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# ON SPECTRAL RADIUS ALGEBRAS AND CONDITIONAL TYPE OPERATORS

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Abstract. In this note, we study both the spectral radius and Deddens algebras associated to the normal weighted conditional type operators on $L^2(\Sigma)$. Also, in this setting, some other special properties of these algebras will be investigated.

## 1. Introduction and Preliminaries

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. If $B \subset X$, let $\mathcal{A}_B = \mathcal{A} \cap B$ denote the relative completion of the sigma-algebra generated by $\{A \cap B : A \in \mathcal{A}\}$. We denote the linear spaces of all complex-valued $\Sigma$-measurable functions on $X$ by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Let $\mathcal{A}$ be a sub-$\sigma$-finite algebra of $\Sigma$ and let $f$ be a non-negative $\Sigma$-measurable function on $X$. By the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$, where $A$ is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Note that $E(f)$ depends both on $\mu$ and $\mathcal{A}$. A real-valued measurable function $f = f^+ - f^-$ is said to be conditionable if $\mu(\{x \in X : E(f^+)(x) = E(f^-)(x) = +\infty\}) = 0$. If $f$ is complex-valued, then $f \in \mathcal{D}(E) = \{f \in L^0(\Sigma) : f \text{ is conditionable}\}$ if the real and imaginary parts of $f$ are conditionable and their respective expectations are not both infinite on the same set of positive measure. For $1 \leq p \leq +\infty$, one can show that every $L^p(\Sigma)$ function is conditionable. We use the notation $L^p(\mathcal{A})$ for $L^p(X, \mathcal{A}, \mu_{|_\mathcal{A}})$ and henceforth we write $\mu$ in place $\mu_{|_\mathcal{A}}$.

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The mapping $E^{\mathcal{A}} : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $f \mapsto E^{\mathcal{A}}(f)$, is called the conditional expectation operator with respect to $\mathcal{A}$. In the case of $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$. For further discussion of the conditional expectation operator see [13].

From now on we assume that $u$ and $w$ are conditionable. Operators of the form $M_wEM_u(f) = wE(uf)$ acting in $L^2(\Sigma)$ with $\mathcal{D}(M_wEM_u) = \{f \in L^2(\Sigma) : wE(uf) \in L^2(\Sigma)\}$ are called weighted conditional type (or weighted Lambert type) operators. Several aspects of this operator were studied in [4, 6–8]. Put $K = E(|u|^2)E(|w|^2)$. Estaremi in [3] proved that $M_wEM_u : \mathcal{D}(T) \to L^2(\Sigma)$ is densely defined if and only if $K - 1$ is finite valued (a.e.). Moreover, $T := M_wEM_u$ is bounded if and only if $\mathcal{D}(T) = L^2(\Sigma)$. In this case $T^* = M_{\bar{u}}EM_{\bar{w}}$ and $\|T\|^2 = \|K\|_\infty$. For a bounded linear operator $T$, $\operatorname{spec}(T)$ denote its spectrum. We say that $\lambda \in \mathbb{C}$ belongs to the essential range of a measurable function $f$ if for each neighborhood $G$ of $\lambda$, $\mu(f^{-1}(G)) > 0$. Positive, self-adjoint and normal bounded weighted conditional type operators and their spectrum have recently been characterized in [7] as follows.

**Lemma 1.1** ([7])**.** *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then the followings hold.*

(*a*) *$T$ is positive if and only if $T = M_{g\bar{u}}EM_u$ for some $0 \leq g \in L^0(\mathcal{A})$.*

(*b*) *$T$ is self-adjoint if and only if $T = M_{g\bar{u}}EM_u$ for some $\bar{g} = g \in L^0(\mathcal{A})$.*

(*c*) *$T$ is normal if and only if $T = M_{g\bar{u}}EM_u$ for some $g \in L^0(\mathcal{A})$.*

(*d*) $\operatorname{spec}(T) \setminus \{0\} = \operatorname{ess\,range}(E(uw)) \setminus \{0\}$.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. We use $A^*$, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, to denote the adjoint, the spectral radius, the range and the null space of $A \in B(\mathcal{H})$. $A$ is normal if $A^*A = AA^*$ and $A$ is positive if $\langle Ax, x\rangle \geq 0$ holds for each $x \in \mathcal{H}$ in which case we write $A \geq 0$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $A \in B(\mathcal{H})$ and let $Pj : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto $\mathcal{H}_j$ for $j = 1, 2$. Then $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ is the operator given by $A_{ij} = P_iAP_j\,|_{\mathcal{H}_j}$. In particular, $A(\mathcal{H}_1) \subseteq \mathcal{H}_1$ if and only if $A_{21} = 0$. Also, $\mathcal{H}_1$ reduces $A$ if and only if $A_{12} = 0 = A_{21}$. Let $A \in B(\mathcal{H})$ with $r(A) \neq 0$ and let $0 < a < r(A)^{-1}$ be an arbitrary but fixed number. Define $K_a(A) = \sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$. Since for all $n \in \mathbb{N}$, $\|a^{2n}A^{*n}A^n\| = a^{2n}\|A^n\|^2$, then we have $\overline{\lim}_{n\to+\infty}\|a^{2n}A^{*n}A^n\|^{\frac{1}{n}} = a^2\left(\overline{\lim}_{n\to+\infty}\|A^n\|^{\frac{1}{n}}\right)^2 = a^2r(A)^2 < 1$. This implies that the series $\sum_{n=0}^{+\infty} a^{2n}A^{*n}A^n$ is convergent in the norm topology of $B(\mathcal{H})$, and hence $K_a(A) \in B(\mathcal{H})$. Thus, the map $f_A$ of $(0, r(A)^{-1})$ to $B(\mathcal{H})$ defined by $f_A(a) = K_a(A)$ is well-define, increasing and continuous. Also, for any $x \in \mathcal{H}$ we have that

$$\|x\|^2 \leq \sum_{n=0}^{+\infty} a^{2n}\|A^n(x)\|^2 = \langle K_a(A)x, x\rangle = \left\|\sqrt{K_a(A)}x\right\|^2 \leq \|K_a(A)\| \cdot \|x\|^2. \tag{1.1}$$

So, $K_a(A) \geq I$ and hence $K_a(A)$ is positive and invertible with $\|K_a(A)\| \geq 1$. Set $R_a(A) = K_a^{-1}(A)$ and $S_a(A) = \sqrt{R_a(A)}$. Replacing $x$ by $(K_a(A))^{\frac{-1}{2}}(x)$ in (1.1) we obtain that $\|S_a(A)\| \leq 1$ and $\|R_a(A)\| = \|S_a^2(A)\| \leq 1$. Consequently, $R_a(A)$ and $S_a(A)$ are positive and invertible elements of $B(\mathcal{H})$ and

$$\|K_a(A)\| = \sup_{\|x\|=1} \langle K_a(A)x, x\rangle = \sum_{n=0}^{+\infty} a^{2n}\|A^n\|^2 \leq \sum_{n=0}^{+\infty} \left(\|aA\|^2\right)^n = \frac{1}{1-\|aA\|^2}. \tag{1.2}$$

Let $\{A_m\} \subseteq \{T \in B(\mathcal{H}) : r(T) \leq r(A)\}$. If $\|A_m - A\| \to 0$, then for each $n \in \mathbb{N}$ and $0 < a < r(A)^{-1}$, $a^{2n}A_m^{*n}A_m^n \to a^{2n}A^{*n}A^n$, and so $\|K_a(A_m) - K_a(A)\| \to 0$ as $m \to +\infty$. But the converse is not true. Indeed, if $A_1$ and $A_2$ are distinct unitary operators on $\mathcal{H}$, then $K_a(A_1) = K_a(A_2) = (1-a^2)^{-1}I$ for all $0 < a < 1$. In [9] A. Lambert and S. Petrović define the spectral radius algebra of a bounded linear operator $A$ with $S_a = S_a(A)$ and $0 < a < r(A)^{-1}$ to be the unital subalgebra

$$\mathcal{B}_A = \{T \in B(\mathcal{H}) : \sup_a \|S_a^{-1}TS_a\| < +\infty\}.$$

Lastly, define

$$\mathcal{Q}_A = \{T \in B(\mathcal{H}) : \lim_{a\to r(A)^{-1}} \|S_a^{-1}TS_a\| = 0\}.$$

In [9] it is shown that, $\|K_a(A)\| \to +\infty$ as $a \to \|A\|^{-1}$ and for any $A$, $\mathcal{Q}_A \subseteq \mathcal{B}_A$ is a two-sided ideal consisting entirely of quasinilpotent operators. Furthermore, if $A$ is quasinilpotent, then $A \in \mathcal{Q}_A$.

We now consider the Deddens algebra $\mathcal{D}_A$ associated with $A \in B(\mathcal{H})$, that is, the family of those operators $T \in B(\mathcal{H})$ for which there is a constant $M > 0$ such that for every $n \in \mathbb{N}$ and for every $x \in \mathcal{H}$, $\|A^nTx\| \leq M\|A^nx\|$. $\mathcal{D}_A$ is indeed a unital subalgebra of $B(\mathcal{H})$ with the property that $\{A\}' \subseteq \mathcal{D}_A \subseteq \mathcal{B}_A$, where $\{A\}'$ is the commutant of $A$ (see [11]).

Let $A \in B(\mathcal{H})$ be normal and $0 < a < \|A\|^{-1}$. Then $A^n$ and $A^{*n}$ commute with $K_a(A)$, $R_a(A)$, $S_a(A)$ and $K_a(A^*) = K_a(A) = K_a(|A|)$, where $|A|^2 = A^*A$. Moreover,

$$\begin{aligned} K_a(A) &= \sum_{n=0}^{+\infty} a^{2n}(A^*A)^n = (I - a^2A^*A)^{-1}, \\ R_a(A) &= I - a^2A^*A, \\ S_a(A) &= \sqrt{I - a^2A^*A}, \\ P_A :&= \lim_{a\to\|A\|^{-1}} S_a(A) = \sqrt{I - \|A\|^{-2}A^*A}. \end{aligned} \tag{1.3}$$

For more details on the Deddens and spectral radius algebras see [1,5,11,12]. In the next section, we investigate the spectral radius and the Deddens algebras related to the bounded weighted conditional type operators on $L^2(\Sigma)$. All of these are basically discussed using the conditional expectation properties.

## 2. $\mathcal{B}_T$ and $\mathcal{D}_T$ Associated with $T = M_wEM_u$

From now on we assume that $E(|u|^2) \in L^\infty(\mathcal{A})$, i.e., $T_1 := M_{\bar{u}}EM_u \in B(L^2(\Sigma))$.

**Lemma 2.1.** *For $0 \le b \in L^0(\mathcal{A})$, let $M_bT_1 \in B(L^2(\Sigma))$. Then the followings hold.*
*(a) If $1 \notin \operatorname{spec}(M_bT_1)$, then $(I - M_bT_1)^{-1} = I + M_{\frac{b}{1-bE(|u|^2)}}T_1$.*
*(b) If $-1 \notin \operatorname{spec}(M_bT_1)$, then $(I + M_bT_1)^{-1} = I - M_{\frac{b}{1+bE(|u|^2)}}T_1$.*

*Proof.* We only proof $(a)$, since $(b)$ follows similarly.
Let $1 \in \operatorname{spec}(M_bT_1)$. Using Lemma 1.1 $(d)$, $1 \notin \operatorname{ess\,range} E(b|u|^2)$ and so $(1 - bE(|u|^2))^{-1} \in L^\infty(\mathcal{A})$. Put $S = I + M_{b(1-bE(|u|^2))^{-1}}T_1$. Then $\|S\| \le 1 + \|(1 - bE(|u|^2))^{-1}\|_\infty \|M_bT_1\| < +\infty$. Also, direct computations show that $S(I - M_bT_1) = (I - M_bT_1)S = I$. Now, the desired conclusion holds. □

Set $\mathcal{N} = \{M_wEM_u \in B(L^2(\Sigma)) : M_wEM_u \text{ is normal}\}$. By Lemma 1.1 $(c)$ we have $\mathcal{N} = \{M_gT_1 \in B(L^2(\Sigma)) : g \in L^0(\mathcal{A}), T_1 = M_{\bar{u}}EM_u, u \in L^0(\Sigma)\}$.

**Corollary 2.1.** *Let $T = M_wEM_u \in \mathcal{N}$ and let $0 < a < r(T)^{-1}$. Then $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$ for some $k, v \in L^0(\mathcal{A})$ and $\|K_a(T)\| = 1 + \|vE(|u|^2)\|_\infty$.*

*Proof.* By Lemma 1.1 $(c)$, $T = M_gT_1$ for some $g \in L^0(\mathcal{A})$. Since $T^*T = M_{|g|^2E(|u|^2)}T_1$, then by (1.3) we get that $K_a(T) = (I - M_kT_1)^{-1}$, where $k = a^2|g|^2E(|u|^2)$. Thus, $R_a(T) = (K_a(T))^{-1} = I - M_kT_1$. Also, since $1/a^2 > (r(T))^2 = r(T^*T)$, then $1/a^2 \notin \operatorname{spec}(T^*T) = \operatorname{ess\,range} |g|^2(E(|u|^2))^2$. Therefore,

$$\frac{1}{1 - kE(|u|^2)} = \frac{1}{a^2\{\frac{1}{a^2} - |g|^2(E(|u|^2))^2\}} \in L^\infty(\mathcal{A})$$

and $1 \notin \operatorname{spec}(M_kT_1)$. Now, by Lemma 2.1, $K_a(T) = I + M_vT_1$, where $v = \frac{k}{1-kE(|u|^2)}$. Moreover, since $M_vT_1$ is positive, then $\|K_a(T)\| = 1 + \|M_vT_1\| = 1 + \|vE(|u|^2)\|_\infty$. This completes the proof. □

**Corollary 2.2.** *Under the assumption of above corollary, $S_a(T) = I - M_sT_1$ and $S_a^{-1}(T) = I + M_{\frac{s}{1-sE(|u|^2)}}T_1$ for some $s \in L^0(\mathcal{A})$.*

*Proof.* Set $s = \frac{1-\sqrt{1-kE(|u|^2)}}{E(|u|^2)}\chi_{\sigma(E(|u|^2))}$. Then, for $f \in L^2(\Sigma)$ we have

$$\begin{aligned}(I - M_{s\bar{u}}EM_u)^2(f) &= (I - M_{s\bar{u}}EM_u)(f - s\bar{u}E(uf)) \\ &= f - s\bar{u}E(uf) - s\bar{u}E(uf - s|u|^2E(uf)) \\ &= f - \bar{u}(-2s + E(|u|^2)s^2)E(uf) \\ &= f - \bar{u}kE(uf) \\ &= (I - M_{k\bar{u}}EM_u)(f).\end{aligned}$$

It follows that $S_a(T) = (R_a(T))^{1/2} = (I + M_kT_1)^{1/2} = I - M_sT_1$. Now, the inverse of $S_a(T)$ follows from Lemma 2.1 $(a)$. □

For $T \in \mathcal{N}$ and $v \in L^0(\mathcal{A})$, it is easy to check that $M_vT_1$ commutes with $S_a(T)$. It follows that $\{M_vT_1 \in B(L^2(\Sigma)) : v \in L^0(\mathcal{A})\} \subseteq \mathcal{B}_T$.

**Lemma 2.2.** *Let $T = M_wEM_u \in B(L^2(\Sigma))$. Then $\mathcal{N}(T) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^\perp$.*

*Proof.* Let $f \in L^2(\Sigma)$. Since $\mathcal{R}(E) = L^2(\mathcal{A})$, then we have

$$\begin{aligned} f \in \mathcal{N}(T) \Leftrightarrow \|Tf\|^2 = 0 &\Leftrightarrow \int_X E(|w|^2)|E(uf)|^2 d\mu = 0 \\ &\Leftrightarrow \int_X \left| E(u\sqrt{E(|w|^2)}f) \right|^2 d\mu = 0 \\ &\Leftrightarrow u\sqrt{E(|w|^2)}f \in \mathcal{N}(E) = L^2(\mathcal{A})^\perp \\ &\Leftrightarrow \left\langle u\sqrt{E(|w|^2)}f, g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow \left\langle f, \bar{u}\sqrt{E(|w|^2)}g \right\rangle = 0, \quad \text{for all } g \in L^2(\mathcal{A}) \\ &\Leftrightarrow f \in \left\{ \bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A}) \right\}^\perp. \end{aligned}$$

□

**Corollary 2.3.** $\overline{\mathcal{R}(M_{\bar{u}}EM_u)} = \overline{\bar{u}\sqrt{E(|u|^2)}L^2(\mathcal{A})} = c.l.s. \left\{ \bar{u}\sqrt{E(|u|^2)}\chi_A : A \in \mathcal{A}_{\sigma(u)} \right\}$, *where c.l.s. stands for closed linear span. In particular,* $\overline{\mathcal{R}(EM_u)} = \overline{\bar{u}L^2(\mathcal{A})}$.

Let $P$ be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$ and let $Q = I - P$. Direct computations show that

$$(I - \alpha P)^{-1} = I + \frac{\alpha}{1-\alpha}P, \quad \alpha \neq 1, \tag{2.1}$$

$$(I - \alpha P)^{\frac{1}{2}} = I - (1 - \sqrt{1-\alpha})P, \quad \alpha \leq 1. \tag{2.2}$$

Let $0 < a < 1$. Then $K_a(P) = \sum_{n=0}^{+\infty} a^{2n} P^{*n} P^n = I + \frac{a^2}{1-a^2}P$. Using (2.1) and (2.2) we obtain that

$$\begin{aligned} R_a(P) &= (K_a(P))^{-1} = I - a^2 P, \\ S_a(P) &= (R_a(P))^{\frac{1}{2}} = I - (1 - \sqrt{1-a^2})P, \\ S_a^{-1}(P) &= I + \frac{1-\sqrt{1-a^2}}{\sqrt{1-a^2}}P. \end{aligned}$$

Note that if we take $P = M_{\bar{u}E(|u|^2)^{-1}}EM_u$, then $P^2 = P = P^*$, with $\mathcal{R}(P) = \overline{\bar{u}E(|u|^2)^{-1/2}L^2(\mathcal{A})}$. Now, let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \mathcal{M} \oplus \mathcal{M}^\perp$. Since

$$S_a(P) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} M_{1-\sqrt{1-a^2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix},$$

then we have

$$\mathcal{P}_a(S) := (S_a^{-1}(P))S(S_a(P)) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} X & YM_{\frac{1}{\sqrt{1-a^2}}} \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}.$$

It follows that $\sup\{\|\mathcal{P}_a(S)\| : 0 < a < 1\} < +\infty$ if and only if $Y = 0$. For some $0 < a < 1$, $\mathcal{P}_a(S) = S$ if and only if $Y = Z = 0$. Also, $\lim_{a\to 1} \|\mathcal{P}_a(S)\| = 0$ if and only if $X = Y = W = 0$. Moreover, we have

$$\mathcal{P}_a(SP) = \begin{pmatrix} M_{\frac{1}{\sqrt{1-a^2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{\sqrt{1-a^2}} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} X & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Thus, $SP \in \mathcal{B}_P$ for all $S \in B(L^2(\Sigma))$. Also if $X = 0$, then $SP \in \mathcal{Q}_P$. Similar computations show that

$$\mathcal{P}_a(QS) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & W \end{pmatrix}, \quad \mathcal{P}_a(QSP) = \begin{pmatrix} 0 & 0 \\ ZM_{\sqrt{1-a^2}} & 0 \end{pmatrix}.$$

Let $\{S_n\} \subseteq \mathcal{B}_P$ and let $S_n := \begin{pmatrix} X_n & 0 \\ Z_n & W_n \end{pmatrix} \to S$ as $n \to +\infty$. Then

$$\|Y\| \leq \|S_n - S\| = \left\| \begin{pmatrix} X_n - X & Y \\ Z_n - Z & W_n - W \end{pmatrix} \right\| \to 0.$$

It follows that $Y = 0$ and hence $\mathcal{B}_P$ is closed in the norm operator topology on $B(L^2(\Sigma))$. Moreover, by definition, $S \in \mathcal{D}_P$ if and only if there exists $M > 0$ such that

$$\begin{aligned} \|PSf\| &= \left\| \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} XPf + YQf \\ 0 \end{pmatrix} \right\| \leq M \left\| \begin{pmatrix} Pf \\ 0 \end{pmatrix} \right\|, \end{aligned}$$

for all $f \in L^2(\Sigma)$. Replacing $f$ by $Qf$ in the above and taking $M = M(S) = \|X\|$, we obtain that $S \in \mathcal{D}_P$ if and only if $Y = 0$ on $\mathcal{N}(P)$. As an easy consequence of these observations, we have the following result.

**Proposition 2.1.** *Let $P$ be an orthogonal projection of $L^2(\Sigma)$ onto $\mathcal{M} = \mathcal{R}(P)$, $0 < a < 1$ and let $Q = I - P$. Set*

$$\begin{aligned} \mathcal{Q}_1 &= \{SP : S \in B(L^2(\Sigma)), PSP = 0\}, \\ \mathcal{Q}_2 &= \{QS : S \in B(L^2(\Sigma)), QSQ = 0\}, \\ \mathcal{Q}_3 &= \{QSP : S \in B(L^2(\Sigma))\}. \end{aligned}$$

*Then*

$$\begin{aligned} \mathcal{B}_P &= \{S \in B(L^2(\Sigma)) : S(\mathcal{N}(P)) \subseteq \mathcal{N}(P)\} = \mathcal{D}_P, \\ \mathcal{Q}_P &= \{S \in B(L^2(\Sigma)) : QSP = T\} \supseteq \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3. \end{aligned}$$

*Moreover, $\mathcal{P}_a(S) = S$ if and only if $\mathcal{M}$ reduces $S$.*

Set $P = E^{\mathcal{A}} = E$, $0 < a < 1$ and $\mathcal{P}_a = \mathcal{E}_a$. Let $S = M_w E M_u \in B(L^2(\Sigma))$. Using Proposition 2.1 and [7, Proposition 2.30] with respect the decomposition $L^2(\Sigma) = L^2(\mathcal{A}) \oplus \mathcal{N}(E)$, we have

$$\begin{aligned} &ESE = 0 \Leftrightarrow M_{E(u)E(w)}|_{L^2(\mathcal{A})} = 0, \\ &ESQ = 0 \Leftrightarrow u\chi_{\sigma(E(w))} \in L^0(\mathcal{A}), \\ &QSE = 0 \Leftrightarrow w\chi_{\sigma(E(u))} \in L^0(\mathcal{A}), \\ &QSQ = 0 \Leftrightarrow L^2(\mathcal{A})^\perp = \mathcal{R}(Q) \subseteq \mathcal{N}(S) = \left\{\bar{u}\sqrt{E(|w|^2)}L^2(\mathcal{A})\right\}^\perp. \end{aligned}$$

So we have the following corollary.

**Corollary 2.4.** *Let* $S = M_w E M_u \in B(L^2(\Sigma))$ *and* $0 < a < 1$. *Then,*

(*a*) $S \in \mathcal{B}_E$ *if and only if* $u\chi_{\sigma(E(w))} \in L^0(\mathcal{A})$;

(*b*) $S \in \mathcal{Q}_E$ *if and only if* $S \in \mathcal{B}_E$, $M_{E(uE(w))}|_{L^2(\mathcal{A})} = 0$ *and* $\overline{\mathcal{R}(S)} \subseteq L^2(\mathcal{A})$;

(*c*) $\mathcal{E}_a(S) = S$ *if and only if* $\{u\chi_{\sigma(E(w))}, w\chi_{\sigma(E(u))}\} \subseteq L^0(\mathcal{A})$.

Let $\mathcal{M}(\mathcal{A}) = \{M_\vartheta : \vartheta \in L^\infty(\mathcal{A})\}$ and let $\mathcal{M}'(\mathcal{A})$ be its commutant. It is known that $\mathcal{M}(\Sigma)$ is a maximal abelian subalgebra of $B(L^2(\Sigma))$. But it is invalid if $\Sigma$ is replaced by $\mathcal{A} \neq \Sigma$. Indeed, for any $\mathcal{A} \subset \mathcal{B}$, $E^{\mathcal{B}} \in \mathcal{M}'(\mathcal{A}) \setminus \mathcal{M}(\mathcal{A})$. Alan Lambert in [10, Theorem 3.2] proved that $S \in \mathcal{M}'(\mathcal{A})$ if and only if there exists $C > 0$ such that $E(|Sf|^2) \leq CE(|f|^2)$ for all $f \in L^2(\Sigma)$. Consequently, if $S \in B(L^2(\Sigma))$ and $\{\vartheta_n, \vartheta_n^{-1}\} \subseteq L^\infty(\mathcal{A})$, then $\sup_n \|M_{\vartheta_n^{-1}} S M_{\vartheta_n}\| < +\infty$ whenever $S \in \mathcal{M}'(\mathcal{A})$.

For a fixed $T = M_g T_1 \in \mathcal{N}$ and $0 < a < \|T\|^{-1}$, put $A := S_a^{-1}(T)$. Then by Corollary 2.2, $A = I + M_\theta T_1$ for some $0 \leq \theta \in L^0(\mathcal{A})$. Since $A$ is bounded, then so is $M_\theta T_1$. Thus, $\theta E(|u|^2) \in L^\infty(\mathcal{A})$ and hence $\theta E(|u|^2)g \in L^2(\mathcal{A})$ for all $f \in L^2(\mathcal{A})$. Relative to the direct sum decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$, the matrix form of $A$ is $(A_{ij})_{1\leq i,j\leq 2}$. Set $P = P_{\overline{\mathcal{R}(T_1)}}$ and $Q = I - P$. Let $f \in L^2(\Sigma)$. Then without loss of generality, we can assume that $Pf = \bar{u}\sqrt{E(|u|^2)}g$, for some $g \in L^2(\mathcal{A})$. Then

$$\begin{aligned} A_{11}f &= P(A(Pf)) = P(Pf + \theta\bar{u}E(uPf)) \\ &= P\left(Pf + \bar{u}\sqrt{E(|u|^2)}\left(\theta E(|u|^2)g\right)\right) \\ &= Pf + \theta E(|u|^2)Pf \\ &= M_{1+\theta E(|u|^2)}Pf, \end{aligned}$$

where $\theta = 1 + \frac{s}{1-sE(|u|^2)}$. By Corollary 2.2, $1+\theta E(|u|^2) = \frac{1}{1-sE(|u|^2)} = \frac{1}{\sqrt{1-kE(|u|^2)}}$ where $k = a^2|g|^2E(|u|^2)$. Thus, $A_{11} = PAP = M_{(1-kE(|u|^2))^{-1/2}}P$. Similar computations show that $A_{12} = A_{21} = 0$ and $A_{22} = I_{|\mathcal{N}(T_1)}$. Let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ be the block matrix representation of $S \in B(L^2(\Sigma))$ with respect the decomposition $L^2(\Sigma) = \overline{\mathcal{R}(T_1)} \oplus \mathcal{N}(T_1)$. Set $\mathcal{L}_a(S) := (S_a^{-1}(T))S(S_a(T))$ and $\vartheta := \sqrt{1 - kE(|u|^2)}$. Then

$\vartheta \to 0$ as $a \to \|T\|^{-1}$ and

$$\mathcal{L}_a(S) = \begin{pmatrix} M_{\frac{1}{\vartheta}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_\vartheta & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_{\frac{1}{\vartheta}} X M_\vartheta & M_{\frac{1}{\vartheta}} Y \\ Z M_\vartheta & W \end{pmatrix}.$$

Since $M_\vartheta \to 0$ as $a \to \|T\|^{-1}$, so $\sup\{\|M_{\vartheta^{-1}}\|;\ 0 < a < \|T\|^{-1}\} = +\infty$. Let $M := \sup\{\|M_{\vartheta^{-1}}Y\| < +\infty,\ 0 < a < \|T\|^{-1}\}$. Then for all unit vector $f \in \mathcal{N}(T_1)$, $\|Y(f)\| = \|M_\vartheta M_{\vartheta^{-1}} Y(f)\| \leq M\|M_\vartheta\|$. It follows that $\|Y(f)\| = 0$ and hence $Y_{|_{\mathcal{N}(T_1)}} = 0$. In particular, if $PSP \in \mathcal{M}'(\mathcal{A})$, then $S \in \mathcal{B}_T$ if and only if $S(\mathcal{N}(T_1)) \subseteq \mathcal{N}(T_1)$. In this case, $\mathcal{B}_{M_{g_1T_1}} = \mathcal{B}_{M_{g_2T_1}}$ for all $\{M_{g_1T_1}, M_{g_1T_1}\} \subseteq \mathcal{N}$. Note that

$$\mathcal{L}_a(T) = S_a^{-1}(T) \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix} S_a(T) = \begin{pmatrix} M_{gE(|u|^2)} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\|\mathcal{L}_a(T)\| = \|gE(|u|^2)\|_\infty = \|T\| = r(T)$. In view of these observations we have the following results.

**Theorem 2.1.** *Let $T = M_gT_1 \in \mathcal{N}$ and let $\vartheta = \sqrt{1 - a^2|g|^2(E(|u|^2))^2}$. Then the followings hold.*

(*a*) *$S \in \mathcal{B}_T$ if and only if $Y = 0$ and* $\sup\{\|M_{\vartheta^{-1}}XM_\vartheta\| :\ 0 < a < \|T\|^{-1}\} < +\infty$. *In particular, if $XM_\vartheta = M_\vartheta X$, then $S \in \mathcal{B}_T$ if and only if $\{\bar{u}\sqrt{E(|u|^2)L^2(\mathcal{A})}\}^\perp$ is an invariant subspace for $S$.*

(*b*) *$S \in \mathcal{Q}_T$ if and only if $Y = W = 0$ and $\|M_{\vartheta^{-1}}XM_\vartheta\| \to 0$, as $a \to \|T\|^{-1}$. Moreover, if $XM_\vartheta = M_\vartheta X$, then $S \in \mathcal{Q}_T$ if and only if $X = Y = W = 0$.*

Let $T = M_gT_1 \in \mathcal{N}$ and $S \in B(L^2(\Sigma))$. Then, for all $n \in \mathbb{N}$ and $f \in L^2(\Sigma)$, $T^n = M_{g^n(E(|u|^2))^{n-1}}T_1$ and

$$T^nSf = \begin{pmatrix} M_{\omega^n} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} Pf \\ Qf \end{pmatrix} = \begin{pmatrix} M_{\omega^n}XPf + M_{\omega^n}YQf \\ 0 \end{pmatrix},$$

where $\omega = gE(|u|^2)$. It follows that $S \in \mathcal{D}_T$ if and only if there exists $M > 0$ such that $\|M_{\omega^n}XPf + M_{\omega^n}YQf\| \leq M\|M_{\omega^n}Pf\|$. If we set $f = Qg$, for some $g \in L^2(\Sigma)$, then we get $\|M_{\omega^n|_{\sigma(\omega)}}YQg\| \leq \|M_{\omega^n}YQg\| = 0$ and hence $Y\ |_{\mathcal{N}(T_1)} = 0$. Now, if $M_\omega X = XM_\omega$, then $\|M_{\omega^n}XPf\| \leq \|X\| \cdot \|M_{\omega^n}Pf\|$. Note that the commutativity of $M_\omega$ and $X$ implies that $M_\vartheta X = XM_\vartheta$. So we have the following result.

**Theorem 2.2.** *Let $T = M_gT_1 \in \mathcal{N}$, $\omega = gE(|u|^2)$ and let $S \in B(L^2(\Sigma))$. Then $S \in \mathcal{D}_T$ if and only if $PSP \in \mathcal{D}_T$ and $PSQ = 0$. Moreover, if $(PSP)M_\omega = M_\omega(PSP)$, then $\mathcal{D}_T = \mathcal{B}_T$.*

**Corollary 2.5.** *Let $\{T, S\} \subseteq \mathcal{N}$. Then $S \in \mathcal{B}_T$ if and only if $PSQ = 0$.*

*Proof.* Let $S = M_{g_1\bar{v}}EM_v \in \mathcal{B}_T$, with $g_1 \in L^0(\mathcal{A})$. Then $PSP = M_\gamma$, where $\gamma = g_1E(u)E(\bar{v})E(\bar{u}v) \in L^0(\mathcal{A})$. Since $PSP$ commutes with $M_\gamma$, then the desired conclusion follows from Theorem 2.2. □

*Example* 2.1. Let $X = \{1, 2, 3\}$, $\Sigma = 2^X$, $\mu(\{n\}) = 1/3$ and let $\mathcal{A}$ be the $\sigma$-algebra generated by the partition $\{\{1, 3\}, \{2\}\}$. Then $L^2(\Sigma) \cong \mathbb{C}^3$ and

$$E(f) = \left(\frac{1}{\mu(A_1)}\int_{A_1} f d\mu\right)\chi_{A_1} + \left(\frac{1}{\mu(A_2)}\int_{A_2} f d\mu\right)\chi_{A_2} = \frac{f_1 + f_3}{2}\chi_{A_1} + f_2\chi_{A_2},$$

where $A_1 = \{1, 3\}$ and $A_2 = \{2\}$. Then matrix representation of $E$ with respect to the standard orthonormal basis is $E = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. It can be easily checked that $E^2 = E = E^*$, $\mathcal{N}_2(E) = \langle (a, 0, -a) : a \in \mathbb{C}\rangle$, $\mathcal{R}(E) = \langle (a, b, a) : a, b \in \mathbb{C}\rangle$. For $1 < a < 1$ we have

$$K_a(E) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{a^2}{1-a^2}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{a^2}{2(1-a^2)} \\ 0 & \frac{1}{1-a^2} & 0 \\ \frac{a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(E) = I - (1 - \sqrt{1-a^2})E = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{\sqrt{1-a^2}-1}{2} \\ 0 & \sqrt{1-a^2} & 0 \\ \frac{\sqrt{1-a^2}-1}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$P_E = \sqrt{I-P} = Q = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Set $u = (1, i, -1)$, $g = (1, 2, 1)$, $0 < a < \frac{1}{2}$ and let $T_1 = M_{\bar{u}}EM_u$. Then

$$k = a^2|g|^2E(|u|^2) = a^2(1, 4, 1)E(1, 1, 1) = (a^2, 4a^2, a^2),$$

$$v = \frac{k}{1 - kE(|u|^2)} = \left(\frac{a^2}{1-a^2}, \frac{4a^2}{1-4a^2}, \frac{a^2}{1-a^2}\right),$$

$$s = \frac{1 - \sqrt{1 - kE(|u|^2)}}{E(|u|^2)} = \left(1 - \sqrt{1-a^2}, 1 - \sqrt{1-4a^2}, 1 - \sqrt{1-a^2}\right),$$

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Take $T = M_gT_1$. Since $K_a(T) = I + M_vT_1$ and $R_a(T) = I - M_kT_1$, then we have

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix},$$

$$R_a(T) = I - \mathrm{diag}(a^2, 4a^2, a^2)T_1 = \begin{bmatrix} \frac{2-a^2}{2} & 0 & \frac{a^2}{2} \\ 0 & 1-4a^2 & 0 \\ \frac{a^2}{2} & 0 & \frac{2-a^2}{2} \end{bmatrix},$$

$$K_a(T) = I + \operatorname{diag}\left(\frac{a^2}{1-a^2}, \frac{4a^2}{1-a^2}, \frac{a^2}{1-a^2}\right) T_1 = \begin{bmatrix} \frac{2-a^2}{2(1-a^2)} & 0 & \frac{-a^2}{2(1-a^2)} \\ 0 & \frac{1}{2(1-4a^2)} & 0 \\ \frac{-a^2}{2(1-a^2)} & 0 & \frac{2-a^2}{2(1-a^2)} \end{bmatrix},$$

$$S_a(T) = I - M_s T_1 = \begin{bmatrix} \frac{1+\sqrt{1-a^2}}{2} & 0 & \frac{1-\sqrt{1-a^2}}{2} \\ 0 & -1+\sqrt{1-4a^2} & 0 \\ \frac{1-\sqrt{1-a^2}}{2} & 0 & \frac{1+\sqrt{1-a^2}}{2} \end{bmatrix},$$

$$S_a^{-1}(T) = I - M_{\frac{s}{1-s}} T_1 = \begin{bmatrix} \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} \\ 0 & \frac{1}{\sqrt{1-4a^2}} & 0 \\ \frac{\sqrt{1-a^2}-1}{2\sqrt{1-a^2}} & 0 & \frac{\sqrt{1-a^2}+1}{2\sqrt{1-a^2}} \end{bmatrix}.$$

Since $\|T\| = \|gE(|u|^2)\|_\infty = \|(1,2,1)\|_\infty = 2$ and $T^*T = M_{|g|^2E(|u|^2)}T_1$, then $P_T^2 = I - \|T\|^{-2}T^*T = I - M_z T_1$, where $z = \frac{|g|^2E(|u|^2)}{4} = (\frac{1}{4}, 1, \frac{1}{4})$. It follows that

$$P_T = I - M_{(1-\sqrt{1-z})}T_1 = I - \operatorname{diag}\left(\frac{2-\sqrt{3}}{2}, 1, \frac{2-\sqrt{3}}{2}\right) T_1 = \begin{bmatrix} \frac{2+\sqrt{3}}{4} & 0 & \frac{2-\sqrt{3}}{4} \\ 0 & 0 & 0 \\ \frac{2-\sqrt{3}}{4} & 0 & \frac{2+\sqrt{3}}{4} \end{bmatrix}.$$

Note that, $r(T) = 2 > 0$ but $P_T \neq 0$ (see [2]). Also, $\mathcal{R}(T) = \bar{u}|g|\sqrt{E(|u|^2)}L^2(\mathcal{A}) = \{(1,-i,-1)(1,2,1)(1,1,1)(a,b,a) : a,b \in \mathbb{C}\} = \{(a,c,-a) : a,c \in \mathbb{C}\}$. Now set $u = (1,0,1)$ and $v = (2,-i,-2)$. Consider the rank-one operator $u \otimes v$ defined by $(u \otimes v)w = \langle w, v\rangle u$, for all $w \in \mathbb{C}^3$. Then $u \otimes v = \begin{pmatrix} 2 & i & -2 \\ 0 & 0 & 0 \\ 2 & i & -2 \end{pmatrix}$ and $(u \otimes v)T \neq T(u \otimes v)$. However, since

$$\sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|S_a(T)v\| \leq \sup_{0<a<\frac{1}{2}} \|S_a^{-1}(T)u\| \cdot \|v\| = \|u\| \cdot \|v\| = 3\sqrt{2},$$

then by [9, Lemma 3.9], $u \otimes v \in \mathcal{B}_T$. Thus, $\mathcal{B}_T$ properly contains $\{T\}'$. In the finite dimensional case, if $\mathcal{A} \neq \Sigma$, then $T$ is not injective and hence the spectral radius algebra $\mathcal{B}_T$ always properly contains the commutant of $T$.

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