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INTEGRAL INVOLVING THE PRODUCT OF MULTIVARIABLE ALEPH-FUNCTION, GENERAL CLASS OF SRIVASTAVA POLYNOMIALS AND ALEPH-FUNCTION OF ONE VARIABLE

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Abstract. In this paper, we derive an integral involving the multivariable Aleph-function, the general class of Srivastva polynomials, and the Aleph-function of one variable, all of which are sufficiently general in nature and are capable of yielding a large number of simpler and more useful results simply by specialization of their parameters. Moreover, we establish certain specific instances.

1. Introduction and Preliminaries

The Aleph ($\aleph$)-function was established by Südland et al. [30], but its notation and complete definition are offered below in terms of the Mellin-Barnes type integral (see also, [2, 3, 7, 13, 23, 25]):

$$\aleph(z) = \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(z \,\middle|\, \begin{matrix} (a_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r'} \\ (b_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r'} \end{matrix}\right)$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)\, z^{-s}\mathrm{d}s, \tag{1.1}$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r'} c_i\left\{\prod_{j=N+1}^{P_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - B_{ji}s)\right\}}, \tag{1.2}$$

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where $|\arg z| < \frac{1}{2}\pi\,\Omega$ and $\Omega = \sum_{j=1}^{M} B_j + \sum_{j=1}^{N} A_j - c_i\left(\sum_{j=M+1}^{Q_i} B_{ji} + \sum_{j=N+1}^{P_i} A_{ji}\right) > 0$ for $i = 1, \ldots, r'$. For convergence conditions and other details of Aleph-function (one variable), see Südland et al. [30] (see also, [23, 24]). The series representation of Aleph-function is given by Chaurasia and Singh [6], defined as

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z) = \sum_{G=1}^{M}\sum_{g=0}^{+\infty}\frac{(-1)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}z^{-s}, \tag{1.3}$$

with $s = \eta_{G,g} = \frac{b_G+g}{B_G}$, $P_i < Q_i$, $|z| < 1$ and $\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)$ is given in (1.2).

The generalized polynomials defined by Srivastava [29], and studied by many authors, e.g., [5, 7, 8, 10–12, 14, 18, 20], is given in the following manner:

$$\begin{aligned} S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}[y_1,\ldots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\times\cdots\times\frac{(-N_s)_{M_sK_s}}{K_s!} \\ \times A[N_1,K_1;\ldots;N_s,K_s]\,y_1^{K_1}\cdots y_s^{K_s}, \end{aligned} \tag{1.4}$$

where $M_1, \ldots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \ldots; N_s, K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation:

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\times\cdots\times\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\ldots;N_s,K_s]. \tag{1.5}$$

The Aleph-function of several variables generalizes the multivariable I-function defined by Sharma and Ahmad [26], which is a generalization of G and H-functions [8, 21] of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariable Aleph-function throughout our present study and will be defined and represented as follows (see also, [4, 15–17, 19]).

$$\begin{aligned} &\aleph(z_1,\ldots,z_r) \\ &=\aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}\left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle|\begin{array}{c} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,\mathfrak{n}}, \\ \ldots, \end{array}\right. \\ &\quad\begin{array}{l} \left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{\mathfrak{n}+1,p_i} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}, \\ \left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{1,q_i} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}, \end{array} \\ &\quad\left.\begin{array}{l} \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r},\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}} \\ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r},\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}} \end{array}\right) \\ &=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\,z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r, \end{aligned} \tag{1.6}$$

with $\omega = \sqrt{-1}$. For more details, see Ayant [1]. The real numbers τ_i are positives for $i = 1, \ldots, R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \ldots, R^{(k)}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.6) can be obtained by extension of the corresponding conditions for multivariable H-function given by $|\arg z_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$- \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \ldots, r,\ i = 1, \ldots, R,\ i^{(k)} = 1, \ldots, R^{(k)}. \tag{1.7}$$

The complex numbers z_i are not zero. Throughout this paper, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the asymptotic expansion in the following convenient form:

$$\aleph(z_1, \ldots, z_r) = 0\left(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}\right), \quad \max\{|z_1|, \ldots, |z_r|\} \to 0,$$

$$\aleph(z_1, \ldots, z_r) = 0\left(|z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}\right), \quad \min\{|z_1|, \ldots, |z_r|\} \to +\infty,$$

where $k = 1, \ldots, r$, $\alpha_k = \min\left\{\operatorname{Re}\left(d_j^{(k)}/\delta_j^{(k)}\right) : j = 1, \ldots, m_k\right\}$ and $\beta_k = \max\left\{\operatorname{Re}\left(\left(c_j^{(k)} - 1\right)/\gamma_j^{(k)}\right) : j = 1, \ldots, n_k\right\}$. We will use these following notations:

$$U = p_i, q_i, \tau_i; R, \quad V = m_1, n_1; \ldots; m_r, n_r, \tag{1.8}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \ldots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}, \tag{1.9}$$

$$A = \left(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,n}, \left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)\right]_{n+1,p_i}, \tag{1.10}$$

$$B = \left[\tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \ldots, \beta_{ji}^{(r)}\right)\right]_{1,q_i}, \tag{1.11}$$

$$C = \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,n_1}, \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_{i^{(1)}}}, \ldots,$$
$$\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}, \left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_{i^{(r)}}}, \tag{1.12}$$

$$D = \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,m_1}, \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_{i^{(1)}}}, \ldots,$$
$$\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}, \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_{i^{(r)}}}. \tag{1.13}$$

We denote the multivariable Aleph-function as

$$\aleph(z_1, \ldots, z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{c} A : C \\ B : D \end{array}\right). \tag{1.14}$$

We have the following required integral [9]:

(1.15)
$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{\alpha+\beta}}\,\mathrm{d}\theta=\frac{1}{2a^{2\beta}b^{2\alpha}}B\left(\alpha,\beta\right),\quad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,$$

where $a,b\in\mathbb{C}\backslash\{0\}$ and $B\left(\cdot,\cdot\right)$ is the Beta function.

2. Main Integral

In this section we evaluate the integral involving multivariable Aleph-function, a class of polynomials of several variables and a Aleph-function of one variable.

Theorem 2.1.

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{\alpha+\beta}}\ \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c+d}}\right)$$
$$\times\ S^{M_1,\dots,M_s}_{N_1,\dots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}}\\ \vdots\\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}}\end{pmatrix}\aleph^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}}\\ \vdots\\ z_r\frac{\sin^{2h_r}\theta\ \cos^{2l_r}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_r+l_r}}\end{pmatrix}\mathrm{d}\theta$$
$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{+\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}\left(\eta_{G,g}\right)}{B_G\,g!}t^{\eta_{G,g}}\,t_1^{K_1}\cdots t_s^{K_s}$$
$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_sd_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_sc_i\right)}$$
$$\times\ \aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{matrix}\frac{z_1}{a^{2l_1}\,b^{2h_1}}\\ \vdots\\ \frac{z_r}{a^{2l_r}\,b^{2h_r}}\end{matrix}\ \middle|\ \begin{matrix}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;h_1,\dots,h_r\right),\\ -\end{matrix}\right.$$
$$\left.\begin{matrix}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^{s}K_id_i;l_1,\dots,l_r\right),A:C\\ \left(1-\alpha-\beta-\left(c+d\right)\eta_{G,g}-\sum_{i=1}^{s}K\left(c_i+d_i\right);h_1+l_1,\dots,h_r+l_r\right),B:D\end{matrix}\right),\tag{2.1}$$

where $U_{21}=p_i+2,\ q_i+1,\ \tau_i,\ R,$ also satisfy the following conditions:

(a) $\min\left\{c,c_i,h_j\right\}\le 0,\ i=1,\dots,s;\ j=1,\dots,r$ *(h_j are not simultaneously zero);*

(b) $\min\left\{d,d_i,l_j\right\}\le 0,\ i=1,\dots,s;\ j=1,\dots,r$ *(l_j are not simultaneously zero);*

(c) $\mathrm{Re}\left(\alpha\right)+c+\sum_{i=1}^{r}c_i\mathrm{Re}\left(\alpha\right)+c\min\limits_{1\leqslant l\leqslant M}\mathrm{Re}\left(\frac{b_j^{(i)}}{\beta_j}\right)+\sum_{i=1}^{r}h_i\min\limits_{1\leqslant j\leqslant m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;$

(d) $\mathrm{Re}\left(\alpha\right)+d\min\limits_{1\leqslant l\leqslant M}\mathrm{Re}\left(\frac{b_j^{(i)}}{\beta_j}\right)+\sum_{i=1}^{r}l_i\min\limits_{1\leqslant j\leqslant m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0;$

(e) $\left|\arg z_k\right|<\frac{1}{2}A_i^{(k)}\pi,$ *where $A_i^{(k)}$ is given in (1.7);*

(f) $\left|\arg t\right|<\frac{1}{2}\pi\Omega,$ *where*

$$\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i\left(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji}\right)>0.$$

Proof. Expressing the Aleph-function of one variable in series form with the help of (1.3), the general class of polynomials of several variables in series with the help of (1.4), and the Aleph-function of r variables in Mellin-Barnes contour integral with the help of (1.6). The conditions (e) and (f) are satisfied, then the integral representing multivariable Aleph function converges uniformly, and we can invert the sums and multiple Mellin-Barnes integrals. Next, by changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and summations involved in the process) and then evaluating the resulting integral with the help of equation (1.15). Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result (2.1). □

3. Multivariable I-function

Corollary 3.1. *If $\tau_i, \tau_{i^{(1)}}, \ldots, \tau_{i^{(r)}} \to 1$, the Aleph-function of several variables renovates to the I-function of several variables. The simple integral has been derived in this section for multivariable I-functions defined by Sharma and Ahmad* [26]

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta \ \ \cos^{2\beta-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\alpha+\beta}} \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c+d}}\right)$$

$$\times\ S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_1+d_1}} \\ \vdots \\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{c_s+d_s}}\end{pmatrix} I^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_1+l_1}} \\ \vdots \\ z_r\frac{\sin^{2h_r}\theta\ \cos^{2l_r}\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{h_r+l_r}}\end{pmatrix} \mathrm{d}\theta$$

$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G\, g!}\ t^{\eta_{G,g}}\, t_1^{K_1}\cdots t_s^{K_s}$$

$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_s d_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_s c_i\right)}$$

$$\times\ I^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{matrix}\frac{z_1}{a^{2l_1}\,b^{2h_1}}\\ \vdots \\ \frac{z_r}{a^{2l_r}\,b^{2h_r}}\end{matrix}\ \middle|\ \begin{matrix}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^{s}K_ic_i;h_1,\ldots,h_r\right), \\ - \end{matrix}\right.$$

(3.1)

$$\left.\begin{matrix}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^{s}K_id_i;l_1,\ldots,l_r\right),A':C' \\ \left(1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^{s}K\,(c_i+d_i)\,;h_1+l_1,\ldots,h_r+l_r\right),B':D'\end{matrix}\right),$$

where $A'=\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n},\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}$, $B'=\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{1,q_i}$, $C'=\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1},\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r},\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}$, $D'=\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1},\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}$, *also under the same conditions that* (2.1).

4. Aleph-function of Two Variables

Corollary 4.1. *If we set* $r = 2$ *in* (1.6)*, then we obtain the Aleph-function of two variables defined by Sharma* [28] *and further generalized by Kumar* [13]*. We have the following simple integral*

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\alpha+\beta}}\ \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c+d}}\right)$$
$$\times\ S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c_1+d_1}}\\ \vdots\\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c_s+d_s}}\end{pmatrix} \aleph^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{h_1+l_1}}\\ z_2\frac{\sin^{2h_2}\theta\ \cos^{2l_2}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{h_2+l_2}}\end{pmatrix}\mathrm{d}\theta$$
$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{+\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-1)^g\ \Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G\, g!}\ t^{\eta_{G,g}}\, t_1^{K_1}\cdots t_s^{K_s}$$
$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^s K_s d_i\right)}\ b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^s K_s c_i\right)}$$
$$\times\ \aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{matrix}\frac{z_1}{a^{2l_1}b^{2h_1}}\\ \frac{z_2}{a^{2l_2}b^{2h_2}}\end{matrix}\ \middle|\ \begin{matrix}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^s K_i c_i; h_1,h_2\right),\\ -\end{matrix}\right.$$
(4.1)
$$\left.\begin{matrix}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^s K_i d_i; l_1,l_2\right), A'':C'';E''\\ \left(1-\alpha-\beta-(c+d)\eta_{G,g}-\sum_{i=1}^s K\left(c_i+d_i\right); h_1+l_1, h_2+l_2\right), B'':D'';F''\end{matrix}\right),$$

where $A''=\left(a_j;\alpha'_j,\alpha''_j\right)_{1,n}$, $\left[\tau_i\left(a_{ji};\alpha'_{ji},\alpha''_{ji}\right)\right]_{n+1,p_i}$; $B''=\left[\tau_i\left(b_{ji};\beta'_{ji},\beta''_{ji}\right)\right]_{1,q_i}$, $C''=(c_j,\gamma_j)_{1,n_1}$, $[\tau_{i'}(c_{ji'},\gamma_{ji'})]_{n_1+1,p_{i'}}$; $D''=(d_j,\delta_j)_{1,m_1}$, $[\tau_{i'}(d_{ji'},\delta_{ji'})]_{m_1+1,q_{i'}}$, $E''=(e_j,\eta_j)_{1,n_2}$, $[\tau_{i''}(e_{ji''},\eta_{ji''})]_{n_2+1,p_{i''}}$; $F''=(f_j,\zeta_j)_{1,m_2}$, $[\tau_{i''}(f_{ji''},\zeta_{ji''})]_{m_2+1,q_{i''}}$, *also satisfy the existence conditions provided in* (2.1)*.*

5. I-function of Two Variables

Corollary 5.1. *If we set* $\tau_i,\tau_{i'},\tau_{i''}\to 1$ *in* (4.1)*, the Aleph-function of two variables reduces to the* I*-function of two variables defined by Sharma and Mishra* [27]*, and we obtain the same formula with the* I*-function of two variables.*

$$\int_0^{\pi/2} \frac{\sin^{2\alpha-1}\theta\ \cos^{2\beta-1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\alpha+\beta}}\ \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(t\frac{\sin^{2c}\theta\ \cos^{2d}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c+d}}\right)$$
$$\times\ S^{M_1,\ldots,M_s}_{N_1,\ldots,N_s}\begin{pmatrix} t_1\frac{\sin^{2c_1}\theta\ \cos^{2d_1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c_1+d_1}}\\ \vdots\\ t_s\frac{\sin^{2c_s}\theta\ \cos^{2d_s}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{c_s+d_s}}\end{pmatrix} I^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix} z_1\frac{\sin^{2h_1}\theta\ \cos^{2l_1}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{h_1+l_1}}\\ z_2\frac{\sin^{2h_2}\theta\ \cos^{2l_2}\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{h_2+l_2}}\end{pmatrix}\mathrm{d}\theta$$

$$=\frac{1}{2}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G\,g!}\,t^{\eta_{G,g}}\,t_1^{K_1}\cdots t_s^{K_s}$$

$$\times\ a^{-2\left(\beta+d\eta_{G,g}+\sum_{i=1}^{s}K_s d_i\right)}\,b^{-2\left(\alpha+c\eta_{G,g}+\sum_{i=1}^{s}K_s c_i\right)}$$

$$\times\, I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\frac{z_1}{a^{2l_1}\,b^{2h_1}}\\ \frac{z_2}{a^{2l_2}\,b^{2h_2}}\end{array}\,\middle|\,\begin{array}{c}\left(1-\alpha-c\eta_{G,g}-\sum_{i=1}^{s}K_i c_i;h_1,h_2\right),\\ -\end{array}\right.$$

(5.1)

$$\left.\begin{array}{c}\left(1-\beta-d\eta_{G,g}-\sum_{i=1}^{s}K_i d_i;l_1,l_2\right),A''':C''';E'''\\ \left(1-\alpha-\beta-(c+d)\,\eta_{G,g}-\sum_{i=1}^{s}K\left(c_i+d_i\right);h_1+l_1,h_2+l_2\right),B''':D''';F'''\end{array}\right),$$

where $A'''=\left(a_j;\alpha'_j,\alpha''_j\right)_{1,n}$, $\left(a_{ji};\alpha'_{ji},\alpha''_{ji}\right)_{n+1,p_i}$, $B'''=\left(b_{ji};\beta'_{ji},\beta''_{ji}\right)_{1,q_i}$, $C'''=\left(c_j,\gamma_j\right)_{1,n_1},\left(c_{ji'},\gamma_{ji'}\right)_{n_1+1,p_{i'}}$; $D'''=\left(d_j,\delta_j\right)_{1,m_1},\left(d_{ji'},\delta_{ji'}\right)_{m_1+1,q_{i'}}$, $E'''=\left(e_j,\eta_j\right)_{1,n_2},\left(e_{ji''},\eta_{ji''}\right)_{n_2+1,p_{i''}}$, $F'''=\left(f_j,\zeta_j\right)_{1,m_2},\left(f_{ji''},\zeta_{ji''}\right)_{m_2+1,q_{i''}}$, *also satisfy the conditions stated in* (2.1).

For more details of I-function of two variables reader can refer to work Kumari et al. [22].

6. Conclusion

In this work, an integral involving the multivariable Aleph-function, a class of polynomials with several variables (Srivastava polynomials), and an Aleph-function with one variable was evaluated. The integral derived in this study is of a highly broad character, since it incorporates the multivariable Aleph-function, which is a generic function of multiple variables previously explored. Consequently, the integral produced by this study would serve as a key formula from which, by adjusting the parameters, as many outcomes as required involving the special functions of one and multiple variables may be generated.

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