

## ON THE LAPLACIAN COEFFICIENTS OF TREES

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**ABSTRACT.** Let  $G$  be a finite simple graph with Laplacian polynomial  $\psi(G, \lambda) = \sum_{k=0}^n (-1)^{n-k} c_k(G) \lambda^k$ . In an earlier paper, we computed the coefficient of  $c_{n-4}$  for trees with respect to some degree-based graph invariant. The aim of this paper is to continue this work by giving an exact formula for the coefficient  $c_{n-5}$  in the polynomial  $\psi(G, \lambda)$ . As a consequence of this work, the Laplacian coefficients  $c_{n-k}$ ,  $k = 2, 3, 4, 5$ , for some known trees were computed.

### 1. DEFINITIONS AND NOTATIONS

Throughout this paper, our graphs will be assumed to be simple, connected and undirected, and the standard notation for such a graph is used. The notations  $n(G)$  and  $m(G)$  stand for the number of elements in the vertex set  $V(G)$  and the edge set  $E(G)$ , respectively. The degree of a vertex  $v$  in  $G$ ,  $\deg_G(v)$ , is the number of edges in  $G$  with one end point  $v$  and the degree of an edge  $e$  in  $G$ ,  $\deg_G(e)$ , is the degree of vertex  $e$  in the line graph of  $G$ . It is easy to see that  $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$ . The *distance* between two vertices  $u$  and  $v$  is defined as the length of a shortest path connecting them. If  $Z \subseteq V(G)$ , then the *induced subgraph*  $G[Z]$  is the graph with vertex set  $Z$  and edge set  $\{uv \in E(G) \mid \{u, v\} \subseteq Z\}$ .

Suppose  $G$  is a graph. The *subdivision graph*  $S(G)$  is a graph obtained from  $G$  by inserting a new vertex on each edge of  $G$ . It is clear from this definition that  $n(S(G)) = n(G) + m(G)$  and  $m(S(G)) = 2m(G)$ .

Suppose  $e = xy$  and  $f$  are two edges of a graph  $G$  and  $v \in V(G)$ , where  $v \neq x, y$ . The *common vertex* of  $e$  and  $f$  is denoted by  $e \cap f$  and  $e \cap f = \emptyset$  means that  $e$  and  $f$  are not incident. If  $e \cap f = \emptyset$  then  $e$  and  $f$  are said to be independent. A subset  $M$

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of  $E(G)$  is called a matching if all pairs of distinct edges in  $M$  are independent. Note that  $M$  is a matching in  $G$  if  $|\{u \mid u \text{ is an end point of an edge in } M\}| = 2|M|$ . If  $M$  is a matching of size  $k$  then we say  $M$  is a  $k$ -matching. Furthermore, the notation  $p(G; k)$ ,  $1 \leq k \leq \frac{n}{2}$ , is used for the number of distinct  $k$ -matchings in  $G$ . The matchings polynomial of  $G$  was first introduced by Godsil and Gutman in [4]. This polynomial is defined as  $p(G; 0) = 1$  and for other values of  $x$ ,  $\alpha(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k p(G; k) x^{n-2k}$ .

Suppose  $G$  is a simple graph with vertex set  $\{a_1, \dots, a_n\}$ . The 0-1 matrix  $A(G) = (a_{ij})$  such that  $a_{ij} = 1$  if and only if  $v_i v_j \in E(G)$  is called the adjacency matrix of  $G$ . The Laplacian matrix of  $G$  is another  $n \times n$  matrix defined as  $L(G) = D(G) - A(G)$ , where  $D(G)$  is the diagonal matrix of  $G$  whose diagonal entry  $d_{ii}$  is the degree of  $a_i$  in  $G$ . It is well-known that the eigenvalues of  $L(G)$  are non-negative real numbers with 0 as the smallest eigenvalue. The characteristic polynomial of  $L(G)$  is called the Laplacian polynomial of  $G$  and its roots are Laplacian eigenvalues of  $G$ . In this paper we write this polynomial in the form of  $\psi(G, x) = \det(xI_n - L(G)) = \sum_{k=0}^n (-1)^{n-k} c_k(G) x^k$ .

The first and second Zagreb indices of a graph  $G$  are two important degree-based graph invariants that was introduced by two pioneers of *Chemical Graph Theory* Gutman and Trinajstić [6]. These invariants are defined as  $M_1(G) = \sum_{v \in V(G)} \deg_G(v)^2$  and  $M_2(G) = \sum_{uv \in E(G)} \deg_G(u) \deg_G(v)$ . We encourage the interested readers to consult the interesting papers [7] and [14], for more information about mathematical properties and chemical applications of these invariants.

Following Milićević et al. [11], the edge counterpart of the first and second Zagreb indices of a graph  $G$  are defined as  $EM_1(G) = \sum_{e \sim f} (\deg_G(e) + \deg_G(f)) = \sum_{e \in E(G)} \deg_G(e)^2$  and  $EM_2(G) = \sum_{e \sim f} \deg_G(e) \deg_G(f)$ , where for  $e = uv$ ,  $\deg_G(e) = \deg_G(u) + \deg_G(v) - 2$  denotes the degree of the edge  $e$ , and  $e \sim f$  means that the edges  $e$  and  $f$  are incident.

Furtula and Gutman [5] studied in details the sum of cubes of degrees of vertices in a graph  $G$  and used the name *forgotten index* for this invariant. They defined  $F(G) = \sum_{v \in V(G)} \deg_G(v)^3 = \sum_{e=uv \in E(G)} (\deg_G(u)^2 + \deg_G(v)^2)$ . The first Zagreb index and the forgotten index can be generalized in the form of  $M_1^\alpha(G) = \sum_{u \in V(G)} \deg_G(u)^\alpha$ , where  $\alpha \neq 0, 1$  is a real number. Zhang and Zhang [17] obtained some extremal values of this invariant in the class of all unicyclic graphs of a given order. An interesting survey of these degree-based indices is given in [8].

Let  $T$  be a tree with Laplacian polynomial

$$\psi(T, x) = \det(xI_n - L(T)) = \sum_{k=0}^n (-1)^{n-k} c_k(T) x^k.$$

Merris [12] and Mohar [13] proved that  $c_0(T) = 0$ ,  $c_1(T) = n$ ,  $c_n = 1$  and  $c_{n-1}(T) = 2(n-1)$ . In [16], it is proved that  $c_2(T) = W(T)$  and in the paper [15], the authors proved that  $c_{n-2}(T) = 2n^2 - 5n + 3 - \frac{1}{2}M_1(T)$  and  $c_{n-3}(T) = \frac{1}{3}(4n^3 - 18n^2 + 24n - 10 + F(T) - 3(n-2)M_1(T))$ .

Suppose  $\lambda$  is an arbitrary real number. We now define three invariants which is useful in simplifying formulas in our results. These are:

$$\begin{aligned}\alpha_\lambda(G) &= \sum_{uv \in E(G)} \deg_G(u) \deg_G(v) \left( \deg_G(u)^\lambda + \deg_G(v)^\lambda \right), \\ \beta(G) &= \sum_{e \sim f} \deg_G(e \cap f) (\deg_G(e) + \deg_G(f)), \\ M_2^\lambda(G) &= \sum_{uv \in E(G)} (\deg_G(u) \deg_G(v))^\lambda.\end{aligned}$$

Note that the second Zagreb index is just the case of  $\lambda = 1$  in  $M_2^\lambda(G)$ .

The girth of a graph  $G$ ,  $g(G)$ , is defined as the length of a shortest cycle of  $G$ . In a recent paper [3], Das et al. proved the following result.

**Theorem 1.1.** *Let  $G$  be a graph with  $m$  edges and  $g(G) \geq 5$ . Then*

$$\begin{aligned}5p(G; 5) &= \frac{1}{24}m(m^4 + 10m^3 + 43m^2 + 54m - 328) + \frac{5}{4}(M_1(G))^2 - \frac{1}{2}\alpha_1(G)(m - 7) \\ &\quad - \frac{5}{6}\alpha_2(G) - \frac{1}{12}M_1(G)(2m^3 + 30m^2 + 61m - 225) + \frac{1}{2}\beta(G) \\ &\quad + \frac{1}{12}M_2(G)(6m^2 + 66m - 239) + \frac{1}{24}F(G)(6m^2 + 24m - 149) \\ &\quad + \frac{1}{12}M_1^4(G)(m + 10) + \frac{1}{4}M_2^2(G) - EM_2(G) - \frac{5}{24}M_1^5(G) \\ &\quad + \frac{1}{8} \sum_{uv \in E(G)} (M_1(G - \{u, v\}))^2 + \frac{1}{3} \sum_{uv \in E(G)} m(G - \{u, v\})F(G - \{u, v\}) \\ &\quad - \frac{1}{4} \sum_{uv \in E(G)} m^2(G - \{u, v\})M_1(G - \{u, v\}) - \sum_{uv \in E(G)} EM_2(G - \{u, v\}) \\ &\quad + \sum_{uv \in E(G)} m(G - \{u, v\})M_2(G - \{u, v\}).\end{aligned}$$

The present authors [1, 2] proved the following formulas for the coefficient  $c_{n-4}(T)$ , when  $T$  is a tree:

$$\begin{aligned}c_{n-4}(T) &= (n - 1) \left( \frac{16}{24}n^3 - 4n^2 + \frac{348}{24}n - \frac{532}{6} \right) + \frac{17}{8}M_1(T)^2 \\ &\quad + \left( \frac{4}{6}n - \frac{412}{24} \right) F(T) + \frac{39}{2}EM_1(T) - \frac{108}{48}M_1^4(T) - 40M_2(T) \\ &\quad - \left( n^2 + \frac{7}{2}n - \frac{1920}{24} \right) M_1(T) - 16 \sum_{\{u, v\} \subset V(T)} \binom{\deg_T(u)}{2} \binom{\deg_T(v)}{2} \\ &= \frac{1}{6}(n - 1)(4n^3 - 24n^2 + 39n - 16) + \frac{1}{3}F(G)(2n - 5) \\ &\quad + \frac{1}{8}M_1(T)(-8n^2 + M_1(T) + 36n - 32) - \frac{1}{4}M_1^4(T) - M_2(T).\end{aligned}$$

In this paper, an exact formula for computing the coefficient  $c_{n-5}(T)$ ,  $T$  is a tree, with respect to some degree-based topological indices is presented.

## 2. MAIN RESULTS

The aim of this section is to present a closed formula for  $c_{n-5}(T)$ , when  $T$  is tree. To do this, we first define five invariants  $\chi_1, \chi_2, \chi_3, \chi_4$  and  $\chi_5$  with respect to the subdivision graph as follows:

$$\begin{aligned}\chi_1(S(G)) &= \sum_{uv \in E(S(G))} (M_1(S(G) - \{u, v\}))^2, \\ \chi_2(S(G)) &= \sum_{uv \in E(S(G))} m(S(G) - \{u, v\})F(S(G) - \{u, v\}), \\ \chi_3(S(G)) &= \sum_{uv \in E(S(G))} m^2(S(G) - \{u, v\})M_1(S(G) - \{u, v\}), \\ \chi_4(S(G)) &= \sum_{uv \in E(S(G))} EM_2(S(G) - \{u, v\}), \\ \chi_5(S(G)) &= \sum_{uv \in E(S(G))} m(S(G) - \{u, v\})M_2(S(G) - \{u, v\}).\end{aligned}$$

**Lemma 2.1.** *Let  $G$  be a graph with  $m$  edges. Then*

$$\begin{aligned}M_1(S(G)) &= M_1(G) + 4m, & F(S(G)) &= F(G) + 8m, \\ M_1^4(S(G)) &= M_1^4(G) + 16m, & M_1^5(S(G)) &= M_1^5(G) + 32m, \\ \alpha_1(S(G)) &= 4M_1(G) + 2F(G), & \alpha_2(S(G)) &= 8M_1(G) + 2M_1^4(G), \\ \beta(S(G)) &= 2M_1(G) + M_1^4(G) - F(G), & M_2(S(G)) &= 2M_1(G), \\ EM_2(S(G)) &= M_2(G) + \frac{1}{2}M_1^4(G) - \frac{1}{2}F(G), & M_2^2(S(G)) &= 4F(G).\end{aligned}$$

*Proof.* By definition of subdivision graph, we have:

$$\begin{aligned}M_1(S(G)) &= \sum_{v \in V(G)} \deg_G(v)^2 + \sum_{uv \in E(G)} 4 = M_1(G) + 4m, \\ F(S(G)) &= \sum_{v \in V(G)} \deg_G(v)^3 + \sum_{uv \in E(G)} 8 = F(G) + 8m, \\ M_1^4(S(G)) &= \sum_{v \in V(G)} \deg_G(v)^4 + \sum_{uv \in E(G)} 16 = M_1^4(G) + 16m, \\ M_1^5(S(G)) &= \sum_{v \in V(G)} \deg_G(v)^5 + \sum_{uv \in E(G)} 32 = M_1^5(G) + 32m, \\ \alpha_1(S(G)) &= \sum_{v \in V(G)} \sum_{uv \in E(G)} 2 \deg_G(v)(2 + \deg_G(v)) \\ &= \sum_{v \in V(G)} \sum_{uv \in E(G)} (4 \deg_G(v) + 2 \deg_G(v)^2) \\ &= \sum_{v \in V(G)} \deg_G(v)(4 \deg_G(v) + 2 \deg_G(v)^2) = 4M_1(G) + 2F(G),\end{aligned}$$

$$\begin{aligned}
\alpha_2(S(G)) &= \sum_{v \in V(G)} \sum_{uv \in E(G)} 2 \deg_G(v)(4 + \deg_G(v)^2) \\
&= \sum_{v \in V(G)} \sum_{uv \in E(G)} (8 \deg_G(v) + 2 \deg_G(v)^3) \\
&= \sum_{v \in V(G)} \deg_G(v)(8 \deg_G(v) + 2 \deg_G(v)^3) = 8M_1(G) + 2M_1^4(G), \\
\beta(S(G)) &= \sum_{uv \in E(G)} 2(\deg_G(u) + \deg_G(v)) \\
&\quad + \sum_{v \in V(G)} \binom{\deg_G(v)}{2} \deg_G(v)(\deg_G(v) + \deg_G(v)) \\
&= 2M_1(G) + \sum_{v \in V(G)} \deg_G(v)(\deg_G(v) - 1) \deg_G(v)^2 \\
&= 2M_1(G) + M_1^4(G) - F(G), \\
M_2(S(G)) &= \sum_{v \in V(G)} \sum_{uv \in E(G)} 2 \deg_G(v) = \sum_{v \in V(G)} 2 \deg_G(v)^2 = 2M_1(G), \\
M_2^2(S(G)) &= \sum_{v \in V(G)} \sum_{uv \in E(G)} 4 \deg_G(v)^2 = \sum_{v \in V(G)} 4 \deg_G(v)^3 = 4F(G), \\
EM_2(S(G)) &= \sum_{uv \in E(G)} \deg_G(u) \deg_G(v) + \sum_{v \in V(G)} \binom{\deg_G(v)}{2} \deg_G(v)^2 \\
&= M_2(G) + \frac{1}{2} \sum_{v \in V(G)} \deg_G(v)(\deg_G(v) - 1) \deg_G(v)^2 \\
&= M_2(G) + \frac{1}{2} M_1^4(G) - \frac{1}{2} F(G),
\end{aligned}$$

proving the lemma.  $\square$

**Lemma 2.2.**  $\chi_1(S(G)) = (2m-10)(M_1(G))^2 + (16m^2-2F(G)-40m)M_1(G) + 32m^3 - 8mF(G) + 13F(G) + 6M_1^4(G) + M_1^5(G) + 24M_2(G) + 4\alpha_1(G)$ .

*Proof.* By definition of the graph  $S(G)$ ,

$$\begin{aligned}
\chi_1(S(G)) &= \sum_{v \in V(G)} \sum_{uv \in E(G)} (M_1(S(G)) - \deg_G(v)^2 - 3 \deg_G(v) - 2 \deg_G(u))^2 \\
&= \sum_{v \in V(G)} \sum_{uv \in E(G)} (\deg_G(v)^4 + 6 \deg_G(v)^3 + 9 \deg_G(v)^2) \\
&\quad - M_1(S(G)) \sum_{v \in V(G)} \sum_{uv \in E(G)} (2 \deg_G(v)^2 + 6 \deg_G(v)) \\
&\quad + \sum_{v \in V(G)} \sum_{uv \in E(G)} M_1(S(G))^2 \\
&\quad + \sum_{v \in V(G)} \sum_{uv \in E(G)} (4 \deg_G(u) \deg_G(v)^2 + 12 \deg_G(u) \deg_G(v))
\end{aligned}$$

$$\begin{aligned}
& + \sum_{v \in V(G)} \sum_{uv \in E(G)} (-4M_1(S(G)) \deg_G(u) + 4 \deg_G(u)^2) \\
& = M_1^5(G) + 6M_1^4(G) + 9F(G) - 2M_1(S(G))F(G) - 6M_1(S(G))M_1(G) \\
& \quad + 2mM_1(S(G))^2 + \sum_{uv \in E(G)} (4 \deg_G(u) \deg_G(v)^2 + 4 \deg_G(u)^2 \deg_G(v) \\
& \quad + 24 \deg_G(u) \deg_G(v)) + \sum_{uv \in E(G)} (-4M_1(S(G)) \deg_G(u) + 4 \deg_G(u)^2 \\
& \quad - 4M_1(S(G)) \deg_G(v) + 4 \deg_G(v)^2) \\
& = M_1^5(G) + 6M_1^4(G) + 13F(G) - 2M_1(S(G))F(G) - 10M_1(S(G))M_1(G) \\
& \quad + 2mM_1(S(G))^2 + 4\alpha_1(G) + 24M_2(G).
\end{aligned}$$

We now apply Lemma 2.1 to deduce that

$$\begin{aligned}
\chi_1(S(G)) = & (2m - 10)(M_1(G))^2 + (16m^2 - 2F(G) - 40m)M_1(G) + 32m^3 - 8mF(G) \\
& + 13F(G) + 6M_1^4(G) + M_1^5(G) + 24M_2(G) + 4\alpha_1(G),
\end{aligned}$$

which completes the proof.  $\square$

**Lemma 2.3.**  $\chi_2(S(G)) = 32m^3 + (4F(G) - 24)m^2 - (8F(G) + 16M_1(G) + 2M_1^4(G))m + 4m - (M_1(G) - 10)F(G) + 6M_1(G) + M_1^4(G) + M_1^5(G) - 6M_2 + 3\alpha_1(G)$ .

*Proof.* It is easy to see that  $\chi_2(S(G)) = \sum_{v \in V(G)} \sum_{uv \in E(G)} (2m - \deg_G(v) - 1)(F(S(G)) - \deg_G(v)^3 - 3 \deg_G(u)^2 + 3 \deg_G(u) - 7 \deg_G(v) - 2)$ . If we expand the summation, this becomes:

$$\begin{aligned}
\chi_2(S(G)) = & \sum_{v \in V(G)} \sum_{uv \in E(G)} (\deg_G(v)^4 + \deg_G(v)^3 + 7 \deg_G(v)^2 + 9 \deg_G(v) - F(S(G)) \\
& + 2(1 - m \deg_G(v)^3 - 7m \deg_G(v) - 2m + mF(S(G))) - F(S(G)) \deg_G(v)) \\
& + \sum_{v \in V(G)} \sum_{uv \in E(G)} (-6m \deg_G(u)^2 + 6m \deg_G(u) + 3 \deg_G(u)^2 - 3 \deg_G(u)) \\
& + \sum_{v \in V(G)} \sum_{uv \in E(G)} (3 \deg_G(u)^2 \deg_G(v) - 3 \deg_G(u) \deg_G(v)) \\
& = M_1^5(G) + M_1^4(G) + 7F(G) + 9M_1(G) + 4m - 2mM_1^4(G) - 14mM_1(G) \\
& \quad - 8m^2 + 4m^2F(S(G)) - F(S(G))M_1(G) - 2mF(S(G)) \\
& \quad + \sum_{uv \in E(G)} (-6m \deg_G(u)^2 + 6m \deg_G(u) + 3 \deg_G(u)^2 - 3 \deg_G(u) \\
& \quad - 6m \deg_G(v)^2 + 6m \deg_G(v) + 3 \deg_G(v)^2 - 3 \deg_G(v)) \\
& \quad + \sum_{uv \in E(G)} (3 \deg_G(u)^2 \deg_G(v) - 6 \deg_G(u) \deg_G(v) + 3 \deg_G(v)^2 \deg_G(u)) \\
& = M_1^5(G) + M_1^4(G) + 10F(G) + 6M_1(G) + 4m - 2mM_1^4(G) - 8mM_1(G) \\
& \quad - 8m^2 + 4m^2F(S(G)) - F(S(G))M_1(G) - 2mF(S(G)) - 6mF(G)
\end{aligned}$$

$$+ 3\alpha_1(G) - 6M_2(G).$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_2(S(G)) = & 32m^3 + (4F(G) - 24)m^2 - (8F(G) + 16M_1(G) + 2M_1^4(G))m + 4m \\ & - (M_1(G) - 10)F(G) + 6M_1(G) + M_1^4(G) + M_1^5(G) - 6M_2 + 3\alpha_1(G).\end{aligned}$$

This completes the proof.  $\square$

**Lemma 2.4.**  $\chi_3(S(G)) = 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 + (20F(G) - 4(M_1(G))^2 + 30M_1(G) + 4M_1^4(G) + 16M_2(G))m + F(G)M_1(G) + 2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G) - 8M_2(G) - 2\alpha_1(G).$

*Proof.* The degree sequence of subdivision graph  $S(G)$  shows that  $\chi_3(S(G)) = \sum_{v \in V(G)} \sum_{uv \in E(G)} (2m - \deg_G(v) - 1)^2 (M_1(S(G)) - \deg_G(v)^2 - 3\deg_G(v) - 2\deg_G(u))$ . By expanding this summation,

$$\begin{aligned}\chi_3(S(G)) = & \sum_{v \in V(G)} \sum_{uv \in E(G)} (4\deg_G(v)^3 m - \deg_G(v)^4 - 4\deg_G(v)^2 m^2 \\ & + M_1(S(G))\deg_G(v)^2 - 4M_1(S(G))\deg_G(v)m + 4M_1(S(G))m^2 \\ & - 5\deg_G(v)^3 + 16\deg_G(v)^2 m - 12\deg_G(v)m^2 + 2M_1(S(G))\deg_G(v) \\ & - 4M_1(S(G))m - 7\deg_G(v)^2 + 12\deg_G(v)m + M_1(S(G)) - 3\deg_G(v)) \\ & + \sum_{v \in V(G)} \sum_{uv \in E(G)} (8\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 8\deg_G(u)m^2 \\ & - 4\deg_G(u)\deg_G(v) + 8\deg_G(u)m - 2\deg_G(u)) \\ = & 4M_1^4(G)m - M_1^5(G) - 4F(G)m^2 + M_1(S(G))F(G) \\ & - 4M_1(S(G))M_1(G)m + 8M_1(S(G))m^3 - 5M_1^4(G) + 16F(G)m \\ & - 12M_1(G)m^2 + 2M_1(S(G))M_1(G) - 8M_1(S(G))m^2 - 7F(G) \\ & + 12M_1(G)m + 2mM_1(S(G)) - 3M_1(G) \\ & + \sum_{uv \in E(G)} (16\deg_G(u)\deg_G(v)m - 2\deg_G(u)\deg_G(v)^2 - 2\deg_G(v)\deg_G(u)^2 \\ & - 8\deg_G(u)m^2 - 8\deg_G(v)m^2 - 8\deg_G(u)\deg_G(v) + 8\deg_G(u)m \\ & + 8\deg_G(v)m - 2\deg_G(u) - 2\deg_G(v)) \\ = & 4M_1^4(G)m - 4F(G)m^2 + M_1(S(G))F(G) - 4M_1(S(G))M_1(G)m \\ & + 8M_1(S(G))m^3 + 16F(G)m - 20M_1(G)m^2 + 2M_1(S(G))M_1(G) \\ & - 8M_1(S(G))m^2 - 7F(G) + 20M_1(G)m + 2mM_1(S(G)) - 5M_1(G) \\ & + 16mM_2(G) - 2\alpha_1(G) - 8M_2(G) - M_1^5(G) - 5M_1^4(G).\end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned}\chi_3(S(G)) = & 32m^4 + (8M_1(G) - 32)m^3 - (4F(G) + 44M_1(G) - 8)m^2 \\ & + (20F(G) - 4(M_1(G))^2 + 30M_1(G) + 4M_1^4(G) + 16M_2(G))m\end{aligned}$$

$$+ F(G)M_1(G) + 2(M_1(G))^2 - 7F(G) - 5M_1(G) - 5M_1^4(G) - M_1^5(G) - 8M_2(G) - 2\alpha_1(G).$$

Hence, the result follows.  $\square$

**Lemma 2.5.**  $\chi_4(S(G)) = \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G) - \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G).$

*Proof.* By relation between adjacencies in  $G$  and  $S(G)$ , we can see that

$$\begin{aligned} \chi_4(S(G)) &= 2mEM_2(S(G)) - \sum_{v \in V(G)} \left( \binom{\deg_G(v)}{2} \deg_G(v)^3 + (\deg_G(v) - 1) \deg_G(v)^3 \right. \\ &\quad \left. + \binom{\deg_G(v) - 1}{2} \deg_G(v)^3 - \binom{\deg_G(v) - 1}{2} (\deg_G(v) - 1)^2 \deg_G(v) \right) \\ &\quad - \sum_{v \in V(G)} \sum_{uv \in E(G)} (\deg_G(u) - 1)(\deg_G(v)^2(\deg_G(v) - 1) - \deg_G(v) \\ &\quad \times (\deg_G(v) - 1)^2) - \sum_{uv \in E(G)} (\deg_G(u) \deg_G(v)(\deg_G(u) + \deg_G(v)) \\ &\quad + \deg_G(u) \deg_G(v)(\deg_G(u) + \deg_G(v) - 2) \\ &\quad - (\deg_G(u) - 1)^2 \deg_G(v) - \deg_G(u)(\deg_G(v) - 1)^2) \\ &= 2mEM_2(S(G)) - \sum_{v \in V(G)} \left( \frac{1}{2} \deg_G(v)^5 + \frac{3}{2} \deg_G(v)^4 - \frac{9}{2} \deg_G(v)^3 \right. \\ &\quad \left. + \frac{7}{2} \deg_G(v)^2 - \deg_G(v) \right) - \sum_{uv \in E(G)} (\deg_G(u) \deg_G(v)^2 + \deg_G(v) \deg_G(u)^2 \\ &\quad - 2 \deg_G(u) \deg_G(v) - \deg_G(u)^2 - \deg_G(v)^2 + \deg_G(u) + \deg_G(v)) \\ &\quad - \sum_{uv \in E(G)} (\deg_G(u)^2 \deg_G(v) + \deg_G(u) \deg_G(v)^2 + 2 \deg_G(u) \deg_G(v) \\ &\quad - \deg_G(u) - \deg_G(v)) \\ &= 2mEM_2(S(G)) - \left( \frac{1}{2}M_1^5(G) + \frac{3}{2}M_1^4(G) - \frac{11}{2}F(G) + \frac{7}{2}M_1(G) - 2m \right) \\ &\quad - 2\alpha_1(G). \end{aligned}$$

Now by Lemma 2.1,

$$\begin{aligned} \chi_4(S(G)) &= \frac{1}{2}m(4M_2(G) - 2F(G) + 2M_1^4(G) + 4) + \frac{11}{2}F(G) - 2\alpha_1(G) - \frac{7}{2}M_1(G) \\ &\quad - \frac{3}{2}M_1^4(G) - \frac{1}{2}M_1^5(G), \end{aligned}$$

which is our goal.  $\square$

**Lemma 2.6.**  $\chi_5(S(G)) = (8M_1(G) + 8)m^2 - (4F(G) + 10M_1(G) + 4M_2(G) + 4)m - 2(M_1(G))^2 + 2F(G) + M_1(G) + 2M_1^4(G) + 8M_2(G) + \alpha_1(G).$



*Proof.* Again definition of subdivision graph,

$$\begin{aligned}
 \chi_5(S(G)) &= \sum_{uv \in E(G)} ((2m - \deg_G(u) - 1)(M_2(S(G)) - 2\deg_G(u)^2 - 2\deg_G(v) \\
 &\quad - \sum_{wu \in E(G)} \deg_G(w) + \deg_G(v) - 2\deg_G(v)(\deg_G(v) - 1) \\
 &\quad + 2(\deg_G(v) - 1)^2) + (2m - \deg_G(v) - 1)(M_2(S(G)) \\
 &\quad - 2\deg_G(v)^2 - 2\deg_G(u) - \sum_{vz \in E(G)} \deg_G(z) \\
 &\quad + \deg_G(u) - 2\deg_G(u)(\deg_G(u) - 1) + 2(\deg_G(u) - 1)^2)) \\
 &= \sum_{uv \in E(G)} (2\deg_G(u)^3 + 2\deg_G(v)^3 - 4\deg_G(u)^2m - 4\deg_G(v)^2m \\
 &\quad - M_2(S(G))\deg_G(u) - M_2(S(G))\deg_G(v) + 4M_2(S(G))m - 2M_2(S(G)) \\
 &\quad + 8m - 4 + 2\deg_G(u)^2 + 2\deg_G(v)^2 + 6\deg_G(u)\deg_G(v) - 6\deg_G(u)m \\
 &\quad - 6\deg_G(v)m + \deg_G(u) + \deg_G(v)) - \sum_{uv \in E(G)} ((2m - \deg_G(u) - 1) \\
 &\quad \sum_{wu \in E(G)} \deg_G(w) - (2m - \deg_G(v) - 1) \sum_{vz \in E(G)} \deg_G(z)) \\
 &= 2M_1^4(G) - 4mF(G) - M_2(S(G))M_1(G) + 4M_2(S(G))m^2 - 2mM_2(S(G)) \\
 &\quad + 8m^2 - 4m + 2F(G) + 6M_2(G) - 6mM_1(G) + M_1(G) \\
 &\quad - \sum_{uv \in E(G)} ((2m - \deg_G(u) - 1)\deg_G(v)\deg_G(u) \\
 &\quad + (2m - \deg_G(v) - 1)\deg_G(u)\deg_G(v)) \\
 &= 2M_1^4(G) - 4mF(G) - M_2(S(G))M_1(G) + 4M_2(S(G))m^2 - 2mM_2(S(G)) \\
 &\quad + 8m^2 - 4m + 2F(G) + 8M_2(G) - 6mM_1(G) + M_1(G) + \alpha_1(G) \\
 &\quad - 4mM_2(G),
 \end{aligned}$$

Now, by Lemma 2.1,

$$\begin{aligned}
 \chi_5(S(G)) &= (8M_1(G) + 8)m^2 - (4F(G) + 10M_1(G) + 4M_2(G) + 4)m - 2(M_1(G))^2 \\
 &\quad + 2F(G) + M_1(G) + 2M_1^4(G) + 8M_2(G) + \alpha_1(G),
 \end{aligned}$$

which proving the lemma.  $\square$

Let  $G$  be a graph. It is easy to see that  $g(S(G)) \geq 6$ . Therefore, by Lemma 2.1, 2.2, 2.3, 2.4, 2.5, 2.6 and Theorem 1.1, we have the following theorem.

**Theorem 2.1.** *Let  $G$  be a graph with  $m$  edges. Then*

$$\begin{aligned}
 p(S(G); 5) &= \frac{1}{15}m^2(4m^3 - 20m^2 + 15m + 15) + \frac{1}{12}m(8F(G)m - 8M_1(G)m^2 \\
 &\quad + 3(M_1(G))^2 + 36M_1(G)m - 28F(G) - 24M_1(G) - 6M_1^4(G) - 24M_2(G))
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G) \\
& + M_1^4(G) + F(G).
\end{aligned}$$

We are now ready to prove our main result. For the sake of completeness, we mention here a useful result of Zhou and Gutman [18].

**Theorem 2.2.** *Let  $G$  be an  $n$ -vertex tree. Then  $c_{n-k}(G) = p(S(G); k)$ , for  $0 \leq k \leq n$ .*

**Theorem 2.3.** *Let  $G$  be an acyclic graph on  $n$  vertices and  $m$  edges. Then*

$$\begin{aligned}
c_{n-5}(G) = & \frac{1}{15}m^2(4m^3 - 20m^2 + 15m + 15) + \frac{1}{12}m(8F(G)m - 8M_1(G)m^2 \\
& + 3(M_1(G))^2 + 36M_1(G)m - 28F(G) - 24M_1(G) - 6M_1^4(G) - 24M_2(G)) \\
& - \frac{1}{6}M_1(G)(3M_1(G) + F(G) + 6) + \alpha_1(G) + 2M_2(G) + \frac{1}{5}M_1^5(G) \\
& + M_1^4(G) + F(G).
\end{aligned}$$

*Proof.* Apply Theorem 2.1 and 2.2. □

**Corollary 2.1.** *Let  $T$  be a tree on  $n$  vertices. Then*

$$\begin{aligned}
c_{n-5}(G) = & \frac{1}{15}(n-1)^2(4n^3 - 32n^2 + 67n - 24) + \frac{1}{12}n(8nF(G) - 8n^2M_1(G) \\
& + 3(M_1(G))^2 + 60nM_1(G) - 44F(G) - 120M_1(G) - 6M_1^4(G) - 24M_2(G)) \\
& - \frac{1}{12}M_1(G)(2F(G) + 9M_1(G) - 56) + \alpha_1(G) + \frac{1}{5}M_1^5(G) + \frac{3}{2}M_1^4(G) \\
& + 4M_2(G) + 4F(G).
\end{aligned}$$

*Proof.* The result follows from Theorem 2.3 and the fact that  $m(T) = n - 1$ . □

### 3. APPLICATIONS

The aim of this section is to apply our results in Section 2 for computing the Laplacian coefficients  $c_{n-k}(G)$ ,  $k = 2, 3, 4, 5$ , when  $G$  is a certain tree. We first assume that  $T(k, t)$  be a rooted tree with degree sequence  $k, k, \dots, k, 1, 1, \dots, 1$  and  $t$  is the distance between the center and any pendent vertex, Figure 1. Then,

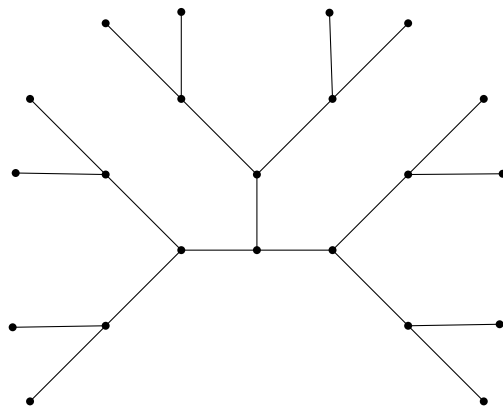
$$\begin{aligned}
c_{n-1}(T(k, t)) &= \frac{k}{k-2}(2(k-1)^t - 1), \\
c_{n-2}(T(k, t)) &= \frac{k}{2(k-2)^2}((k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k^2 - 2k(k-1)^t + k - 2)), \\
c_{n-3}(T(k, t)) &= -\frac{k}{3(k-2)^3}((k-1)(k-2)^2(k^2 - 3k(k-1)^t + 5k - 8)(k-1)^{t-1} \\
& - 2k((k-1)^t - 1)(k^2 - 2k(k-1)^t + 3k - 6)(k - (k-1)^t - 1)), \\
c_{n-4}(T(k, t)) &= \frac{k}{4(k-2)^4}((1/2)k(k-1)^2(k-2)^4((k-1)^{t-1})^2 + 4((k-1)^t)^2k^2 \\
& + (-(14/3)k^3 - (34/3)k^2 + (76/3)k)(k-1)^t + k^4 + (29/3)k^3 - (53/3)k^2
\end{aligned}$$

$$\begin{aligned}
& - (52/3)k + 28)(k-1)(k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1) \\
& - (4/3)((k-1)^t)^3 k^3 + (4k^4 - 8k^2)((k-1)^t)^2 + (-(11/3)k^5 - (10/3)k^4 \\
& + (85/3)k^3 - 20k^2 - 4k)(k-1)^t + (k^5 + (14/3)k^4 - 18k^3 + 6k^2 \\
& + 16k - 8)(k-1))), \\
c_{n-5}(T(k, t)) = & - \frac{k}{5(k-2)^5} ((5/6)(k^2 - (3/2)k(k-1)^t + (7/2)k - 8)(k-1)^2(k-2)^4 k( \\
& (k-1)^{t-1})^2 + (-(10/3)((k-1)^t)^3 k^3 + (25/3)(k^2 + (7/5)k - (22/5))k^2 \\
& \times ((k-1)^t)^2 - (35/6)(k^4 + 5k^3 - (99/7)k^2 - (32/7)k + (116/7))k(k-1)^t \\
& + k^6 + (95/6)k^5 - (223/6)k^4 - (341/6)k^3 + 158k^2 - (118/3)k - 48)(k-1) \\
& (k-2)^2(k-1)^{t-1} - 2((k-1)^t - 1)(k-3/2 - (1/2)(k-1)^t)k(-(4/3)((k-1)^t)^3 k^3 \\
& + (4k^4 + (8/3)k^3 - (40/3)k^2)((k-1)^t)^2 + (-(11/3)k^5 \\
& - 10k^4 + (137/3)k^3 - 20k^2 - 20k)(k-1)^t + k^6 + (23/3)k^5 - 32k^4 \\
& + (310/3)k^2 - 120k + 40)), \\
c_{n-1}(T(3, t)) = & 6 \times 2^t - 6, \\
c_{n-2}(T(3, t)) = & - \frac{93}{2} 2^t + 18 \times 2^{2t} + 30, \\
c_{n-3}(T(3, t)) = & 272 \times 2^t - 171 \times 2^{2t} + 36 \times 2^{3t} - 144, \\
c_{n-4}(T(3, t)) = & - \frac{5799}{4} 2^t + \frac{9177}{8} 2^{2t} - 405 \times 2^{3t} + 54 \times 2^{4t} + 687, \\
c_{n-5}(T(3, t)) = & \frac{74427}{10} 2^t - \frac{26967}{4} 2^{2t} + \frac{12267}{4} 2^{3t} - 702 \times 2^{4t} + \frac{324}{5} 2^{5t} - 3294, \\
c_{n-1}(T(4, t)) = & 4 \times 3^t - 4, \\
c_{n-2}(T(4, t)) = & - 24 \times 3^t + 8 \times 3^{2t} + 18, \\
c_{n-3}(T(4, t)) = & \frac{392}{3} 3^t - 64 \times 3^{2t} + \frac{32}{3} 3^{3t} - 88, \\
c_{n-4}(T(4, t)) = & - \frac{2132}{3} 3^t + \frac{1232}{3} 3^{2t} - \frac{320}{3} 3^{3t} + \frac{32}{3} 3^{4t} + 457, \\
c_{n-5}(T(4, t)) = & \frac{19644}{5} 3^t - 2480 \times 3^{2t} + \frac{2368}{3} 3^{3t} - 128 \times 3^{4t} + \frac{128}{15} 3^{5t} - 2484.
\end{aligned}$$

Our second class of trees are known as Kragujevac trees. To define, we assume that  $B_1, B_2, B_3, \dots, B_k$  are branches whose structure is depicted in Figure 2. A proper Kragujevac tree is a tree possessing a central vertex of degree at least 3, to which branches of the form  $B_1$  and/or  $B_2$  and/or  $B_3$  and/or  $\dots$  are attached [10].

Let  $G_i$ , for  $i = 1, 2, \dots, 7$ , be the proper Kragujevac tree on  $n$  vertices in Figure 3. Then

$$\begin{aligned}
c_{n-2}(G_1) &= \frac{3}{98}n(65n - 231) - 3, \quad c_{n-3}(G_1) = \frac{8}{1029}n(169n^2 - 1302n + 1127) + 60, \\
c_{n-4}(G_1) &= \frac{1}{57624}n(37349n^3 - 503594n^2 + 1625575n + 4758782) - 462, \\
c_{n-5}(G_1) &= \frac{3}{336140}n(28561n^4 - 597415n^3 + 3893785n^2 + 1016995n - 107579206) + 2868,
\end{aligned}$$

FIGURE 1. The rooted tree  $T(3, 3)$ .

$$\begin{aligned}
 c_{n-2}(G_2) &= \frac{15}{98}(n-1)(13n-34), \quad c_{n-3}(G_2) = \frac{2}{1029}(n-1)(676n^2 - 4649n + 8481), \\
 c_{n-4}(G_2) &= \frac{1}{57624}(n-1)(37349n^3 - 478413n^2 + 2146954n - 3432552), \\
 c_{n-5}(G_2) &= \frac{1}{336140}(n-1)(85683n^4 - 1750502n^3 + 13991793n^2 - 52528222n + 79270320), \\
 c_{n-2}(G_3) &= \frac{1}{98}(15n-16)(13n-33), \\
 c_{n-3}(G_3) &= \frac{1}{1029}n(1352n^2 - 10611n + 26563) - \frac{6480}{343}, \\
 c_{n-4}(G_3) &= \frac{1}{57624}n(37349n^3 - 513734n^2 + 2635015n - 5871574) + \frac{184469}{2401}, \\
 c_{n-5}(G_3) &= \frac{1}{1008420}n(257049n^4 - 5486585n^3 + 47226105n^2 - 204551395n + 437870586) \\
 &\quad - \frac{5694446}{16807}, \\
 c_{n-2}(G_4) &= \frac{1}{98}n(195n-701) + \frac{125}{49}, \quad c_{n-3}(G_4) = \frac{4}{1029}n(n-4)(338n-1291) + \frac{3008}{343}, \\
 c_{n-4}(G_4) &= \frac{1}{57624}n(37349n^3 - 511706n^2 + 2298967n - 2158546) - \frac{291540}{2401}, \\
 c_{n-5}(G_4) &= \frac{1}{1008420}n(257049n^4 - 5464615n^3 + 43233545n^2 - 130350725n - 74554454) \\
 &\quad + \frac{15573272}{16807}, \\
 c_{n-2}(G_5) &= \frac{1}{98}n(195n-713) + \frac{405}{49}, \\
 c_{n-3}(G_5) &= \frac{2}{1029}n(676n^2 - 5403n + 16460) - \frac{14418}{343}, \\
 c_{n-4}(G_5) &= \frac{1}{57624}n(37349n^3 - 523874n^2 + 3028255n - 9178570) + \frac{540746}{2401}, \\
 c_{n-5}(G_5) &= \frac{1}{1008420}n(257049n^4 - 5596435n^3 + 52360845n^2 - 275208485n + 846144906)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{20524022}{16807}, \\
c_{n-2}(G_6) &= \frac{1}{98}(39n+1)(5n-18), \quad c_{n-3}(G_6) = \frac{2}{1029}n(676n^2 - 5247n + 7454) + \frac{11758}{343}, \\
c_{n-4}(G_6) &= \frac{1}{57624}n(37349n^3 - 507650n^2 + 1965703n + 1289150) - \frac{709627}{2401}, \\
c_{n-2}(G_6) &= \frac{1}{1008420}n(257049n^4 - 5420675n^3 + 39172605n^2 - 60754765n - 529616214) \\
& \quad + \frac{33001040}{16807}, \\
c_{n-2}(G_7) &= \frac{1}{98}n(195n - 709) + \frac{332}{49}, \quad c_{n-3}(G_7) = \frac{2}{1029}n(676n^2 - 5364n + 14831) - \frac{10096}{343}, \\
c_{n-4}(G_7) &= \frac{1}{57624}n(37349n^3 - 519818n^2 + 2830243n - 7399558) + \frac{340472}{2401}, \\
c_{n-2}(G_7) &= \frac{1}{1008420}n(257049n^4 - 5552495n^3 + 49827665n^2 - 237728905n + 619570486) \\
& \quad - \frac{11978116}{16807}.
\end{aligned}$$

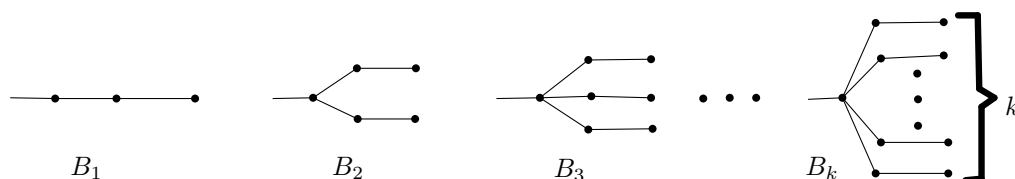


FIGURE 2. The branches of proper Kragujevac trees.

Our third class of trees are caterpillar trees. A caterpillar is a tree in which all the vertices are within distance 1 of a central path [9]. Let  $T_i$ , for  $i = 1, 2, \dots, 5$ , be the caterpillar tree on  $n \geq 8$  vertices, see Figure 4. Then,

$$\begin{aligned}
c_{n-2}(T_1) &= \frac{1}{2}(4n-7)(n-2), \quad c_{n-3}(T_1) = \frac{1}{3}(n-2)(4n^2 - 25n + 42), \\
c_{n-4}(T_1) &= \frac{1}{24}(n-4)(16n^3 - 168n^2 + 611n - 726), \\
c_{n-5}(T_1) &= \frac{1}{60}(n-4)(16n^4 - 296n^3 + 2111n^2 - 6811n + 8250), \\
c_{n-2}(T_2) &= \frac{1}{2}n(4n-15) + \frac{15}{2}, \quad c_{n-3}(T_2) = \frac{1}{3}(n-3)(4n^2 - 21n + 32), \\
c_{n-4}(T_2) &= \frac{1}{24}n(16n^3 - 232n^2 + 1307n - 3404) + \frac{1155}{8}, \\
c_{n-5}(T_2) &= \frac{1}{60}(n-5)(16n^4 - 280n^3 + 1935n^2 - 6270n + 8079), \\
c_{n-2}(T_3) &= 2(n-2)^2, \quad c_{n-3}(T_3) = \frac{4}{3}(n-2)(n^2 - 7n + 14), \\
c_{n-4}(T_3) &= \frac{2}{3}n(n^3 - 16n^2 + 100n - 281) + \frac{575}{3},
\end{aligned}$$

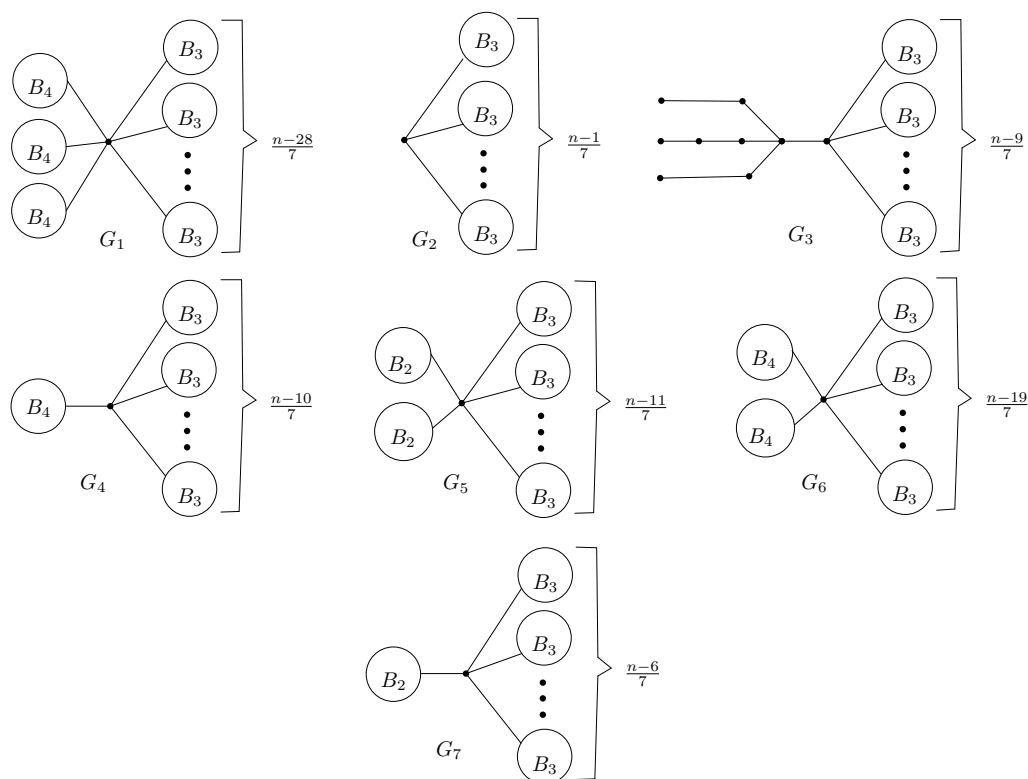


FIGURE 3. The proper Kragujevac trees that have illustrated in [10, Conjecture 3].

$$\begin{aligned}
 c_{n-5}(T_3) &= \frac{2}{15}(n-5)(2n^4 - 40n^3 + 320n^2 - 1170n + 1573), \\
 c_{n-2}(T_4) &= 2(n-2)^2 + 1, \quad c_{n-3}(T_4) = \frac{2}{3}(n-3)(2n^2 - 12n + 23), \\
 c_{n-4}(T_4) &= \frac{2}{3}(n-4)(n^3 - 12n^2 + 55n - 93), \\
 c_{n-5}(T_4) &= \frac{4}{15}n(n^4 - 25n^3 + 265n^2 - 1480n + 4314) - 1386, \\
 c_{n-2}(T_5) &= 2(n-2)^2 + 1, \quad c_{n-3}(T_5) = \frac{2}{3}n(2n^2 - 18n + 59) - \frac{140}{3}, \\
 c_{n-4}(T_5) &= \frac{2}{3}n(n^3 - 16n^2 + 103n - 315) + \frac{769}{3}, \\
 c_{n-5}(T_5) &= \frac{2}{15}n(2n^4 - 50n^3 + 530n^2 - 2970n + 8773) - 1456.
 \end{aligned}$$

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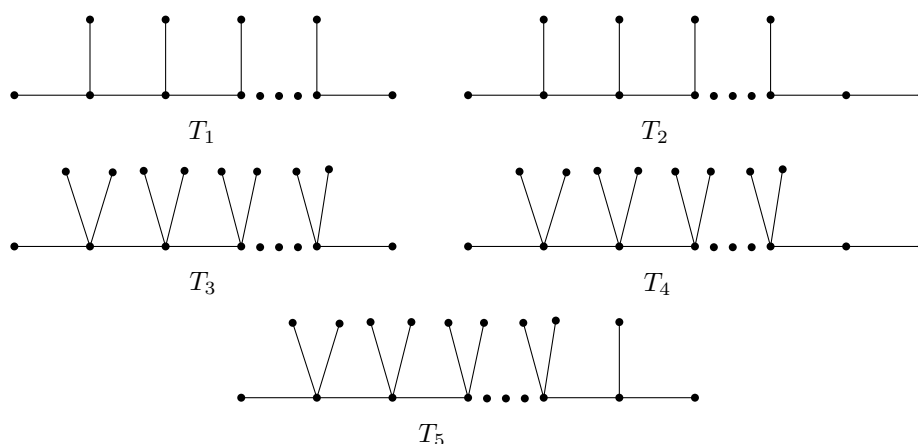


FIGURE 4. The caterpillar trees.

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