+\infty}(1-aq^l).$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C},$$

and

$$[n]_q! = [1]_q \times [2]_q \times \dots \times [n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad n = 0, 1, 2, \dots$$

The q-gamma function is given by (see [1])

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, \dots$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1,$$
$$\lim_{q\to1^-}\Gamma_q(x) = \Gamma(x), \quad \mathrm{Re}(x) > 0.$$

The q-derivative $\mathcal{D}_q f$ of a function f is given by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0.$$

$\mathcal{D}_q f(0) = f'(0)$ provided $f'(0)$ exists.

The q-Jackson integrals are defined by (see [16])

$$\int_0^a f(t)d_qt = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_a^b f(t)d_qt = (1-q)\sum_{n=0}^{+\infty} q^n \left[bf(bq^n) - af(aq^n)\right],$$

$$\int_0^{+\infty} f(t)d_qt = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(t)d_qt = (1-q)a\sum_{n=-\infty}^{-1} q^n f(aq^n).$$

Note that

$$\mathcal{D}_q\left(\int_x^a f(t)d_qt\right) = -f(x) \tag{2.2}$$

and

$$\int_a^b \mathcal{D}_q f(t)d_qt = f(b) - f(a).$$

The q-integration by parts formula is given by

$$\int_a^b g(t)\mathcal{D}_q f(t)d_qt = [f(b)g(b) - f(a)g(a)] - \int_a^b f(qt)\mathcal{D}_q g(t)d_qt. \tag{2.3}$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b f\left(\frac{\lambda}{s}\right)\lambda^{2\nu+1}d_q\lambda = s^{2\nu+2}\int_{\frac{a}{s}}^{\frac{b}{s}} f(t)t^{2\nu+1}d_qt, \quad \text{for all } s \in \mathbb{R}_q^+.$$

Let $\nu \geq -1/2$. The normalized third Jackson q-Bessel function of order ν is defined by (see [1])

$$j_\nu(x,q^2) = \sum_{n=0}^{+\infty}(-1)^n \frac{\Gamma_{q^2}(\nu+1)q^{n(n+1)}}{\Gamma_{q^2}(\nu+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{2.4}$$

For $\lambda \in \mathbb{C}$, the function $x \mapsto j_\nu(\lambda x, q^2)$ is a solution of the following q-differential equation

$$\begin{cases} \Delta_{q,\nu} f(x) = -\lambda^2 f(x), \\ f(0) = 1, \end{cases}$$

where $\Delta_{q,\nu}$ is the q-Bessel operator defined by

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - (1+q^{2\nu})f(x) + q^{2\nu}f(qx)\right], \quad x \in \mathbb{R}_q^+.$$

Lemma 2.1. *i) The following inequalities are valid for a q-Bessel function:*

$$j_\nu(t,q^2) = \mathcal{O}(1), \quad \textit{if } t \geq 0 \textit{ and } t \in \mathbb{R}_q^+, \tag{2.5}$$

$$1 - j_\nu(t,q^2) = \mathcal{O}(1), \quad \textit{if } t \geq 1 \textit{ and } t \in \mathbb{R}_q^+,$$

$$1 - j_\nu(t,q^2) = \mathcal{O}(t^2), \quad \textit{if } t \leq 1 \textit{ and } t \in \mathbb{R}_q^+. \tag{2.6}$$

ii) The inequality

$$|1 - j_\nu(t,q^2)| \geq c \tag{2.7}$$

is true with $t \geq 1$, $t \in \mathbb{R}_q^+$, where $c > 0$ is a certain constant.

Proof. See [12, Lemma 3.1]. □

Moreover, by the relation (2.4), a simple calculation yields

$$\lim_{t\to 0}\frac{1-j_\nu(t,q^2)}{t^2}=\frac{1}{[\nu+1]_{q^2}}\left(\frac{q}{q+1}\right)^2\neq 0, \tag{2.8}$$

hence, there exist $c'>0$ and $\eta>0$ satisfying

$$|t|<\eta \;\Rightarrow\; |1-j_\nu(t,q^2)|\geq c't^2. \tag{2.9}$$

For $1\leq p<+\infty$, we denote by $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ the set of all real functions on $\mathbb{R}^+_q$ for which

$$\|f\|_{q,p,\nu}=\left(\int_0^{+\infty}|f(t)|^p t^{2\nu+1}d_qt\right)^{1/p}<+\infty,$$

and $\mathcal{C}_{q,0}(\mathbb{R}^+_q)$, for the space of functions defined on $\mathbb{R}^+_q$ tending to 0 as $t\to+\infty$ and continuous at 0. The space $\mathcal{C}_{q,0}(\mathbb{R}^+_q)$, when equipped with the topology of uniform convergence, is a complete normed linear space with norm

$$\|f\|_{q,\infty}=\sup_{t\in\mathbb{R}^+_q}|f(t)|.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ associated with the q-Bessel operator $\Delta_{q,\nu}$ is defined for every function f in $\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$ by

$$\mathcal{F}_{q,\nu}(f)(\lambda)=C_{q,\nu}\int_0^{+\infty}f(x)j_\nu(\lambda x,q^2)x^{2\nu+1}d_qx,\quad \text{for all } \lambda\in\mathbb{R}^+_q,$$

where

$$C_{q,\nu}=\frac{(1+q)^{-\nu}}{\Gamma_{q^2}(\nu+1)}.$$

Moreover, it was shown in [8, 13], that q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ satisfies the following properties.

(i) *Riemann-Lebesgue lemma.* Let $f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$, then $\mathcal{F}_{q,\nu}(f)\in\mathcal{C}_{q,0}(\mathbb{R}^+_q)$ and we have

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty}\leq\mathcal{B}_{q,\nu}\|f\|_{q,1,\nu},$$

where

$$\mathcal{B}_{q,\nu}=\frac{1}{1-q}\cdot\frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

(ii) *q-Inversion formula.* If $f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$ such that $\mathcal{F}_{q,\nu}f\in\mathcal{L}^1_{q,\nu}(\mathbb{R}^+_q)$, then for all $x\in\mathbb{R}^+$, we have

$$f(x)=C_{q,\nu}\int_0^{+\infty}\mathcal{F}_{q,\nu}(f)(\lambda)j_\nu(\lambda x,q^2)\lambda^{2\nu+1}d_q\lambda,\quad\text{for all } \lambda\in\mathbb{R}^+_q.$$

(iii) *q-Plancherel formula.* The q-Bessel transform $\mathcal{F}_{q,\nu}$ can be uniquely extended to an isometric isomorphism on $\mathcal{L}^2_{q,\nu}(\mathbb{R}^+_q)$ with

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}=\|f\|_{q,2,\nu}. \tag{2.10}$$

(iv) *q-Hausdorff-Young inequality.* Let $1 < p \leq 2$. If $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, then $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}^{\overline{p}}_{q,\nu}(\mathbb{R}^+_q)$ and

$$\|\mathcal{F}_{q,\nu}(f)\|_{q,\overline{p},\nu} \leq \mathcal{B}_{q,\nu}^{\frac{2}{p}-1}\|f\|_{q,p,\nu}, \tag{2.11}$$

where the numbers p and $\overline{p}$ above are conjugate exponents:

$$\frac{1}{p} + \frac{1}{\overline{p}} = 1.$$

The q-generalized translation operator associated with the q-Bessel transform $T^\nu_{q,h}$, $h \in \mathbb{R}^+_q$ was introduced in [13] and rectified in [8], where it is defined by the use of Jackson's q-integral and the q-shifted factorial as

$$T^\nu_{q,h}f(x) = \int_0^{+\infty} f(t)\mathcal{K}_{q,\nu}(h,x,t)t^{2\nu+1}d_qt,$$

where

$$\mathcal{K}_{q,\nu}(h,x,y) = C^2_{q,\nu}\int_0^{+\infty} j_\nu(ht,q^2)j_\nu(xt,q^2)j_\nu(yt,q^2)t^{2\nu+1}d_qt.$$

In particular the product formula

$$T^\nu_{q,h}j_\nu(x,q^2) = j_\nu(h,q^2)j_\nu(x,q^2)$$

holds.

Lemma 2.2. *For $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $p \geq 1$, we have $T^\nu_{q,h}f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ and*

$$\|T^\nu_{q,h}f\|_{q,p,\nu} \leq \|f\|_{q,p,\nu}.$$

Let $\mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, be the Sobolev space constructed by the q-Bessel operator $\Delta_{q,\nu}$, that is,

$$\mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q) := \{f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q) : \Delta^j_{q,\nu}f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q), j = 1,2,\ldots,m\},$$

where

$$\Delta^0_{q,\nu}f = f, \quad \Delta^j_{q,\nu}f = \Delta_{q,\nu}(\Delta^{j-1}_{q,\nu}f), \quad j = 1,2,\ldots,m.$$

Lemma 2.3. *For $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, we have*

$$\mathcal{F}_{q,\nu}(T^\nu_{q,h}f)(\lambda) = j_\nu(\lambda h,q^2)\mathcal{F}_{q,\nu}(f)(\lambda), \tag{2.12}$$

and if $f \in \mathcal{W}^m_{q,p,\nu}(\mathbb{R}^+_q)$, we get

$$\mathcal{F}_{q,\nu}(\Delta^m_{q,\nu}f)(\lambda) = (-1)^m\lambda^{2m}\mathcal{F}_{q,\nu}(f)(\lambda).$$

Proof. See [9]. □

For every $f \in \mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$, we define the differences $\mathcal{Z}^m_h f$ of order m, $m = 1,2,\ldots$, with step $h > 0$, $h \in \mathbb{R}^+_q$ by:

$$\mathcal{Z}^1_h f(x) = \mathcal{Z}_h f(x) := T^\nu_{q,h}f(x) - f(x),$$
$$\mathcal{Z}^m_h f(x) = \mathcal{Z}_h(\mathcal{Z}^{m-1}_h f(x)), \quad \text{for } m \geq 2.$$

Also, we can write that

$$\mathcal{Z}_h^m f(x) = (T_{q,h}^{\nu} - \mathcal{I})^m f(x),$$

where $\mathcal{I}$ is the identity operator in $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$.

3. Lipschitz and Dini-Lipschitz Condition in the q-Bessel Setting on the Space $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$

In this section, we prove Titchmarsh's theorem [19, Theorem 84] for the q-Bessel transform for functions satisfying the q-Bessel-Lipschitz condition and Younis's theorem [21, Theorem 3.3] on the image under the q-Bessel transform of a class of functions satisfying the q-Bessel-Dini-Lipschitz condition in the space $\mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+, t^{2\nu+1}d_qt)$, where $1 < p \leq 2$. We begin with auxiliary results interesting in themselves.

Lemma 3.1. *Let $f \in \mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ and $h > 0$ with $h \in \mathbb{R}_q^+$, then*

$$\int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{m\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h^m f\|_{q,p,\nu}^{\overline{p}}, \tag{3.1}$$

where K is a positive constant and $m = 0, 1, 2, \ldots$

Proof. By formula (2.12), we obtain

$$\begin{aligned} \mathcal{F}_{q,\nu}(\mathcal{Z}_h f)(\lambda) &= \mathcal{F}_{q,\nu}(T_{q,h}^{\nu} f)(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda) \\ &= j_\nu(\lambda h, q^2)\mathcal{F}_{q,\nu}(f)(\lambda) - \mathcal{F}_{q,\nu}(f)(\lambda) \\ &= (j_\nu(\lambda h, q^2) - 1)\mathcal{F}_{q,\nu}(f)(\lambda). \end{aligned}$$

Using the proof of recurrence for m, we have

$$\mathcal{F}_{q,\nu}(\mathcal{Z}_h^m f)(\lambda) = (j_\nu(\lambda h, q^2) - 1)^m \mathcal{F}_{q,\nu}(f)(\lambda), \quad \text{for all } h \in \mathbb{R}_q^+.$$

Now by q-Hausdorff-Young inequality (2.11), we have the result. □

Remark 3.1. If $f \in \mathcal{W}_{q,p,\nu}^m(\mathbb{R}_q^+)$, from (3.1) we get

$$\int_0^{+\infty} \lambda^{2k\overline{p}} |1 - j_\nu(\lambda h, q^2)|^{m\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h^m (\Delta_{q,\nu}^k f)\|_{q,p,\nu}^{\overline{p}}, \tag{3.2}$$

where $k = 0, 1, \ldots, m$.

Definition 3.1. Let $0 < \alpha < 1$. A function $f \in \mathcal{L}_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ is said to be in the q-Bessel-Lipschitz class, denoted by q-Lip$(\alpha; p, \nu)$, if

$$\|\mathcal{Z}_h f\|_{q,p,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

Lipschitz classes have been constantly employed in Fourier analysis, although they appear in the realm of trigonometric series, more than they occur in Fourier transforms. Now, we are going to give some results associated with Lipschitz functions in the space $L_{q,\nu}^p(\mathbb{R}_q^+)$, $1 < p \leq 2$ for q-Bessel transform. We here prove the following theorem.

Theorem 3.1. *Let f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$ and let f also belong to q-$\mathcal{L}ip(\alpha; p, \nu)$. Then, $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^\beta_{q,\nu}(\mathbb{R}^+_q)$, where*

$$\frac{2p\nu + 2p}{2p + 2\nu(p-1) + \alpha p - 2} < \beta \leq \overline{p} = \frac{p}{p-1}. \tag{3.3}$$

Proof. By using the Hausdorff-Young formula (2.11), we note that the theorem is proved in the case where $\beta = \overline{p}$.

Indeed, we have $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}^{\overline{p}}_{q,\nu}(\mathbb{R}^+_q)$ and for $\nu > -1/2$, $0 < \alpha < 1$, we get

$$\begin{aligned}
1 + \frac{\alpha\overline{p}}{2\nu+2} > 1 \ &\Rightarrow \frac{p-1}{p} \cdot \frac{2\nu + 2 + \alpha\overline{p}}{2\nu+2} > \frac{p-1}{p} \\
&\Rightarrow \frac{(2\nu+2)(p-1) + \alpha p}{p(2\nu+2)} > \frac{1}{\overline{p}} \\
&\Rightarrow \frac{p(2\nu+2)}{(2\nu+2)(p-1) + \alpha p} < \beta = \overline{p} = \frac{p}{p-1}.
\end{aligned}$$

Then, we get

$$\frac{2p\nu + 2p}{2p + 2\nu(p-1) + \alpha p - 2} < \beta = \overline{p} = \frac{p}{p-1}.$$

Assume now that $\beta < \overline{p}$. If f belong to q-$\mathcal{L}ip(\alpha; p, \nu)$, then we have

$$\|\mathcal{Z}_h f\|_{q,p,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

It follows from the formula (3.1) that

$$\int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda \leq K \|\mathcal{Z}_h f\|^{\overline{p}}_{q,p,\nu} \leq K' h^{\alpha\overline{p}},$$

where K' is a positive constant, being the last inequality valid for sufficiently small values of h.

If $0 \leq \lambda \leq \frac{\eta}{h}$, then $0 \leq \lambda h \leq \eta$ and inequality (2.9) imply that

$$|1 - j_\nu(\lambda h, q^2)| \geq c' \lambda^2 h^2.$$

From this, we get

$$\int_0^{\eta/h} h^{2\overline{p}} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{\alpha\overline{p}}).$$

Then

$$\int_0^{\eta/h} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{(\alpha-2)\overline{p}}) \quad \text{as} \quad h \to 0.$$

Thus,

$$\int_0^{\xi} \lambda^{2\overline{p}} |\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}} \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(\xi^{(2-\alpha)\overline{p}}) \quad \text{as} \quad \xi \to +\infty.$$

We consider the function ψ defined by

$$\psi(\xi) = \int_1^{\xi} |\lambda^2 \mathcal{F}_{q,\nu}(f)(\lambda)|^\beta \lambda^{(2\nu+1)\frac{\beta}{\overline{p}}} d_q\lambda.$$

Then, by Hölder inequality we obtain

$$\begin{aligned}\psi(\xi) &\leq \left(\int_1^\xi |\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda\right)^{\beta/\overline{p}}\left(\int_1^\xi d_q\lambda\right)^{(\overline{p}-\beta)/\overline{p}}\\ &= \mathcal{O}\left(\xi^{(2-\alpha)\overline{p}\frac{\beta}{\overline{p}}}\xi^{\frac{\overline{p}-\beta}{\overline{p}}}\right)\\ &= \mathcal{O}\left(\xi^{2\beta-\alpha\beta+1-\frac{\beta}{\overline{p}}}\right).\end{aligned}$$

Furthermore, by (2.2), we can see that

$$\mathcal{D}_q\psi(\lambda) = \left(|\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\right)\lambda^{(2\nu+1)\frac{\beta}{\overline{p}}}. \tag{3.4}$$

According to the q-integration by parts formula (2.3), we get

$$\begin{aligned}&\int_1^\xi |\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\lambda^{2\nu+1}d_q\lambda = \int_1^\xi \lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}}\lambda^{2\nu+1}\mathcal{D}_q\psi(\lambda)d_q\lambda\\ =&\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+1}\psi(\xi) - \int_1^\xi \psi(q\lambda)\mathcal{D}_q(\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+1})d_q\lambda\\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right) - [-2\beta-(2\nu+1)\beta/\overline{p}+2\nu+1]_q\\ &\times\int_1^\xi \psi(q\lambda)\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu}d_q\lambda\\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right) + \mathcal{O}\left(\int_1^\xi \lambda^{1-\alpha\beta+\beta(\frac{p+1}{p})}\lambda^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu}d_q\lambda\right)\\ =&\mathcal{O}\left(\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}\right)\end{aligned}$$

and this is bounded as $\xi \to +\infty$ if

$$-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta\left(\frac{p+1}{p}\right) < 0,$$

that is

$$\beta > \frac{2p\nu+2p}{2p+2\nu(p-1)+\alpha p-2}.$$

We do the same proof for the integral over $(-\xi,-1)$, this proves Theorem 3.1. □

We now generalize Theorem 3.1 as follows.

Corollary 3.1. *Let f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$ and if*

$$\|\mathcal{Z}_h^m f\|_{q,p,\nu} = \mathcal{O}(h^\alpha), \quad 0 < \alpha < m \quad \text{as} \quad h \to 0,$$

then $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^\beta_{q,\nu}(\mathbb{R}^+_q)$, where (3.3) holds.

Remark 3.2. The condition (3.3) can be written in a simple and easier to understand way when it is replaced by

$$\left(\frac{p-1}{p}+\frac{\alpha}{2(\nu+1)}\right)^{-1}<\beta\leq\overline{p}=\frac{p}{p-1}.$$

This also shows directly that the width of the interval for β shrinks as ν increases.

In 1986, Younis studied the same phenomena "Younis's thoerem [21, Theorem 84]" for the wider Dini-Lipschitz class, he replaced $O(h^\alpha)$ by Younes Dini-Lipschitz condition $O\left(h^\alpha(\log(\frac{1}{|h|}))^{-\gamma}\right)$, $\gamma\geq 0$ as $h\to 0$. We now show that Theorem 3.1 could be extended. We begin to define the q-Bessel-Dini Lipschitz class.

Definition 3.2. Let $0<\alpha<1$ and $\gamma\geq 0$, we define the q-Bessel-Dini-Lipschitz class and we denote q-$\mathfrak{D}\operatorname{Lip}((\alpha,\gamma);p,\nu)$, the set of functions f belonging to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$ satisfying

$$\|\mathcal{Z}_h f\|_{q,p,\nu}=O\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right)\quad\text{as}\quad h\to 0.$$

Theorem 3.2. *If $\alpha>2$, $\gamma\geq 0$ and f belong to q-$\mathfrak{D}\operatorname{Lip}((\alpha,\gamma);p,\nu)$, then f is null almost everywhere on $\mathbb{R}^+_q$.*

Proof. Assume that $f\in q$-$\mathfrak{D}\operatorname{Lip}((\alpha,\gamma);p,\nu)$. Then we have

$$\|\mathcal{Z}_h f\|_{q,p,\nu}\leq C\frac{h^\alpha}{(\log\frac{1}{h})^\gamma},\quad \gamma\geq 0.$$

where C is a positive constant, being the last inequality valid for sufficiently small values of h.

From the relation (3.1), we get

$$\int_0^{+\infty}|1-j_\nu(\lambda h,q^2)|^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda\leq KC^{\overline{p}}\frac{h^{\alpha\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}.$$

Then

$$\frac{1}{h^{2\overline{p}}}\int_0^{+\infty}|1-j_\nu(\lambda h,q^2)|^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda\leq KC^{\overline{p}}\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}.$$

Since $\alpha>2$, we have

$$\lim_{h\to 0}\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}=0.$$

Thus,

$$\lim_{h\to 0}\int_0^{+\infty}\lambda^{2\overline{p}}\left(\frac{|1-j_\alpha(\lambda h,q^2)|}{\lambda^2h^2}\right)^{\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda=0.$$

Hence, from relation (2.8), one gets

$$\|\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)\|^{\overline{p}}_{q,\overline{p},\nu}=\int_0^{+\infty}\lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda=0.$$

Hence $\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda) = 0$ for all $\lambda \in \mathbb{R}_q^+$. The injectivity of the q-Bessel transform yields to the wanted result. □

Remark 3.3. The same conclusion holds if we consider a function f such that

$$\|\mathcal{Z}_h^m f\|_{q,p,\nu} = \mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right) \quad \text{as} \quad h \to 0,$$

provided that $\alpha > 2m$, $\gamma \geq 0$ and $1 < p \leq 2$.

Since the same technics previously are available, then we remove details in the proofs of the theorems below.

Theorem 3.3. *Let f belong to q-$\mathcal{D}\,\mathrm{Lip}((\alpha,\gamma);p,\nu)$, $1 < p \leq 2$. Then $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}_{q,\nu}^\beta(\mathbb{R}_q^+)$, where* (3.3) *holds.*

Proof. By analogy with the proof of Theorem 3.1, we can establish the following result:

$$\int_0^{\eta/h} \lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{h^{(\alpha-2)\overline{p}}}{(\log\frac{1}{h})^{\gamma\overline{p}}}\right),$$

hence

$$\int_0^{\xi} \lambda^{2\overline{p}}|\mathcal{F}_{q,\nu}(f)(\lambda)|^{\overline{p}}\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{\xi^{(2-\alpha)\overline{p}}}{(\log\xi)^{\gamma\overline{p}}}\right).$$

Let

$$\psi(\xi) = \int_1^{\xi} |\lambda^2\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\lambda^{(2\nu+1)\frac{\beta}{p}}d_q\lambda.$$

Then, if $\beta < \overline{p}$, by Hölder inequality we obtain

$$\psi(\xi) = \mathcal{O}\left(\frac{\xi^{2\beta-\alpha\beta+1-\frac{\beta}{\overline{p}}}}{(\log\xi)^{\gamma\beta}}\right) \quad \text{as} \quad \xi \to +\infty.$$

By using (3.4), an q-integration by parts yields

$$\int_1^{\xi} |\mathcal{F}_{q,\nu}(f)(\lambda)|^\beta\lambda^{2\nu+1}d_q\lambda = \mathcal{O}\left(\frac{\xi^{-2\beta-(2\nu+1)\frac{\beta}{\overline{p}}+2\nu+2-\alpha\beta+\beta(\frac{p+1}{p})}}{(\log\xi)^{\gamma\beta}}\right)$$

and for the right hand of this estimate to be bounded as $\xi \to +\infty$ one must have

$$-2\beta - (2\nu+1)\frac{\beta}{\overline{p}} + 2\nu + 2 - \alpha\beta + \beta\left(\frac{p+1}{p}\right) < 0,$$

therefore

$$\frac{2p\nu+2p}{2p+2\nu(p-1)+\delta p-2} < \beta \leq \overline{p} = \frac{p}{p-1},$$

and we do the same proof for the integral over $(-\xi,-1)$, this ends the proof of this theorem. □

We shall also generalize Theorem 3.3 to the following corollary.

Corollary 3.2. *Let $\gamma \geq 0$ and f belong to $\mathcal{L}^p_{q,\nu}(\mathbb{R}^+_q)$, $1 < p \leq 2$ such that*

$$\|\mathcal{Z}^m_h f\|_{q,p,\nu} = \mathcal{O}\left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma}\right), \quad 0 < \alpha < m \quad as \quad h \to 0.$$

Then, $\mathcal{F}_{q,\nu}(f)$ belongs to $\mathcal{L}^\beta_{q,\nu}(\mathbb{R}^+_q)$, where (3.3) *holds.*

4. An Equivalence Theorem for the q-Bessel-Lipschitz Class Functions

In this section, we consider $p = 2$. We try to put the previous theorem, Theorem 3.1, into form in which it is reversible. More precisely, we will give a generalization of Titchmarsh's theorem [19, Theorem 85] on the image under the q-Bessel transform of a class functions satisfying a generalized Lipschitz condition in the space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$. The q-Hausdorff-Young inequality (2.11) will likewise be replaced by the q-Plancherel formula (2.10).

We need first to define the q-Bessel-Lipschitz class in the space $\mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$.

Definition 4.1. Let $0 < \alpha < m$, $m \in \mathbb{N}$. A function $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$ is said to be in the q-Bessel-Lipschitz class, denoted by q-$\mathcal{L}ip(\alpha; 2, m, \nu)$, if

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|_{q,2,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

Lemma 4.1. *Let $f \in \mathcal{L}^2_{q,\nu}(\mathbb{R}^+_q)$ and $h > 0$ with $h \in \mathbb{R}^+_q$, then*

$$\|\mathcal{Z}^m_h f\|^2_{q,2,\nu} = \int_0^{+\infty} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda, \tag{4.1}$$

where $m = 0, 1, 2, \ldots$

Proof. The result follows easily by using the q-Plancherel formula (2.10), (2.12) and an induction on m. ☐

Remark 4.1. If $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$, from (4.1) we get

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|^2_{q,2,\nu} = \int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda, \tag{4.2}$$

where $k = 0, 1, \ldots, m$.

Theorem 4.1. *Let $0 < \alpha < m$ and assume that $f \in \mathcal{W}^m_{q,2,\nu}(\mathbb{R}^+_q)$. Then the following statements are equivalent:*

(1) $f \in q$-Lip$(\alpha; 2, m, \nu)$.

(2) $\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha})$ *as $N \to +\infty$.*

Proof. (1) ⇒ (2) Assume that $f \in q$-$\mathcal{L}ip(\alpha; 2, m, \nu)$. Then we have:

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|_{q,2,\nu} = \mathcal{O}(h^\alpha) \quad \text{as} \quad h \to 0.$$

From (4.2), we have

$$\|\mathcal{Z}^m_h(\Delta^k_{q,\nu} f)\|^2_{q,2,\nu} = \int_0^{\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda.$$

If $\frac{1}{h} \leq \lambda \leq \frac{2}{h}$, then $\lambda h \geq 1$ and inequality (2.7) implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_\nu(\lambda h, q^2)|^{2m}.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda &\leq \frac{1}{c^{2m}} \int_{\frac{1}{h}}^{\frac{2}{h}} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq \frac{1}{c^{2m}} \int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &= \frac{1}{c^{2m}} \|\mathcal{Z}_h^m(\Delta_{q,\nu}^k f)\|_{q,2,\nu}^2 \\ &= \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h \to 0.\end{aligned}$$

So, we obtain

$$\int_N^{2N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty.$$

Thus there exists $C > 0$ such that

$$\int_N^{2N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \leq C N^{-2\alpha}.$$

Furthermore, we have

$$\begin{aligned}\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda &= \sum_{l=0}^{+\infty} \int_{2^l N}^{2^{l+1} N} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq C \sum_{l=0}^{+\infty} (2^l N)^{-2\alpha} \\ &= C_\alpha N^{-2\alpha},\end{aligned}$$

where $C_\alpha = C(1 - 2^{-2\alpha})^{-1}$. This proves that

$$\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty.$$

(2) $\Rightarrow$ (1) Suppose now that

$$\int_N^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(N^{-2\alpha}) \quad \text{as} \quad N \to +\infty,$$

we have to show that

$$\int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h \to 0.$$

We write

$$\int_0^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda = \mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 = \int_0^{1/h} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda$$

and

$$\mathcal{J}_2 = \int_{1/h}^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda.$$

Let us estimate the summands $\mathcal{J}_1$ and $\mathcal{J}_2$ from above. From inequality (2.5) of Lemma 2.1, there exists a constant c_1 such that

$$|j_\nu(\lambda h, q^2)| \leq c_1.$$

Hence, from this we conclude that

$$\begin{aligned}\mathcal{J}_2 &= \int_{1/h}^{+\infty} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq (1 + c_1)^{2m} \int_{1/h}^{+\infty} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &= \mathcal{O}(h^{2\alpha}) \quad \text{as} \quad h \to 0. \qquad (4.3)\end{aligned}$$

Now, let us estimate $\mathcal{J}_1$. From inequality (2.6) of Lemma 2.1, there exists a constant c_2 such that

$$|1 - j_\nu(\lambda h, q^2)| \leq c_2 \lambda^2 h^2.$$

Then

$$\begin{aligned}\mathcal{J}_1 &= \int_0^{1/h} \lambda^{4k} |1 - j_\nu(\lambda h, q^2)|^{2m} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq c_2^{2m} \int_0^{1/h} h^{4m} \lambda^{4m} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda \\ &\leq c_2^{2m} h^{2m} \int_0^{1/h} \lambda^{2m} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda. \qquad (4.4)\end{aligned}$$

Now, we need to introduce the function φ defined by

$$\varphi(\lambda) = \int_\lambda^{+\infty} t^{4k} |\mathcal{F}_{q,\nu}(f)(t)|^2 t^{2\nu+1} d_q t.$$

Therefore, it follows from (2.2) that

$$\mathcal{D}_q \varphi(\lambda) = -\lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1}.$$

Now, we apply the q-integration by parts formula (2.3). We obtain

$$\begin{aligned}\int_0^{1/h} \lambda^{2m} \lambda^{4k} |\mathcal{F}_{q,\nu}(f)(\lambda)|^2 \lambda^{2\nu+1} d_q\lambda &= \int_0^{1/h} -\lambda^{2m} \mathcal{D}_q \varphi(\lambda) d_q\lambda \\ &= -\frac{1}{h^{2m}} \varphi\left(\frac{1}{h}\right) + \int_0^{1/h} \varphi(q\lambda) \mathcal{D}_q \lambda^{2m} d_q\lambda \\ &\leq \int_0^{1/h} \varphi(q\lambda) \mathcal{D}_q \lambda^{2m} d_q\lambda \\ &= [2m]_q \int_0^{1/h} \varphi(q\lambda) \lambda^{2m-1} d_q\lambda \\ &= [2m]_q \int_0^{1/h} \mathcal{O}((q\lambda)^{-2\alpha}) \lambda^{2m-1} d_q\lambda\end{aligned}$$

$$=\mathcal{O}\left(\int_0^{1/h}\lambda^{2m-2\alpha-1}d_q\lambda\right).$$

Since

$$\int_0^{1/h}\lambda^{2m-2\alpha-1}d_q\lambda=\left((1-q)\sum_{n=0}^{+\infty}q^{2n(m-\alpha)}\right)h^{2(\alpha-m)}.$$

Then, we conclude that

$$\int_0^{1/h}\lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda=\mathcal{O}(h^{2(\alpha-m)}). \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\mathcal{J}_1=\mathcal{O}\left(h^{2m}\int_0^{1/h}\lambda^{2m}\lambda^{4k}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda\right)=\mathcal{O}(h^{2\alpha})\quad\text{as}\quad h\to 0.$$

Finally, from this and (4.3), we deduce that

$$\int_0^{+\infty}\lambda^{4k}|1-j_\nu(\lambda h,q^2)|^{2m}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda=\mathcal{O}(h^{2\alpha})\quad\text{as}\quad h\to 0,$$

which completes the proof of this theorem. □

Corollary 4.1. *Let $f\in\mathcal{W}^m_{q,2,\nu}(\mathbb{R}_q^+)$ and let $f\in q$-Lip$(\alpha;2,m,\nu)$. Then*

$$\int_N^{+\infty}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda=\mathcal{O}(N^{-4k-2\alpha})\quad as\quad N\to+\infty.$$

We do the same technique of the proof of Theorem 4.1. We get the following theorem.

Theorem 4.2. *Let $0<\alpha<m$, $\gamma\geq 0$ and assume that $f\in\mathcal{W}^m_{q,2,\nu}(\mathbb{R}_q^+)$. Then*

$$\|\mathcal{Z}_h^m(\Delta_{q,\nu}^k f)\|_{q,2,\nu}=\mathcal{O}\left(\frac{h^\alpha}{(\log\frac{1}{h})^\gamma}\right)\quad as\quad h\to 0$$

is equivalent to

$$\int_N^{+\infty}|\mathcal{F}_{q,\nu}(f)(\lambda)|^2\lambda^{2\nu+1}d_q\lambda=\mathcal{O}\left(\frac{N^{-2\alpha-4k}}{(\log N)^{2\gamma}}\right)\quad as\quad N\to+\infty.$$

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References

[1] N. Bettaibi and R. H. Bettaieb, *q-Analogue of the Dunkl transform on the real line*, Tamsui Oxf. J. Math. Sci. **25**(2) (2007), 117–205.
[2] R. Daher, M. El Hamma and A. El Houasni, *Titchmarsh's theorem for the Bessel transform*, Matematika (Johor) **28**(2) (2012), 127–131.
[3] R. Daher, J. Delgado and M. Ruzhansky, *Titchmarsh theorems for Fourier transforms of Hölder-Lipschitz functions on compact homogeneous manifolds*, Monatsh. Math. **189** (2019), 23–49.
[4] R. Daher and O. Tyr, *An analog of Titchmarsh's theorem for the q-Dunkl transform in the space* $L^2_{q,\alpha}(\mathbb{R}_q)$, J. Pseudo-Differ. Oper. Appl. **11** (2020), 1933–1949. `https://doi.org/10.1007/s11868-020-00330-6`
[5] R. Daher and O. Tyr, *Growth properties of the q-Dunkl transform in the space* $L^p_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$, Ramanujan J. **57** (2022), 119–134. `https://doi.org/10.1007/s11139-021-00387-x`
[6] R. Daher and O. Tyr, *Integrability of the Fourier-Jacobi transform of functions satisfying Lipschitz and Dini–Lipschitz-type estimates*, Integral Transforms Spec. Funct. **33**(2) (2021), 115–126. `https://doi.org/10.1080/10652469.2021.1913414`
[7] R. Daher and O. Tyr, *Weighted approximation for the generalized discrete Fourier–Jacobi transform on space* $L_p(\mathbb{T})$, J. Pseudo-Differ. Oper. Appl. **33** (2020), 1685–1697.
[8] L. Dhaouadi, A. Fitouhi and J. El Kamel, *Inequalities in q-Fourier analysis*, Journal of Inequalities in Pure and Applied Mathematics **7**(5) (2006), 1–14
[9] L. Dhaoudi, *On the q-Bessel Fourier transform*, Bull. Math. Anal. Appl. **5**(2) (2013), 42–60.
[10] L. Dhaouadi, *Positivity of the generalized translation associated with the q-Hankel transform and applications*, Integral Transforms Spec. Funct. **26**(2) (2015), 102–117.
[11] S. El Ouadih and R. Daher, *Jacobi-Dunkl Dini Lipschitz functions in the space* $L^p(\mathbb{R}, A_{\alpha,\beta}dx)$, Appl. Math. E-Notes **16** (2016), 88–98.
[12] S. El Ouadih, R. Daher and M. El Hamma, *Moduli of smoothness and K-functional in* $L_2(\mathbb{R}_q^+)$ *space with power weight*, J. Math. Anal. Appl. **45**(3) (2019), 491–503.
[13] A. Fitouhi, M. Hamza and F. Bouzeffour, *The* q-j_α*-Bessel function*, J. Approx. Theory. **115** (2002), 144–166.
[14] R. Ghabi and M. Mili, *Lipschitz conditions for the generalised Fourier transform associated with the Jacobi-Cherednik operator on* $\mathbb{R}$, Adv. Pure Appl. Math. **7**(1) (2016), 51–62.
[15] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications **35**, Cambridge Univ. Press, Cambridge, 1990.
[16] F. H. Jackson, *On a q-definite Integrals*, J. Pure Appl. Math. **41** (1910), 193–203.
[17] V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.
[18] T. H. Koornwinder and R. F. Swarttouw, *On q-analogues of the Hankel and Fourier transform*, Trans. Amer. Math. Soc. **333** (1992), 445–461.
[19] E. C. Titchmarsh, *Introduction of the Theory of Fourier Integrals*, Oxford University Press, Oxford, 1937.
[20] M. S. Younis, *Fourier transforms in* L^p *spaces*, M. Phil. Thesis, Chelsea College, 1970.
[21] M. S. Younis, *Fourier transforms of Dini-Lipschitz functions*, Int. J. Math. Math. Sci. **9**(2) (1986), 301–312.
[22] M. S. Younis, *Fourier transforms of Lipschitz functions on compact groups*, Ph. D. Thesis, McMaster University, Hamilton, 1974.

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ON THE LAPLACIAN COEFFICIENTS OF TREES

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ABSTRACT. Let G be a finite simple graph with Laplacian polynomial $\psi(G,\lambda) = \sum_{k=0}^{n}(-1)^{n-k}c_k(G)\lambda^k$. In an earlier paper, we computed the coefficient of c_{n-4} for trees with respect to some degree-based graph invariant. The aim of this paper is to continue this work by giving an exact formula for the coefficient c_{n-5} in the polynomial $\psi(G,\lambda)$. As a consequence of this work, the Laplacian coefficients c_{n-k}, $k = 2, 3, 4, 5$, for some know trees were computed.

1. DEFINITIONS AND NOTATIONS

Throughout this paper, our graphs will be assumed to be simple, connected and undirected, and the standard notation for such a graphs is used. The notations $n(G)$ and $m(G)$ stand for the number of elements in the vertex set $V(G)$ and the edge set $E(G)$, respectively. The degree of a vertex v in G, $\deg_G(v)$, is the number of edges in G with one end point v and the degree of an edge e in G, $\deg_G(e)$, is the degree of vertex e in the line graph of G. It is easy to see that $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$. The *distance* between two vertices u and v is defined as the length of a sort path connecting them. If $Z \subseteq V(G)$, then the *induced subgraph* $G[Z]$ is the graph with vertex set Z and edge set $\{uv \in E(G) \mid \{u,v\} \subseteq Z\}$.

Suppose G is a graph. The *subdivision graph* $S(G)$ is a graph obtained from G by inserting a new vertex on each edge of G. It is clear from this definition that $n(S(G)) = n(G) + m(G)$ and $m(S(G)) = 2m(G)$.

Suppose $e = xy$ and f are two edges of a graph G and $v \in V(G)$, where $v \neq x, y$. The *common vertex* of e and f is denoted by $e \cap f$ and $e \cap f = \emptyset$ means that e and f are not incident. If $e \cap f = \emptyset$ then e and f are said to be independent. A subset M

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of $E(G)$ is called a matching if all pairs of distinct edges in M are independent. Note that M is a matching in G if $|\{u \mid u \text{ is an end point of an edge in } M\}| = 2|M|$. If M is a matching of size k then we say M is a $k-$matching. Furthermore, the notation $p(G;k)$, $1 \leq k \leq \frac{n}{2}$, is used for the number of distinct $k-$matchings in G. The matchings polynomial of G was first introduced by Godsil and Gutman in [4]. This polynomial is defined as $p(G;0) = 1$ and for other values of x, $\alpha(G,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k p(G;k)x^{n-2k}$.

Suppose G is a simple graph with vertex set $\{a_1, \ldots, a_n\}$. The $0-1$ matrix $A(G) = (a_{ij})$ such that $a_{ij} = 1$ if and only if $v_iv_j \in E(G)$ is called the adjacency matrix of G. The Laplacian matrix of G is another $n \times n$ matrix defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of G whose diagonal entry dii is the degree of a_i in G. It is well-known that the eigenvalues of $L(G)$ are non-negative real numbers with 0 as the smallest eigenvalue. The characteristic polynomial of $L(G)$ is called the Laplacian polynomial of G and its roots are Laplacian eigenvalues of G. In this paper we write this polynomial in the form of $\psi(G,x) = \det(xI_n - L(G)) = \sum_{k=0}^{n}(-1)^{n-k}c_k(G)x^k$.

The first and second Zagreb indices of a graph G are two important degree-based graph invariants that was introduced by two pioneers of *Chemical Graph Theory* Gutman and Trinajstić [6]. These invariants are defined as $M_1(G) = \sum_{v \in V(G)} \deg_G(v)^2$ and $M_2(G) = \sum_{uv \in E(G)} \deg_G(u)\deg_G(v)$. We encourage the interested readers to consult the interesting papers [7] and [14], for more information about mathematical properties and chemical applications of these invariants.

Following Milićević et al. [11], the edge counterpart of the first and second Zagreb indices of a graph G are defined as $EM_1(G) = \sum_{e \sim f}(\deg_G(e) + \deg_G(f)) = \sum_{e \in E(G)} \deg_G(e)^2$ and $EM_2(G) = \sum_{e \sim f} \deg_G(e)\deg_G(f)$, where for $e = uv$, $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$ denotes the degree of the edge e, and $e \sim f$ means that the edges e and f are incident.

Furtula and Gutman [5] studied in details the sum of cubes of degrees of vertices in a graph G and used the name *forgotten index* for this invariant. They defined $F(G) = \sum_{v \in V(G)} \deg_G(v)^3 = \sum_{e=uv \in E(G)}(\deg_G(u)^2 + \deg_G(v)^2)$. The first Zagreb index and the forgotten index can be generalized in the form of $M_1^{\alpha}(G) = \sum_{u \in V(G)} \deg_G(u)^{\alpha}$, where $\alpha \neq 0, 1$ is a real number. Zhang and Zhang [17] obtained some extremal values of this invariant in the class of all unicyclic graphs of a given order. An interesting survey of these degree-based indices is given in [8].

Let T be a tree with Laplacian polynomial

$$\psi(T,x) = \det(xI_n - L(T)) = \sum_{k=0}^{n}(-1)^{n-k}c_k(T)x^k.$$

Merris [12] and Mohar [13] proved that $c_0(T) = 0$, $c_1(T) = n$, $c_n = 1$ and $c_{n-1}(T) = 2(n-1)$. In [16], it is proved that $c_2(T) = W(T)$ and in the paper [15], the authors proved that $c_{n-2}(T) = 2n^2 - 5n + 3 - \frac{1}{2}M_1(T)$ and $c_{n-3}(T) = \frac{1}{3}(4n^3 - 18n^2 + 24n - 10 + F(T) - 3(n-2)M_1(T))$.

Suppose λ is an arbitrary real number. We now define three invariants which is useful in simplifying formulas in our results. These are:

$$\alpha_\lambda(G) = \sum_{uv\in E(G)} \deg_G(u)\deg_G(v)\left(\deg_G(u)^\lambda + \deg_G(v)^\lambda\right),$$
$$\beta(G) = \sum_{e\sim f} \deg_G(e\cap f)\left(\deg_G(e) + \deg_G(f)\right),$$
$$M_2^\lambda(G) = \sum_{uv\in E(G)} \left(\deg_G(u)\deg_G(v)\right)^\lambda .$$

Note that the second Zagreb index is just the case of $\lambda = 1$ in $M_2^\lambda(G)$.

The girth of a graph G, $g(G)$, is defined as the length of a shortest cycle of G. In a recent paper [3], Das et al. proved the following result.

Theorem 1.1. *Let G be a graph with m edges and $g(G) \geq 5$. Then*

$$\begin{aligned} 5p(G;5) =& \frac{1}{24}m(m^4 + 10m^3 + 43m^2 + 54m - 328) + \frac{5}{4}(M_1(G))^2 - \frac{1}{2}\alpha_1(G)(m-7) \\ &- \frac{5}{6}\alpha_2(G) - \frac{1}{12}M_1(G)(2m^3 + 30m^2 + 61m - 225) + \frac{1}{2}\beta(G) \\ &+ \frac{1}{12}M_2(G)(6m^2 + 66m - 239) + \frac{1}{24}F(G)(6m^2 + 24m - 149) \\ &+ \frac{1}{12}M_1^4(G)(m+10) + \frac{1}{4}M_2^2(G) - EM_2(G) - \frac{5}{24}M_1^5(G) \\ &+ \frac{1}{8}\sum_{uv\in E(G)}(M_1(G-\{u,v\}))^2 + \frac{1}{3}\sum_{uv\in E(G)} m(G-\{u,v\})F(G-\{u,v\}) \\ &- \frac{1}{4}\sum_{uv\in E(G)} m^2(G-\{u,v\})M_1(G-\{u,v\}) - \sum_{uv\in E(G)} EM_2(G-\{u,v\}) \\ &+ \sum_{uv\in E(G)} m(G-\{u,v\})M_2(G-\{u,v\}). \end{aligned}$$

The present authors [1,2] proved the following formulas for the coefficient $c_{n-4}(T)$, when T is a tree:

$$\begin{aligned} c_{n-4}(T) =& (n-1)\left(\frac{16}{24}n^3 - 4n^2 + \frac{348}{24}n - \frac{532}{6}\right) + \frac{17}{8}M_1(T)^2 \\ &+ \left(\frac{4}{6}n - \frac{412}{24}\right)F(T) + \frac{39}{2}EM_1(T) - \frac{108}{48}M_1^4(T) - 40M_2(T) \\ &- \left(n^2 + \frac{7}{2}n - \frac{1920}{24}\right)M_1(T) - 16\sum_{\{u,v\}\subset V(T)}\binom{\deg_T(u)}{2}\binom{\deg_T(v)}{2} \\ =& \frac{1}{6}(n-1)(4n^3 - 24n^2 + 39n - 16) + \frac{1}{3}F(G)(2n-5) \\ &+ \frac{1}{8}M_1(T)(-8n^2 + M_1(T) + 36n - 32) - \frac{1}{4}M_1^4(T) - M_2(T). \end{aligned}$$

In this paper, an exact formula for computing the coefficient $c_{n-5}(T)$, T is a tree, with respect to some degree-based topological indices is presented.

2. Main Results

The aim of this section is to present a closed formula for $c_{n-5}(T)$, when T is tree. To do this, we first define five invariants $\chi_1, \chi_2, \chi_3, \chi_4$ and χ_5 with respect to the subdivision graph as follows:

$$\begin{aligned}
\chi_1(S(G)) &= \sum_{uv\in E(S(G))} (M_1(S(G)-\{u,v\}))^2,\\
\chi_2(S(G)) &= \sum_{uv\in E(S(G))} m(S(G)-\{u,v\})F(S(G)-\{u,v\}),\\
\chi_3(S(G)) &= \sum_{uv\in E(S(G))} m^2(S(G)-\{u,v\})M_1(S(G)-\{u,v\}),\\
\chi_4(S(G)) &= \sum_{uv\in E(S(G))} EM_2(S(G)-\{u,v\}),\\
\chi_5(S(G)) &= \sum_{uv\in E(S(G))} m(S(G)-\{u,v\})M_2(S(G)-\{u,v\}).
\end{aligned}$$

Lemma 2.1. *Let G be a graph with m edges. Then*

$$\begin{array}{ll}
M_1(S(G)) = M_1(G)+4m, & F(S(G)) = F(G)+8m,\\
M_1^4(S(G)) = M_1^4(G)+16m, & M_1^5(S(G)) = M_1^5(G)+32m,\\
\alpha_1(S(G)) = 4M_1(G)+2F(G), & \alpha_2(S(G)) = 8M_1(G)+2M_1^4(G),\\
\beta(S(G)) = 2M_1(G)+M_1^4(G)-F(G), & M_2(S(G)) = 2M_1(G),\\
EM_2(S(G)) = M_2(G)+\frac{1}{2}M_1^4(G)-\frac{1}{2}F(G), & M_2^2(S(G)) = 4F(G).
\end{array}$$

Proof. By definition of subdivision graph, we have:

$$\begin{aligned}
M_1(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^2 + \sum_{uv\in E(G)} 4 = M_1(G)+4m,\\
F(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^3 + \sum_{uv\in E(G)} 8 = F(G)+8m,\\
M_1^4(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^4 + \sum_{uv\in E(G)} 16 = M_1^4(G)+16m,\\
M_1^5(S(G)) &= \sum_{v\in V(G)} \deg_G(v)^5 + \sum_{uv\in E(G)} 32 = M_1^5(G)+32m,\\
\alpha_1(S(G)) &= \sum_{v\in V(G)} \sum_{uv\in E(G)} 2\deg_G(v)(2+\deg_G(v))\\
&= \sum_{v\in V(G)} \sum_{uv\in E(G)} (4\deg_G(v)+2\deg_G(v)^2)\\
&= \sum_{v\in V(G)} \deg_G(v)(4\deg_G(v)+2\deg_G(v)^2) = 4M_1(G)+2F(G),
\end{aligned}$$

$$\begin{aligned}
\alpha_2(S(G)) &= \sum_{v\in V(G)}\sum_{uv\in E(G)} 2\deg_G(v)(4+\deg_G(v)^2)\\
&= \sum_{v\in V(G)}\sum_{uv\in E(G)} (8\deg_G(v)+2\deg_G(v)^3)\\
&= \sum_{v\in V(G)} \deg_G(v)(8\deg_G(v)+2\deg_G(v)^3) = 8M_1(G)+2M_1^4(G),\\
\beta(S(G)) &= \sum_{uv\in E(G)} 2(\deg_G(u)+\deg_G(v))\\
&\quad + \sum_{v\in V(G)} \binom{\deg_G(v)}{2} \deg_G(v)(\deg_G(v)+\deg_G(v))\\
&= 2M_1(G) + \sum_{v\in V(G)} \deg_G(v)(\deg_G(v)-1)\deg_G(v)^2\\
&= 2M_1(G)+M_1^4(G)-F(G),\\
M_2(S(G)) &= \sum_{v\in V(G)}\sum_{uv\in E(G)} 2\deg_G(v) = \sum_{v\in V(G)} 2\deg_G(v)^2 = 2M_1(G),\\
M_2^2(S(G)) &= \sum_{v\in V(G)}\sum_{uv\in E(G)} 4\deg_G(v)^2 = \sum_{v\in V(G)} 4\deg_G(v)^3 = 4F(G),\\
EM_2(S(G)) &= \sum_{uv\in E(G)} \deg_G(u)\deg_G(v) + \sum_{v\in V(G)} \binom{\deg_G(v)}{2}\deg_G(v)^2\\
&= M_2(G) + \frac{1}{2}\sum_{v\in V(G)} \deg_G(v)(\deg_G(v)-1)\deg_G(v)^2\\
&= M_2(G)+\frac{1}{2}M_1^4(G)-\frac{1}{2}F(G),
\end{aligned}$$

proving the lemma. □

Lemma 2.2. $\chi_1(S(G)) = (2m-10)(M_1(G))^2+(16m^2-2F(G)-40m)M_1(G)+32m^3-8mF(G)+13F(G)+6M_1^4(G)+M_1^5(G)+24M_2(G)+4\alpha_1(G)$.

Proof. By definition of the graph $S(G)$,

$$\begin{aligned}
\chi_1(S(G)) &= \sum_{v\in V(G)}\sum_{uv\in E(G)} (M_1(S(G))-\deg_G(v)^2-3\deg_G(v)-2\deg_G(u))^2\\
&= \sum_{v\in V(G)}\sum_{uv\in E(G)} (\deg_G(v)^4+6\deg_G(v)^3+9\deg_G(v)^2)\\
&\quad - M_1(S(G))\sum_{v\in V(G)}\sum_{uv\in E(G)} (2\deg_G(v)^2+6\deg_G(v))\\
&\quad + \sum_{v\in V(G)}\sum_{uv\in E(G)} M_1(S(G))^2\\
&\quad + \sum_{v\in V(G)}\sum_{uv\in E(G)} (4\deg_G(u)\deg_G(v)^2+12\deg_G(u)\deg_G(v))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(-4M_1(S(G))\deg_G(u)+4\deg_G(u)^2)\\
=&M_1^5(G)+6M_1^4(G)+9F(G)-2M_1(S(G))F(G)-6M_1(S(G))M_1(G)\\
&+2mM_1(S(G))^2+\sum_{uv\in E(G)}(4\deg_G(u)\deg_G(v)^2+4\deg_G(u)^2\deg_G(v)\\
&+24\deg_G(u)\deg_G(v))+\sum_{uv\in E(G)}(-4M_1(S(G))\deg_G(u)+4\deg_G(u)^2\\
&-4M_1(S(G))\deg_G(v)+4\deg_G(v)^2)\\
=&M_1^5(G)+6M_1^4(G)+13F(G)-2M_1(S(G))F(G)-10M_1(S(G))M_1(G)\\
&+2mM_1(S(G))^2+4\alpha_1(G)+24M_2(G).
\end{aligned}
$$

We now apply Lemma 2.1 to deduce that

$$
\begin{aligned}
\chi_1(S(G))=&(2m-10)(M_1(G))^2+(16m^2-2F(G)-40m)M_1(G)+32m^3-8mF(G)\\
&+13F(G)+6M_1^4(G)+M_1^5(G)+24M_2(G)+4\alpha_1(G),
\end{aligned}
$$

which completes the proof. □

Lemma 2.3. $\chi_2(S(G))=32m^3+(4F(G)-24)m^2-(8F(G)+16M_1(G)+2M_1^4(G))m + 4m-(M_1(G)-10)F(G)+6M_1(G)+M_1^4(G)+M_1^5(G)-6M_2+3\alpha_1(G)$.

Proof. It is easy to see that $\chi_2(S(G))=\sum_{v\in V(G)}\sum_{uv\in E(G)}(2m-\deg_G(v)-1)(F(S(G)) - \deg_G(v)^3-3\deg_G(u)^2+3\deg_G(u)-7\deg_G(v)-2)$. If we expand the summation, this becomes:

$$
\begin{aligned}
\chi_2(S(G))=&\sum_{v\in V(G)}\sum_{uv\in E(G)}(\deg_G(v)^4+\deg_G(v)^3+7\deg_G(v)^2+9\deg_G(v)-F(S(G))\\
&+2(1-m\deg_G(v)^3-7m\deg_G(v)-2m+mF(S(G)))-F(S(G))\deg_G(v))\\
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(-6m\deg_G(u)^2+6m\deg_G(u)+3\deg_G(u)^2-3\deg_G(u))\\
&+\sum_{v\in V(G)}\sum_{uv\in E(G)}(3\deg_G(u)^2\deg_G(v)-3\deg_G(u)\deg_G(v))\\
=&M_1^5(G)+M_1^4(G)+7F(G)+9M_1(G)+4m-2mM_1^4(G)-14mM_1(G)\\
&-8m^2+4m^2F(S(G))-F(S(G))M_1(G)-2mF(S(G))\\
&+\sum_{uv\in E(G)}(-6m\deg_G(u)^2+6m\deg_G(u)+3\deg_G(u)^2-3\deg_G(u)\\
&-6m\deg_G(v)^2+6m\deg_G(v)+3\deg_G(v)^2-3\deg_G(v))\\
&+\sum_{uv\in E(G)}(3\deg_G(u)^2\deg_G(v)-6\deg_G(u)\deg_G(v)+3\deg_G(v)^2\deg_G(u))\\
=&M_1^5(G)+M_1^4(G)+10F(G)+6M_1(G)+4m-2mM_1^4(G)-8mM_1(G)\\
&-8m^2+4m^2F(S(G))-F(S(G))M_1(G)-2mF(S(G))-6mF(G)
\end{aligned}
$$

$$+ 3\alpha_1(G) - 6M_2(G).$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_2(S(G)) =& 32m^3 + (4F(G) - 24)m^2 - (8F(G) + 16M_1(G) + 2M_1^4(G))m + 4m \\ & - (M_1(G) - 10)F(G) + 6M_1(G) + M_1^4(G) + M_1^5(G) - 6M_2 + 3\alpha_1(G).\end{aligned}$$

This completes the proof. □

Lemma 2.4. *$\chi_3(S(G)) = 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 +(20F(G) -4(M_1(G))^2+30M_1(G)+4M_1^4(G)+16M_2(G))m+F(G)M_1(G) +2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G) - 8M_2(G) - 2\alpha_1(G)$.*

Proof. The degree sequence of subdivision graph $S(G)$ shows that $\chi_3(S(G)) = \sum_{v\in V(G)} \sum_{uv\in E(G)}(2m-\deg_G(v)-1)^2(M_1(S(G))-\deg_G(v)^2-3\deg_G(v)-2\deg_G(u))$. By expanding this summation,

$$\begin{aligned}\chi_3(S(G)) =& \sum_{v\in V(G)} \sum_{uv\in E(G)} (4\deg_G(v)^3m - \deg_G(v)^4 - 4\deg_G(v)^2m^2 \\ & + M_1(S(G))\deg_G(v)^2 - 4M_1(S(G))\deg_G(v)m + 4M_1(S(G))m^2 \\ & - 5\deg_G(v)^3 + 16\deg_G(v)^2m - 12\deg_G(v)m^2 + 2M_1(S(G))\deg_G(v) \\ & - 4M_1(S(G))m - 7\deg_G(v)^2 + 12\deg_G(v)m + M_1(S(G)) - 3\deg_G(v)) \\ & + \sum_{v\in V(G)} \sum_{uv\in E(G)} (8\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 8\deg_G(u)m^2 \\ & - 4\deg_G(u)\deg_G(v) + 8\deg_G(u)m - 2\deg_G(u)) \\ =& 4M_1^4(G)m - M_1^5(G) - 4F(G)m^2 + M_1(S(G))F(G) \\ & - 4M_1(S(G))M_1(G)m + 8M_1(S(G))m^3 - 5M_1^4(G) + 16F(G)m \\ & - 12M_1(G)m^2 + 2M_1(S(G))M_1(G) - 8M_1(S(G))m^2 - 7F(G) \\ & + 12M_1(G)m + 2mM_1(S(G)) - 3M_1(G) \\ & + \sum_{uv\in E(G)} (16\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 2\deg_G(v)\deg_G(u)^2 \\ & - 8\deg_G(u)m^2 - 8\deg_G(v)m^2 - 8\deg_G(u)\deg_G(v) + 8\deg_G(u)m \\ & + 8\deg_G(v)m - 2\deg_G(u) - 2\deg_G(v)) \\ =& 4M_1^4(G)m - 4F(G)m^2 + M_1(S(G))F(G) - 4M_1(S(G))M_1(G)m \\ & + 8M_1(S(G))m^3 + 16F(G)m - 20M_1(G)m^2 + 2M_1(S(G))M_1(G) \\ & - 8M_1(S(G))m^2 - 7F(G) + 20M_1(G)m + 2mM_1(S(G)) - 5M_1(G) \\ & + 16mM_2(G) - 2\alpha_1(G) - 8M_2(G) - M_1^5(G) - 5M_1^4(G).\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_3(S(G)) =& 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 \\ & + (20F(G) - 4(M_1(G))^2 + 30M_1(G) + 4M_1^4(G) + 16M_2(G))m\end{aligned}$$

$$+ F(G)M_1(G) + 2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G)$$
$$- 8M_2(G) - 2\alpha_1(G).$$

Hence, the result follows. □

Lemma 2.5. $\chi_4(S(G)) = \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G) - \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G)$.

Proof. By relation between adjacencies in G and $S(G)$, we can see that

$$\begin{aligned}
\chi_4(S(G)) =& 2mEM_2(S(G)) - \sum_{v\in V(G)} \left(\binom{\deg_G(v)}{2} \deg_G(v)^3 + (\deg_G(v) - 1)\deg_G(v)^3 \right. \\
& \left. + \binom{\deg_G(v) - 1}{2} \deg_G(v)^3 - \binom{\deg_G(v) - 1}{2} (\deg_G(v) - 1)^2 \deg_G(v) \right) \\
& - \sum_{v\in V(G)} \sum_{uv\in E(G)} (\deg_G(u) - 1)(\deg_G(v)^2(\deg_G(v) - 1) - \deg_G(v) \\
& \times (\deg_G(v) - 1)^2) - \sum_{uv\in E(G)} (\deg_G(u)\deg_G(v)(\deg_G(u) + \deg_G(v)) \\
& + \deg_G(u)\deg_G(v)(\deg_G(u) + \deg_G(v) - 2) \\
& - (\deg_G(u) - 1)^2 \deg_G(v) - \deg_G(u)(\deg_G(v) - 1)^2) \\
=& 2mEM_2(S(G)) - \sum_{v\in V(G)} \left(\frac{1}{2}\deg_G(v)^5 + \frac{3}{2}\deg_G(v)^4 - \frac{9}{2}\deg_G(v)^3 \right. \\
& \left. + \frac{7}{2}\deg_G(v)^2 - \deg_G(v) \right) - \sum_{uv\in E(G)} (\deg_G(u)\deg_G(v)^2 + \deg_G(v)\deg_G(u)^2 \\
& - 2\deg_G(u)\deg_G(v) - \deg_G(u)^2 - \deg_G(v)^2 + \deg_G(u) + \deg_G(v)) \\
& - \sum_{uv\in E(G)} (\deg_G(u)^2\deg_G(v) + \deg_G(u)\deg_G(v)^2 + 2\deg_G(u)\deg_G(v) \\
& - \deg_G(u) - \deg_G(v)) \\
=& 2mEM_2(S(G)) - \left(\frac{1}{2}M_1^5(G) + \frac{3}{2}M_1^4(G) - \frac{11}{2}F(G) + \frac{7}{2}M_1(G) - 2m \right) \\
& - 2\alpha_1(G).
\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}
\chi_4(S(G)) =& \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G) \\
& - \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G),
\end{aligned}$$

which is our goal. □

Lemma 2.6. $\chi_5(S(G)) = (8M_1(G) + 8)m^2 - (4F(G) + 10M_1(G) + 4M_2(G) + 4)m - 2(M_1(G))^2 + 2F(G) + M_1(G) + 2M_1^4(G) + 8M_2(G) + \alpha_1(G)$.

Proof. Again definition of subdivision graph,

$$\begin{aligned}
\chi_5(S(G)) = &\sum_{uv\in E(G)} ((2m - \deg_G(u) - 1)(M_2(S(G)) - 2\deg_G(u)^2 - 2\deg_G(v)\\
&- \sum_{wu\in E(G)} \deg_G(w) + \deg_G(v) - 2\deg_G(v)(\deg_G(v) - 1)\\
&+ 2(\deg_G(v) - 1)^2) + (2m - \deg_G(v) - 1)(M_2(S(G))\\
&- 2\deg_G(v)^2 - 2\deg_G(u) - \sum_{vz\in E(G)} \deg_G(z)\\
&+ \deg_G(u) - 2\deg_G(u)(\deg_G(u) - 1) + 2(\deg_G(u) - 1)^2))\\
= &\sum_{uv\in E(G)} (2\deg_G(u)^3 + 2\deg_G(v)^3 - 4\deg_G(u)^2 m - 4\deg_G(v)^2 m\\
&- M_2(S(G))\deg_G(u) - M_2(S(G))\deg_G(v) + 4M_2(S(G))m - 2M_2(S(G))\\
&+ 8m - 4 + 2\deg_G(u)^2 + 2\deg_G(v)^2 + 6\deg_G(u)\deg_G(v) - 6\deg_G(u)m\\
&- 6\deg_G(v)m + \deg_G(u) + \deg_G(v)) - \sum_{uv\in E(G)} ((2m - \deg_G(u) - 1)\\
&\sum_{wu\in E(G)} \deg_G(w) - (2m - \deg_G(v) - 1) \sum_{vz\in E(G)} \deg_G(z))\\
=&2M_1^4(G) - 4mF(G) - M_2(S(G))M_1(G) + 4M_2(S(G))m^2 - 2mM_2(S(G))\\
&+ 8m^2 - 4m + 2F(G) + 6M_2(G) - 6mM_1(G) + M_1(G)\\
&- \sum_{uv\in E(G)} ((2m - \deg_G(u) - 1)\deg_G(v)\deg_G(u)\\
&+ (2m - \deg_G(v) - 1)\deg_G(u)\deg_G(v))\\
=&2M_1^4(G) - 4mF(G) - M_2(S(G))M_1(G) + 4M_2(S(G))m^2 - 2mM_2(S(G))\\
&+ 8m^2 - 4m + 2F(G) + 8M_2(G) - 6mM_1(G) + M_1(G) + \alpha_1(G)\\
&- 4mM_2(G),
\end{aligned}$$

Now, by Lemma 2.1,

$$\begin{aligned}
\chi_5(S(G)) =&(8M_1(G) + 8)m^2 - (4F(G) + 10M_1(G) + 4M_2(G) + 4)m - 2(M_1(G))^2\\
&+ 2F(G) + M_1(G) + 2M_1^4(G) + 8M_2(G) + \alpha_1(G),
\end{aligned}$$

which proving the lemma. □

Let G be a graph. It is easy to see that $g(S(G)) \geq 6$. Therefore, by Lemma 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 and Theorem 1.1, we have the following theorem.

Theorem 2.1. *Let G be a graph with m edges. Then*

$$\begin{aligned}
p(S(G);5) =&\frac{1}{15}m^2(4m^3 - 20m^2 + 15m + 15) + \frac{1}{12}m(8F(G)m - 8M_1(G)m^2\\
&+ 3(M_1(G))^2 + 36M_1(G)m - 28F(G) - 24M_1(G) - 6M_1^4(G) - 24M_2(G))
\end{aligned}$$

$$- \frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G)$$
$$+ M_1^4(G) + F(G).$$

We are now ready to prove our main result. For the sake of completeness, we mention here a useful result of Zhou and Gutman [18].

Theorem 2.2. *Let G be an n-vertex tree. Then $c_{n-k}(G) = p(S(G); k)$, for $0 \leq k \leq n$.*

Theorem 2.3. *Let G be an acyclic graph on n vertices and m edges. Then*

$$\begin{aligned} c_{n-5}(G) =& \frac{1}{15}m^2(4m^3 - 20m^2 + 15m + 15) + \frac{1}{12}m(8F(G)m - 8M_1(G)m^2 \\ &+ 3(M_1(G))^2 + 36M_1(G)m - 28F(G) - 24M_1(G) - 6M_1^4(G) - 24M_2(G)) \\ &- \frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G) \\ &+ M_1^4(G) + F(G). \end{aligned}$$

Proof. Apply Theorem 2.1 and 2.2. □

Corollary 2.1. *Let T be a tree on n vertices. Then*

$$\begin{aligned} c_{n-5}(G) =& \frac{1}{15}(n-1)^2(4n^3 - 32n^2 + 67n - 24) + \frac{1}{12}n(8nF(G) - 8n^2M_1(G) \\ &+ 3(M_1(G))^2 + 60nM_1(G) - 44F(G) - 120M_1(G) - 6M_1^4(G) - 24M_2(G)) \\ &- \frac{1}{12}M_1(G)(2F(G) + 9M_1(G) - 56) + \alpha_1(G) + \frac{1}{5}M_1^5(G) + \frac{3}{2}M_1^4(G) \\ &+ 4M_2(G) + 4F(G). \end{aligned}$$

Proof. The result follows from Theorem 2.3 and the fact that $m(T) = n - 1$. □

3. Applications

The aim of this section is to apply our results in Section 2 for computing the Laplacian coefficients $c_{n-k}(G)$, $k = 2, 3, 4, 5$, when G is a certain tree. We first assume that $T(k, t)$ be a rooted tree with degree sequence $k, k, \ldots, k, 1, 1, \ldots, 1$ and t is the distance between the center and any pendent vertex, Figure 1. Then,

$$\begin{aligned} c_{n-1}(T(k,t)) =& \frac{k}{k-2}(2(k-1)^t - 1)), \\ c_{n-2}(T(k,t)) =& \frac{k}{2(k-2)^2}((k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k^2 - 2k(k-1)^t + k - 2)), \\ c_{n-3}(T(k,t)) =& -\frac{k}{3(k-2)^3}((k-1)(k-2)^2(k^2 - 3k(k-1)^t + 5k - 8)(k-1)^{t-1} \\ &- 2k((k-1)^t - 1)(k^2 - 2k(k-1)^t + 3k - 6)(k - (k-1)^t - 1)), \\ c_{n-4}(T(k,t)) =& \frac{k}{4(k-2)^4}((1/2)k(k-1)^2(k-2)^4((k-1)^{t-1})^2 + (4((k-1)^t)^2k^2 \\ &+ (-(14/3)k^3 - (34/3)k^2 + (76/3)k)(k-1)^t + k^4 + (29/3)k^3 - (53/3)k^2 \end{aligned}$$

$$
\begin{aligned}
&- (52/3)k + 28)(k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)\\
&- (4/3)((k-1)^t)^3k^3 + (4k^4 - 8k^2)((k-1)^t)^2 + (-(11/3)k^5 - (10/3)k^4\\
&+ (85/3)k^3 - 20k^2 - 4k)(k-1)^t + (k^5 + (14/3)k^4 - 18k^3 + 6k^2\\
&+ 16k - 8)(k-1))),\\
c_{n-5}(T(k,t)) =& - \frac{k}{5(k-2)^5}((5/6)(k^2 - (3/2)k(k-1)^t + (7/2)k - 8)(k-1)^2(k-2)^4k(\\
&(k-1)^{t-1})^2 + (-(10/3)((k-1)^t)^3k^3 + (25/3)(k^2 + (7/5)k - (22/5))k^2\\
&\times ((k-1)^t)^2 - (35/6)(k^4 + 5k^3 - (99/7)k^2 - (32/7)k + (116/7))k(k-1)^t\\
&+ k^6 + (95/6)k^5 - (223/6)k^4 - (341/6)k^3 + 158k^2 - (118/3)k - 48)(k-1)\\
&(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k - 3/2 - (1/2)(k-1)^t)k((-(4/3)((k-1)^t)^3k^3\\
&+ (4k^4 + (8/3)k^3 - (40/3)k^2)((k-1)^t)^2 + (-(11/3)k^5\\
&- 10k^4 + (137/3)k^3 - 20k^2 - 20k)(k-1)^t + k^6 + (23/3)k^5 - 32k^4\\
&+ (310/3)k^2 - 120k + 40)),\\
c_{n-1}(T(3,t)) =& 6 \times 2^t - 6,\\
c_{n-2}(T(3,t)) =& - \frac{93}{2}2^t + 18 \times 2^{2t} + 30,\\
c_{n-3}(T(3,t)) =& 272 \times 2^t - 171 \times 2^{2t} + 36 \times 2^{3t} - 144,\\
c_{n-4}(T(3,t)) =& - \frac{5799}{4}2^t + \frac{9177}{8}2^{2t} - 405 \times 2^{3t} + 54 \times 2^{4t} + 687,\\
c_{n-5}(T(3,t)) =& \frac{74427}{10}2^t - \frac{26967}{4}2^{2t} + \frac{12267}{4}2^{3t} - 702 \times 2^{4t} + \frac{324}{5}2^{5t} - 3294,\\
c_{n-1}(T(4,t)) =& 4 \times 3^t - 4,\\
c_{n-2}(T(4,t)) =& - 24 \times 3^t + 8 \times 3^{2t} + 18,\\
c_{n-3}(T(4,t)) =& \frac{392}{3}3^t - 64 \times 3^{2t} + \frac{32}{3}3^{3t} - 88,\\
c_{n-4}(T(4,t)) =& - \frac{2132}{3}3^t + \frac{1232}{3}3^{2t} - \frac{320}{3}3^{3t} + \frac{32}{3}3^{4t} + 457,\\
c_{n-5}(T(4,t)) =& \frac{19644}{5}3^t - 2480 \times 3^{2t} + \frac{2368}{3}3^{3t} - 128 \times 3^{4t} + \frac{128}{15}3^{5t} - 2484.
\end{aligned}
$$

Our second class of trees are known as Kragujevac trees. To define, we assume that $B_1, B_2, B_3, \dots, B_k$ are branches whose structure is depicted in Figure 2. A proper Kragujevac tree is a tree possessing a central vertex of degree at least 3, to which branches of the form B_1 and/or B_2 and/or B_3 and/or ... are attached [10].

Let G_i, for $i = 1, 2, \dots, 7$, be the proper Kragujevac tree on n vertices in Figure 3. Then

$$
\begin{aligned}
c_{n-2}(G_1) &= \frac{3}{98}n(65n - 231) - 3, \;\; c_{n-3}(G_1) = \frac{8}{1029}n(169n^2 - 1302n + 1127) + 60,\\
c_{n-4}(G_1) &= \frac{1}{57624}n(37349n^3 - 503594n^2 + 1625575n + 4758782) - 462,\\
c_{n-5}(G_1) &= \frac{3}{336140}n(28561n^4 - 597415n^3 + 3893785n^2 + 1016995n - 107579206) + 2868,
\end{aligned}
$$

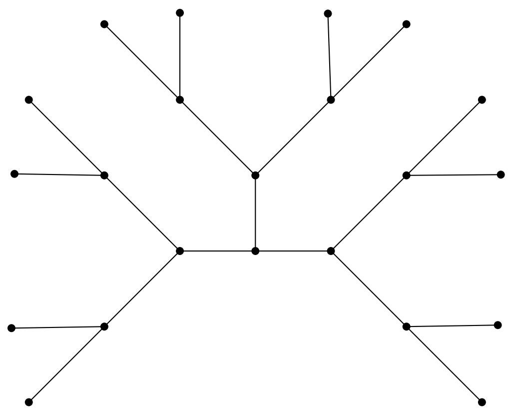

FIGURE 1. The rooted tree $T(3,3)$.

$$c_{n-2}(G_2) = \frac{15}{98}(n-1)(13n-34),\ c_{n-3}(G_2) = \frac{2}{1029}(n-1)(676n^2-4649n+8481),$$
$$c_{n-4}(G_2) = \frac{1}{57624}(n-1)(37349n^3-478413n^2+2146954n-3432552),$$
$$c_{n-5}(G_2) = \frac{1}{336140}(n-1)(85683n^4-1750502n^3+13991793n^2-52528222n+79270320),$$
$$c_{n-2}(G_3) = \frac{1}{98}(15n-16)(13n-33),$$
$$c_{n-3}(G_3) = \frac{1}{1029}n(1352n^2-10611n+26563)-\frac{6480}{343},$$
$$c_{n-4}(G_3) = \frac{1}{57624}n(37349n^3-513734n^2+2635015n-5871574)+\frac{184469}{2401},$$
$$c_{n-5}(G_3) = \frac{1}{1008420}n(257049n^4-5486585n^3+47226105n^2-204551395n+437870586) -\frac{5694446}{16807},$$
$$c_{n-2}(G_4) = \frac{1}{98}n(195n-701)+\frac{125}{49},\ c_{n-3}(G_4) = \frac{4}{1029}n(n-4)(338n-1291)+\frac{3008}{343},$$
$$c_{n-4}(G_4) = \frac{1}{57624}n(37349n^3-511706n^2+2298967n-2158546)-\frac{291540}{2401},$$
$$c_{n-5}(G_4) = \frac{1}{1008420}n(257049n^4-5464615n^3+43233545n^2-130350725n-74554454) +\frac{15573272}{16807},$$
$$c_{n-2}(G_5) = \frac{1}{98}n(195n-713)+\frac{405}{49},$$
$$c_{n-3}(G_5) = \frac{2}{1029}n(676n^2-5403n+16460)-\frac{14418}{343},$$
$$c_{n-4}(G_5) = \frac{1}{57624}n(37349n^3-523874n^2+3028255n-9178570)+\frac{540746}{2401},$$
$$c_{n-2}(G_5) = \frac{1}{1008420}n(257049n^4-5596435n^3+52360845n^2-275208485n+846144906)$$

$$
\begin{aligned}
&- \frac{20524022}{16807},\\
c_{n-2}(G_6) &= \frac{1}{98}(39n+1)(5n-18),\ \ c_{n-3}(G_6) = \frac{2}{1029}n(676n^2-5247n+7454)+\frac{11758}{343},\\
c_{n-4}(G_6) &= \frac{1}{57624}n(37349n^3-507650n^2+1965703n+1289150)-\frac{709627}{2401},\\
c_{n-2}(G_6) &= \frac{1}{1008420}n(257049n^4-5420675n^3+39172605n^2-60754765n-529616214)\\
&\quad+\frac{33001040}{16807},\\
c_{n-2}(G_7) &= \frac{1}{98}n(195n-709)+\frac{332}{49},\ \ c_{n-3}(G_7) = \frac{2}{1029}n(676n^2-5364n+14831)-\frac{10096}{343},\\
c_{n-4}(G_7) &= \frac{1}{57624}n(37349n^3-519818n^2+2830243n-7399558)+\frac{340472}{2401},\\
c_{n-2}(G_7) &= \frac{1}{1008420}n(257049n^4-5552495n^3+49827665n^2-237728905n+619570486)\\
&\quad-\frac{11978116}{16807}.
\end{aligned}
$$

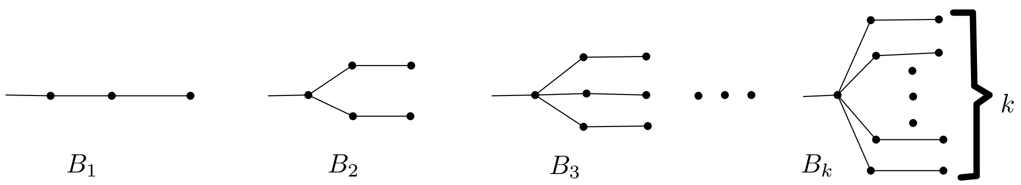

FIGURE 2. The branches of proper Kragujevac trees.

Our third class of trees are caterpillar trees. A caterpillar is a tree in which all the vertices are within distance 1 of a central path [9]. Let T_i, for $i = 1, 2, \ldots, 5$, be the caterpillar tree on $n \geq 8$ vertices, see Figure 4. Then,

$$
\begin{aligned}
c_{n-2}(T_1) &= \frac{1}{2}(4n-7)(n-2), \quad c_{n-3}(T_1) = \frac{1}{3}(n-2)(4n^2-25n+42),\\
c_{n-4}(T_1) &= \frac{1}{24}(n-4)(16n^3-168n^2+611n-726),\\
c_{n-5}(T_1) &= \frac{1}{60}(n-4)(16n^4-296n^3+2111n^2-6811n+8250),\\
c_{n-2}(T_2) &= \frac{1}{2}n(4n-15)+\frac{15}{2}, \quad c_{n-3}(T_2) = \frac{1}{3}(n-3)(4n^2-21n+32),\\
c_{n-4}(T_2) &= \frac{1}{24}n(16n^3-232n^2+1307n-3404)+\frac{1155}{8},\\
c_{n-5}(T_2) &= \frac{1}{60}(n-5)(16n^4-280n^3+1935n^2-6270n+8079),\\
c_{n-2}(T_3) &= 2(n-2)^2, \quad c_{n-3}(T_3) = \frac{4}{3}(n-2)(n^2-7n+14),\\
c_{n-4}(T_3) &= \frac{2}{3}n(n^3-16n^2+100n-281)+\frac{575}{3},
\end{aligned}
$$

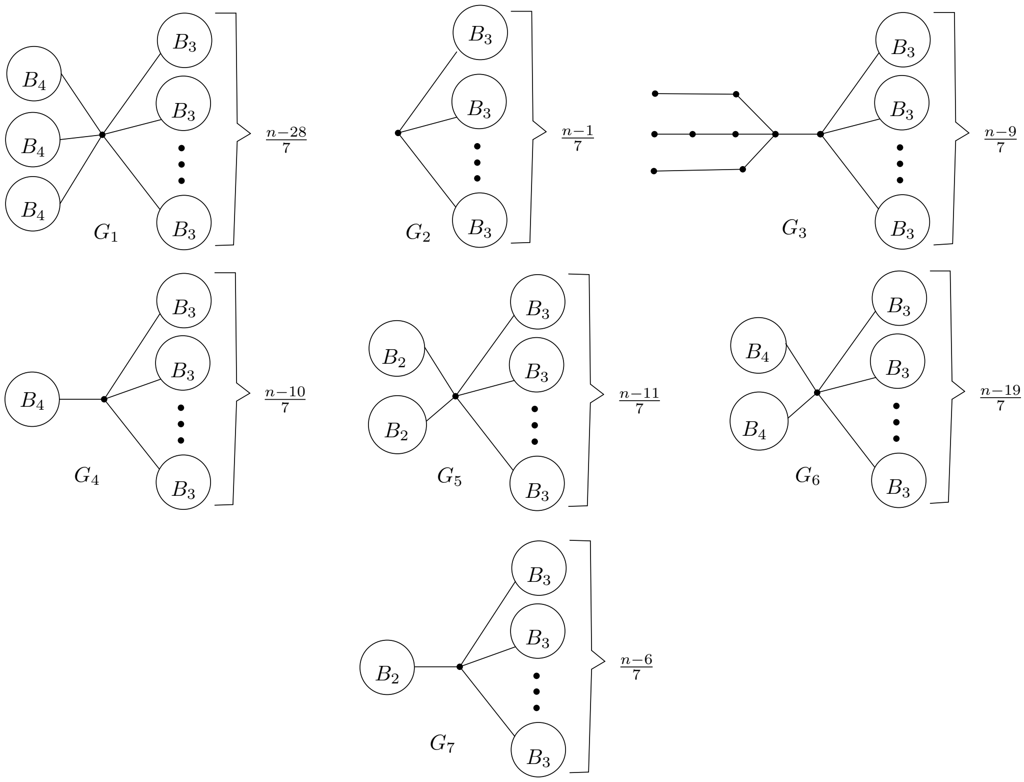

FIGURE 3. The proper Kragujevac trees that have illustrated in [10, Conjecture 3].

$$
\begin{aligned}
c_{n-5}(T_3) &= \frac{2}{15}(n-5)(2n^4-40n^3+320n^2-1170n+1573),\\
c_{n-2}(T_4) &= 2(n-2)^2+1, \quad c_{n-3}(T_4) = \frac{2}{3}(n-3)(2n^2-12n+23),\\
c_{n-4}(T_4) &= \frac{2}{3}(n-4)(n^3-12n^2+55n-93),\\
c_{n-5}(T_4) &= \frac{4}{15}n(n^4-25n^3+265n^2-1480n+4314)-1386,\\
c_{n-2}(T_5) &= 2(n-2)^2+1, \quad c_{n-3}(T_5) = \frac{2}{3}n(2n^2-18n+59)-\frac{140}{3},\\
c_{n-4}(T_5) &= \frac{2}{3}n(n^3-16n^2+103n-315)+\frac{769}{3},\\
c_{n-5}(T_5) &= \frac{2}{15}n(2n^4-50n^3+530n^2-2970n+8773)-1456.
\end{aligned}
$$

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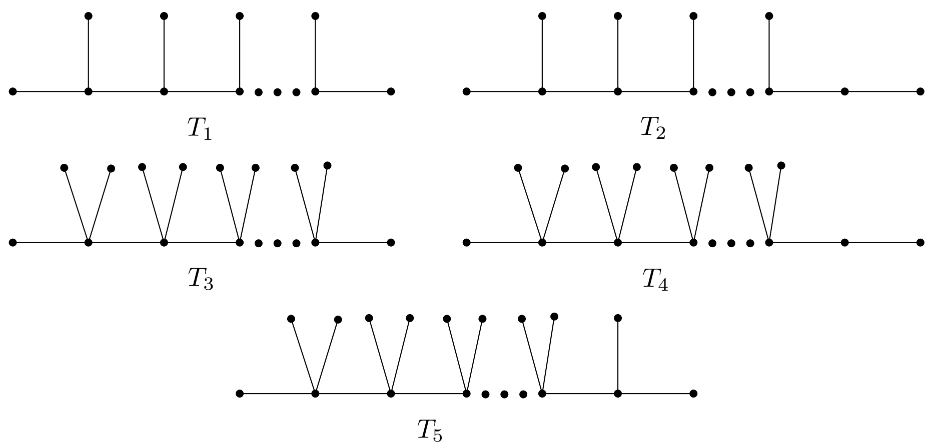

Figure 4. The caterpillar trees.

References

[1] A. R. Ashrafi, M. Eliasi and A. Ghalavand, *Laplacian coefficients and Zagreb indices of trees*, Linear Multilinear Algebra **67**(9) (2019), 1736–1749. https://doi.org/10.1080/03081087.2018.1469599

[2] A. Ghalavand and A. R. Ashrafi, *Laplacian coefficients of trees*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **24**(542) (2020), 1–14. https://doi.org/10.21857/m16wjcev89

[3] K. C. Das, A. Ghalavand and A. R. Ashrafi, *On the number of k-matchings in graphs*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences **92**(4) (2022), 563–570. https://doi.org/10.1007/s40010-022-00771-2.

[4] C. D. Godsil and I. Gutman, *On the theory of the matching polynomial*, J. Graph Theory **5** (1981), 137–144. https://doi.org/10.1002/jgt.3190050203

[5] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z

[6] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), 535–538. https://doi.org/10.1016/0009-2614(72)85099-1

[7] I. Gutman and K. C. Das, *The first Zagreb index* 30 *years after*, MATCH Commun. Math. Comput. Chem. **50** (2004), 83–92.

[8] I. Gutman, E. Milovanović and I. Milovanović, *Beyond the Zagreb indices*, AKCE Int. J. Graphs Comb. **17**(1) (2020), 74–85. https://doi.org/10.1016/j.akcej.2018.05.002

[9] F. Harary and A. J. Schwenk, *The number of caterpillars*, Discrete Mathematics **6**(4) (1973), 359–365. https://doi.org/10.1016/0012-365X(73)90067-8

[10] S. A. Hosseini, M. B. Ahmadi and I. Gutman, *Kragujevac trees with minimal atom-bond connectivity index*, MATCH Commun. Math. Comput. Chem. **71** (2014), 5–20.

[11] A. Milićević, S. Nikolić and N. Trinajstić, *On reformulated Zagreb indices*, Molecular Diversity **8** (2004), 393–399. https://doi.org/10.1023/B:MODI.0000047504.14261.2a

[12] R. Merris, *A survey of graph Laplacians*, Linear Multilinear Algebra **39** (1995), 19–31. https://doi.org/10.1080/03081089508818377

[13] B. Mohar, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Ollermann, A. J. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Wiley, New York, 1991, 871–898.

[14] S. Nikolić, G. Kovačević, A. Milićević and N. Trinajstić, *The Zagreb indices* 30 *years after*, Croat. Chem. Acta **76**(2) (2003), 113–124.
[15] C. S. Oliveira, N. M. M. de Abreu and S. Jurkewicz, *The characteristic polynomial of the Laplacian of graphs in* (a, b)*-linear classes*, Linear Algebra Appl. **356** (2002), 113–121. https://doi.org/10.1016/S0024-3795(02)00357-9
[16] W. Yan and Y.-N. Yeh, *Connections between Wiener index and matchings*, J. Math. Chem. **39** (2006), 389–399. https://doi.org/10.1007/s10910-005-9026-0
[17] S. Zhang and H. Zhang, *Unicyclic graphs with the first three smallest and largest first general Zagreb index*, MATCH Commun. Math. Comput. Chem. **55**(2) (2006), 427–438.
[18] B. Zhou and I. Gutman, *A connection between ordinary and Laplacian spectra of bipartite graphs*, Linear Multilinear Algebra **56**(3) (2008), 305–310. https://doi.org/10.1080/03081080601002254

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INTEGRAL INVOLVING THE PRODUCT OF MULTIVARIABLE ALEPH-FUNCTION, GENERAL CLASS OF SRIVASTAVA POLYNOMIALS AND ALEPH-FUNCTION OF ONE VARIABLE

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Abstract. In this paper, we derive an integral involving the multivariable Aleph-function, the general class of Srivastva polynomials, and the Aleph-function of one variable, all of which are sufficiently general in nature and are capable of yielding a large number of simpler and more useful results simply by specialization of their parameters. Moreover, we establish certain specific instances.

1. Introduction and Preliminaries

The Aleph ($\aleph$)-function was established by Südland et al. [30], but its notation and complete definition are offered below in terms of the Mellin-Barnes type integral (see also, [2, 3, 7, 13, 23, 25]):

$$\aleph(z) = \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(z \,\middle|\, \begin{matrix} (a_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r'} \\ (b_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r'} \end{matrix}\right)$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)\, z^{-s} \mathrm{d}s, \tag{1.1}$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r'} c_i\left\{\prod_{j=N+1}^{P_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - B_{ji}s)\right\}}, \tag{1.2}$$

Key words and phrases. Aleph-function of several variables, general class of Srivastava polynomials, Aleph-function of one and two variables, I-function of two and several variables.

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where $|\arg z| < \frac{1}{2}\pi\, \Omega$ and $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i \left(\sum_{j=M+1}^{Q_i} B_{ji} + \sum_{j=N+1}^{P_i} A_{ji} \right) > 0$ for $i = 1, \ldots, r'$. For convergence conditions and other details of Aleph-function (one variable), see Südland et al. [30] (see also, [23, 24]). The series representation of Aleph-function is given by Chaurasia and Singh [6], defined as

$$\aleph^{M,N}_{P_i,Q_i,c_i;r'}(z) = \sum_{G=1}^{M} \sum_{g=0}^{+\infty} \frac{(-1)^g \Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)}{B_G g!} z^{-s}, \tag{1.3}$$

with $s = \eta_{G,g} = \frac{b_G + g}{B_G}$, $P_i < Q_i$, $|z| < 1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)$ is given in (1.2).

The generalized polynomials defined by Srivastava [29], and studied by many authors, e.g., [5, 7, 8, 10–12, 14, 18, 20], is given in the following manner:

$$\begin{aligned} S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}[y_1,\ldots,y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \times \cdots \times \frac{(-N_s)_{M_sK_s}}{K_s!} \\ \times A[N_1,K_1;\ldots;N_s,K_s]\, y_1^{K_1} \cdots y_s^{K_s}, \end{aligned} \tag{1.4}$$

where $M_1, \ldots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \ldots; N_s, K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation:

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!} \times \cdots \times \frac{(-N_s)_{M_sK_s}}{K_s!} A[N_1,K_1;\ldots;N_s,K_s]. \tag{1.5}$$

The Aleph-function of several variables generalizes the multivariable I-function defined by Sharma and Ahmad [26], which is a generalization of G and H-functions [8, 21] of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariable Aleph-function throughout our present study and will be defined and represented as follows (see also, [4, 15–17, 19]).

$$\begin{aligned} &\aleph(z_1,\ldots,z_r) \\ &= \aleph^{0,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,\mathfrak{n}}, \\ \ldots, \end{array} \right. \\ &\quad \begin{array}{l} \left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{\mathfrak{n}+1,p_i} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}, \\ \left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{1,q_i} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}, \end{array} \\ &\quad \left. \begin{array}{l} \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}, \left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}} \\ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}, \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}} \end{array} \right) \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\ldots,s_r) \prod_{k=1}^{r} \theta_k(s_k)\, z_k^{s_k}\, \mathrm{d}s_1 \cdots \mathrm{d}s_r, \end{aligned} \tag{1.6}$$

with $\omega = \sqrt{-1}$. For more details, see Ayant [1]. The real numbers τ_i are positives for $i = 1, \ldots, R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \ldots, R^{(k)}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.6) can be obtained by extension of the corresponding conditions for multivariable H-function given by $|\arg z_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$- \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \ldots, r,\ i = 1, \ldots, R,\ i^{(k)} = 1, \ldots, R^{(k)}. \tag{1.7}$$

The complex numbers z_i are not zero. Throughout this paper, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the asymptotic expansion in the following convenient form:

$$\aleph(z_1, \ldots, z_r) = 0\left(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}\right), \quad \max\{|z_1|, \ldots, |z_r|\} \to 0,$$

$$\aleph(z_1, \ldots, z_r) = 0\left(|z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}\right), \quad \min\{|z_1|, \ldots, |z_r|\} \to +\infty,$$

where $k = 1, \ldots, r$, $\alpha_k = \min\left\{\mathrm{Re}\left(d_j^{(k)}/\delta_j^{(k)}\right) : j = 1, \ldots, m_k\right\}$ and $\beta_k = \max\left\{\mathrm{Re}\left(\left(c_j^{(k)} - 1\right)/\gamma_j^{(k)}\right) : j = 1, \ldots, n_k\right\}$. We will use these following notations:

$$U = p_i, q_i, \tau_i; R, \quad V = m_1, n_1; \ldots; m_r, n_r, \tag{1.8}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \ldots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}, \tag{1.9}$$

$$A = \left(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,\mathfrak{n}}, \ \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)\right]_{\mathfrak{n}+1,p_i}, \tag{1.10}$$

$$B = \left[\tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \ldots, \beta_{ji}^{(r)}\right)\right]_{1,q_i}, \tag{1.11}$$

$$C = \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,n_1}, \ \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_{i^{(1)}}}, \ldots,$$
$$\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}, \left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_{i^{(r)}}}, \tag{1.12}$$

$$D = \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,m_1}, \ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_{i^{(1)}}}, \ldots,$$
$$\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}, \ \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_{i^{(r)}}}. \tag{1.13}$$

We denote the multivariable Aleph-function as

$$\aleph(z_1, \ldots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} A : C \\ B : D \end{array}\right). \tag{1.14}$$

We have the following required integral [9]:

(1.15)
$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{\alpha+\beta}}\,\mathrm{d}\theta = \frac{1}{2a^{2\beta}b^{2\alpha}}B\left(\alpha,\beta\right), \quad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0,$$

where $a, b \in \mathbb{C}\backslash\{0\}$ and $B\left(\cdot,\cdot\right)$ is the Beta function.

2. Main Integral

In this section we evaluate the integral involving multivariable Aleph-function, a class of polynomials of several variables and a Aleph-function of one variable.

Theorem 2.1.

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{\alpha+\beta}}\ \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c+d}}\right)$$
$$\times\ S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}} \\ \vdots \\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}} \end{pmatrix} \aleph^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}} \\ \vdots \\ z_r\frac{\sin^{2h_r}\theta\ \cos^{2l_r}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_r+l_r}} \end{pmatrix}\mathrm{d}\theta$$
$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{+\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}\left(\eta_{G,g}\right)}{B_G\, g!}t^{\eta_{G,g}}\,t_1^{K_1}\cdots t_s^{K_s}$$
$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^s K_s d_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^s K_s c_i\right)}$$
$$\times\ \aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{matrix} \frac{z_1}{a^{2l_1}\,b^{2h_1}} \\ \vdots \\ \frac{z_r}{a^{2l_r}\,b^{2h_r}} \end{matrix}\ \middle|\ \begin{matrix} \left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^s K_i c_i; h_1,\ldots,h_r\right), \\ \\ - \end{matrix}\right.$$

(2.1)
$$\left.\begin{matrix} \left(1-\beta-d\eta_{G,g}-\sum_{i=1}^s K_i d_i; l_1,\ldots,l_r\right), A : C \\ \left(1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^s K\left(c_i+d_i\right); h_1+l_1,\ldots,h_r+l_r\right), B : D \end{matrix}\right),$$

where $U_{21}=p_i+2,\ q_i+1,\ \tau_i,\ R,$ *also satisfy the following conditions:*

(a) $\min\{c, c_i, h_j\} \le 0,\ i=1,\ldots,s;\ j=1,\ldots,r$ (h_j *are not simultaneously zero*);

(b) $\min\{d, d_i, l_j\} \le 0,\ i=1,\ldots,s;\ j=1,\ldots,r$ (l_j *are not simultaneously zero*);

(c) $\mathrm{Re}\left(\alpha\right)+c+\sum_{i=1}^r c_i\mathrm{Re}\left(\alpha\right)+c\min\limits_{1\leqslant l\leqslant M}\mathrm{Re}\left(\frac{b_j^{(i)}}{\beta_j}\right)+\sum_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;$

(d) $\mathrm{Re}\left(\alpha\right)+d\min\limits_{1\leqslant l\leqslant M}\mathrm{Re}\left(\frac{b_j^{(i)}}{\beta_j}\right)+\sum_{i=1}^r l_i\min\limits_{1\leqslant j\leqslant m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;$

(e) $|\arg z_k| < \frac{1}{2}A_i^{(k)}\,\pi,$ *where* $A_i^{(k)}$ *is given in* (1.7)*;*

(f) $|\arg t| < \frac{1}{2}\pi\Omega,$ *where*

$$\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i\left(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji}\right)>0.$$

Proof. Expressing the Aleph-function of one variable in series form with the help of (1.3), the general class of polynomials of several variables in series with the help of (1.4), and the Aleph-function of r variables in Mellin-Barnes contour integral with the help of (1.6). The conditions (e) and (f) are satisfied, then the integral representing multivariable Aleph function converges uniformly, and we can invert the sums and multiple Mellin-Barnes integrals. Next, by changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and summations involved in the process) and then evaluating the resulting integral with the help of equation (1.15). Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result (2.1). □

3. Multivariable I-function

Corollary 3.1. *If $\tau_i, \tau_{i^{(1)}}, \ldots, \tau_{i^{(r)}} \to 1$, the Aleph-function of several variables renovates to the I-function of several variables. The simple integral has been derived in this section for multivariable I-functions defined by Sharma and Ahmad* [26]

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta \ \cos^{2\beta-1}\theta}{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\alpha+\beta}} \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta \ \cos^{2d}\theta}{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{c+d}}\right)$$

$$\times\ S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}} \\ \vdots \\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}}\end{pmatrix} I^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}} \\ \vdots \\ z_r\frac{\sin^{2h_r}\theta\ \cos^{2l_r}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_r+l_r}}\end{pmatrix} \mathrm{d}\theta$$

$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}\left(\eta_{G,g}\right)}{B_G\, g!}\ t^{\eta_{G,g}}\, t_1^{K_1}\cdots t_s^{K_s}$$

$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_sd_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_sc_i\right)}$$

$$\times\ I^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c} \frac{z_1}{a^{2l_1}\,b^{2h_1}} \\ \vdots \\ \frac{z_r}{a^{2l_r}\,b^{2h_r}}\end{array}\ \middle|\ \begin{array}{c}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;h_1,\ldots,h_r\right), \\ -\end{array}\right.$$

(3.1)

$$\left.\begin{array}{c}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^{s}K_id_i;l_1,\ldots,l_r\right),A':C' \\ \left(1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^{s}K\left(c_i+d_i\right);h_1+l_1,\ldots,h_r+l_r\right),B':D'\end{array}\right),$$

where $A'=\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n},\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}$, $B'=\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{1,q_i}$, $C'=\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1},\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r},\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}$, $D'=\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1},\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}$, *also under the same conditions that* (2.1).

4. Aleph-function of Two Variables

Corollary 4.1. *If we set* $r = 2$ *in* (1.6)*, then we obtain the Aleph-function of two variables defined by Sharma* [28] *and further generalized by Kumar* [13]*. We have the following simple integral*

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\alpha+\beta}} \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c+d}}\right)$$
$$\times\ S^{M_1,\dots,M_s}_{N_1,\dots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}} \\ \vdots \\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}}\end{pmatrix} \aleph^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}} \\ z_2\frac{\sin^{2h_2}\theta\ \cos^{2l_2}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_2+l_2}}\end{pmatrix} \mathrm{d}\theta$$
$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{+\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G\, g!}\ t^{\eta_{G,g}}\ t_1^{K_1}\cdots t_s^{K_s}$$
$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^s K_s d_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^s K_s c_i\right)}$$
$$\times\ \aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c} \frac{z_1}{a^{2l_1}\,b^{2h_1}} \\ \frac{z_2}{a^{2l_2}\,b^{2h_2}}\end{array}\middle|\begin{array}{c}(1-\alpha-c\eta_{G,g}-\sum_{i=1}^s K_i c_i;h_1,h_2)\,, \\ -\end{array}\right.$$
$$\left.\begin{array}{l}(1-\beta-d\eta_{G,g}-\sum_{i=1}^s K_i d_i;l_1,l_2)\,,A'':C'';E'' \\ (1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^s K\,(c_i+d_i)\,;h_1+l_1,h_2+l_2)\,,B'':D'';F''\end{array}\right), \tag{4.1}$$

where $A''=\left(a_j;\alpha'_j,\alpha''_j\right)_{1,n}$, $\left[\tau_i\left(a_{ji};\alpha'_{ji},\alpha''_{ji}\right)\right]_{n+1,p_i}$; $B''=\left[\tau_i\left(b_{ji};\beta'_{ji},\beta''_{ji}\right)\right]_{1,q_i}$, $C''=(c_j,\gamma_j)_{1,n_1}$, $[\tau_{i'}(c_{ji'},\gamma_{ji'})]_{n_1+1,p_{i'}}$; $D''=(d_j,\delta_j)_{1,m_1}$, $[\tau_{i'}(d_{ji'},\delta_{ji'})]_{m_1+1,q_{i'}}$, $E''=(e_j,\eta_j)_{1,n_2}$, $[\tau_{i''}(e_{ji''},\eta_{ji''})]_{n_2+1,p_{i''}}$; $F''=(f_j,\zeta_j)_{1,m_2}$, $[\tau_{i''}(f_{ji''},\zeta_{ji''})]_{m_2+1,q_{i''}}$*, also satisfy the existence conditions provided in* (2.1)*.*

5. *I*-function of Two Variables

Corollary 5.1. *If we set* $\tau_i,\tau_{i'},\tau_{i''}\to 1$ *in* (4.1)*, the Aleph-function of two variables reduces to the I-function of two variables defined by Sharma and Mishra* [27]*, and we obtain the same formula with the I-function of two variables.*

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\alpha+\beta}} \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c+d}}\right)$$
$$\times\ S^{M_1,\dots,M_s}_{N_1,\dots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}} \\ \vdots \\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}}\end{pmatrix} I^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}} \\ z_2\frac{\sin^{2h_2}\theta\ \cos^{2l_2}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_2+l_2}}\end{pmatrix} \mathrm{d}\theta$$

$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-1)^g\, \Omega^{M,N}_{P_i,Q_i,c_i,r'}\left(\eta_{G,g}\right)}{B_G\, g!}\, t^{\eta_{G,g}}\, t_1^{K_1}\cdots t_s^{K_s}$$
$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_s d_i\right)}\, b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_s c_i\right)}$$
$$\times I^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c}\frac{z_1}{a^{2l_1}\,b^{2h_1}}\\ \frac{z_2}{a^{2l_2}\,b^{2h_2}}\end{array}\left|\begin{array}{c}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;h_1,h_2\right),\\ -\end{array}\right.\right.$$
$$\left.\begin{array}{c}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^{s}K_id_i;l_1,l_2\right),A''':C''';E'''\\ \left(1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^{s}K\left(c_i+d_i\right);h_1+l_1,h_2+l_2\right),B''':D''';F'''\end{array}\right), \tag{5.1}$$

where $A'''=\left(a_j;\alpha'_j,\alpha''_j\right)_{1,n}$, $\left(a_{ji};\alpha'_{ji},\alpha''_{ji}\right)_{n+1,p_i}$, $B'''=\left(b_{ji};\beta'_{ji},\beta''_{ji}\right)_{1,q_i}$, $C'''=\left(c_j,\gamma_j\right)_{1,n_1},\left(c_{ji'},\gamma_{ji'}\right)_{n_1+1,p_{i'}}$; $D'''=\left(d_j,\delta_j\right)_{1,m_1},\left(d_{ji'},\delta_{ji'}\right)_{m_1+1,q_{i'}}$, $E'''=\left(e_j,\eta_j\right)_{1,n_2},\left(e_{ji''},\eta_{ji''}\right)_{n_2+1,p_{i''}}$, $F'''=\left(f_j,\zeta_j\right)_{1,m_2},\left(f_{ji''},\zeta_{ji''}\right)_{m_2+1,q_{i''}}$, *also satisfy the conditions stated in* (2.1).

For more details of I-function of two variables reader can refer to work Kumari et al. [22].

6. Conclusion

In this work, an integral involving the multivariable Aleph-function, a class of polynomials with several variables (Srivastava polynomials), and an Aleph-function with one variable was evaluated. The integral derived in this study is of a highly broad character, since it incorporates the multivariable Aleph-function, which is a generic function of multiple variables previously explored. Consequently, the integral produced by this study would serve as a key formula from which, by adjusting the parameters, as many outcomes as required involving the special functions of one and multiple variables may be generated.

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References

[1] F. Y. Ayant, *An integral associated with the Aleph-functions of several variables*, Int. J. Math. Trends Tech. **31**(3) (2016), 142–154.

[2] F. Y. Ayant and D. Kumar, *A unified study of Fourier series involving the Aleph-function and the Kampé de Fériet's function*, Int. J. Math. Trends Tech. **35**(1) (2016), 40–48. https://doi.org/10.14445/22315373/IJMTT/V35P507

[3] F. Y. Ayant and D. Kumar, *Certain finite double integrals involving the hypergeometric function and Aleph-function*, Int. J. Math. Trends Tech. **35**(1) (2016), 49–55. https://doi.org/10.14445/22315373/IJMTT/V35P508

[4] F. Y. Ayant and D. Kumar, *Generating relations and multivariable Aleph-function*, Analysis **38**(3) (2018), 137–143. https://doi.org/10.1515/anly-2017-0054

[5] D. Baleanu, D. Kumar and S. D. Purohit, *Generalized fractional integrals of product of two H-functions and a general class of polynomials*, Int. J. Comput. Math. **93**(8) (2016), 1320–1329. https://doi.org/10.1080/00207160.2015.1045886

[6] V. B. L. Chaurasia and Y. Singh, *New generalization of integral equations of fredholm type using Aleph-function*, International Journal of Modern Mathematical Sciences **9**(3) (2014), 208–220.

[7] J. Choi and D. Kumar, *Certain unified fractional integrals and derivatives for a product of Aleph function and a general class of multivariable polynomials*, J. Inequal. Appl. **2014** (2014), 15 pages. https://doi.org/10.1186/1029-242X-2014-499

[8] J. Daiya, J. Ram and D. Kumar, *The multivariable H-function and the general class of Srivastava polynomials involving the generalized Mellin-Barnes contour integrals*, FILOMAT **30**(6) (2016), 1457–1464. https://doi.org/10.2298/FIL1606457D

[9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 1996.

[10] R. K. Gupta, B. S. Shaktawat and D. Kumar, *On generalized fractional differentials involving product of two H-functions and a general class of polynomials*, J. Rajasthan Acad. Phys. Sci. **15**(4) (2016), 327–344.

[11] R. K. Gupta, B. S. Shaktawat and D. Kumar, *Generalized fractional integrals and derivative formulas for $\overline{H}$-function and a general class of polynomials*, Journal of Chemical, Biological and Physical Sciences, Section C **6**(3) (2016), 1081–1097.

[12] D. Kumar, *Fractional calculus formulas involving $\overline{H}$-function and Srivastava polynomials*, Commun. Korean Math. Soc. **31**(4) (2016), 827–844. https://doi.org/10.4134/CKMS.c150251

[13] D. Kumar, *Generalized fractional differintegral operators of the Aleph-function of two variables*, Journal of Chemical, Biological and Physical Sciences, Section C **6**(3) (2016), 1116–1131.

[14] D. Kumar, P. Agarwal and S. D. Purohit, *Generalized fractional integration of the $\overline{H}$-function involving general class of polynomials*, Walailak Journal of Science and Technology **11**(12) (2014), 1019–1030. https://doi.org/10.14456/WJST.2014.57

[15] D. Kumar and F. Y. Ayant, *Application of Jacobi polynomial and multivariable Aleph-function in heat conduction in non-homogeneous moving rectangular parallelepiped*, Kragujevac J. Math. **45**(3) (2021), 439–447. https://doi.org/10.46793/KgJMat2103.439K

[16] D. Kumar and F. Y. Ayant, *Fractional calculus operators pertaining to multivariable Aleph-function*, Bulletin of Parana's Mathematical Society **40** (2022), 1–10. https://doi.org/10.5269/bspm.42491

[17] D. Kumar, F. Y. Ayant, S. Asawasamrit and J. Tariboon, *Certain finite integrals related to the product of special functions*, Symmetry **13** (2021), Article ID 2013, 11 pages. https://doi.org/10.3390/sym13112013

[18] D. Kumar, F. Y. Ayant and J. Choi, *Application of product of the multivariable A-function and the multivariable Srivastava polynomials*, East Asian Math. J. **34**(3) (2018), 295–303. https://doi.org/10.7858/eamj.2018.021

[19] D. Kumar, F. Y. Ayant and D. Kumar, *A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function*, Kragujevac J. Math. **44**(4) (2020), 539–550. https://doi.org/10.46793/KgJMat2004.539K

[20] D. Kumar, F. Y. Ayant and A. Prakash, *Certain integral involving the product of Srivastava polynomials and special functions*, Afr. Mat. **32** (2021), 1–9. https://doi.org/10.1007/s13370-021-00885-7

[21] D. Kumar, S. D. Purohit and J. Choi, *Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials*, J. Nonlinear Sci. Appl. **9** (2016), 8–21. https://doi.org/10.1007/s13370-021-00885-7

[22] S. K. Kumari, V. T. M. Nambisan and A. K. Rathie, *A study of I-functions of two variables*, Le Matematiche **69**(1) (2014), 285–305.
[23] J. Ram and D. Kumar, *Generalized fractional integration of the $\aleph$-function*, J. Rajasthan Acad. Phys. Sci. **10**(4) (2011), 373–382.
[24] R. K. Saxena and D. Kumar, *Generalized fractional calculus of the Aleph-function involving a general class of polynomials*, Acta Math. Sci. Ser. B (Engl. Ed.) **35**(5) (2015), 1095–1110. https://doi.org/10.1016/S0252-9602(15)30042-4
[25] R. K. Saxena, J. Ram and D. Kumar, *Generalized fractional integral of the product of two Aleph-functions*, Appl. Appl. Math. **8**(2) (2013), 631–646.
[26] C. K. Sharma and S. S. Ahmad, *On the multivariable I-function*, Acta Ciencia Indica (Mathematics) **20**(2) (1994), 113–116.
[27] C. K. Sharma and P. L. Mishra, *On the I-function of two variables and its properties*, Acta Ciencia Indica (Mathematics) **17** (1991), 667–672.
[28] K. Sharma, *On the integral representation and applications of the generalized function of two variables*, Int. J. Math. Eng. Sci. **3** (2014), 1–13.
[29] H. M. Srivastava, *A multi-linear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial*, Pacific. J. Math. **177** (1985), 183–191.
[30] N. Südland, B. Baumann and T. F. Nonnenmacher, *Open problem: who knows about the Aleph-functions?*, Fract. Calc. Appl. Anal. **1**(4) (1998), 401–402.

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PROPERTIES OF (C, r)-HANKEL OPERATORS AND (R, r)-HANKEL OPERATORS ON HILBERT SPACES

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Abstract. We introduce the operators which are generalizations of Hankel-type operators, called the (C, r)-Hankel operator and (R, r)-Hankel operator on general Hilbert spaces. Our main result is to obtain characterizations for a bounded operator on general Hilbert spaces to be a (C, r)-Hankel operator (or (R, r)-Hankel operator). We also discuss some algebraic properties like boundedness (for $|r| \neq 1$) of these operators and the relationship between them. Moreover, some characterizations for the commutativity of these operators are explored.

1. Introduction

The notion of Hankel matrices made its first appearance in 1861 when Hankel began the study of finite matrices with entries being a function of the sum of the coordinates only [6], the Hilbert matrix being the most prominent example of the same [2]. Hankel, Kronecker, Nehari and Hartman are the most celebrated names in this area for their contribution towards the most classical results about Hankel operators. For a pivot study on Hankel operators, one can refer [3, 5, 13].

Since inception, a lot of research has been done on this class of matrices, corresponding operators and associated variants due to their high scores of applications in the fields of perturbation theory, interpolation process, rational approximation, probability, moment problems, theory of systems and control etc. (refer [11–13]). The rapid development of this domain has led to numerous generalizations both in terms of twists in the operator form as well as the space of play. To adduce a few, Hankel operators,

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slant Hankel operators, essentially Hankel operators, λ-Hankel operators, weighted Hankel operators, small Hankel operators, slant little-Hankel operators, essentially slant-Hankel operators, kth-order slant Hankel operators etc. have been studied on different spaces like Hardy spaces, Bergmann spaces, Fock spaces, weighted Fock spaces, Harmonic Dirichlet spaces and so on [1,3,4,8–10,14] and references therein.

Recently, Mirotin et al. introduced the idea of μ-Hankel operators on Hilbert spaces in the following way and discussed this class on Hardy space in particular [10]: Let μ be a complex number, $\alpha = (\alpha_n)_{n\geq 0}$ be a sequence of complex numbers, H and H' be separable Hilbert spaces. The operator $A_{\mu,\alpha} : H \to H'$ is called μ-Hankel operator if for some orthonormal bases $(e_k)_{k\geq 0} \subset H$ and $(e'_j)_{j\geq 0} \subset H'$, the matrix $(a_{jk})_{k,j\geq 0}$ of this operator consists of elements of the form $a_{jk} = \mu^k \alpha_{j+k}$. All these developments motivated the authors to define two new classes of operators on general Hilbert spaces that are closely related to Hankel operators in the sense that these classes result in Hankel-type operators if alternate columns of one or alternate rows of the other are deleted. Interesting results are established to derive the connection between these classes, over and above the discussion of their algebraic properties. Characterizations are obtained for which these operators commute. It is also proved that these classes neither contain any Fredholm operator nor unitary operator.

We begin with the following preliminaries.

A bounded linear operator T on a Hilbert space H is said to be Hilbert-Schmidt operator if the Hilbert-Schmidt norm $\|T\|^2_{HS} = \sum_n \|T(u_n)\|^2 < +$ for an orthonormal basis $(u_n)_{n\in\mathbb{N}_0}$ of H, where $\|\cdot\|$ represents the norm of H. A bounded operator T on H is said to be a Fredholm operator if Range of T is closed, dimension of kernel T and dimension of kernel T^* are finite. In this case, index of T is defined as

$$\operatorname{index} T = \dim\ker T - \dim\ker T^*.$$

A bounded operator T on H is said to be isometry if $T^*T = I_H$, and unitary if T is bijective and $T^*T = TT^* = I_H$, where I_H denotes the identity operator on H. Throughout the paper, we restrict the symbols H_1 and H_2 for any separable Hilbert spaces. If $H_1 = H_2$, then it is denoted by H. We denote by $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$, the orthonormal bases for H_1 and H_2, respectively. The symbols U_1 and U_2 denote the right shift operators on H_1 and H_2, respectively and are defined as $U_1(u_i) = u_{i+1}$ and $U_2(v_i) = v_{i+1}$ for all $i \in \mathbb{N}_0$. The symbols $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}_0$ denote the set of all complex numbers, integers and non-negative integers, respectively.

2. The (C,r)-Hankel Operator and (R,r)-Hankel Operator

We now introduce (C,r)-Hankel operators and (R,r)-Hankel operators on general Hilbert spaces as under.

Definition 2.1. Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers. Then the operator (C,r)-Hankel operator, $C_{r,\alpha}$ from a Hilbert space H_1 to Hilbert space H_2 is defined as

$$C_{r,\alpha}(u_i) = \sum_{j=0}^{+\infty} r^i \alpha_{i+2j} v_j, \qquad \text{for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$ are orthonormal bases for H_1 and H_2, respectively.

For $i, j \in \mathbb{N}_0$, the $(i,j)^{\text{th}}$-entry of the matrix representation of $C_{r,\alpha}$ with respect to the orthonormal bases is $C_{i,j}$, where

$$C_{i,j} = \langle C_{r,\alpha}(u_j), v_i\rangle = \left\langle \sum_{l=0}^{+\infty} r^j \alpha_{j+2l} v_l, v_i \right\rangle = \sum_{l=0}^{+\infty} r^j \alpha_{j+2l}\langle v_l, v_i\rangle = r^j \alpha_{j+2i},$$

and hence, the corresponding matrix is given as:

$$[C_{r,\alpha}] = \begin{bmatrix} \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & r^4\alpha_4 & \cdots \\ \alpha_2 & r\alpha_3 & r^2\alpha_4 & r^3\alpha_5 & r^4\alpha_6 & \cdots \\ \alpha_4 & r\alpha_5 & r^2\alpha_6 & r^3\alpha_7 & r^4\alpha_8 & \cdots \\ \alpha_6 & r\alpha_7 & r^2\alpha_8 & r^3\alpha_9 & r^4\alpha_{10} & \cdots \\ \alpha_8 & r\alpha_9 & r^2\alpha_{10} & r^3\alpha_{11} & r^4\alpha_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Definition 2.2. Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers. Then the operator (R,r)-Hankel operator, $R_{r,\alpha}$ from a Hilbert space H_1 to Hilbert space H_2 is defined as

$$R_{r,\alpha}(u_i) = \sum_{j=0}^{+\infty} r^i \alpha_{2i+j} v_j, \qquad \text{for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$ are orthonormal bases for H_1 and H_2, respectively.

Observe that for $i, j \in \mathbb{N}_0$, if $R_{i,j}$ is the $(i,j)^{\text{th}}$-entry of the matrix representation of $R_{r,\alpha}$ with respect to the orthonormal bases, then

$$R_{i,j} = \langle R_{r,\alpha}(u_j), v_i\rangle = r^j \alpha_{2j+i},$$

and the corresponding matrix is given as:

$$[R_{r,\alpha}] = \begin{bmatrix} \alpha_0 & r\alpha_2 & r^2\alpha_4 & r^3\alpha_6 & r^4\alpha_8 & \cdots \\ \alpha_1 & r\alpha_3 & r^2\alpha_5 & r^3\alpha_7 & r^4\alpha_9 & \cdots \\ \alpha_2 & r\alpha_4 & r^2\alpha_6 & r^3\alpha_8 & r^4\alpha_{10} & \cdots \\ \alpha_3 & r\alpha_5 & r^2\alpha_7 & r^3\alpha_9 & r^4\alpha_{11} & \cdots \\ \alpha_4 & r\alpha_6 & r^2\alpha_8 & r^3\alpha_{10} & r^4\alpha_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note 2.1. **(A)** A (C,r)-Hankel operator becomes r^2-Hankel operator if its alternate columns are deleted and a (R,r)-Hankel operator becomes r-Hankel operator if its alternate rows are deleted.

(B) For every non-zero complex number r and complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, the (C,r)-Hankel operator, $C_{r,\alpha}$ and (R,r)-Hankel operator, $R_{r,\alpha}$ may not correspond to bounded linear operators.

Example 2.1. Take $r = 1+i$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, for all $n \in \mathbb{N}_0$, and $x = \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} u_n \in H$. Then,

$$\|x\|^2 = \sum_{n=0}^{+\infty} |x_n|^2 = \sum_{n=0}^{+\infty} \left| \frac{1}{(1+i)^n} \right|^2$$

is finite whereas

$$\|C_{r,\alpha}(x)\|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} r^n \alpha_{n+2j} \right|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n+2j+1}} \right|^2 \to +\infty$$

and

$$\|R_{r,\alpha}(x)\|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} r^n \alpha_{2n+j} \right|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{\sqrt{2n+j+1}} \right|^2 \to +\infty.$$

3. Boundedness of (C,r)-Hankel Operators and (R,r)-Hankel Operators

In this section, we study conditions under which these operators become bounded. Characterizations of these operators are also derived.

Theorem 3.1. *Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a complex sequence. Then the following hold.*

(A) *The operator $C_{r,\alpha} : H_1 \to H_2$ is bounded if and only if $\sum_{n\in\mathbb{N}_0} |\beta_n|^2 < +\infty$ where*

$$\beta_n = \begin{cases} \alpha_n, & \text{if } n \text{ is even}, \\ r\alpha_n, & \text{if } n \text{ is odd}. \end{cases} \tag{3.1}$$

(B) *The operator $R_{r,\alpha} : H_1 \to H_2$ is bounded if and only if $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$.*

Proof. **(A)** Let $|r| < 1$. If $C_{r,\alpha}$ is bounded, then there exists a positive constant C such that $\|C_{r,\alpha}(x)\|^2 \le C\|x\|^2$ for every $x \in H_1$. Take in particular $x = u_0$, we get $\sum_{n\in\mathbb{N}_0} |\alpha_{2n}|^2 = \|C_{r,\alpha}(u_0)\|^2 \le C\|u_0\|^2 = C$. Again, taking $x = u_1$, it follows that $|r|^2 \sum_{n\in\mathbb{N}_0} |\alpha_{2n+1}|^2 = \|C_{r,\alpha}(u_1)\|^2 \le C\|u_1\|^2 = C$. Therefore, $\sum_{n\in\mathbb{N}_0} |\beta_n|^2 = \sum_{n\in\mathbb{N}_0} |\alpha_{2n}|^2 + |r|^2 \sum_{n\in\mathbb{N}_0} |\alpha_{2n+1}|^2$ is finite.

Conversely, suppose that (3.1) holds. Consider

$$\begin{aligned} \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |C_{i,j}|^2 =& \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \left(1 + |r^2|^2 + |r^4|^2 + \cdots + |r^{2n}|^2\right) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \\ & \times \left(|r|^2 + |r^3|^2 + |r^5|^2 + \cdots + |r^{2n+1}|^2\right) \\ =& \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \left(1 + |r|^4 + |r|^8 + \cdots + |r|^{4n}\right) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \Big(|r|^2 + |r|^6 \end{aligned}$$

$$
\begin{aligned}
& + |r|^{10} + \cdots + |r|^{2(2n+1)}\Big) \\
&= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \Big(1 + |r|^4 + (|r|^4)^2 + \cdots + (|r|^4)^n\Big) + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \Big(1 + |r|^4 \\
& + (|r|^4)^2 + \cdots + (|r|^4)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) (1 + |r|^4 + \Big(|r|^4)^2 + \cdots + (|r|^4)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \left(\frac{1 - |r|^{4n}}{1 - |r|^4}\right) \\
&\leq \left(\frac{1}{1 - |r|^4}\right) \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \\
&= \left(\frac{1}{1 - |r|^4}\right) \left(\sum_{n=0}^{+\infty} |\beta_n|^2\right).
\end{aligned}
$$

Using (3.1), it follows that $\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |C_{i,j}|^2 < +\infty$. Therefore, the operator $C_{r,\alpha}$ is Hilbert-Schmidt and hence bounded.

(B) Let $|r| < 1$ and $R_{r,\alpha}$ be bounded, then there exists a positive constant C such that $\|R_{r,\alpha}(x)\|^2 \leq C\|x\|^2$ for every $x \in H_1$. Taking in particular $x = u_0$, we get $\sum_{n\in\mathbb{N}_0} |\alpha_n|^2 = \|C_{r,\alpha}(u_0)\|^2 \leq C\|u_0\|^2 = C$.

Conversely, suppose that $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$. Consider

$$
\begin{aligned}
\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |R_{i,j}|^2 &= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \Big(1 + |r|^2 + |r^2|^2 + \cdots + |r^n|^2\Big) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \\
&\quad \times \Big(1 + |r|^2 + |r^2|^2 + \cdots + |r^n|^2\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \Big(1 + |r|^2 + |r|^4 + \cdots + |r|^{2n}\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \Big(1 + |r|^2 + (|r|^2)^2 + \cdots + (|r|^2)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \left(\frac{1 - |r|^{2n}}{1 - |r|^2}\right) \\
&\leq \left(\frac{1}{1 - |r|^2}\right) \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \\
&= \left(\frac{1}{1 - |r|^2}\right) \left(\sum_{n=0}^{+\infty} |\alpha_n|^2\right).
\end{aligned}
$$

Using $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$, it follows that $\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |R_{i,j}|^2 < +\infty$. Therefore, the operator $R_{r,\alpha}$ is Hilbert-Schmidt and hence bounded. □

The next theorem gives characterizations of bounded linear (C, r)-Hankel and (R, r)-Hankel operators in terms of operator equations involving shift operator.

Theorem 3.2. *Let U_1 and U_2 be the right shift operators on H_1 and H_2, respectively. Let r be a non-zero complex number. Then the following hold.*

- **(A)** *A bounded operator $T : H_1 \to H_2$ is a (C, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1^2 = r^2U_2^*T$.*
- **(B)** *A bounded operator $T : H_1 \to H_2$ is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1^2 = r^2(U_2^4)^*T$ and $T_{i,1} = rT_{i+2,0}$ for all $i \in \mathbb{N}_0$, where $(T_{i,j})$ represents matrix representation of T with respect to orthonormal bases of H_1 and H_2, respectively.*

Proof. **(A)** Suppose $T : H_1 \to H_2$ is a (C, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. For each $i, j \in \mathbb{N}_0$,

$$\langle TU_1^2(u_i), v_j\rangle = \langle T(u_{i+2}), v_j\rangle = r^{i+2}\alpha_{i+2+2j}$$

and

$$\langle r^2U_2^*T(u_i), v_j\rangle = r^2\langle T(u_i), U_2(v_j)\rangle = r^2\langle T(u_i), v_{j+1}\rangle = r^2r^i\alpha_{i+2+2j} = r^{i+2}\alpha_{i+2+2j}.$$

Using the boundedness of T, it follows that $TU_1^2 = r^2U_2^*T$.

Conversely, let $TU_1^2 = r^2U_2^*T$. We define a complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ as follows:

$$\alpha_n = \begin{cases} \langle T(u_0), v_{n/2}\rangle, & \text{if } n \text{ is even,} \\ (1/r)\langle T(u_1), v_{(n-1)/2}\rangle, & \text{elsewhere.} \end{cases} \tag{3.2}$$

Then, for all non-negative integers i, j such that $i \geq 2$,

$$\begin{aligned} \langle T(u_i), v_j\rangle &= \langle TU_1^2(u_{i-2}), v_j\rangle = \langle r^2U_2^*T(u_{i-2}), v_j\rangle = r^2\langle T(u_{i-2}), U_2(v_j)\rangle \\ &= r^2\langle T(u_{i-2}), v_{j+1}\rangle = \cdots = r^4\langle T(u_{i-4}), v_{j+2}\rangle = \cdots = \\ &= \begin{cases} r^i\langle T(u_0), v_{j+i/2}\rangle, & \text{if } i \text{ is even,} \\ r^{i-1}\langle T(u_1), v_{j+(i-1)/2}\rangle, & \text{if } i \text{ is odd,} \end{cases} \\ &= \begin{cases} r^i\alpha_{2j+i}, & \text{if } i \text{ is even,} \\ r^{i-1}r\alpha_{2j+i}, & \text{if } i \text{ is odd,} \end{cases} \\ &= r^i\alpha_{2j+i}. \end{aligned}$$

Hence, $T = C_{r,\alpha}$ for the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ defined in (3.2).

(B) Suppose $T : H_1 \to H_2$ is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Clearly, $T_{i,1} = rT_{i+2,0}$ for all $i \in \mathbb{N}_0$. Now, for each $i, j \in \mathbb{N}_0$,

$$\langle TU_1^2(u_i), v_j\rangle = \langle T(u_{i+2}), v_j\rangle = r^{i+2}\alpha_{2i+4+j}$$

and

$$\langle r^2(U_2^4)^*T(u_i), v_j\rangle = r^2\langle T(u_i), U_2^4(v_j)\rangle = r^2\langle T(u_i), v_{j+4}\rangle = r^2r^i\alpha_{2i+j+4} = r^{i+2}\alpha_{2i+4+j}.$$

Using the boundedness of T, it follows that $TU_1^2 = r^2(U_2^4)^*T$.

Conversely, suppose that $TU_1^2 = r^2(U_2^4)^*T$ and

$$T_{i,1} = rT_{i+2,0}, \quad \text{for all } i \in \mathbb{N}_0, \tag{3.3}$$

where $(T_{i,j})$ represents matrix representation of the operator T. For each $n \in \mathbb{N}_0$, let

$$\alpha_n = \begin{cases} \langle T(u_0), v_n \rangle, & \text{if } n \text{ is even}, \\ \langle T(u_0), v_1 \rangle, & \text{if } n = 1, \\ (1/r)\langle T(u_1), v_{n-2} \rangle, & \text{elsewhere}. \end{cases} \tag{3.4}$$

Then $(\alpha_n)_{n\in\mathbb{N}_0}$ is a sequence in the complex plane. Using (3.3) and (3.4), for all non-negative integers i, j such that $i \geq 2$, evaluating

$$\begin{aligned} \langle T(u_i), v_j \rangle &= \langle TU_1^2(u_{i-2}), v_j \rangle = \langle r^2(U_2^4)^*T(u_{i-2}), v_j \rangle = r^2\langle T(u_{i-2}), U_2^4(v_j) \rangle \\ &= r^2\langle T(u_{i-2}), v_{j+4} \rangle = \cdots = r^4\langle T(u_{i-4}), v_{j+8} \rangle = \cdots = \\ &= \begin{cases} r^i\langle T(u_0), v_{j+2i} \rangle, & \text{if } i \text{ is even}, \\ r^{i-1}\langle T(u_1), v_{j+2(i-1)} \rangle, & \text{if } i \text{ is odd}, \end{cases} \\ &= \begin{cases} r^i\alpha_{j+2i}, & \text{if } i, j \text{ both are even}, \\ r^{i-1}\langle T(u_1), v_{j+2i-2)} \rangle, & \text{if } i \text{ is even and } j \text{ is odd}, \\ r^{i-1}r\alpha_{j+2i}, & \text{and } i, j \text{ both are odd}, \\ r^{i-1}r\langle T(u_0), v_{j+2i} \rangle, & \text{if } i \text{ is odd and } j \text{ is even}, \end{cases} \\ &= r^i\alpha_{j+2i}. \end{aligned}$$

Hence, $T = R_{r,\alpha}$ for complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. □

Proposition 3.1. *Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ be a sequence. Then the adjoint of bounded (C, r)-Hankel operator, $C_{r,\alpha} : H_1 \to H_2$ is the (R, s)-Hankel operator, $R_{s,\beta}$ from H_2 to H_1, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\,\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$.*

Proof. Let $i, j \in \mathbb{N}_0$. Evaluating

$$\langle C_{r,\alpha}^*(v_j), u_i \rangle = \langle v_j, C_{r,\alpha}(u_i) \rangle = \overline{\langle C_{r,\alpha}(u_i), v_j \rangle} = \overline{r^i}\,\overline{\alpha_{i+2j}} = \overline{r}^i\,\overline{\alpha_{i+2j}}$$

and

$$\langle R_{s,\beta}(v_j), u_i \rangle = s^j\beta_{i+2j} = \left(\frac{1}{\overline{r^2}}\right)^j \overline{r^{i+2j}}\,\overline{\alpha_{i+2j}} = \overline{r}^i\,\overline{\alpha_{i+2j}}.$$

Hence, $C_{r,\alpha}^* = R_{s,\beta}$, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\,\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. □

Theorem 3.3. *Let U_1 and U_2 be the right shift operators on H_1 and H_2, respectively. Let r be a non-zero complex number. Then a bounded operator $T : H_1 \to H_2$ is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1 = r(U_2^*)^2T$.*

Proof. Suppose that T is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Using Proposition 3.1, it follows that $T = R_{r,\alpha} = C_{s,\beta}^*$, where $C_{s,\beta} : H_2 \to H_1$ is (C, s)-Hankel operator, $s = \left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}$ and $\beta_n = \left(\frac{1}{\overline{r^n}}\right)^{\frac{1}{2}}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Now, Theorem 3.2 gives $C_{s,\beta}U_2^2 = s^2U_1^*C_{s,\beta}$. Taking adjoint on both sides, it follows that $(U_2^*)^2C_{s,\beta}^* = \overline{s}^2C_{s,\beta}^*U_1$. That is, $TU_1 = r(U_2^*)^2T$.

Conversely, if an operator T is such that $TU_1 = r(U_2^*)^2T$, then, by reversing the steps above and by using Theorem 3.2 and Proposition 3.1, we can conclude that T is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. □

Corollary 3.1. *The kernel of (R, r)-Hankel operator is an invariant subspace of shift operator.*

Proposition 3.2. *For a non-zero complex number $r \in \mathbb{C}$, if an operator T is a (C, r)-Hankel operator as well as (R, r)-Hankel operator on H for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, then U^*T is r-Toeplitz operator on H, where U is the right shift operator on H.*

Proof. Suppose that T is a (C, r)-Hankel operator as well as (R, r)-Hankel operator on H for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Since T is (C, r)-Hankel operator, therefore, by using Theorem 3.2, it follows that

$$TU^2 = r^2U^*T. \tag{3.5}$$

Also, T is (R, r)-Hankel operator, therefore, Proposition 3.3 gives

$$TU = r(U^*)^2T. \tag{3.6}$$

Using (3.5) and (3.6), we obtain that

$$r^2U^*T = TU^2 = (TU)U = r(U^*)^2TU = rU^*(U^*T)U.$$

This implies that $U^*(U^*T)U = r(U^*T)$ which means that U^*T is r-Toeplitz operator [7] on H. □

In Theorem 3.1, boundedness conditions of these operators for the case $|r| < 1$ have been discussed. We discuss boundedness of these operators for $|r| > 1$ in the next result.

Theorem 3.4. *Let r be a non-zero complex number such that $|r| > 1$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a complex sequence. Then the following hold.*

(A) *The operator $C_{r,\alpha} : H_1 \to H_2$ is bounded if and only if*

$$\sum_{n=0}^{+\infty} |r|^{2n}|\alpha_n|^2 < +\infty.$$

(B) *Then the operator $R_{r,\alpha} : H_1 \to H_2$ is bounded if and only if*

$$\sum_{n=0}^{+\infty} |\gamma_n|^2 < +\infty,$$

where

$$\gamma_n = \begin{cases} \left(\frac{1}{r^n}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is even,} \\ \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $|r| > 1$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a complex sequence.

(A) Let $s = \frac{1}{r^2}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be a sequence, where $\beta_n = \overline{r^n}\,\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. The operator $C_{r,\alpha}$ is bounded if and only if $C^*_{r,\alpha}$ is bounded. Using Proposition 3.1, it follows that the operator $C^*_{r,\alpha}$ is bounded if and only if $R_{s,\beta}$ is bounded. Since $|s| < 1$, therefore, using Theorem 3.1 (B), it is concluded that $R_{s,\beta}$ is bounded if and only if $\sum_{n=0}^{+\infty} |\beta_n|^2 < +\infty$, that is, $\sum_{n=0}^{+\infty} |r|^{2n}|\alpha_n|^2 < +\infty$.

(B) Let $s = \left(\frac{1}{r}\right)^{\frac{1}{2}}$ and $\beta_n = \left(\frac{1}{r^n}\right)^{\frac{1}{2}} \overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Since $|r| > 1$, so $|s| < 1$. The operator $R_{r,\alpha}$ is bounded if and only if $R^*_{r,\alpha}$ is bounded. Using Proposition 3.1, it follows that the operator $R^*_{r,\alpha}$ is bounded if and only if $C_{s,\beta}$ is bounded. Since $|s| < 1$, therefore, using Theorem 3.1 (A), it gives $C_{s,\beta}$ is bounded if and only if $\sum_{n=0}^{+\infty} |\gamma_n|^2 < +\infty$, where

$$\gamma_n = \begin{cases} \beta_n, & \text{if } n \text{ is even,} \\ s\beta_n, & \text{if } n \text{ is odd.} \end{cases}$$

That is,

$$\gamma_n = \begin{cases} \left(\frac{1}{\overline{r^n}}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is even,} \\ \left(\frac{1}{\overline{r^{n+1}}}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is odd.} \end{cases}$$ □

For $r \in \mathbb{C}\backslash\{0\}$, let $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$ denote the classes of all bounded (C, r)-Hankel operators and (R, r)-Hankel operators, respectively defined from H_1 to H_2. They are denoted by $\mathcal{C}_r(H)$ and $\mathcal{R}_r(H)$ if $H_1 = H_2$. It can easily be seen that the classes $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$ are weakly closed and hence strongly closed, vector subspaces of the space $\mathcal{B}(H_1, H_2)$, where $\mathcal{B}(H_1, H_2)$ is the class of all bounded linear operators from H_1 to H_2.

Proposition 3.3. *Let $r \in \mathbb{C}\backslash\{0\}$. Then there does not exist any Fredholm operator in the classes $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$.*

Proof. Suppose that there exist a Fredholm (C, r)-Hankel operator, $C_{r,\alpha}$ in $\mathcal{C}_r(H_1, H_2)$ for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, whose index is n. Using Theorem 3.2 (A), it follows that $C_{r,\alpha}U_1^2 = r^2U_2^*C_{r,\alpha}$, where U_1 and U_2 are right shift operators on H_1 and H_2, respectively. Since $C_{r,\alpha}$ is Fredholm of index n, this implies that $C_{r,\alpha}U_1^2$ is Fredholm of index $n - 2$. On the other hand, $r^2U_2^*C_{r,\alpha}$ is Fredholm of index $n + 1$. This means that $n - 2 = n + 1$ which is a contradiction. Hence, there does not exist any Fredholm operator in the class $\mathcal{C}_r(H_1, H_2)$.

Similarly, using Theorem 3.2 (B), one can obtain that there does not exist any Fredholm operator in the class $\mathcal{R}_r(H_1, H_2)$. □

4. Commutativity of (C, r)-Hankel Operators and (R, r)-Hankel Operators

This section is devoted to explore the characterizations for commutativity of operators in $\mathcal{C}_r(H)$ and $\mathcal{R}_r(H)$.

Theorem 4.1. *Let r and s be non-zero complex numbers and $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be two complex sequences. Then the following hold.*

(A) *The bounded operators $C_{r,\alpha}$ and $C_{s,\beta}$ on Hilbert space H commute if and only if*

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i, k \in \mathbb{N}_0$, provided the series converge.

(B) *The bounded operators $C_{r,\alpha}$ and $R_{s,\beta}$ on H commute if and only if*

$$\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k},$$

for all $i, k \in \mathbb{N}_0$, provided the series converge.

Proof. **(A)** For each $i \in \mathbb{N}_0$, consider

$$\begin{aligned} C_{r,\alpha}C_{s,\beta}(u_i) &= C_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}u_j\right) = \left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}C_{r,\alpha}(u_j)\right) \\ &= \left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}\left(\sum_{k=0}^{+\infty} r^j\alpha_{j+2k}u_k\right)\right) \\ &= \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k}\right)u_k\right). \end{aligned} \tag{4.1}$$

Similarly, we obtain that

$$C_{s,\beta}C_{r,\alpha}(u_i) = \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k}\right)u_k\right). \tag{4.2}$$

Since $(u_i)_{i\in\mathbb{N}_0}$ is an orthonormal basis for H, therefore, using (4.1) and (4.2), it follows that the bounded operators $C_{r,\alpha}$ and $C_{s,\beta}$ commute if and only if

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i, k \in \mathbb{N}_0$.

(B) For each $i \in \mathbb{N}_0$, evaluate

$$\begin{aligned} C_{r,\alpha}R_{s,\beta}(u_i) &= C_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}u_j\right) = \left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}C_{r,\alpha}(u_j)\right) \\ &= \left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}\left(\sum_{k=0}^{+\infty} r^j\alpha_{j+2k}u_k\right)\right) \\ &= \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k}\right)u_k\right). \end{aligned} \tag{4.3}$$

Similarly, it is obtained that

$$R_{s,\beta}C_{r,\alpha}(u_i) = \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k}\right)u_k\right). \tag{4.4}$$

Using (4.3) and (4.4), it follows that the bounded operators $C_{r,\alpha}$ and $R_{s,\beta}$ commute if and only if

$$\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k},$$

for all $i, k \in \mathbb{N}_0$. □

The following example demonstrates commuting operators in $\mathcal{C}_r(H)$.

Example 4.1. If $r = s = \frac{i}{2}$, $\alpha(n) = (\frac{i}{2})^n$ and $\beta(n) = \frac{i^n}{2^{n+1}}$ for all $n \in \mathbb{N}_0$, then one can easily see that the operators $C_{r,\alpha}$ and $C_{s,\beta}$ are bounded (using Theorem 3.1) and they satisfy the following expression:

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i, k \in \mathbb{N}_0$. Hence, the operators $C_{r,\alpha}$ and $C_{s,\beta}$ commute on H.

Let $\mathscr{C}_{0,0}$ denote the set of all complex sequences whose only finitely many terms are non-zero.

Theorem 4.2. *Let $r, s \in \mathbb{C}\backslash\{0\}$ and $\alpha, \beta \in \mathscr{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j\in\mathbb{N}_0}$ and $\beta = (\beta_j)_{j\in\mathbb{N}_0}$. Let n and m be the largest non-negative integers such that*

$$\alpha_n \neq 0 \quad \text{and} \quad \beta_m \neq 0.$$

Then the operators $R_{r,\alpha}$ and $R_{s,\beta}$ on Hilbert space H commute if and only if $n = m$, $r = s$ and there exists $\lambda \in \mathbb{C}$ such that $\beta_j = \lambda\alpha_j$ for all $j \in \mathbb{N}_0$.

Proof. Let the operators $R_{r,\alpha}$ and $R_{s,\beta}$ commute. That is,

$$R_{r,\alpha}R_{s,\beta}(x) = R_{s,\beta}R_{r,\alpha}(x), \tag{4.5}$$

for all $x \in H$. Two cases arise.

Case 1. If $n = m$. Let $n = 2p + r_1$ where $p \in \mathbb{N}_0$ and $r_1 = 0$ or 1. In particular, take $x = u_p$ in (4.5), we have

$$R_{r,\alpha}R_{s,\beta}(u_p) = R_{s,\beta}R_{r,\alpha}(u_p). \tag{4.6}$$

Subcase 1. If $r_1 = 0$. Consider

$$R_{r,\alpha}R_{s,\beta}(u_p) = R_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^p\beta_{2p+j}u_j\right) = s^p\beta_{2p}R_{r,\alpha}(u_0) = s^p\beta_{2p}\left(\sum_{j=0}^{+\infty}\alpha_j u_j\right)$$

$$= s^p\beta_{2p}\left(\sum_{j=0}^{n}\alpha_j u_j\right) = \sum_{j=0}^{n}(s^p\beta_{2p}\alpha_j)u_j. \tag{4.7}$$

Similarly, we can obtain that

$$R_{s,\beta}R_{r,\alpha}(u_p) = \sum_{j=0}^{n}(r^p\alpha_{2p}\beta_j)u_j. \tag{4.8}$$

Since $(u_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of H, therefore, using (4.6), (4.7) and (4.8), we get $s^p\beta_{2p}\alpha_j = r^p\alpha_{2p}\beta_j$ for all $0 \leq j \leq n$. Let $\lambda = \frac{\beta_n}{\alpha_n}$. This implies that $r^p = s^p$ and $\beta_j = \lambda\alpha_j$ for all $0 \leq j \leq n$.

Subcase 2. If $r_1 = 1$. Consider

$$\begin{aligned}
R_{r,\alpha}R_{s,\beta}(u_p) &= R_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^p\beta_{2p+j}u_j\right) = R_{r,\alpha}\left(\sum_{j=0}^{1} s^p\beta_{2p+j}u_j\right)\\
&= s^p\beta_{2p}\left(\sum_{j=0}^{+\infty}\alpha_j u_j\right) + s^p\beta_{2p+1}\left(\sum_{j=0}^{+\infty} r\alpha_{2+j}u_j\right)\\
&= s^p\beta_{2p}\left(\sum_{j=0}^{n}\alpha_j u_j\right) + s^p\beta_{2p+1}\left(\sum_{j=0}^{n-2} r\alpha_{2+j}u_j\right)\\
&= \left(\sum_{j=0}^{n-2} s^p\left(\beta_{2p}\alpha_j + \beta_{2p+1}r\alpha_{2+j}\right)u_j\right) + s^p\beta_{2p}\alpha_{n-1}u_{n-1} + s^p\beta_{2p}\alpha_n u_n. \qquad (4.9)
\end{aligned}$$

Similarly, we can obtain that

$$R_{s,\beta}R_{r,\alpha}(u_p) = \left(\sum_{j=0}^{n-2} r^p\left(\alpha_{2p}\beta_j + \alpha_{2p+1}s\beta_{2+j}\right)u_j\right) + r^p\alpha_{2p}\beta_{n-1}u_{n-1} + r^p\alpha_{2p}\beta_n u_n. \tag{4.10}$$

Again using the fact that the set $(u_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of H, therefore, using (4.6), (4.9) and (4.10), we get $s^p\beta_{2p}\alpha_n = r^p\alpha_{2p}\beta_n, s^p\beta_{2p}\alpha_{n-1} = r^p\alpha_{2p}\beta_{n-1}$ and $s^p\left(\beta_{2p}\alpha_j + \beta_{2p+1}r\alpha_{2+j}\right) = r^p\left(\alpha_{2p}\beta_j + \alpha_{2p+1}s\beta_{2+j}\right)$ for each $0 \leq j \leq n-2$. Let $\lambda = \frac{\beta_n}{\alpha_n}$. On solving successively, it follows that $s^p = r^p$ and $\beta_j = \lambda\alpha_j$ for all $0 \leq j \leq n$.

Using (4.5) at $x = u_1$, together with $s^p = r^p$ and $\beta_j = \frac{\beta_n}{\alpha_n}\alpha_j$ for all $0 \leq j \leq n$, one can obtain $s = r$ in both the subcases.

Case 2. If $n \neq m$. Without loss of generality, we can assume that $n > m$. Let $n = 2p + r_1$ and $m = 2q + r_2$, where $p, q \in \mathbb{N}_0$ and $r_1, r_2 \in \{0, 1\}$. In this case, we claim that the operators $R_{r,\alpha}$ and $R_{s,\beta}$ do not commute. Assume on the contrary that $R_{r,\alpha}$ and $R_{s,\beta}$ commute.

Subcase 1. If $m = 0$. Using (4.5), we get $R_{r,\alpha}R_{s,\beta}(u_0) = R_{s,\beta}R_{r,\alpha}(u_0)$ which gives $\beta_0\left(\sum_{j=0}^{n}\alpha_j u_j\right) = \beta_0\alpha_0 u_0$. On comparing the coefficients of u_n, it follows that $\beta_0\alpha_n = 0$, which is not possible as $\beta_0 \neq 0$ and $\alpha_n \neq 0$.

Subcase 2. If $m = 1$. Again using (4.5) for $x = u_0$, we get $\beta_0\left(\sum_{j=0}^{n}\alpha_j u_j\right) + \beta_1\left(\sum_{j=0}^{n-2} r\alpha_{j+2}u_j\right) = \sum_{j=0}^{1}\alpha_0\beta_j u_j$. On comparing the coefficients of u_n and u_{n-2}, we get $\beta_0\alpha_n = 0$ and $\beta_0\alpha_{n-2} + \beta_1 r\alpha_n = 0$, which is not possible as $\beta_1 \neq 0$ and $\alpha_n \neq 0$.

Subcase 3. If $m = 2q$. Take $x = u_q$ in (4.5), we get

$$s^q \beta_{2q} \sum_{j=0}^{n} \alpha_j u_j = \sum_{j=0}^{\min(n-m,q)} \sum_{k=0}^{m-2j} r^q s^j \alpha_{m+j} \beta_{2j+k} u_k.$$

On comparing the coefficients of u_n, it follows that $s^q \beta_{2q} \alpha_n = 0$. It follows that $\alpha_n = 0$, which is not true.

Subcase 4. If $m = 2q+1$. Take $x = u_q$ in (4.5), we get $s^q \beta_{2q} \sum_{j=0}^{n} \alpha_j u_j + s^q \beta_{2q+1} \sum_{j=0}^{n-2} \alpha_{j+2} u_j = \sum_{j=0}^{\min(n-2q,q)} \sum_{k=0}^{m-2j} r^q s^j \alpha_{2q+j} \beta_{2j+k} u_k$. On comparing the coefficients of u_n, it follows that $s^q \beta_{2q} \alpha_n = 0$. It follows that $\alpha_n = 0$, which is not true.

Hence, from all the subcases, it follows that the operators $R_{r,\alpha}$ and $R_{s,\beta}$ can not commute. □

As a consequence of this result and by using Proposition 3.1, we get the following result.

Corollary 4.1. *Let $r, s \in \mathbb{C} \backslash \{0\}$ and $\alpha, \beta \in \mathscr{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j \in \mathbb{N}_0}$ and $\beta = (\beta_j)_{j \in \mathbb{N}_0}$. Let n and m be the largest non-negative integers such that*

$$\alpha_n \neq 0 \quad \text{and} \quad \beta_m \neq 0.$$

Then the operators $C_{r,\alpha}$ and $C_{s,\beta}$ on Hilbert space H commute if and only if $n = m$, $r^2 = s^2$ and there exists $\lambda \in \mathbb{C}$ such that $s^j \beta_j = \lambda r^j \alpha_j$ for all $j \in \mathbb{N}_0$.

Now, we show that the class $\mathscr{C}_r(H)$ and hence, $\mathscr{R}_r(H)$ does not contain any unitary operator.

Proposition 4.1. *The class $\mathscr{C}_r(H)$ does not contain any unitary operator for any non-zero $r \in \mathbb{C}$.*

Proof. Suppose there exists unitary operator $C_{r,\alpha}$ in $\mathscr{C}_r(H)$ for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. This implies that

$$\|C_{r,\alpha}(x)\|^2 = \|x\|^2 = \|C^*_{r,\alpha}(x)\|^2, \tag{4.11}$$

for all $x \in H$.

Case 1. If $|r| < 1$. For $x = u_0$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |\alpha_{2j}|^2 = 1. \tag{4.12}$$

Now take $x = u_2$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |r|^4 |\alpha_{2j+2}|^2 = 1. \tag{4.13}$$

On solving (4.12) and (4.13), we obtain that $|\alpha_0|^2 = 1 - \frac{1}{|r|^4} < 0$, a contradiction.

Case 2. If $|r| > 1$. Using Proposition 3.1, it follows that $C^*_{r,\alpha} = R_{s,\beta}$, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. For $x = u_0$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |\beta_j|^2 = 1. \tag{4.14}$$

Now take $x = u_1$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |s|^2|\beta_{j+2}|^2 = 1. \tag{4.15}$$

On solving (4.14) and (4.15), it follows that $|\beta_0|^2+|\beta_1|^2 = 1-\frac{1}{|s|^2} < 0$ (a contradiction), since $|r| > 1$ implies $|s| < 1$.

Case 3. If $|r| = 1$. For each $i \in \mathbb{N}_0$, take $x = u_i$ in (4.11), we get $\sum_{j=0}^{+\infty} |\alpha_{2j}|^2 = 1, \sum_{j=0}^{+\infty} |r|^2|\alpha_{2j+1}|^2 = 1, \sum_{j=0}^{+\infty} |r|^4|\alpha_{2j+2}|^2 = 1, \ldots$ On solving these equations, we get $\alpha_i = 0$ for all $i \in \mathbb{N}_0$, a contradiction.

Hence, there does not exist any unitary operator in the class $\mathscr{C}_r(H)$ for any non-zero complex number r. □

As a consequence of this result, we get the following result.

Corollary 4.2. *Let r be a non-zero complex number, then the following hold.*

(A) *If $|r| < 1$, then the class $\mathscr{C}_r(H)$ does not contain any isometry.*
(B) *If $|r| > 1$, then the class $\mathscr{R}_r(H)$ does not contain any isometry.*

References

[1] S. C. Arora and J. Bhola, *Spectrum of a kth-order slant Hankel operator*, Bull. Math. Anal. Appl. **3** (2011), 175–183.

[2] M. D. Choi, *Tricks or treats with the Hilbert matrix*, Amer. Math. Monthly **90** (1983), 301–312. https://doi.org/10.2307/2975779

[3] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Pure and Applied Mathematics, Vol. 49, Academic Press, New York, London, 1972.

[4] A. Gupta and B. Gupta, *Commutativity and spectral properties of k^{th}-order slant little Hankel operators on the Bergman space*, Oper. Matrices **13** (2019), 209–220. https://doi.org/10.7153/oam-2019-13-14

[5] P. R. Halmos, *A Hilbert Space Problem Book*, Second Edition, *Encyclopedia of Mathematics and its Applications*, Vol. 17, Springer-Verlag, New York, Berlin, 1982.

[6] H. Hankel, *Ueber eine besondere classe der symmetrishchen determinanten*, (Leipziger) Dissertation, Göttingen, 1861.

[7] M. C. Ho, *On the rotational invariance for the essential spectrum of λ-Toeplitz operators*, J. Math. Anal. Appl. **413** (2014), 557–565. https://doi.org/10.1016/j.jmaa.2013.11.056

[8] R. A. Martínez-Avendaño, *Essentially Hankel operators*, J. London Math. Soc. (2) **66** (2002), 741–752. https://doi.org/10.1112/S002461070200368X

[9] R. A. Martínez-Avendaño, *A generalization of Hankel operators*, J. Funct. Anal. **190** (2002), 418–446. https://doi.org/10.1006/jfan.2001.3869

[10] A. Mirotin and E. Kuzmenkova, *μ-Hankel operators on Hilbert spaces*, Opuscula Math. **41** (2021), 881–898. https://doi.org/10.7494/opmath.2021.41.6.881

[11] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, Vol. 1, Mathematical Surveys and Monographs, Vol. 92, American Mathematical Society, Providence, RI, 2002 (Translated from the French by Andreas Hartmann).

[12] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, Vol. 2, Mathematical Surveys and Monographs, Vol. 93, American Mathematical Society, Providence, RI, 2002 (Translated from the French by Andreas Hartmann and revised by the author).

[13] V. V. Peller, *Hankel Operators and their Applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003. `https://doi.org/10.1007/978-0-387-21681-2`

[14] E. Wang and Z. Hu, *Small Hankel operators between Fock spaces*, Complex Var. Elliptic Equ. **64** (2019), 409–419. `https://doi.org/10.1080/17476933.2018.1441832`

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ON SOME COMBINATORIAL PROPERTIES OF GENERALIZED COMMUTATIVE PELL AND PELL-LUCAS QUATERNIONS

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Abstract. Generalized commutative quaternions generalize elliptic, parabolic and hyperbolic quaternions, bicomplex numbers, complex hyperbolic numbers and hyperbolic complex numbers. In this paper, we study generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. We present some properties of these numbers and relations between them.

1. Introduction

Let $n \geq 0$ be an integer. The nth Pell number P_n is defined in the following way $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0,\ P_1 = 1$. Solving the above recurrence equation we obtain the direct formula of the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}},$$

named also as the Binet formula for Pell numbers.

The nth Pell-Lucas number Q_n is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$, with $Q_0 = Q_1 = 2$. The Binet formula for Pell-Lucas numbers has the form

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

The first six terms of the Pell sequence and Pell-Lucas sequence are $0, 1, 2, 5, 12, 29$ and $2, 2, 6, 14, 34, 82$, respectively.

The Pell and Pell-Lucas numbers belong to the class of numbers of the Fibonacci type and have applications also in the theory of hypercomplex numbers (see [1–3,9–12]).

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In this paper, we use the Pell and Pell-Lucas numbers in the theory of generalized commutative quaternions.

Let $\mathbb{H}^c_{\alpha\beta}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.1}$$

$$e_1 e_2 = e_2 e_1 = e_3, \quad e_2 e_3 = e_3 e_2 = \beta e_1 \quad \text{and} \quad e_3 e_1 = e_1 e_3 = \alpha e_2, \tag{1.2}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$.

The generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Generalized commutative quaternions were introduced in [8]. The authors defined generalized commutative quaternions of the Fibonacci type – generalized commutative Horadam quaternions.

For integers p, q, n and $n \geq 0$ Horadam defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$, for $n \geq 2$, with fixed real numbers W_0, W_1. Let t_1, t_2 be the two distinct real roots of the equation $t^2 - pt + q = 0$. Then, the Binet type formula for the Horadam numbers has the form $W_n = At_1^n + Bt_2^n$, where $t_1 = \frac{p-\sqrt{p^2-4q}}{2}$, $t_2 = \frac{p+\sqrt{p^2-4q}}{2}$, $A = \frac{W_1 - W_0 t_2}{t_1 - t_2}$, $B = \frac{W_0 t_1 - W_1}{t_1 - t_2}$. We have $P_n = W_n(0, 1; 2, -1)$ and $Q_n = W_n(2, 2; 2, -1)$, so the Pell and Pell-Lucas numbers are special cases of Horadam numbers.

The nth generalized commutative Horadam quaternion $gc\mathcal{H}_n$ is defined as

$$gc\mathcal{H}_n = W_n + W_{n+1} e_1 + W_{n+2} e_2 + W_{n+3} e_3.$$

In [8], it was presented the following result.

Theorem 1.1 (Binet type formula for generalized commutative Horadam quaternions [8])**.** *Let $n \geq 0$ be an integer. Then*

$$gc\mathcal{H}_n = At_1^n \left(1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3\right) + Bt_2^n \left(1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3\right).$$

2. Main Results

Let $n \geq 0$ be an integer. The nth generalized commutative Pell quaternion $gc\mathcal{P}_n$ and the nth generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ are defined as

$$\begin{aligned} gc\mathcal{P}_n &= P_n + P_{n+1} e_1 + P_{n+2} e_2 + P_{n+3} e_3, \\ gc\mathcal{Q}_n &= Q_n + Q_{n+1} e_1 + Q_{n+2} e_2 + Q_{n+3} e_3, \end{aligned}$$

respectively, where P_n is the nth Pell number, Q_n is the nth Pell-Lucas number and e_1, e_2, e_3 are units which satisfy (1.1) and (1.2).

Using the above definitions we can give initial generalized commutative Pell and Pell-Lucas quaternions, i.e.,

$$\begin{aligned} gc\mathcal{P}_0 &= e_1 + 2e_2 + 5e_3, \\ gc\mathcal{P}_1 &= 1 + 2e_1 + 5e_2 + 12e_3, \\ gc\mathcal{P}_2 &= 2 + 5e_1 + 12e_2 + 29e_3, \\ gc\mathcal{Q}_0 &= 2 + 2e_1 + 6e_2 + 14e_3, \\ gc\mathcal{Q}_1 &= 2 + 6e_1 + 14e_2 + 34e_3, \\ gc\mathcal{Q}_2 &= 6 + 14e_1 + 34e_2 + 82e_3. \end{aligned}$$

Proposition 2.1. *Let $n \geq 0$ be an integer. Then the generalized commutative Pell quaternions satisfy the recurrence relation*

$$gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}, \quad \text{for } n \geq 2, \tag{2.1}$$

with initial conditions

$$gc\mathcal{P}_0 = e_1 + 2e_2 + 5e_3, \quad gc\mathcal{P}_1 = 1 + 2e_1 + 5e_2 + 12e_3.$$

Proposition 2.2. *Let $n \geq 0$ be an integer. The generalized commutative Pell-Lucas quaternions satisfy*

$$gc\mathcal{Q}_n = 2gc\mathcal{Q}_{n-1} + gc\mathcal{Q}_{n-2}, \quad \text{for } n \geq 2,$$

with $gc\mathcal{Q}_0 = 2 + 2e_1 + 6e_2 + 14e_3$, $gc\mathcal{Q}_1 = 2 + 6e_1 + 14e_2 + 34e_3$.

In this paper, we will focus on properties of generalized commutative Pell-Lucas quaternions and we will show some dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. As a special case of Theorem 1.1 we get the following remark.

Remark 2.1. Let $n \geq 0$ be an integer. Then

$$\begin{aligned} gc\mathcal{P}_n =& \frac{\left(1+\sqrt{2}\right)^n}{2\sqrt{2}} \left(1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3\right) \\ &- \frac{\left(1-\sqrt{2}\right)^n}{2\sqrt{2}} \left(1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3\right) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} gc\mathcal{Q}_n =& \left(1+\sqrt{2}\right)^n \left(1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3\right) \\ &+ \left(1-\sqrt{2}\right)^n \left(1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3\right). \end{aligned} \tag{2.3}$$

For simplicity of notation let

$$t_1 = 1-\sqrt{2}, \quad t_2 = 1+\sqrt{2}, \quad A = -\frac{1}{2\sqrt{2}}, \quad B = \frac{1}{2\sqrt{2}},$$

$$\hat{t_1} = 1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3, \tag{2.4}$$

$$\hat{t_2} = 1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3.$$

Then we can write (2.2) and (2.3) as

$$gc\mathcal{P}_n = At_1^n\hat{t_1} + Bt_2^n\hat{t_2} \tag{2.5}$$

and

$$gc\mathcal{Q}_n = t_1^n\hat{t_1} + t_2^n\hat{t_2}, \tag{2.6}$$

respectively, where t_1, t_2, A, B, $\hat{t_1}$, $\hat{t_2}$ are given by (2.4).

Theorem 2.1 (General bilinear index-reduction formula for generalized commutative Pell-Lucas quaternions)**.** *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d = \left(t_1^at_2^b + t_2^at_1^b - t_1^ct_2^d - t_2^ct_1^d\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, $\hat{t_1}$, $\hat{t_2}$ are given by (2.4)*.*

Proof. Using (2.6) we have

$$\begin{aligned}
&gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d \\
=&\left(t_1^a\hat{t_1} + t_2^a\hat{t_2}\right)\left(t_1^b\hat{t_1} + t_2^b\hat{t_2}\right) - \left(t_1^c\hat{t_1} + t_2^c\hat{t_2}\right)\left(t_1^d\hat{t_1} + t_2^d\hat{t_2}\right) \\
=&t_1^a\hat{t_1}t_2^b\hat{t_2} + t_2^a\hat{t_2}t_1^b\hat{t_1} - t_1^c\hat{t_1}t_2^d\hat{t_2} - t_2^c\hat{t_2}t_1^d\hat{t_1} \\
=&\left(t_1^at_2^b + t_2^at_1^b - t_1^ct_2^d - t_2^ct_1^d\right)\hat{t_1}\hat{t_2},
\end{aligned}$$

which ends the proof. □

Moreover, $t_1t_2 = -1$ and

$$\hat{t_1}\hat{t_2} = \hat{t_2}\hat{t_1} = 1 - \alpha + \beta - \alpha\beta + (2+2\beta)e_1 + (6-6\alpha)e_2 + 12e_3. \tag{2.7}$$

For special values of a, b, c, d we obtain Catalan, Cassini, Halton, Vajda and d'Ocagne type identities.

Corollary 2.1 (Catalan type identity for generalized commutative Pell-Lucas quaternions)**.** *Let $n \geq 0$, $k \geq 0$ be integers such that $n \geq k$. Then*

$$gc\mathcal{Q}_{n+k} \cdot gc\mathcal{Q}_{n-k} - (gc\mathcal{Q}_n)^2 = (-1)^n\left(\left(\frac{t_1}{t_2}\right)^k + \left(\frac{t_2}{t_1}\right)^k - 2\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.2 (Cassini type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 1$ be an integer. Then*

$$gc\mathcal{Q}_{n+1} \cdot gc\mathcal{Q}_{n-1} - (gc\mathcal{Q}_n)^2 = 8(-1)^{n+1}\hat{t_1}\hat{t_2},$$

where $\hat{t_1}\hat{t_2}$ is given by (2.7)*.*

Corollary 2.3 (The first Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$gc\mathcal{Q}_{m+r} \cdot gc\mathcal{Q}_n - gc\mathcal{Q}_r \cdot gc\mathcal{Q}_{m+n} = (-1)^r \left(t_2^{n-r} - t_1^{n-r}\right)\left(t_1^m - t_2^m\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.4 (The second Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $k \geq 0$, $s \geq 0$ be integers such that $n \geq k$, $n \geq s$. Then*

$$gc\mathcal{Q}_{n+k} \cdot gc\mathcal{Q}_{n-k} - gc\mathcal{Q}_{n+s} \cdot gc\mathcal{Q}_{n-s} = (-1)^n \left(\left(\frac{t_1}{t_2}\right)^k + \left(\frac{t_2}{t_1}\right)^k - \left(\frac{t_1}{t_2}\right)^s - \left(\frac{t_2}{t_1}\right)^s\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.5 (Vajda type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$, $n \geq m$. Then*

$$\begin{aligned} &gc\mathcal{Q}_{m+k} \cdot gc\mathcal{Q}_{n-k} - gc\mathcal{Q}_m \cdot gc\mathcal{Q}_n \\ =&(-1)^m \left(t_2^{n-m}\left(\left(\frac{t_1}{t_2}\right)^k - 1\right) + t_1^{n-m}\left(\left(\frac{t_2}{t_1}\right)^k - 1\right)\right)\hat{t_1}\hat{t_2}, \end{aligned}$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.6 (d'Ocagne type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$gc\mathcal{Q}_n \cdot gc\mathcal{Q}_{m+1} - gc\mathcal{Q}_{n+1} \cdot gc\mathcal{Q}_m = 2\sqrt{2}(-1)^m \left(t_1^{n-m} - t_2^{n-m}\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Theorem 2.2 (General bilinear index-reduction formula for generalized commutative Pell and Pell-Lucas quaternions). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$gc\mathcal{P}_a \cdot gc\mathcal{Q}_b - gc\mathcal{P}_c \cdot gc\mathcal{Q}_d = \left(At_1^a t_2^b + Bt_2^a t_1^b - At_1^c t_2^d - Bt_2^c t_1^d\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, A, B and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Proof. Using (2.5) and (2.6) we have

$$\begin{aligned} & gc\mathcal{P}_a \cdot gc\mathcal{Q}_b - gc\mathcal{P}_c \cdot gc\mathcal{Q}_d \\ =& \left(At_1^a\hat{t_1} + Bt_2^a\hat{t_2}\right) \cdot \left(t_1^b\hat{t_1} + t_2^b\hat{t_2}\right) - \left(At_1^c\hat{t_1} + Bt_2^c\hat{t_2}\right) \cdot \left(t_1^d\hat{t_1} + t_2^d\hat{t_2}\right) \\ =& At_1^a\hat{t_1}t_2^b\hat{t_2} + Bt_2^a\hat{t_2}t_1^b\hat{t_1} - At_1^c\hat{t_1}t_2^d\hat{t_2} - Bt_2^c\hat{t_2}t_1^d\hat{t_1} \\ =& \left(At_1^at_2^b + Bt_2^at_1^b - At_1^ct_2^d - Bt_2^ct_1^d\right)\hat{t_1}\hat{t_2}, \end{aligned}$$

which ends the proof. □

For special values of a, b, c, d we can obtain another dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions, for example a dependency similar to $P_kQ_{n+j} - P_jQ_{n+k}$ of Pell and Pell-Lucas numbers from [7].

Corollary 2.7. *Let $n \geq 0$, $j \geq 0$, $k \geq 0$ be integers. Then*

$$gc\mathcal{P}_k \cdot gc\mathcal{Q}_{n+j} - gc\mathcal{P}_j \cdot gc\mathcal{Q}_{n+k} = (At_2^n - Bt_1^n)\left(t_1^kt_2^j - t_1^jt_2^k\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, A, B and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7), *respectively.*

We recall some well-known properties of Pell and Pell-Lucas numbers which can be found in [5, 6]

$$P_{n+1} + P_{n-1} = Q_n, \tag{2.8}$$

$$P_{n+1} - P_{n-1} = 2P_n, \tag{2.9}$$

$$Q_{n+1} + Q_{n-1} = 8P_n, \tag{2.10}$$

$$Q_{n+1} - Q_{n-1} = 2Q_n, \tag{2.11}$$

$$P_n + P_{n-1} = \frac{Q_n}{2}, \tag{2.12}$$

$$Q_n + Q_{n-1} = 4P_n, \tag{2.13}$$

$$\sum_{l=0}^{n} P_l = \frac{Q_{n+1} - 2}{4}, \tag{2.14}$$

$$\sum_{l=0}^{n} Q_l = 2P_{n+1}. \tag{2.15}$$

Using (2.8)–(2.13) it immediately follows

Theorem 2.3. *Let $n \geq 0$. Then*

(i) $gc\mathcal{P}_{n+1} + gc\mathcal{P}_{n-1} = gc\mathcal{Q}_n$;
(ii) $gc\mathcal{P}_{n+1} - gc\mathcal{P}_{n-1} = 2gc\mathcal{P}_n$;
(iii) $gc\mathcal{Q}_{n+1} + gc\mathcal{Q}_{n-1} = 8gc\mathcal{P}_n$;
(iv) $gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_{n-1} = 2gc\mathcal{Q}_n$;
(v) $gc\mathcal{P}_n + gc\mathcal{P}_{n-1} = \frac{gc\mathcal{Q}_n}{2}$;
(vi) $gc\mathcal{Q}_n + gc\mathcal{Q}_{n-1} = 4gc\mathcal{P}_n$.

Now we give formulae for the sum of generalized commutative Pell and Pell-Lucas quaternions.

Theorem 2.4. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} gc\mathcal{P}_l = \frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4}.$$

Proof. Using (2.14), we have

$$\begin{aligned}
\sum_{l=0}^{n} gc\mathcal{P}_l = & gc\mathcal{P}_0 + gc\mathcal{P}_1 + \cdots + gc\mathcal{P}_n \\
=&(P_0 + P_1e_1 + P_2e_2 + P_3e_3) + (P_1 + P_2e_1 + P_3e_2 + P_4e_3) \\
&+ \cdots + (P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3) \\
=&(P_0 + P_1 + \cdots + P_n) \\
&+ (P_1 + P_2 + \cdots + P_{n+1} + P_0 - P_0)e_1 \\
&+ (P_2 + P_3 + \cdots + P_{n+2} + P_0 + P_1 - P_0 - P_1)e_2 \\
&+ (P_3 + P_4 + \cdots + P_{n+3} + P_0 + P_1 + P_2 - P_0 - P_1 - P_2)e_3 \\
=&\frac{Q_{n+1} - 2}{4} + \left(\frac{Q_{n+2} - 2}{4} - 0\right) e_1 \\
&+ \left(\frac{Q_{n+3} - 2}{4} - 1\right) e_2 + \left(\frac{Q_{n+4} - 2}{4} - 3\right) e_3 \\
=&\frac{Q_{n+1} + Q_{n+2}e_1 + Q_{n+3}e_2 + Q_{n+4}e_3}{4} - \frac{2 + 2e_1 + 6e_2 + 14e_3}{4} \\
=&\frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4},
\end{aligned}$$

which ends the proof. □

In the same way, using (2.15), we can prove the following result.

Theorem 2.5. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} gc\mathcal{Q}_l = 2\left(gc\mathcal{P}_{n+1} - gc\mathcal{P}_0\right).$$

Now, we give a matrix representation of the generalized commutative Pell and Pell-Lucas quaternions. In [4], Ercolano gave a matrix representation of the Pell sequence defining the matrix generator $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Then $M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, for integer $n \geq 1$.

Let $R(n) = \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix}$ be a matrix of order 2 with entries being generalized commutative Pell quaternions.

Theorem 2.6. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2}.$$

Proof. If $n = 2$ then by simple calculations the result immediately follows. Assume that the equality holds for all integers $2, 3, \ldots, n$. We shall prove that the equation is true for integer $n + 1$. Using our assumption and formula (2.1) we obtain

$$\begin{aligned}
\begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2} \cdot M &= \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2gc\mathcal{P}_n + gc\mathcal{P}_{n-1} & gc\mathcal{P}_n \\ 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2} & gc\mathcal{P}_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} gc\mathcal{P}_{n+1} & gc\mathcal{P}_n \\ gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \end{bmatrix},
\end{aligned}$$

which ends the proof. □

In the same way we can obtain the matrix generator for the generalized commutative Pell-Lucas quaternions.

Let $S(n) = \begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix}$ be a matrix with entries being the generalized commutative Pell-Lucas quaternions.

Theorem 2.7. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{Q}_2 & gc\mathcal{Q}_1 \\ gc\mathcal{Q}_1 & gc\mathcal{Q}_0 \end{bmatrix} \cdot M^{n-2}.$$

At the end, we give the generating functions for $gc\mathcal{P}_n$ and $gc\mathcal{Q}_n$.

Theorem 2.8. *The generating function for the generalized commutative Pell quaternion $gc\mathcal{P}_n$ is*

$$g(t) = \frac{gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t}{1 - 2t - t^2} = \frac{e_1 + 2e_2 + 5e_3 + (1 + e_2 + 2e_3)t}{1 - 2t - t^2}.$$

Proof. Assuming that the generating function of the generalized commutative Pell quaternion sequence $\{gc\mathcal{P}_n\}$ has the form $g(t) = \sum_{n=0}^{\infty} gc\mathcal{P}_n t^n$, we obtain

$$\begin{aligned}
(1 - 2t - t^2)g(t) =& (1 - 2t - t^2)(gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \cdots) \\
=& gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \cdots \\
& - 2gc\mathcal{P}_0 t - gc\mathcal{P}_1 t^2 - 2gc\mathcal{P}_2 t^3 - \cdots \\
& - gc\mathcal{P}_0 t^2 - gc\mathcal{P}_1 t^3 - gc\mathcal{P}_2 t^4 - \cdots \\
=& gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t,
\end{aligned}$$

since $gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}$ and the coefficients of t^n, for $n \geq 2$, are equal to zero. □

Theorem 2.9. *The generating function for the generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ is*

$$g(t) = \frac{gc\mathcal{Q}_0 + (gc\mathcal{Q}_1 - 2gc\mathcal{Q}_0)t}{1-2t-t^2} = \frac{(2+2e_1+6e_2+14e_3) + (-2+2e_1+2e_2+6e_3)t}{1-2t-t^2}.$$

References

[1] F. T. Aydın, *On bicomplex Pell and Pell-Lucas numbers*, Communications in Advanced Mathematical Sciences **1**(2) (2018), 142–155. https://doi.org/10.33434/cams.439752

[2] G. Bilgici and P. Catarino, *Unrestricted Pell and Pell-Lucas quaternions*, International Journal of Mathematics and System Science **1**(3) (2018), 1–4. http://dx.doi.org/10.24294/ijmss.v1i3.816

[3] C. B. Çimen and A. İpek, *On Pell quaternions and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebr. **26** (2016), 39–51. https://doi.org/10.1007/s00006-015-0571-8

[4] J. Ercolano, *Matrix generators of Pell sequences*, Fibonacci Quart. **17(1)** (1979), 71–77.

[5] A. F. Horadam, *Minmax sequences for Pell numbers*, Applications of Fibonacci Numbers **6** (1996), 231–249.

[6] A. F. Horadam, *Pell identities*, Fibonacci Quart. **9.3** (1971), 245–263.

[7] N. Patel and P. Shrivastava, *Pell and Pell-Lucas identities*, Global Journal of Mathematical Sciences: Theory and Practical **5**(4) (2013), 229–236.

[8] A. Szynal-Liana and I. Włoch, *Generalized commutative quaternions of the Fibonacci type*, Bol. Soc. Mat. Mex. **28**1 (2022). https://doi.org/10.1007/s40590-021-00386-4

[9] A. Szynal-Liana and I. Włoch, *Hypercomplex Numbers of the Fibonacci Type*, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów, 2019.

[10] A. Szynal-Liana and I. Włoch, *On Pell and Pell-Lucas hybrid numbers*, Comment. Math. **58**(1–2) (2018), 11–17. http://dx.doi.org/10.14708/cm.v58i1-2.6364

[11] A. Szynal-Liana and I. Włoch, *The Pell quaternions and the Pell octonions*, Adv. Appl. Clifford Algebr. **26** (2016), 435–440. https://doi.org/10.1007/s00006-015-0570-9

[12] Ü.Tokeşer, Z. Ünal and G. Bilgici, *Split Pell and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebr. **27** (2017), 1881–1893. https://doi.org/10.1007/s00006-016-0747-x

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DISCRETE LOCAL FRACTIONAL HILBERT-TYPE INEQUALITIES

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Abstract. The main objective of this paper is a study of some new discrete local fractional Hilbert-type inequalities. We apply our general results to homogeneous kernels. Also, the obtained results have the best possible constants.

1. Introduction

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^{+\infty} f^2(x)dx < +\infty$ and $0 < \int_0^{+\infty} g^2(x)dx < +\infty$, then we have (see [1]):

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx \int_0^{+\infty} g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant π is the best possible. The inequality (1.1) is well known as Hilbert's integral inequality, which is important in mathematical analysis and its applications.

Over the last ten years, by using the kinds of generalized fractional integral operators, a great deal of fractional integral inequalities have been presented [2–5]. Recently, local fractional calculus has caused widespread attention from many scholars, we give basic definitions and results of the local fractional calculus (see [6–13]). Based on the local fractal identity and the generalized p-convexity, some novel Newton's type variants for the local differentiable functions were obtained in the paper [14]. Sarikaya et al. [15] established the generalized Grüss type inequality and some generalized Čebyšev type inequalities for local fractional integrals on fractal sets. Acorrding to the identity

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involving local fractional integrals, Iftikhar et al. [16] presented some new Newton-type inequalities for functions with the local fractional derivatives. By employing the local fractional integrals, Akkurt et al. [17] investigated the generalized Ostrowski type integral inequalities involving moments of continuous random variables. Sarikaya and Budak [18] gave a generalized Ostrowski inequality and some new inequalities using the generalized convex function for local fractional integrals on fractal sets. Based on two local fractional integral operators with Mittag-Leffler kernel, Sun [19] obtained some Hermite-Hadamard and Hermite-Hadamard-Fejér-type local fractional integral inequalities for generalized preinvex functions on Yang's fractal sets.

For the sake of convenience, we recall Yang's fractal set Ω^α, where the set Ω is called base set of fractional set, and α denotes the dimension of cantor set, $0 < \alpha \leq 1$. The α-type set of integers $\mathbb{Z}^\alpha$ is defined by (see [6–8])

$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha : m \in \mathbb{N}\}.$$

The α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by

$$\mathbb{Q}^\alpha := \{q^\alpha : q \in \mathbb{Q}\} = \left\{\left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N}\right\}.$$

The α-type set of irrational numbers $\mathbb{J}^\alpha$ is defined by

$$\mathbb{J}^\alpha := \{r^\alpha : r \in \mathbb{J}\} = \left\{r^\alpha \neq \left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z}, n \in \mathbb{N}\right\}.$$

The α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by

$$\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha.$$

Some basic operation rules on $\mathbb{R}^\alpha$ are presented as follows: If a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$, then

(a1) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$, $a^\alpha b^\alpha \in \mathbb{R}^\alpha$;
(a2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
(a3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
(a4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(a5) $a^\alpha(b^\alpha c^\alpha) = (a^\alpha b^\alpha)c^\alpha$;
(a6) $a^\alpha(b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(a7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$;
(a8) for each $a^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-a^\alpha)$ may be written as $-a^\alpha$; for each $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/b)^\alpha$ may be written as $1^\alpha/b^\alpha$ but not as $1/b^\alpha$;
(a9) $a^\alpha < b^\alpha$ if and only if $a < b$;
(a10) $a^\alpha = b^\alpha$ if and only if $a = b$.

Further, we define the local fractional derivative and integral.

Definition 1.1. A non-differentiable function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$, there exists for $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha,$$

holds for $0 < |x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_\alpha(a, b)$.

Definition 1.2. Let $f(x) \in C_\alpha[a, b]$. Local fractional derivative of the function $f(x)$ at $x = x_0$ is given by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha}.$$

Definition 1.3. Let $f(x) \in C_\alpha[a, b]$ and let $P = \{t_0, t_1, \dots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Further, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \dots, N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$. Then the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^\alpha f(x)$) is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.$$

The above definition implies that ${}_aI_b^{(\alpha)} f(x) = 0$ if $a = b$, and ${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{(\alpha)} f(x)$, then we denote by $f(x) \in I_x^{(\alpha)}[a, b]$.

At the end of this summary, we give some useful formulas:

(b1) $\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$, $k > 0$;

(b2) $\frac{d^\alpha E_\alpha((cx)^\alpha)}{dx^\alpha} = c^\alpha E_\alpha((cx)^\alpha)$, where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function given by $E_\alpha(x^\alpha) = \sum_{k=0}^{+\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$;

(b3) If $y(x) = (f \circ g)(x)$, then $\frac{d^\alpha y(x)}{dx^\alpha} = f^{(\alpha)}(g(x))(g'(x))^\alpha$;

(b4) $\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha)$;

(b5) $\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k > 0$;

(b6) $B_\alpha(a, b) = \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} \frac{x^{\alpha(b-1)}}{(1^\alpha + x^\alpha)^{a+b}} (dx)^\alpha$, where $B_\alpha(a, b)$ denotes local fractional Beta function.

In this paper, by using the way of weight functions and the technique of local fractional calculus, a new Hilbert-type discrete inequality with homogeneous kernel and a best constant is built. As applications, the equivalent form and some particular cases are obtained.

2. Main Results

The starting point in the researching Hilbert-type inequalities is the well-known Hölder's inequality. A fractal version of Hölder's inequality is presented in the following lemma.

Lemma 2.1 ([8]). *Let $1/p + 1/q = 1$, $p > 1$, and let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be non-negative real sequences. Then*

$$\sum_{i=1}^{n} a_i^{\alpha} b_i^{\alpha} \le \left(\sum_{i=1}^{n} a_i^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^{\alpha q}\right)^{\frac{1}{q}}.$$

If $\sum_{i=1}^{+\infty} a_i^{\alpha p} < +\infty$ and $\sum_{i=1}^{+\infty} b_i^{\alpha q} < +\infty$, then the following inequalitiy holds

$$\sum_{i=1}^{+\infty} a_i^{\alpha} b_i^{\alpha} \le \left(\sum_{i=1}^{+\infty} a_i^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{+\infty} b_i^{\alpha q}\right)^{\frac{1}{q}}.$$

In particular, a two-variable version of the fractal Hölder's inequality is given in the next lemma.

Lemma 2.2. *Let $1/p + 1/q = 1$, $p > 1$, and let h, F, $G \in C_{\alpha}(\mathbb{R}_+^2)$ be non-negative functions. If*

$$0 < \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n) < +\infty, \quad 0 < \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n) < +\infty,$$

then the following inequality holds

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F(m,n)G(m,n) \le \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)^{\frac{1}{p}} \times \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right)^{\frac{1}{p}}. \tag{2.1}$$

Proof. The inequality (2.1) is trivially true in the case when h or F or G is identically zero. Suppose that

$$\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right) \neq 0.$$

Applying the following α-Young's inequality

$$x_i^{\frac{\alpha}{p}} y_i^{\frac{\alpha}{q}} \le \frac{x_i^{\alpha}}{p^{\alpha}} + \frac{y_i^{\alpha}}{q^{\alpha}}, \quad x_i, y_i \ge 0, \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

to

$$x^{\alpha} := \frac{h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)}$$

and

$$y^{\alpha} := \frac{h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)},$$

we can obtain

$$\frac{[h(m,n)]^{\frac{1}{p}} F(m,n)\, [h(m,n)]^{\frac{1}{q}} G(m,n)}{\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)^{\frac{1}{p}} \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right)^{\frac{1}{p}}}$$
$$\leq \frac{1}{p^\alpha}\cdot\frac{h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)} + \frac{1}{q^\alpha}\cdot\frac{h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)}.$$

Summarizing both side of the obtained inequality, we have

$$\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F(m,n)G(m,n)}{\left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)\right)^{\frac{1}{p}} \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)\right)^{\frac{1}{p}}}$$
$$\leq \frac{1}{p^\alpha}\cdot\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)F^p(m,n)} + \frac{1}{q^\alpha}\cdot\frac{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)}{\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} h(m,n)G^q(m,n)}$$
$$= \frac{1}{p^\alpha} + \frac{1}{q^\alpha} = 1^\alpha.$$

This directly gives the desired inequality (2.1). The proof is completed. □

Besides, we introduce the following notation and definition (see [21]).

Definition 2.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. If the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) \tag{2.2}$$

holds, for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, then f is said to be a generalized convex function on I.

Mo et al. [21] proved the following generalized Hermite-Hadamard inequality for local fractional integral. Let $f \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^{(\alpha)} f \leq \frac{f(a)+f(b)}{2^\alpha}. \tag{2.3}$$

Applying above inequality we can prove next lemma.

Lemma 2.3. *If $f \in I_x^{(\alpha)}(\mathbb{R}_+)$, $f^{(\alpha)}(t) < 0$, $f^{(2\alpha)}(t) > 0$, $t \in (1/2, +\infty)$, then we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} f(t)(dt)^\alpha \leq \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{+\infty} f(n) \leq \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} f(t)(dt)^\alpha. \tag{2.4}$$

Proof. Setting $a = n - \frac{1}{2}$, $b = n + \frac{1}{2}$, the generalized Hermite-Hadamard inequality (2.3) yields

$$\frac{f(n)}{\Gamma(1+\alpha)} \leq \frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha. \tag{2.5}$$

Similarly, for $a = n$, $b = n+1$, from (2.3) we get

$$\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha \leq \frac{f(n)}{\Gamma(1+\alpha)}. \tag{2.6}$$

Now, from (2.5) and (2.6) we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha \le \frac{f(n)}{\Gamma(1+\alpha)} \le \frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha.$$

Furthermore, we can obtain

$$\begin{aligned}&\sum_{n=1}^{+\infty}\frac{1}{\Gamma(1+\alpha)}\int_n^{n+1} f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} f(t)(dt)^\alpha\\ \le&\frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{+\infty} f(t) \le \sum_{n=1}^{+\infty}\frac{1}{\Gamma(1+\alpha)}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} f(t)(dt)^\alpha,\end{aligned}$$

which implies (2.4) holds. This completes the proof. □

Suppose that $r > 0$ and $K(x,y)$ is strictly decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Using chain rule for local fractional derivative (the formula (b3) from Introduction) yields

$$\frac{\partial^\alpha}{\partial x^\alpha}K(x,n)x^{-\alpha r} = \frac{1}{x^{\alpha r}}\cdot\frac{\partial^\alpha}{\partial x^\alpha}\left[K(x,n)\right] - \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r-1)\alpha)}\cdot\frac{K(x,n)}{x^{\alpha(r+1)}} < 0$$

and

$$\begin{aligned}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}K(x,n)x^{-\alpha r} =&\frac{1}{x^{\alpha r}}\cdot\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[K(x,n)\right] - \frac{\Gamma(1+r\alpha)}{\Gamma(1+(r-1)\alpha)}\cdot\frac{K(x,n)}{x^{\alpha(r+1)}}\\ &\times\frac{\partial^\alpha}{\partial x^\alpha}\left[K(x,n)\right] > 0,\end{aligned}$$

for $x > 0$ and $n \in \mathbb{N}$. In this way (see also [22], Corollary 1) we obtain the following result.

Lemma 2.4. *Let $r > 0$, $m,n \in \mathbb{N}$, and $K(x,y)$ be strictly decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Then*

$$K(m,y)y^{-\alpha r} \quad \textit{and} \quad K(x,n)x^{-\alpha r}$$

are strictly decreasing and generalized convex function on $\mathbb{R}_+$.

In what follows we suppose that $K \in C_\alpha(\mathbb{R}_+^2)$ is a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. Further, we define

$$k(\beta) = \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} K(1,t)t^{-\alpha\beta}(dt)^\alpha, \tag{2.7}$$

under assumption $k(\beta) < +\infty$.

To prove our main results we need some technical lemma.

Lemma 2.5. *Let $1/p + 1/q = 1$, $p > 1$, and let $K \in C_\alpha(\mathbb{R}_+^2)$ be a non-negative homogeneous function of degree $-\alpha s$, $s > 0$. If K is strictly decreasing and generalized* *convex function in both variables on* $\mathbb{R}_+$, *then*

$$\omega_m(pA_2) := \sum_{n=1}^{+\infty} K(m,n)\left(\frac{m}{n}\right)^{\alpha p A_2} \leq \Gamma(1+\alpha) m^{\alpha(1-s)} k(pA_2) \tag{2.8}$$

and

$$\overline{\omega}_n(qA_1) := \sum_{n=1}^{+\infty} K(m,n)\left(\frac{n}{m}\right)^{\alpha q A_1} \leq \Gamma(1+\alpha) n^{\alpha(1-s)} k(2-s-qA_1), \tag{2.9}$$

where $A_1 \in (\max\{(1-s)/q, 0\}, 1/q)$ *and* $A_2 \in (\max\{(1-s)/p, 0\}, 1/p)$.

Proof. Applying Lemma 2.2 and Lemma 2.4 we get

$$\omega_m(pA_2) \leq \Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} K(m,x)\left(\frac{x}{m}\right)^{-\alpha p A_2} (dx)^\alpha.$$

Further, using homogeneity of function K and substituting $u = x/m$, we have

$$\begin{aligned}\omega_m(pA_2) &\leq \Gamma(1+\alpha) m^{\alpha(1-s)} \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} K(1,u) u^{-\alpha p A_2} (du)^\alpha \\ &= \Gamma(1+\alpha) m^{\alpha(1-s)} k(pA_2),\end{aligned}$$

which implies (2.8), where we used the definition of $k(\beta)$ in equation (2.7). Similarly, we obtain (2.9). □

The main results are stated below.

Theorem A. *Let* $1/p + 1/q = 1$, $p > 1$, *and let* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be non-negative real sequences. If* $K(x,y)$, A_1, A_2 *are defined as in Lemma* 2.5, *then the following inequalities hold and are equivalent*

$$\begin{aligned} I := \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n) a_m^\alpha b_n^\alpha \leq & L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}} \\ & \times \left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} J := & \left(\sum_{n=1}^{+\infty} n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)} \left(\sum_{m=1}^{+\infty} K(m,n) a_m^\alpha\right)^p\right)^{\frac{1}{p}} \\ \leq & L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}}, \end{aligned} \tag{2.11}$$

where $L = \Gamma(1+\alpha) k(pA_2)^{1/p} k(2-s-qA_1)^{1/q}$.

Proof. By using the local fractional Hölder's inequality (2.1), we have

$$\begin{aligned}
I =& \sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\; a_m^{\alpha}\frac{m^{\alpha A_1}}{n^{\alpha A_2}}\; b_n^{\alpha}\frac{n^{\alpha A_2}}{m^{\alpha A_1}} \\
\le& \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\frac{m^{\alpha p A_1}}{n^{\alpha p A_2}} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\frac{n^{\alpha q A_2}}{m^{\alpha q A_1}} b_n^{\alpha q}\right)^{\frac{1}{q}} \\
=& \left(\sum_{m=1}^{+\infty}\left(\sum_{n=1}^{+\infty} K(m,n)\left(\frac{m}{n}\right)^{\alpha p A_2}\right) m^{\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}} \\
=& \left(\sum_{n=1}^{+\infty}\left(\sum_{m=1}^{+\infty} K(m,n)\left(\frac{n}{m}\right)^{\alpha q A_1}\right) n^{\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, from Lemma 2.5, we get the inequality (2.10).

We suppose that the inequality (2.10) is valid. To obtain (2.11), we set

$$b_n^{\alpha} := n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)^{p-1}.$$

It follows that

$$J^p = \sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}.$$

By using the inequality (2.10), we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)^{p} = J^p = I \\
\le& L\left(\sum_{m=1}^{+\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}},
\end{aligned}$$

which implies the inequality (2.11) holds. By using the two dimensional Hölder's inequality in Lemma 2.1, we have

$$\begin{aligned}
I =& \sum_{n=1}^{+\infty}\left(n^{\alpha(s-1)\frac{1}{q}+\alpha(A_1-A_2)}\left(\sum_{m=1}^{+\infty} K(m,n) a_m^{\alpha}\right)\right) n^{\alpha(1-s)\frac{1}{q}+\alpha(A_2-A_1)} b_n^{\alpha} \\
\le& J\left(\sum_{n=1}^{+\infty} n^{\alpha(1-s)+\alpha q(A_2-A_1)} b_n^{\alpha q}\right)^{\frac{1}{q}}.
\end{aligned}$$

From (2.11) and the above inequality, we have (2.10). Therefore, the inequalities (2.11) and (2.10) are equivalent. □

Now, we consider some special choises of the parameters A_1 and A_2. More precisely, let the parameters A_1 and A_2 satisfy condition

$$pA_2 + qA_1 = 2 - s. \tag{2.12}$$

Then, the constant L from Theorem A becomes

$$L^* = \Gamma(1+\alpha)k(pA_2). \tag{2.13}$$

Further, the inequalities (2.10) and (2.11) take form

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)a_m^\alpha b_n^\alpha \le L^* \left(\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} n^{-\alpha+\alpha pqA_2} b_n^{\alpha q}\right)^{\frac{1}{q}} \tag{2.14}$$

and

$$\left(\sum_{n=1}^{+\infty} n^{\alpha(p-1)(1-pqA_2)} \left(\sum_{m=1}^{+\infty} K(m,n)a_m^\alpha\right)^p\right)^{\frac{1}{p}} \le L^* \left(\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1} a_m^{\alpha p}\right)^{\frac{1}{p}}. \tag{2.15}$$

In the following theorem we show that, if the parameters A_1 and A_2 satisfy condition (2.12), then one obtains the best possible constant.

Theorem B. *Let s, A_1, A_2 and $K(x,y)$ be defined as in Theorem* A. *If the parameters A_1 and A_2 satisfy condition $pA_2+qA_1 = 2-s$, then the constant $L^* = \Gamma(1+\alpha)k(pA_2)$ in inequalities* (2.14) *and* (2.15) *is the best possible.*

Proof. For this purpose, set $\widetilde{a}_m^\alpha = m^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}$ and $\widetilde{b}_n^\alpha = n^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}$ where $0<\varepsilon<\frac{1-pA_2}{q}$. Now, let us suppose that the inequality (2.14) is valid. By using Lemma 2.2, we have

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)\varepsilon^\alpha} &= \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha-\alpha\varepsilon}\\ &= \frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}\widetilde{a}_m^{\alpha p}\\ &\le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty} u^{-\alpha-\alpha\varepsilon}(du)^\alpha.\end{aligned}$$

Hence, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} m^{-\alpha+\alpha pqA_1}\widetilde{a}_m^{\alpha p} \le \frac{1}{\varepsilon^\alpha\Gamma(1+\alpha)} + O(1), \tag{2.16}$$

and similarly

$$\frac{1}{\Gamma(1+\alpha)}\sum_{m=1}^{+\infty} n^{-\alpha+\alpha pqA_2}\widetilde{b}_n^{\alpha q} \le \frac{1}{\varepsilon^\alpha\Gamma(1+\alpha)} + O(1). \tag{2.17}$$

Suppose that the constant L^*, defined by (2.13), is not the best possible in inequality (2.14). That implies that there exits constant M, smaller than L^*, such that the inequality (2.14) is still valid if we replace L^* with M. Hence, if we insert relations (2.16) and (2.17) in inequality (2.14), with the constant M instead of L^*, we have

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty} K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha \le \frac{1}{\varepsilon^\alpha}(M+o(1)). \tag{2.18}$$

Further, we estimate the left-hand side of inequality (2.14). Namely, if we insert the above defined sequences $(\widetilde{a}_m^\alpha)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n^\alpha)_{n\in\mathbb{N}}$ in the left-hand side of (2.14), we easily obtain the inequality

(2.19)
$$\begin{aligned} J_\varepsilon :=&\frac{1}{\Gamma^2(1+\alpha)}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha\\ \geq&\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}x^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}K(x,y)y^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(dy)^\alpha\right)(dx)^\alpha, \end{aligned}$$

where we used Lemma 2.2. By using the substitution $u=y/x$, we obtain

$$J_\varepsilon\geq\frac{1}{\Gamma(1+\alpha)}\int_1^{+\infty}x^{-\alpha-\alpha\varepsilon}\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{x}}^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha\right)(dx)^\alpha. \tag{2.20}$$

Further, since the kernel K is strictly decreasing in both variables, it follows that $K(1,0)>K(1,t)$, for $t>0$, so we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{x}}^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha\\ >&\frac{1}{\Gamma(1+\alpha)}\int_0^{+\infty}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha-\frac{K(1,0)}{\Gamma(1+\alpha)}\int_0^{\frac{1}{x}}K(1,u)u^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}(du)^\alpha\\ =&k\left(pA_2+\frac{\varepsilon}{q}\right)-\frac{K(1,0)}{\Gamma(1+\alpha)(1-pA_2-\frac{\varepsilon}{q})^\alpha}x^{\alpha pA_2+\frac{\alpha\varepsilon}{q}-\alpha} \end{aligned}$$

and, consequently,

$$J_\varepsilon\geq\frac{1}{\varepsilon^\alpha}\cdot\frac{k\left(pA_2+\frac{\varepsilon}{q}\right)}{\Gamma(1+\alpha)}+\frac{K(1,0)}{\Gamma^2(1+\alpha)}\cdot\frac{1}{(1-pA_2-\frac{\varepsilon}{q})^\alpha(pA_2-\frac{\varepsilon}{p}-1)^\alpha}. \tag{2.21}$$

Now, the relations (2.19), (2.20) and (2.21) yield the estimate for the left-hand side of inequality (2.14):

$$\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}K(m,n)\widetilde{a}_m^\alpha\widetilde{b}_n^\alpha>\frac{1}{\varepsilon^\alpha}(L^*+o(1)). \tag{2.22}$$

Finally, by comparing (2.18) and (2.22), and by letting $\varepsilon\longrightarrow 0^+$, we get that $L^*\leq M$, which contradicts with the assumption that the constant M is smaller than L^*.

The equivalence of inequalities (2.14) and (2.15) means that the constant L^* is the best possible in the inequality (2.15). The proof is now completed. □

As corollaries of Theorem B we have the following results. We processed with the kernel $K_1(x,y)=(x+y)^{-\alpha s}$, $s>0$. By using local fractional calculus, we have

$$\frac{\partial^\alpha}{\partial x^\alpha}\cdot\frac{1}{(m+x)^{\alpha s}}=-\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\cdot\frac{1}{(m+x)^{\alpha(s+1)}}<0,\quad x>0,$$

and similarly

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \cdot \frac{1}{(m+x)^{\alpha s}} = \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s-1)\alpha)} \cdot \frac{1}{(m+x)^{\alpha(s+2)}} > 0, \quad x > 0.$$

Now, by applying Lemma 2.4 we obtain

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} K_1(x,y) x^{-\alpha r} < 0 \quad \text{and} \quad \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} K_1(x,y) x^{-\alpha r} > 0$$

and

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}} K_1(x,y) y^{-\alpha r} < 0 \quad \text{and} \quad \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} K_1(x,y) y^{-\alpha r} > 0,$$

for $r > 0$.

In what follows we suppose that

$$A_1 = \frac{2-s}{2q}, \quad A_2 = \frac{2-s}{2p}. \tag{2.23}$$

Then, based on equation (2.23), the constant L^* from Theorem B becomes

$$\begin{aligned} L^* =& \Gamma(1+\alpha) k(pA_2) = \Gamma(1+\alpha) k\left(1 - \frac{s}{2}\right) \\ =& \Gamma(1+\alpha) \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} \frac{u^{-\alpha - \frac{\alpha s}{2}}}{(1+u)^{\alpha s}} (du)^{\alpha} = \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right). \end{aligned}$$

Now, from Theorem B, we get the following result.

Corollary 2.1. *Let $1/p + 1/q = 1$, $p > 1$, $0 < s < 2$, and $(a_m)_{m \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be non-negative real sequences. Then the following inequalities hold and are equivalent*

$$\begin{aligned} \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{a_m^{\alpha} b_n^{\alpha}}{(m+n)^{\alpha s}} \leq & \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right) \\ & \times \left(\sum_{m=1}^{+\infty} m^{\alpha p(1-\frac{s}{2}) - \alpha} a_m^{\alpha p}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} n^{\alpha p(1-\frac{s}{2}) - \alpha} b_n^{\alpha q}\right)^{\frac{1}{q}} \end{aligned}$$

and

$$\left(\sum_{n=1}^{+\infty} n^{\frac{\alpha p s}{2} - \alpha} \left(\sum_{m=1}^{+\infty} \frac{a_m^{\alpha}}{(m+n)^{\alpha s}}\right)^p\right)^{\frac{1}{p}} \leq \Gamma(1+\alpha) B_{\alpha}\left(\frac{s}{2}, \frac{s}{2}\right) \left(\sum_{m=1}^{+\infty} m^{\alpha p(1-\frac{s}{2}) - \alpha} a_m^{\alpha p}\right)^{\frac{1}{p}},$$

where the constant $\Gamma(1+\alpha) B_{\alpha}(s/2, s/2)$ is the best possible.

3. Conclusion

In this paper, we have firstly obtained a fractal Hölder's inequality and some related inequalities. According to the basic results, some new discrete local fractional Hilbert-type inequalities have been investigated. At the same time, some new fractional Hilbert-type inequalities with homogeneous kernels have been given.

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References

[1] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Second Edition, Cambridge University Press, Cambridge, 1967.

[2] F. Jarad, T. Abdeljawad and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives*, European Physical Journal Special Topics **226** (2017), 3457–3471. https://doi.org/10.1140/epjst/e2018-00021-7

[3] J. V. C. Sousa, D. S. Oliveira and E. C. Oliveira, *Grüss-type inequalities by means of generalized fractional integrals*, Bull. Braz. Math. Soc. (N.S.) **50** (2019), 1029–1047. https://doi.org/10.1007/s00574-019-00138-z

[4] W. Yang, *Certain new Chebyshev and Grüss-type inequalities for unified fractional integral operators via an extended generalized Mittag-Leffler function*, Fractal and Fractional **6**(4) (2022), Paper ID 182, 27 pages. https://doi.org/10.3390/fractalfract6040182

[5] W. Yang, *Certain new weighted Young and Pólya-Szegö-type inequalities for unified fractional integral operators via an extended generalized Mittag-Leffler function with applications*, Fractals **30**(6) (2022), Paper ID 2250106, 37 pages. https://doi.org/10.1142/S0218348X22501067

[6] X. J. Yang, *Local Fractional Functional Analysis and its Applications*, Asian Academic Publisher Limited, Hong Kong, 2011.

[7] X. J. Yang, *Advanced Local Fractional Calculus and its Applications*, World Science Publisher, New York, 2012.

[8] X. J. Yang, D. Baleanu and H. M. Srivastava, *Advanced analysis of local fractional calculus applied to the rice theory in fractal fracture mechanics*, in: J. Singh, H. Dutta, D. Kumar, D. Baleanu and J. Hristov (Eds.), *Methods of Mathematical Modelling and Computation for Complex Systems*, Studies in Systems, Decision and Control, Vol. 373. Springer, Cham, 2022.

[9] M. Krnić and P. Vuković, *Multidimensional Hilbert-type inequalities obtained via local fractional calculus*, Acta Appl. Math. **169**(1) (2020), 667–680. https://doi.org/10.1007/s10440-020-00317-x

[10] G. Jumarie, *Fractional Euler's integral of first and second kinds. Application to fractional Hermite's polynomials and to probability density of fractional order*, J. Appl. Math. Inform. **28** (2010), 257–273.

[11] Q. Liu, *A Hilbert-type fractional integral inequality with the kernel of Mittag-Leffler function and its applications*, Math. Inequal. Appl. **21**(3) (2018), 729–737. https://dx.doi.org/10.7153/mia-2018-21-52

[12] Q. Liu and D. Chen, *A Hilbert-type integral inequality on the fractal spaces*, Integral Transforms Spec. Funct. **28**(10) (2017), 772–780. https://doi.org/10.1080/10652469.2017.1359588

[13] T. Batbold, M. Krnić and P. Vuković, *A unified approach to fractal Hilbert-type inequalities*, J. Inequal. Appl. **2019** (2019), Paper ID 117, 13 pages. https://doi.org/10.1186/s13660-019-2076-9

[14] Y. Li, S. Rashid, Z. Hammouch, D. Baleanu and Y. Chu, *New Newton's type estimates pertaining to local fractional integral via generalized p-convexity with applications*, Fractals **29**(5) (2021), Paper ID 2140018, 20 pages. https://doi.org/10.1142/S0218348X21400181

[15] M. Z. Sarikaya, T. Tunc and H. Budak, *On generalized some integral inequalities for local fractional integrals*, Appl. Math. Comput. **276** (2016), 316–323. https://doi.org/10.1016/j.amc.2015.11.096

[16] S. Iftikhar, P. Kumam and S. Erden, *Newton's-type integral inequalities via local fractional integrals*, Fractals **28**(3) (2020), Paper ID 2050037, 13 pages. https://doi.org/10.1142/S0218348X20500371

[17] A. Akkurt, M. Z. Sarikaya, H. Budak and H. Yildirim, *Generalized Ostrowski type integral inequalities involving generalized moments via local fractional integrals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **111**(3) (2017), 797–807. https://doi.org/10.1007/s13398-016-0336-9

[18] M. Z. Sarikaya and H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc. **145**(4) (2017), 1527–1538. http://dx.doi.org/10.1090/proc/13488

[19] W. Sun, *Hermite-Hadamard type local fractional integral inequalities with Mittag-Leffler kernel for generalized preinvex functions*, Fractals **29**(8) (2021), Paper ID 2150253, 13 pages. https://doi.org/10.1142/S0218348X21502534

[20] J. Choi, E. Set and M. Tomar, *Certain generalized Ostrowski type inequalities for local fractional integrals*, Commun. Korean Math. Soc. **32**(3) (2017), 601–617. https://doi.org/10.4134/CKMS.c160145

[21] H. Mo, X. Sui and D. Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstr. Appl. Anal. **2014** (2014), Paper ID 636751, 7 pages. https://doi.org/10.1155/2014/254737

[22] H. Budak, M. Z. Sarikaya and H. Yildirim, *New inequalities for local fractional integrals*, Iran. J. Sci. Technol. Trans. A Sci. **41**(4) (2017), 1039–1046. https://doi.org/10.1007/s40995-017-0315-9

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THE PERFECT CODES OF NON-COPRIME AND COPRIME GRAPHS

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Abstract. In this paper, we focus on the perfect and total perfect codes of the non-coprime and coprime graphs associated to the dihedral groups and finite Abelian groups. We used the advantage of independent sets and tried to present the independent polynomial for them.

1. Introduction

The birth of coding theory was established by Claude Shannon in 1948 (see [10]). In his paper, he showed for a noisy communication channel, there is a number, called the capacity of the channel. If proper encoding and decoding techniques are used, the reliable communication can be achieved at any rate below the channel capacity. Coding theory is concerned with successfully transmitting data through a noisy channel and correcting errors in corrupted messages [5]. Let Γ be a graph with vertex and edge set $V(\Gamma)$ and $E(\Gamma)$, repectively. Suppose $r \geq 1$ is an integer. The ball with center $v \in V(\Gamma)$ and radius r is the set of vertices of Γ with distance at most r to v in Γ. A code in Γ is simply a subset of $V(\Gamma)$. A code $C \subseteq V(\Gamma)$ is called a perfect r-code in Γ if the balls with centers in C and radius r form a partition of $V(\Gamma)$, that is, every vertex of Γ is at distance no more than r to exactly one vertex of C [4]. If $r = 1$, then we call perfect r-code, perfect code, for abbreviation. Consequently, in order to find a perfect code, we should search among all independent sets and check if every vertex of $V(\Gamma) \setminus C$ is adjacent to exactly one vertex of C. A code C is said to be a total perfect code in Γ if every vertex of Γ has exactly one neighbor in C [3]. The existence of perfect codes is a classical problem which was started in a vector

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space. One can replace the vector space by a graph, whose vertices are vectors and whose edges join vectors which differ in precisely one coordinate. It is clear that we may pose the perfect code question for any graph [1]. There are several papers in this area we refer the readers to [6, 7]. The non-coprime graph associated to the group G was introduced in [9]. Suppose G is a group and e its identity element. The non-coprime graph of G is a graph with vertex set $G \setminus \{e\}$ and if $\gcd(|x|, |y|) \neq 1$, then two distinct vertices x, y are adjacent. Denote this graph by Π_G. The authors verified some numerical invariants like diameter, girth, dominating number, independence and chromatic numbers of non-coprime graph. Moreover, they characterized its planarity. X. Ma et al. defined the coprime graph of a finite group [8]. The coprime graph Γ_G which is associated to the finite group G is a graph with G as the vertex set and join two distinct vertices x and y if $\gcd(|x|, |y|) = 1$. They gave some properties of coprime graph on diameter, planarity, partition, clique number, etc. Moreover, some groups whose coprime graphs are complete, planar, a star, or regular were characterized. There are other papers about the properties of coprime graph, see for instance [2]. In this research, we investigate the existence of the perfect and total perfect code for non-coprime and coprime graph of certain groups beside to present the independent polynomials for them.

2. The Perfect and Total Perfect Codes of Non-coprime Graph of Certain Groups

In this section we verify the perfect and total perfect codes of non-coprime graph of dihedral groups and finite Abelian groups. Let $D_{2n} = \langle a, b : a^n = b^2 = 1,\ a^b = a^{-1} \rangle$ be the dihedral group of order $2n$ and $n \geq 4$.

(i) Suppose $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers and α_i's are positive integers. It is clear that the independence number for this graph is $k+1$. For instance, a set $\mathfrak{I}$ contain an element of order two and k elements of order $p_i^{\beta_i}$, form an independent set of the largest size, $1 \leq i \leq k$, $1 \leq \alpha_i \leq \beta_i$. The number of singleton independent sets is $2\prod_{i=1}^{k} p_i^{\alpha_i} - 1$. The number of two-element independent sets is,

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\sum_{i=1}^{k}\sum_{1\leq\beta_i\leq\alpha_i}\varphi(p_i^{\beta_i})\right)+\sum_{i=1}^{k}\left(\sum_{1\leq\beta_i\leq\alpha_i}\varphi(p_i^{\beta_i})\left(\sum_{j=i+1}^{k}\sum_{1\leq\beta_j\leq\alpha_j}\varphi(p_j^{\beta_j})\right)\right), \tag{2.1}$$

where φ is the Euler function. Consequently, the number of independent sets with ℓ elements is equal to the sum of all possible ℓ-multiplications of elements in the set

$$\mathfrak{A} = \left\{\sum_{\beta_i=1}^{\alpha_i}\varphi(p_i^{\beta_i}) : 1 \leq i \leq k\right\} \cup \left\{\prod_{i=1}^{k} p_i^{\alpha_i}\right\},$$

which are arranged similar to the equation (2.1), where $1 \leq \ell \leq k+1$. Note that ℓ-multiplications of elements in $\mathfrak{A}$ means choosing ℓ elements of the set $\mathfrak{A}$ randomly and compute their multiplications. Inside the set $\mathfrak{A}$ are the numbers of elements of order 2 and $p_i^{\beta_i}$ in the group D_{2n}, $1 \leq i \leq k$, $1 \leq \beta_i \leq \alpha_i$. Obviously, the number of independent sets with $k+1$ elements is

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\prod_{i=1}^{k} \sum_{1\leq\beta_i\leq\alpha_i} \varphi(p_i^{\beta_i})\right).$$

(ii) Let $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i is an odd prime. Then similar to the first case the number of independent sets of different sizes can be computed. The number of singleton independent sets is $2^{s+1}\prod_{i=1}^{k} p_i^{\alpha_i} - 1$. The number of independent sets with ℓ elements is equal to the sum of all possible suitable ℓ-multiplications of elements in the set

$$\mathfrak{A} = \left\{\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) : 1 \leq i \leq k\right\} \cup \left\{2^s \prod_{i=1}^{k} p_i^{\alpha_i} + 1 + \sum_{\beta=2}^{s} \varphi(2^{\beta})\right\},$$

where $1 \leq \ell \leq k+1$ (similar to (2.1)). Note that inside the set $\mathfrak{A}$ are the numbers of elements of order 2^{β} and $p_i^{\beta_i}$ in the group D_{2n}, $1 \leq i \leq k$, $1 \leq \beta_i \leq \alpha_i$ and $1 \leq \beta \leq s$.

(iii) Assume $n = 2^s$. Then the independence number is one and the number of independent sets of size one is $2^{s+1} - 1$.

It is not hard to write the independent polynomial for the non-coprime graph of D_{2n} by use of the above results.

Theorem 2.1. *For the perfect codes of the non-coprime graph of D_{2n}, we have the following cases.*

(i) If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then the perfect codes of non-coprime graph of D_{2n} are sets of two-elements, they contained one element of order 2 and an element of order $\prod_{i=1}^{k} p_i^{\beta_i}$, where p_i are odd prime numbers, α_i, β_i are positive integers and $1 \leq \beta_i \leq \alpha_i$. Moreover, the number of these codes is $\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\sum_{t=1}^{n} \psi(a^t)\right)$, where

$$\psi(a^t) = \begin{cases} 1, & \prod_{i=1}^{k} p_i \mid |a^t|, \quad 1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}, \\ 0, & \textit{otherwise}. \end{cases}$$

More practical formula for the number of perfect codes is

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\begin{matrix} \ulcorner M \urcorner \\ \llcorner\, k \,\lrcorner \end{matrix}\right),$$

where M' is the set of all prime numbers which divides $\prod_{i=1}^{k} p_i^{\alpha_i}$, M is the set of all prime power numbers which are chosen from M' and their powers are more or equal than one and less or equal than α_i and the notation

$$\begin{matrix} \ulcorner M \urcorner \\ \llcorner\, j \,\lrcorner \end{matrix},$$

means the sum of Euler functions of multiply of j prime power numbers which are chosen randomly from M and these multiplications are multiplying of distinct prime numbers.

(ii) *If $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, then the perfect codes of non-coprime graph of D_{2n} are sets of singletons. The singletons contain elements of order $2^{\beta} \prod_{i=1}^{k} p_i^{\beta_i}$, where $1 \leq \beta \leq s$ and $1 \leq \beta_i \leq \alpha_i$. Further, the number of singleton perfect codes is*

$$\ulcorner M \urcorner$$
$$\llcorner k+1 \lrcorner,$$

where the notation is the same as part (i) *with the difference that M contains the possible powers of* 2.

(iii) *If $n = 2^s$, then singleton subsets of whole vertices are perfect codes and the number of them is $2^{s+1} - 1$.*

Proof. (i) By definition, for a perfect code of a graph, we must search among its independent subsets of vertices. Secondly, every vertex out of the perfect code is adjacent to exactly one vertex in the code. By these tools, and the way that two vertices are adjacent in the non-coprime graph it is clear that a perfect codes for the $\Pi_{D_{2n}}$ are sets of two-elements, they contained one element of order 2 and an element of order $\prod_{i=1}^{k} p_i^{\beta_i}$, $1 \leq \beta_i \leq \alpha_i$. Furthermore, the number of elements of order 2 in dihedral group of order $2n$ is $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. It is enough to count the number of elements of order $\prod_{i=1}^{k} p_i^{\beta_i}$, $1 \leq \beta_i \leq \alpha_i$. It is obvious that all the elements $a^t b$ are of order 2, $1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}$. Thus, for a fixed β_i, $|a^t| = \prod_{i=1}^{k} p_i^{\beta_i}$ whenever $\gcd(t, \prod_{i=1}^{k} p_i^{\alpha_i}) = \prod_{i=1}^{k} p_i^{\alpha'_i}$, where $0 \leq \alpha'_i \leq \alpha_i - 1$, $1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}$ and $\beta_i = \alpha_i - \alpha'_i$. Count the number of such t's. Consequently, there are $\varphi(\prod_{i=1}^{k} p_i^{\beta_i})$, a^t of order $\prod_{i=1}^{k} p_i^{\beta_i}$, where φ is the Euler function. Now, when the power β_i changed through the possible cases we require the sum of such Euler functions which described in the statement of proposition.

The proof of (ii) is very similar to (i) so we omit it and the third part is straightforward. □

Proposition 2.1. *The non-coprime graph of D_{2n} does not have any total perfect code.*

Proof. By definition of total perfect code and the fact that all the n vertices of order 2 in dihedral group of order $2n$ are adjacent in $\Pi_{D_{2n}}$, we deduce that $n \leq 2$ and a contradiction. □

Theorem 2.2. *Let $\mathbb{Z}_n$ be the cyclic group of order n.*

(i) *The non-coprime graph $\Pi_{\mathbb{Z}_p^s}$ has $p^s - 1$ singleton subsets of the vertices which are perfect codes, where p is a prime number and s a positive integer.*

(ii) *The non-coprime graph $\Pi_{\mathbb{Z}_{pq}}$ has $\varphi(pq)$ singleton perfect codes, where p, q are prime numbers and φ is the Euler function.*

(iii) *The non-coprime graph of cyclic group $\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}$ has $\varphi(\prod_{i=1}^{k} p_i^{\alpha_i}) + {\ulcorner M \urcorner \atop \llcorner k \lrcorner}$ singleton perfect codes, where M is a set which is defined the same as in Theorem* 2.1 (i),

p_i's are prime numbers and α_i's are positive integers, $1 \leq i \leq k$. The independence number of $\Pi_{\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}}$ is k and independent sets with more than 2 elements are not perfect codes. Furthermore, the number of independent sets with s elements is equal to the sum of all possible s-multiplications of elements in the set

$$\mathfrak{A} = \left\{ \sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) : 1 \leq i \leq k \right\},$$

which are arranged similar to the equation (2.2), where $1 \leq s \leq k$. Note that s-multiplications of elements in $\mathfrak{A}$ means choosing s elements of the set $\mathfrak{A}$ randomly and compute their multiplications. In particular, the number of two-element and k element independent sets are

$$\sum_{i=1}^{k} \left[\left(\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) \right) \left(\sum_{j=i+1}^{k} \sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j}) \right) \right] \tag{2.2}$$

and

$$\prod_{i=1}^{k} \left(\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) \right),$$

respectively.

(iv) *Every singleton subset of vertices of the non-coprime graph of the group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ is a perfect code, where p is a prime number.*

Proof. (i) The independence number for this graph is one and all the possible subsets of the vertices with one element are perfect codes.

(ii) This graph has $pq - 1$, $\varphi(p)\varphi(q)$ independent sets with one and two elements, respectively. Clearly, singleton subsets which contains an element of order p (or q) are not perfect codes because there exists an element out of it which does not join to the vertex inside that singleton. If we consider singletons which contains generators of order pq, they clearly are perfect codes. The independent sets with two elements are not perfect codes, since the generators join to both vertices inside them.

(iii) It is clear that this graph has $\prod_{i=1}^{k} p_i^{\alpha_i} - 1$ independent sets with one element. It is obvious among these independent sets, the only singletons which contain a generator element or an element of order $\prod_{i=1}^{k} p_i^{\omega_i}$ are perfect codes, where $1 \leq \omega_i \leq \alpha_i$. Moreover, for an independent set with more than one element, there is a generator out of it which joins to both of it and so it is not a perfect code. □

By considering the third part of Theorem 2.2, one can present independent polynomial for that graph.

Proposition 2.2. *The non-coprime of the cyclic group $\mathbb{Z}_n$ admits a total perfect code if and only if $n = 3$.*

Proof. Suppose $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where there exists an index $i \in \{1, 2, \ldots, k\}$ such that $\alpha_i \geq 2$ and p_i is a prime number. Clearly, there are vertices v_1, v_2 of order $p_i^{\alpha_i}$. Let

C be a total perfect code which contain v_1, then by definition v_1 has exactly one neighborhood inside C. This neighborhood can be v_2 or an element of order p_i. If each of them is inside C, then the other one is outside of C has two neighborhood in C and it is a contradiction. Thus, all α_i are equal to 1. Clearly all p_i are greater than 2, because otherwise an element of order 2 does not have any neighborhood. Again by definition for each vertex of order p_i, there is a unique neighborhood in C. Therefore for every vertex of order p_i out of C, there are two neighborhood. Hence the number of elements of order of p_i must be 2, i.e., $\varphi(p_i) = 2$, $p_i = 3$ and the assertion is clear. □

3. The Perfect and Total Perfect Codes of Coprime Graph of Certain Groups

In this section, we present perfect and total perfect codes of coprime graph of dihedral groups and finite Abelian groups. For coprime graphs, if we consider the identity element of the group as a vertex, then it is meaningless. Thus, it is nature to consider the induced subgraph by non-trivial elements. Let us consider the coprime graph of the group G, with vertex set $G \setminus \{e\}$ and denote it by Γ^*_G. We refer the readers to see [6] for its interesting results. Consider the dihedral group D_{2n}.

(i) Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers. Consider A_0 which is the set of all elements of order two and A_j's are the sets of elements of order $p_j^{\beta_j}$, and elements of order $\prod_{i=1}^{q} p_i^{\omega_i}$, where p_j surly exists in the multiplication, $1 \leq \beta \leq \alpha_j$, $1 \leq j \leq k$, $1 \leq q \leq k$ and $0 \leq \omega_i \leq \alpha_i$. Then A_0 and A_j's are the samples of independent sets for $\Gamma_{D_{2n}}$. For $1 < t \leq k$, construct A_t somehow it does not have any common elements with all the sets A_s, with $s < t$. The number of elements in A_0 and A_j are $\prod_{i=1}^{k} p_i^{\alpha_i}$ and $\sum_{1\leq\beta\leq\alpha_j} \varphi(p_j^{\beta}) + \sum_{u=2}^{k} \ulcorner M_j \urcorner_{\llcorner u_j \lrcorner}$, respectively. The notations were defined in Theorem 2.1, note that M_j is the set of all prime powers in M' such that powers of $p_z \neq p_j$ is more or equal than zero and less or equal than α_z, and the powers of the prime p_j must be at least one and less or equal than α_j, where $1 \leq z \leq k$. In computing of Euler function of multiplication, the prime number p_j always must be selected and it is possible that the power of the other prime numbers be zero, in other words it is possible that some prime numbers distinct from p_j do not appear in the multiplication. Moreover, this is very significant to obtain the independent polynomial, in order to compute $\ulcorner M_t \urcorner_{\llcorner u_t \lrcorner,}$ the sum of Euler function on u_t-multiplication of elements in M_t is somehow that it does not have any common summands with all the summands in $\ulcorner M_s \urcorner_{\llcorner u_s \lrcorner,}$ for $1 < t \leq k$, for all $s < t$. Suppose $\alpha(\Gamma_{D_{2n}})$ is independence number of the graph. The independent polynomial is,

$$f(x) = \left(2\prod_{i=1}^{k} p_i^{\alpha_i}\right) x + \sum_{\omega=2}^{\alpha(\Gamma_{D_{2n}})} \left[\binom{\prod_{i=1}^{k} p_i^{\alpha_i}}{\omega} + \sum_{j=1}^{k} \binom{\sum_{1\leq\beta\leq\alpha_j} \varphi(p_j^{\beta}) + \sum_{u=2}^{k} \ulcorner M_j \urcorner_{\llcorner u_j \lrcorner}}{\omega} \right] x^{\omega},$$

and if $\omega > \sum_{1\le\beta_j\le\alpha_j}\varphi(p_j^{\beta}) + \sum_{u=2}^{k} \ulcorner \frac{M_j}{u_j} \urcorner$, or $\omega > \prod_{i=1}^{k} p_i^{\alpha_i}$, then put zero instead of

$$\binom{\sum_{1\le\beta\le\alpha_j}\varphi(p_j^{\beta}) + \sum_{u=2}^{k} \ulcorner \frac{M_j}{u_j} \urcorner}{\omega} \quad \text{or} \quad \binom{\prod_{i=1}^{k} p_i^{\alpha_i}}{\omega}$$

in $f(x)$.

(ii) For $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, suppose

$$\begin{aligned}
A_{0j} =& \{\text{All the elements of order power of } 2\} \\
& \cup \Big\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\le\eta_i\le\alpha_i,\ 1\le\beta\le s, \\
& q\le k-1,\ \text{certainly } p_j \text{ exists in the multiplication}\Big\} \\
& \cup \Big\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\le\eta_i\le\alpha_i,\ 1\le\beta\le s, \\
& q\le k-1,\ \text{certainly } p_j \text{ does not exist in the multiplication}\Big\} \\
& \cup \Big\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{k} p_i^{\eta_i},\ \text{where } 1\le\eta_i\le\alpha_i,\ 1\le\beta\le s\Big\},
\end{aligned}$$

with $a_1,\ a_{2j},\ a_{3j}$ and a_4 as first, second, third and forth set sizes which constructed A_{0j}, respectively. Moreover, assume

$$\begin{aligned}
A_j =& \{\text{All the elements of order power of } p_j,\ 1\le j\le k\} \\
& \cup \Big\{\text{All the elements of order } \prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\le\eta_i\le\alpha_i,\ q\le k, \\
& \text{certainly } p_j \text{ exists in the multiplication}\Big\} \\
& \cup \Big\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{k} p_i^{\eta_i},\ \text{where } 1\le\eta_i\le\alpha_i,\ 1\le\beta\le s\Big\} \\
& \cup \Big\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\le\eta_i\le\alpha_i,\ 1\le\beta\le s, \\
& q\le k-1 \text{ certainly } p_j \text{ exists in the multiplication}\Big\},
\end{aligned}$$

with $b_{1j},\ b_{2j},\ b_{3j} = a_4$ and $b_{j4} = a_{j2}$ as first, second, third, forth set sizes which constructed A_j, respectively. Obviously A_{0j} and A_j's are independent sets $1\le j\le k$. For $1 < t \le k$, construct A_{0t} and A_t somehow they do not have any common elements with all the sets A_{0s} and A_s, with $s < t$. The value of a_i and b_{ij} can be obtained similar to the previous parts. Since replacing of them by their values makes the appearance of the computations more complicated so we work with a_i and b_{ij}. By these hypothesis, the independent polynomial is

$$\begin{aligned}
f(x) =& 2^{s+1}\prod_{i=1}^{k} p_i^{\alpha_i}x + \sum_{\omega=2}^{\alpha(\Gamma_{D_{2n}})}\Bigg[\binom{a_1}{\omega}+\binom{a_4}{\omega}+\sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}}{\omega}+\binom{a_1+a_4}{\omega}\\
&+\sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}+a_1}{\omega}+\sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}+a_4}{\omega}\\
&+\sum_{j=1}^{k}\left[\binom{a_1+a_{2j}+a_{3j}}{\omega}+\binom{a_4+a_{2j}+a_{3j}}{\omega}+\binom{a_1+a_4+a_{ij}}{\omega}\right]\\
&+\sum_{j=1}^{k}\binom{a_1+a_4+a_{2j}+a_{3j}}{\omega}\\
&+\sum_{j=1}^{k}\Bigg[\sum_{i=1}^{2}\binom{b_{ij}}{\omega}+\sum_{i=1}^{3}\sum_{\ell=i+1}^{4}\binom{b_{ij}+b_{lj}}{\omega}-\binom{b_{3j}+b_{4j}}{\omega}\\
&+\sum_{i=1}^{2}\sum_{\ell=i+1}^{3}\sum_{k=\ell+1}^{4}\binom{b_{ij}+b_{lj}+b_{kj}}{\omega}+\binom{b_{1j}+b_{2j}+b_{3j}+b_{4j}}{\omega}\Bigg]\Bigg]x^{\omega},
\end{aligned}$$

such that in the selection $\binom{\sum y_i}{\omega}$ at least one element choose from each sets with sizes y_i. Moreover, if $\omega > \binom{\sum y_i}{\omega}$, then put zero for $\binom{\sum y_i}{\omega}$.

(iii) Assume $n = 2^s$. In this case, the largest independent set for $\Gamma_{D_{2n}}$ is $D_{2n} \setminus \{1\}$. The independent polynomial is,

$$f(x) = 2^{s+1}x + \sum_{\ell=2}^{2^{s+1}-1}\binom{2^{s+1}-1}{\ell}x^{\ell}.$$

It is clear that the singleton contains the identity element is a perfect code for $\Gamma_{D_{2n}}$ and independent sets with more than one elements are not perfect codes.

Proposition 3.1. $\Gamma^*_{D_{2n}}$ *does not have any perfect code.*

Proof. Initially, let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers. Consider the singleton subset of vertices $X = \{x\}$. If the vertex x is of order $p_j^{\beta_j}$ (or 2), where $1 \le j \le k$ and $1 \le \beta_j \le \alpha_j$. Thus, elements of order $p_j^{\beta'_j}$ (or 2) outside of X, are not adjacent to x, $1 \le \beta'_j \le \alpha_j$. Note that the existence of such elements outside of X is clear, as $n \ge 4$. Suppose $|x| = \prod_{i=1}^{q} p_i^{\beta_i}$, $q \le k$. If p_j appear in the multiplication of prime numbers in the order of x, then there is an element of order $p_j^{\omega_j}$ does not join to x. Therefore, the independent sets with one element are not perfect codes. According to the first part of argument before the proposition, if an independent set have more than one element, then it contains just elements of order 2 or just elements of order power of p_j and multiplication of prime powers that include p_j. In both cases, there is a vertex that join to more than one vertex inside them not exactly one. Thus, independent set with more than one element is not a perfect code. If $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$ or 2^s, the assertion follows similarly. $\square$

Proposition 3.2. *For the total perfect code of* $\Gamma^*_{D_{2n}}$ *, we have the following cases.*

(i) *If* $n = \prod_{i=1}^{k} p_i^{\alpha_i}$*, then two-element subset of vertices which contain an element of order* 2 *and an element of order* $\prod_{i=1}^{k} p_i^{\beta_i}$ *is a total perfect code, where* p_i*'s are odd prime numbers and* $1 \leq \beta_i \leq \alpha_i$*. The number of such total perfect codes are* $\prod_{i=1}^{k} p_i^{\alpha_i} \binom{\ulcorner M \urcorner}{\llcorner k \lrcorner}$*, where notations were defined in Theorem* 2.1 (i).

(ii) *If* $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$*, then* $\Gamma^*_{D_{2n}}$ *does not have any total perfect code.*

(iii) *For* $n = 2^s$*,* $\Gamma^*_{D_{2n}}$ *does not have any total perfect code.*

Proof. The proof of first and third part is clear, let us prove the second part.

(ii) Let $T \subseteq V(\Gamma_G)$ be a total perfect code for the graph. If $x \in T$ and the order of x is power of 2, then it has a unique neighborhood $y \in T$, by definition of total perfect code. Thus the order of y is the multiplication of prime numbers which divide $\prod_{i=1}^{k} p_i^{\alpha_i}$. All the vertices outside T have a unique neighborhood inside T except a vertex v of order $2^\beta \prod_{i=1}^{k} p_i^{\beta_i}$, $1 \leq \beta \leq s$ and $1 \leq \beta_i \leq \alpha_i$. We can not consider v inside T, since there is no neighborhood for it inside T. Let us construct T, by use of other vertices. If $x \in T$ of order $\prod_{i=1}^{q} p_i^{\omega_i}$ and $y \in T$ is its neighborhood, then $|y| = \prod_{i=1}^{q'} p_i'^{\omega_i'}$ or $2^\beta \prod_{i=1}^{q'} p_i'^{\omega_i'}$, where $p_i^{\omega_i}$ and $p_i'^{\omega_i'}$ divides $\prod_{i=1}^{k} p_i^{\alpha_i}$ and $\gcd(\prod_{i=1}^{q} p_i^{\omega_i}, \prod_{i=1}^{q'} p_i'^{\omega_i'}) = 1$, $q, q' \leq k$. An element of order $\prod_{i=1}^{k} p_i^{\alpha_i}$ outside T does not join to any vertex inside T. $\square$

Consider the coprime graph of the cyclic group $\mathbb{Z}_n$.

(i) If $n = p^s$, then $\Gamma_{\mathbb{Z}_n}$ has p^s singleton independent sets. Clearly, its independence number is $p^s - 1$ and it has $\binom{p^s-1}{\ell}$ independent sets with ℓ elements.

(ii) Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. The coprime graph of $\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}$ has $\prod_{i=1}^{k} p_i^{\alpha_i}$ singleton independent sets. Moreover, every subsets of the following sets are samples of independent sets of $\Gamma_{\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}}$. Let

$$A_{0j} = \left\{\text{All the elements of order } p_j^{\beta_j}\right\},$$

$$A_{1j} = \left\{\text{All the elements of order } \prod_{i=1}^{k} p_i^{\alpha_i}\right\} \cup \left\{\text{All the elements of order } p_j^{\beta_j}\right\},$$

$$A_2 = \left\{\text{All the elements of order } \prod_{i=1}^{k} p_i^{\alpha_i}\right\} \cup \left\{\text{All the elements of order } \prod_{i=1}^{q} p_i^{\beta_i}\right\},$$

where $1 \leq j \leq k$, $1 \leq \beta_j \leq \alpha_j$, $2 \leq q \leq k-1$ and in A_2, $\prod_{i=1}^{q} p_i^{\beta_i}$, it is possible that some β_i be zero and the order of elements in the set {All the elements of order $\prod_{i=1}^{q} p_i^{\beta_i}$} include at least one prime number p_i in its multiplication.

Now, let $2 \leq \ell \leq \alpha\left(\Gamma_{\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}}\right)$,

$$\mathcal{A}_0 = \sum_{j=1}^{k} \binom{\sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j})}{\ell}, \quad \mathcal{A}_1 = \sum_{j=1}^{k} \binom{\varphi(\prod_{i=1}^{k} p_i^{\alpha_i})}{w_1}\binom{\sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j})}{w_2},$$

where $w_1 \geq 1,\ 2 \leq w_1 + w_2 \leq \ell$ and $\mathcal{A}_2 = \sum_{r=1}^{k} \binom{\varphi(\prod_{i=1}^{k} p_i^{\alpha_i})}{u_1}\binom{\sum_{j=2}^{k-1} \ulcorner M_{p_r} \urcorner \llcorner j \lrcorner}{u_2}$, where $u_2 \geq 1,\ 2 \leq u_1 + u_2 \leq \ell$. Furthermore, the notation $\ulcorner M_{p_r} \urcorner \llcorner j \lrcorner$ is defined similar to Theorem 2.1(i), such that in choosing the prime power numbers from the set M, some power of p_r is selected. By this hypothesis, the number of independent sets with ℓ elements is more than $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$.

Suppose $\Gamma_{\mathbb{Z}_n}$ has a total perfect code T. Since the greatest common divisor of order of identity element (zero) with respect to all other element orders is one so $0 \in T$. By definition 0 has a unique neighborhood inside T, say x. As every vertex of total perfect codes cover exactly one vertex of the graph, there is just one other vertex outside of T. Hence, $n = 2, 3$. The coprime graph of $\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}$ does not have any perfect and total perfect code. Clearly $\Gamma_{\mathbb{Z}_2}$ has a singleton perfect code and $\Gamma_{\mathbb{Z}_{p^s}}$ does not have any perfect code, where p is a prime and s a positive integer ($p \geq 3$ and $s \geq 1$ or $p = 2$ and $s \geq 2$). Note that if we consider the induces subgraph of $\Gamma_{\mathbb{Z}_{p^s}}$ by omitting the identity, then the set of all vertices, largest independent set, is a perfect code. There are $\binom{p^{(\alpha_1 + \cdots + \alpha_k)}}{\ell}$ independent sets of size ℓ for the coprime graph of the group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$, where $1 \leq \ell \leq \alpha(\Gamma_{\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}})$. Clearly, the coprime graph of $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ does not have any perfect and total perfect code.

Theorem 3.1. *Suppose Γ_G^* has a total perfect code with two elements $T = \{g_1, g_2\}$ such that $|g_1| = \prod_{i=1}^{k} p_i^{\alpha_i}$, $|g_2| = \prod_{j=1}^{k'} q_j^{\beta_j}$, where p_i and q_j's are distinct prime numbers. Then G is non-cyclic and the set of prime divisor of order of G is $\Pi(|G|) = \{p_i, q_j : 1 \leq i \leq k,\ 1 \leq j \leq k'\}$. Moreover, G does not contain an element of order $\prod_{i=1}^{s} p_i^{\alpha'_i} \prod_{j=1}^{s'} q_j^{\beta'_j}$, where $0 \leq \alpha'_i \leq \alpha_i,\ 0 \leq \beta'_i \leq \beta_i$ and note that α'_i (and also β'_i) are not all zero, simultaneously.*

Proof. Let $x \in G$, $|x| = r$, where r is a prime number distinct from p_i, q_j's. Then x join to both g_1 and g_2 which is a contradiction. Therefore, the only prime numbers that divide the order of G are the prime numbers that divide the order of g_i's, $i = 1, 2$. Now, if the group G contain an element of order $\prod_{i=1}^{k} p_i^{\alpha_i} \prod_{j=1}^{k'} q_j^{\beta_j}$, then this element is not in the neighborhood of g_i's and again this is against the definition of total perfect code. Hence G is a non-cyclic group. □

The definition of total perfect code and coprime graph signify that the coprime graph does not have any singleton total perfect codes. Moreover, if Γ_G^* has a total perfect code with more than two elements, then similar result as Theorem 3.1 will be obtained.

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References

[1] N. Biggs, *Perfect codes in graphs*, J. Combin. Theory Ser. B **15** (1973), 289–296. http://dx.doi.org/10.1016/0095-8956(73)90042-7

[2] H. Dorbidi, *A note on the coprime graph of a group*, Int. J. Group Theory **5**(4) (2016), 17–22.

[3] A-A. Ghidewon, R. H. Hammack and D. T. Taylor, *Total perfect codes in tensor products of graphs*, Ars Combin. **88** (2008), 129–134.

[4] J. Kratochví, *Perfect codes over graphs*, J. Combin. Theory Ser. B **4**0(2) (1986), 224–228. http://dx.doi.org/10.1016/0095-8956(86)90079-1

[5] S. Ling and C. Xing, *Coding Theory a First Course*, Cambridge University Press, 2004.

[6] X. Ma, *Perfect codes in proper reduced power graphs of finite groups*, Commun. Algebra **4**8(9) (2020), 3881–3890. http://dx.doi.org/10.1080/00927872.2020.1749845

[7] X. Ma, G. L. Walls, K. Wang and S. Zhou, *Subgroup perfect codes in Cayley graphs*, SIAM J. Discrete Math. **3**4(3) (2020), 1909–1921. http://dx.doi.org/10.1137/19M1258013

[8] X. Ma, H. Wei and L. Yang, *The coprime graph of a group*, Int. J. Group Theory **3**(3) (2014), 13–23.

[9] F. Mansoori, A. Erfanian and B. Tolue, *Non-coprime graph of a finite group*, AIP Conference Proceedings **1750**(1) (2016), Article ID 050017. https://doi.org/10.1063/1.4954605

[10] C. E. Shannon, *A mathematical theory of communication*, The Bell System Technical Journal **27** (1948), 379–423, 623–656. http://dx.doi.org/10.1002/j.1538-7305.1948.tb01338.x, http://dx.doi.org/10.1002/j.1538-7305.1948.tb00917.x

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A NEW CHARACTERIZATION OF PROJECTIVE SPECIAL LINEAR GROUPS $\mathbf{L_2(q)}$

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Abstract. In this paper, we prove that projective special linear groups $L_2(q)$, where $q \equiv \pm 2 \pmod 5$ (q is an odd prime) can be uniquely determined by $|L_2(q)|$ and $nse(L_2(q))$.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of the order of G and $\pi_e(G)$ be the set of the order of elements in G. If $k \in \pi_e(G)$, then we denote the number of elements of order k in G by $m_k(G)$ and the set of the numbers of elements with the same order in G by $nse(G)$. In other words, $nse(G) = \{m_k(G) \mid k \in \pi_e(G)\}$. Also we denote a sylow p-subgroup of G by G_p and the number of sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

One of the important problems in finite groups theory is, group characterization by specific property. Properties such as, elements order, set of elements with the same order, the largest elements order, etc. One of this methods is group characterization by using the order of group and $nse(G)$. In other words, we say the group G is characterizable by using the order of G and $nse(G)$, if there exists the group H, so that $nse(G) = nse(H)$ and $|G| = |H|$, then $G \cong H$. Next, in [1,2,4–8,11,12,16,17,20–22] was proved the groups such as, $PGL_2(q)$, suzuki groups, sporadic groups, $PSL(3,q)$,

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Symmetric group, $U_4(2)$, $PSU(3,3)$ and projective special linear group $PSL(3,3)$, $L_2(p)$ with $p \in \{19, 23\}$, $L_2(q)$ where $q \in \{17, 27, 29\}$, the projective special linear group $L_2(2^a)$, where either $2^a - 1$ or $2^a + 1$ is a prime number, symplectic groups $C_2(3^n)$, projective special linear group $l_3(q)$, projective special unitary groups $U_3(3^n)$ and ${}^2G_2(q)$, where $q \pm \sqrt{3q} + 1$ are prime numbers are characterizable by using $nse(G)$ and the order of G. In this paper, we prove that projective special linear groups $L_2(q)$, where q is an odd prime can be uniquely determined by $|L_2(q)|$ and $nse(L_2(q))$. In fact, we prove the following main theorem.

Theorem 1.1 (Main Theorem). *Let $L_2(q)$ be projective special linear groups, where $q \equiv \pm 2 \pmod 5$ (q is an odd prime) and G be a group with $nse(G) = nse(L_2(q))$, $|G| = |L_2(q)|$. Then $G \cong L_2(q)$.*

2. Notation and Preliminaries

Lemma 2.1 ([13]). *Let H be a finite soluble group all of whose elements are of a power prime order. Then $|\pi(H)| \leq 2$.*

Lemma 2.2 ([10]). *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(b) *$|H|$ divides $|K| - 1$;*
(c) *K is nilpotent.*

Definition 2.1. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\frac{G}{H}$ and K are Frobenius groups with kernels $\frac{K}{H}$ and H, respectively.

Lemma 2.3 ([14]). *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi(H) \cup \pi(\frac{G}{K}) = \pi_1$ and $\pi(\frac{K}{H}) = \pi_2$;*
(b) *$\frac{G}{K}$ and $\frac{K}{H}$ are cyclic groups satisfying $|\frac{G}{K}|$ divides $|Aut(\frac{K}{H})|$. In particular, every 2-Frobenius group is soluble group.*

Lemma 2.4 ([25]). *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*
(b) *G is a 2-Frobenius group;*
(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and $\frac{G}{K}$ are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|\frac{G}{K}|$ divides $|Out(\frac{K}{H})|$.*

Lemma 2.5 ([9]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. *Let G be a finite group. Then, for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 2.5, the proof is straightforward. □

Lemma 2.7 ([24])**.** *Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(a) $L_n(q)$, $n = 2, 3$, $q \equiv \pm 2 \pmod 5$;
(b) $G_2(q)$, $q \equiv \pm 2 \pmod 5$;
(c) ${}^2A_2(q)$, $q \equiv \pm 2 \pmod 5$;
(d) ${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;
(e) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.

Remark 2.1. In previous lemma, we consider the projective special linear groups by $A_n(q) \cong L_{n+1}(q)$ and the projective special unitary groups by ${}^2A_{n-1}(q) \cong U_n(q)$.

Lemma 2.8 ([26])**.** *Let q, k, l be natural numbers. Then*

(a) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;

(b) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1, & \textit{if both } \frac{k}{(k,l)} \textit{ and } \frac{l}{(k,l)} \textit{ are odd,} \\ (2, q+1,) & \textit{otherwise}; \end{cases}$

(c) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1, & \textit{if } \frac{k}{(k,l)} \textit{ is even and } \frac{l}{(k,l)} \textit{ is odd,} \\ (2, q+1), & \textit{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.9. *Let L be projective special linear group $L_2(q)$, where q is an odd prime number and p is an isolated vertex in $\Gamma(G)$. Then $m_p(G) = \frac{(p-1)|G|}{(4p)}$ and for every $i \in \pi_e(G) - \{1, p\}$, p divides $m_i(G)$.*

Proof. Since that $|G_p| = p$, it follows that G_p is a cyclic group of order p. Thus $m_p(G) = \phi(G)n_p(G)$. Now, it is enough to show $n_p(G) = \frac{|G|}{(4p).}$ By [25], p is an isolated vertex of $\Gamma(G)$. Hence, $|C_G(G_p)| = p$ and $|N_G(G_p)| = xp$ for a natural number x. On the other hand, we know that $\frac{N_G(G_p)}{C_G(G_p)}$ embed in $Aut(G_p)$, which implies $x \mid p - 1$. Furthermore, by Sylow's theorem, $n_p(G) = |G : N_G(G_p)|$ and $n_p(G) \equiv 1 \pmod p$. Thus, p divides $\frac{|G|}{xp} - 1$ thus $\frac{q \pm 1}{2}$ divides $\frac{q(q^2-1)}{xp} - 1$. It follows that $\frac{q \pm 1}{2}$ divides $q^2 \pm q - x$ so we have $p \mid 4 - x$ and since $x \mid p - 1$ we deduce that $x = 4$, and the proof is finished. We prove that p is an isolated vertex of $\Gamma(G)$. Opposite there is $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $m_{tp}(G) = \phi(tp)n_p(G)k$, where k is the number of cyclic subgroups of order t in $C_G(G_p)$ and since $n_p(G) = n_p(L)$, it follows that $m_{tp}(G) = \frac{(t-1)(p-1)|L|k}{(4p)}$. If $m_{tp}(G) = m_p(L)$, then $t = 2$ and $k = 1$. Furthermore lemma 2.6 yields $p \mid m_2(G) + m_{2p}(G)$ and since $m_2(G) = m_2(L)$ and $p \mid m_2(L)$, we deduce that $p \mid m_{2p}(G)$, which is a contradiction. So Lemma 2.5 implies that $p \mid m_{tp}(G)$. Hence $p \mid t - 1$ and since $m_{tp}(G) < |G|$ we deduce $t - 1 \leq 6$. As a result $t \in \{2, 3, 5, 7\}$, where this is a contradiction. Let $r \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, it follows that $p \nmid r$ and $pr \notin \pi_e(G)$. Thus, G_p acts fixed point freely on the set of elements of order r by conjugation and hence $|G_p| \mid m_r(G)$. So, we conclude that $p \mid m_r(G)$. □

3. Proof of the Main Theorem

In this section, we prove the main theorem in the following lemmas. We denote the Projective specialy unitary group $L_2(q)$, where q is an prime number by L and prime number $\frac{q\pm1}{2}$ by p. Recall that G is a group with $|G| = |L|$ and $nse(G) = nse(L)$.

Lemma 3.1. *$m_2(G) = m_2(L)$, $m_p(G) = m_p(L)$, $n_p(G) = n_p(L)$, and $p \mid m_k(G)$ for every $k \in \pi_e(G) - \{1, p\}$.*

Proof. By Lemma 2.6, for every $1 \neq r \in \pi_e(G)$, $r = 2$ if and only if $m_r(G)$ is odd. Thus it follows that $m_2(G) = m_2(L)$. According to Lemma 2.6, $(m_p(G), p) = 1$. Thus $p \nmid m_p(G)$ and hence Lemma 2.9 implies that $m_p(G) \in \{m_1(L), m_2(L), m_p(L)\}$. Moreover, $m_p(G)$ is even, so we deduce that $m_p(G) = m_p(L)$. Since G_p and L_p are cyclic groups of order p and $m_p(G) = m_p(L)$, we deduce that $m_p(G) = \varphi(p)n_p(G) = \varphi(p)n_p(L) = m_p(L)$, so $n_p(G) = n_p(L)$. Let $k \in \pi_e(G) - \{1, p\}$. Since p is an isolated vertex of $\Gamma(G)$, $p \nmid k$ and $pk \notin \pi_e(G)$. Thus, G_p acts fixed point freely on the set of elements of order k by conjugation and hence $|G_p| \mid m_k(G)$. So, we conclude that $p \mid m_k(G)$. □

Lemma 3.2. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 3.2, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now, by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. It follows that (i) $|H| = p$ and $|K| = \frac{|G|}{p}$ or (ii) $|H| = \frac{|G|}{p}$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we deduce that the case (ii) cannot occur. So, $|H| = p$ and $|K| = \frac{|G|}{p}$. Hence, $\frac{q\pm1}{2} \mid \frac{q(q^2-1)}{q\pm1} - 1$. So, we have $q \pm 1 \mid 2q^2 \pm 2q - 2$. It follows that $p \mid 2$. Thus, $p \mid 2$, which is impossible. □

Lemma 3.3. *The group G is not soluble group.*

Proof. Let $r \neq 2$ and s be a prime divisor of $\frac{q+1}{2}$ and let t be a prime divosor $\frac{q-1}{2}$. If G were soluble, then there would exist a $\{2, r, s\}$-Hall subgroup H of G. Since H does not contain any elements of orders $2r$, $2s$, rs. Thus, all of elements of H would be of prime power order. But this contradicts Lemma 2.1. So, G is not soluble group. □

Lemma 3.4. *The group G is not a 2-Frobenius group.*

Proof. By previous lemma, we have that G is not soluble group. On the other hand by Lemma 2.3 since that every 2-Frobenius group is soluble group, so G is not a 2-Frobenius group. □

Lemma 3.5. *The group G is isomorphic to the group L.*

Proof. By Lemma 2.9, p is an isolated vertex of $\Gamma(G)$. Thus, $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. Now, Lemma 3.2 and Lemma 3.4 imply that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occurs. So, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |\frac{K}{H}|$. On the other hand, we know that $5 \nmid |G|$. Thus, $\frac{K}{H}$ is isomorphic to one of the groups in Lemma 2.7. Hence, we consider the following isomorphisms.

(1) If $\frac{K}{H} \cong G_2(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [25], $p = q'^2 \pm q' + 1$. On the other hand, we know $|G_2(q')|$ divides $|G|$, in other words $q'^6(q'^6-1)(q'-1) \mid q(q^2-1)$. Thus, $\frac{q\pm1}{2} = q'^2 \pm q' + 1$ as a result $q \pm 1 = 2q'^2 + 2q' + 2$. So, we deduce $q = 2q'^2 + 2q' + 1$ and $q = 2q'^2 + 2q' + 3$.

(2) If $\frac{K}{H} \cong {}^2A_2(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [25], $p = \frac{q'^2-q'+1}{(3,q'+1)}$. On the other hand, we know $|{}^2A_2(q')| \mid |G|$, in other words $\frac{q'^3(q'^3+1)(q'^2-1)}{(3,q'+1)} \mid \frac{q(q^2-1)}{2}$. First, if $(3, q'+1) = 1$, then similar to part (1) we deduce a contradiction. Now let $(3, q'+1) = 3$. Then, $\frac{q\pm1}{2} = \frac{q'^2-q'+1}{3}$. Thus $3q \pm 3 = 2q'^2 - 2q' + 2$. As a result $q = \frac{2q'^2-2q'-1}{3}$ and $q = \frac{2q'^2-2q'+5}{3}$. Since that $|{}^2A_2(q')| \nmid |G|$, so we have a contradiction.

(3) If $\frac{K}{H} \cong {}^2G_2(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [25] $p = q' \pm \sqrt{3q'} + 1$. On the other hand, we know $|{}^2G_2(q')| \mid |G|$, in other words $q'^3(q'^3+1)(q'-1) \mid \frac{q(q^2-1)}{2}$. Now, we consider $\frac{q\pm1}{2} = q' \pm \sqrt{3q'} + 1$ as a result $q = 2(3^{2m+1}) + 2(3^{m+1}+1)$ and $q = 2(3^{2m+1}) - 2(3^{m+1}) + 3$. Since that $|{}^2G_2(q')| \nmid |G|$, so we have a contradiction.

(4) If $\frac{K}{H} \cong L_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [25], $p = \frac{q'^2+q'+1}{(3,q'-1)}$. On the other hand, we know $|L_3(q')| \mid |G|$, in other words $\frac{q'^2+q'+1}{(3,q'-1)} \mid \frac{q(q^2-1)}{2}$. First if $\frac{q\pm1}{2} = q'^2 + q' + 1$, then $q \pm 1 = 2q'^2 + 2q' + 2$. So, we deduce $q = 2q'^2 + 2q' + 1$and $q = 2q'^2 + 2q' + 3$. Now since $|L_3(q')| \nmid |G|$, which is a contradiction. Now, if $(3; q'-1) = 3$, then $\frac{q\pm1}{2} = \frac{q'^2+q'+1}{3}$, so $3q \pm 3 = 2q'^2 + 2q' + 2$. It follows that $q = \frac{2q'^2+2q'-1}{3}$ and $q = \frac{2q'^2+2q'+5}{3}$. But $|L_3(q')| \nmid |G|$, which is a contradiction.

Now, let $(3; q'+1) = 3$. So, $3q \pm 3 = 2q'^2 - 2q' + 2$. As a result $q = \frac{2q'^2-2q'-1}{3}$ and $q = \frac{2q'^2-2q'+5}{3}$. But $|L_3(q')| \nmid |G|$, which is a contradiction.

(5) If $\frac{K}{H} \cong {}^3D_4(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [25], $p = q'^4 - q'^2 + 1$. On the other hand, we know $|{}^3D_4(q')| \mid |G|$, in other words $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{q(q^2-1)}{2}$. As a result $\frac{q\pm1}{2} = q'^4 - q'^2 + 1$, so $q = 2q'^4 - 2q'^2 + 1$ and $q = 2q'^4 - 2q'^2 + 3$. Since $|{}^3D_4(q')| \nmid |G|$, which is a contradiction.

(6) Hence, $\frac{K}{H} \cong L_2(q')$. As a result $|\frac{K}{H}| = |L_2(q')|$. On the other hand, we have $p \mid \pi(\frac{K}{H})$, so $\frac{q\pm1}{2} = \frac{q'\pm1}{2}$. It follows that $q = q'$. Now, on the other hand, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and $\frac{G}{K}$ are π_1-groups, so $|H| = 1$ and $K = L_2(q') \cong G$. □

References

[1] S. Asgary and N. Ahanjideh, *Characterization of $PSL(3, q)$ by nse*, Boletim da Sociedade Paranaense de Matemática **19**(4) (2017), 425–438. https://doi.org/10.5269/bspm.39391

[2] A. Babai and Z. Akhlaghi, *A new characterization of symmetric group by NSE*, Czech. Math. J **67**(2) (2017), 427-437. https://doi.org/10.21136/CMJ.2017.0700-15

[3] G. Y. Chen, *On the structure of Frobenius groups and 2-Frobenius groups*, Journal of Southwest China Normal University **20**(5) (1995), 485–487.

[4] B. Ebrahimzadeh, A. Iranmanesh and H. Parvizi Mosaed, *A new characterization of Ree group* $^2G_2(q)$ *by the order of group and number of elements with same order*, Int. J. Group Theory **6**(4) (2017), 1–6. https://doi.org/10.22108/ijgt.2017.21233

[5] B. Ebrahimzadeh and R. Mohammadyari, *A new characterization of Suzuki groups*, Arch. Math. **55**(1) (2019), 17–21. https://doi.org/10.5817/AM2019-1-17

[6] B. Ebrahimzadeh and R. Mohammadyari, *A new characterization of symplectic groups* $C_2(3^n)$, Acta Comment. Univ. Tartu. Math. **23**(1) (2019), 117–124. https://doi.org/10.12697/ACUTM.2019.23.12

[7] B. Ebrahimzadeh, *A new characterization of projective special linear group* $L_3(q)$, Algebra Discrete Math. **31**(2) (2021), 212–218. https://doi.org/10.12958/adm1235

[8] B. Ebrahimzadeh and A. Iranmanesh, *A new characterization of projective special unitary groups* $U_3(3^n)$ *by the order of group and the number of elements with the same order*, Algebraic Structures and Their Applications **9**(2) (2022), 113–120. https://doi.org/10.22034/as.2022.2675

[9] G. Frobenius, *Verallgemeinerung des Sylowschen Satze*, Berliner sitz, (1895), 981–983.

[10] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1980.

[11] F. Hajati, A. Iranmanesh and A. Tehranian, *A characterization of* $U_4(2)$ *by* NSE, Facta Univ. Ser. Math. Inform. **34**(4) (2019), 641–649. https://doi.org/10.22190/FUMI1904641H

[12] F. Hajati, A. Iranmanesh and A. Tehranian, *A characterization of projective special unitary group* $PSU(3,3)$ *and projective special linear group* $PSL(3,3)$ *by* NSE, Mathematics **6**(7) (2018), Paper ID 120. https://doi.org/10.3390/math6070120

[13] G. Higman, *Finite groups in which every element has prime power order*, J. London Math. Soc **32** (1957), 335–342.

[14] Q. Jiang and C. Shao, *A new characterization of* $L_2(p)$ *with* $p \in \{19, 23\}$ *by* NSE, Ital. J. Pure Appl. Math. **38** (2017), 624–630.

[15] A. Khalili Asboei and A. Iranmanesh, *Characterization of the linear groups* $L_2(p)$, Czechoslovak Math. J. **64**(139) (2014), 459–464. https://doi.org10.1007/s10587-014-0112-y

[16] A. Khalili Asboei, S. S. Amiri, A. Iranmanesh and A. Tehranian, *A new characterization of sporadic simple groups by nse and order*, J. Algebra Appl. **12**(2) (2013), Paper ID 1250158. https://doi.org/10.1142/S0219498812501587

[17] A. Khalili Asboei, *Characterization of projective general linear groups*, Int. J. Group Theory **5**(1) (2016), 17–28. https://doi.org/10.22108/ijgt.2016.5634

[18] A. S. Kondratev, *Prime graph components of finite simple groups*, Mathematics of the USSR-Sbornik **67**(1) (1990), 235–247. https://doi.org/10.1070/SM1990v067n01ABEH001363

[19] V. D. Mazurov and E. I. Khukhro, *Unsolved Problems in Group Theory*, The Kourovka Notebook, Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk, 2006.

[20] H. Parvizi Mosaed, A. Iranmanesh and A. Tehranian, *Characterization of Suzuki group by nse and order of group*, Bull. Korean Math. Soc. **53**(3) (2016), 651–656. https://doi.org/10.4134/BKMS.b140564

[21] C. Shao and Q. Jiang, *Characterization of* $L_2(q)$ *by nse where* $q \in \{17, 27, 29\}$, Chin. Ann. Math. Ser. B **13**(2) (2016), 103–110. https://doi.org/10.1007/s11401-015-0953-1

[22] C. Shao and Q. Jiang, *A new characterization of some linear groups by nse*, J. Algebra Appl. **37**(1) (2014), Paper ID 1350094. https://doi.org/10.1142/S0219498813500941

[23] W. J. Shi, *A new characterization of the sporadic simple groups*, J. Group Theory, Proceedings of the 1987 Singapore Conference, Berlin, Walter de Gruyter, 1989, 531–540. https://doi.org/10.1515/9783110848397-040

[24] W. J. Shi, *A characterization of* $U_3(2^n)$ *by their element orders*, Journal of Southwest China Normal University **25**(4) (2000), 353–360. https://doi.org/10.13718/j.cnki.xsxb.2000.04.001

[25] J. S. Williams, *Prime graph components of finite groups*, J. Algebra **69**(2) (1981), 487–513. https://doi.org/10.1016/0021-8693(81)90218-0

[26] A. V. Zavarnitsine, *Recognition of the simple groups $L_3(q)$ by element orders*, J. Group Theory **7**(1) (2004), 81–97. https://doi.org/10.1515/jgth.2003.044

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INEQUALITIES FOR STRONGLY r-CONVEX FUNCTIONS ON TIME SCALES

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Abstract. In this paper, first we establish the Hermite-Hadamard type inequality based on diamond-α integral for a subset of strongly r-convex functions. Then we prove several new inequalities for n-times continuously differentiable strongly r-convex functions on time scales by virtue of some techniques and introducing new quantities.

1. Introduction

The analysis on time scales is a relatively new area of mathematics that unifies and generalizes discrete and continuous theories. Moreover, it is a crucial tool in many computational and numerical applications.

The differential calculus on time scales generalizes classical both continuous and discrete differential calculus depending on the structure of the time scale under consideration. There are several common generalizations of classical derivative to time scales. For example, one of them is the so-called Δ-derivative, which simultaneously generalizes the forward divided difference of the first order, while the first-order backward divided difference is generalized by the ∇-derivative. There is also the so-called diamond-α dynamic derivative or, shortly, $\Diamond_\alpha$ -derivative being, in turn, the linear combination of Δ and ∇-derivatives with the coefficients α and $1-\alpha$, respectively, for some $\alpha \in [0,1]$. For each type of derivatives on time scales there is its own notion of the integral. Thus, the diamond-α integral corresponds to the $\Diamond_\alpha$-derivative.

Key words and phrases. Time scales, strongly r-convex functions, Hermite-Hadamard inequality, delta differentiable function, diamond integral.

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The main purpose of this discussions is to reflect some certain inequalities for strongly r-convex functions and it is inspired by the papers [3,7–10] where the authors focused on to obtain several new integral inequalities for different class of convex functions which are n-times differentiable on an interval in $\mathbb{R}$. Since many continuous models in biology, physics, chemistry and etc. have discrete analogues, our aim in this paper is to unify these inequalities in the discrete and continuous case.

This paper is organized as follows. In the next section, we briefly recall key notions and notations on time scales and then we introduce diamond-α derivatives by recalling the basic property of this combined dynamic derivatives. We also present definition of diamond-α integral and several theorems concerning the properties of it. In Section 3, which is devoted to our main results, we deduce some integral inequalities by applying the definition of strongly r-convexity and the integral identity which we prove in the sequel. We also introduce some new quantities and use well-known inequalities to present our results.

2. Time Scales Revisited

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The forward jump and backward jump operators σ and ρ can be defined respectively by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Note that for any $t \in \mathbb{T}$, $\sigma(t) \geq t$ and $\rho(t) \leq t$. Moreover, for $t \in \mathbb{T}$, we say the graininess function $\mu : \mathbb{T} \to [0, +\infty)$ to be as follows

$$\mu(t) = \sigma(t) - t.$$

We define the interval $[a, b]_{\mathbb{T}}$ in $\mathbb{T}$ as follows

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half open intervals etc. are defined accordingly.

For $t \in \mathbb{T}$, we have the following cases.

- If $\sigma(t) > t$, then we say that t is right-scattered.
- If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then we say that t is right-dense.
- If $\rho(t) < t$, then we say that t is left-scattered.
- If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that t is left-dense.

We define $\mathbb{T}^k = \mathbb{T}$ if $\sup \mathbb{T}$ is left-dense and $\mathbb{T}^k = \mathbb{T} \setminus \{\sup \mathbb{T}\}$ if $\sup \mathbb{T}$ is left-scattered. Similarly, we define $\mathbb{T}_k = \mathbb{T}$ if $\inf \mathbb{T}$ is right-dense and $\mathbb{T}_k = \mathbb{T} \setminus \{\inf \mathbb{T}\}$ if $\inf \mathbb{T}$ is right-scattered. We denote $\mathbb{T}^k \cap \mathbb{T}_k = \mathbb{T}^k_k$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. We define $f^\Delta(t)$ to be a number, provided it exists, as follows: for any $\epsilon > 0$ there is a neighbourhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We say $f^\Delta(t)$ is the delta or Hilger derivative of f at t. Also, we say f is delta differentiable on $\mathbb{T}^k$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. Similarly, we say that a function f defined on $\mathbb{T}$ is ∇ differentiable at $t \in \mathbb{T}_k$ if for $\varepsilon > 0$ there is a neighborhood V of t such that for some γ the following inequality holds:

$$|f(\rho(t)) - f(s) - \gamma(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in V$ and in this case, we write $f^{\nabla}(t) = \gamma$. We say that f is ∇ differentiable on $\mathbb{T}_k$ if $f^{\nabla}(t)$ exists for any $t \in \mathbb{T}_k$.

Definition 2.1 ([12,13])**.** Let $\alpha \in [0,1]$ and $f : \mathbb{T} \to \mathbb{R}$ be Δ and ∇ differentiable at $t \in \mathbb{T}$. Define the diamond-α dynamic derivative $f^{\Diamond_\alpha}$ of f at t as follows

$$f^{\Diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t).$$

Thus, f is diamond-α-differentiable at $t \in \mathbb{T}$ if and only if f is Δ and ∇ differentiable at t. When $\alpha = 1$, we have

$$f\Diamond_\alpha(t) = f^{\Delta}(t)$$

and for $\alpha = 0$, we have

$$f\Diamond_\alpha(t) = f^{\nabla}(t).$$

In [12], they proved the following criteria for $\diamond_\alpha$-differentiability of a function.

Theorem 2.1 ([12])**.** *Let* $\alpha \in [0,1]$.

(a) *If* $t \in \mathbb{T}$ *is dense and* $f'(t)$ *exists, then*

$$f^{\diamond_\alpha}(t) = f^{\Delta}(t) = f^{\nabla}(t) = f'(t).$$

(b) *If* $t \in \mathbb{T}$ *is isolated, then* $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

(c) *If* $t \in \mathbb{T}$ *is left-scattered and right-dense, and*

$$f'(t^+) = \lim_{h \to 0+} \frac{f(t+h) - f(t)}{h}$$

exists, then $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \alpha f'(t^+) + (1-\alpha)\frac{f^{\rho}(t) - f(t)}{\rho(t) - t}.$$

(d) *If* $t \in I$ *is right-scattered and left-dense, and*

$$f'(t^-) = \lim_{h \to 0-} \frac{f(t+h) - f(t)}{h}$$

exists, then $f^{\diamond_\alpha}(t)$ *exists and*

$$f^{\diamond_\alpha}(t) = \alpha\frac{f^{\sigma}(t) - f(t)}{\sigma(t) - t} + (1-\alpha)f'(t^-).$$

Below we will list some of the properties of the diamond-α derivative. Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$.

Theorem 2.2 ([12, 13])**.** *$f+g$ is diamond-α differentiable at t and*

$$(f+g)^{\Diamond_\alpha}(t) = f^{\Diamond_\alpha}(t) + g^{\Diamond_\alpha}(t).$$

Theorem 2.3 ([12, 13])**.** *For any $c \in \mathbb{C}$, we have cf is diamond-α differentiable at t and*

$$(cf)^{\Diamond_\alpha}(t) = cf^{\Diamond_\alpha}(t).$$

Theorem 2.4 ([12, 13])**.** *fg is diamond-α differentiable at t and*

$$\begin{aligned}(fg)^{\Diamond_\alpha}(t) &= f^{\Diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t)\\ &= f(t)g^{\Diamond_\alpha}(t) + \alpha f^{\Delta}(t)g^{\sigma}(t) + (1-\alpha)f^{\nabla}(t)g^{\rho}(t).\end{aligned}$$

Definition 2.2 ([2])**.** Let $a, t \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ derivative of f provided that $F^{\Delta}(t) = f(t)$ holds for $t \in \mathbb{T}$. We define the Δ integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

Let $g : \mathbb{T} \to \mathbb{R}$. A function $G : \mathbb{T} \to \mathbb{R}$ is called a ∇ derivative of g provided that $G^{\nabla}(t) = g(t)$ holds for $t \in \mathbb{T}$. We define the ∇ integral of g by

$$\int_a^t g(s)\nabla s = G(t) - G(a), \quad t \in \mathbb{T}.$$

Definition 2.3 ([12, 13])**.** Let $\alpha \in [0,1]$, $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Define diamond-$\alpha$ integral of h as follows

$$\int_a^t h(s)\Diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s.$$

Remark 2.1. Note that

$$\begin{aligned}\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Diamond_\alpha} =& \alpha\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Delta} + (1-\alpha)\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\nabla}\\ =& \alpha\left(\alpha\int_a^t f(s)\Delta s + (1-\alpha)\int_a^t f(s)\nabla s\right)^{\Delta}\\ &+ (1-\alpha)\left(\alpha\int_a^t f(s)\Delta s + (1-\alpha)\int_a^t f(s)\nabla s\right)^{\nabla}\\ =& \alpha^2 f(t) + \alpha(1-\alpha)f(\sigma(t)) + \alpha(1-\alpha)f(\rho(t)) + (1-\alpha)^2 f(t)\\ =& \left(2\alpha^2 - 2\alpha + 1\right)f(t) + \alpha(1-\alpha)(f(\sigma(t)) + f(\rho(t))), \quad t \in \mathbb{T}.\end{aligned}$$

Thus, in the general case we do not have

$$\left(\int_a^t f(s)\Diamond_\alpha s\right)^{\Diamond_\alpha} = f(t).$$

In [11], they proved the following criteria for $\diamond_\alpha$-integrability of a function.

Theorem 2.5 ([11])**.** (a) *Every monotone function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a,b]_{\mathbb{T}}$ *is* $\diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$.

(b) *Every continuous function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a,b]_{\mathbb{T}}$ *is* $\diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$.

(c) *Every regulated function* $f : \mathbb{T} \to \mathbb{R}$ *on* $[a,b]_{\mathbb{T}}$ *is* $\diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$.

Below we suppose that $f, g : \mathbb{T} \to \mathbb{R}$ are diamond-α integrable over $[a,b]_{\mathbb{T}}$.

Theorem 2.6 ([12,13])**.** *For any* $c \in \mathbb{C}$*, the function* cf *is diamond-*α *integrable over* $[a,b]_{\mathbb{T}}$ *and*

$$\int_a^b (cf)(s)\Diamond_\alpha s = c\int_a^b f(s)\Diamond_\alpha s.$$

Theorem 2.7 ([12,13])**.** $f+g$ *is diamond-*α *integrable over* $[a,b]_{\mathbb{T}}$ *and*

$$\int_a^b (f+g)(s)\Diamond_\alpha s = \int_a^b f(s)\Diamond_\alpha s + \int_a^b g(s)\Diamond_\alpha s.$$

Theorem 2.8 ([12,13])**.** *We have*

$$\int_a^b f(s)\Diamond_\alpha s = \int_a^t f(s)\Diamond_\alpha s + \int_t^b f(s)\Diamond_\alpha s,$$

for any $t \in [a,b]_{\mathbb{T}}$.

For $a, b \in \mathbb{T}$, $a < b$, denote

$$\begin{aligned}
x_\alpha &= \frac{1}{b-a}\int_a^b t \diamond_\alpha t,\\
x_{\alpha,\alpha} &= \frac{1}{b-a}\int_a^b t^2 \diamond_\alpha t,\\
x_{\alpha,r,-} &= \left(\frac{1}{b-a}\int_a^b (b-t)^{\frac{1}{r}} \diamond_\alpha t\right)^r,\\
x_{\alpha,r,+} &= \left(\frac{1}{b-a}\int_a^b (t-a)^{\frac{1}{r}} \diamond_\alpha t\right)^r,\\
h_0(x,a) &= 1,\\
h_k(x,a) &= \int_a^x h_{k-1}(\tau,a)\Delta\tau, \quad k \in \mathbb{N}, x \in [a,b]_{\mathbb{T}}.
\end{aligned}$$

We have

$$h_n(x,a) \le \frac{(x-a)^n}{n!}, \quad n \in \mathbb{N}, x \in [a,b]_{\mathbb{T}}.$$

Definition 2.4 ([2])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{C}_{rd}(\mathbb{T})$.

The set of functions $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ that are n-times rd-continuously Δ-differentiable on $[a,b]$ is denoted by $\mathcal{C}^n_{rd}([a,b]_{\mathbb{T}})$.

For some of our main results we will use Taylor's formula.

Theorem 2.9 ([2], Taylor's formula). *Let* $f \in \mathcal{C}_{rd}^{n+1}([a,b]_{\mathbb{T}})$. *Then*

$$f(x) = \sum_{k=0}^{n} h_k(x,a) f^{\Delta^k}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) f^{\Delta^{n+1}}(\tau)\Delta\tau, \quad x \in [a,b]_{\mathbb{T}}.$$

We also need the following well-known inequality for proving our results.

Theorem 2.10 ([4], Hölder's inequality). *Let* $a, b \in \mathbb{T}$, $a < b$. *For rd-continuous functions* $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *we have*

$$\int_a^b |f(t)g(t)|\Delta\tau \leq \left(\int_a^b |f(t)|^p \Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Delta t\right)^{\frac{1}{q}},$$

where $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main Results

In this section, we attempt to establish several new inequalities for strongly r-convex functions on time scales by virtue of some notions and results and by introducing new quantities.

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds, for all $x, y \in I$ and $\lambda \in [0,1]$. Also, for the convex function $f : [a,b] \to \mathbb{R}$ the following inequality is known as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Definition 3.1. Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \to \mathbb{R}$ is called strongly r-convex function with modulus c, if

$$f(\lambda x + (1-\lambda)y) \leq (\lambda (f(x))^r + (1-\lambda)(f(y))^r)^{\frac{1}{r}} - c\lambda(1-\lambda)(x-y)^2,$$

for any pair of $x, y \in I$, $t \in [0,1]$ and $r \neq 0$. If we take $c = 0$, we have the definition of r-convexity of the function f.

We can extend the above definition on any time scale $\mathbb{T}$. From now on, we suppose that $[a,b]_{\mathbb{T}}$ is an interval in $\mathbb{T}$. Note that, if $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is positive strongly r-convex function with modulus c, we have

$$f(t) \leq \left(\frac{b-t}{b-a}(f(a))^r + \frac{t-a}{b-a}(f(b))^r\right)^{\frac{1}{r}} - c(b-t)(t-a), \quad t \in [a,b]_{\mathbb{T}}. \tag{3.1}$$

Now we are in a position to present our first result.

Theorem 3.1. *Suppose that* $0 < r \leq 1$ *and* $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *is a positive strongly* r*-convex function and* $\diamond_\alpha$*-integrable on* $[a,b]_{\mathbb{T}}$. *Then*

$$\frac{1}{b-a}\int_a^b f(t) \diamond_\alpha t \leq \frac{1}{(b-a)^{\frac{1}{r}}}\Big(x_{\alpha,r,-}\big(f(a)\big)^r + x_{\alpha,r,+}(f(b))^r\Big)^{\frac{1}{r}}$$

$$+ c\left(ab - (a+b)x_\alpha + x_{\alpha,\alpha}\right).$$

Proof. By taking the diamond-α integral side by side in (3.1), we have

$$\begin{aligned}
\int_a^b f(t) \diamond_\alpha t \leq& \int_a^b \left(\frac{b-t}{b-a}(f(a))^r + \frac{t-a}{b-a}(f(b))^r\right)^{\frac{1}{r}} \diamond_\alpha t \\
&- c\int_a^b (b-t)(t-a) \diamond_\alpha t \\
=& \left(\int_a^b \left(\left(\left(\frac{b-t}{b-a}\right)^{\frac{1}{r}} f(a)\right)^r + \left(\left(\frac{t-a}{b-a}\right)^{\frac{1}{r}} f(b)\right)^r\right)^{\frac{1}{r}} \diamond_\alpha t\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a) \diamond_\alpha t.
\end{aligned}$$

Now, by applying Minkowski's inequality, we find

$$\begin{aligned}
\int_a^b f(t) \diamond_\alpha t \leq& \left(\left(\int_a^b \left(\frac{b-t}{b-a}\right)^{\frac{1}{r}} f(a) \diamond_\alpha t\right)^r + \left(\int_a^b \left(\frac{t-a}{b-a}\right)^{\frac{1}{r}} f(b) \diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a) \diamond_\alpha t \\
=& \left((f(a))^r \left(\int_a^b \left(\frac{b-t}{b-a}\right)^{\frac{1}{r}} \diamond_\alpha t\right)^r + (f(b))^r \left(\int_a^b \left(\frac{t-a}{b-a}\right)^{\frac{1}{r}} \diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a) \diamond_\alpha t \\
=& \frac{1}{(b-a)^{\frac{1}{r}}} \left((f(a))^r \left(\int_a^b (b-t)^{\frac{1}{r}} \diamond_\alpha t\right)^r + (f(b))^r \left(\int_a^b (t-a)^{\frac{1}{r}} \diamond_\alpha t\right)^r\right)^{\frac{1}{r}} \\
&- c\int_a^b (b-t)(t-a) \diamond_\alpha t \\
=& \frac{b-a}{(b-a)^{\frac{1}{r}}}\left(x_{\alpha,r,-}(f(a))^r + x_{\alpha,r,+}(f(b))^r\right)^{\frac{1}{r}} - c\int_a^b \left(bt - ab - t^2 + at\right) \diamond_\alpha t \\
=& \frac{b-a}{(b-a)^{\frac{1}{r}}}\left(x_{\alpha,r,-}(f(a))^r + x_{\alpha,r,+}(f(b))^r\right)^{\frac{1}{r}} \\
&- (b-a)c\left(-ab + (a+b)x_\alpha - x_{\alpha,\alpha}\right),
\end{aligned}$$

whereupon we obtain the desired inequality. This completes the proof. □

Before we establish the next result, we provide the following integral identity.

Lemma 3.1. *Let $f \in \mathcal{C}_{rd}^n([a,b]_{\mathbb{T}})$. Then*

$$\int_a^b f(s)\Delta s = \sum_{k=1}^n h_k(b,a) f^{\Delta^{k-1}}(a) + \int_a^{\rho^n(b)} h_n(b,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau. \tag{3.2}$$

Proof. Let
$$g(x) = \int_a^x f(s)\Delta s, \quad x \in [a,b]_{\mathbb{T}}.$$
By Taylor's formula, we have
$$g(x) = \sum_{k=0}^{n} h_k(x,a) g^{\Delta^k}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) g^{\Delta^{n+1}}(\tau)\Delta\tau,$$
for $x \in [a,b]_{\mathbb{T}}$. Hence,
$$\int_a^x f(s)\Delta s = \sum_{k=1}^{n} h_k(x,a) f^{\Delta^{k-1}}(a) + \int_a^{\rho^n(x)} h_n(x,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau,$$
for $x \in [a,b]_{\mathbb{T}}$. By the last equality, for $x = b$, we find (3.2). This completes the proof. □

Set
$$I(a,b,n,f) = \int_a^b f(s)\Delta s - \sum_{k=1}^{n} h_k(b,a) f^{\Delta^{k-1}}(a),$$
$$y_\alpha = \frac{1}{b-a}\int_a^b t\Delta t,$$
$$y_{\alpha,\alpha} = \frac{1}{b-a}\int_a^b t^2\Delta t,$$
$$y_{\alpha,r,-} = \left(\frac{1}{b-a}\int_a^b (b-t)^{\frac{1}{r}}\Delta t\right)^r,$$
$$y_{\alpha,r,+} = \left(\frac{1}{b-a}\int_a^b (t-a)^{\frac{1}{r}}\Delta t\right)^r.$$

Theorem 3.2. *Let $f \in \mathcal{C}_{rd}^n([a,b]_{\mathbb{T}})$, $r > 0$, $q > 1$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$. Then*
$$\frac{1}{b-a}|I(a,b,n,f)| \leq \frac{(b-a)^{n-\frac{1}{rq}}}{n!} 2^{\frac{1}{q}} \Bigg(2^{\frac{1}{rq}} |f^{\Delta^n}(b)| y_{\alpha,r,+}^{\frac{1}{rq}} + 2^{\frac{1}{rq}} |f^{\Delta^n}(a)| y_{\alpha,r,-}^{\frac{1}{rq}} + c^{\frac{1}{q}} (b-a)^{\frac{1}{rq}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{q}} \Bigg).$$

Proof. Since $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$, applying (3.1), we have

(3.3)
$$\left|f^{\Delta^n}(x)\right|^q \leq \left(\frac{b-x}{b-a}\left|f^{\Delta^n}(a)\right|^{qr} + \frac{x-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}} - c(b-x)(x-a), \quad x \in [a,b]_{\mathbb{T}}.$$

Hence, by Lemma 3.1 and the inequality
$$(x+y)^k \leq 2^k(x^k + y^k), \quad x, y, k > 0,$$

we get

$$\begin{aligned}
|I(a,b,n,f)| =& \left|\int_a^{\rho^n(b)} h_n(b,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau\right| \\
\leq& \int_a^{\rho^n(b)} |h_n(b,\sigma(\tau))| \left|f^{\Delta^n}(\tau)\right| \Delta\tau \quad \text{(By Hölder's inequality)} \\
\leq& \left(\int_a^{\rho^n(b)} |h_n(b,\sigma(\tau))|^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^{\rho^n(b)} \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& \left(\int_a^{\rho^n(b)} \left(\frac{(b-\sigma(\tau))^n}{n!}\right)^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^{\rho^n(b)} \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left(\left(\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right.\right.\right. \\
&\left.\left.\left. + \frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}\right)^{\frac{1}{r}} - c(b-\tau)(\tau-a)\right)\Delta\tau\right)^{\frac{1}{q}} \\
=& (b-a)^{\frac{1}{p}} \left(\frac{(b-a)^n}{n!}\right) \left(\int_a^b \left(\left(\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right.\right.\right. \\
&\left.\left.\left. + \frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}\right)^{\frac{1}{r}} - c(b\tau-ab-\tau^2+a\tau)\right)\Delta\tau\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}} \frac{(b-a)^n}{n!} \left(\frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}} \int_a^b (\tau-a)^{\frac{1}{r}}\Delta\tau\right. \\
&\left. + \frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}} \int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau\right)^{\frac{1}{q}} \\
&\left. + c\int_a^b \tau^2\Delta\tau - c(a+b)\int_a^b \tau\Delta\tau + abc(b-a)\right)^{\frac{1}{q}} \\
=& (b-a)^{\frac{1}{q}} \frac{(b-a)^n}{n!} \left(\frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a) y_{\alpha,r,+}^{\frac{1}{r}}\right. \\
&\left. + \frac{2^{\frac{1}{r}}\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a) y_{\alpha,r,-}^{\frac{1}{r}} + c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab)\right)^{\frac{1}{q}} \\
\leq& (b-a)^{\frac{1}{p}+\frac{1}{q}} \frac{(b-a)^n}{n!} 2^{\frac{1}{q}} \left(\frac{2^{\frac{1}{rq}}\left|f^{\Delta^n}(b)\right|}{(b-a)^{\frac{1}{rq}}} y_{\alpha,r,+}^{\frac{1}{rq}}\right.
\end{aligned}$$

$$+\frac{2^{\frac{1}{rq}}|f^{\Delta^n}(a)|}{(b-a)^{\frac{1}{rq}}}y_{\alpha,r,-}^{\frac{1}{rq}}+c^{\frac{1}{q}}|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|^{\frac{1}{q}}\Bigg),$$

whereupon we get the desired result. This completes the proof. □

The next result reads as follows.

Theorem 3.3. *Let $r>0$, $q\geq 1$, $f\in \mathcal{C}_{rd}^n([a,b]_{\mathbb{T}})$, $\left|f^{\Delta^n}\right|\geq 1$ on $[a,b]_{\mathbb{T}}$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$. Then*

$$\frac{1}{b-a}|I(a,b,n,f)|\leq\frac{(b-a)^n}{n!}\Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\Bigg(\left|f^{\Delta^n}(b)\right|^q y_{\alpha,r,+}^{\frac{1}{r}}+\left|f^{\Delta^n}(a)\right|^q y_{\alpha,r,-}^{\frac{1}{r}}\Bigg)$$
$$+c\left(y_{\alpha,\alpha}-(a+b)y_\alpha+ab\right)\Bigg).$$

Proof. Because $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_{\mathbb{T}}$, the inequality (3.3) holds. Hence,

$$\begin{aligned}|I(a,b,n,f)|=&\left|\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau\right|\\
\leq&\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\int_a^b h_n(b,\tau)\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\leq h_n(b,a)\int_a^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\frac{(b-a)^n}{n!}\int_a^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\\
\leq&\frac{(b-a)^n}{n!}\Bigg(\int_a^b\left(\frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr}+\frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}}\Delta\tau\\
&-c\int_a^b(b-\tau)(\tau-a)\Delta\tau\Bigg).\end{aligned}$$

Now, using the inequality

$$(x+y)^k\leq 2^k\left(x^k+y^k\right),\quad x,y>0,k>0,$$

we have

$$\begin{aligned}|I(a,b,n,f)|\leq&\frac{(b-a)^n}{n!}\Bigg(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}\int_a^b(b-\tau)^{\frac{1}{r}}\Delta\tau+2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}\int_a^b(\tau-a)^{\frac{1}{r}}\Delta\tau\\
&-c\int_a^b\left(-ab+(a+b)\tau-\tau^2\right)\Delta\tau\Bigg)\\
=&\frac{(b-a)^n}{n!}\Bigg(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a)y_{\alpha,r,-}^{\frac{1}{r}}+2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}(b-a)y_{\alpha,r,+}^{\frac{1}{r}}\end{aligned}$$

$$+ (b-a)c\left(y_{\alpha,\alpha} - (a+b)y_\alpha + ab\right)\Bigg).$$

From here,

$$\frac{1}{b-a}|I(a,b,n,f)| \leq \frac{(b-a)^n}{n!}\left(2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(a)\right|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}} + 2^{\frac{1}{r}}\frac{\left|f^{\Delta^n}(b)\right|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right.$$
$$\left. + c\left(y_{\alpha,\alpha} - (a+b)y_\alpha + ab\right)\right).$$

This completes the proof. □

Now we present another inequality for this class of functions by different approach.

Theorem 3.4. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $r>0$, $f \in \mathcal{C}_{rd}^n([a,b]_\mathbb{T})$ and $\left|f^{\Delta^n}\right|^q$ is a strongly r-convex function with modulus c on $[a,b]_\mathbb{T}$. Then*

$$\frac{1}{b-a}|I(a,b,n,f)| \leq \left(\frac{1}{p} - r_0\right)\left(\frac{(b-a)^n}{n!}\right)^p + \left(\frac{1}{q} - r_0\right)\left(2^{\frac{1}{r}}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}}\right.\right.$$
$$\left.\left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right) + c(y_{\alpha,\alpha} - (a+b)y_\alpha + ab)\right)$$
$$+ 2^{\frac{3}{2}}\frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}}\left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}}\left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}}\right)^{\frac{1}{2}}\right.$$
$$\left. + c^{\frac{1}{2}}|y_{\alpha,\alpha} - (a+b)y_\alpha + ab|^{\frac{1}{2}}\right),$$

where $r_0 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof. Since $\left|f^{\Delta^n}\right|^q$ is strongly r-convex with modulus c on $[a,b]_\mathbb{T}$, we have (3.3). Now, using the refinement of Young inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} - r_0(x^{\frac{p}{2}} - y^{\frac{q}{2}})^2, \quad x,y \geq 0,$$

we have

$$\begin{aligned}
|I(a,b,n,f)| &= \left|\int_a^{\rho^n(b)} h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau\right| \\
&\leq \int_a^{\rho^n(b)} h_n(b,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|\Delta\tau \\
&\leq \int_a^b h_n(b,\tau)\left|f^{\Delta^n}(\tau)\right|\Delta\tau \\
&\leq \frac{1}{p}\int_a^b (h_n(b,\tau))^p\,\Delta\tau + \frac{1}{q}\int_a^b \left|f^{\Delta^n}(\tau)\right|^q\Delta\tau \\
&\quad - r_0\int_a^b\left(((h_n(b,\tau))^{\frac{p}{2}} - \left|f^{\Delta^n}(\tau)\right|^{\frac{q}{2}}\right)^2\Delta\tau
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)\int_a^b \left(\frac{b-\tau}{b-a}\left|f^{\Delta^n}(a)\right|^{qr} + \frac{\tau-a}{b-a}\left|f^{\Delta^n}(b)\right|^{qr}\right)^{\frac{1}{r}}\Delta\tau \\
&\quad - c\left(\frac{1}{q} - r_0\right)\int_a^b (b-\tau)(\tau-a)\Delta\tau + 2r_0\int_a^b (h_n(b,\tau))^{\frac{p}{2}}|f^{\Delta^n}(\tau)|^{\frac{q}{2}}\Delta\tau \\
&\le \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}\int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau \right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}\int_a^b (\tau-a)^{\frac{1}{r}}\Delta\tau\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\left(\int_a^b (h_n(b,\tau))^p\Delta\tau\right)^{\frac{1}{2}}\left(\int_a^b |f^{\Delta^n}(\tau)|^q\Delta\tau\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}(b-a)\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\left(\int_a^b \left(\frac{b-\tau}{b-a}|f^{\Delta^n}(a)|^{qr} + \frac{\tau-a}{b-a}|f^{\Delta^n}(b)|^{qr}\right)^{\frac{1}{r}}\right. \\
&\quad \left. - c\int_a^b (b-\tau)(\tau-a)\Delta\tau\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{p} - r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p \\
&\quad + \left(\frac{1}{q} - r_0\right)2^{\frac{1}{r}}(b-a)\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,-}^{\frac{1}{r}} + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y_{\alpha,r,+}^{\frac{1}{r}}\right) \\
&\quad + \left(\frac{1}{q} - r_0\right)c(b-a)(y_{\alpha,\alpha} - (a+b)y_\alpha + ab) \\
&\quad + 2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\left(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\left(|f^{\Delta^n}(a)|^q\int_a^b (b-\tau)^{\frac{1}{r}}\Delta\tau\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+|f^{\Delta^n}(b)|^q\int_a^b(\tau-a)^{\frac{1}{r}}\Delta\tau\Bigg)\\
&+c(b-a)|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|\Bigg)^{\frac{1}{2}}\\
=&\left(\frac{1}{p}-r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p\\
&+\left(\frac{1}{q}-r_0\right)2^{\frac{1}{r}}(b-a)\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,-}+\frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,+}\right)\\
&+\left(\frac{1}{q}-r_0\right)c(b-a)(y_{\alpha,\alpha}-(a+b)y_\alpha+ab)\\
&+2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\Bigg(|f^{\Delta^n}(a)|^q(b-a)y^{\frac{1}{r}}_{\alpha,r,-}\\
&+|f^{\Delta^n}(b)|^q(b-a)y^{\frac{1}{r}}_{\alpha,r,+}\Bigg)\\
&+c(b-a)|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|\Bigg)^{\frac{1}{r}}\\
=&\left(\frac{1}{p}-r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p\\
&+\left(\frac{1}{q}-r_0\right)(b-a)\Bigg(2^{\frac{1}{r}}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,-}+\frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,+}\right)\\
&+c(y_{\alpha,\alpha}-(a+b)y_\alpha+ab)\Bigg)\\
&+2r_0\frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}}\Bigg(\frac{2^{\frac{1}{r}}}{(b-a)^{\frac{1}{r}}}\Bigg(|f^{\Delta^n}(a)|^q(b-a)y^{\frac{1}{r}}_{\alpha,r,-}\\
&+|f^{\Delta^n}(b)|^q(b-a)y^{\frac{1}{r}}_{\alpha,r,+}\Bigg)\\
&+c(b-a)|y_{\alpha,\alpha}-(a+b)y_\alpha+ab|\Bigg)^{\frac{1}{2}}\\
\leq&\left(\frac{1}{p}-r_0\right)(b-a)\left(\frac{(b-a)^n}{n!}\right)^p\\
&+\left(\frac{1}{q}-r_0\right)(b-a)\Bigg(2^{\frac{1}{r}}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,-}+\frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}}y^{\frac{1}{r}}_{\alpha,r,+}\right)\\
&+c(y_{\alpha,\alpha}-(a+b)y_\alpha+ab)\Bigg)
\end{aligned}
$$

$$+ 2^{\frac{3}{2}} r_0 \frac{(b-a)^{\frac{np+1}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q (b-a) y_{\alpha,r,-}^{\frac{1}{r}} \right.\right.$$

$$\left. + |f^{\Delta^n}(b)|^q (b-a) y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}}$$

$$\left. + c^{\frac{1}{2}} (b-a)^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).$$

This completes the proof. □

Remark 3.1. According to Theorem 3.4 we conclude that:

- If $p > q$, then we have

$$\frac{1}{b-a} |I(a,b,n,f)| \leq \frac{p-q}{pq} \left(2^{\frac{1}{r}} \left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,-}^{\frac{1}{r}} \right.\right.$$

$$\left.\left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{\frac{1}{r}}} y_{\alpha,r,+}^{\frac{1}{r}} \right) + c(y_{\alpha,\alpha} - (a+b) y_\alpha + ab) \right)$$

$$+ 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right.$$

$$\left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).$$

- If $p < q$, then we obtain

$$\frac{1}{b-a} |I(a,b,n,f)| \leq \frac{q-p}{pq} \left(\frac{(b-a)^n}{n!} \right)^p$$

$$+ 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right.$$

$$\left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).$$

- If $p = q = 2$, then we have

$$\frac{1}{b-a} |I(a,b,n,f)| \leq 2^{\frac{3}{2}} \frac{(b-a)^{\frac{np}{2}}}{(n!)^{\frac{p}{2}}} \left(\frac{2^{\frac{1}{2r}}}{(b-a)^{\frac{1}{2r}}} \left(|f^{\Delta^n}(a)|^q y_{\alpha,r,-}^{\frac{1}{r}} + |f^{\Delta^n}(b)|^q y_{\alpha,r,+}^{\frac{1}{r}} \right)^{\frac{1}{2}} \right.$$

$$\left. + c^{\frac{1}{2}} |y_{\alpha,\alpha} - (a+b) y_\alpha + ab|^{\frac{1}{2}} \right).$$

Remark 3.2. If $\mathbb{T} = \mathbb{R}$, then the delta and nabla derivatives coincide with the classical derivative. Hence, diamond-α integral from a to t of f will be reduced to classical integral.

References

[1] N. Atasever and B. Kaymakçalan, G. Lešaja and K. Taş, *Generalized diamond-α dynamic opial inequalities*, Adv. Difference Equ. **2012**(109) (2012), 1–9. https://doi.org/10.1186/1687-1847-2012-109

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston and Berlin, 2001. https://doi.org/10.1007/978-1-4612-0201-1

[3] F. Chen and X. Liu, *Refinements on the Hermite-Hadamard inequalities for r-convex functions*, J. Appl. Math. **2013** (2013), Article ID 978493, 5 pages. https://doi.org/10.1155/2013/978493

[4] S. Georgiev, *Integral Inequalities on Time Scales*, De Gruyter, 2020. https://doi.org/10.1515/9783110705553

[5] G. S. Guseinov and B. Kaymakçalan, *Basics of Riemann delta and nabla integration on time scales*, J. Difference Equ. Appl. **8**(11) (2002), 1001–1017. https://doi.org/10.1080/10236190290015272

[6] A. F. Güvenilir, B. Kaymakçalan and N. N. Pelen, *Constantin's inequality for nabla and diamond-alpha derivative*, J. Inequal. Appl. **2015**(167) (2015), 1–17. https://doi.org/10.1186/s13660-015-0681-9

[7] İ. İscan, H. Kadakal and M. Kadakal, *Some new integral inequalities for n-times differentiable log-convex functions*, New Trends in Mathematical Sciences **5**(2) (2017), 10–15. http://dx.doi.org/10.20852/ntmsci.2017.150

[8] H. Kadakal, *New inequalities for strongly r-convex functions*, J. Funct. Spaces **2019** (2019), Article ID 1219237, 10 pages. https://doi.org/10.1155/2019/1219237

[9] M. Kadakal, H. Kadakal and İ. İscan, *Some new integral inequalities for n-times differentiable s-convex functions in the first sense*, Turkish Journal of Analysis and Number Theory **5**(2) (2017), 63–6. https://doi.org/10.12691/tjant-5-2-4

[10] M. Kadakal, H. Kadakal and İ. İscan, *Some new integral inequalities for n-times differentiable strongly convex functions*, Karaelmas Science and Engineering Journal **8**(1) (2018), 147–150. https://doi.org/10.7212/2Fzkufbd.v8i1.919

[11] A. Malinowska and D. Torres, *On the diamond-α Riemann integral and mean value theorems on time scales*, Dynamic Systems and Applications **18** (2009), 469–482.

[12] J. W. Rogers Jr. and Q. Sheng, *Notes on the diamond-α dynamic derivative on time scales*, J. Math. Anal. Appl. **326**(1) (2007), 228–241. https://doi.org/10.1016/j.jmaa.2006.03.004

[13] Q. Sheng, M. Fadag, J. Henderson and J. M. Davis, *An exploration of combined dynamic derivatives on time scales and their applications*, Nonlinear Anal. Real World Appl. **7**(3) (2006), 395–413. https://doi.org/10.1016/j.nonrwa.2005.03.008

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ON THE EXISTENCE AND UNIQUENESS OF FUZZY MILD SOLUTION OF FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, the nonlocal Cauchy problem is discussed for the fuzzy fractional evolution equations in an arbitrary Banach space for order $q \in (1, 2)$ and the criteria on the existence and uniqueness of mild fuzzy solutions are obtained by using Schauder's fixed point theorem. An example to illustrate the applications of main results is also given.

1. Introduction

Fuzzy set theory has been attracting increasing interest in recent years as it is widely used in several fields such as mechanics, electrical engineering, signal processing, etc. As a result, in recent decades, fuzzy set theory has become a hot and current topic and has received much attention from researchers (see for instance [16, 17]).

Note that Kaleva [11] discussed the properties of differentiable fuzzy set-valued mappings by means of the concept of H-differentiability due to Puri and Ralescu [12], gave the existence and uniqueness theorem for a solution of the fuzzy differential equation

$$u'(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1.1}$$

when $f : I \times E^n \to E^n$ satisfies the Lipschitz condition.

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In [13] Bhaskar Dubey and Raju K. George studied the linear time-invariant systems with fuzzy initial condition

$$u'(t) = Au(t) + Bc(t), \quad u(t_0) = u_0, \tag{1.2}$$

where $c(t) \in (E^1)^m$ a control and A, B, are $n \times n, n \times m$ real matrices, respectively, $t_0 \geq 0$.

In [14] Nguyen Thi Kim Son demonstrate the efficiency of theoretical results by studying the existence of fuzzy mild solutions of nonlinear fuzzy fractional evolution equations

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^q x(t) = Ax(t) + f(t, x(t)), \quad t \in [0, a], \\ x(0) = \varphi_0, \end{cases} \tag{1.3}$$

where ${}_{gH}^{C}\mathcal{D}^q$ is the fuzzy Caputo fractional derivative of order $q \in (0, 1)$, and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $\mathcal{T}$ the set of all triangular fuzzy numbers.

Building on this work, we have opted for the fuzzy Caputo derivative to prove the existence and uniqueness of the soft solution of the fuzzy initial value problem of the fractional evolution equation of order $q \in (1, 2)$

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^q x(t) = Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathcal{D}^{q-1} x(t)\right), \quad t \in [0, a], 1 < q < 2, \\ x(0) = x_0, \\ {}_{gH}^{C}\mathcal{D}^{q-i} x(0) = x_i, \quad i = 1, \dots, |q|, \end{cases} \tag{1.4}$$

where A is a linear operator and f is a continuous function.

The purpose of this study is to develop an original kind of fuzzy initial value problem of the fractional evolution equation of order $q \in (1, 2)$ utilizing fuzzy Caputo derivative of order $q \in (1, 2)$, and also to show the existence and uniqueness of its mild solutions.

The following is a breakdown of the paper's structure. After this Introduction we give Preliminaries which will be used throughout this paper, Fractional Integral Equation can be found in Section 3, The existence of mild solutions of the Cauchy problem for fractional evolution equations is studied in Section 4. In Section 5,we provide an example to present the applications of the results obtained in the abstract.

2. Preliminaries

In this part we recall some basic notions that will be useful in the rest of our article.

2.1. The metric space E^1.

Definition 2.1. A fuzzy number is a fuzzy set $x : \mathbb{R} \to [0, 1]$ that satisfies the following conditions:

1. x is normal, i.e., there is a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$;
2. x is a fuzzy convex set;

3. x is upper semi-continuous;
4. x closure of $\{t \in \mathbb{R} : x(t) > 0\}$ is compact.

We denote by E^1 the space of all fuzzy numbers on $\mathbb{R}$.

$$E^1 = \{x : \mathbb{R} \to [0,1] : x \text{ satisfies 1–4. below}\}.$$

For all $\alpha \in (0,1]$ the α-cut of an element of E^1 is defined by

$$x^\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}.$$

By the former parcels we can write

$$x^\alpha = [\underline{x}(\alpha), \bar{x}(\alpha)]. \tag{2.1}$$

The distance between two element of E^1 is given by (see [1])

$$d(x,y) = \sup_{\alpha \in (0,1]} \max\{|\bar{x}(\alpha) - \bar{y}(\alpha)|, |\underline{x}(\alpha) - \underline{y}(\alpha)|\}. \tag{2.2}$$

And the following properties are valid:

1. $d(x+z, y+z) = d(x,y)$;
2. $d(\lambda x, \lambda y) = |\lambda| d(x,y)$;
3. $d(x+y, w+z) \leq d(x,w) + d(y,z)$.

The operations of addition and scalar multiplication of fuzzy numbers on $\mathbb{R}_{\mathcal{F}}$ have the form

$$[x \oplus y]^\alpha = [x]^\alpha + [y]^\alpha \quad \text{and} \quad [\lambda \odot x]^\alpha = \lambda [x]^\alpha, \quad \lambda \in \mathbb{R}, \tag{2.3}$$

where

$$[x]^\alpha + [y]^\alpha = \{a + b : a \in [x]^\alpha, b \in [y]^\alpha\} \tag{2.4}$$

is the Minkowski sum of $[x]^\alpha$ and $[y]^\alpha$ and

$$\lambda [x]^\alpha = \{\lambda a : a \in [x]^\alpha\}. \tag{2.5}$$

For $x, y \in \mathbb{R}_{\mathcal{F}}$, the gH difference [2] of x and y, denoted by $x \ominus_{gH} y$, is defined as the element $z \in \mathbb{R}_{\mathcal{F}}$ such that

$$x \ominus_{gH} y = z \Leftrightarrow \{\text{(i) } x = y + z \quad \text{or} \quad \text{(ii) } y = x + (-1)z\}. \tag{2.6}$$

In terms of α-levels we have

$$(x \ominus_{gH} y)^\alpha = [\min\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}, \max\{\underline{x}(\alpha) - \underline{y}(\alpha), \bar{x}(\alpha) - \bar{y}(\alpha)\}].$$

And the conditions for the existence of $z = x \ominus_g y \in E^1$ are:

case (i):

$$\begin{cases} \underline{z}(\alpha) = \underline{x}(\alpha) - \underline{y}(\alpha) \text{ and } \overline{z}(\alpha) = \overline{x}(\alpha) - \overline{y}(\alpha), \\ \text{with } \underline{z}(\alpha) \text{ increasing, } \overline{z}(\alpha) \text{ decreasing, } \underline{z}(\alpha) \leq \overline{z}(\alpha); \end{cases} \tag{2.7}$$

case (ii):

$$\begin{cases} \underline{z}(\alpha) = \overline{x}(\alpha) - \overline{y}(\alpha) \text{ and } \overline{z}(\alpha) = \underline{x}(\alpha) - \underline{y}(\alpha), \\ \text{with } \underline{z}(\alpha) \text{ increasing, } \overline{z}(\alpha) \text{ decreasing, } \underline{z}(\alpha) \leq \overline{z}(\alpha), \end{cases} \tag{2.8}$$

for all $\alpha \in [0,1]$.

In general, with $x \in \mathbb{R}_{\mathcal{F}}$, there does not exist $y \in \mathbb{R}_{\mathcal{F}}$ such that $x \oplus y = 0$. Then, unfortunately, $\mathbb{R}_{\mathcal{F}}$ is not a linear space with addition and scalar multiplication. Consequently, $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|)$ is not a Banach space, where $\|x\| = d_\infty(x, \hat{0})$, $x \in \mathbb{R}_{\mathcal{F}}$.

Denote $\mathcal{T}$ by the set of all triangular fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$. $(\mathcal{T}, d_\infty)$ is a subspace of the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$. It is a complete metric space. Moreover, Bede [3] showed that if $x, y \in \mathcal{T}$, then the difference $x \ominus_{gH} y$ always exists in $\mathcal{T}$ and $x \ominus_{gH} y = (-1) \odot (y \ominus_{gH} x)$.

Let X be a subset of $\mathbb{R}_{\mathcal{F}}$, $J \subset \mathbb{R}$, and denote $\mathcal{C}(J, X)$ by the set of all continuous mappings $f : J \to X$.

2.2. **Hukuhara's derivative.** Let $f : [a,b] \subset \mathbb{R} \to E^1$ a fuzzy-valued function. The α-level of f is given by

$$f(t,\alpha) = \left[\underline{f}(t,\alpha), \overline{f}(t,\alpha)\right], \quad \text{for all } t \in [a,b], \alpha \in [0,1].$$

Definition 2.2 ([4]). Let $t_0 \in (a,b)$ and h be such that $t_0 + h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy value function $f : (a,b) \to E^1$ at t_0 is defined as

$$\lim_{h \to 0} \left\| \frac{f\left(t_0 + h\right) -_g f\left(t_0\right)}{h} -_g f'_{gH}\left(t_0\right) \right\|_1 = 0. \tag{2.9}$$

If $f_{gH}\left(t_0\right) \in E^1$ satisfying (3.4) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.3 ([4]). Let $f : [a,b] \to E^1$ and $t_0 \in (a,b)$, with $\underline{f}(t,\alpha)$ and $\bar{f}(t,\alpha)$ both differentiable at t_0.

We say that

1. f is $[(i) - gH]$-differentiable at t_0 if

$$f'_{i,gH}\left(t_0\right) = \left[\underline{f}'(t,\alpha), \bar{f}'(t,\alpha)\right]; \tag{2.10}$$

2. f is $[(ii) - gH]$-differentiable at t_0 if

$$f'_{ii,gH}\left(t_0\right) = \left[\bar{f}'(t,\alpha), \underline{f}'(t,\alpha)\right]. \tag{2.11}$$

Theorem 2.1 ([6]). *Let $f : J \subset \mathbb{R} \to E^1$ and $\phi : J \to \mathbb{R}$ and $t \in J$. Suppose that $\phi(t)$ is differentiable function at t and the fuzzy-valued function $f(t)$ is gH-differentiable at t. So,*

$$(f\phi)'_g(t) = \left(f'\phi\right)_g(t) + \left(f\phi'\right)_g(t). \tag{2.12}$$

Definition 2.4 ([5]). Let $f : [a,b] \to E^1$ and $f'_g H(t)$ be gH-differentiable at $t_0 \in (a,b)$, moreover there isn't any switching point on (a,b) and $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ both differentiable at t_0. We say that

- f' is $[(i) - gH]$-differentiable at t_0, if

$$f''_{i,gH}(t_0) = \left[\underline{f}''(t,\alpha), \overline{f}''(x,\alpha)\right];$$

- f' is $[(ii)-gH]$-differentiable at t_0, if

$$f''_{ii,gH}(t_0)=\left[\overline{f}''(t,\alpha),\underline{f}''(t,\alpha)\right].$$

2.3. **Fuzzy fractional derivative.** We present generalized fuzzy fractional derivative and their properties.

Definition 2.5 ([9])**.** Let $f\in L^{E^1}([a,b])$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$I^q_{RL}f(t)=\frac{1}{\Gamma(q)}\odot\int_a^t(t-s)^{q-1}\odot f(s)ds,\quad a<s<t,0<q<1. \tag{2.13}$$

Definition 2.6 ([6], Riemann-Liouville fractional derivative-RL)**.** Let us consider $f\in L^{E^1}([a,b])$ is a fuzzy number valued function,

$$D^q_{RL_{gH}}f(s)=\begin{cases}\frac{1}{\Gamma(n-q)}\odot\left(\frac{d}{ds}\right)^n\int_a^s(s-t)^{n-q-1}\odot f(t)dt, & n-1<q<n,\\ \left(\frac{d}{ds}\right)^{n-1}f(s), & q=n-1.\end{cases} \tag{2.14}$$

Definition 2.7 ([6])**.** In the definition of RL fractional derivative, suppose the integer order of the derivative is an operator inside of the integral and operating on operand function $f(t)\in E^1, t\in[a,b]$. We get the definition of Caputo gH derivative of $f(t)$

$${}^{C}_{gH}D^qf(s)=\begin{cases}\frac{1}{\Gamma(n-q)}\odot\int_a^s(s-t)^{n-q-1}\odot f^{(n)}_{gH}(t)dt, & n-1<q<n,\\ \left(\frac{d}{ds}\right)^{n-1}f(s), & q=n-1.\end{cases} \tag{2.15}$$

Also we say that f is $[(i)-gH]$-differentiable at t_0, if

$${}_{gH}D^q_tf(x,t;\alpha)=\left[D^q\underline{f}(x,t;\alpha),D^q\bar{f}(x,t;\alpha)\right],\quad \text{for all } q\in(0,1), \tag{2.16}$$

and f is $[(ii)-gH]$-differentiable at t_0, if

$${}_{gH}D^q_tf(x,t;\alpha)=\left[D^q\bar{f}(x,t;\alpha),\underline{f}(x,t;\alpha)\right],\quad \text{for all } q\in(0,1). \tag{2.17}$$

Definition 2.8 ([10])**.** Let $f:[0,+\infty)\to X\subset\mathbb{R}_{\mathcal{F}}$ be a continuous function such that $e^{-st}\odot f(t)$ is integrable. Then the fuzzy Laplace transform of f, denoted by $\mathcal{L}[f(t)]$, is

$$\mathcal{L}[f(t)]:=F(s)=\int_0^{+\infty}e^{-st}\odot f(t)dt,\quad s>0. \tag{2.18}$$

A fuzzy-valued function f is exponent bounded of order β if there exists $M>0$ similar that

$$(\exists t_0>0)\, d_{+\infty}(f(t),\hat{0})\le Me^{\beta t},\quad \text{for all } t\ge t_0.$$

Proposition 2.1. *If $x(t)$ is a fuzzy peace-wise continuous function on $[0,+\infty]$ and of exponential order a, then*

$$\boldsymbol{L}((x\star y)(t))=\boldsymbol{L}(x(t))\odot\boldsymbol{L}(y(t)), \tag{2.19}$$

where $y(t)$ is a peace-wise continuous real function on $[0,+\infty)$.

Proof. We have

$$\begin{aligned}\boldsymbol{L}(x(t)) \odot \boldsymbol{L}(y(t)) &= \left(\int_0^{+\infty} e^{-s\tau} \odot x(\tau)d\tau\right) \odot \left(\int_0^{+\infty} e^{-s\sigma} \odot y(\sigma)d\sigma\right)\\ &= \int_0^{+\infty} \left(\int_0^{+\infty} e^{-s(\tau+\sigma)} \odot x(\tau)d\tau\right) \odot y(\sigma)d\sigma.\end{aligned}$$

Let us to hold τ fixed in the interior integral, substituting $t = \tau + \sigma$ and $d\sigma = dt$, we obtain

$$\begin{aligned}\boldsymbol{L}(x(t)) \odot \boldsymbol{L}(y(t)) &= \int_0^{+\infty} \left(\int_\sigma^{+\infty} e^{-st} \odot x(\tau) \odot y(t-\tau)dt\right) d\tau\\ &= \int_0^{+\infty} \int_\sigma^{\infty} e^{-st} \odot x(\tau) \odot y(t-\tau)dtd\tau\\ &= \int_0^{+\infty} e^{-st} \odot \left(\int_0^t x(t-\sigma) \odot y(\sigma)d\tau\right) d\sigma\\ &= \boldsymbol{L}((x \star y)(t)).\end{aligned}$$

□

Definition 2.9 ([15])**.** 1. The Gamma function is given by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \text{for all } x > 0. \tag{2.20}$$

2. The $\mathbb{B}$ function is defined by

$$\mathbb{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}, \quad \text{for all } x, y > 0. \tag{2.21}$$

Proposition 2.2 ([15])**.** 1. *For all $x, y \in \mathbb{R}_+^*$, $\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.*

2. *For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$. It is easy to show the following lemma.*

Proposition 2.3. *For all $\alpha > 0$, we get the following result*

$$\int_0^t E_{\alpha,1}\left(As^\alpha\right) ds = tE_{\alpha,2}\left(At^\alpha\right). \tag{2.22}$$

Proof.

$$\begin{aligned}\int_0^t E_{\alpha,1}\left(As^\alpha\right) ds &= \int_0^t \sum_{n=0}^{+\infty} \frac{s^{n\alpha}}{\Gamma(n\alpha+1)} A^n ds\\ &= \sum_{n=0}^{+\infty} \frac{\int_0^t s^{n\alpha}ds}{\Gamma(n\alpha+1)} A^n\\ &= \sum_{n=0}^{+\infty} \frac{t^{n\alpha+1}}{(n\alpha+1)\Gamma(n\alpha+1)} A^n\\ &= \sum_{n=0}^{+\infty} \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} A^n\\ &= tE_{\alpha,2}\left(At^\alpha\right).\end{aligned}$$

□

Lemma 2.1. *For all $\alpha \in [1,2]$ and $s > 0$, we have*
1. $s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)$;
2. $s^{\alpha-2}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(tE_{\alpha,2}\left(At^{\alpha}\right)\right)(s)$;
3. $\left(s^{\alpha}-A\right)^{-1} = \frac{1}{\Gamma(\alpha-1)}\mathscr{L}\left(\int_0^t (t-s)^{\alpha-2}E_{\alpha,1}\left(As^{\alpha}\right)ds\right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned}\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s) &= \mathscr{L}\left(\sum_{n=0}^{+\infty}\frac{t^{\alpha n}A^n}{\Gamma(\alpha n+1)}\right)\\ &= \sum_{n=0}^{+\infty}\mathscr{L}\left(t^{\alpha n}\right)\frac{A^n}{\Gamma(\alpha n+1)}\\ &= \sum_{n=0}^{+\infty}\frac{1}{s^{n\alpha+1}}A^n\\ &= s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}.\end{aligned}$$

2. For $s > 0$, $s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1} = \mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)$, then

$$\begin{aligned}s^{\alpha-2}\left(s^{\alpha}-A\right)^{-1} &= s^{-1}s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}\\ &= \mathscr{L}(1)(s)\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(1 * E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\int_0^t E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(tE_{\alpha,2}\left(t^{\alpha}A\right)\right)(s).\end{aligned}$$

3. From (1), we get

$$\begin{aligned}\left(s^{\alpha}-A\right)^{-1} &= s^{1-\alpha}\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\right)\mathscr{L}\left(E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * E_{\alpha,1}\left(At^{\alpha}\right)\right)(s)\\ &= \mathscr{L}\left(\int_0^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}E_{\alpha,1}\left(A\delta^{\alpha}\right)d\delta\right)(s),\end{aligned}$$

hence the desired result. □

Lemma 2.2 ([10]). (1) *Let $f, g : [0,+\infty) \to X$ be continuous functions, $c_1, c_2 \in \mathbb{R}^+$. Then*

$$\mathcal{L}\left[c_1 \odot f(t) + c_2 \odot g(t)\right] = c_1 \odot \mathcal{L}[f(t)] + c_2 \odot \mathcal{L}[g(t)].$$

(2) *Let $f : [0,+\infty) \to X$ be a continuous function. Then*

$$\mathcal{L}\left[e^{at} \odot f(t)\right] = F(s-a), \quad s-a > 0.$$

(3) *Let $f \in C^1([0,+\infty), X)$ be exponent bounded of order β. Then*

(i) *if f is $[(i)-gH]$ differentiable, then* $\mathcal{L}\left[\mathcal{D}^{i}_{gH}f(t)\right] = s\odot\mathcal{L}[f(t)]\ominus f(0)$;

(ii) *if f is $[(ii)-gH]$ differentiable, then* $\mathcal{L}\left[\mathcal{D}^{ii}_{gH}f(t)\right] = (-1)\odot f(0)\ominus(-s)\odot\mathcal{L}[f(t)]$.

Now, we recall Schauder's fixed point theorem and the Ascoli-Arzela theorem as follows.

Theorem 2.2 (Schauder fixed point theorem)**.** *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

Theorem 2.3 (Ascoli-Arzela)**.** *Let $\phi_n(t)$ be a sequence of functions from $[a,b]$ to $\mathbb{R}$ which is uniformly bounded and equicontinuous. Then, $\phi_n(t)$ has a uniformly convergent subsequence.*

3. Fuzzy Fractional Integral Equation

In this section, we have performed the Cauchy problem for fuzzy fractional evolution equations involving Caputo gH derivatives

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^q x(t) = Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right), \quad t\in[0,a], \\ x(0)=x_0, \quad 1<q<2, \\ {}^{C}_{gH}\mathcal{D}^{q-i}x(0) = x_i, \quad i=1,\dots,|q|, \end{cases} \tag{3.1}$$

where ${}^{C}_{gH}\mathcal{D}^q$ is the fuzzy Caputo fractional derivative of order $q\in(1,2)$, and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{T}$.

Using Caputo's fuzzy fractional derivative definition, applying the Riemann-Liouville mixed fractional integral operator ${}^{RL}\mathcal{I}^q_{0+}$ member to member in (3.1) and using the Newton-Leibnitz formula for the gH derivative [3], we get

$$x(t)\ominus_{gH} x(0)\ominus_{gH} t\odot x'(0) = {}^{RL}\mathcal{I}^q_{0+}(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right). \tag{3.2}$$

From the definition of gH difference and (3.1), we get the following assertions.

(i) If x is Caputo $[(i)-\text{gH}]$ differentiable, then

$$x(t) = x(0) + t\odot x'(0) + {}^{RL}\mathcal{I}^q_{0+}\left(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right)\right). \tag{3.3}$$

(ii) If x is Caputo $[(ii)-gH]$ differentiable, then

$$x(t) = x(0) + t\odot x'(0)\ominus(-1)\odot {}^{RL}\mathcal{I}^q_{0+}\left(Ax(t) + f\left(t, x(t), {}^{C}_{gH}\mathcal{D}^{q-1}x(t)\right)\right). \tag{3.4}$$

By applying the fuzzy Laplace transform in [10], we obtain the precise integral formula of the Cauchy problem (3.1) as follows.

Lemma 3.1. (*i*) *If x is Caputo $[(i)-gH]$ differentiable satisfying the Cauchy problem* (3.1)*, then*

$$x(t) = E_{q,1}\left(At^q\right)\odot x_0 + t\odot E_{q,2}\left(At^q\right)\odot x_1 \tag{3.5}$$

$$+\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s, x(s), {}^{C}_{gH}\mathcal{D}^{q-1}x(s)\right)d\delta ds. \tag{3.6}$$

(ii) If x is Caputo $[(ii)-gH]$ differentiable satisfying the Cauchy problem (3.1), then

$$x(t) = E_{q,1}(At^q) \odot x_0 + t \odot E_{q,2}(At^q) \odot x_1 \tag{3.7}$$

$$\ominus (-1) \odot \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(A(\delta-s)^q) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds. \tag{3.8}$$

Here, $E_{q,1}(At^q)$ is the Mittag-Leffler function.

Proof. Set

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty e^{-st} \odot x(t)dt \tag{3.9}$$

and

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} \odot f\left(t, x(t), {}_{gH}^{C}\mathcal{D}^{q-1}x(t)\right) dt. \tag{3.10}$$

Case 1. Assume that x is Caputo $[(i)-gH]$ differentiable on $[0,+\infty)$. Then from (3.3) and [10] we have

$$\begin{aligned}
x(t) &= x(0) + t \odot x'(0) + {}^{RL}\mathfrak{I}_{0+}^{q}\left(Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathcal{D}^{q-1}x(t)\right)\right),\\
X(s) &= \mathcal{L}(x(0)) + \mathcal{L}(t \odot x'(0)) + \mathcal{L}\left({}^{RL}\mathfrak{I}_{0+}^{q}\left(Ax(t) + f\left(t, x(t), {}_{gH}^{C}\mathcal{D}^{q-1}x(t)\right)\right)\right),\\
&= \frac{1}{s} \odot x_0 + \frac{1}{s^2} \odot x_1 + \frac{1}{s^q} \odot AX(s) + \frac{1}{s^q} \odot F(s),
\end{aligned}$$

which implies

$$\begin{aligned}
s^q \odot X(s) &= s^{q-1} \odot x_0 + s^{q-2} \odot x_1 + AX(s) + F(s),\\
(s^q \odot Id \ominus A) \odot X(s) &= s^{q-1} \odot x_0 + s^{q-2} \odot x_1 + F(s),\\
X(s) &= (s^q \odot Id \ominus A)^{-1} \odot s^{q-1} \odot x_0 + (s^q \odot Id \ominus A)^{-1} \odot s^{q-2} \odot x_1\\
&\quad + (s^q \odot Id \ominus A)^{-1} \odot F(s)
\end{aligned}$$

or

$$\begin{aligned}
(s^q \odot Id \ominus A)^{-1} \odot s^{q-1} &= \mathcal{L}\left(E_{q,1}(At^q)\right)(s),\\
(s^q \odot Id \ominus A)^{-1} \odot s^{q-2} &= \mathcal{L}\left(t \odot E_{q,2}(At^q)\right)(s),\\
(s^q \odot Id \ominus A)^{-1} &= \frac{1}{\Gamma(q-1)} \odot \mathcal{L}\left(\int_0^t (t-\tau)^{q-2} E_{q,1}(A\tau^q)\, d\tau\right),
\end{aligned}$$

which give

$$X(s) = \mathcal{L}\left(E_{q,1}(At^q)\right) \odot x_0 + \mathcal{L}\left(t \odot E_{q,2}(At^q)\right) \odot x_1 + \mathcal{L}(g * f), \tag{3.11}$$

with $g(t) = \int_0^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(A\tau^q)\, d\tau$.

Applying the inverse Laplace transformation, we get:

$$x(t) = E_{q,1}(At^q) \odot x_0 + t \odot E_{q,2}(At^q) \odot x_1 \tag{3.12}$$

$$+\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds. \tag{3.13}$$

Case 2. Assume that x is Caputo $[(ii)-gH]$ differentiable on $[0,+\infty)$. Then from (3.4) we have in the same way

(3.14)

$$x(t) = E_{q,1}\left(At^q\right) \odot x_0 + t \odot E_{q,2}\left(At^q\right) \odot x_1$$

$$\ominus (-1)\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds. \tag{3.15}$$

This completes the proof. □

4. Existence and Uniqueness of Mild Solutions

Definition 4.1. By a mild fuzzy solution in type 1 of the Cauchy problem (3.1), we mean a function $x \in \mathcal{C}([0,a],\mathcal{T})$ that satisfies (3.5). By a mild fuzzy solution in type 2 of the Cauchy problem (3.1), we mean a function $x \in \mathcal{C}([0,a],\mathcal{T})$ that satisfies (3.7).

The following hypotheses will be used in the next results.

(H1) For almost all $t \in [0,a]$, the function $f \in \mathcal{C}([0,a]\times\mathcal{T}\times\mathcal{T},\mathcal{T})$ is continuous and for each $z \in \mathcal{C}([0,a],\mathcal{T})$, the function $f(\cdot, z, {}_{gH}^{C}\mathcal{D}^{q-1}z) : [0,a] \to \mathcal{T}$ is strongly measurable.

(H2) There exist $q_2 \in [0,q)$, $B_r := \left\{x \in \mathcal{T} : d_\infty(x,\hat{0}) \le r\right\} \subset \mathcal{T}$, $r > 0$, and $\rho(\cdot) \in L^{\frac{1}{q_2}}([0,a],\mathbb{R}^+)$ such that for any $x, y \in \mathcal{C}([0,a],B_r)$ we have

(4.1)

$$d_\infty(f(t,x(t),{}_{gH}^{C}\mathcal{D}^{q-1}x(t)), f(t,y(t),{}_{gH}^{C}\mathcal{D}^{q-1}y(t))) \le \rho(t) d_\infty(x(t),y(t)), \quad t \in [0,a].$$

(H3) There exists a constant $q_1 \in [0,q)$ and $m \in L^{\frac{1}{q_1}}([0,a],\mathbb{R}^+)$ such that

$$d_\infty(f(t,z(t),{}_{gH}^{C}\mathcal{D}^{q-1}z(t)),\hat{0}) \le m(t), \tag{4.2}$$

for all $z \in \mathcal{C}([0,a],\mathcal{T})$ and for almost all $t \in [0,a]$.

(H4) $E_{q,n}\left(At^q\right)$ is a compact operator for every $t > 0$ and $n \in \mathbb{N}$.

Theorem 4.1. *Under hypotheses* (H_1)-(H_4) *the Cauchy problem* (3.1) *has a mild fuzzy solution in type* 1 *in space* $\mathcal{C}([0,a],\mathcal{T})$.

Proof. Let $x \in \mathcal{C}([0,a],\mathcal{T})$. Since x is continuous with respect to t and hypothesis (H_1), $f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s))$ is a measurable function on $[0,a]$. Let

$$b = \frac{q-1}{1-q_1}, \quad M_1 = \|m\|_{L^{\frac{1}{q_1}}[0,a]}. \tag{4.3}$$

For $t \in [0,a]$, by applying Holder's inequality and (H_3), we have

$$d_\infty\left(\int_0^t (t-s)^{q-1} \odot f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s))ds, \hat{0}\right)$$

$$\leq \int_0^t (t-s)^{q-1} \odot d_\infty(f(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)),\hat{0})ds$$
$$\leq \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1} \|m\|_{L^{\frac{1}{q_1}}}[0,t]$$
$$\leq \frac{M_1 a^{(1+b)(1-q_1)}}{(1+b)^{1-q_1}}.$$

Therefore, we obtain

$$d_\infty\left(\int_0^t\int_s^t\left(\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\right.$$
$$\leq \int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot d_\infty\left(E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right),\hat{0}\right)d\delta ds$$
$$\leq \frac{M}{\Gamma(q)}\odot\int_0^t (t-s)^{q-1}d_\infty(f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right),\hat{0})ds$$
$$\leq \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}, \quad \text{for all } t\in[0,a].$$

Then

$$\int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds$$

is bounded for all $t\in[0,a]$.

For $x\in\mathcal{C}([0,a],\mathcal{T})$, we define

$$(F_1x)(t) = E_{q,1}(At^q)\odot x_0 + t\odot E_{q,2}(At^q)\odot x_1, \quad t\in[0,a],$$
$$(F_2x)(t) = \int_0^t\int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds.$$

Set

$$k_0 = M\left(\|x_0\|+a\|x_1\|\right)+\frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}$$

and $\mathcal{B}_{k_0} := \left\{x(\cdot)\in C([0,a],\mathcal{T}) : d_\infty(x(t),\hat{0})\leq k_0 \text{ for all } t\in[0,a]\right\}$. We will prove that F_1x+F_2x has a fixed point on $\mathcal{B}_{k_0}$.

Step 1. We show for every $x\in\mathcal{B}_{k_0}$, $F_1x+F_2x\in\mathcal{B}_{k_0}$. Indeed, with $0\leq t_1\leq t_2\leq a$ we have

$$d_\infty\left((F_2x)(t_2),(F_2x)(t_1)\right)$$
$$=d_\infty\left(\int_0^{t_2}\int_s^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\right.$$
$$\left.\int_0^{t_1}\int_s^{t_1}\frac{(t_1-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\right)$$

$$
\begin{aligned}
=&d_\infty\Bigg(\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_{0}^{t_1}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&-\int_{0}^{t_1}\int_{s}^{t_1}\frac{(t_1-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
=&d_\infty\Bigg(\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_{0}^{t_1}\int_{s}^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&+\int_{0}^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
\leq&d_\infty\left(\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
&+d_\infty\Bigg(\int_{0}^{t_1}\int_{s}^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
&+d_\infty\left(\int_{0}^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
=&I_1+I_2+I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&d_\infty\left(\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
I_2=&d_\infty\Bigg(\int_{0}^{t_1}\int_{s}^{t_1}\frac{[(t_2-\delta)^{q-2}-(t_1-\delta)^{q-2}]}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\Bigg)\\
I_3=&d_\infty\left(\int_{0}^{t_1}\int_{t_1}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right).
\end{aligned}
$$

We have:

$$
I_1=d_\infty\left(\int_{t_1}^{t_2}\int_{s}^{t_2}\frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)
$$

$$\leq \frac{MM_1}{\Gamma(q)} \left(\int_{t_1}^{t_2} (t_2 - s)^{1-q_1} \right)$$
$$\leq \frac{MM_1}{(1+b)^{1-q_1}\Gamma(q)} (t_2 - t_1)^{(b+1)(1-q_1)},$$

also

$$\begin{aligned} I_2 =& d_\infty \Bigg(\int_0^{t_1} \int_s^{t_1} \frac{[(t_2-\delta)^{q-2} - (t_1-\delta)^{q-2}]}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta - s)^q\right) \\ & \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds, \hat{0} \Bigg) \\ \leq & \frac{M}{\Gamma(q)} \Bigg(\int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1} - (t_2-t_1)^{q-1}\right] \\ & \odot d_\infty \left(f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) ds, \hat{0}\right) \Bigg) \\ \leq & \frac{M}{\Gamma(q)} \Bigg(\int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] \odot d_\infty \left(f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) ds, \hat{0}\right) \\ & - \int_0^{t_1} (t_2-t_1)^{q-1} \odot d_\infty \left(f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) ds, \hat{0}\right) \Bigg) \\ \leq & \frac{M}{\Gamma(q)} \left(\int_0^{t_1} \left[(t_2-s)^b - (t_1-s)^b\right]^{1-q_1} \odot M_1 - (t_2-t_1)^{q-1} \odot t_1^{1-q_1} M_1 \right) \\ \leq & \frac{MM_1}{(b+1)^{1-q_1}\Gamma(q)} \left(-(t_2-t_1)^{b+1} + t_2^{b+1} - t_1^{b+1} - (t_2-t_1)^{q-1}(b+1)^{1-q_1} \right). \end{aligned}$$

Likewise

$$\begin{aligned} I_3 &= d_\infty \left(\int_0^{t_1} \int_{t_1}^{t_2} \frac{(t_2-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds, \hat{0} \right) \\ &\leq \frac{M}{\Gamma(q)} \left(\int_0^{t_1} (t_2-t_1)^{q-1} \odot d_\infty \left(f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) ds, \hat{0}\right) \right) \\ &\leq \frac{MM_1(t_2-t_1)^{q-1}}{\Gamma(q)} t_1^{1-q_1}. \end{aligned}$$

Then it is straightforward that I_1, I_2 and I_3 tend to 0 as $t_2 - t_1 \to 0$. So, $(F_2x)(t)$ is continuous in $t \in [0, a]$. It is easy to see that $(F_1x)(t)$ is also continuous in $t \in [0, a]$. Now, for any $x \in \mathcal{B}_{k_0}$ and $t \in [0, a]$, we have

$$d_\infty \left((F_1x)(t) + (F_2x)(t), \hat{0} \right) \leq M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}} \leq k_0. \tag{4.4}$$

Then $F_1 + F_2$ is an operator from $\mathcal{B}_{k_0}$ into $\mathcal{B}_{k_0}$.

Step 2. We prove that F_2 is a fully continuous operator that can be decomposed into several small steps.

First, we show that F_2 is continuous in $\mathcal{B}_{k_0}$.

Let $\{x_n\} \subseteq \mathcal{B}_{k_0}$ with $x_n \to x$ on $\mathcal{B}_{k_0}$. Applying hypothesis (H_2), we get

$$f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right) \to f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right), \quad \text{as } n \to +\infty, \tag{4.5}$$

almost everywhere $t \in [0, a]$.

From the hypothesis (H_3),

$$d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) \leq 2m(s).$$

Therefore, by the domination convergence theorem, we get

$$\begin{aligned}
&d_\infty\left(\left(F_2x_n\right)(t), \left(F_2x\right)(t)\right)\\
\leq&\int_0^t \int_s^t \frac{M(t-\delta)^{q-2}}{\Gamma(q-1)} \odot d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) d\delta ds\\
\leq&\int_0^t (t-s)^{q-1}\frac{qM}{\Gamma(1+q)} \odot d_\infty\left(f\left(s, x_n(s), {}_{gH}^{C}\mathcal{D}^{q-1}x_n(s)\right), f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)\right) ds\\
\to&0,
\end{aligned}$$

when $n \to +\infty$, This means F_2 is continuous.

Next, we show that $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact. This is the family of functions $\{F_2x : x \in \mathcal{B}_{k_0}\}$ and $\{\left(F_2x\right)(t)\}$ relative compactness: $x \in \mathcal{B}_{k_0}$, where $t \in [0, a]$.

We proved this for all $x \in \mathcal{B}_{k_0}$ and $0 \leq t_1 \leq t_2 \leq a$

$$d_\infty\left(\left(F_2x\right)(t_2), \left(F_2x\right)(t_1)\right) \leq I_1 + I_2 + I_3.$$

We now have

$$\begin{aligned}
I_1 \leq& \frac{MM_1}{(1+b)^{1-q_1}\Gamma(q)}(t_2-t_1)^{(b+1)(1-q_1)},\\
I_2 \leq& \frac{MM_1}{(b+1)^{1-q_1}\Gamma(q)}\left(-(t_2-t_1)^{b+1} + t_2^{b+1} - t_1^{b+1} - (t_2-t_1)^{q-1}(b+1)^{1-q_1}\right),\\
I_3 \leq& \frac{MM_1(t_2-t_1)^{q-1}}{\Gamma(q)}t_1^{1-q_1}.
\end{aligned}$$

From **Step 1**, it is easy to see that $F_2\left(\mathcal{B}_{k_0}\right)$ is equicontinuous.

Proving this is enough for each $t \in [0, a]$, $V(t) = \{\left(F_2x\right)(t) : x \in \mathcal{B}_{k_0}\}$ is relatively compact. For any fixed $0 < t \leq a$, for all $\epsilon \in (0, t)$ and for all $\delta > 0$, let the operator $F_{\epsilon,\delta}$ be define as

$$\begin{aligned}
&\left(F_{\epsilon,\delta}x\right)(t)\\
=&\int_0^{t-\epsilon}\int_{s+\eta}^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q\right) \odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds\\
=&\int_0^{t-\epsilon}\int_{s+\eta}^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(A(\delta-s)^q - A(\eta-\epsilon) + A(\eta-\epsilon)\right)\\
&\odot f\left(s, x(s), {}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right) d\delta ds
\end{aligned}$$

$$
\begin{aligned}
=&E_{q,1}\left(A(\eta-\epsilon)\right)\int_0^{t-\epsilon}\int_{s+\eta}^{t}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q-A(\eta-\epsilon)\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,
\end{aligned}
$$

where $x\in\mathcal{B}_{k_0}$. From hypothesis $(H4)$, $E_{q,1}\left(A(\eta-\epsilon)\right)$ is the compact operator, then $V_{\epsilon,\delta}(t)=\left\{\left(F_{\epsilon,\delta}x\right)(t):x\in\mathcal{B}_{k_0}\right\}$ is relatively compact. Moreover, for all $x\in\mathcal{B}_{k_0}$, and we have

$$
\begin{aligned}
&d_\infty\left(\left(F_2x\right)(t),\left(F_{\epsilon,\delta}x\right)(t)\right)\\
=&d_\infty\left(\int_0^t\int_s^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\right.\\
&\left.-\int_0^{t-\epsilon}\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\right)\\
=&d_\infty\left(\int_0^t\int_s^{s+\eta}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\right.\\
&+\int_0^t\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds\\
&\left.-\int_0^{t-\epsilon}\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
\leq&d_\infty\left(\int_0^t\int_s^{s+\eta}\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
&+d_\infty\left(\int_{t-\epsilon}^t\int_{s+\eta}^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds,\hat{0}\right)\\
\leq&\frac{M_1M}{(b+1)\Gamma(q)}\left[\left((-\eta)^{b+1}-(a-\eta)^{b+1}+a^{b+1}\right)^{1-q_1}\left(-(-\eta)^{b+1}+(-\epsilon-\eta)^{b+1}\right)^{1-q_1}\right]\\
\to&0,
\end{aligned}
$$

when $\eta,\epsilon\to0$. Then we have a relatively compact set arbitrarily close to $V(t)$, $t>0$, which means that $V(t)$, $t>0$, is also relatively compact.

Applying the Ascoli-Arzela theorem shows that $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact. Since F_2 is continuous and $F_2\left(\mathcal{B}_{k_0}\right)$ is relatively compact, F_2 is a fully continuous operator.

According to Schauder's fixed point theorem, F_1+F_2 has a fixed point at $\mathcal{B}_{k_0}$. So the nonlocal Cauchy problem (3.1) has a mild fuzzy solution of type 1. □

Set

$$
\begin{aligned}
\hat{F}[x](t)=&E_{q,1}\left(At^q\right)\odot x_0+tE_{q,2}\left(At^q\right)\odot x_1\\
&\ominus(-1)\odot\int_0^t\int_s^t\frac{(t-\delta)^{q-2}}{\Gamma(q-1)}\odot E_{q,1}\left(A(\delta-s)^q\right)\\
&\odot f\left(s,x(s),{}_{gH}^{C}\mathcal{D}^{q-1}x(s)\right)d\delta ds
\end{aligned}
$$

and

$$\hat{C}([0,a],\mathcal{T}) = \left\{x \in \mathcal{C}([0,a],\mathcal{T}) : \hat{F}[x](t) \text{ exists for all } t \in [0,a]\right\}. \tag{4.6}$$

The following results show that there exists a mild fuzzy solution for type 2 in the space $\mathcal{C}([0,a],\mathcal{T})$.

Theorem 4.2. *The hypothesises* $(H1)$-$(H4)$ *are true and*

(Q2) $\hat{C}([0,a],\mathcal{T}) \neq \emptyset$;

(Q3) *if* $x \in \hat{C}([0,a],\mathcal{T})$, *hence* $\hat{F}[x] \in \hat{C}([0,a],\mathcal{T})$.

In this case the Cauchy problem (3.1) *has a mild fuzzy solution of type* 2 *in space* $\mathcal{C}([0,a],\mathcal{T})$.

Proof. For $x \in \hat{C}([0,a],\mathcal{T})$, $\hat{F}[x](t) = (F_1x)(t) \ominus (-1) \odot (F_2x)(t)$.

Set

$$k_0 = M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}}.$$

Using a similar method as before with the Caputo $[(i)-\text{gH}]$ derivative, we get: $F_1x \ominus (-1) \odot F_2y \in \mathcal{B}_{k_0}$ for any pair $x, y \in \mathcal{B}_{k_0} \subset \hat{C}([0,a],\mathcal{T})$, where $(F_1x)(t)$ and $(F_2x)(t)$ are continuous in $t \in [0,a]$.

Now for any $x, y \in \mathcal{B}_{k_0}$, we have

$$\begin{aligned} d_\infty\left((F_1x)(t) \ominus (-1) \odot (F_2x)(t), \hat{0}\right) &\leq d_\infty\left((F_1x)(t), \hat{0}\right) + d_\infty\left((F_2x)(t), \hat{0}\right) \\ &\leq M\left(\|x_0\| + a\|x_1\|\right) + \frac{M_1 M a^{(1+b)(1-q_1)}}{\Gamma(q)(1+b)^{1-q_1}} \\ &= k_0, \end{aligned}$$

which means that $F_1 \ominus (-1) \odot F_2$ is an operator from $\mathcal{B}_{k_0}$ into $\mathcal{B}_{k_0}$.

Since F_2 is a fully continuous operator, according to the Schauder fixed point theorem $F_1 \ominus (-1) \odot F_2$ has a fixed point on $\mathcal{B}_{k_0}$, this means that the Cauchy problem (3.1) has a mild fuzzy solution of type 2. □

5. An Example

Consider the following equations

(5.1)
$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} u(t,x) = \dfrac{\partial}{\partial t} u(t,\ x) + \dfrac{e^{-t}}{9+e^t}\left(\dfrac{|u(t,\ x)|}{1+|u(t,\ x)|}\right), \quad (t,x) \in]0,1[\times]0,1[, \\ u(t,0) = u(t,1) = 0, \quad t \in]0,1[, \\ u(0,x) = \psi(x), \quad x \in]0,1[, \\ {}^C D_{0+}^{\frac{1}{2}} u(0,x) = \phi(x), \quad x \in]0,1[. \end{cases}$$

We choose $\mathbb{X} = \mathrm{C}([0,\ 1] \times \mathcal{T}, \mathcal{T})$ and we do not forget the operator $\mathrm{A} : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$D(A) = \left\{u \in \mathbb{X} : \frac{\partial}{\partial t} u \in \mathbb{X} \text{ and } u(0,0) = u(0,1) = 0\right\}, \quad Au = \frac{\partial}{\partial t} u.$$

Then, we get

$$\overline{D(A)} = \{u \in \mathbb{X} : u(t,0) = u(t,1) = 0\}. \tag{5.2}$$

This implies that A satisfies (H4).

As is well known that A generates a compact C_0-semigroup $E_{q,n}(At^q)$ on $\overline{D(A)}$. Let's pose $X(t) = u(t,\cdot)$, that is $X(t)(x) = u(t,x)$, for all $(t,x) \in]0,1[\times]0,1[$.

In this example, we have the function $f :]0,1[\times\mathcal{T} \longrightarrow \mathcal{T}$ is given by

$$f(t,X(t)) = \frac{e^{-t}}{9+e^t}\left(\frac{|X(t)|}{1+|X(t)|}\right).$$

It is clear that for all $X, Y \in \mathcal{C}([0,\ 1], B_r)$ we have

$$d_\infty\left(f(t,X(t), f(t,Y(t))\right) \le \rho(t) d_\infty(X(t),Y(t)), \quad \text{with } \rho(t) = \frac{e^{-t}}{9+e^t} \in L^1,$$

for all $t \in]0,1[$, and that

$$d_\infty\left(f(t,X(t),\tilde{0}\right) \le m(t), \quad \text{with } m(t) = \frac{1}{9+e^t} \in L^1, \text{ for all } t \in]0,1[.$$

Moreover, f is continuos, therefore it is strongly measurable. Hence, according to Theorem 4.1 and Definition 4.1, problem (5.1) admits two types of solutions expressed as follow

$$\begin{aligned} X(t) =& E_{q,1}\left(\frac{\partial}{\partial t}t^q\right) \odot x_0 + t \odot E_{q,2}\left(\frac{\partial}{\partial t}t^q\right) \odot x_1 \\ &+ \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\frac{\partial}{\partial s}(\delta - s)^q\right) \odot \frac{e^{-s}}{9+e^s}\left(\frac{|X(s)|}{1+|X(s)|}\right) d\delta ds \end{aligned}$$

and

$$\begin{aligned} X(t) =& E_{q,1}\left(\frac{\partial}{\partial t}t^q\right) \odot x_0 + t \odot E_{q,2}\left(\frac{\partial}{\partial t}t^q\right) \odot x_1 \\ &\ominus (-1) \int_0^t \int_s^t \frac{(t-\delta)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}\left(\frac{\partial}{\partial s}(\delta - s)^q\right) \odot \frac{e^{-s}}{9+e^s}\left(\frac{|X(s)|}{1+|X(s)|}\right) d\delta ds. \end{aligned}$$

6. Conclusion

In this work, the nonlocal Cauchy problem of fuzzy evolutionary equations in arbitrary Banach spaces of order $q \in (1,2)$ is discussed, and the existence and uniqueness criteria for mild fuzzy solutions are determined utilizing the Schauder fixed point theorem. An example illustrating the application of the main results is also provided.

References

[1] G. A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Studies in Fuzziness and Soft Computing, Springer, Berlin Heidelberg, 2010. https://doi.org/10.1007/978-3-642-11220-1

[2] B. Bede and S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems **151**(3) (2005), 581–599. https://doi.org/10.1016/j.fss.2004.08.001

[3] B. Bede, *Fuzzy sets*, in: *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, Berlin, Heidelberg, 2013, 1–12. https://doi.org/10.1007/978-3-642-35221-8_1

[4] B. Bede and L. Stefanini, *Generalized differentiability of fuzzy valued functions*, Fuzzy Sets and Sysstems **230**(0) (2013), 119–141. https://doi.org/10.1016/j.fss.2012.10.003

[5] T. Allahviranloo, Z. Gouyandeh, A. Armand and A. Hasanoglu, *On fuzzy solutions for heat equation based on generalized Hukuhara differentiability*, Fuzzy Sets and Systems **265** (2015), 1–23. https://doi.org/10.1016/j.fss.2014.11.009

[6] T. Allahviranloo, *Fuzzy fractional differential operators and equations*, Studies in Fuzziness and Soft Computing **397** (2021). https://doi.org/10.1007/978-3-030-51272-9

[7] H. C. Wu, *The improper fuzzy Riemann integral and its numerical integration*, Inf. Sci. **111**(14) (1998), 109–137. https://doi.org/10.1016/S0020-0255(98)00016-4

[8] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010. https://doi.org/10.1007/978-0-387-70914-7

[9] A. Armand and Z. Gouyandeh, *Fuzzy fractional integro-differential equations under generalized Caputo differentiability*, Annals of Fuzzy Mathematics and Informatics **10**(5) (2015), 789–798.

[10] T. Allahviranloo and M. B. Ahmadi, *Fuzzy Laplace transforms*, Soft Computing **14**(3) (2010), 235–243. https://doi.org/10.1007/s00500-008-0397-6

[11] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems **24**(3) (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7

[12] M. L. Puri, D. A. Ralescu and L. Zadeh, *Fuzzy random variables*, in: *Readings in Fuzzy Sets for Intelligent Systems*, Morgan Kaufmann, 1993, 265–271. https://doi.org/10.1016/B978-1-4832-1450-4.50029-8

[13] B. Dubey and R. K. George, *Controllability of linear time-invariant dynamical systems with fuzzy initial condition*, Proceedings of the World Congress on Engineering and Computer Science (2) (2013).

[14] N. T. K. Son, *A foundation on semigroups of operators defined on the set of triangular fuzzy numbers and its application to fuzzy fractional evolution equations*, Fuzzy Sets and Systems **347** (2018), 1–28. https://doi.org/10.1016/j.fss.2018.02.003

[15] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittag-Leffler Functions*, Springer, New York, 2014. https://doi.org/10.1007/978-3-662-43930-2

[16] N. Iqbal, A. U. K. Niazi, R. Shafqat and S. Zaland, *Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation*, Journal of Function Spaces **2021** (2021), 1–8. https://doi.org/10.1155/2021/5795065

[17] A. U. K. Niazi, J. He, R. Shafqat and B. Ahmed, *Existence, uniqueness, and E_q-Ulam-type stability of fuzzy fractional differential equation*, Fractal and Fractional **5**(3) (2021), paper ID 66. https://doi.org/10.3390/fractalfract5030066

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ON SPECTRAL RADIUS ALGEBRAS AND CONDITIONAL TYPE OPERATORS

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ABSTRACT. In this note, we study both the spectral radius and Deddens algebras associated to the normal weighted conditional type operators on $L^2(\Sigma)$. Also, in this setting, some other special properties of these algebras will be investigated.

1. INTRODUCTION AND PRELIMINARIES

Let (X, Σ, μ) be a complete σ-finite measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. If $B \subset X$, let $\mathcal{A}_B = \mathcal{A} \cap B$ denote the relative completion of the sigma-algebra generated by $\{A \cap B : A \in \mathcal{A}\}$. We denote the linear spaces of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\mathcal{A}$ be a sub-σ-finite algebra of Σ and let f be a non-negative Σ-measurable function on X. By the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$, where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Note that $E(f)$ depends both on μ and $\mathcal{A}$. A real-valued measurable function $f = f^+ - f^-$ is said to be conditionable if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = +\infty\}) = 0$. If f is complex-valued, then $f \in \mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$ if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. For $1 \leq p \leq +\infty$, one can show that every $L^p(\Sigma)$ function is conditionable. We use the notation $L^p(\mathcal{A})$ for $L^p(X, \mathcal{A}, \mu_{|_\mathcal{A}})$ and henceforth we write μ in place $\mu_{|_\mathcal{A}}$.

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The mapping $E^{\mathcal{A}} : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. In the case of $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. For further discussion of the conditional expectation operator see [13].

From now on we assume that u and w are conditionable. Operators of the form $M_wEM_u(f) = wE(uf)$ acting in $L^2(\Sigma)$ with $\mathcal{D}(M_wEM_u) = \{f \in L^2(\Sigma) : wE(uf) \in L^2(\Sigma)\}$ are called weighted conditional type (or weighted Lambert type) operators. Several aspects of this operator were studied in [4, 6–8]. Put $K = E(|u|^2)E(|w|^2)$. Estaremi in [3] proved that $M_wEM_u : \mathcal{D}(T) \to L^2(\Sigma)$ is densely defined if and only if $K - 1$ is finite valued (a.e.). Moreover, $T := M_wEM_u$ is bounded if and only if $\mathcal{D}(T) = L^2(\Sigma)$. In this case $T^* = M_{\bar{u}}EM_{\bar{w}}$ and $\|T\|^2 = \|K\|_\infty$. For a bounded linear operator T, $\operatorname{spec}(T)$ denote its spectrum. We say that $\lambda \in \mathbb{C}$ belongs to the essential range of a measurable function f if for each neighborhood G of λ, $\mu(f^{-1}(G)) > 0$. Positive, self-adjoint and normal bounded weighted conditional type operators and their spectrum have recently been characterized in [7] as follows.

Lemma 1.1 ([7])**.** *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then the followings hold.*
(*a*) *T is positive if and only if $T = M_{g\bar{u}}EM_u$ for some $0 \le g \in L^0(\mathcal{A})$.*
(*b*) *T is self-adjoint if and only if $T = M_{g\bar{u}}EM_u$ for some $\bar{g} = g \in L^0(\mathcal{A})$.*
(*c*) *T is normal if and only if $T = M_{g\bar{u}}EM_u$ for some $g \in L^0(\mathcal{A})$.*
(*d*) $\operatorname{spec}(T) \setminus \{0\} = \operatorname{ess\,range}(E(uw)) \setminus \{0\}$.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. We use A^*, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the adjoint, the spectral radius, the range and the null space of $A \in B(\mathcal{H})$. A is normal if $A^*A = AA^*$ and A is positive if $\langle Ax, x \rangle \ge 0$ holds for each $x \in \mathcal{H}$ in which case we write $A \ge 0$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $A \in B(\mathcal{H})$ and let $Pj : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto $\mathcal{H}_j$ for $j = 1, 2$. Then $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ is the operator given by $A_{ij} = P_iAP_j \mid_{\mathcal{H}_j}$. In particular, $A(\mathcal{H}_1) \subseteq \mathcal{H}_1$ if and only if $A_{21} = 0$. Also, $\mathcal{H}_1$ reduces A if and only if $A_{12} = 0 = A_{21}$. Let $A \in B(\mathcal{H})$ with $r(A) \neq 0$ and let $0 < a < r(A)^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$. Since for all $n \in \mathbb{N}$, $\|a^{2n}A^{*n}A^n\| = a^{2n}\|A^n\|^2$, then we have $\overline{\lim}_{n\to+\infty}\|a^{2n}A^{*n}A^n\|^{\frac{1}{n}} = a^2 \left(\overline{\lim}_{n\to+\infty}\|A^n\|^{\frac{1}{n}}\right)^2 = a^2r(A)^2 < 1$. This implies that the series $\sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$ is convergent in the norm topology of $B(\mathcal{H})$, and hence $K_a(A) \in B(\mathcal{H})$. Thus, the map f_A of $(0, r(A)^{-1})$ to $B(\mathcal{H})$ defined by $f_A(a) = K_a(A)$ is well-define, increasing and continuous. Also, for any $x \in \mathcal{H}$ we have that

$$\|x\|^2 \le \sum_{n=0}^{+\infty} a^{2n}\|A^n(x)\|^2 = \langle K_a(A)x, x \rangle = \left\|\sqrt{K_a(A)}x\right\|^2 \le \|K_a(A)\| \cdot \|x\|^2. \tag{1.1}$$

So, $K_a(A) \ge I$ and hence $K_a(A)$ is positive and invertible with $\|K_a(A)\| \ge 1$. Set $R_a(A) = K_a^{-1}(A)$ and $S_a(A) = \sqrt{R_a(A)}$. Replacing x by $(K_a(A))^{\frac{-1}{2}}(x)$ in (1.1) we obtain that $\|S_a(A)\| \leq 1$ and $\|R_a(A)\| = \|S_a^2(A)\| \leq 1$. Consequently, $R_a(A)$ and $S_a(A)$ are positive and invertible elements of $B(\mathcal{H})$ and

$$\|K_a(A)\| = \sup_{\|x\|=1} \langle K_a(A)x, x\rangle = \sum_{n=0}^{+\infty} a^{2n}\|A^n\|^2 \leq \sum_{n=0}^{+\infty} \left(\|aA\|^2\right)^n = \frac{1}{1-\|aA\|^2}. \tag{1.2}$$

Let $\{A_m\} \subseteq \{T \in B(\mathcal{H}) : r(T) \leq r(A)\}$. If $\|A_m - A\| \to 0$, then for each $n \in \mathbb{N}$ and $0 < a < r(A)^{-1}$, $a^{2n}A_m^{*n}A_m^n \to a^{2n}A^{*n}A^n$, and so $\|K_a(A_m) - K_a(A)\| \to 0$ as $m \to +\infty$. But the converse is not true. Indeed, if A_1 and A_2 are distinct unitary operators on $\mathcal{H}$, then $K_a(A_1) = K_a(A_2) = (1-a^2)^{-1}I$ for all $0 < a < 1$. In [9] A. Lambert and S. Petrović define the spectral radius algebra of a bounded linear operator A with $S_a = S_a(A)$ and $0 < a < r(A)^{-1}$ to be the unital subalgebra

$$\mathcal{B}_A = \{T \in B(\mathcal{H}) : \sup_a \|S_a^{-1}TS_a\| < +\infty\}.$$

Lastly, define

$$\mathcal{Q}_A = \{T \in B(\mathcal{H}) : \lim_{a\to r(A)^{-1}} \|S_a^{-1}TS_a\| = 0\}.$$

In [9] it is shown that, $\|K_a(A)\| \to +\infty$ as $a \to \|A\|^{-1}$ and for any A, $\mathcal{Q}_A \subseteq \mathcal{B}_A$ is a two-sided ideal consisting entirely of quasinilpotent operators. Furthermore, if A is quasinilpotent, then $A \in \mathcal{Q}_A$.

We now consider the Deddens algebra $\mathcal{D}_A$ associated with $A \in B(\mathcal{H})$, that is, the family of those operators $T \in B(\mathcal{H})$ for which there is a constant $M > 0$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathcal{H}$, $\|A^nTx\| \leq M\|A^nx\|$. $\mathcal{D}_A$ is indeed a unital subalgebra of $B(\mathcal{H})$ with the property that $\{A\}' \subseteq \mathcal{D}_A \subseteq \mathcal{B}_A$, where $\{A\}'$ is the commutant of A (see [11]).

Let $A \in B(\mathcal{H})$ be normal and $0 < a < \|A\|^{-1}$. Then A^n and A^{*n} commute with $K_a(A)$, $R_a(A)$, $S_a(A)$ and $K_a(A^*) = K_a(A) = K_a(|A|)$, where $|A|^2 = A^*A$. Moreover,

$$\begin{aligned} K_a(A) &= \sum_{n=0}^{+\infty} a^{2n}(A^*A)^n = (I - a^2A^*A)^{-1}, \\ R_a(A) &= I - a^2A^*A, \\ S_a(A) &= \sqrt{I - a^2A^*A}, \\ P_A := &\lim_{a\to\|A\|^{-1}} S_a(A) = \sqrt{I - \|A\|^{-2}A^*A}. \end{aligned} \tag{1.3}$$

For more details on the Deddens and spectral radius algebras see [1,5,11,12]. In the next section, we investigate the spectral radius and the Deddens algebras related to the bounded weighted conditional type operators on $L^2(\Sigma)$. All of these are basically discussed using the conditional expectation properties.

2. $\mathcal{B}_T$ and $\mathcal{D}_T$ Associated with $T = M_wEM_u$

From now on we assume that $E(|u|^2) \in L^\infty(\mathcal{A})$, i.e., $T_1 := M_{\bar{u}}EM_u \in B(L^2(\Sigma))$.

Lemma 2.1. *For $0 \leq b \in L^0(\mathcal{A})$, let $M_bT_1 \in B(L^2(\Sigma))$. Then the followings hold.*
(a) If $1 \notin \operatorname{spec}(M_bT_1)$, then $(I - M_bT_1)^{-1} = I + M_{\frac{b}{1-bE(|u|^2)}}T_1$.
(b) If $-1 \notin \operatorname{spec}(M_bT_1)$, then $(I + M_bT_1)^{-1} = I - M_{\frac{b}{1+bE(|u|^2)}}T_1$.

Proof. We only proof (a), since (b) follows similarly.

Let $1 \in \operatorname{spec}(M_bT_1)$. Using Lemma 1.1 (d), $1 \notin \operatorname{ess\,range} E(b|u|^2)$ and so $(1 - bE(|u|^2))^{-1} \in L^\infty(\mathcal{A})$. Put $S = I + M_{b(1-bE(|u|^2))^{-1}}T_1$. Then $\|S\| \leq 1 + \|(1 - bE(|u|^2))^{-1}\|_\infty \|M_bT_1\| < +\infty$. Also, direct computations show that $S(I - M_bT_1) = (I - M_bT_1)S = I$. Now, the desired conclusion holds. □

Set $\mathcal{N} = \{M_wEM_u \in B(L^2(\Sigma)) : M_wEM_u \text{ is normal}\}$. By Lemma 1.1 (c) we have $\mathcal{N} = \{M_gT_1 \in B(L^2(\Sigma)) : g \in L^0(\mathcal{A}), T_1 = M_{\bar{u}}EM_u, u \in L^0(\Sigma)\}$.

Corollary 2.1. *Let $T = M_wEM_u \in \mathcal{N}$ and let $0 < a < r(T)^{-1}$. Then $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$ for some $k, v \in L^0(\mathcal{A})$ and $\|K_a(T)\| = 1 + \|vE(|u|^2)\|_\infty$.*

Proof. By Lemma 1.1 (c), $T = M_gT_1$ for some $g \in L^0(\mathcal{A})$. Since $T^*T = M_{|g|^2E(|u|^2)}T_1$, then by (1.3) we get that $K_a(T) = (I - M_kT_1)^{-1}$, where $k = a^2|g|^2E(|u|^2)$. Thus, $R_a(T) = (K_a(T))^{-1} = I - M_kT_1$. Also, since $1/a^2 > (r(T))^2 = r(T^*T)$, then $1/a^2 \notin \operatorname{spec}(T^*T) = \operatorname{ess\,range} |g|^2(E(|u|^2))^2$. Therefore,

$$\frac{1}{1 - kE(|u|^2)} = \frac{1}{a^2\{\frac{1}{a^2} - |g|^2(E(|u|^2))^2\}} \in L^\infty(\mathcal{A})$$

and $1 \notin \operatorname{spec}(M_kT_1)$. Now, by Lemma 2.1, $K_a(T) = I + M_vT_1$, where $v = \frac{k}{1-kE(|u|^2)}$. Moreover, since M_vT_1 is positive, then $\|K_a(T)\| = 1 + \|M_vT_1\| = 1 + \|vE(|u|^2)\|_\infty$. This completes the proof. □

Corollary 2.2. *Under the assumption of above corollary, $S_a(T) = I - M_sT_1$ and $S_a^{-1}(T) = I + M_{\frac{s}{1-sE(|u|^2)}}T_1$ for some $s \in L^0(\mathcal{A})$.*

Proof. Set $s = \frac{1-\sqrt{1-kE(|u|^2)}}{E(|u|^2)}\chi_{\sigma(E(|u|^2))}$. Then, for $f \in L^2(\Sigma)$ we have

$$\begin{aligned}(I - M_{s\bar{u}}EM_u)^2(f) &= (I - M_{s\bar{u}}EM_u)(f - s\bar{u}E(uf)) \\ &= f - s\bar{u}E(uf) - s\bar{u}E(uf - s|u|^2E(uf)) \\ &= f - \bar{u}(-2s + E(|u|^2)s^2)E(uf) \\ &= f - \bar{u}kE(uf) \\ &= (I - M_{k\bar{u}}EM_u)(f).\end{aligned}$$

It follows that $S_a(T) = (R_a(T))^{1/2} = (I + M_kT_1)^{1/2} = I - M_sT_1$. Now, the inverse of $S_a(T)$ follows from Lemma 2.1 (a). □

For $T \in \mathcal{N}$ and $v \in L^0(\mathcal{A})$, it is easy to check that M_vT_1 commutes with $S_a(T)$. It follows that $\{M_vT_1 \in B(L^2(\Sigma)) : v \in L^0(\mathcal{A})\} \subseteq \mathcal{B}_T$.

Lemma 2.2. *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then $\mathcal{N}(T) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^\perp$.*

Proof. Let $f \in L^2(\Sigma)$. Since $\mathcal{R}(E) = L^2(\mathcal{A})$, then we have

$$\begin{aligned} f \in \mathcal{N}(T) \Leftrightarrow \|Tf\|^2 = 0 &\Leftrightarrow \int_X E(|w|^2)|E(uf)|^2 d\mu = 0 \\ &\Leftrightarrow \int_X \left| E(u\sqrt{E(|w|^2)}f) \right|^2 d\mu = 0 \\ &\Leftrightarrow u\sqrt{E(|w|^2)}f \in \mathcal{N}(E) = L^2(\mathcal{A})^\perp \\ &\Leftrightarrow \left\langle u\sqrt{E(|w|^2)}f, g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow \left\langle f, \bar{u}\sqrt{E(|w|^2)}g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow f \in \left\{ \bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A}) \right\}^\perp . \end{aligned}$$

□

Corollary 2.3. $\overline{\mathcal{R}(M_{\bar{u}}EM_u)} = \overline{\bar{u}\sqrt{E(|u|^2)}L^2(\mathcal{A})} = c.l.s. \left\{ \bar{u}\sqrt{E(|u|^2)}\chi_A : A \in \mathcal{A}_{\sigma(u)} \right\}$, *where c.l.s. stands for closed linear span. In particular,* $\overline{\mathcal{R}(EM_u)} = \overline{\bar{u}L^2(\mathcal{A})}$.

Let P be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$ and let $Q = I - P$. Direct computations show that

$$(I - \alpha P)^{-1} = I + \frac{\alpha}{1-\alpha}P, \quad \alpha \neq 1, \tag{2.1}$$

$$(I - \alpha P)^{\frac{1}{2}} = I - (1 - \sqrt{1-\alpha})P, \quad \alpha \leq 1. \tag{2.2}$$

Let $0 < a < 1$. Then $K_a(P) = \sum_{n=0}^{+\infty} a^{2n} P^{*n} P^n = I + \frac{a^2}{1-a^2}P$. Using (2.1) and (2.2) we obtain that

$$\begin{aligned} R_a(P) &= (K_a(P))^{-1} = I - a^2 P, \\ S_a(P) &= (R_a(P))^{\frac{1}{2}} = I - (1 - \sqrt{1-a^2})P, \\ S_a^{-1}(P) &= I + \frac{1 - \sqrt{1-a^2}}{\sqrt{1-a^2}}P. \end{aligned}$$

Note that if we take $P = M_{\bar{u}E(|u|^2)^{-1}}EM_u$, then $P^2 = P = P^*$, with $\mathcal{R}(P) = \overline{\bar{u}E(|u|^2)^{-1/2}L^2(\mathcal{A})}$. Now, let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \mathcal{M} \oplus \mathcal{M}^\perp$. Since

$$S_a(P) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} M_{1-\sqrt{1-a^2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix},$$

then we have

$$\mathcal{P}_a(S) := (S_a^{-1}(P))S(S_a(P)) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} X & YM_{\frac{1}{\sqrt{1-a^2}}} \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}.$$

It follows that $\sup\{\|\mathcal{P}_a(S)\| : 0 < a < 1\} < +\infty$ if and only if $Y = 0$. For some $0 < a < 1$, $\mathcal{P}_a(S) = S$ if and only if $Y = Z = 0$. Also, $\lim_{a\to 1} \|\mathcal{P}_a(S)\| = 0$ if and only if $X = Y = W = 0$. Moreover, we have

$$\mathcal{P}_a(SP) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} X & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Thus, $SP \in \mathcal{B}_P$ for all $S \in B(L^2(\Sigma))$. Also if $X = 0$, then $SP \in \mathcal{Q}_P$. Similar computations show that

$$\mathcal{P}_a(QS) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}, \quad \mathcal{P}_a(QSP) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Let $\{S_n\} \subseteq \mathcal{B}_P$ and let $S_n := \begin{pmatrix} X_n & 0 \\ Z_n & W_n \end{pmatrix} \to S$ as $n \to +\infty$. Then

$$\|Y\| \le \|S_n - S\| = \left\| \begin{pmatrix} X_n - X & Y \\ Z_n - Z & W_n - W \end{pmatrix} \right\| \to 0.$$

It follows that $Y = 0$ and hence $\mathcal{B}_P$ is closed in the norm operator topology on $B(L^2(\Sigma))$. Moreover, by definition, $S \in \mathcal{D}_P$ if and only if there exists $M > 0$ such that

$$\begin{aligned} \|PSf\| &= \left\| \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} XPf + YQf \\ 0 \end{pmatrix} \right\| \le M \left\| \begin{pmatrix} Pf \\ 0 \end{pmatrix} \right\|, \end{aligned}$$

for all $f \in L^2(\Sigma)$. Replacing f by Qf in the above and taking $M = M(S) = \|X\|$, we obtain that $S \in \mathcal{D}_P$ if and only if $Y = 0$ on $\mathcal{N}(P)$. As an easy consequence of these observations, we have the following result.

Proposition 2.1. *Let P be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$, $0 < a < 1$ and let $Q = I - P$. Set*

$$\begin{aligned} \mathcal{Q}_1 &= \{SP : S \in B(L^2(\Sigma)), PSP = 0\}, \\ \mathcal{Q}_2 &= \{QS : S \in B(L^2(\Sigma)), QSQ = 0\}, \\ \mathcal{Q}_3 &= \{QSP : S \in B(L^2(\Sigma))\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{B}_P &= \{S \in B(L^2(\Sigma)) : S(\mathcal{N}(P)) \subseteq \mathcal{N}(P)\} = \mathcal{D}_P, \\ \mathcal{Q}_P &= \{S \in B(L^2(\Sigma)) : QSP = T\} \supseteq \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3. \end{aligned}$$

Moreover, $\mathcal{P}_a(S) = S$ if and only if $\mathcal{M}$ reduces S.

Set $P = E^{\mathcal{A}} = E$, $0 < a < 1$ and $\mathcal{P}_a = \mathcal{E}_a$. Let $S = M_w E M_u \in B(L^2(\Sigma))$. Using Proposition 2.1 and [7, Proposition 2.30] with respect the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, we have

$$\begin{aligned} ESE = 0 &\Leftrightarrow M_{E(u)E(w)}|_{L^2(\mathcal{A})} = 0, \\ ESQ = 0 &\Leftrightarrow u\chi_{\sigma(E(w))} \in L^0(\mathcal{A}), \\ QSE = 0 &\Leftrightarrow w\chi_{\sigma(E(u))} \in L^0(\mathcal{A}), \\ QSQ = 0 &\Leftrightarrow L^2(\mathcal{A})^{\perp} = \mathcal{R}(Q) \subseteq \mathcal{N}(S) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^{\perp}. \end{aligned}$$

So we have the following corollary.

Corollary 2.4. *Let* $S = M_w E M_u \in B(L^2(\Sigma))$ *and* $0 < a < 1$. *Then,*
(*a*) $S \in \mathcal{B}_E$ *if and only if* $u\chi_{\sigma(E(w))} \in L^0(\mathcal{A})$;
(*b*) $S \in \mathcal{Q}_E$ *if and only if* $S \in \mathcal{B}_E$, $M_{E(uE(w))}|_{L^2(\mathcal{A})} = 0$ *and* $\overline{\mathcal{R}(S)} \subseteq L^2(\mathcal{A})$;
(*c*) $\mathcal{E}_a(S) = S$ *if and only if* $\{u\chi_{\sigma(E(w))}, w\chi_{\sigma(E(u))}\} \subseteq L^0(\mathcal{A})$.

Let $\mathcal{M}(\mathcal{A}) = \{M_\vartheta : \vartheta \in L^\infty(\mathcal{A})\}$ and let $\mathcal{M}'(\mathcal{A})$ be its commutant. It is known that $\mathcal{M}(\Sigma)$ is a maximal abelian subalgebra of $B(L^2(\Sigma))$. But it is invalid if Σ is replaced by $\mathcal{A} \neq \Sigma$. Indeed, for any $\mathcal{A} \subset \mathcal{B}$, $E^{\mathcal{B}} \in \mathcal{M}'(\mathcal{A}) \setminus \mathcal{M}(\mathcal{A})$. Alan Lambert in [10, Theorem 3.2] proved that $S \in \mathcal{M}'(\mathcal{A})$ if and only if there exists $C > 0$ such that $E(|Sf|^2) \leq CE(|f|^2)$ for all $f \in L^2(\Sigma)$. Consequently, if $S \in B(L^2(\Sigma))$ and $\{\vartheta_n, \vartheta_n^{-1}\} \subseteq L^\infty(\mathcal{A})$, then $\sup_n \|M_{\vartheta_n^{-1}} S M_{\vartheta_n}\| < +\infty$ whenever $S \in \mathcal{M}'(\mathcal{A})$.

For a fixed $T = M_g T_1 \in \mathcal{N}$ and $0 < a < \|T\|^{-1}$, put $A := S_a^{-1}(T)$. Then by Corollary 2.2, $A = I + M_\theta T_1$ for some $0 \leq \theta \in L^0(\mathcal{A})$. Since A is bounded, then so is $M_\theta T_1$. Thus, $\theta E(|u|^2) \in L^\infty(\mathcal{A})$ and hence $\theta E(|u|^2)g \in L^2(\mathcal{A})$ for all $f \in L^2(\mathcal{A})$. Relative to the direct sum decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$, the matrix form of A is $(A_{ij})_{1\leq i,j\leq 2}$. Set $P = P_{\overline{\mathcal{R}(T_1)}}$ and $Q = I - P$. Let $f \in L^2(\Sigma)$. Then without loss of generality, we can assume that $Pf = \bar{u}\sqrt{E(|u|^2)}g$, for some $g \in L^2(\mathcal{A})$. Then

$$\begin{aligned} A_{11}f &= P(A(Pf)) = P(Pf + \theta\bar{u}E(uPf)) \\ &= P(Pf + \bar{u}\sqrt{E(|u|^2)}\left(\theta E(|u|^2)g)\right) \\ &= Pf + \theta E(|u|^2)Pf \\ &= M_{1+\theta E(|u|^2)}Pf, \end{aligned}$$

where $\theta = 1 + \frac{s}{1-sE(|u|^2)}$. By Corollary 2.2, $1 + \theta E(|u|^2) = \frac{1}{1-sE(|u|^2)} = \frac{1}{\sqrt{1-kE(|u|^2)}}$ where $k = a^2|g|^2E(|u|^2)$. Thus, $A_{11} = PAP = M_{(1-kE(|u|^2))^{-1/2}}P$. Similar computations show that $A_{12} = A_{21} = 0$ and $A_{22} = I_{|\mathcal{N}(T_1)}$. Let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$. Set $\mathcal{L}_a(S) := (S_a^{-1}(T))S(S_a(T))$ and $\vartheta := \sqrt{1 - kE(|u|^2)}$. Then

$\vartheta \to 0$ as $a \to \|T\|^{-1}$ and

$$\mathcal{L}_a(S) = \begin{pmatrix} M_{\frac{1}{\vartheta}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_\vartheta & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_{\frac{1}{\vartheta}} X M_\vartheta & M_{\frac{1}{\vartheta}} Y \\ Z M_\vartheta & W \end{pmatrix}.$$

Since $M_\vartheta \to 0$ as $a \to \|T\|^{-1}$, so $\sup\{\|M_{\vartheta^{-1}}\|;\ 0 < a < \|T\|^{-1}\} = +\infty$. Let $M := \sup\{\|M_{\vartheta^{-1}} Y\| < +\infty,\ 0 < a < \|T\|^{-1}\}$. Then for all unit vector $f \in \mathcal{N}(T_1)$, $\|Y(f)\| = \|M_\vartheta M_{\vartheta^{-1}} Y(f)\| \leq M \|M_\vartheta\|$. It follows that $\|Y(f)\| = 0$ and hence $Y_{|_{\mathcal{N}(T_1)}} = 0$. In particular, if $PSP \in \mathcal{M}'(\mathcal{A})$, then $S \in \mathcal{B}_T$ if and only if $S(\mathcal{N}(T_1)) \subseteq \mathcal{N}(T_1)$. In this case, $\mathcal{B}_{M_{g_1 T_1}} = \mathcal{B}_{M_{g_2 T_1}}$ for all $\{M_{g_1 T_1}, M_{g_1 T_1}\} \subseteq \mathcal{N}$. Note that

$$\mathcal{L}_a(T) = S_a^{-1}(T) \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix} S_a(T) = \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\|\mathcal{L}_a(T)\| = \|gE(|u|^2)\|_\infty = \|T\| = r(T)$. In view of these observations we have the following results.

Theorem 2.1. *Let $T = M_g T_1 \in \mathcal{N}$ and let $\vartheta = \sqrt{1 - a^2|g|^2(E(|u|^2))^2}$. Then the followings hold.*

(*a*) *$S \in \mathcal{B}_T$ if and only if $Y = 0$ and* $\sup\{\|M_{\vartheta^{-1}} X M_\vartheta\| :\ 0 < a < \|T\|^{-1}\} < +\infty$. *In particular, if $X M_\vartheta = M_\vartheta X$, then $S \in \mathcal{B}_T$ if and only if $\{\bar{u}\sqrt{E(|u|^2)L^2(\mathcal{A})}\}^\perp$ is an invariant subspace for S.*

(*b*) *$S \in \mathcal{Q}_T$ if and only if $Y = W = 0$ and $\|M_{\vartheta^{-1}} X M_\vartheta\| \to 0$, as $a \to \|T\|^{-1}$. Moreover, if $X M_\vartheta = M_\vartheta X$, then $S \in \mathcal{Q}_T$ if and only if $X = Y = W = 0$.*

Let $T = M_g T_1 \in \mathcal{N}$ and $S \in B(L^2(\Sigma))$. Then, for all $n \in \mathbb{N}$ and $f \in L^2(\Sigma)$, $T^n = M_{g^n (E(|u|^2))^{n-1}} T_1$ and

$$T^n S f = \begin{pmatrix} M_{\omega^n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} = \begin{pmatrix} M_{\omega^n} X P f + M_{\omega^n} Y Q f \\ 0 \end{pmatrix},$$

where $\omega = gE(|u|^2)$. It follows that $S \in \mathcal{D}_T$ if and only if there exists $M > 0$ such that $\|M_{\omega^n} X P f + M_{\omega^n} Y Q f\| \leq M \|M_{\omega^n} P f\|$. If we set $f = Qg$, for some $g \in L^2(\Sigma)$, then we get $\|M_{\omega^n|_{\sigma(\omega)}} Y Q g\| \leq \|M_{\omega^n} Y Q g\| = 0$ and hence $Y\ |_{\mathcal{N}(T_1)} = 0$. Now, if $M_\omega X = X M_\omega$, then $\|M_{\omega^n} X P f\| \leq \|X\| \cdot \|M_{\omega^n} P f\|$. Note that the commutativity of M_ω and X implies that $M_\vartheta X = X M_\vartheta$. So we have the following result.

Theorem 2.2. *Let $T = M_g T_1 \in \mathcal{N}$, $\omega = gE(|u|^2)$ and let $S \in B(L^2(\Sigma))$. Then $S \in \mathcal{D}_T$ if and only if $PSP \in \mathcal{D}_T$ and $PSQ = 0$. Moreover, if $(PSP)M_\omega = M_\omega(PSP)$, then $\mathcal{D}_T = \mathcal{B}_T$.*

Corollary 2.5. *Let $\{T, S\} \subseteq \mathcal{N}$. Then $S \in \mathcal{B}_T$ if and only if $PSQ = 0$.*

Proof. Let $S = M_{g_1 \bar{v}} E M_v \in \mathcal{B}_T$, with $g_1 \in L^0(\mathcal{A})$. Then $PSP = M_\gamma$, where $\gamma = g_1 E(u) E(\bar{v}) E(\bar{u} v) \in L^0(\mathcal{A})$. Since PSP commutes with M_γ, then the desired conclusion follows from Theorem 2.2. □

Example 2.1. Let $X = \{1, 2, 3\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/3$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{\{1, 3\}, \{2\}\}$. Then $L^2(\Sigma) \cong \mathbb{C}^3$ and

$$E(f) = \left(\frac{1}{\mu(A_1)}\int_{A_1} f d\mu\right)\chi_{A_1} + \left(\frac{1}{\mu(A_2)}\int_{A_2} f d\mu\right)\chi_{A_2} = \frac{f_1 + f_3}{2}\chi_{A_1} + f_2\chi_{A_2},$$

where $A_1 = \{1, 3\}$ and $A_2 = \{2\}$. Then matrix representation of E with respect to the standard orthonormal basis is $E = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. It can be easily checked that $E^2 = E = E^*$, $\mathcal{N}_2(E) = \langle (a, 0, -a) : a \in \mathbb{C}\rangle$, $\mathcal{R}(E) = \langle (a, b, a) : a, b \in \mathbb{C}\rangle$. For $1 < a < 1$ we have

$$K_a(E) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{a^2}{1 - a^2}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{a^2}{2(1-a^2)} \\ 0 & \frac{1}{1-a^2} & 0 \\ \frac{a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(E) = I - (1 - \sqrt{1 - a^2})E = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{\sqrt{1-a^2}-1}{2} \\ 0 & \sqrt{1 - a^2} & 0 \\ \frac{\sqrt{1-a^2}-1}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$P_E = \sqrt{I - P} = Q = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Set $u = (1, i, -1)$, $g = (1, 2, 1)$, $0 < a < \frac{1}{2}$ and let $T_1 = M_{\bar{u}}EM_u$. Then

$$k = a^2|g|^2E(|u|^2) = a^2(1, 4, 1)E(1, 1, 1) = (a^2, 4a^2, a^2),$$

$$v = \frac{k}{1 - kE(|u|^2)} = \left(\frac{a^2}{1 - a^2}, \frac{4a^2}{1 - 4a^2}, \frac{a^2}{1 - a^2}\right),$$

$$s = \frac{1 - \sqrt{1 - kE(|u|^2)}}{E(|u|^2)} = \left(1 - \sqrt{1 - a^2}, 1 - \sqrt{1 - 4a^2}, 1 - \sqrt{1 - a^2}\right),$$

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Take $T = M_gT_1$. Since $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$, then we have

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_a(T) = I - \operatorname{diag}(a^2, 4a^2, a^2)T_1 = \begin{bmatrix} \frac{2-a^2}{2} & 0 & \frac{a^2}{2} \\ 0 & 1 - 4a^2 & 0 \\ \frac{a^2}{2} & 0 & \frac{2-a^2}{2} \end{bmatrix},$$

$$K_a(T) = I + \operatorname{diag}\left(\frac{a^2}{1-a^2}, \frac{4a^2}{1-a^2}, \frac{a^2}{1-a^2}\right) T_1 = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{-a^2}{2(1-a^2)} \\ 0 & \frac{1}{2(1-4a^2)} & 0 \\ \frac{-a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(T) = I - M_s T_1 = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{1-\sqrt{1-a^2}}{2} \\ 0 & -1+\sqrt{1-4a^2} & 0 \\ \frac{1-\sqrt{1-a^2}}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$S_a^{-1}(T) = I - M_{\frac{s}{1-s}} T_1 = \begin{bmatrix} \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} \\ 0 & \frac{1}{\sqrt{1-4a^2}} & 0 \\ \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} \end{bmatrix}.$$

Since $\|T\| = \|gE(|u|^2)\|_\infty = \|(1,2,1)\|_\infty = 2$ and $T^*T = M_{|g|^2E(|u|^2)}T_1$, then $P_T^2 = I - \|T\|^{-2}T^*T = I - M_z T_1$, where $z = \frac{|g|^2E(|u|^2)}{4} = (\frac{1}{4}, 1, \frac{1}{4})$. It follows that

$$P_T = I - M_{(1-\sqrt{1-z})}T_1 = I - \operatorname{diag}\left(\frac{2-\sqrt{3}}{2}, 1, \frac{2-\sqrt{3}}{2}\right) T_1 = \begin{bmatrix} \frac{2+\sqrt{3}}{4} & 0 & \frac{2-\sqrt{3}}{4} \\ 0 & 0 & 0 \\ \frac{2-\sqrt{3}}{4} & 0 & \frac{2+\sqrt{3}}{4} \end{bmatrix}.$$

Note that, $r(T) = 2 > 0$ but $P_T \neq 0$ (see [2]). Also, $\mathcal{R}(T) = \bar{u}|g|\sqrt{E(|u|^2)}L^2(\mathcal{A}) = \{(1,-i,-1)(1,2,1)(1,1,1)(a,b,a) : a,b \in \mathbb{C}\} = \{(a,c,-a) : a,c \in \mathbb{C}\}$. Now set $u = (1,0,1)$ and $v = (2,-i,-2)$. Consider the rank-one operator $u \otimes v$ defined by $(u \otimes v)w = \langle w, v\rangle u$, for all $w \in \mathbb{C}^3$. Then $u \otimes v = \begin{pmatrix} 2 & i & -2 \\ 0 & 0 & 0 \\ 2 & i & -2 \end{pmatrix}$ and $(u \otimes v)T \neq T(u \otimes v)$. However, since

$$\sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|S_a(T)v\| \leq \sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|v\| = \|u\| \cdot \|v\| = 3\sqrt{2},$$

then by [9, Lemma 3.9], $u \otimes v \in \mathcal{B}_T$. Thus, $\mathcal{B}_T$ properly contains $\{T\}'$. In the finite dimensional case, if $\mathcal{A} \neq \Sigma$, then T is not injective and hence the spectral radius algebra $\mathcal{B}_T$ always properly contains the commutant of T.

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References

[1] A. Biswas, A. Lambert and S. Petrović, *On spectral radius algebras and normal operators*, Indiana Univ. Math. J. **56**(4) (2007), 1661–1674. `https://doi.org/10.1512/iumj.2007.56.2907`

[2] A. Biswas, A. Lambert, S. Petrović and B. Weinstock, *On spectral radius algebras*, Oper. Matrices **2**(2) (2008), 167–176. `https://doi.org/10.7153/oam-02-11`

[3] Y. Estaremi, *Unbounded weighted conditional expectation operators*, Complex Anal. Oper. Theory **10**(3) (2016), 567–580. `https://doi.org/10.1007/s11785-015-0499-y`

[4] Y. Estaremi and M. R. Jabbarzadeh, *Spectral radius algebras of WCE operators*, Oper. Matrices **11**(2) (2017), 337–346. https://doi.org/10.7153/oam-2017-11-21
[5] J. D. Herron, *Spectral radius algebras of idempotents*, Integral Equations Operator Theory **64**(2) (2009), 193–201. https://doi.org/10.1007/s00020-009-1684-z
[6] M. R. Jabbarzadeh and S. Khalil Sarbaz, *Lambert multipliers between L^p spaces*, Czechoslovak Math. J. **60**(1) (2010), 31–43. https://doi.org/10.1007/s10587-010-0012-8
[7] M. R. Jabbarzadeh and M. H. Sharifi, *Lambert conditional type operators on $L^2(\Sigma)$*, Linear Multilinear Algebra **67**(10) (2019), 2030–2047. https://doi.org/10.1080/03081087.2018.1479372
[8] M. R. Jabbarzadeh and M. H. Sharifi, *Operators whose ranges are contained in the null space of conditional expectations*, Math. Nachrichten **292**(11) (2019), 2427–2440. https://doi.org/10.1002/mana.201800066
[9] A. Lambert and S. Petrović, *Beyond hyperinvariance for compact operators*, J. Funct. Anal. **219**(1) (2005), 93–108. https://doi.org/10.1016/j.jfa.2004.06.001
[10] A. Lambert, *Conditional expectation related characterizations of the commutant of an abelian W^*-algebra*, Far East J. Math. Sci. **2**(1) (1994), 1–8.
[11] S. Petrović, *Spectral radius algebras, Deddens algebras, and weighted shifts*, Bull. Lond. Math. Soc. **43**(3) (2011), 513–522. https://doi.org/10.1112/blms/bdq118
[12] S. Petrović, *On the structure of the spectral radius algebras*, J. Operator Theory **60**(1) (2008), 137–148.
[13] M. M. Rao, *Conditional Measure and Applications*, Marcel Dekker, New York, 1993.

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