Kragujevac Journal of Mathematics
Volume 49(6) (2025), Pages 967–977.

ON SPECTRAL RADIUS ALGEBRAS AND CONDITIONAL TYPE OPERATORS

MOHAMMAD REZA JABBARZADEH AND BAHMAN MINAYI

Abstract. In this note, we study both the spectral radius and Deddens algebras associated to the normal weighted conditional type operators on $L^2(\Sigma)$. Also, in this setting, some other special properties of these algebras will be investigated.

1. Introduction and Preliminaries

Let (X, Σ, μ) be a complete σ-finite measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. If $B \subset X$, let $\mathcal{A}_B = \mathcal{A} \cap B$ denote the relative completion of the sigma-algebra generated by $\{A \cap B : A \in \mathcal{A}\}$. We denote the linear spaces of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\mathcal{A}$ be a sub-σ-finite algebra of Σ and let f be a non-negative Σ-measurable function on X. By the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$, where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Note that $E(f)$ depends both on μ and $\mathcal{A}$. A real-valued measurable function $f = f^+ - f^-$ is said to be conditionable if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = +\infty\}) = 0$. If f is complex-valued, then $f \in \mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$ if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. For $1 \leq p \leq +\infty$, one can show that every $L^p(\Sigma)$ function is conditionable. We use the notation $L^p(\mathcal{A})$ for $L^p(X, \mathcal{A}, \mu_{|_{\mathcal{A}}})$ and henceforth we write μ in place $\mu_{|_{\mathcal{A}}}$.

Key words and phrases. Deddens algebra, spectral radius algebra, conditional expectation, invariant subspace.

2020 *Mathematics Subject Classification.* Primary: 47A65. Secondary: 47L30.

DOI 10.46793/KgJMat2506.967J

Received: February 18, 2022.

Accepted: April 25, 2023.

The mapping $E^{\mathcal{A}} : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. In the case of $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. For further discussion of the conditional expectation operator see [13].

From now on we assume that u and w are conditionable. Operators of the form $M_wEM_u(f) = wE(uf)$ acting in $L^2(\Sigma)$ with $\mathcal{D}(M_wEM_u) = \{f \in L^2(\Sigma) : wE(uf) \in L^2(\Sigma)\}$ are called weighted conditional type (or weighted Lambert type) operators. Several aspects of this operator were studied in [4, 6–8]. Put $K = E(|u|^2)E(|w|^2)$. Estaremi in [3] proved that $M_wEM_u : \mathcal{D}(T) \to L^2(\Sigma)$ is densely defined if and only if $K - 1$ is finite valued (a.e.). Moreover, $T := M_wEM_u$ is bounded if and only if $\mathcal{D}(T) = L^2(\Sigma)$. In this case $T^* = M_{\bar{u}}EM_{\bar{w}}$ and $\|T\|^2 = \|K\|_\infty$. For a bounded linear operator T, $\operatorname{spec}(T)$ denote its spectrum. We say that $\lambda \in \mathbb{C}$ belongs to the essential range of a measurable function f if for each neighborhood G of λ, $\mu(f^{-1}(G)) > 0$. Positive, self-adjoint and normal bounded weighted conditional type operators and their spectrum have recently been characterized in [7] as follows.

Lemma 1.1 ([7]). *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then the followings hold.*

(*a*) *T is positive if and only if $T = M_{g\bar{u}}EM_u$ for some $0 \leq g \in L^0(\mathcal{A})$.*

(*b*) *T is self-adjoint if and only if $T = M_{g\bar{u}}EM_u$ for some $\bar{g} = g \in L^0(\mathcal{A})$.*

(*c*) *T is normal if and only if $T = M_{g\bar{u}}EM_u$ for some $g \in L^0(\mathcal{A})$.*

(*d*) $\operatorname{spec}(T) \setminus \{0\} = \operatorname{ess\,range}(E(uw)) \setminus \{0\}$.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. We use A^*, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the adjoint, the spectral radius, the range and the null space of $A \in B(\mathcal{H})$. A is normal if $A^*A = AA^*$ and A is positive if $\langle Ax, x\rangle \geq 0$ holds for each $x \in \mathcal{H}$ in which case we write $A \geq 0$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $A \in B(\mathcal{H})$ and let $Pj : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto $\mathcal{H}_j$ for $j = 1, 2$. Then $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ is the operator given by $A_{ij} = P_iAP_j\,|_{\mathcal{H}_j}$. In particular, $A(\mathcal{H}_1) \subseteq \mathcal{H}_1$ if and only if $A_{21} = 0$. Also, $\mathcal{H}_1$ reduces A if and only if $A_{12} = 0 = A_{21}$. Let $A \in B(\mathcal{H})$ with $r(A) \neq 0$ and let $0 < a < r(A)^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$. Since for all $n \in \mathbb{N}$, $\|a^{2n}A^{*n}A^n\| = a^{2n}\|A^n\|^2$, then we have $\overline{\lim}_{n\to+\infty}\|a^{2n}A^{*n}A^n\|^{\frac{1}{n}} = a^2\left(\overline{\lim}_{n\to+\infty}\|A^n\|^{\frac{1}{n}}\right)^2 = a^2r(A)^2 < 1$. This implies that the series $\sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$ is convergent in the norm topology of $B(\mathcal{H})$, and hence $K_a(A) \in B(\mathcal{H})$. Thus, the map f_A of $(0, r(A)^{-1})$ to $B(\mathcal{H})$ defined by $f_A(a) = K_a(A)$ is well-define, increasing and continuous. Also, for any $x \in \mathcal{H}$ we have that

$$\|x\|^2 \leq \sum_{n=0}^{+\infty} a^{2n}\|A^n(x)\|^2 = \langle K_a(A)x, x\rangle = \left\|\sqrt{K_a(A)}x\right\|^2 \leq \|K_a(A)\| \cdot \|x\|^2. \tag{1.1}$$

So, $K_a(A) \geq I$ and hence $K_a(A)$ is positive and invertible with $\|K_a(A)\| \geq 1$. Set $R_a(A) = K_a^{-1}(A)$ and $S_a(A) = \sqrt{R_a(A)}$. Replacing x by $(K_a(A))^{\frac{-1}{2}}(x)$ in (1.1) we obtain that $\|S_a(A)\| \leq 1$ and $\|R_a(A)\| = \|S_a^2(A)\| \leq 1$. Consequently, $R_a(A)$ and $S_a(A)$ are positive and invertible elements of $B(\mathcal{H})$ and

$$\|K_a(A)\| = \sup_{\|x\|=1} \langle K_a(A)x, x\rangle = \sum_{n=0}^{+\infty} a^{2n}\|A^n\|^2 \leq \sum_{n=0}^{+\infty} \left(\|aA\|^2\right)^n = \frac{1}{1-\|aA\|^2}. \tag{1.2}$$

Let $\{A_m\} \subseteq \{T \in B(\mathcal{H}) : r(T) \leq r(A)\}$. If $\|A_m - A\| \to 0$, then for each $n \in \mathbb{N}$ and $0 < a < r(A)^{-1}$, $a^{2n}A_m^{*n}A_m^n \to a^{2n}A^{*n}A^n$, and so $\|K_a(A_m) - K_a(A)\| \to 0$ as $m \to +\infty$. But the converse is not true. Indeed, if A_1 and A_2 are distinct unitary operators on $\mathcal{H}$, then $K_a(A_1) = K_a(A_2) = (1-a^2)^{-1}I$ for all $0 < a < 1$. In [9] A. Lambert and S. Petrović define the spectral radius algebra of a bounded linear operator A with $S_a = S_a(A)$ and $0 < a < r(A)^{-1}$ to be the unital subalgebra

$$\mathcal{B}_A = \{T \in B(\mathcal{H}) : \sup_a \|S_a^{-1}TS_a\| < +\infty\}.$$

Lastly, define

$$\mathcal{Q}_A = \{T \in B(\mathcal{H}) : \lim_{a\to r(A)^{-1}} \|S_a^{-1}TS_a\| = 0\}.$$

In [9] it is shown that, $\|K_a(A)\| \to +\infty$ as $a \to \|A\|^{-1}$ and for any A, $\mathcal{Q}_A \subseteq \mathcal{B}_A$ is a two-sided ideal consisting entirely of quasinilpotent operators. Furthermore, if A is quasinilpotent, then $A \in \mathcal{Q}_A$.

We now consider the Deddens algebra $\mathcal{D}_A$ associated with $A \in B(\mathcal{H})$, that is, the family of those operators $T \in B(\mathcal{H})$ for which there is a constant $M > 0$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathcal{H}$, $\|A^nTx\| \leq M\|A^nx\|$. $\mathcal{D}_A$ is indeed a unital subalgebra of $B(\mathcal{H})$ with the property that $\{A\}' \subseteq \mathcal{D}_A \subseteq \mathcal{B}_A$, where $\{A\}'$ is the commutant of A (see [11]).

Let $A \in B(\mathcal{H})$ be normal and $0 < a < \|A\|^{-1}$. Then A^n and A^{*n} commute with $K_a(A)$, $R_a(A)$, $S_a(A)$ and $K_a(A^*) = K_a(A) = K_a(|A|)$, where $|A|^2 = A^*A$. Moreover,

$$\begin{aligned} K_a(A) &= \sum_{n=0}^{+\infty} a^{2n}(A^*A)^n = (I - a^2A^*A)^{-1}, \\ R_a(A) &= I - a^2A^*A, \\ S_a(A) &= \sqrt{I - a^2A^*A}, \\ P_A := \lim_{a\to\|A\|^{-1}} S_a(A) &= \sqrt{I - \|A\|^{-2}A^*A}. \end{aligned} \tag{1.3}$$

For more details on the Deddens and spectral radius algebras see [1,5,11,12]. In the next section, we investigate the spectral radius and the Deddens algebras related to the bounded weighted conditional type operators on $L^2(\Sigma)$. All of these are basically discussed using the conditional expectation properties.

2. $\mathcal{B}_T$ and $\mathcal{D}_T$ Associated with $T = M_wEM_u$

From now on we assume that $E(|u|^2) \in L^\infty(\mathcal{A})$, i.e., $T_1 := M_{\bar{u}}EM_u \in B(L^2(\Sigma))$.

Lemma 2.1. *For $0 \le b \in L^0(\mathcal{A})$, let $M_bT_1 \in B(L^2(\Sigma))$. Then the followings hold.*
(a) *If $1 \notin \operatorname{spec}(M_bT_1)$, then $(I - M_bT_1)^{-1} = I + M_{\frac{b}{1-bE(|u|^2)}}T_1$.*
(b) *If $-1 \notin \operatorname{spec}(M_bT_1)$, then $(I + M_bT_1)^{-1} = I - M_{\frac{b}{1+bE(|u|^2)}}T_1$.*

Proof. We only proof (a), since (b) follows similarly.
Let $1 \in \operatorname{spec}(M_bT_1)$. Using Lemma 1.1 ($d$), $1 \notin \operatorname{ess\,range} E(b|u|^2)$ and so $(1 - bE(|u|^2))^{-1} \in L^\infty(\mathcal{A})$. Put $S = I + M_{b(1-bE(|u|^2))^{-1}}T_1$. Then $\|S\| \le 1 + \|(1 - bE(|u|^2))^{-1}\|_\infty \|M_bT_1\| < +\infty$. Also, direct computations show that $S(I - M_bT_1) = (I - M_bT_1)S = I$. Now, the desired conclusion holds. $\square$

Set $\mathcal{N} = \{M_wEM_u \in B(L^2(\Sigma)) : M_wEM_u \text{ is normal}\}$. By Lemma 1.1 ($c$) we have $\mathcal{N} = \{M_gT_1 \in B(L^2(\Sigma)) : g \in L^0(\mathcal{A}), T_1 = M_{\bar{u}}EM_u, u \in L^0(\Sigma)\}$.

Corollary 2.1. *Let $T = M_wEM_u \in \mathcal{N}$ and let $0 < a < r(T)^{-1}$. Then $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$ for some $k, v \in L^0(\mathcal{A})$ and $\|K_a(T)\| = 1 + \|vE(|u|^2)\|_\infty$.*

Proof. By Lemma 1.1 (c), $T = M_gT_1$ for some $g \in L^0(\mathcal{A})$. Since $T^*T = M_{|g|^2E(|u|^2)}T_1$, then by (1.3) we get that $K_a(T) = (I - M_kT_1)^{-1}$, where $k = a^2|g|^2E(|u|^2)$. Thus, $R_a(T) = (K_a(T))^{-1} = I - M_kT_1$. Also, since $1/a^2 > (r(T))^2 = r(T^*T)$, then $1/a^2 \notin \operatorname{spec}(T^*T) = \operatorname{ess\,range} |g|^2(E(|u|^2))^2$. Therefore,

$$\frac{1}{1 - kE(|u|^2)} = \frac{1}{a^2\{\frac{1}{a^2} - |g|^2(E(|u|^2))^2\}} \in L^\infty(\mathcal{A})$$

and $1 \notin \operatorname{spec}(M_kT_1)$. Now, by Lemma 2.1, $K_a(T) = I + M_vT_1$, where $v = \frac{k}{1-kE(|u|^2)}$. Moreover, since M_vT_1 is positive, then $\|K_a(T)\| = 1 + \|M_vT_1\| = 1 + \|vE(|u|^2)\|_\infty$. This completes the proof. $\square$

Corollary 2.2. *Under the assumption of above corollary, $S_a(T) = I - M_sT_1$ and $S_a^{-1}(T) = I + M_{\frac{s}{1-sE(|u|^2)}}T_1$ for some $s \in L^0(\mathcal{A})$.*

Proof. Set $s = \frac{1-\sqrt{1-kE(|u|^2)}}{E(|u|^2)}\chi_{\sigma(E(|u|^2))}$. Then, for $f \in L^2(\Sigma)$ we have

$$\begin{aligned}(I - M_{s\bar{u}}EM_u)^2(f) &= (I - M_{s\bar{u}}EM_u)(f - s\bar{u}E(uf)) \\ &= f - s\bar{u}E(uf) - s\bar{u}E(uf - s|u|^2E(uf)) \\ &= f - \bar{u}(-2s + E(|u|^2)s^2)E(uf) \\ &= f - \bar{u}kE(uf) \\ &= (I - M_{k\bar{u}}EM_u)(f).\end{aligned}$$

It follows that $S_a(T) = (R_a(T))^{1/2} = (I + M_kT_1)^{1/2} = I - M_sT_1$. Now, the inverse of $S_a(T)$ follows from Lemma 2.1 (a). $\square$

For $T \in \mathcal{N}$ and $v \in L^0(\mathcal{A})$, it is easy to check that M_vT_1 commutes with $S_a(T)$. It follows that $\{M_vT_1 \in B(L^2(\Sigma)) : v \in L^0(\mathcal{A})\} \subseteq \mathcal{B}_T$.

Lemma 2.2. *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then $\mathcal{N}(T) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^\perp$.*

Proof. Let $f \in L^2(\Sigma)$. Since $\mathcal{R}(E) = L^2(\mathcal{A})$, then we have

$$\begin{aligned} f \in \mathcal{N}(T) \Leftrightarrow \|Tf\|^2 = 0 &\Leftrightarrow \int_X E(|w|^2)|E(uf)|^2 d\mu = 0 \\ &\Leftrightarrow \int_X \left| E(u\sqrt{E(|w|^2)}f)\right|^2 d\mu = 0 \\ &\Leftrightarrow u\sqrt{E(|w|^2)}f \in \mathcal{N}(E) = L^2(\mathcal{A})^{\perp} \\ &\Leftrightarrow \left\langle u\sqrt{E(|w|^2)}f, g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow \left\langle f, \bar{u}\sqrt{E(|w|^2)}g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow f \in \left\{ \bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^{\perp}. \end{aligned}$$

□

Corollary 2.3. $\overline{\mathcal{R}(M_{\bar{u}}EM_u)} = \overline{\bar{u}\sqrt{E(|u|^2)}L^2(\mathcal{A})} = c.l.s.\left\{\bar{u}\sqrt{E(|u|^2)}\chi_A : A \in \mathcal{A}_{\sigma(u)}\right\}$, *where c.l.s. stands for closed linear span. In particular,* $\overline{\mathcal{R}(EM_u)} = \overline{\bar{u}L^2(\mathcal{A})}$.

Let P be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$ and let $Q = I - P$. Direct computations show that

$$(I - \alpha P)^{-1} = I + \frac{\alpha}{1-\alpha}P, \quad \alpha \neq 1, \tag{2.1}$$

$$(I - \alpha P)^{\frac{1}{2}} = I - (1 - \sqrt{1-\alpha})P, \quad \alpha \leq 1. \tag{2.2}$$

Let $0 < a < 1$. Then $K_a(P) = \sum_{n=0}^{+\infty} a^{2n}P^{*n}P^n = I + \frac{a^2}{1-a^2}P$. Using (2.1) and (2.2) we obtain that

$$\begin{aligned} R_a(P) &= (K_a(P))^{-1} = I - a^2P, \\ S_a(P) &= (R_a(P))^{\frac{1}{2}} = I - (1 - \sqrt{1-a^2})P, \\ S_a^{-1}(P) &= I + \frac{1-\sqrt{1-a^2}}{\sqrt{1-a^2}}P. \end{aligned}$$

Note that if we take $P = M_{\bar{u}E(|u|^2)^{-1}}EM_u$, then $P^2 = P = P^*$, with $\mathcal{R}(P) = \overline{\bar{u}E(|u|^2)^{-1/2}L^2(\mathcal{A})}$. Now, let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Since

$$S_a(P) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} M_{1-\sqrt{1-a^2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix},$$

then we have

$$\mathcal{P}_a(S) := (S_a^{-1}(P))S(S_a(P)) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} X & YM_{\frac{1}{\sqrt{1-a^2}}} \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}.$$

It follows that $\sup\{\|\mathcal{P}_a(S)\| : 0 < a < 1\} < +\infty$ if and only if $Y = 0$. For some $0 < a < 1$, $\mathcal{P}_a(S) = S$ if and only if $Y = Z = 0$. Also, $\lim_{a\to 1} \|\mathcal{P}_a(S)\| = 0$ if and only if $X = Y = W = 0$. Moreover, we have

$$\mathcal{P}_a(SP) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} X & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Thus, $SP \in \mathcal{B}_P$ for all $S \in B(L^2(\Sigma))$. Also if $X = 0$, then $SP \in \mathcal{Q}_P$. Similar computations show that

$$\mathcal{P}_a(QS) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}, \quad \mathcal{P}_a(QSP) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Let $\{S_n\} \subseteq \mathcal{B}_P$ and let $S_n := \begin{pmatrix} X_n & 0 \\ Z_n & W_n \end{pmatrix} \to S$ as $n \to +\infty$. Then

$$\|Y\| \le \|S_n - S\| = \left\| \begin{pmatrix} X_n - X & Y \\ Z_n - Z & W_n - W \end{pmatrix} \right\| \to 0.$$

It follows that $Y = 0$ and hence $\mathcal{B}_P$ is closed in the norm operator topology on $B(L^2(\Sigma))$. Moreover, by definition, $S \in \mathcal{D}_P$ if and only if there exists $M > 0$ such that

$$\begin{aligned} \|PSf\| &= \left\| \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} XPf + YQf \\ 0 \end{pmatrix} \right\| \le M \left\| \begin{pmatrix} Pf \\ 0 \end{pmatrix} \right\|, \end{aligned}$$

for all $f \in L^2(\Sigma)$. Replacing f by Qf in the above and taking $M = M(S) = \|X\|$, we obtain that $S \in \mathcal{D}_P$ if and only if $Y = 0$ on $\mathcal{N}(P)$. As an easy consequence of these observations, we have the following result.

Proposition 2.1. *Let P be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$, $0 < a < 1$ and let $Q = I - P$. Set*

$$\begin{aligned} \mathcal{Q}_1 &= \{SP : S \in B(L^2(\Sigma)), PSP = 0\}, \\ \mathcal{Q}_2 &= \{QS : S \in B(L^2(\Sigma)), QSQ = 0\}, \\ \mathcal{Q}_3 &= \{QSP : S \in B(L^2(\Sigma))\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{B}_P &= \{S \in B(L^2(\Sigma)) : S(\mathcal{N}(P)) \subseteq \mathcal{N}(P)\} = \mathcal{D}_P, \\ \mathcal{Q}_P &= \{S \in B(L^2(\Sigma)) : QSP = T\} \supseteq \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3. \end{aligned}$$

Moreover, $\mathcal{P}_a(S) = S$ if and only if $\mathcal{M}$ reduces S.

Set $P = E^{\mathcal{A}} = E$, $0 < a < 1$ and $\mathcal{P}_a = \mathcal{E}_a$. Let $S = M_w E M_u \in B(L^2(\Sigma))$. Using Proposition 2.1 and [7, Proposition 2.30] with respect the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, we have

$$\begin{aligned}
&ESE = 0 \Leftrightarrow M_{E(u)E(w)}|_{L^2(\mathcal{A})} = 0,\\
&ESQ = 0 \Leftrightarrow u\chi_{\sigma(E(w))} \in L^0(\mathcal{A}),\\
&QSE = 0 \Leftrightarrow w\chi_{\sigma(E(u))} \in L^0(\mathcal{A}),\\
&QSQ = 0 \Leftrightarrow L^2(\mathcal{A})^{\perp} = \mathcal{R}(Q) \subseteq \mathcal{N}(S) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^{\perp}.
\end{aligned}$$

So we have the following corollary.

Corollary 2.4. *Let $S = M_w E M_u \in B(L^2(\Sigma))$ and $0 < a < 1$. Then,*
(a) $S \in \mathcal{B}_E$ if and only if $u\chi_{\sigma(E(w))} \in L^0(\mathcal{A})$;
(b) $S \in \mathcal{Q}_E$ if and only if $S \in \mathcal{B}_E$, $M_{E(uE(w))}|_{L^2(\mathcal{A})} = 0$ and $\overline{\mathcal{R}(S)} \subseteq L^2(\mathcal{A})$;
(c) $\mathcal{E}_a(S) = S$ if and only if $\{u\chi_{\sigma(E(w))}, w\chi_{\sigma(E(u))}\} \subseteq L^0(\mathcal{A})$.

Let $\mathcal{M}(\mathcal{A}) = \{M_\vartheta : \vartheta \in L^\infty(\mathcal{A})\}$ and let $\mathcal{M}'(\mathcal{A})$ be its commutant. It is known that $\mathcal{M}(\Sigma)$ is a maximal abelian subalgebra of $B(L^2(\Sigma))$. But it is invalid if Σ is replaced by $\mathcal{A} \neq \Sigma$. Indeed, for any $\mathcal{A} \subset \mathcal{B}$, $E^{\mathcal{B}} \in \mathcal{M}'(\mathcal{A}) \setminus \mathcal{M}(\mathcal{A})$. Alan Lambert in [10, Theorem 3.2] proved that $S \in \mathcal{M}'(\mathcal{A})$ if and only if there exists $C > 0$ such that $E(|Sf|^2) \leq CE(|f|^2)$ for all $f \in L^2(\Sigma)$. Consequently, if $S \in B(L^2(\Sigma))$ and $\{\vartheta_n, \vartheta_n^{-1}\} \subseteq L^\infty(\mathcal{A})$, then $\sup_n \|M_{\vartheta_n^{-1}} S M_{\vartheta_n}\| < +\infty$ whenever $S \in \mathcal{M}'(\mathcal{A})$.

For a fixed $T = M_g T_1 \in \mathcal{N}$ and $0 < a < \|T\|^{-1}$, put $A := S_a^{-1}(T)$. Then by Corollary 2.2, $A = I + M_\theta T_1$ for some $0 \leq \theta \in L^0(\mathcal{A})$. Since A is bounded, then so is $M_\theta T_1$. Thus, $\theta E(|u|^2) \in L^\infty(\mathcal{A})$ and hence $\theta E(|u|^2)g \in L^2(\mathcal{A})$ for all $f \in L^2(\mathcal{A})$. Relative to the direct sum decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$, the matrix form of A is $(A_{ij})_{1 \leq i,j \leq 2}$. Set $P = P_{\overline{\mathcal{R}(T_1)}}$ and $Q = I - P$. Let $f \in L^2(\Sigma)$. Then without loss of generality, we can assume that $Pf = \bar{u}\sqrt{E(|u|^2)}g$, for some $g \in L^2(\mathcal{A})$. Then

$$\begin{aligned}
A_{11}f &= P(A(Pf)) = P(Pf + \theta\bar{u}E(uPf))\\
&= P(Pf + \bar{u}\sqrt{E(|u|^2)}\left(\theta E(|u|^2)g)\right)\\
&= Pf + \theta E(|u|^2)Pf\\
&= M_{1+\theta E(|u|^2)}Pf,
\end{aligned}$$

where $\theta = 1 + \frac{s}{1 - sE(|u|^2)}$. By Corollary 2.2, $1 + \theta E(|u|^2) = \frac{1}{1 - sE(|u|^2)} = \frac{1}{\sqrt{1 - kE(|u|^2)}}$ where $k = a^2|g|^2E(|u|^2)$. Thus, $A_{11} = PAP = M_{(1-kE(|u|^2))^{-1/2}}P$. Similar computations show that $A_{12} = A_{21} = 0$ and $A_{22} = I_{|\mathcal{N}(T_1)}$. Let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$. Set $\mathcal{L}_a(S) := (S_a^{-1}(T))S(S_a(T))$ and $\vartheta := \sqrt{1 - kE(|u|^2)}$. Then

$\vartheta \to 0$ as $a \to \|T\|^{-1}$ and

$$\mathcal{L}_a(S) = \begin{pmatrix} M_{\frac{1}{\vartheta}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_\vartheta & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_{\frac{1}{\vartheta}} X M_\vartheta & M_{\frac{1}{\vartheta}} Y \\ Z M_\vartheta & W \end{pmatrix}.$$

Since $M_\vartheta \to 0$ as $a \to \|T\|^{-1}$, so $\sup\{\|M_{\vartheta^{-1}}\|;\ 0 < a < \|T\|^{-1}\} = +\infty$. Let $M := \sup\{\|M_{\vartheta^{-1}} Y\| < +\infty,\ 0 < a < \|T\|^{-1}\}$. Then for all unit vector $f \in \mathcal{N}(T_1)$, $\|Y(f)\| = \|M_\vartheta M_{\vartheta^{-1}} Y(f)\| \leq M\|M_\vartheta\|$. It follows that $\|Y(f)\| = 0$ and hence $Y_{|_{\mathcal{N}(T_1)}} = 0$. In particular, if $PSP \in \mathcal{M}'(\mathcal{A})$, then $S \in \mathcal{B}_T$ if and only if $S(\mathcal{N}(T_1)) \subseteq \mathcal{N}(T_1)$. In this case, $\mathcal{B}_{M_{g_1T_1}} = \mathcal{B}_{M_{g_2T_1}}$ for all $\{M_{g_1T_1}, M_{g_1T_1}\} \subseteq \mathcal{N}$. Note that

$$\mathcal{L}_a(T) = S_a^{-1}(T) \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix} S_a(T) = \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\|\mathcal{L}_a(T)\| = \|gE(|u|^2)\|_\infty = \|T\| = r(T)$. In view of these observations we have the following results.

Theorem 2.1. *Let $T = M_gT_1 \in \mathcal{N}$ and let $\vartheta = \sqrt{1 - a^2|g|^2(E(|u|^2))^2}$. Then the followings hold.*

(*a*) *$S \in \mathcal{B}_T$ if and only if $Y = 0$ and* $\sup\{\|M_{\vartheta^{-1}} X M_\vartheta\| :\ 0 < a < \|T\|^{-1}\} < +\infty$. *In particular, if $XM_\vartheta = M_\vartheta X$, then $S \in \mathcal{B}_T$ if and only if $\{\bar{u}\sqrt{E(|u|^2)}L^2(\mathcal{A})\}^\perp$ is an invariant subspace for S.*

(*b*) *$S \in \mathcal{Q}_T$ if and only if $Y = W = 0$ and $\|M_{\vartheta^{-1}} X M_\vartheta\| \to 0$, as $a \to \|T\|^{-1}$. Moreover, if $XM_\vartheta = M_\vartheta X$, then $S \in \mathcal{Q}_T$ if and only if $X = Y = W = 0$.*

Let $T = M_gT_1 \in \mathcal{N}$ and $S \in B(L^2(\Sigma))$. Then, for all $n \in \mathbb{N}$ and $f \in L^2(\Sigma)$, $T^n = M_{g^n(E(|u|^2))^{n-1}} T_1$ and

$$T^n S f = \begin{pmatrix} M_{\omega^n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} = \begin{pmatrix} M_{\omega^n} XPf + M_{\omega^n} YQf \\ 0 \end{pmatrix},$$

where $\omega = gE(|u|^2)$. It follows that $S \in \mathcal{D}_T$ if and only if there exists $M > 0$ such that $\|M_{\omega^n} XPf + M_{\omega^n} YQf\| \leq M\|M_{\omega^n} Pf\|$. If we set $f = Qg$, for some $g \in L^2(\Sigma)$, then we get $\|M_{\omega^n|_{\sigma(\omega)}} YQg\| \leq \|M_{\omega^n} YQg\| = 0$ and hence $Y\ |_{\mathcal{N}(T_1)} = 0$. Now, if $M_\omega X = XM_\omega$, then $\|M_{\omega^n} XPf\| \leq \|X\| \cdot \|M_{\omega^n} Pf\|$. Note that the commutativity of M_ω and X implies that $M_\vartheta X = XM_\vartheta$. So we have the following result.

Theorem 2.2. *Let $T = M_gT_1 \in \mathcal{N}$, $\omega = gE(|u|^2)$ and let $S \in B(L^2(\Sigma))$. Then $S \in \mathcal{D}_T$ if and only if $PSP \in \mathcal{D}_T$ and $PSQ = 0$. Moreover, if $(PSP)M_\omega = M_\omega(PSP)$, then $\mathcal{D}_T = \mathcal{B}_T$.*

Corollary 2.5. *Let $\{T, S\} \subseteq \mathcal{N}$. Then $S \in \mathcal{B}_T$ if and only if $PSQ = 0$.*

Proof. Let $S = M_{g_1\bar{v}} E M_v \in \mathcal{B}_T$, with $g_1 \in L^0(\mathcal{A})$. Then $PSP = M_\gamma$, where $\gamma = g_1E(u)E(\bar{v})E(\bar{u}v) \in L^0(\mathcal{A})$. Since PSP commutes with M_γ, then the desired conclusion follows from Theorem 2.2. $\square$

Example 2.1. Let $X = \{1,2,3\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/3$ and let $\mathcal{A}$ be the σ-algebra generated by the partition $\{\{1,3\},\{2\}\}$. Then $L^2(\Sigma) \cong \mathbb{C}^3$ and

$$E(f) = \left(\frac{1}{\mu(A_1)}\int_{A_1} f d\mu\right)\chi_{A_1} + \left(\frac{1}{\mu(A_2)}\int_{A_2} f d\mu\right)\chi_{A_2} = \frac{f_1+f_3}{2}\chi_{A_1} + f_2\chi_{A_2},$$

where $A_1 = \{1,3\}$ and $A_2 = \{2\}$. Then matrix representation of E with respect to the standard orthonormal basis is $E = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. It can be easily checked that $E^2 = E = E^*$, $\mathcal{N}_2(E) = \langle (a,0,-a) : a \in \mathbb{C}\rangle$, $\mathcal{R}(E) = \langle (a,b,a) : a,b \in \mathbb{C}\rangle$. For $1 < a < 1$ we have

$$K_a(E) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{a^2}{1-a^2}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{a^2}{2(1-a^2)} \\ 0 & \frac{1}{1-a^2} & 0 \\ \frac{a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(E) = I - (1-\sqrt{1-a^2})E = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{\sqrt{1-a^2}-1}{2} \\ 0 & \sqrt{1-a^2} & 0 \\ \frac{\sqrt{1-a^2}-1}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$P_E = \sqrt{I-P} = Q = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Set $u = (1,i,-1)$, $g = (1,2,1)$, $0 < a < \frac{1}{2}$ and let $T_1 = M_{\bar{u}}EM_u$. Then

$$k = a^2|g|^2E(|u|^2) = a^2(1,4,1)E(1,1,1) = (a^2,4a^2,a^2),$$

$$v = \frac{k}{1-kE(|u|^2)} = \left(\frac{a^2}{1-a^2}, \frac{4a^2}{1-4a^2}, \frac{a^2}{1-a^2}\right),$$

$$s = \frac{1-\sqrt{1-kE(|u|^2)}}{E(|u|^2)} = \left(1-\sqrt{1-a^2}, 1-\sqrt{1-4a^2}, 1-\sqrt{1-a^2}\right),$$

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Take $T = M_gT_1$. Since $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$, then we have

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_a(T) = I - \operatorname{diag}(a^2,4a^2,a^2)T_1 = \begin{bmatrix} \frac{2-a^2}{2} & 0 & \frac{a^2}{2} \\ 0 & 1-4a^2 & 0 \\ \frac{a^2}{2} & 0 & \frac{2-a^2}{2} \end{bmatrix},$$

$$K_a(T) = I + \operatorname{diag}\left(\frac{a^2}{1-a^2}, \frac{4a^2}{1-a^2}, \frac{a^2}{1-a^2}\right) T_1 = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{-a^2}{2(1-a^2)} \\ 0 & \frac{1}{2(1-4a^2)} & 0 \\ \frac{-a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(T) = I - M_s T_1 = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{1-\sqrt{1-a^2}}{2} \\ 0 & -1+\sqrt{1-4a^2} & 0 \\ \frac{1-\sqrt{1-a^2}}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$S_a^{-1}(T) = I - M_{\frac{s}{1-s}} T_1 = \begin{bmatrix} \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} \\ 0 & \frac{1}{\sqrt{1-4a^2}} & 0 \\ \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} \end{bmatrix}.$$

Since $\|T\| = \|gE(|u|^2)\|_\infty = \|(1,2,1)\|_\infty = 2$ and $T^*T = M_{|g|^2E(|u|^2)}T_1$, then $P_T^2 = I - \|T\|^{-2}T^*T = I - M_z T_1$, where $z = \frac{|g|^2E(|u|^2)}{4} = (\frac{1}{4}, 1, \frac{1}{4})$. It follows that

$$P_T = I - M_{(1-\sqrt{1-z})}T_1 = I - \operatorname{diag}\left(\frac{2-\sqrt{3}}{2}, 1, \frac{2-\sqrt{3}}{2}\right) T_1 = \begin{bmatrix} \frac{2+\sqrt{3}}{4} & 0 & \frac{2-\sqrt{3}}{4} \\ 0 & 0 & 0 \\ \frac{2-\sqrt{3}}{4} & 0 & \frac{2+\sqrt{3}}{4} \end{bmatrix}.$$

Note that, $r(T) = 2 > 0$ but $P_T \neq 0$ (see [2]). Also, $\mathcal{R}(T) = \bar{u}|g|\sqrt{E(|u|^2)}L^2(\mathcal{A}) = \{(1,-i,-1)(1,2,1)(1,1,1)(a,b,a) : a,b \in \mathbb{C}\} = \{(a,c,-a) : a,c \in \mathbb{C}\}$. Now set $u = (1,0,1)$ and $v = (2,-i,-2)$. Consider the rank-one operator $u \otimes v$ defined by $(u \otimes v)w = \langle w, v\rangle u$, for all $w \in \mathbb{C}^3$. Then $u \otimes v = \begin{pmatrix} 2 & i & -2 \\ 0 & 0 & 0 \\ 2 & i & -2 \end{pmatrix}$ and $(u \otimes v)T \neq T(u \otimes v)$. However, since

$$\sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|S_a(T)v\| \leq \sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|v\| = \|u\| \cdot \|v\| = 3\sqrt{2},$$

then by [9, Lemma 3.9], $u \otimes v \in \mathcal{B}_T$. Thus, $\mathcal{B}_T$ properly contains $\{T\}'$. In the finite dimensional case, if $\mathcal{A} \neq \Sigma$, then T is not injective and hence the spectral radius algebra $\mathcal{B}_T$ always properly contains the commutant of T.

Acknowledgements. The authors would like to sincerely thank the anonymous referee for carefully reading the paper and for very useful comments.

References

[1] A. Biswas, A. Lambert and S. Petrović, *On spectral radius algebras and normal operators*, Indiana Univ. Math. J. **56**(4) (2007), 1661–1674. `https://doi.org/10.1512/iumj.2007.56.2907`

[2] A. Biswas, A. Lambert, S. Petrović and B. Weinstock, *On spectral radius algebras*, Oper. Matrices **2**(2) (2008), 167–176. `https://doi.org/10.7153/oam-02-11`

[3] Y. Estaremi, *Unbounded weighted conditional expectation operators*, Complex Anal. Oper. Theory **10**(3) (2016), 567–580. `https://doi.org/10.1007/s11785-015-0499-y`

[4] Y. Estaremi and M. R. Jabbarzadeh, *Spectral radius algebras of WCE operators*, Oper. Matrices **11**(2) (2017), 337–346. https://doi.org/10.7153/oam-2017-11-21

[5] J. D. Herron, *Spectral radius algebras of idempotents*, Integral Equations Operator Theory **64**(2) (2009), 193–201. https://doi.org/10.1007/s00020-009-1684-z

[6] M. R. Jabbarzadeh and S. Khalil Sarbaz, *Lambert multipliers between L^p spaces*, Czechoslovak Math. J. **60**(1) (2010), 31–43. https://doi.org/10.1007/s10587-010-0012-8

[7] M. R. Jabbarzadeh and M. H. Sharifi, *Lambert conditional type operators on $L^2(\Sigma)$*, Linear Multilinear Algebra **67**(10) (2019), 2030–2047. https://doi.org/10.1080/03081087.2018.1479372

[8] M. R. Jabbarzadeh and M. H. Sharifi, *Operators whose ranges are contained in the null space of conditional expectations*, Math. Nachrichten **292**(11) (2019), 2427–2440. https://doi.org/10.1002/mana.201800066

[9] A. Lambert and S. Petrović, *Beyond hyperinvariance for compact operators*, J. Funct. Anal. **219**(1) (2005), 93–108. https://doi.org/10.1016/j.jfa.2004.06.001

[10] A. Lambert, *Conditional expectation related characterizations of the commutant of an abelian W^*-algebra*, Far East J. Math. Sci. **2**(1) (1994), 1–8.

[11] S. Petrović, *Spectral radius algebras, Deddens algebras, and weighted shifts*, Bull. Lond. Math. Soc. **43**(3) (2011), 513–522. https://doi.org/10.1112/blms/bdq118

[12] S. Petrović, *On the structure of the spectral radius algebras*, J. Operator Theory **60**(1) (2008), 137–148.

[13] M. M. Rao, *Conditional Measure and Applications*, Marcel Dekker, New York, 1993.

Faculty of Mathematical Sciences,
University of Tabriz,
P. O. Box: 5166615648, Tabriz, Iran
Email address: mjabbar@tabrizu.ac.ir
Email address: b.minayi@tabrizu.ac.ir