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ON SOME COMBINATORIAL PROPERTIES OF GENERALIZED COMMUTATIVE PELL AND PELL-LUCAS QUATERNIONS

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Abstract. Generalized commutative quaternions generalize elliptic, parabolic and hyperbolic quaternions, bicomplex numbers, complex hyperbolic numbers and hyperbolic complex numbers. In this paper, we study generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. We present some properties of these numbers and relations between them.

1. Introduction

Let $n \geq 0$ be an integer. The nth Pell number P_n is defined in the following way $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0,\ P_1 = 1$. Solving the above recurrence equation we obtain the direct formula of the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}},$$

named also as the Binet formula for Pell numbers.

The nth Pell-Lucas number Q_n is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$, with $Q_0 = Q_1 = 2$. The Binet formula for Pell-Lucas numbers has the form

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

The first six terms of the Pell sequence and Pell-Lucas sequence are $0, 1, 2, 5, 12, 29$ and $2, 2, 6, 14, 34, 82$, respectively.

The Pell and Pell-Lucas numbers belong to the class of numbers of the Fibonacci type and have applications also in the theory of hypercomplex numbers (see [1–3,9–12]).

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In this paper, we use the Pell and Pell-Lucas numbers in the theory of generalized commutative quaternions.

Let $\mathbb{H}_{\alpha\beta}^{c}$ be the set of generalized commutative quaternions $\mathbf{x}$ of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \tag{1.1}$$

$$e_1 e_2 = e_2 e_1 = e_3, \quad e_2 e_3 = e_3 e_2 = \beta e_1 \quad \text{and} \quad e_3 e_1 = e_1 e_3 = \alpha e_2, \tag{1.2}$$

and $x_0, x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$.

The generalized commutative quaternions generalize elliptic quaternions ($\alpha < 0$, $\beta = 1$), parabolic quaternions ($\alpha = 0$, $\beta = 1$), hyperbolic quaternions ($\alpha > 0$, $\beta = 1$), bicomplex numbers ($\alpha = -1$, $\beta = -1$), complex hyperbolic numbers ($\alpha = -1$, $\beta = 1$) and hyperbolic complex numbers ($\alpha = 1$, $\beta = -1$).

Generalized commutative quaternions were introduced in [8]. The authors defined generalized commutative quaternions of the Fibonacci type – generalized commutative Horadam quaternions.

For integers p, q, n and $n \geq 0$ Horadam defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$, for $n \geq 2$, with fixed real numbers W_0, W_1. Let t_1, t_2 be the two distinct real roots of the equation $t^2 - pt + q = 0$. Then, the Binet type formula for the Horadam numbers has the form $W_n = At_1^n + Bt_2^n$, where $t_1 = \frac{p-\sqrt{p^2-4q}}{2}$, $t_2 = \frac{p+\sqrt{p^2-4q}}{2}$, $A = \frac{W_1 - W_0 t_2}{t_1 - t_2}$, $B = \frac{W_0 t_1 - W_1}{t_1 - t_2}$. We have $P_n = W_n(0, 1; 2, -1)$ and $Q_n = W_n(2, 2; 2, -1)$, so the Pell and Pell-Lucas numbers are special cases of Horadam numbers.

The nth generalized commutative Horadam quaternion $gc\mathcal{H}_n$ is defined as

$$gc\mathcal{H}_n = W_n + W_{n+1} e_1 + W_{n+2} e_2 + W_{n+3} e_3.$$

In [8], it was presented the following result.

Theorem 1.1 (Binet type formula for generalized commutative Horadam quaternions [8])**.** *Let $n \geq 0$ be an integer. Then*

$$gc\mathcal{H}_n = At_1^n \left(1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3\right) + Bt_2^n \left(1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3\right).$$

2. Main Results

Let $n \geq 0$ be an integer. The nth generalized commutative Pell quaternion $gc\mathcal{P}_n$ and the nth generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ are defined as

$$\begin{aligned} gc\mathcal{P}_n =& P_n + P_{n+1} e_1 + P_{n+2} e_2 + P_{n+3} e_3, \\ gc\mathcal{Q}_n =& Q_n + Q_{n+1} e_1 + Q_{n+2} e_2 + Q_{n+3} e_3, \end{aligned}$$

respectively, where P_n is the nth Pell number, Q_n is the nth Pell-Lucas number and e_1, e_2, e_3 are units which satisfy (1.1) and (1.2).

Using the above definitions we can give initial generalized commutative Pell and Pell-Lucas quaternions, i.e.,

$$\begin{aligned} gc\mathcal{P}_0 &= e_1 + 2e_2 + 5e_3, \\ gc\mathcal{P}_1 &= 1 + 2e_1 + 5e_2 + 12e_3, \\ gc\mathcal{P}_2 &= 2 + 5e_1 + 12e_2 + 29e_3, \\ gc\mathcal{Q}_0 &= 2 + 2e_1 + 6e_2 + 14e_3, \\ gc\mathcal{Q}_1 &= 2 + 6e_1 + 14e_2 + 34e_3, \\ gc\mathcal{Q}_2 &= 6 + 14e_1 + 34e_2 + 82e_3. \end{aligned}$$

Proposition 2.1. *Let $n \geq 0$ be an integer. Then the generalized commutative Pell quaternions satisfy the recurrence relation*

$$gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}, \quad \text{for } n \geq 2, \tag{2.1}$$

with initial conditions

$$gc\mathcal{P}_0 = e_1 + 2e_2 + 5e_3, \quad gc\mathcal{P}_1 = 1 + 2e_1 + 5e_2 + 12e_3.$$

Proposition 2.2. *Let $n \geq 0$ be an integer. The generalized commutative Pell-Lucas quaternions satisfy*

$$gc\mathcal{Q}_n = 2gc\mathcal{Q}_{n-1} + gc\mathcal{Q}_{n-2}, \quad \text{for } n \geq 2,$$

with $gc\mathcal{Q}_0 = 2 + 2e_1 + 6e_2 + 14e_3$, $gc\mathcal{Q}_1 = 2 + 6e_1 + 14e_2 + 34e_3$.

In this paper, we will focus on properties of generalized commutative Pell-Lucas quaternions and we will show some dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions. As a special case of Theorem 1.1 we get the following remark.

Remark 2.1. Let $n \geq 0$ be an integer. Then

$$\begin{aligned} gc\mathcal{P}_n =& \frac{\left(1+\sqrt{2}\right)^n}{2\sqrt{2}} \left(1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3\right) \\ &- \frac{\left(1-\sqrt{2}\right)^n}{2\sqrt{2}} \left(1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3\right) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} gc\mathcal{Q}_n =& \left(1+\sqrt{2}\right)^n \left(1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3\right) \\ &+ \left(1-\sqrt{2}\right)^n \left(1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3\right). \end{aligned} \tag{2.3}$$

For simplicity of notation let

$$\begin{aligned}
&t_1 = 1-\sqrt{2}, \quad t_2 = 1+\sqrt{2}, \quad A = -\frac{1}{2\sqrt{2}}, \quad B = \frac{1}{2\sqrt{2}},\\
&\hat{t_1} = 1 + (1-\sqrt{2})e_1 + (3-2\sqrt{2})e_2 + (7-5\sqrt{2})e_3,\\
&\hat{t_2} = 1 + (1+\sqrt{2})e_1 + (3+2\sqrt{2})e_2 + (7+5\sqrt{2})e_3.
\end{aligned} \tag{2.4}$$

Then we can write (2.2) and (2.3) as

$$gc\mathcal{P}_n = At_1^n\hat{t_1} + Bt_2^n\hat{t_2} \tag{2.5}$$

and

$$gc\mathcal{Q}_n = t_1^n\hat{t_1} + t_2^n\hat{t_2}, \tag{2.6}$$

respectively, where t_1, t_2, A, B, $\hat{t_1}$, $\hat{t_2}$ are given by (2.4).

Theorem 2.1 (General bilinear index-reduction formula for generalized commutative Pell-Lucas quaternions)**.** *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d = \left(t_1^a t_2^b + t_2^a t_1^b - t_1^c t_2^d - t_2^c t_1^d\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, $\hat{t_1}$, $\hat{t_2}$ are given by (2.4).

Proof. Using (2.6) we have

$$\begin{aligned}
&gc\mathcal{Q}_a \cdot gc\mathcal{Q}_b - gc\mathcal{Q}_c \cdot gc\mathcal{Q}_d\\
&= \left(t_1^a\hat{t_1} + t_2^a\hat{t_2}\right)\left(t_1^b\hat{t_1} + t_2^b\hat{t_2}\right) - \left(t_1^c\hat{t_1} + t_2^c\hat{t_2}\right)\left(t_1^d\hat{t_1} + t_2^d\hat{t_2}\right)\\
&= t_1^a\hat{t_1}t_2^b\hat{t_2} + t_2^a\hat{t_2}t_1^b\hat{t_1} - t_1^c\hat{t_1}t_2^d\hat{t_2} - t_2^c\hat{t_2}t_1^d\hat{t_1}\\
&= \left(t_1^a t_2^b + t_2^a t_1^b - t_1^c t_2^d - t_2^c t_1^d\right)\hat{t_1}\hat{t_2},
\end{aligned}$$

which ends the proof. □

Moreover, $t_1 t_2 = -1$ and

$$\hat{t_1}\hat{t_2} = \hat{t_2}\hat{t_1} = 1 - \alpha + \beta - \alpha\beta + (2+2\beta)e_1 + (6-6\alpha)e_2 + 12e_3. \tag{2.7}$$

For special values of a, b, c, d we obtain Catalan, Cassini, Halton, Vajda and d'Ocagne type identities.

Corollary 2.1 (Catalan type identity for generalized commutative Pell-Lucas quaternions)**.** *Let $n \geq 0$, $k \geq 0$ be integers such that $n \geq k$. Then*

$$gc\mathcal{Q}_{n+k} \cdot gc\mathcal{Q}_{n-k} - (gc\mathcal{Q}_n)^2 = (-1)^n\left(\left(\frac{t_1}{t_2}\right)^k + \left(\frac{t_2}{t_1}\right)^k - 2\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7), *respectively.*

Corollary 2.2 (Cassini type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 1$ be an integer. Then*

$$gc\mathcal{Q}_{n+1} \cdot gc\mathcal{Q}_{n-1} - (gc\mathcal{Q}_n)^2 = 8(-1)^{n+1}\hat{t_1}\hat{t_2},$$

where $\hat{t_1}\hat{t_2}$ is given by (2.7)*.*

Corollary 2.3 (The first Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $r \geq 0$ be integers such that $n \geq r$. Then*

$$gc\mathcal{Q}_{m+r} \cdot gc\mathcal{Q}_n - gc\mathcal{Q}_r \cdot gc\mathcal{Q}_{m+n} = (-1)^r \left(t_2^{n-r} - t_1^{n-r}\right)\left(t_1^m - t_2^m\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.4 (The second Halton type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $k \geq 0$, $s \geq 0$ be integers such that $n \geq k$, $n \geq s$. Then*

$$gc\mathcal{Q}_{n+k} \cdot gc\mathcal{Q}_{n-k} - gc\mathcal{Q}_{n+s} \cdot gc\mathcal{Q}_{n-s} = (-1)^n \left(\left(\frac{t_1}{t_2}\right)^k + \left(\frac{t_2}{t_1}\right)^k - \left(\frac{t_1}{t_2}\right)^s - \left(\frac{t_2}{t_1}\right)^s\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.5 (Vajda type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$, $k \geq 0$ be integers such that $n \geq k$, $n \geq m$. Then*

$$\begin{aligned} &gc\mathcal{Q}_{m+k} \cdot gc\mathcal{Q}_{n-k} - gc\mathcal{Q}_m \cdot gc\mathcal{Q}_n \\ =&(-1)^m \left(t_2^{n-m}\left(\left(\frac{t_1}{t_2}\right)^k - 1\right) + t_1^{n-m}\left(\left(\frac{t_2}{t_1}\right)^k - 1\right)\right)\hat{t_1}\hat{t_2}, \end{aligned}$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Corollary 2.6 (d'Ocagne type identity for generalized commutative Pell-Lucas quaternions). *Let $n \geq 0$, $m \geq 0$ be integers such that $n \geq m$. Then*

$$gc\mathcal{Q}_n \cdot gc\mathcal{Q}_{m+1} - gc\mathcal{Q}_{n+1} \cdot gc\mathcal{Q}_m = 2\sqrt{2}(-1)^m \left(t_1^{n-m} - t_2^{n-m}\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2 and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Theorem 2.2 (General bilinear index-reduction formula for generalized commutative Pell and Pell-Lucas quaternions). *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a + b = c + d$. Then*

$$gc\mathcal{P}_a \cdot gc\mathcal{Q}_b - gc\mathcal{P}_c \cdot gc\mathcal{Q}_d = \left(At_1^a t_2^b + Bt_2^a t_1^b - At_1^c t_2^d - Bt_2^c t_1^d\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, A, B and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

Proof. Using (2.5) and (2.6) we have

$$\begin{aligned} & gc\mathcal{P}_a \cdot gc\mathcal{Q}_b - gc\mathcal{P}_c \cdot gc\mathcal{Q}_d \\ =& \left(At_1^a\hat{t_1} + Bt_2^a\hat{t_2}\right) \cdot \left(t_1^b\hat{t_1} + t_2^b\hat{t_2}\right) - \left(At_1^c\hat{t_1} + Bt_2^c\hat{t_2}\right) \cdot \left(t_1^d\hat{t_1} + t_2^d\hat{t_2}\right) \\ =& At_1^a\hat{t_1}t_2^b\hat{t_2} + Bt_2^a\hat{t_2}t_1^b\hat{t_1} - At_1^c\hat{t_1}t_2^d\hat{t_2} - Bt_2^c\hat{t_2}t_1^d\hat{t_1} \\ =& \left(At_1^a t_2^b + Bt_2^a t_1^b - At_1^c t_2^d - Bt_2^c t_1^d\right)\hat{t_1}\hat{t_2}, \end{aligned}$$

which ends the proof. □

For special values of a, b, c, d we can obtain another dependencies between generalized commutative Pell quaternions and generalized commutative Pell-Lucas quaternions, for example a dependency similar to $P_k Q_{n+j} - P_j Q_{n+k}$ of Pell and Pell-Lucas numbers from [7].

Corollary 2.7. *Let $n \geq 0$, $j \geq 0$, $k \geq 0$ be integers. Then*

$$gc\mathcal{P}_k \cdot gc\mathcal{Q}_{n+j} - gc\mathcal{P}_j \cdot gc\mathcal{Q}_{n+k} = (At_2^n - Bt_1^n)\left(t_1^k t_2^j - t_1^j t_2^k\right)\hat{t_1}\hat{t_2},$$

where t_1, t_2, A, B and $\hat{t_1}\hat{t_2}$ are given by (2.4) *and* (2.7)*, respectively.*

We recall some well-known properties of Pell and Pell-Lucas numbers which can be found in [5,6]

$$P_{n+1} + P_{n-1} = Q_n, \tag{2.8}$$

$$P_{n+1} - P_{n-1} = 2P_n, \tag{2.9}$$

$$Q_{n+1} + Q_{n-1} = 8P_n, \tag{2.10}$$

$$Q_{n+1} - Q_{n-1} = 2Q_n, \tag{2.11}$$

$$P_n + P_{n-1} = \frac{Q_n}{2}, \tag{2.12}$$

$$Q_n + Q_{n-1} = 4P_n, \tag{2.13}$$

$$\sum_{l=0}^{n} P_l = \frac{Q_{n+1} - 2}{4}, \tag{2.14}$$

$$\sum_{l=0}^{n} Q_l = 2P_{n+1}. \tag{2.15}$$

Using (2.8)–(2.13) it immediately follows

Theorem 2.3. *Let $n \geq 0$. Then*

(i) $gc\mathcal{P}_{n+1} + gc\mathcal{P}_{n-1} = gc\mathcal{Q}_n$;
(ii) $gc\mathcal{P}_{n+1} - gc\mathcal{P}_{n-1} = 2gc\mathcal{P}_n$;
(iii) $gc\mathcal{Q}_{n+1} + gc\mathcal{Q}_{n-1} = 8gc\mathcal{P}_n$;
(iv) $gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_{n-1} = 2gc\mathcal{Q}_n$;
(v) $gc\mathcal{P}_n + gc\mathcal{P}_{n-1} = \frac{gc\mathcal{Q}_n}{2}$;
(vi) $gc\mathcal{Q}_n + gc\mathcal{Q}_{n-1} = 4gc\mathcal{P}_n$.

Now we give formulae for the sum of generalized commutative Pell and Pell-Lucas quaternions.

Theorem 2.4. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} gc\mathcal{P}_l = \frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4}.$$

Proof. Using (2.14), we have

$$\begin{aligned}\sum_{l=0}^{n} gc\mathcal{P}_l &= gc\mathcal{P}_0 + gc\mathcal{P}_1 + \cdots + gc\mathcal{P}_n \\ &= (P_0 + P_1e_1 + P_2e_2 + P_3e_3) + (P_1 + P_2e_1 + P_3e_2 + P_4e_3) \\ &\quad + \cdots + (P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3) \\ &= (P_0 + P_1 + \cdots + P_n) \\ &\quad + (P_1 + P_2 + \cdots + P_{n+1} + P_0 - P_0)e_1 \\ &\quad + (P_2 + P_3 + \cdots + P_{n+2} + P_0 + P_1 - P_0 - P_1)e_2 \\ &\quad + (P_3 + P_4 + \cdots + P_{n+3} + P_0 + P_1 + P_2 - P_0 - P_1 - P_2)e_3 \\ &= \frac{Q_{n+1} - 2}{4} + \left(\frac{Q_{n+2} - 2}{4} - 0\right)e_1 \\ &\quad + \left(\frac{Q_{n+3} - 2}{4} - 1\right)e_2 + \left(\frac{Q_{n+4} - 2}{4} - 3\right)e_3 \\ &= \frac{Q_{n+1} + Q_{n+2}e_1 + Q_{n+3}e_2 + Q_{n+4}e_3}{4} - \frac{2 + 2e_1 + 6e_2 + 14e_3}{4} \\ &= \frac{gc\mathcal{Q}_{n+1} - gc\mathcal{Q}_0}{4},\end{aligned}$$

which ends the proof. □

In the same way, using (2.15), we can prove the following result.

Theorem 2.5. *Let $n \geq 0$. Then*

$$\sum_{l=0}^{n} gc\mathcal{Q}_l = 2\left(gc\mathcal{P}_{n+1} - gc\mathcal{P}_0\right).$$

Now, we give a matrix representation of the generalized commutative Pell and Pell-Lucas quaternions. In [4], Ercolano gave a matrix representation of the Pell sequence defining the matrix generator $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Then $M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, for integer $n \geq 1$.

Let $R(n) = \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix}$ be a matrix of order 2 with entries being generalized commutative Pell quaternions.

Theorem 2.6. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2}.$$

Proof. If $n = 2$ then by simple calculations the result immediately follows. Assume that the equality holds for all integers $2, 3, \ldots, n$. We shall prove that the equation is true for integer $n + 1$. Using our assumption and formula (2.1) we obtain

$$\begin{aligned} \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot M^{n-2} \cdot M &= \begin{bmatrix} gc\mathcal{P}_2 & gc\mathcal{P}_1 \\ gc\mathcal{P}_1 & gc\mathcal{P}_0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \\ gc\mathcal{P}_{n-1} & gc\mathcal{P}_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2gc\mathcal{P}_n + gc\mathcal{P}_{n-1} & gc\mathcal{P}_n \\ 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2} & gc\mathcal{P}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} gc\mathcal{P}_{n+1} & gc\mathcal{P}_n \\ gc\mathcal{P}_n & gc\mathcal{P}_{n-1} \end{bmatrix}, \end{aligned}$$

which ends the proof. □

In the same way we can obtain the matrix generator for the generalized commutative Pell-Lucas quaternions.

Let $S(n) = \begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix}$ be a matrix with entries being the generalized commutative Pell-Lucas quaternions.

Theorem 2.7. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} gc\mathcal{Q}_n & gc\mathcal{Q}_{n-1} \\ gc\mathcal{Q}_{n-1} & gc\mathcal{Q}_{n-2} \end{bmatrix} = \begin{bmatrix} gc\mathcal{Q}_2 & gc\mathcal{Q}_1 \\ gc\mathcal{Q}_1 & gc\mathcal{Q}_0 \end{bmatrix} \cdot M^{n-2}.$$

At the end, we give the generating functions for $gc\mathcal{P}_n$ and $gc\mathcal{Q}_n$.

Theorem 2.8. *The generating function for the generalized commutative Pell quaternion $gc\mathcal{P}_n$ is*

$$g(t) = \frac{gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t}{1 - 2t - t^2} = \frac{e_1 + 2e_2 + 5e_3 + (1 + e_2 + 2e_3)t}{1 - 2t - t^2}.$$

Proof. Assuming that the generating function of the generalized commutative Pell quaternion sequence $\{gc\mathcal{P}_n\}$ has the form $g(t) = \sum_{n=0}^{\infty} gc\mathcal{P}_n t^n$, we obtain

$$\begin{aligned} (1 - 2t - t^2)g(t) =& (1 - 2t - t^2)(gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \cdots) \\ =& gc\mathcal{P}_0 + gc\mathcal{P}_1 t + gc\mathcal{P}_2 t^2 + \cdots \\ & - 2gc\mathcal{P}_0 t - gc\mathcal{P}_1 t^2 - 2gc\mathcal{P}_2 t^3 - \cdots \\ & - gc\mathcal{P}_0 t^2 - gc\mathcal{P}_1 t^3 - gc\mathcal{P}_2 t^4 - \cdots \\ =& gc\mathcal{P}_0 + (gc\mathcal{P}_1 - 2gc\mathcal{P}_0)t, \end{aligned}$$

since $gc\mathcal{P}_n = 2gc\mathcal{P}_{n-1} + gc\mathcal{P}_{n-2}$ and the coefficients of t^n, for $n \geq 2$, are equal to zero. □

Theorem 2.9. *The generating function for the generalized commutative Pell-Lucas quaternion $gc\mathcal{Q}_n$ is*

$$g(t) = \frac{gc\mathcal{Q}_0 + (gc\mathcal{Q}_1 - 2gc\mathcal{Q}_0)t}{1 - 2t - t^2}$$
$$= \frac{(2 + 2e_1 + 6e_2 + 14e_3) + (-2 + 2e_1 + 2e_2 + 6e_3)t}{1 - 2t - t^2}.$$

References

[1] F. T. Aydın, *On bicomplex Pell and Pell-Lucas numbers*, Communications in Advanced Mathematical Sciences **1**(2) (2018), 142–155. `https://doi.org/10.33434/cams.439752`

[2] G. Bilgici and P. Catarino, *Unrestricted Pell and Pell-Lucas quaternions*, International Journal of Mathematics and System Science **1**(3) (2018), 1–4. `http://dx.doi.org/10.24294/ijmss.v1i3.816`

[3] C. B. Çimen and A. İpek, *On Pell quaternions and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebr. **26** (2016), 39–51. `https://doi.org/10.1007/s00006-015-0571-8`

[4] J. Ercolano, *Matrix generators of Pell sequences*, Fibonacci Quart. **17(1)** (1979), 71–77.

[5] A. F. Horadam, *Minmax sequences for Pell numbers*, Applications of Fibonacci Numbers **6** (1996), 231–249.

[6] A. F. Horadam, *Pell identities*, Fibonacci Quart. **9.3** (1971), 245–263.

[7] N. Patel and P. Shrivastava, *Pell and Pell-Lucas identities*, Global Journal of Mathematical Sciences: Theory and Practical **5**(4) (2013), 229–236.

[8] A. Szynal-Liana and I. Włoch, *Generalized commutative quaternions of the Fibonacci type*, Bol. Soc. Mat. Mex. **28**1 (2022). `https://doi.org/10.1007/s40590-021-00386-4`

[9] A. Szynal-Liana and I. Włoch, *Hypercomplex Numbers of the Fibonacci Type*, Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów, 2019.

[10] A. Szynal-Liana and I. Włoch, *On Pell and Pell-Lucas hybrid numbers*, Comment. Math. **58**(1–2) (2018), 11–17. `http://dx.doi.org/10.14708/cm.v58i1-2.6364`

[11] A. Szynal-Liana and I. Włoch, *The Pell quaternions and the Pell octonions*, Adv. Appl. Clifford Algebr. **26** (2016), 435–440. `https://doi.org/10.1007/s00006-015-0570-9`

[12] Ü.Tokeşer, Z. Ünal and G. Bilgici, *Split Pell and Pell-Lucas quaternions*, Adv. Appl. Clifford Algebr. **27** (2017), 1881–1893. `https://doi.org/10.1007/s00006-016-0747-x`

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