

ON SOME COMBINATORIAL PROPERTIES OF GENERALIZED COMMUTATIVE PELL AND PELL-LUCAS QUATERNIONS

DOROTA BRÓD AND ANETTA SZYNAL-LIANA

ABSTRACT. Generalized commutative quaternions generalize elliptic, parabolic and hyperbolic quaternions, bicomplex numbers, complex hyperbolic numbers and hyperbolic complex numbers. In this paper, we study generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. We present some properties of these numbers and relations between them.

1. INTRODUCTION

Let $n \geq 0$ be an integer. The n th Pell number P_n is defined in the following way $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$. Solving the above recurrence equation we obtain the direct formula of the form

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}},$$

named also as the Binet formula for Pell numbers.

The n th Pell-Lucas number Q_n is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$, with $Q_0 = Q_1 = 2$. The Binet formula for Pell-Lucas numbers has the form

$$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

The first six terms of the Pell sequence and Pell-Lucas sequence are 0, 1, 2, 5, 12, 29 and 2, 2, 6, 14, 34, 82, respectively.

The Pell and Pell-Lucas numbers belong to the class of numbers of the Fibonacci type and have applications also in the theory of hypercomplex numbers (see [1–3, 9–12]).

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In this paper, we use the Pell and Pell-Lucas numbers in the theory of generalized commutative quaternions.

Let $\mathbb{H}_{\alpha\beta}^c$ be the set of generalized commutative quaternions \mathbf{x} of the form

$$\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$(1.1) \quad e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta,$$

$$(1.2) \quad e_1e_2 = e_2e_1 = e_3, \quad e_2e_3 = e_3e_2 = \beta e_1 \quad \text{and} \quad e_3e_1 = e_1e_3 = \alpha e_2,$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$.

The generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0, \beta = 1$), parabolic quaternions ($\alpha = 0, \beta = 1$), hyperbolic quaternions ($\alpha > 0, \beta = 1$), bicomplex numbers ($\alpha = -1, \beta = -1$), complex hyperbolic numbers ($\alpha = -1, \beta = 1$) and hyperbolic complex numbers ($\alpha = 1, \beta = -1$).

Generalized commutative quaternions were introduced in [8]. The authors defined generalized commutative quaternions of the Fibonacci type – generalized commutative Horadam quaternions.

For integers p, q, n and $n \geq 0$ Horadam defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$, for $n \geq 2$, with fixed real numbers W_0, W_1 . Let t_1, t_2 be the two distinct real roots of the equation $t^2 - pt + q = 0$. Then, the Binet type formula for the Horadam numbers has the form $W_n = At_1^n + Bt_2^n$, where $t_1 = \frac{p + \sqrt{p^2 - 4q}}{2}$, $t_2 = \frac{p - \sqrt{p^2 - 4q}}{2}$, $A = \frac{W_1 - W_0t_2}{t_1 - t_2}$, $B = \frac{W_0t_1 - W_1}{t_1 - t_2}$. We have $P_n = W_n(0, 1; 2, -1)$ and $Q_n = W_n(2, 2; 2, -1)$, so the Pell and Pell-Lucas numbers are special cases of Horadam numbers.

The n th generalized commutative Horadam quaternion $gc\mathcal{H}_n$ is defined as

$$gc\mathcal{H}_n = W_n + W_{n+1}e_1 + W_{n+2}e_2 + W_{n+3}e_3.$$

In [8], it was presented the following result.

Theorem 1.1 (Binet type formula for generalized commutative Horadam quaternions [8]). *Let $n \geq 0$ be an integer. Then*

$$gc\mathcal{H}_n = At_1^n (1 + t_1e_1 + t_1^2e_2 + t_1^3e_3) + Bt_2^n (1 + t_2e_1 + t_2^2e_2 + t_2^3e_3).$$

2. MAIN RESULTS

Let $n \geq 0$ be an integer. The n th generalized commutative Pell quaternion $gc\mathcal{P}_n$ and the n th generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ are defined as

$$\begin{aligned} gc\mathcal{P}_n &= P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3, \\ gc\mathcal{Q}_n &= Q_n + Q_{n+1}e_1 + Q_{n+2}e_2 + Q_{n+3}e_3, \end{aligned}$$

respectively, where P_n is the n th Pell number, Q_n is the n th Pell-Lucas number and e_1, e_2, e_3 are units which satisfy (1.1) and (1.2).

Using the above definitions we can give initial generalized commutative Pell and Pell-Lucas quaternions, i.e.,

$$\begin{aligned}gc\mathcal{P}_0 &= e_1 + 2e_2 + 5e_3, \\gc\mathcal{P}_1 &= 1 + 2e_1 + 5e_2 + 12e_3, \\gc\mathcal{P}_2 &= 2 + 5e_1 + 12e_2 + 29e_3, \\gc\mathcal{Q}_0 &= 2 + 2e_1 + 6e_2 + 14e_3, \\gc\mathcal{Q}_1 &= 2 + 6e_1 + 14e_2 + 34e_3, \\gc\mathcal{Q}_2 &= 6 + 14e_1 + 34e_2 + 82e_3.\end{aligned}$$

Proposition 2.1. *Let $n \geq 0$ be an integer. Then the generalized commutative Pell quaternions satisfy the recurrence relation*

$$(2.1) \quad gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}, \quad \text{for } n \geq 2,$$

with initial conditions

$$gc\mathcal{P}_0 = e_1 + 2e_2 + 5e_3, \quad gc\mathcal{P}_1 = 1 + 2e_1 + 5e_2 + 12e_3.$$

Proposition 2.2. *Let $n \geq 0$ be an integer. The generalized commutative Pell-Lucas quaternions satisfy*

$$gc\mathcal{Q}_n = 2gc\mathcal{Q}_{n-1} + gc\mathcal{Q}_{n-2}, \quad \text{for } n \geq 2,$$

with $gc\mathcal{Q}_0 = 2 + 2e_1 + 6e_2 + 14e_3$, $gc\mathcal{Q}_1 = 2 + 6e_1 + 14e_2 + 34e_3$.

In this paper, we will focus on properties of generalized commutative Pell-Lucas quaternions and we will show some dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. As a special case of Theorem 1.1 we get the following remark.

Remark 2.1. Let $n \geq 0$ be an integer. Then

$$(2.2) \quad gc\mathcal{P}_n = \frac{(1 + \sqrt{2})^n}{2\sqrt{2}} \left(1 + (1 + \sqrt{2})e_1 + (3 + 2\sqrt{2})e_2 + (7 + 5\sqrt{2})e_3 \right) \\ - \frac{(1 - \sqrt{2})^n}{2\sqrt{2}} \left(1 + (1 - \sqrt{2})e_1 + (3 - 2\sqrt{2})e_2 + (7 - 5\sqrt{2})e_3 \right)$$

and

$$(2.3) \quad gc\mathcal{Q}_n = (1 + \sqrt{2})^n \left(1 + (1 + \sqrt{2})e_1 + (3 + 2\sqrt{2})e_2 + (7 + 5\sqrt{2})e_3 \right) \\ + (1 - \sqrt{2})^n \left(1 + (1 - \sqrt{2})e_1 + (3 - 2\sqrt{2})e_2 + (7 - 5\sqrt{2})e_3 \right).$$

For simplicity of notation let

$$(2.4) \quad \begin{aligned} t_1 &= 1 - \sqrt{2}, \quad t_2 = 1 + \sqrt{2}, \quad A = -\frac{1}{2\sqrt{2}}, \quad B = \frac{1}{2\sqrt{2}}, \\ \hat{t}_1 &= 1 + (1 - \sqrt{2})e_1 + (3 - 2\sqrt{2})e_2 + (7 - 5\sqrt{2})e_3, \\ \hat{t}_2 &= 1 + (1 + \sqrt{2})e_1 + (3 + 2\sqrt{2})e_2 + (7 + 5\sqrt{2})e_3. \end{aligned}$$

Then we can write (2.2) and (2.3) as

$$(2.5) \quad gc\mathcal{P}_n = At_1^n \hat{t}_1 + Bt_2^n \hat{t}_2$$

and

$$(2.6) \quad gc\mathcal{Q}_n = t_1^n \hat{t}_1 + t_2^n \hat{t}_2,$$

respectively, where $t_1, t_2, A, B, \hat{t}_1, \hat{t}_2$ are given by (2.4).

Theorem 2.1 (General bilinear index-reduction formula for generalized commutative Pell-Lucas quaternions). *Let $a \geq 0, b \geq 0, c \geq 0, d \geq 0$ be integers such that $a + b = c + d$. Then*

$$gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d = (t_1^a t_2^b + t_2^a t_1^b - t_1^c t_2^d - t_2^c t_1^d) \hat{t}_1 \hat{t}_2,$$

where $t_1, t_2, \hat{t}_1, \hat{t}_2$ are given by (2.4).

Proof. Using (2.6) we have

$$\begin{aligned} &gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d \\ &= (t_1^a \hat{t}_1 + t_2^a \hat{t}_2) (t_1^b \hat{t}_1 + t_2^b \hat{t}_2) - (t_1^c \hat{t}_1 + t_2^c \hat{t}_2) (t_1^d \hat{t}_1 + t_2^d \hat{t}_2) \\ &= t_1^a \hat{t}_1 t_2^b \hat{t}_2 + t_2^a \hat{t}_2 t_1^b \hat{t}_1 - t_1^c \hat{t}_1 t_2^d \hat{t}_2 - t_2^c \hat{t}_2 t_1^d \hat{t}_1 \\ &= (t_1^a t_2^b + t_2^a t_1^b - t_1^c t_2^d - t_2^c t_1^d) \hat{t}_1 \hat{t}_2, \end{aligned}$$

which ends the proof. □

Moreover, $t_1 t_2 = -1$ and

$$(2.7) \quad \hat{t}_1 \hat{t}_2 = \hat{t}_2 \hat{t}_1 = 1 - \alpha + \beta - \alpha\beta + (2 + 2\beta)e_1 + (6 - 6\alpha)e_2 + 12e_3.$$

For special values of a, b, c, d we obtain Catalan, Cassini, Halton, Vajda and d'Ocagne type identities.

Corollary 2.1 (Catalan type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0, k \geq 0$ be integers such that $n \geq k$. Then*

$$gc\mathcal{Q}_{n+k} \cdot gc\mathcal{Q}_{n-k} - (gc\mathcal{Q}_n)^2 = (-1)^n \left(\left(\frac{t_1}{t_2} \right)^k + \left(\frac{t_2}{t_1} \right)^k - 2 \right) \hat{t}_1 \hat{t}_2,$$

where t_1, t_2 and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Corollary 2.2 (Cassini type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 1$ be an integer. Then*

$$gcQ_{n+1} \cdot gcQ_{n-1} - (gcQ_n)^2 = 8(-1)^{n+1} \hat{t}_1 \hat{t}_2,$$

where $\hat{t}_1 \hat{t}_2$ is given by (2.7).

Corollary 2.3 (The first Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$gcQ_{m+r} \cdot gcQ_n - gcQ_r \cdot gcQ_{m+n} = (-1)^r (t_2^{n-r} - t_1^{n-r}) (t_1^m - t_2^m) \hat{t}_1 \hat{t}_2,$$

where t_1 , t_2 and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Corollary 2.4 (The second Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $k \geq 0$, $s \geq 0$ be integers such that $n \geq k$, $n \geq s$. Then*

$$gcQ_{n+k} \cdot gcQ_{n-k} - gcQ_{n+s} \cdot gcQ_{n-s} = (-1)^n \left(\left(\frac{t_1}{t_2} \right)^k + \left(\frac{t_2}{t_1} \right)^k - \left(\frac{t_1}{t_2} \right)^s - \left(\frac{t_2}{t_1} \right)^s \right) \hat{t}_1 \hat{t}_2,$$

where t_1 , t_2 and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Corollary 2.5 (Vajda type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$, $n \geq m$. Then*

$$\begin{aligned} & gcQ_{m+k} \cdot gcQ_{n-k} - gcQ_m \cdot gcQ_n \\ &= (-1)^m \left(t_2^{n-m} \left(\left(\frac{t_1}{t_2} \right)^k - 1 \right) + t_1^{n-m} \left(\left(\frac{t_2}{t_1} \right)^k - 1 \right) \right) \hat{t}_1 \hat{t}_2, \end{aligned}$$

where t_1 , t_2 and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Corollary 2.6 (d'Ocagne type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$gcQ_n \cdot gcQ_{m+1} - gcQ_{n+1} \cdot gcQ_m = 2\sqrt{2}(-1)^m (t_1^{n-m} - t_2^{n-m}) \hat{t}_1 \hat{t}_2,$$

where t_1 , t_2 and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Theorem 2.2 (General bilinear index-reduction formula for generalized commutative Pell and Pell-Lucas quaternions). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$gcP_a \cdot gcQ_b - gcP_c \cdot gcQ_d = (At_1^a t_2^b + Bt_2^a t_1^b - At_1^c t_2^d - Bt_2^c t_1^d) \hat{t}_1 \hat{t}_2,$$

where t_1 , t_2 , A , B and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

Proof. Using (2.5) and (2.6) we have

$$\begin{aligned} & gc\mathcal{P}_a \cdot gc\mathcal{Q}_b - gc\mathcal{P}_c \cdot gc\mathcal{Q}_d \\ &= (At_1^a \hat{t}_1 + Bt_2^a \hat{t}_2) \cdot (t_1^b \hat{t}_1 + t_2^b \hat{t}_2) - (At_1^c \hat{t}_1 + Bt_2^c \hat{t}_2) \cdot (t_1^d \hat{t}_1 + t_2^d \hat{t}_2) \\ &= At_1^a \hat{t}_1 t_2^b \hat{t}_2 + Bt_2^a \hat{t}_2 t_1^b \hat{t}_1 - At_1^c \hat{t}_1 t_2^d \hat{t}_2 - Bt_2^c \hat{t}_2 t_1^d \hat{t}_1 \\ &= (At_1^a t_2^b + Bt_2^a t_1^b - At_1^c t_2^d - Bt_2^c t_1^d) \hat{t}_1 \hat{t}_2, \end{aligned}$$

which ends the proof. \square

For special values of a, b, c, d we can obtain another dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions, for example a dependency similar to $P_k Q_{n+j} - P_j Q_{n+k}$ of Pell and Pell-Lucas numbers from [7].

Corollary 2.7. *Let $n \geq 0, j \geq 0, k \geq 0$ be integers. Then*

$$gc\mathcal{P}_k \cdot gc\mathcal{Q}_{n+j} - gc\mathcal{P}_j \cdot gc\mathcal{Q}_{n+k} = (At_2^n - Bt_1^n) (t_1^k t_2^j - t_1^j t_2^k) \hat{t}_1 \hat{t}_2,$$

where t_1, t_2, A, B and $\hat{t}_1 \hat{t}_2$ are given by (2.4) and (2.7), respectively.

We recall some well-known properties of Pell and Pell-Lucas numbers which can be found in [5, 6]

$$(2.8) \quad P_{n+1} + P_{n-1} = Q_n,$$

$$(2.9) \quad P_{n+1} - P_{n-1} = 2P_n,$$

$$(2.10) \quad Q_{n+1} + Q_{n-1} = 8P_n,$$

$$(2.11) \quad Q_{n+1} - Q_{n-1} = 2Q_n,$$

$$(2.12) \quad P_n + P_{n-1} = \frac{Q_n}{2},$$

$$(2.13) \quad Q_n + Q_{n-1} = 4P_n,$$

$$(2.14) \quad \sum_{l=0}^n P_l = \frac{Q_{n+1} - 2}{4},$$

$$(2.15) \quad \sum_{l=0}^n Q_l = 2P_{n+1}.$$

Using (2.8)–(2.13) it immediately follows

Theorem 2.3. *Let $n \geq 0$. Then*

- (i) $gc\mathcal{P}_{n+1} + gc\mathcal{P}_{n-1} = gc\mathcal{Q}_n$;
- (ii) $gc\mathcal{P}_{n+1} - gc\mathcal{P}_{n-1} = 2gc\mathcal{P}_n$;
- (iii) $gc\mathcal{Q}_{n+1} + gc\mathcal{Q}_{n-1} = 8gc\mathcal{P}_n$;
- (iv) $gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_{n-1} = 2gc\mathcal{Q}_n$;
- (v) $gc\mathcal{P}_n + gc\mathcal{P}_{n-1} = \frac{gc\mathcal{Q}_n}{2}$;
- (vi) $gc\mathcal{Q}_n + gc\mathcal{Q}_{n-1} = 4gc\mathcal{P}_n$.

Now we give formulae for the sum of generalized commutative Pell and Pell-Lucas quaternions.

Theorem 2.4. *Let $n \geq 0$. Then*

$$\sum_{l=0}^n gc\mathcal{P}_l = \frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4}.$$

Proof. Using (2.14), we have

$$\begin{aligned} \sum_{l=0}^n gc\mathcal{P}_l &= gc\mathcal{P}_0 + gc\mathcal{P}_1 + \cdots + gc\mathcal{P}_n \\ &= (P_0 + P_1e_1 + P_2e_2 + P_3e_3) + (P_1 + P_2e_1 + P_3e_2 + P_4e_3) \\ &\quad + \cdots + (P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3) \\ &= (P_0 + P_1 + \cdots + P_n) \\ &\quad + (P_1 + P_2 + \cdots + P_{n+1} + P_0 - P_0)e_1 \\ &\quad + (P_2 + P_3 + \cdots + P_{n+2} + P_0 + P_1 - P_0 - P_1)e_2 \\ &\quad + (P_3 + P_4 + \cdots + P_{n+3} + P_0 + P_1 + P_2 - P_0 - P_1 - P_2)e_3 \\ &= \frac{Q_{n+1} - 2}{4} + \left(\frac{Q_{n+2} - 2}{4} - 0 \right) e_1 \\ &\quad + \left(\frac{Q_{n+3} - 2}{4} - 1 \right) e_2 + \left(\frac{Q_{n+4} - 2}{4} - 3 \right) e_3 \\ &= \frac{Q_{n+1} + Q_{n+2}e_1 + Q_{n+3}e_2 + Q_{n+4}e_3}{4} - \frac{2 + 2e_1 + 6e_2 + 14e_3}{4} \\ &= \frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4}, \end{aligned}$$

which ends the proof. \square

In the same way, using (2.15), we can prove the following result.

Theorem 2.5. *Let $n \geq 0$. Then*

$$\sum_{l=0}^n gc\mathcal{Q}_l = 2(gc\mathcal{P}_{n+1} - gc\mathcal{P}_0).$$

Now, we give a matrix representation of the generalized commutative Pell and Pell-Lucas quaternions. In [4], Ercolano gave a matrix representation of the Pell sequence defining the matrix generator $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Then $M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, for integer $n \geq 1$.

Let $R(n) = \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix}$ be a matrix of order 2 with entries being generalized commutative Pell quaternions.

Theorem 2.6. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2}.$$

Proof. If $n = 2$ then by simple calculations the result immediately follows. Assume that the equality holds for all integers $2, 3, \dots, n$. We shall prove that the equation is true for integer $n + 1$. Using our assumption and formula (2.1) we obtain

$$\begin{aligned} \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2} \cdot M &= \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2gc\mathcal{P}_n + gc\mathcal{P}_{n-1} & gc\mathcal{P}_n \\ 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2} & gc\mathcal{P}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} gc\mathcal{P}_{n+1} & gc\mathcal{P}_n \\ gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \end{bmatrix}, \end{aligned}$$

which ends the proof. \square

In the same way we can obtain the matrix generator for the generalized commutative Pell-Lucas quaternions.

Let $S(n) = \begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix}$ be a matrix with entries being the generalized commutative Pell-Lucas quaternions.

Theorem 2.7. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{Q}_2 & gc\mathcal{Q}_1 \\ gc\mathcal{Q}_1 & gc\mathcal{Q}_0 \end{bmatrix} \cdot M^{n-2}.$$

At the end, we give the generating functions for $gc\mathcal{P}_n$ and $gc\mathcal{Q}_n$.

Theorem 2.8. *The generating function for the generalized commutative Pell quaternion $gc\mathcal{P}_n$ is*

$$g(t) = \frac{gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t}{1 - 2t - t^2} = \frac{e_1 + 2e_2 + 5e_3 + (1 + e_2 + 2e_3)t}{1 - 2t - t^2}.$$

Proof. Assuming that the generating function of the generalized commutative Pell quaternion sequence $\{gc\mathcal{P}_n\}$ has the form $g(t) = \sum_{n=0}^{\infty} gc\mathcal{P}_n t^n$, we obtain

$$\begin{aligned} (1 - 2t - t^2)g(t) &= (1 - 2t - t^2)(gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \dots) \\ &= gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \dots \\ &\quad - 2gc\mathcal{P}_0 t - gc\mathcal{P}_1 t^2 - 2gc\mathcal{P}_2 t^3 - \dots \\ &\quad - gc\mathcal{P}_0 t^2 - gc\mathcal{P}_1 t^3 - gc\mathcal{P}_2 t^4 - \dots \\ &= gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t, \end{aligned}$$

since $gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}$ and the coefficients of t^n , for $n \geq 2$, are equal to zero. \square

Theorem 2.9. *The generating function for the generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ is*

$$\begin{aligned} g(t) &= \frac{gc\mathcal{Q}_0 + (gc\mathcal{Q}_1 - 2gc\mathcal{Q}_0)t}{1 - 2t - t^2} \\ &= \frac{(2 + 2e_1 + 6e_2 + 14e_3) + (-2 + 2e_1 + 2e_2 + 6e_3)t}{1 - 2t - t^2}. \end{aligned}$$

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FACULTY OF MATHEMATICS AND APPLIED PHYSICS
 RZESZOW UNIVERSITY OF TECHNOLOGY,
 AL. POWSTAŃCÓW WARSZAWY 12, 35-959 RZESZÓW, POLAND
Email address: dorotab@prz.edu.pl
Email address: aszynal@prz.edu.pl