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THE PERFECT CODES OF NON-COPRIME AND COPRIME GRAPHS

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Abstract. In this paper, we focus on the perfect and total perfect codes of the non-coprime and coprime graphs associated to the dihedral groups and finite Abelian groups. We used the advantage of independent sets and tried to present the independent polynomial for them.

1. Introduction

The birth of coding theory was established by Claude Shannon in 1948 (see [10]). In his paper, he showed for a noisy communication channel, there is a number, called the capacity of the channel. If proper encoding and decoding techniques are used, the reliable communication can be achieved at any rate below the channel capacity. Coding theory is concerned with successfully transmitting data through a noisy channel and correcting errors in corrupted messages [5]. Let Γ be a graph with vertex and edge set $V(\Gamma)$ and $E(\Gamma)$, repectively. Suppose $r \geq 1$ is an integer. The ball with center $v \in V(\Gamma)$ and radius r is the set of vertices of Γ with distance at most r to v in Γ. A code in Γ is simply a subset of $V(\Gamma)$. A code $C \subseteq V(\Gamma)$ is called a perfect r-code in Γ if the balls with centers in C and radius r form a partition of $V(\Gamma)$, that is, every vertex of Γ is at distance no more than r to exactly one vertex of C [4]. If $r = 1$, then we call perfect r-code, perfect code, for abbreviation. Consequently, in order to find a perfect code, we should search among all independent sets and check if every vertex of $V(\Gamma) \setminus C$ is adjacent to exactly one vertex of C. A code C is said to be a total perfect code in Γ if every vertex of Γ has exactly one neighbor in C [3]. The existence of perfect codes is a classical problem which was started in a vector

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space. One can replace the vector space by a graph, whose vertices are vectors and whose edges join vectors which differ in precisely one coordinate. It is clear that we may pose the perfect code question for any graph [1]. There are several papers in this area we refer the readers to [6, 7]. The non-coprime graph associated to the group G was introduced in [9]. Suppose G is a group and e its identity element. The non-coprime graph of G is a graph with vertex set $G \setminus \{e\}$ and if $\gcd(|x|, |y|) \neq 1$, then two distinct vertices x, y are adjacent. Denote this graph by Π_G. The authors verified some numerical invariants like diameter, girth, dominating number, independence and chromatic numbers of non-coprime graph. Moreover, they characterized its planarity. X. Ma et al. defined the coprime graph of a finite group [8]. The coprime graph Γ_G which is associated to the finite group G is a graph with G as the vertex set and join two distinct vertices x and y if $\gcd(|x|, |y|) = 1$. They gave some properties of coprime graph on diameter, planarity, partition, clique number, etc. Moreover, some groups whose coprime graphs are complete, planar, a star, or regular were characterized. There are other papers about the properties of coprime graph, see for instance [2]. In this research, we investigate the existence of the perfect and total perfect code for non-coprime and coprime graph of certain groups beside to present the independent polynomials for them.

2. The Perfect and Total Perfect Codes of Non-coprime Graph of Certain Groups

In this section we verify the perfect and total perfect codes of non-coprime graph of dihedral groups and finite Abelian groups. Let $D_{2n} = \langle a, b : a^n = b^2 = 1,\ a^b = a^{-1} \rangle$ be the dihedral group of order $2n$ and $n \geq 4$.

(i) Suppose $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers and α_i's are positive integers. It is clear that the independence number for this graph is $k + 1$. For instance, a set $\mathfrak{I}$ contain an element of order two and k elements of order $p_i^{\beta_i}$, form an independent set of the largest size, $1 \leq i \leq k$, $1 \leq \alpha_i \leq \beta_i$. The number of singleton independent sets is $2\prod_{i=1}^{k} p_i^{\alpha_i} - 1$. The number of two-element independent sets is,

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\sum_{i=1}^{k}\sum_{1\leq\beta_i\leq\alpha_i}\varphi(p_i^{\beta_i})\right)+\sum_{i=1}^{k}\left(\sum_{1\leq\beta_i\leq\alpha_i}\varphi(p_i^{\beta_i})\left(\sum_{j=i+1}^{k}\sum_{1\leq\beta_j\leq\alpha_j}\varphi(p_j^{\beta_j})\right)\right), \tag{2.1}$$

where φ is the Euler function. Consequently, the number of independent sets with ℓ elements is equal to the sum of all possible ℓ-multiplications of elements in the set

$$\mathfrak{A} = \left\{\sum_{\beta_i=1}^{\alpha_i}\varphi(p_i^{\beta_i}) : 1 \leq i \leq k\right\} \cup \left\{\prod_{i=1}^{k} p_i^{\alpha_i}\right\},$$

which are arranged similar to the equation (2.1), where $1 \leq \ell \leq k+1$. Note that ℓ-multiplications of elements in $\mathfrak{A}$ means choosing ℓ elements of the set $\mathfrak{A}$ randomly and compute their multiplications. Inside the set $\mathfrak{A}$ are the numbers of elements of order 2 and $p_i^{\beta_i}$ in the group D_{2n}, $1 \leq i \leq k$, $1 \leq \beta_i \leq \alpha_i$. Obviously, the number of independent sets with $k+1$ elements is

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\prod_{i=1}^{k} \sum_{1\leq\beta_i\leq\alpha_i} \varphi(p_i^{\beta_i})\right).$$

(ii) Let $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i is an odd prime. Then similar to the first case the number of independent sets of different sizes can be computed. The number of singleton independent sets is $2^{s+1}\prod_{i=1}^{k} p_i^{\alpha_i} - 1$. The number of independent sets with ℓ elements is equal to the sum of all possible suitable ℓ-multiplications of elements in the set

$$\mathfrak{A} = \left\{\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) : 1 \leq i \leq k\right\} \cup \left\{2^s \prod_{i=1}^{k} p_i^{\alpha_i} + 1 + \sum_{\beta=2}^{s} \varphi(2^\beta)\right\},$$

where $1 \leq \ell \leq k+1$ (similar to (2.1)). Note that inside the set $\mathfrak{A}$ are the numbers of elements of order 2^β and $p_i^{\beta_i}$ in the group D_{2n}, $1 \leq i \leq k$, $1 \leq \beta_i \leq \alpha_i$ and $1 \leq \beta \leq s$.

(iii) Assume $n = 2^s$. Then the independence number is one and the number of independent sets of size one is $2^{s+1} - 1$.

It is not hard to write the independent polynomial for the non-coprime graph of D_{2n} by use of the above results.

Theorem 2.1. *For the perfect codes of the non-coprime graph of D_{2n}, we have the following cases.*

(i) If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then the perfect codes of non-coprime graph of D_{2n} are sets of two-elements, they contained one element of order 2 and an element of order $\prod_{i=1}^{k} p_i^{\beta_i}$, where p_i are odd prime numbers, α_i, β_i are positive integers and $1 \leq \beta_i \leq \alpha_i$. Moreover, the number of these codes is $\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\sum_{t=1}^{n} \psi(a^t)\right)$, where

$$\psi(a^t) = \begin{cases} 1, & \prod_{i=1}^{k} p_i \mid |a^t|, \quad 1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}, \\ 0, & \textit{otherwise.} \end{cases}$$

More practical formula for the number of perfect codes is

$$\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right)\left(\begin{matrix} \ulcorner M \urcorner \\ \llcorner\ k\ \lrcorner \end{matrix}\right),$$

where M' is the set of all prime numbers which divides $\prod_{i=1}^{k} p_i^{\alpha_i}$, M is the set of all prime power numbers which are chosen from M' and their powers are more or equal than one and less or equal than α_i and the notation

$$\begin{matrix} \ulcorner M \urcorner \\ \llcorner\ j\ \lrcorner, \end{matrix}$$

means the sum of Euler functions of multiply of j prime power numbers which are chosen randomly from M and these multiplications are multiplying of distinct prime numbers.

(ii) *If $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, then the perfect codes of non-coprime graph of D_{2n} are sets of singletons. The singletons contain elements of order $2^\beta \prod_{i=1}^{k} p_i^{\beta_i}$, where $1 \leq \beta \leq s$ and $1 \leq \beta_i \leq \alpha_i$. Further, the number of singleton perfect codes is*

$$\begin{matrix} \ulcorner & M & \urcorner \\ \llcorner & k+1 & \lrcorner \end{matrix},$$

where the notation is the same as part (i) *with the difference that M contains the possible powers of* 2.

(iii) *If $n = 2^s$, then singleton subsets of whole vertices are perfect codes and the number of them is $2^{s+1} - 1$.*

Proof. (i) By definition, for a perfect code of a graph, we must search among its independent subsets of vertices. Secondly, every vertex out of the perfect code is adjacent to exactly one vertex in the code. By these tools, and the way that two vertices are adjacent in the non-coprime graph it is clear that a perfect codes for the $\Pi_{D_{2n}}$ are sets of two-elements, they contained one element of order 2 and an element of order $\prod_{i=1}^{k} p_i^{\beta_i}$, $1 \leq \beta_i \leq \alpha_i$. Furthermore, the number of elements of order 2 in dihedral group of order $2n$ is $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. It is enough to count the number of elements of order $\prod_{i=1}^{k} p_i^{\beta_i}$, $1 \leq \beta_i \leq \alpha_i$. It is obvious that all the elements $a^t b$ are of order 2, $1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}$. Thus, for a fixed β_i, $|a^t| = \prod_{i=1}^{k} p_i^{\beta_i}$ whenever $\gcd(t, \prod_{i=1}^{k} p_i^{\alpha_i}) = \prod_{i=1}^{k} p_i^{\alpha'_i}$, where $0 \leq \alpha'_i \leq \alpha_i - 1$, $1 \leq t \leq \prod_{i=1}^{k} p_i^{\alpha_i}$ and $\beta_i = \alpha_i - \alpha'_i$. Count the number of such t's. Consequently, there are $\varphi(\prod_{i=1}^{k} p_i^{\beta_i})$, a^t of order $\prod_{i=1}^{k} p_i^{\beta_i}$, where φ is the Euler function. Now, when the power β_i changed through the possible cases we require the sum of such Euler functions which described in the statement of proposition.

The proof of (ii) is very similar to (i) so we omit it and the third part is straightforward. □

Proposition 2.1. *The non-coprime graph of D_{2n} does not have any total perfect code.*

Proof. By definition of total perfect code and the fact that all the n vertices of order 2 in dihedral group of order $2n$ are adjacent in $\Pi_{D_{2n}}$, we deduce that $n \leq 2$ and a contradiction. □

Theorem 2.2. *Let $\mathbb{Z}_n$ be the cyclic group of order n.*

(i) *The non-coprime graph $\Pi_{\mathbb{Z}_p^s}$ has $p^s - 1$ singleton subsets of the vertices which are perfect codes, where p is a prime number and s a positive integer.*

(ii) *The non-coprime graph $\Pi_{\mathbb{Z}_{pq}}$ has $\varphi(pq)$ singleton perfect codes, where p, q are prime numbers and φ is the Euler function.*

(iii) *The non-coprime graph of cyclic group $\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}$ has $\varphi(\prod_{i=1}^{k} p_i^{\alpha_i}) + \begin{smallmatrix} \ulcorner M \urcorner \\ \llcorner\, k\, \lrcorner \end{smallmatrix}$, singleton perfect codes, where M is a set which is defined the same as in Theorem* 2.1 (i),

p_i's are prime numbers and α_i's are positive integers, $1 \leq i \leq k$. The independence number of $\Pi_{\mathbb{Z}_{\prod_{i=1}^k p_i^{\alpha_i}}}$ is k and independent sets with more than 2 elements are not perfect codes. Furthermore, the number of independent sets with s elements is equal to the sum of all possible s-multiplications of elements in the set

$$\mathfrak{A} = \left\{ \sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) : 1 \leq i \leq k \right\},$$

which are arranged similar to the equation (2.2), where $1 \leq s \leq k$. Note that s-multiplications of elements in $\mathfrak{A}$ means choosing s elements of the set $\mathfrak{A}$ randomly and compute their multiplications. In particular, the number of two-element and k element independent sets are

$$\sum_{i=1}^{k} \left[\left(\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) \right) \left(\sum_{j=i+1}^{k} \sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j}) \right) \right] \tag{2.2}$$

and

$$\prod_{i=1}^{k} \left(\sum_{\beta_i=1}^{\alpha_i} \varphi(p_i^{\beta_i}) \right),$$

respectively.

(iv) *Every singleton subset of vertices of the non-coprime graph of the group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ is a perfect code, where p is a prime number.*

Proof. (i) The independence number for this graph is one and all the possible subsets of the vertices with one element are perfect codes.

(ii) This graph has $pq - 1$, $\varphi(p)\varphi(q)$ independent sets with one and two elements, respectively. Clearly, singleton subsets which contains an element of order p (or q) are not perfect codes because there exists an element out of it which does not join to the vertex inside that singleton. If we consider singletons which contains generators of order pq, they clearly are perfect codes. The independent sets with two elements are not perfect codes, since the generators join to both vertices inside them.

(iii) It is clear that this graph has $\prod_{i=1}^k p_i^{\alpha_i} - 1$ independent sets with one element. It is obvious among these independent sets, the only singletons which contain a generator element or an element of order $\prod_{i=1}^k p_i^{\omega_i}$ are perfect codes, where $1 \leq \omega_i \leq \alpha_i$. Moreover, for an independent set with more than one element, there is a generator out of it which joins to both of it and so it is not a perfect code. □

By considering the third part of Theorem 2.2, one can present independent polynomial for that graph.

Proposition 2.2. *The non-coprime of the cyclic group $\mathbb{Z}_n$ admits a total perfect code if and only if $n = 3$.*

Proof. Suppose $n = \prod_{i=1}^k p_i^{\alpha_i}$, where there exists an index $i \in \{1, 2, \ldots, k\}$ such that $\alpha_i \geq 2$ and p_i is a prime number. Clearly, there are vertices v_1, v_2 of order $p_i^{\alpha_i}$. Let

C be a total perfect code which contain v_1, then by definition v_1 has exactly one neighborhood inside C. This neighborhood can be v_2 or an element of order p_i. If each of them is inside C, then the other one is outside of C has two neighborhood in C and it is a contradiction. Thus, all α_i are equal to 1. Clearly all p_i are greater than 2, because otherwise an element of order 2 does not have any neighborhood. Again by definition for each vertex of order p_i, there is a unique neighborhood in C. Therefore for every vertex of order p_i out of C, there are two neighborhood. Hence the number of elements of order of p_i must be 2, i.e., $\varphi(p_i) = 2$, $p_i = 3$ and the assertion is clear. $\square$

3. The Perfect and Total Perfect Codes of Coprime Graph of Certain Groups

In this section, we present perfect and total perfect codes of coprime graph of dihedral groups and finite Abelian groups. For coprime graphs, if we consider the identity element of the group as a vertex, then it is meaningless. Thus, it is nature to consider the induced subgraph by non-trivial elements. Let us consider the coprime graph of the group G, with vertex set $G \setminus \{e\}$ and denote it by Γ_G^*. We refer the readers to see [6] for its interesting results. Consider the dihedral group D_{2n}.

(i) Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers. Consider A_0 which is the set of all elements of order two and A_j's are the sets of elements of order $p_j^{\beta_j}$, and elements of order $\prod_{i=1}^{q} p_i^{\omega_i}$, where p_j surly exists in the multiplication, $1 \leq \beta \leq \alpha_j$, $1 \leq j \leq k$, $1 \leq q \leq k$ and $0 \leq \omega_i \leq \alpha_i$. Then A_0 and A_j's are the samples of independent sets for $\Gamma_{D_{2n}}$. For $1 < t \leq k$, construct A_t somehow it does not have any common elements with all the sets A_s, with $s < t$. The number of elements in A_0 and A_j are $\prod_{i=1}^{k} p_i^{\alpha_i}$ and $\sum_{1 \leq \beta \leq \alpha_j} \varphi(p_j^{\beta}) + \sum_{u=2}^{k} \genfrac{}{}{0pt}{}{\ulcorner M_j \urcorner}{\llcorner u_j \lrcorner}$, respectively. The notations were defined in Theorem 2.1, note that M_j is the set of all prime powers in M' such that powers of $p_z \neq p_j$ is more or equal than zero and less or equal than α_z, and the powers of the prime p_j must be at least one and less or equal than α_j, where $1 \leq z \leq k$. In computing of Euler function of multiplication, the prime number p_j always must be selected and it is possible that the power of the other prime numbers be zero, in other words it is possible that some prime numbers distinct from p_j do not appear in the multiplication. Moreover, this is very significant to obtain the independent polynomial, in order to compute $\genfrac{}{}{0pt}{}{\ulcorner M_t \urcorner}{\llcorner u_t \lrcorner},$ the sum of Euler function on u_t-multiplication of elements in M_t is somehow that it does not have any common summands with all the summands in $\genfrac{}{}{0pt}{}{\ulcorner M_s \urcorner}{\llcorner u_s \lrcorner},$ for $1 < t \leq k$, for all $s < t$. Suppose $\alpha(\Gamma_{D_{2n}})$ is independence number of the graph. The independent polynomial is,

$$f(x) = \left(2 \prod_{i=1}^{k} p_i^{\alpha_i}\right) x + \sum_{\omega=2}^{\alpha(\Gamma_{D_{2n}})} \left[\binom{\prod_{i=1}^{k} p_i^{\alpha_i}}{\omega} + \sum_{j=1}^{k} \binom{\sum_{1 \leq \beta \leq \alpha_j} \varphi(p_j^{\beta}) + \sum_{u=2}^{k} \genfrac{}{}{0pt}{}{\ulcorner M_j \urcorner}{\llcorner u_j \lrcorner}}{\omega} \right] x^{\omega},$$

and if $\omega > \sum_{1\leq\beta_j\leq\alpha_j}\varphi(p_j^{\beta}) + \sum_{u=2}^{k} {}^{\ulcorner M_j \urcorner}_{\llcorner u_j \lrcorner}$, or $\omega > \prod_{i=1}^{k} p_i^{\alpha_i}$, then put zero instead of

$$\binom{\sum_{1\leq\beta\leq\alpha_j}\varphi(p_j^{\beta}) + \sum_{u=2}^{k} {}^{\ulcorner M_j \urcorner}_{\llcorner u_j \lrcorner}}{\omega} \quad \text{or} \quad \binom{\prod_{i=1}^{k} p_i^{\alpha_i}}{\omega}$$

in $f(x)$.

(ii) For $n = 2^s \prod_{i=1}^{k} p_i^{\alpha_i}$, suppose

$$\begin{aligned}
A_{0j} =& \{\text{All the elements of order power of } 2\} \\
&\cup \Bigg\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\leq\eta_i\leq\alpha_i,\ 1\leq\beta\leq s, \\
&q\leq k-1,\ \text{certainly } p_j \text{ exists in the multiplication}\Bigg\} \\
&\cup \Bigg\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\leq\eta_i\leq\alpha_i,\ 1\leq\beta\leq s, \\
&q\leq k-1,\ \text{certainly } p_j \text{ does not exist in the multiplication}\Bigg\} \\
&\cup \Bigg\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{k} p_i^{\eta_i},\ \text{where } 1\leq\eta_i\leq\alpha_i,\ 1\leq\beta\leq s\Bigg\},
\end{aligned}$$

with $a_1,\ a_{2j},\ a_{3j}$ and a_4 as first, second, third and forth set sizes which constructed A_{0j}, respectively. Moreover, assume

$$\begin{aligned}
A_j =& \{\text{All the elements of order power of } p_j,\ 1\leq j\leq k\} \\
&\cup \Bigg\{\text{All the elements of order } \prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\leq\eta_i\leq\alpha_i,\ q\leq k, \\
&\text{certainly } p_j \text{ exists in the multiplication}\Bigg\} \\
&\cup \Bigg\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{k} p_i^{\eta_i},\ \text{where } 1\leq\eta_i\leq\alpha_i,\ 1\leq\beta\leq s\Bigg\} \\
&\cup \Bigg\{\text{All the elements of order } 2^{\beta}\prod_{i=1}^{q} p_i^{\eta_i},\ \text{where } 0\leq\eta_i\leq\alpha_i,\ 1\leq\beta\leq s, \\
&q\leq k-1 \text{ certainly } p_j \text{ exists in the multiplication}\Bigg\},
\end{aligned}$$

with $b_{1j},\ b_{2j},\ b_{3j} = a_4$ and $b_{j4} = a_{j2}$ as first, second, third, forth set sizes which constructed A_j, respectively. Obviously A_{0j} and A_j's are independent sets $1\leq j\leq k$. For $1 < t \leq k$, construct A_{0t} and A_t somehow they do not have any common elements with all the sets A_{0s} and A_s, with $s < t$. The value of a_i and b_{ij} can be obtained similar to the previous parts. Since replacing of them by their values makes the appearance of the computations more complicated so we work with a_i and b_{ij}. By these hypothesis, the independent polynomial is

$$\begin{aligned} f(x) =& 2^{s+1}\prod_{i=1}^{k} p_i^{\alpha_i} x + \sum_{\omega=2}^{\alpha(\Gamma_{D_{2n}})} \left[\binom{a_1}{\omega} + \binom{a_4}{\omega} + \sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}}{\omega} + \binom{a_1+a_4}{\omega} \right. \\ & + \sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}+a_1}{\omega} + \sum_{j=1}^{k}\sum_{i=2}^{3}\binom{a_{ij}+a_4}{\omega} \\ & + \sum_{j=1}^{k}\left[\binom{a_1+a_{2j}+a_{3j}}{\omega} + \binom{a_4+a_{2j}+a_{3j}}{\omega} + \binom{a_1+a_4+a_{ij}}{\omega}\right] \\ & + \sum_{j=1}^{k}\binom{a_1+a_4+a_{2j}+a_{3j}}{\omega} \\ & + \sum_{j=1}^{k}\left[\sum_{i=1}^{2}\binom{b_{ij}}{\omega} + \sum_{i=1}^{3}\sum_{\ell=i+1}^{4}\binom{b_{ij}+b_{lj}}{\omega} - \binom{b_{3j}+b_{4j}}{\omega} \right. \\ & \left.\left. + \sum_{i=1}^{2}\sum_{\ell=i+1}^{3}\sum_{k=\ell+1}^{4}\binom{b_{ij}+b_{lj}+b_{kj}}{\omega} + \binom{b_{1j}+b_{2j}+b_{3j}+b_{4j}}{\omega}\right]\right] x^{\omega}, \end{aligned}$$

such that in the selection $\binom{\sum y_i}{\omega}$ at least one element choose from each sets with sizes y_i. Moreover, if $\omega > \binom{\sum y_i}{\omega}$, then put zero for $\binom{\sum y_i}{\omega}$.

(iii) Assume $n = 2^s$. In this case, the largest independent set for $\Gamma_{D_{2n}}$ is $D_{2n} \setminus \{1\}$. The independent polynomial is,

$$f(x) = 2^{s+1}x + \sum_{\ell=2}^{2^{s+1}-1}\binom{2^{s+1}-1}{\ell}x^{\ell}.$$

It is clear that the singleton contains the identity element is a perfect code for $\Gamma_{D_{2n}}$ and independent sets with more than one elements are not perfect codes.

Proposition 3.1. *$\Gamma^*_{D_{2n}}$ does not have any perfect code.*

Proof. Initially, let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are odd prime numbers. Consider the singleton subset of vertices $X = \{x\}$. If the vertex x is of order $p_j^{\beta_j}$ (or 2), where $1 \le j \le k$ and $1 \le \beta_j \le \alpha_j$. Thus, elements of order $p_j^{\beta'_j}$ (or 2) outside of X, are not adjacent to x, $1 \le \beta'_j \le \alpha_j$. Note that the existence of such elements outside of X is clear, as $n \ge 4$. Suppose $|x| = \prod_{i=1}^{q} p_i^{\beta_i}$, $q \le k$. If p_j appear in the multiplication of prime numbers in the order of x, then there is an element of order $p_j^{\omega_j}$ does not join to x. Therefore, the independent sets with one element are not perfect codes. According to the first part of argument before the proposition, if an independent set have more than one element, then it contains just elements of order 2 or just elements of order power of p_j and multiplication of prime powers that include p_j. In both cases, there is a vertex that join to more than one vertex inside them not exactly one. Thus, independent set with more than one element is not a perfect code. If $n = 2^s \prod_{i=1}^k p_i^{\alpha_i}$ or 2^s, the assertion follows similarly. $\square$

Proposition 3.2. *For the total perfect code of $\Gamma^*_{D_{2n}}$, we have the following cases.*

(i) *If $n = \prod_{i=1}^k p_i^{\alpha_i}$, then two-element subset of vertices which contain an element of order 2 and an element of order $\prod_{i=1}^k p_i^{\beta_i}$ is a total perfect code, where p_i's are odd prime numbers and $1 \leq \beta_i \leq \alpha_i$. The number of such total perfect codes are $\prod_{i=1}^k p_i^{\alpha_i} \binom{\ulcorner M \urcorner}{\llcorner k \lrcorner}$, where notations were defined in Theorem 2.1 (i).*

(ii) *If $n = 2^s \prod_{i=1}^k p_i^{\alpha_i}$, then $\Gamma^*_{D_{2n}}$ does not have any total perfect code.*

(iii) *For $n = 2^s$, $\Gamma^*_{D_{2n}}$ does not have any total perfect code.*

Proof. The proof of first and third part is clear, let us prove the second part.

(ii) Let $T \subseteq V(\Gamma_G)$ be a total perfect code for the graph. If $x \in T$ and the order of x is power of 2, then it has a unique neighborhood $y \in T$, by definition of total perfect code. Thus the order of y is the multiplication of prime numbers which divide $\prod_{i=1}^k p_i^{\alpha_i}$. All the vertices outside T have a unique neighborhood inside T except a vertex v of order $2^\beta \prod_{i=1}^k p_i^{\beta_i}$, $1 \leq \beta \leq s$ and $1 \leq \beta_i \leq \alpha_i$. We can not consider v inside T, since there is no neighborhood for it inside T. Let us construct T, by use of other vertices. If $x \in T$ of order $\prod_{i=1}^q p_i^{\omega_i}$ and $y \in T$ is its neighborhood, then $|y| = \prod_{i=1}^{q'} p_i'^{\omega_i'}$ or $2^\beta \prod_{i=1}^{q'} p_i'^{\omega_i'}$, where $p_i^{\omega_i}$ and $p_i'^{\omega_i'}$ divides $\prod_{i=1}^k p_i^{\alpha_i}$ and $\gcd(\prod_{i=1}^q p_i^{\omega_i}, \prod_{i=1}^{q'} p_i'^{\omega_i'}) = 1$, $q, q' \leq k$. An element of order $\prod_{i=1}^k p_i^{\alpha_i}$ outside T does not join to any vertex inside T. $\square$

Consider the coprime graph of the cyclic group $\mathbb{Z}_n$.

(i) If $n = p^s$, then $\Gamma_{\mathbb{Z}_n}$ has p^s singleton independent sets. Clearly, its independence number is $p^s - 1$ and it has $\binom{p^s-1}{\ell}$ independent sets with ℓ elements.

(ii) Let $n = \prod_{i=1}^k p_i^{\alpha_i}$. The coprime graph of $\mathbb{Z}_{\prod_{i=1}^k p_i^{\alpha_i}}$ has $\prod_{i=1}^k p_i^{\alpha_i}$ singleton independent sets. Moreover, every subsets of the following sets are samples of independent sets of $\Gamma_{\mathbb{Z}_{\prod_{i=1}^k p_i^{\alpha_i}}}$. Let

$$A_{0j} = \left\{\text{All the elements of order } p_j^{\beta_j}\right\},$$

$$A_{1j} = \left\{\text{All the elements of order } \prod_{i=1}^k p_i^{\alpha_i}\right\} \cup \left\{\text{All the elements of order } p_j^{\beta_j}\right\},$$

$$A_2 = \left\{\text{All the elements of order } \prod_{i=1}^k p_i^{\alpha_i}\right\} \cup \left\{\text{All the elements of order } \prod_{i=1}^q p_i^{\beta_i}\right\},$$

where $1 \leq j \leq k$, $1 \leq \beta_j \leq \alpha_j$, $2 \leq q \leq k-1$ and in A_2, $\prod_{i=1}^q p_i^{\beta_i}$, it is possible that some β_i be zero and the order of elements in the set {All the elements of order $\prod_{i=1}^q p_i^{\beta_i}$} include at least one prime number p_i in its multiplication.

Now, let $2 \leq \ell \leq \alpha\left(\Gamma_{\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}}\right)$,

$$\mathcal{A}_0 = \sum_{j=1}^{k} \binom{\sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j})}{\ell}, \quad \mathcal{A}_1 = \sum_{j=1}^{k} \binom{\varphi(\prod_{i=1}^{k} p_i^{\alpha_i})}{w_1} \binom{\sum_{\beta_j=1}^{\alpha_j} \varphi(p_j^{\beta_j})}{w_2},$$

where $w_1 \geq 1$, $2 \leq w_1 + w_2 \leq \ell$ and $\mathcal{A}_2 = \sum_{r=1}^{k} \binom{\varphi(\prod_{i=1}^{k} p_i^{\alpha_i})}{u_1} \binom{\sum_{j=2}^{k-1} \ulcorner M_{p_r} \urcorner \llcorner j \lrcorner}{u_2}$, where $u_2 \geq 1$, $2 \leq u_1 + u_2 \leq \ell$. Furthermore, the notation $\ulcorner M_{p_r} \urcorner \llcorner j \lrcorner$ is defined similar to Theorem 2.1(i), such that in choosing the prime power numbers from the set M, some power of p_r is selected. By this hypothesis, the number of independent sets with ℓ elements is more than $\mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$.

Suppose $\Gamma_{\mathbb{Z}_n}$ has a total perfect code T. Since the greatest common divisor of order of identity element (zero) with respect to all other element orders is one so $0 \in T$. By definition 0 has a unique neighborhood inside T, say x. As every vertex of total perfect codes cover exactly one vertex of the graph, there is just one other vertex outside of T. Hence, $n = 2, 3$. The coprime graph of $\mathbb{Z}_{\prod_{i=1}^{k} p_i^{\alpha_i}}$ does not have any perfect and total perfect code. Clearly $\Gamma_{\mathbb{Z}_2}$ has a singleton perfect code and $\Gamma_{\mathbb{Z}_{p^s}}$ does not have any perfect code, where p is a prime and s a positive integer ($p \geq 3$ and $s \geq 1$ or $p = 2$ and $s \geq 2$). Note that if we consider the induces subgraph of $\Gamma_{\mathbb{Z}_{p^s}}$ by omitting the identity, then the set of all vertices, largest independent set, is a perfect code. There are $\binom{p^{(\alpha_1+\cdots+\alpha_k)}}{\ell}$ independent sets of size ℓ for the coprime graph of the group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$, where $1 \leq \ell \leq \alpha(\Gamma_{\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}})$. Clearly, the coprime graph of $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_k}}$ does not have any perfect and total perfect code.

Theorem 3.1. *Suppose Γ_G^* has a total perfect code with two elements $T = \{g_1, g_2\}$ such that $|g_1| = \prod_{i=1}^{k} p_i^{\alpha_i}$, $|g_2| = \prod_{j=1}^{k'} q_j^{\beta_j}$, where p_i and q_j's are distinct prime numbers. Then G is non-cyclic and the set of prime divisor of order of G is $\Pi(|G|) = \{p_i, q_j : 1 \leq i \leq k,\ 1 \leq j \leq k'\}$. Moreover, G does not contain an element of order $\prod_{i=1}^{s} p_i^{\alpha'_i} \prod_{j=1}^{s'} q_j^{\beta'_j}$, where $0 \leq \alpha'_i \leq \alpha_i$, $0 \leq \beta'_i \leq \beta_i$ and note that α'_i (and also β'_i) are not all zero, simultaneously.*

Proof. Let $x \in G$, $|x| = r$, where r is a prime number distinct from p_i, q_j's. Then x join to both g_1 and g_2 which is a contradiction. Therefore, the only prime numbers that divide the order of G are the prime numbers that divide the order of g_i's, $i = 1, 2$. Now, if the group G contain an element of order $\prod_{i=1}^{k} p_i^{\alpha_i} \prod_{j=1}^{k'} q_j^{\beta_j}$, then this element is not in the neighborhood of g_i's and again this is against the definition of total perfect code. Hence G is a non-cyclic group. $\square$

The definition of total perfect code and coprime graph signify that the coprime graph does not have any singleton total perfect codes. Moreover, if Γ_G^* has a total perfect code with more than two elements, then similar result as Theorem 3.1 will be obtained.

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