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WELL-POSEDNESS AND EXPONENTIAL DECAY OF ENERGY FOR THE SOLUTION OF A WAVE EQUATION WITH NONLINEAR SOURCE AND LOCALIZED DAMPING TERMES

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Abstract. We consider the wave equation with a locally damping and a nonlinear source term in a bounded domain. $y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y$, where $p > 2$. The damping is nonlinear and is effective only in a neighborhood of a suitable subset of the boundary. We show, for certain initial data and suitable conditions on g, a and p that this solution is global we use the Faedo-Galerkin method. Also we established the exponential decay of the energy when the nonlinear damping grows linearly by introducing a suitable Lyapunov functional.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, having a boundary $\Gamma = \partial\Omega$ of class C^2. We denote by ν the unit normal pointing into the exterior of Ω. We fix $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$ and we set

$$\Gamma(x^0) = \{x \in \Gamma : m(x)\nu(x) > 0\} \tag{1.1}$$

and

$$m(x) = x - x^0. \tag{1.2}$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider δ sufficiently small such that

$$\mathcal{M}_0 = \Big\{x \in \Omega : d(x, \Gamma(x^0)) < \delta\Big\} \subset \omega, \tag{1.3}$$

$$\mathcal{M}_1 = \Big\{x \in \Omega : d(x, \Gamma(x^0)) < 2\delta\Big\} \subset \omega. \tag{1.4}$$

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If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have

$$d(x;A) = \inf_{y \in A}(|x-y|).$$

and $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \omega$.

Now consider with the following initial-boundary value problem of damped wave equation

$$\begin{cases} y_{tt} - \Delta y + a(x)g(y_t) = f(y), \quad x \in \Omega \times [0,+\infty[, \\ y = 0, \quad x \in \Gamma \times [0,\infty[, \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad x \in \Omega \times [0,+\infty[, \end{cases} \tag{1.5}$$

where $f(y) = |y|^{p-2}y$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function with $g(0) = 0$ and $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded function.

In the absence of nonlinear source term (i.e., if $f = 0$), Tebou [12] has used the multipliers techniques to prove the decay estimates of global solutions for the problem (1.5) for certain initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and g having a polynomial growth near the origin. Precisely, he showed that the rate of decay of the energy is exponential or polynomial depending on exponents of the damping terms. This method is based on new integral inequality that generalizes a result of Haraux [6] and Komornik [7]. Tebou [14] studied (1.5) for a localized nonlinear strong damping. He proved that for certain initial data the global existence by using the Fadeo-Galerkin approximations and the semigroup methods, he used and also showed that the energy of the system decays exponentially by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [9].

When $f = 0$ and the feedback term depends on the velocity in a linear way, as in the present paper, Zuazua [15] proved that the energy related to problem (1.5) decays exponentially if the damping region contains a neighbourhood of the boundary Γ or, at least, contains a neibourhood of the particular part given by (1.5).

When $g(y_t) = \operatorname{div}(a(x)\nabla y_t)$, where $a(x) = d1_\omega(x), d > 0$, Ammari et al. [2] consider the problem (1.5) without the source term $f(y)$. They obtained a logarithmic decay of energy. Their idea is to transform the resolvent problem to a transmission system to easily use the so-called Carleman estimate.

When $g(\Delta y_t) = |\Delta y_t|^{p-2}\Delta y_t$ and the source term is absent, Tebou [13] investigates the global existence of solution with initial-boundary value conditions. Meanwhile, he proved that the rate of decay of the energy is exponential or polynomial depending on exponents of the damping terms.

In the presence of the viscoelastic term Cavalcanti et al. [5] studied (1.5) in the presence of a linear localised frictional damping $(a(x)y_t)$. They obtained an exponential rate of decay by assuming that the kernel term is decaying exponentially. This work was later improved by Berrimi and Messaoudi [4] by introducing a different functional which allowed them to weaken the conditions on viscoelastic damping.

Motivated by previous works, it is interesting to investigate the global existence and decay of solutions to problem (1.5). Firstly, we show that, under suitable conditions on the functions g and a, the parameter p and certain initial data in the stable set, the existence of regular and weak solutions to problem (1.5).

After that, we establish the rate of decay of solutions by the perturbed energy method. Precisely, we show that the decay rate of energy function is exponential. In this way, we can extend the results of [14] where the authors considered (1.5) without source term and the results of [10] and [11] in the linear damping term.

This article is organized as follows. In the next section, we give some preliminaries. In Section 3, we prove the existence and uniqueness for regular and weak solutions. Then in Section 4, we are devoted to the proof of decay estimate.

2. Preliminaries

To state and prove our result, we need some assumptions.

(A1) $g : \mathbb{R} \to \mathbb{R}$ is non decreasing function of class C^1 functions such that $g(0) = 0$ and

$$(\exists \tau_0, \tau_1 > 0) \quad \tau_0 \leq g'(s) \leq \tau_1, \quad \text{for all } s \in \mathbb{R}.$$

(A2) The nonnegative function $a : \Omega \to [0, +\infty)$ is assumed bounded such that

$$\begin{aligned} &(\exists a_0 > 0) \quad a(x) \geq a_0 > 0, \quad \text{a.e. in } \omega, \\ &a(x) \in W^{1,\infty}(\Omega). \end{aligned} \tag{2.1}$$

(A3) Let p be a number with $2 \leq p < +\infty$, $n = 1, 2$, and $2 \leq p \leq \frac{2n-2}{n-2}$, $n \geq 3$.

Now, we define the following functionals

$$\begin{aligned} I(y(t)) &= \|\nabla y(t)\|^2 - \|y(t)\|_p^p, \\ J(y(t)) &= \frac{1}{2}\|\nabla y(t)\|^2 - \frac{1}{p}\|y(t)\|_p^p. \end{aligned}$$

We define the energy as

$$E(t) = \frac{1}{2}\|y_t(t)\|^2 + \frac{1}{2}\|\nabla y(t)\|^2 - \frac{1}{p}\|y(t)\|_p^p = \frac{1}{2}\|y_t(t)\|^2 + J(y(t)), \quad \text{for all } t \geq 0. \tag{2.2}$$

The energy E is a nonincreasing function of the time variable t, and its derivative satisfies

$$E'(t) = -\int_\Omega a(x) y_t g(y_t)\, dx \leq 0, \quad \text{for all } t \geq 0. \tag{2.3}$$

We can define the stable set as

$$\mathcal{W} = \Big\{ y \mid y \in H_0^1(\Omega), I(y) > 0 \Big\} \cup \{0\}.$$

For later applications, we list up some lemmas.

Lemma 2.1 ([1]). *Let q be a number with $2 \leq q < +\infty$, $n = 1, 2$, or $2 \leq q \leq 2n/(n-2)$, $n \geq 3$, then there exists a constant $C_s = C(\Omega, q)$ such that*

$$\|y\|_q \leq C_s \|\nabla y\|, \quad \textit{for } y \in H_0^1(\Omega).$$

Lemma 2.2 ([8]). *Let Ω a bounded domain of $\mathbb{R}_x \times \mathbb{R}_t$, φ_m and φ functions of $L^q(\Omega)$, $1 < q < +\infty$, such that*

$$\|\varphi_m\|_{L^q(\Omega)} \leq C, \quad \varphi_m \to \varphi, \quad a.e.\ in\ \Omega.$$

Then,

$$\varphi_m \to \varphi \quad in\ L^q\ weak.$$

Lemma 2.3. *Suppose that $n \geq 3$ and $p \leq \frac{2n}{n-2}$. Let $y(t)$ be a local solution on $[0, t_m]$ with the initial data $y_0 \in \mathcal{W}$ such that*

$$C_s^p \left(\frac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}} < 1.$$

Then, $y(t) \in \mathcal{W}$ for all $t \in [0, t_m]$.

Proof. We introduce

$$t^* = \inf\{t \in [0, T^*] \mid y(t) \notin \mathcal{W}\} \neq \emptyset.$$

For continuity in time of $y(t)$, $y(t) \in \mathcal{W}$ for all $0 \leq t \leq t^*$ and $y(t^*) \notin \mathcal{W}$, then we have $y(t^*) \neq 0$.

From the continuity of y and the definition of t^*

$$I(y(t^*)) = 0. \tag{2.4}$$

Hence, we get

$$J(y(t)) = \frac{p-2}{2p}\|\nabla y(t)\|^2 + \frac{1}{p} I(y(t)) \geq \frac{p-2}{2p}\|\nabla y(t)\|^2, \quad \text{on } [0, t^*]. \tag{2.5}$$

By the energy identity (2.2) and (2.5), we get

$$\|\nabla y(t)\|^2 \leq \frac{2p}{p-2} J(y(t)) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \quad \text{on } [0, t^*]. \tag{2.6}$$

Hence, from the Sobolev-Poincaré inequality, we get

$$\begin{aligned} \|y(t)\|_p^p &\leq C_s^p \|\nabla y(t)\|^p \leq C_s^p \|\nabla y(t)\|^{p-2} \|\nabla y(t)\|^2 \\ &\leq C_s^p \left(\frac{2p}{p-2} E(0)\right)^{\frac{p-2}{2}} \|\nabla y(t)\|^2, \quad \text{on } [0, t^*]. \end{aligned} \tag{2.7}$$

As $t \to t^*$ and $\alpha < 1$, we obtain

$$\|y(t^*)\|_p^p \leq \alpha \|\nabla y(t^*)\|^2 \quad \text{and} \quad \|\nabla y(t^*)\|^2 \neq 0.$$

Then

$$\|y(t^*)\|_p^p \leq \|\nabla y(t^*)\|^2.$$

As a result, we obtain $I(y(t^*)) > 0$, which contradicts to (2.4). Thus, we conclude that $u(t) \in \mathcal{W}$, on $[0, t^*]$. This ends the proof of Lemma 2.3. □

3. Well-Posedness

In this section we prove the existence of regular solutions to problem (1.5) and for this purpose we employ Galerkin method. Then, using a density argument we extend the same result to weak solutions.

Theorem 3.1. *Let $y_0 \in H^2(\Omega) \cap \mathcal{W}$, $y_1 \in H_0^1(\Omega)$. Assume that* **(A1)-(A3)** *hold. Then problem* (1.5) *admits a unique regular solution $y(x,t)$ in the class*

$$y \in L^\infty([0,\infty); H^2(\Omega) \cap \mathcal{W}), \quad y_t \in L^\infty([0,\infty); H_0^1(\Omega)), \quad y_{tt} \in L^\infty([0,\infty); L^2(\Omega)).$$

Theorem 3.2. *Let $y_0 \in \mathcal{W}$, $y_1 \in L^2(\Omega)$. Assume that* **(A1)-(A3)** *hold. Then problem* (1.5) *possesses a weak solution in the class*

$$y \in C^0([0,\infty); \mathcal{W}) \cap C^1([0,\infty); L^2(\Omega)).$$

Proof. We employ the Faedo-Galerkin approximation method to construct a global solution, let $\{w^i \mid i \in \mathbb{N}\}$ be the Hilbert basis of $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^2(\Omega)$ given by

$$\begin{cases} -\Delta w^i = \lambda^i w^i, & \text{in } \Omega, \\ w^i = 0, & \text{on } \Gamma. \end{cases}$$

Set V^m the space generated by $\{w^1, w^2, \dots, w^i\}$ and we construct approximate solutions y^m, $m = 1, 2, 3, \dots$, in the form

$$y^m(t,x) = \sum_{j=1}^{m} c^{j,m}(t) w^j(x),$$

where $c^{j,m}$ is determined by the ordinary differential equations

$$(y_{tt}^m(t), v) - (\Delta y^m(t), v) + (a(x) g(y_t^m), v) = (|y^m|^{p-2} y^m, v), \quad \text{for all } v \in V^m, \tag{3.1}$$

on some interval $[0, t_m)$. Let y_0^m and y_1^m in V^m be such that

$$y^m(0) = y_0^m = \sum_{j=1}^{m} (y_0, w^j) w^j \to y_0, \quad \text{in } H^2(\Omega) \cap \mathcal{W} \text{ as } m \to +\infty, \tag{3.2}$$

$$y_t^m(0) = y_1^m = \sum_{j=1}^{m} (y_1, w^j) w^j \to y_1, \quad \text{in } H_0^1(\Omega) \text{ as } m \to +\infty, \tag{3.3}$$

and

$$\Delta y_0^m - a(x) g(y_1^m) + |y_0^m|^{p-2} y_0^m \to \Delta y_0 - a(x) g(y_1) + |y_0|^{p-2} y_0, \quad \text{in } L^2(\Omega) \text{ as } m \to +\infty. \tag{3.4}$$

3.1. **A priori estimates.**

3.1.1. *The first estimate.* We are going to use some a priori estimates to show that $t_m = +\infty$.

Choosing $v = 2y_t^m$ in (3.1), using Green's formula and then integrating over $(0,t)$, we find

$$\|y_t^m(t)\|^2 + 2J(y^m(t)) + 2\int_0^t \int_\Omega a(x) y_t^m(s) g(y_t^m(s))\, dx\, ds = \|y_1^m\|^2 + 2J(y_0^m),$$

for all $t \in [0, t_m)$. Using (3.2) and (3.3), we obtain

$$\|y_t^m(t)\|^2 + 2J(y^m(t)) + 2\int_0^t \int_\Omega a(x) y_t^m(s) g(y_t^m(s))\, dx\, ds \leq C_0, \tag{3.5}$$

where $J(y^m(t)) = \frac{1}{2}\|\nabla y(^m t)\|^2 - \frac{1}{p}\|y^m(t)\|_p^p$, for some C_0 independent of m. These estimates imply that the solution y^m exists globally in $[0, +\infty[$.

Estimate (3.5) yields

$$y^m \text{ is bounded in } L^\infty(0, T, \mathcal{W}), \tag{3.6}$$

$$y_t^m \text{ is bounded in } L^\infty(0, T, L^2(\Omega)), \tag{3.7}$$

$$a(x) y_t^m g(y_t^m) \text{ is bounded in } L^1(\Omega \times (0,T)). \tag{3.8}$$

We prove that $a(x)g(y_t^m(t))$ is bounded, using (**A1**) and (3.8), we have

$$\int_0^T \int_\Omega a^2(x) g^2(y_t^m)\, dx\, dt \leq \tau_1 \|a\|_\infty \int_0^T \int_\Omega a(x) |y_t^m g(y_t^m)|\, dx\, dt \leq K.$$

Then

$$a(x) g(y_t^m) \text{ is bounded in } L^2(\Omega \times (0,T)). \tag{3.9}$$

3.1.2. *The second estimate.* We now proceed with further a priori estimates. In doing so, differentiating (3.1) with respect to t, we get

$$(y_{ttt}^m(t) - \Delta y_t^m(t) + a(x) y_{tt}^m g'(y_t^m(t)), v) = ((p-1)|y^m(t)|^{p-2} y_t^m(t), v).$$

Choosing $v = y_{tt}^m$, we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (|y_{tt}^m(t)|^2 + |\nabla y_t^m(t)|^2)\, dx + 2\int_\Omega a(x) |y_{tt}^m(t)|^2 g'(y_t^m(t))\, dx \\ =&2(p-1)\int_\Omega |y^m(t)|^{p-2} y_t^m(t) y_{tt}^m(t)\, dx. \end{aligned} \tag{3.10}$$

From Hölder's, Young's inequalities and (3.6), we have

(3.11)
$$\begin{aligned}
&\left|\int_\Omega |y^m(t)|^{p-2}y_t^m(t)y_{tt}^m(t)\,dx\right|\\
\le& C(\Omega)\Big(\int_\Omega 1^{\frac{p-1}{p-2}}\,dx\Big)^{\frac{p-2}{p-1}}\Big(\int_\Omega |y^m(t)|^{2(p-1)}\,dx\Big)^{\frac{p-2}{2(p-1)}}\Big(\int_\Omega |y_t^m(t)|^{2(p-1)}\Big)^{\frac{1}{2(p-1)}}\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C_s\|\nabla y^m(t)\|^{p-2}\|\nabla y_t^m(t)\|\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C\|\nabla y_t^m(t)\|\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C(\varepsilon)\|\nabla y_t^m(t)\|^2+\varepsilon\|y_{tt}^m(t)\|^2.
\end{aligned}$$

Integrating (3.10) over $(0,t)$ and using (3.11), we have

$$\begin{aligned}
&\int_\Omega(|y_{tt}^m(t)|^2+|\nabla y_t^m(t)|^2)\,dx+2\tau_0\int_0^t\int_\Omega a(x)|y_{tt}^m(s)|^2\,dx\,ds\\
\le&\|y_{tt}^m(0)\|^2+\|\nabla y_1^m\|^2+C\int_0^t\|y_{tt}^m(s)\|^2+\|\nabla y_t^m(s)\|^2\,ds.
\end{aligned}\tag{3.12}$$

We shall estimate $\|y_{tt}^m(0)\|$. To this end, choose $v=y_{tt}^m$ in (3.1) and set $t=0$ to derive

$$\|y_{tt}^m(0)\|^2=\int_\Omega y_{tt}^m(0)(\Delta y_0^m-a(x)g(y_1^m)+|y_0^m|^{p-2}y_0^m)\,dx,$$

from which, thanks to (3.4) and Cauchy-Schwarz inequality, we find $\|y_{tt}^m(0)\|\le C_1$, where C_1 is a positive constant independent of m.

We gain from (3.12) and Gronwall's lemma that

$$\|y_{tt}^m(t)\|^2+\|\nabla y_t^m(t)\|^2\le C_2,\tag{3.13}$$

for all $t\in[0,T]$, and C_2 is a positive constant independent of m. We conclude from (3.13) that

$$y_t^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)),\tag{3.14}$$

$$y_{tt}^m \text{ is bounded in } L^\infty(0,T,L^2(\Omega)).\tag{3.15}$$

3.1.3. *The third estimate.* Choosing $v=-\Delta y_t^m$ in (3.1) and then integrating over $[0,t]$ for all $t\in[0,T]$, we obtain

$$\begin{aligned}
&\int_\Omega(|\nabla y_t^m(t)|^2+|\Delta y^m(t)|^2)\,dx-2\int_0^t\int_\Omega a(x)\Delta y_t^m g(y_t^m)\,dx\,ds\\
=&\|\nabla y_1^m\|^2+\|\Delta y_0^m\|^2-2\int_0^t\int_\Omega |y^m(s)|^{p-2}y^m(s)\Delta y_t^m(s)\,dx\,ds.
\end{aligned}\tag{3.16}$$

Since $g(0)=0$ and $y_t^m=0$ on Γ, applying the Green formula, we obtain

$$-\int_\Omega a(x)\Delta y_t^m g(y_t^m)\,dx=\int_\Omega \nabla a(x)\nabla y_t^m g(y_t^m)\,dx+\int_\Omega a(x)|\nabla y_t^m|^2 g'(y_t^m)\,dx,$$

using (**A1**), we obtain

$$\int_\Omega \nabla a(x)\nabla y_t^m g(y_t^m)\,dx \le C_s\tau_1\|\nabla a\|_\infty \int_\Omega |\nabla y_t^m|^2\,dx. \tag{3.17}$$

Thanks to Green's formula, Hölder's inequality, we have

$$\begin{aligned} -\int_\Omega |y^m(t)|^{p-2}y^m(t)\Delta y_t^m(t)\,dx &= \int_\Omega \nabla(|y^m(t)|^{p-2}y^m(t))\nabla y_t^m(t)\,dx \\ &\le \frac{1}{2}\|\nabla y^m(t)\|^{2(p-1)} + \frac{1}{2}\|\nabla y_t^m(t)\|^2. \end{aligned} \tag{3.18}$$

Reporting estimate (3.17) and (3.18) in (3.16), we find

$$\begin{aligned} &\|\nabla y_t^m(t)\|^2 + \|\Delta y^m(t)\|^2 + 2\tau_0\int_0^t\int_\Omega a(x)|\nabla y_t^m(s)|^2\,dx\,ds \\ \le &\|\nabla y_1^m\|^2 + \|\Delta y_0^m\|^2 + \int_0^t \|\nabla y^m(s)\|^{2(p-1)}\,ds + \left(\frac{1}{2} + C_s\tau_1\|\nabla a\|_\infty\right)\int_0^t \|\nabla y_t^m(s)\|^2\,ds. \end{aligned}$$

By Gronwall lemma, we obtain

$$\|\nabla y_t^m(t)\|^2 + \|\Delta y^m(t)\|^2 \le C_3, \tag{3.19}$$

where C_3 is a positive constant independent of m. We conclude from (3.19) that

$$y^m \text{ is bounded in } L^\infty(0,T,H^2(\Omega)). \tag{3.20}$$

Furthermore, we have from **(A3)**, Lemma (2.1) and (3.6) that

$$|y^m|^{p-2}y^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)). \tag{3.21}$$

3.2. **Solvability of** (1.5)**.** Applying the Dunford-Pettis theorem and the Riesz lemma we conclude from (3.6), (3.7), (3.9), (3.14), (3.15), (3.20) and (3.21), replacing the sequence y^m with a subsequence if needed, that

$$y^m \rightharpoonup y \text{ weakly star in } L^\infty(0,T,H^2(\Omega)\cap\mathcal{W}), \tag{3.22}$$

$$y_t^m \rightharpoonup y_t \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.23}$$

$$y_{tt}^m \rightharpoonup y_{tt} \text{ weakly star in } L^\infty(0,T,L^2(\Omega)), \tag{3.24}$$

$$|y^m|^{p-2}y^m \rightharpoonup \chi \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.25}$$

$$a(x)g(y_t^m) \rightharpoonup \varphi \text{ weakly star in } L^2(\Omega\times(0,T)). \tag{3.26}$$

3.2.1. *Analysis of the nonlinear terms.* From (3.6), we see that

$$y^m \text{ is bounded in } L^2(0,T,H^1(\Omega)). \tag{3.27}$$

Then, we have y^m is bounded in $H^1(\mathcal{Q})$, where $\mathcal{Q} = [0,T]\times\Omega$ and the injection $H^1(\mathcal{Q}) \hookrightarrow L^2(\mathcal{Q})$ is compact, and there exists a subsequence of y^m still denoted by the same notation such that

$$y^m \to y, \quad \text{a.e. in } L^2(\mathcal{Q}) \tag{3.28}$$

$$y_t^m \to y_t, \quad \text{a.e. in } L^2(\mathcal{Q}) \tag{3.29}$$

We deduce from (3.28) that

$$|y^m|^{p-2}y^m \to |y|^{p-2}y, \quad \text{a.e. in } \mathcal{Q}.$$

From Lemma (2.2), we deduce

$$|y^m|^{p-2}y^m \rightharpoonup |y|^{p-2}y, \quad \text{weakly star in } L^\infty(0,T,H_0^1(\Omega)). \tag{3.30}$$

By (3.26) and (3.30), we obtain $\chi = |y|^{p-2}y$. It remains now to prove that

$$\int_0^T \int_\Omega a(x)g(y_t^m)v\,dx\,dt \to \int_0^T \int_\Omega a(x)g(y_t)v\,dx\,dt, \quad \text{for all } v \in L^2(0,T,L^2(\Omega)).$$

We have $a(x)g(y_t) \in L^1(\mathcal{Q})$. Since g is continuous, we deduce from (3.29), that

$$a(x)g(y_t^m) \to a(x)g(y_t, \quad \text{a.e. in } \mathcal{Q}. \tag{3.31}$$

$$a(x)y_t^m g(y_t^m) \to a(x)y_t g(y_t), \quad \text{a.e. in } \mathcal{Q}.$$

Using (3.8) and Fatou's Lemma, we deduce that

$$\int_0^T \int_\Omega a(x)y_t g(y_t)\,dx\,dt \le K.$$

By using Cauchy-Schwarz's inequality, we obtain

$$\int_0^T \int_\Omega |a(x)g(y_t)|\,dx\,dt \le c|\mathcal{Q}|^{\frac{1}{2}}\Big(\int_0^T \int_\Omega |a(x)g(y_t)|^2\,dx\,dt\Big)^{\frac{1}{2}} \le \widetilde{K}.$$

Let $Q \subset [0,T]\times\Omega$. We set

$$Q_1 = \Big\{(t,x) \in [0,T]\times\Omega \mid |g(y_t^m)| \le |Q|^{-1/2}\Big\}, \quad Q_2 = Q \setminus Q_1$$

and $J(r) = \inf\Big\{|s| \mid s \in \mathbb{R}, |g(s)| \ge r\Big\}$. Then, we have

$$\begin{aligned}\int_Q a(x)g(y_t^m)\ dx\ dt &= \int_{Q_1} a(x)g(y_t^m)\,dx\,dt + \int_{Q_2} a(x)g(y_t^m)\,dx\,dt\\ &\le \|a\|_\infty |Q|^{1/2} + J(|Q|^{\frac{-1}{2}})^{-1}\int_{Q_2} a(x)|y_t^m g(y_t^m)|\,dx\,dt.\end{aligned}$$

Applying (3.8), we find

$$\sup_m \int_Q a(x)g(y_t^m)\,dx\,dt \to 0, \quad \text{when } |Q| \to 0,$$

and from (3.31), we deduce thanks to Vitali's Theorem that

$$a(x)g(y_t^m) \to a(x)g(y_t), \quad \text{in } L^1([0,T]\times\Omega).$$

Hence, (3.26) yields $a(x)g(y_t) = \varphi \in L^2(\mathcal{Q})$ and

$$a(x)g(y_t^m) \rightharpoonup a(x)g(y_t), \quad \text{in } L^2(\mathcal{Q}).$$

We deduce, for all $v \in L^2([0,T]\times L^2(\Omega))$, that

$$\int_0^T \int_\Omega a(x)g(y_t^m)v\,dx\,dt \to \int_0^T \int_\Omega a(x)g(y_t)v\,dx\,dt. \tag{3.32}$$

Convergences (3.22)–(3.26), (3.30) and (3.32) permit us to pass to the limit in the (3.1). As w^j is a basis of $H^2(\Omega)$, then, for all $T > 0$, for all $\theta \in D(0,T)$ and for all $v \in L^2([0,T] \times L^2(\Omega))$, after passing to the limit we obtain

$$\begin{aligned}\int_0^T \int_\Omega (y_{tt}(t), v(t))\theta(t)\,dt - \int_0^T (\Delta y(t), v(t))\theta(t)\,dt \\ + \int_0^T (a(x)(g(y_t), v(t))\theta(t)\,dt - \int_0^T (|y|^{p-2}(t)y(t), v(t))\theta(t)\,dt = 0.\end{aligned} \tag{3.33}$$

From (3.33) and taking $v \in D(0,T))$, we show that

$$y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y, \quad \text{in } D'(\Omega \times (0,T))$$

Now, since y_{tt}, $a(x)g(y_t)$, $|y|^{p-2}y \in L^2(0,\infty, L^2(\Omega))$ we have $\Delta y \in L^2(0,\infty, L^2(\Omega))$ and therefore

$$y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y, \quad \text{in } L^\infty(0,\infty, L^2(\Omega))$$

3.3. **Uniqueness.** Let y_1 and y_2 be solutions to problem (1.5). Then, defining $z = y_1 - y_2$, we obtain

$$(z_{tt}, v) + (\nabla z, \nabla v) + (a(x)(g(y_{1,t}) - g(y_{2,t})), v) = (|y_1|^{p-2}y_1 - |y_2|^{p-2}y_2, v),$$

for all $v \in H_0^1(\Omega)$. Substituting $v = z_t(t)$ in the above equality and observing that g is nondecreasing, it results that

$$\frac{d}{dt}\Big\{\|z_t\|^2 + \|\nabla z\|^2\Big\} + 2\int_\Omega a(x)(g(y_{1,t}) - g(y_{2,t}))z_t\,dx = 2\int_\Omega (|y_1|^{p-2}y_1 - |y_2|^{p-2}y_2)z_t(t)\,dx. \tag{3.34}$$

It follows from the mean value theorem that

$$\begin{aligned}&\Big|y_1(x,t)|^{p-2}y_1(x,t) - |y_2(x,t)|^{p-2}y_2(x,t)\Big| \\ &\le (p-1)(|y_1(x,t)| + |y_2(x,t)|)^{p-2}|y_1(x,t) - y_2(x,t)|,\end{aligned}$$

from (3.34) and using the monotonicity of g a hence, we conclude that

$$\frac{d}{dt}\Big\{\|z_t\|^2 + \|\nabla z\|^2\Big\} \le 2(p-1)\int_\Omega (|y_1(x,t)| + |y_2(x,t)|)^{p-2}|z(t)||z_t(t)|\,dx.$$

Using analogous arguments like those used in the second estimate, we obtain

$$\frac{d}{dt}\Big\{\|z_t\|^2 + \|\nabla z\|^2\Big\} + 2\int_\Omega a(x)(g(y_{1,t}) - g(y_{2,t}))z_t\,dx \le C(\|z_t\|^2 + \|\nabla z\|^2). \tag{3.35}$$

Integrating the inequality (3.35) over $(0,t)$ and making use of Gronwall's lemma we conclude that $\|z_t\|^2 = \|\nabla z\|^2 = 0$. This concludes the first part of the proof.

3.4. **Weak solutions.** In order to obtain existence for weak solutions we use standard arguments of density. Indeed, let us assume that $\{y_0, y_1\} \in \mathcal{W} \times L^2(\Omega)$. So, let $\{y_0^\mu, y_1^\mu\} \in \mathcal{W} \times L^2(\Omega)$ be such that

$$y_0^\mu \to y_0, \quad \text{in } \mathcal{W}, \quad \text{and} \quad y_1^\mu \to y_1, \quad \text{in } L^2(\Omega). \tag{3.36}$$

Then, for each $\mu \in \mathbb{N}$ there exists y^μ regular solution of (1.5) belonging to the class of Theorem (3.1). Repeating the same arguments used in the first estimate we obtain

$$\|y_t^\mu(t)\|^2 + \|\nabla y(^\mu t)\|^2 - \frac{2}{p}\|y^\mu(t)\|_p^p + 2\int_0^t \int_\Omega a(x) y_t^\mu(s) g(y_t^\mu(s))\, dx\, ds \leq C, \tag{3.37}$$

where C is a positive constant independent of μ.

Defining $z^{\mu,\sigma} = y^\mu - y^\sigma$, $\mu, \sigma \in \mathbb{N}$, where y^μ and y^σ are smooth solutions of (1.5), we obtain by the monotonicity of g that

$$\frac{1}{2} \cdot \frac{d}{dt}\left\{\|z_t^{\mu,\sigma}\|^2 + \|\nabla z^{\mu,\sigma}\|^2\right\} \leq K(p) \int_\Omega (|y^\mu(x,t)| + |y^\sigma(x,t)|)^{p-2} |z^{\mu,\sigma}(t)||z_t^{\mu,\sigma}(t)|\, dx. \tag{3.38}$$

Combining (3.37) and (3.38) we obtain, after integrating over $(0,t)$ and using Gronwall's lemma, that

$$\|y_t^\mu(t) - y_t^\sigma(t)\|^2 + \|\nabla y^\mu(t) - \nabla y^\sigma(t)\|^2 \leq K(p,T)(\|y_1^\mu - y_1^\sigma\|^2 + \|\nabla y_0^\mu - \nabla y_0^\sigma\|^2), \tag{3.39}$$

where $K(p,T)$ is a positive constant independent of $\mu, \sigma \in \mathbb{N}$.

From (3.36) and (3.39), we conclude that there exists a function y such that, for all $T > 0$, we have

$$y^\mu \to y \text{ strongly in } C^0(0,T,\mathcal{W}), \tag{3.40}$$

$$y_t^\mu \to y_t \text{ strongly in } C^0(0,T,L^2(\Omega)). \tag{3.41}$$

From (3.37), (3.40) and (3.41) we also have,

$$\begin{aligned} y_t^\mu &\rightharpoonup y_t \text{ weakly star in } L^2_{loc}(0,\infty,L^2(\Omega)), \\ |y^\mu|^{p-2} y^\mu &\rightharpoonup |y|^{p-2} y \text{ weakly star in } L^2_{loc}(0,\infty,L^2(\Omega)), \\ a(x) g(y_t^\mu) &\rightharpoonup a(x) g(y_t) \text{ weakly star in } L^2(\Omega \times (0,T)). \end{aligned} \tag{3.42}$$

The weak convergences from the estimate given by (3.37) and the convergences obtained in (3.40)–(3.42) are sufficient to pass to the limit in order to obtain a weak solution to problem (1.5). □

4. Stability Result

In this section, we state and prove the stability result for the energy of the problem (1.5). The stability result reads as follows.

Theorem 4.1. *Let $y_0 \in H^2(\Omega) \cap \mathcal{W}$, $y_1 \in H_0^1(\Omega)$. Assume that* **(A1)-(A3)** *hold. The energy of the unique solution of the problem* (1.5)*, given by* (2.2)*, decays exponentially to zero, there exist positive constants K and λ, independent of the initial data, with*

$$E(t) \leq K E(0) e^{-\lambda t}, \quad \textit{for all } t \geq 0. \tag{4.1}$$

We first consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} 0 \le \psi \le 1, & \\ \psi = 1, & \text{in } \bar{\Omega}\backslash\mathcal{M}_1, \\ \psi = 0, & \text{in } \mathcal{M}_0. \end{cases} \tag{4.2}$$

For $M > 0$ and $\mu > 0$, define the perturbed energy

$$\widehat{E}(t) = M.E(t) + E^\mu(t)\rho(t), \tag{4.3}$$

where

$$\rho(t) = 2\int_\Omega y_t(h.\nabla y)\,dx + \theta\int_\Omega y_t y\,dx, \tag{4.4}$$

$$h(x) = m(x)\psi(x), \tag{4.5}$$

and $\theta \in]n-2, n[$.

Lemma 4.1. *There exist two positive constants λ_1 and λ_2 such that*

$$\lambda_1 E(t) \le \widehat{E}(t) \le \lambda_2 E(t), \quad \textit{for all}\, t \ge 0. \tag{4.6}$$

Proof. Thanks to Cauchy-Schwarz's inequality, we have

$$|\rho(t)| \le 2\mathcal{R}(x^0)\|\nabla y\|\|y_t\| + \theta\sqrt{C_s}\|\nabla y\|\|y_t\|, \tag{4.7}$$

where

$$\mathcal{R}(x^0) = \max_{x\in\overline{\Omega}} |x - x^0|. \tag{4.8}$$

From (4.7) we obtain

$$\begin{aligned} |\rho(t)| &\le (\theta\sqrt{C_s} + 2\mathcal{R}(x^0))\left\{\frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|\nabla y\|^2\right\} \\ &\le (\theta\sqrt{C_s} + 2\mathcal{R}(x^0))E(t). \end{aligned}$$

Then, for M large enough, we obtain (4.6), where $\lambda_1 = M - E^\mu(0)(\theta\sqrt{C_s} + 2\mathcal{R}(x^0))$ and $\lambda_2 = M + E^\mu(0)(\theta\sqrt{C_s} + 2\mathcal{R}(x^0))$. □

Lemma 4.2. *The functional $\rho(t)$ defined in* (4.4) *satisfies*

$$\begin{aligned} \rho'(t) = &\int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma - (n-\theta)\int_\Omega |y_t|^2\,dx - (\theta - n + 2)\int_\Omega |\nabla y|^2\,dx \\ &- \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi y_t^2\,dx + n\int_{\mathcal{M}_1}(1-\psi)y_t^2\,dx + (n-2)\int_{\mathcal{M}_1}(\psi - 1)|\nabla y|^2 dx \\ &+ \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\backslash\mathcal{M}_0} m_i\frac{\partial\psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx \\ &- \theta\int_\Omega y.a(x)g(y_t)\,dx - \int_\Omega 2(h.\nabla y)a(x)g(y_t)\,dx \\ &+ 2\int_\Omega h.\nabla y|y|^{p-2}y\,dx + \theta\int_\Omega |y|^p\,dx. \end{aligned} \tag{4.9}$$

Proof. Taking the derivative of $\rho(t)$ with respect to t,

$$\begin{aligned}\rho'(t) =& 2\int_\Omega y_{tt}(h\nabla y)\,dx + 2\int_\Omega y_t(h\nabla y_t)\,dx + \theta\int_\Omega y_{tt}y\,dx + \theta\int_\Omega y_t^2\,dx \\ =& 2\int_\Omega y_t(h\nabla y_t)\,dx + +\theta\int_\Omega y_{tt}y\,dx + 2\int_\Omega h.\nabla y.\Delta y\,dx \\ &- 2\int_\Omega h.\nabla y.a(x)g(y_t)\,dx + 2\int_\Omega h.\nabla y|y|^p y\,dx + \theta\int_\Omega |y_t|^2\,dx.\end{aligned} \tag{4.10}$$

Using (1.1)–(1.4), (4.2), (4.5) and Green formulas the first term of the right hand side of (4.10), we have

$$\begin{aligned}2\int_\Omega y_t(h\nabla y_t)dx &= -\int_\Omega \operatorname{div}(h)y_t^2\,dx \\ &= -\int_{\Omega\backslash\mathcal{M}_1}\operatorname{div}(\psi.m)y_t^2dx - \int_{\mathcal{M}_1}\operatorname{div}(\psi.m)y_t^2dx \\ &= -n\int_{\Omega\backslash\mathcal{M}_1} y_t^2dx - \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi y_t^2\,dx - n\int_{\mathcal{M}_1}\psi y_t^2dx.\end{aligned}$$

Then

$$2\int_\Omega y_t(h\nabla y_t) = -n\int_\Omega y_t^2dx + n\int_{\mathcal{M}_1}(1-\psi)y_t^2dx - \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi y_t^2\,dx. \tag{4.11}$$

Using the first equation of (1.5) and applying the Green formula, the second term of the right hand side of (4.10), we obtain

$$\theta\int_\Omega y_{tt}y\,dx = -\theta\int_\Omega |\nabla y|^2\,dx - \theta\int_\Omega a(x)yg(y_t)\,dx + \theta\int_\Omega |y|^p\,dx. \tag{4.12}$$

We have $\frac{\partial y}{\partial x_k} = \frac{\partial y}{\partial \nu}\nu_k$, which implies

$$h\nabla y = (h.\nu)\frac{\partial y}{\partial \nu} \quad \text{and} \quad |\nabla y|^2 = \left(\frac{\partial y}{\partial \nu}\right)^2 \quad \text{on } \Gamma.$$

From the above expressions and using Green's formulas, the third term of the right hand side of (4.10) can be rewritten as follows

$$\begin{aligned}&2\int_\Omega (h\nabla y)\Delta y\,dx \\ =& 2\int_\Gamma (h.\nu)|\nabla y|^2\,d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx - 2\int_\Omega h(\nabla y)\nabla(\nabla y)\,dx \\ =& 2\int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx - \int_\Omega h\nabla(|\nabla y|^2)\,dx \\ =& \int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx + \int_\Omega \operatorname{div}(h)|\nabla y|^2\,dx.\end{aligned} \tag{4.13}$$

So, by using (1.2), (4.2) and (4.5), the second term of (4.13) gives

$$\begin{aligned}
(4.14)\quad & -2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx \\
&= -2\sum_{i,k=1}^{n}\int_{\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx - 2\sum_{i,k=1}^{n}\int_{\Omega\setminus\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx \\
&= -2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\psi_i\frac{\partial m_i}{\partial x_k}\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1} m_i\frac{\partial \psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx \\
&\quad -2\sum_{i,k=0}^{n}\int_{\Omega\setminus\mathcal{M}_1}\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_k}\,dx \\
&= -2\int_{\mathcal{M}_1}\psi|\nabla y|^2\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m_i\frac{\partial \psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx - 2\int_{\Omega\setminus\mathcal{M}_1}|\nabla y|^2\,dx.
\end{aligned}$$

Similarly, the third term of (4.13) can be rewritten as follows

$$\begin{aligned}
(4.15)\quad & \int_{\Omega}(\mathrm{div}h)|\nabla y|^2\,dx = \int_{\Omega\setminus\mathcal{M}_1}\mathrm{div}(\psi m)|\nabla y|^2\,dx + \int_{\mathcal{M}_1}\mathrm{div}(\psi m)|\nabla y|^2\,dx \\
&= n\int_{\Omega\setminus\mathcal{M}_1}|\nabla y|^2\,dx + \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx + n\int_{\mathcal{M}_1}\psi|\nabla y|^2\,dx.
\end{aligned}$$

Inserting (4.14) and (4.15) in (4.13), we arrive at

$$\begin{aligned}
(4.16)\quad 2\int_{\Omega}(h\nabla y)\Delta y\,dx &= \int_{\Gamma}(h.\nu)\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma + (n-2)\int_{\Omega}|\nabla y|^2\,dx \\
&\quad + (n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla y|^2\,dx \\
&\quad - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m_i\frac{\partial \psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx \\
&\quad + \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx.
\end{aligned}$$

Simple substitution of (4.11), (4.12) and (4.16) give (4.9) ends the proof of Lemma 4.2. □

Lemma 4.3. *We have*

$$\begin{aligned}
(4.17)\quad |\rho'(t)| \le & -K_nE(t) + B\int_{\Omega}|\nabla y|^2\,dx + A\int_{\omega}|y_t|^2\,dx \\
& -\theta\int_{\Omega}a(x)yg(y_t)\,dx - 2\int_{\Omega}(h\nabla y)a(x)g(y_t)\,dx \\
& + 2\int_{\Omega}h.\nabla y|y|^{p-2}y\,dx + \left(\theta + \frac{K_n}{p}\right)\int_{\Omega}|y|^p dx,
\end{aligned}$$

where

$$K_n = \min\Big\{2(n-\theta), 2(\theta-n+2)\Big\}, \quad A = \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + n$$

and

$$B = 3\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + (n-2).$$

Proof. Next, we estimate some terms on the RHS of identity (4.9).

Taking (1.1)–(1.4), (4.2) and (4.5), we have

(4.18)

$$\int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma = \int_{\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma + \int_{\Gamma\setminus\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma \le 0,$$

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|y_t|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\omega |y_t|^2\,dx, \tag{4.19}$$

$$n\int_{\mathcal{M}_1}(1-\psi)|y_t|^2\,dx \le n\int_\omega |y_t|^2\,dx, \tag{4.20}$$

$$2\left|\sum_{i,k=0}^n \int_{\mathcal{M}_1\setminus\mathcal{M}_0}\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}m_i\frac{\partial \psi_i}{\partial x_i}\,dx\right| \le 2\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla y|^2\,dx, \tag{4.21}$$

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla y|^2\,dx \tag{4.22}$$

and

$$(n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla y|^2\,dx \le (n-2)\int_\Omega|\nabla y|^2\,dx. \tag{4.23}$$

Taking into account (4.18)–(4.23) into (4.9) we obtain (4.17). The proof of Lemma 4.3 is completed. □

Proof. (of Theorem 4.1) Taking the derivative of (4.3) with respective to t, we have

$$\widehat{E'}(t) = M\,E'(t) + \mu E'(t)E^{\mu-1}(t)\rho(t) + E^\mu(t)\rho'(t).$$

Using (2.2) and (4.17), we have

$$\begin{aligned}\widehat{E'}(t) \le& M\,E'(t) + C_\mu E^\mu(0)|E'(t)| - K_n.E^{\mu+1}(t)\\ &+ A\,E^\mu(t)\int_\omega|y_t|^2\,dx + B\,E^\mu(t)\int_\Omega|\nabla y|^2\,dx\\ &+ 2E^\mu(t)\int_\Omega(h\nabla y)a(x)g(y_t)\,dx - \theta E^\mu(t)\int_\Omega y\,a(x)g(y_t)\,dx\\ &+ 2E^\mu(t)\int_\Omega h\nabla y|y|^{p-2}y\,dx + \left(\theta+\frac{K_n}{p}\right)E^\mu(t)\int_\Omega|y|^p dx.\end{aligned} \tag{4.24}$$

Next, we will estimate some terms on the right-hand side of identity (4.24). Using (2.3), we get

$$A\,E^{\mu}(t)\int_{\omega}|y_t|^2\,dx \le \frac{1}{\tau_1}\frac{A}{a_0}E^{\mu}(t)\int_{\Omega}a(x)y_t g(y_t)\,dx \le CE^{\mu}(t)(-E'(t)) \\ \le CE^{\mu}(0)|E'(t)|. \tag{4.25}$$

By (2.2), we have

$$B.E^{\mu}(t)\int_{\Omega}|\nabla y|^2\;dx \le BE^{\mu+1}(t). \tag{4.26}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}2.E^{\mu}(t)\int_{\Omega}h.a(x)\nabla y g(y_t)\,dx &\le 2\mathcal{R}(x^0)E^{\mu}(t)\|\nabla y\|\Big(\int_{\Omega}a^2(x)g^2(y_t)\,dx\Big)^{\frac{1}{2}}\\ &\le 2c\mathcal{R}(x^0)\sqrt{\|a\|_{\infty}}E^{\mu+\frac{1}{2}}(t)\Big(\int_{\omega}a(x)y_t(t)g(y_t)\,dx\Big)^{\frac{1}{2}}\\ &\le 2c\mathcal{R}(x^0)\sqrt{\|a\|_{\infty}}E^{\mu+\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}}.\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}2.E^{\mu}(t)\int_{\Omega}h.a(x)\nabla y g(y_t)\,dx &\le c\mathcal{R}(x^0)\|a\|_{\infty}E^{2\mu+1}(t)+c\mathcal{R}(x^0)|E'(t)|\\ &\le c\mathcal{R}(x^0)\|a\|_{\infty}E^{\mu}(0)E^{\mu+1}(t)+c\mathcal{R}(x^0)|E'(t)|\\ &\le \frac{K_n}{6}E^{\mu+1}(t)+c\mathcal{R}(x^0)|E'(t)|.\end{aligned} \tag{4.27}$$

Using Cauchy-Schwarz, Young's and Sobolev-Poincares inequalities, we get

$$\begin{aligned}\theta E^{\mu}(t)\int_{\Omega}y.a(x)g(y_t)\,dx &\le \theta C'_s E^{\mu}(t)\|\nabla y\|\left(\int_{\omega}a^2(x)g^2(y_t)\,dx\right)^{\frac{1}{2}}\\ &\le C\frac{\|a\|_{\infty}}{2}E^{\mu}(0)E^{\mu+1}(t)+C'\frac{\|a\|_{\infty}}{2}|E'(t)|\\ &\le \frac{K_n}{6}E^{\mu+1}(t)+C'\frac{\|a\|_{\infty}}{2}|E'(t)|.\end{aligned} \tag{4.28}$$

By Cauchy-Schwarz and Young's inequalities, we find

$$\begin{aligned}2E^{\mu}(t)\int_{\Omega}h.\nabla y|y|^{p-2}y\,dx &\le 2.E^{\mu}(t)\mathcal{R}(x^0)\|\nabla u\|\left(\int_{\Omega}|y|^{2(p-1)}\,dx\right)^{\frac{1}{2}}\\ &\le 2c\mathcal{R}(x^0)E^{\mu+\frac{1}{2}}(t)\|y\|_{2(p-1)}^{p-1}\\ &\le 2c\mathcal{R}(x^0)E^{\mu+\frac{1}{2}}(t)\|\nabla y\|_2^{p-1},\end{aligned}$$

where

$$p \le \frac{2n-2}{n-2},$$

we obtain

$$
(4.29) \qquad 2.E^{\mu}(t)\int_{\Omega} h\nabla y|y|^{p-2}y\,dx \le 2c\Re(x^0)E^{\mu+\frac{1}{2}}(t)E^{\frac{p-1}{2}}(t) \le 2c\Re(x^0)E^{\mu+\frac{p}{2}}(t) \le 2c\Re(x^0)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0).
$$

Using Sobolev-Poincaré and Young's inequalities, we get

$$
\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\,dx \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\|\nabla y\|^p,
$$

where

$$
p \le \frac{2n}{n-2},
$$

we obtain

$$
(4.30) \qquad \left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\;dx \le 2C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)E^{\frac{K_n}{p}}(t) \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(t) \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0).
$$

Combining (4.26), (4.29) and (4.30), we get

(4.31)

$$
2.E^{\mu}(t)\int_{\Omega} h.\nabla y|y|^{p-2}y\,dx + B\,E^{\mu}(t)\int_{\Omega}|\nabla y|^2\;dx + \left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\;dx
$$

$$
\le 2c\Re(x^0)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0) + BE^{\mu+1}(t) + C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0)
$$

$$
\le \frac{K_n}{6}E^{(\mu+1)}.
$$

Reporting (4.25), (4.27), (4.28) and (4.31) in (4.24), we find

$$
\widehat{E'}(t) \le M.E'(t) + CE^{\mu}(0)|E'(t)| + C|E'(t)| - \frac{K_n}{2}E^{\mu+1}(t).
$$

Choosing $\mu = 0$ and M large enough to obtain

$$
(4.32) \qquad \widehat{E'}(t) \le -\frac{K_n}{2}E(t) \le -\frac{K_n}{2\lambda_1}\widehat{E}(t).
$$

Finally, by combining (4.6) and (4.32) we obtain (4.1), which complete the proof. □

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MULTIPLE POSITIVE SOLUTIONS OF DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this work, we deal with the following two-point non-linear Dirichlet boundary value problem for a finite nabla fractional difference equation:

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) = f(u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0. \end{cases}$$

Here a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3$, $1 < \alpha < 2$, $f : \mathbb{R} \to \mathbb{R}^{+} \cup \{0\}$ is a continuous function, and $\nabla^{\alpha}_{\rho(a)}$ denotes the α^{th} order Riemann-Liouville nabla difference operator. First, we construct an associated Green's function and obtain some of its properties. Under suitable conditions on the non-linear part of the difference equation, we deduce some results for at least two and at least three positive solutions of the considered problem. For this purpose, we use a few prominent conical shell fixed point theorems.

1. Introduction

In the year 1695, "L'Hospital inquires Leibniz on the differential operator $\frac{d^n}{dt^n}$: What if the order will be $\frac{1}{2}$? To which Leibniz replied: It will lead to a paradox from which one day useful consequences will be drawn". This question gave birth to a branch of mathematics that we know today as fractional calculus. Although it started around the same time as differential calculus, most of the early developments of fractional calculus were confined to the basement for a long time. Today fractional calculus has been successfully used for mathematical modelling in medical sciences, computational biology, economics, physics and several areas of engineering. For further applications

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and historical literature, we refer here to a few classical texts on fractional calculus [34, 36, 37] and [31].

On the other hand, discrete fractional calculus deals with arbitrary order differences and sums defined on a discrete domain in either a forward (delta) or backward (nabla) sense. The theory of discrete fractional calculus is relatively new, with the most notable works done in the past decade. The notions of the nabla fractional difference and sum can be traced back to the work [14] and [35]. In this line, Atici and Eloe [**?**] developed the nabla fractional Riemann-Liouville difference operator, initiated the study of nabla fractional initial value problem and established the exponential law, product rule and nabla Laplace transform. Following their works, the contributions of several mathematicians have made the theory of discrete fractional calculus a fruitful field of research in science and engineering. We refer here to a recent monograph [12] and the references therein.

The study of boundary value problems has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the fractional side, there is a sudden growth in interest for the development of nabla fractional boundary value problems. Many authors have studied nabla fractional boundary value problems recently. To name a few, [2, 16] and [19] worked with self-adjoint Caputo nabla boundary value problem. Brackins [9] studied a particular class of self-adjoint Riemann-Liouville nabla boundary value problem and derived the Green's function associated with it along with a few of its properties. Gholami et al. [17] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla boundary value problem with Dirichlet boundary conditions. Jonnalagadda [13, 20, 21, 23–25] analysed some qualitative properties of two-point non-linear Riemann-Liouville nabla boundary value problem associated with various types of boundary conditions.

There has been an increasing interest in multiple fixed-point theorems and their applications to boundary value problems for differential equations and finite difference equations. Interest in triple solutions was born from the Leggett–Williams multiple fixed-point theorem [33]. Following this, two triple fixed-point theorems by Avery [5], and Avery and Peterson [7] have been developed and applied to specific boundary value problems for ordinary differential equations as well as for their discrete analogues [3, 7]. Also, Avery and Henderson [6] have established twin fixed-point theorem by dual application of Krasnosel'skii fixed-point theorem. The applications of the above fixed-point theorems in discrete fractional calculus are scarce. To the best of our knowledge, there has been no progress in this line, in the domain of nabla fractional calculus.

Our purpose of this article is to establish sufficient conditions for the existence of multiple positive solutions of the following standard two-point non-linear nabla fractional boundary value problem with Dirichlet boundary conditions

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) = f(u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$, with $b-a \in \mathbb{N}_3, 1 < \alpha < 2$ and $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, using conical shell fixed-point theorems such as Leggett-Williams [33] and Avery-Henderson [6].

The present article is organized as follows. Section 2 contains a few preliminaries on nabla fractional calculus. In Sections 3, we present sufficient conditions on three and two positive solutions of (1.1) using fixed-point theorems by Leggett-Williams [33] and Avery-Henderson [6], respectively, on a suitable cone.

2. Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^+$, respectively. We use the following notations, definitions and known results of nabla fractional calculus [12]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}_a^b$, where $b-a \in \mathbb{Z}^+$, are defined by

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\}, \quad \mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}.$$

Definition 2.2. We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$, by

$$\rho(t) = t-1, \quad t \in \mathbb{N}_{a+1}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\Big(\nabla u\Big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\Big(\nabla^N u\Big)(t) = \Big(\nabla\Big(\nabla^{N-1}u\Big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.3 ([12])**.** Let $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$. The generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.4 (See [12])**.** Let t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided the right-hand side exists.

We observe the following properties of the nabla fractional Taylor monomials.

Lemma 2.1 ([19, 24])**.** *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold.*

(a) *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$ and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.*
(b) *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.*
(c) *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.*
(d) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.*
(e) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a non-decreasing function of t.*

(f) *If* $t \in \mathbb{N}_s$ *and* $\mu > 0$*, then* $H_\mu(t, \rho(s))$ *is an increasing function of* t*.*
(g) *If* $0 < v \leq \mu$*, then* $H_v(t, a) \leq H_\mu(t, a)$*, for each fixed* $t \in \mathbb{N}_a$*.*

Definition 2.5 ([12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by

$$\Big(\nabla_a^{-\nu} u\Big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 ([12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by

$$\Big(\nabla_a^{\nu} u\Big)(t) = \Big(\nabla^N \Big(\nabla_a^{-(N-\nu)} u\Big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Now, we write the expression for the Green's function corresponding to (1.1) and state a few properties of the same, which will be used later.

Theorem 2.1 ([9, 17, 25])**.** *Let* $1 < \alpha < 2$ *and* $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$*. The equivalent form of* (1.1) *is given by*

$$u(t) = \sum_{s=a+2}^{b} G(t, s) f(u(s)), \quad t \in \mathbb{N}_a^b, \tag{2.1}$$

where the Green's function is given by

$$G(t, s) = \begin{cases} G_1(t, s) = \frac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)), & t \in \mathbb{N}_a^{s-1}, \\ G_2(t, s) = \frac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)) - H_{\alpha-1}(t, \rho(s)), & t \in \mathbb{N}_s^b. \end{cases} \tag{2.2}$$

Theorem 2.2 ([9, 17, 25])**.** *The Green's function* $G(t, s)$ *defined in* (2.2) *satisfies the following properties:*

(a) $G(a, s) = G(b, s) = 0$*, for all* $s \in \mathbb{N}_{a+1}^b$*;*
(b) $G(t, a + 1) = 0$*, for all* $t \in \mathbb{N}_a^b$*;*
(c) $G(t, s) > 0$*, for all* $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$*;*
(d) $\max_{t \in \mathbb{N}_{a+1}^{b-1}} G(t, s) = G(s - 1, s)$*, for all* $s \in \mathbb{N}_{a+2}^b$*;*
(e) $\sum_{s=a+1}^{b} G(t, s) \leq \lambda$*, for all* $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$*, where*

$$\lambda = \left(\frac{b - a - 1}{\alpha\Gamma(\alpha + 1)}\right) \left(\frac{(\alpha - 1)(b - a) + 1}{\alpha}\right)^{\overline{\alpha-1}}. \tag{2.3}$$

3. Multiple Positive Solutions

In this section, we establish sufficient conditions on the existence of at least two and three positive solutions of (1.1) using Avery-Henderson [6] and Leggett-Williams [33] fixed-point theorems respectively, on a suitable cone, by suitably constructing the growth conditions on the non-linear part of the boundary value problem.

Definition 3.1 ([1])**.** Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed non-empty convex set $K \subset \mathcal{B}$ is called a cone provided,

(i) $eu + iv \in K$, for all $u, v \in K$ and all $e, i \geq 0$;
(ii) $u \in K$ and $-u \in K$ implies $u = 0$.

Definition 3.2 ([28])**.** An operator $T : \mathcal{B} \to \mathcal{B}$ is called completely continuous, if it is continuous and maps bounded sets into pre-compact sets.

Definition 3.3 ([1])**.** A functional α_1 is said to be a non-negative continuous concave functional on a cone K of a real Banach space $\mathcal{B}$, if $\alpha_1 : K \to [0, +\infty)$ is continuous and

$$\alpha_1(tx + (1-t)y) \geq t\alpha_1(x) + (1-t)\alpha_1(y),$$

for all $x, y \in K$ and $t \in [0, 1]$.

The following theorems which are useful for the main results has appeared in [13] and the same has been proved here for the completeness of the article.

Lemma 3.1. *Let a, b be two real numbers such that $0 < a < b$ and $1 < \alpha < 2$. Then $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. It is enough to show that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}} \right) < 0$. Consider

$$\begin{aligned}
&\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}} \right) \\
=&\frac{-(b-s)^{\overline{\alpha-1}}(\alpha-1)(a-\rho(s))^{\overline{\alpha-2}} + (a-s)^{\overline{\alpha-1}}(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)\left((a-s)(a-\rho(s))^{\overline{\alpha-2}}(b-\rho(s))^{\overline{\alpha-2}} - (b-s)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}\right)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(-b+s+a-s)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(a-b)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}.
\end{aligned}$$

Since $b > a$, it follows from Lemma 2.1 that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}} \right) < 0$. The proof is complete. ☐

Lemma 3.2. *There exits a number $\gamma \in (0, 1)$, such that*

$$\min_{t \in \mathbb{N}_c^d} G(t, s) \geq \gamma \max_{t \in \mathbb{N}_a^b} G(t, s) = \gamma G(s-1, s), \tag{3.1}$$

where $c, d \in \mathbb{N}_{a+1}^{b-1}$, such that $c = a + \left\lceil \frac{b-a+1}{4} \right\rceil$ and $d = a + 3 \left\lfloor \frac{b-a+1}{4} \right\rfloor$.

Proof. We make use Definition 2.4 and properties of Taylor monomials and Green's function from Lemma 2.1 and Theorem 2.2, respectively.

Consider, for $s \in \mathbb{N}_{a+2}^{b}$,

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s > t, \\ \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(t-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \le t. \end{cases}$$

Now, for $s > t$ and $c \le t \le d$, $G_1(t,s)$ is an increasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_1(t,s) = G_1(c,s) = \frac{(c-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)}.$$

For $t > s$ and $c \le t \le d$, $G_2(t,s)$ is a decreasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_2(t,s) = G_2(d,s) = \frac{(d-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)} - \frac{(d-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Thus,

$$\begin{aligned} \min_{t\in\mathbb{N}_c^d} G(t,s) &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{c}, \\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s \in \mathbb{N}_{c+1}^{d-1}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_{d}^{b}, \end{cases} \\ &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{r}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_{r}^{b}, \end{cases} \end{aligned}$$

where $c < r < d$. Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{a+2}^{r}, \\ \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{r}^{b}. \end{cases}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \ge \gamma(s) \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.2}$$

where

$$\gamma(s) = \min\left\{ \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}} \right\}.$$

For $s \in \mathbb{N}_r^b$, denote by

$$\gamma_1(s) = \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} \ge \frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}.$$

Similarly, for $s \in \mathbb{N}_{a+2}^{r}$, we take

$$\gamma_2(s) = \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}\right).$$

By Lemma 3.1, we see that $\frac{(d-s+1)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}$ is a decreasing function for $s \in \mathbb{N}_{a+2}^{r}$. Then

$$\begin{aligned}\gamma_2(s) &\geq \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right)\\ &> \frac{1}{(d-a)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right).\end{aligned}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.3}$$

where

$$\gamma = \min\left\{\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}\right\}. \tag{3.4}$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{a+2}^{b}$, implying that $\gamma > 0$. It would be suffice to prove that one of the terms $\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}$, $1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}$ is less than 1. It follows from Lemma 2.1 that

$$\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} < 1.$$

Therefore, we conclude that $\gamma \in (0,1)$. The proof is complete. □

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector of vector space $\mathbb{R}^{b-a+1}$. Denote by

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ is a Banach space equipped with the maximum norm, i.e.,

$$\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|.$$

Define the operator $T : \mathcal{B} \to \mathcal{B}$ by

$$(Tu)(t) = \sum_{s=a+2}^{b} G(t,s) f(u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.5}$$

Since T is defined on a discrete finite domain, it is trivially completely continuous. We also observe from (2.1) and (3.5), that u is a fixed point of T, if and only if u is a solution of (1.1).

Define the cone

$$K = \left\{u \in \mathcal{B} \mid u(t) \geq 0, \text{ for } t \in \mathbb{N}_a^b \text{ and } \min_{t\in\mathbb{N}_c^d} u(t) \geq \gamma\|u\|\right\}.$$

First, we show that $T : K \to K$. Let $u \in K$. Clearly, $(Tu)(t) \geq 0$, for $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned}\min_{t \in \mathbb{N}_c^d} (Tu)(t) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\ &\geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} (G(t,s)) f(u(s)) \geq \sum_{s=a+2}^{b} \gamma \max_{t \in \mathbb{N}_a^b} (G(t,s)) f(u(s)) \\ &\geq \gamma \max_{t \in \mathbb{N}_a^b} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\ &= \gamma \|Tu\|.\end{aligned}$$

Thus, we have $T : K \to K$. Take

$$D = \sum_{s=a+2}^{b} G(s-1, s). \tag{3.6}$$

We define the following sets

$$\begin{aligned}K_{c'} &= \{u \in K \mid \|u\| < c'\}, \\ K_{\alpha_2}(a', b') &= \{u \in K \mid a' \leq \alpha_2(u), \|u\| \leq b'\},\end{aligned}$$

where $\alpha_2 : K \to [0, +\infty)$ is a non-negative continuous concave functional. We state here the Leggett-Williams fixed-point theorem as follows. The proof of the same can be found in [33] and applications can be found in [3, 8]. Also, we would like to refer here to a paper by Kwong [30], which talks about the geometrical view of the Leggett-Williams fixed point theorem.

Theorem 3.1. *Let $T : \bar{K_{c'}} \to \bar{K_{c'}}$ be completely continuous and α_2 be a non-negative continuous concave functional on K, such that $\alpha_2(x) \leq \|u\|$, for all $u \in \bar{K_{c'}}$. Suppose there exist $0 < d' < a' < b' \leq c'$, such that*

(a) *$\{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\} \neq \emptyset$ and $\alpha_2(Tu) > a'$, for $u \in K_{\alpha_2}(a', b')$;*
(b) *$\|Tu\| < d'$, for $\|u\| \leq d'$;*
(c) *$\alpha_2(Tu) > a'$, for $u \in K_{\alpha_2}(a', c')$ with $\|Tu\| > b'$.*

Then, T has at least three fixed points u_1, u_2, u_3 satisfying

$$\begin{aligned}\|u_1\| &< d', \quad a' < \alpha_2(u_2), \\ \|u_3\| &> d' \quad \text{and} \quad \alpha_2(u_3) < a'.\end{aligned}$$

We introduce here the growth conditions on the non-linear function f, in line with [3].

Theorem 3.2. *Suppose there exist numbers a', b', d', where $0 < d' < a' < \gamma b' < b'$, such that f satisfies the following*

(a) *$f(u) > \frac{a'}{\gamma D}$, if $u \in [a', b']$;*
(b) *$f(u) < \frac{d'}{D}$, if $u \in [0, d']$;*

(c) *There exists* c' *such that* $c' > b'$ *and if* $u \in [0, c']$ *then* $f(u) < \frac{c'}{D}$.

Then, the boundary value problem (1.1) *has at least three positive solutions.*

Proof. Define a non-negative continuous concave functional $\alpha_2 : K \to [0, \infty)$ with $\alpha_2(u) \le \|u\|$, for all $u \in K$, by

$$\alpha_2(u) = \min_{t \in \mathbb{N}_c^d} u(t).$$

Claim 1. If there exists a positive number r such that $u \in [0, r]$ implies $f(u) < \frac{r}{D}$, then $T : K_r \to K_r$. Suppose that $u \in K_r$. Then,

$$\begin{aligned}\|Tu\| &= \max_{t \in \mathbb{N}_a^b} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\ &\le \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} [G(t,s)] \, f(u(s)) \\ &= \sum_{s=a+2}^{b} G(s-1,s) f(u(s)) \\ &< \frac{r}{D} \sum_{s=a+2}^{b} G(s-1,s) = r.\end{aligned}$$

Thus, $T : K_r \to K_r$. Hence, we have that if condition (c) holds, then there exists a number c' such that $c' > b'$ and $T : K_{c'} \to K_{c'}$. Note that with $r = d'$ and using condition (b), we get that $T : K_{d'} \to K_{d'}$.

Claim 2. $\{u \in K_{\alpha_2}(a', b') \mid \alpha_2(u) > a'\} \ne \emptyset$ and $\alpha_2(Tu) > a'$ for $u \in K_{\alpha_2}(a', b')$.

Since $u = \frac{a'+b'}{2} \in \{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\}$, it is non-empty. Let $u \in K_{\alpha_2}(a', b')$. By using condition (a), we have

$$\begin{aligned}\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\ &\ge \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} [G(t,s)] \, f(u(s)) \ge \gamma \sum_{s=a+2}^{b} G(s-1,s) f(u(s)) \\ &> a'.\end{aligned}$$

Thus, if $u \in K_{\alpha_2}(a', b')$, then $\alpha_2(Tu) > a'$.

Claim 3. If $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$, then $\alpha_2(Tu) > a'$. Suppose $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$. Then,

$$\begin{aligned}\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\ &\ge \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} (G(t,s)) \, f(u(s)) \ge \gamma \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} (G(t,s)) \, f(u(s))\end{aligned}$$

$$\geq \gamma \max_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right)$$
$$= \gamma \|Tu\|$$
$$> \gamma b' > a'.$$

Thus, $\alpha_2(Ax) > a'$. Hence, all the hypothesis of Theorem 3.1 are satisfied. Therefore, the boundary value problem (1.1) has at least three positive solutions. □

It has been observed that the flexibility of suitable choice of functionals over norms is the main advantage of Avery-type fixed-point theorems over Leggett–Williams fixed-point theorem [7, 30]. We define here the following subset of K for a positive number q:

$$K(\theta, q) = \{u \in K \mid \theta(u) < q\},$$

and the set $\partial K(\theta, q) = \{u \in K : \theta(u) = q\}$, where θ is a non-negative continuous functional on K.

The following is a twin fixed point theorem by Avery and Henderson [6].

Theorem 3.3. *Let K be a cone in a real Banach space $\mathcal{B}$. Let α_1 and γ_1 be increasing, non-negative continuous functionals on K. Let θ be a non-negative continuous functional on K with $\theta(0) = 0$ such that for some positive constants r and M,*

$$\alpha_1(u) \leq \theta(u) \leq \gamma_1(u) \quad \textit{and} \quad \|u\| \leq M\alpha_1(u),$$

for all $u \in \overline{K(\alpha_1, r)}$. Assume that there exist two positive numbers p and q with $p < q < r$, such that

$$\theta(ku) \leq k\theta(u), \quad \textit{for } 0 \leq k \leq 1 \textit{ and } u \in \partial K(\theta, q).$$

Suppose there exist a completely continuous operator $T : \overline{K(\alpha_1, r)} \to K$, satisfying

(a) *$\alpha_1(Tu) > r$, for all $u \in \partial K(\alpha_1, r)$;*
(b) *$\theta(Tu) < q$, for all $u \in \partial K(\theta, q)$;*
(c) *$K(\gamma_1, p) \neq \emptyset$ and $\gamma_1(Tu) > p$, for all $u \in \partial K(\gamma_1, p)$.*

Then, T has at least two fixed points u_1 and u_2 belonging to $\overline{K(\alpha_1, r)}$, such that

$$p < \gamma_1(u_1), \quad \textit{with } \theta(u_1) < q,$$

and

$$q < \theta(u_2), \quad \textit{with } \alpha_1(u_2) < r.$$

We introduce growth conditions on the non-linear function f here in line with [10]. Set $l = b - a + 1$.

Theorem 3.4. *Suppose that there exist positive constants p, q and r, such that $p < q < r$ and assume that function f satisfies the following conditions:*

(a) *$f(u) > \frac{r}{\gamma l G(s-1,s)}$, for all $u \in [r, \frac{r}{\gamma}]$;*
(b) *$f(u) < \frac{q}{l G(s-1,s)}$, for all $u \in [q, \frac{q}{\gamma}]$;*
(c) *$f(u) > \frac{p}{l G(s-1,s)}$, for all $u \in [\gamma p, p]$.*

Then, the operator T has at least two fixed points, u_1 and u_2, such that

$$p < \gamma_1(u_1), \quad \text{with } \theta(u_1) < q,$$

and

$$q < \theta(u_2), \quad \text{with } \alpha_1(u_2) < r.$$

Proof. We need to verify that the completely continuous operator T satisfies the hypothesis of Theorem 3.3. Denote by

$$\alpha_1(u) = \min_{t \in \mathbb{N}_c^d} u(t), \quad \theta(u) = \max_{t \in \mathbb{N}_c^d} u(t), \quad \gamma_1(u) = \|u\|.$$

For all $u \in K$, we have $\alpha_1(u) \leq \theta(u) \leq \gamma_1(u)$. Let $u \in K$. Then,

$$\alpha_1(u) = \min_{t \in \mathbb{N}_c^d} u(t) \geq \gamma \max_{t \in \mathbb{N}_a^b} u(t) = \gamma\gamma_1(u) = \gamma\|u\|.$$

Hence, for all $k \geq 0$ and $u \in K$, we have

$$\theta(ku) = \max_{t \in \mathbb{N}_c^d} (ku(t)) = k \max_{t \in \mathbb{N}_c^d} u(t) = k\theta(u).$$

Claim 1. If $u \in \partial K(\alpha_1, r)$, then $\alpha_1(Tu) > r$. Let $u \in \partial K(\alpha_1, r)$, i.e., $\min_{t \in \mathbb{N}_c^d} u(t) = r$. Then, $\alpha_1(u) = r \geq \gamma\|u\|$, implying that

$$r \leq \|u\| \leq \frac{r}{\gamma}, \quad \text{for } u \in \partial K(\alpha_1, r).$$

Using condition (a), we have

$$\begin{aligned}
\alpha_1(Tu) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\
&\geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} (G(t,s)) f(u(s)) \geq \gamma \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} (G(t,s)) f(u(s)) \\
&> \gamma \frac{r}{\gamma l G(s-1,s)} \max_{t \in \mathbb{N}_a^b} (G(t,s)) \, l \\
&= r.
\end{aligned}$$

Thus, condition (a) of Theorem 3.3 is satisfied.

Claim 2. If $u \in \partial K(\theta, q)$, then $\theta(Tu) < q$. Let $u \in \partial K(\theta, q)$, i.e., $\max_{t \in \mathbb{N}_c^d} u(t) = q$. We have

$$\theta(u) = q \geq \alpha_1(u) \geq \gamma\|u\| \quad \text{and} \quad \|u\| \geq \theta(u) = q,$$

implying that

$$q \leq \|u\| \leq \frac{q}{\gamma}, \quad \text{for } u \in \partial K(\theta, q).$$

Using condition (b), for $u \in \partial K(\theta, q)$, we have

$$\theta(Tu) = \max_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right)$$

$$\begin{aligned}
&\le \max_{t\in\mathbb{N}_a^b}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right) \le \sum_{s=a+2}^{b}\max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right)f(u(s))\\
&< \frac{q}{lG(s-1,s)}\max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right)l\\
&= q.
\end{aligned}$$

Thus, condition (b) of Theorem 3.3 is satisfied. Now, since $K(\gamma_1,p)=\{u\in K \mid \|u\|<p\}\neq\emptyset$, we observe that $p\ge\gamma_1(u)\ge\alpha_1(u)\ge\gamma p$, for $u\in\partial K(\gamma_1,p)$. Using condition (c), we have

$$\begin{aligned}
\gamma_1(Tu) &= \max_{t\in\mathbb{N}_a^b}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)\\
&\ge \sum_{s=a+2}^{b}\max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right)f(u(s))\\
&> \frac{p}{lG(s-1,s)}\max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right)l\\
&= p.
\end{aligned}$$

Thus, all the conditions of Theorem 3.3 are satisfied. Hence, T has at least two fixed points. The proof is complete. □

Conclusion

In the present article, we have established sufficient conditions for the existence of multiple positive solutions of the standard two-point non-linear nabla fractional boundary value problem with Dirichlet boundary conditions using fixed-point theorems such as Leggett–Williams and Avery–Henderson on a suitable constructed cone. To the best of our knowledge use of above conical shell fixed point theorem in nabla fractional calculus is unknown.

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A NOVEL SHIFTED JACOBI OPERATIONAL MATRIX METHOD FOR LINEAR MULTI-TERMS DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL VARIABLE-ORDER WITH PERIODIC AND ANTI-PERIODIC CONDITIONS

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Abstract. This paper investigates the generalized linear multi-terms delay fractional differential equation of variable order with periodic and anti-periodic conditions. In this work, a novel shifted Jacobi operational matrix technique is applied to solve a class of these equations, so that the original problem becomes a system of algebraic equations that can be solved by numerical methods. The proposed technique is successfully applied to the aforementioned problem. Sufficient and complete numerical tests are presented to demonstrate the accuracy, generality, efficiency of presented technique and the flexibility of this scheme. The numerical results of this method are compared with other existing methods such as fractional backward differential formulas ($FBDF$). Comparing the outcomes of these schemes, as well as comparing the current technique ($NSJOM$) with the exact solution, demonstrates the efficiency and validity of this method. It should be noted that the implementation of current method is considered very easy and general for many numerical techniques. Furthermore, the error and its bound are estimated.

1. Introduction

In the last three decades, analysis and applications of fractional calculus have been the fastest growing active area of research. Currently, it has become an important tool because of its vast applications in different scientific fields for example, physics, chemistry, blood circulation phenomena, electrodynamics, biophysics, capacitor theory,

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Complex environment, polymer rheology, experimental data fitting, dynamic systems, etc. (see [4–7, 11] and references therein). The increasing development of efficient and suitable methods with high accuracy to solve FDEs has caused the interest of many researchers to increase in this field. There are many important and popular methods for estimating of numerical solution of FDEs which can be implied to both linear and nonlinear FDEs, namely fractional linear multi-steps methods and convolution quadrature are presented by Lubich [12]. Galeone and Garrappa presented Fractional Adams-Molton methods for FDEs [13]. Trapezoidal methods to solve FDEs is proposed via Garrappa in [15]. The numerical solution to solve linear multi-term FDEs: systems of equations have presented by Edwards et al. [16]. Ford and Diethelm have suggested the multi-order FDE and their numerical solution in [17] and the numerical analysis for distributed-order DEs is given by these authors [18] and etc.

Incorporating the delay into FDEs creates new perspectives, especially in the field of bioengineering[10],because the realization of dynamics occurring in biological tissues is improved in bioengineering by fractional derivatives [8, 10].

In mathematical sciences, the DDEs are a kind of DEs in that the derivative of an unknown function at a definite time is presented in terms of the values of the function at prior times. The DDEs are also called time-delay systems, systems of deed-time or systems of aftereffect, differential-difference type equations, hereditary systems, deviating arguments equations [21].

Fractional DDEs differ from the ordinary type in which the derivative at any time depends on the solution (and when the equations are neutral then related to the derivative) at previous times. Many real-world happenings can be modeled as the $FDDE$s [11]. The $FDDE$s have many usages in different scientific areas by modeling different problems like electro dynamics, economy, biology, finance, control, physics, chemistry and etc. [21–27].

In the past years, numerical solution of the $FDDE$s analyzed and approximated by Margado et al. in [28]. Cermak et al. in [29] examined the stability areas of systems of $FDDE$s. Lazarovic and Spansic in [30] analyzed the stability for systems of $FDDE$s by means of Gr$\ddot{u}$nwalds approach. A New Predictor-Corrector method ($NPCM$) and new iteration technique have proposed in [31, 32], to numerically solve FDEs. A predictor-corrector method for solving nonlinear $FDDE$s in [14] have peresented via Bhalekar and Daftardar-Gejji. In [9], the algorithm of Adams-Bashforth-moulton which was peresented in [6, 20, 33], is proposed for solving the $FDDE$s. A new technique to solve nonlinear $FDDE$s have presented by Varsha et al. [10]. The Reproducing kernel Hilbert Space method to solve nonlinear $FDDE$s have employed via Ghasemi et al. [21]. In have [8] authors provided a new numerical method for solving FDDEs and Khodabandehlo et al. in [1–3] have proposed a $NSJOM$ technique for nonlinear variable-order $FDDE$s.

Furthermore, the spectral techniques that depend on an orthogonal polynomials set, are applied to solve the FDEs. The classical Jacobi polynomials are one of the most famous, which are as follows:

$$P_n^{(\alpha,\beta)}(t), \quad \beta > -1, \alpha > -1,\ n \geq 0.$$

These polynomials have been used widely in mathematical analysis and practical applications owning to they have the benefits of getting the numeric solutions in parameters β and α. Then, the systematic study of Jacobi polynomials with general indexes α and β will be useful and obviously, this case, in addition to extending the time interval $t \in [0, I]$, can be considered as one of the goals and novelties of this version [19]. Moreover, in recent years interest of researchers has increased in this area (area of variable FDEs) [34–38].

In this paper, generalize the orthogonal polynomials in the base of solution is the our goal. In fact, we present a $NSJOM$ method for the fractional derivatives to solve a class of linear multi-terms variable $FDDE$s with periodic condition which as follow:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z(t) \ + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = z_T, \end{aligned} \tag{1.1}$$

where $z_T = z(T)$. Also, the linear multi-terms variable $FDDE$s with anti-periodic condition is:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z(t) \ + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = -z_T, \end{aligned} \tag{1.2}$$

where $\beta_s \in \mathbb{R}$, $s = 1, 2, \ldots, n+1$, $\beta_{n+1} \neq 0$, $0 < T$ and D^{ζ_s}, $s = 1, 2, \ldots, n$, are the Caputo's derivative of variable-order fractional.

Note 1. If $\zeta_s(t)$, $s = 1, 2, \ldots, n$, are constants, then (1.1) and (1.2) will be as follow:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s} z(t) + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = z_T, \end{aligned}$$

and

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s} z(t) + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = -z_T. \end{aligned}$$

Also note that: we can use many polynomials such as Gegenbauer, Legendre, Fibonacci, all Chebyshev, Lucas, Vieta-Lucas polynomials, and etc. in our novel suggestion scheme.

The numerical outcomes gained for the mentioned equation in this paper reveal that the current technique has high efficiency and accuracy. By comparing numerical results getted via this technique with other available methods, and focusing on them, we find out that the suggested method capable of solving the variable-order $FDDE$, playing role of a powerful effective and practical numerical technique.

2. Fundamentals and Preliminaries

In this section, some of the mos basic fractional calculus theory properties will be mentioned. Then, some important features of Jacobi polynomials, that are relevant for the development of the proposed technique, will be presented [39, 42, 43].

Definition 2.1. The left and right-sided Caputo fractional derivatives of order ζ, $q-1<\zeta\leq q$, are determined as

$$D_{-}^{\zeta}z(t)=\frac{(-1)^{q}}{\Gamma(q-\zeta)}\int_{t}^{T}\frac{z'(s)}{(s-t)^{\zeta-q+1}}ds,$$

$$D_{+}^{\zeta}z(t)=\frac{1}{\Gamma(q-\zeta)}\int_{0}^{t}\frac{z'(s)}{(t-s)^{\zeta-q+1}}ds.$$

that

$$D_{+}^{\zeta}t^{m}=\begin{cases}0, & \text{for } m\in\mathbb{N}_0 \text{ and } m<\lceil\zeta\rceil,\\ \dfrac{\Gamma(m+1)}{\Gamma(m-\zeta+1)}t^{m-\zeta}, & \text{for } m\in\mathbb{N}_0 \text{ and } m>\lceil\zeta\rceil.\end{cases}$$

and

$$D_{-}^{\zeta}(T-t)^{m}=\begin{cases}0, & \text{for } m\in\mathbb{N}_0 \text{ and } m<\lceil\zeta\rceil,\\ \dfrac{(-1)^{m}\Gamma(m+1)}{\Gamma(m-\zeta+1)}(T-t)^{m-\eta}, & \text{for } m\in\mathbb{N}_0 \text{ and } m>\lceil\zeta\rceil.\end{cases}$$

where $\lceil\cdot\rceil$ is the ceiling function and $N_0=\{0,1,2,\dots\}$. And for constants δ and γ, we will have $D_{\pm}^{\zeta}(\delta\psi(t)\ +\gamma\eta(t))=\delta D_{\pm}^{\zeta}(\psi(t))+\gamma D_{\pm}^{\zeta}(\eta(t))$.

Definition 2.2. The Caputo derivative with fractional variable-order $\zeta(t)$ for $z(t)\in C^{m}[0,T]$ is as follows [35, 40]:

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(s)}{(t-s)^{\zeta(t)}}ds+\frac{z(0^+)-z(0^-)}{\Gamma(1-\zeta(t))}t^{-\zeta(t)}. \tag{2.1}$$

At the starting point and for $0<\zeta(t)<1$, we have:

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(s)}{(t-s)^{\zeta(t)}}ds.$$

Also, for constants a and b we have $D^{\zeta(t)}(a\,z_1(t)+b\,z_2(t))=a\,D^{\zeta(t)}z_1(t)+b\,D^{\zeta(t)}z_2(t)$. Using (2.1), then: $D^{\zeta(t)}K=0$, K is a constant.

On the other hand

$$D^{\zeta(t)}t^m = \begin{cases} 0, & \text{for } m = 0, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\zeta(t))}t^{m-\zeta(t)}, & \text{for } m = 1, 2, \dots \end{cases} \tag{2.2}$$

2.1. **Shifted Jacobi polynomials and their properties.** Suppose $P_n^{(\alpha,\beta)}(s)$, $\beta > -1$, $\alpha > -1$ as the n-th degree Jacobi orthogonal polynomial in $-1 \leq s \leq 1$.

As any classical orthogonal polynomials, $P_n^{(\alpha,\beta)}(s)$ form an orthogonal system with respect to weight function $\omega^{(\alpha,\beta)}(s) = (1-s)^\alpha(1+s)^\beta$, namely [39]:

$$\int_{-1}^{1} P_\ell^{(\alpha,\beta)}(s)P_k^{\alpha,\beta)}(s)\omega^{(\alpha,\beta)}ds = h_k^{(\alpha,\beta)}\delta_{\ell,k},$$

where

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\beta+\alpha+1)},$$

$\delta_{\ell,k}$ is the Kronecker function and

$$P_\ell^{(\alpha,\beta)}(s) = \sum_{j=0}^{\ell} \frac{\Gamma(\alpha+\ell+1)\Gamma(\alpha+\ell+1+\beta+j)}{\Gamma(\alpha+\beta+\ell+1)\Gamma(\alpha+1+j)\Gamma(j+1)\Gamma(\ell-j+1)} \left(\frac{s-1}{2}\right)^j, \tag{2.3}$$

is the analytical form of the ℓ-th order Jacobi polynomial [19]. The polynomials given in (2.3) can be obtained as follow:

$$y_{1,\ell}^{\alpha,\beta}P_\ell^{(\alpha,\beta)}(s) = y_{2,\ell}^{\alpha,\beta}P_{\ell-1}^{(\alpha,\beta)}(s) - y_{3,\ell}^{\alpha,\beta}P_{\ell-2}^{(\alpha,\beta)}(s), \quad \ell = 2, 3, \dots,$$

where

$$\begin{aligned} y_{1,\ell}^{\alpha,\beta} &= 2l(\alpha+\ell+\beta)(\alpha+2l-2+\beta), \\ y_{2,\ell}^{\alpha,\beta} &= (\alpha+2l-1+\beta)(\alpha^2-\beta^2+(\alpha+2l+\beta)(\alpha+2l+\beta-2)s), \\ y_{3,\ell}^{\alpha,\beta} &= 2(\alpha+\ell-1)(\beta+\ell-1)(\alpha+2l+\beta). \end{aligned}$$

That start values as follow

$$P_0^{(\alpha,\beta)}(s) = 1 \quad \text{and} \quad P_1^{(\alpha,\beta)}(s) = \frac{1}{2}[(\alpha+\beta+2)s + (\alpha-\beta)].$$

In order to use the polynomial of (2.3) on the interval $0 \leq t \leq T$, we need to change the variable $s = \left(\frac{2t}{T} - 1\right)$. Therefore, the shifted Jacobi orthogonal polynomials $P_j^{(\alpha,\beta)}\left(\frac{2t}{T} - 1\right)$ which marked by $P_{T,j}^{(\alpha,\beta)}(t)$ will be constructed . Then $P_{T,j}^{(\alpha,\beta)}(t)$ form an orthogonal system with $\omega_T^{(\alpha,\beta)}(t) = t^\beta(T-t)^\alpha$ as the weight function for $0 \leq t \leq T$ with the following orthogonal feature:

$$\int_0^T P_{T,\ell}^{(\alpha,\beta)}(t)P_{T,k}^{(\alpha,\beta)}(t)\omega_T^{(\alpha,\beta)}dt = h_{T,k}^{(\alpha,\beta)}\delta_{\ell,k},$$

where

$$h_{T,k}^{(\alpha,\beta)} = \left(\frac{T}{2}\right)^{\alpha+\beta+1} h_k^{(\alpha,\beta)}.$$

Also,

$$P_{T,\ell}^{(\alpha,\beta)}(t)=\sum_{j=0}^{\ell}(-1)^{\ell+j}\frac{\Gamma(\beta+1+\ell)\Gamma(\alpha+j+\ell+1+\beta)}{\Gamma(\alpha+\ell+1+\beta)\Gamma(\beta+1+j)\Gamma(j+1)\Gamma(\ell-j+1)T^j}t^j$$
$$=\sum_{j=0}^{\ell}\frac{\Gamma(\ell+1+\alpha)\Gamma(\alpha+j+\ell+\beta+1)}{\Gamma(\alpha+\ell+1+\beta)\Gamma(\alpha+1+j)\Gamma(j+1)\Gamma(\ell-j+1)T^j}(T-t)^j$$

is the analytical form of the ℓ-th order Shifted Jacobi polynomial [19] and we have

$$P_{T,\ell}^{(\alpha,\beta)}(0)=(-1)^{\ell}\frac{\Gamma(\beta+\ell+1)}{\Gamma(\beta+1)\Gamma(j+1)},$$
$$P_{T,\ell}^{(\alpha,\beta)}(T)=\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\Gamma(j+1)},$$

in the endpoint values.

Note 2. The Jacobi's shifted orthogonal polynomials constitute infinite number of orthogonal polynomials such as the shifted Chebyshev polynomials of the first, second, third and fourth kinds $T_{T,\ell}(t)$, $U_{T,\ell}(t)$, $V_{T,\ell}(t)$ and $W_{T,\ell}(t)$, respectively; the shifted Gegenbauer polynomials $G_{T,\ell}^{(\alpha,\beta)}(t)$ and the shifted Legendre polynomials $\ell_{T,\ell}(t)$. These polynomials, which are all orthogonal, are related to $P_{T,\ell}^{(\alpha,\beta)}(t)$ as follow:

$$\ell_{T,\ell}(t)=P_{T,\ell}^{(0,0)}(t),$$
$$G_{T,\ell}^{(\alpha,\beta)}(t)=\frac{\Gamma(\ell+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}+\ell\right)}P_{T,\ell}^{\left(\alpha-\frac{1}{2},\beta-\frac{1}{2}\right)}(t),$$
$$T_{T,\ell}(t)=\frac{\Gamma(\ell+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\ell\right)}P_{T,\ell}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t),$$
$$U_{T,\ell}(t)=\frac{\Gamma(\ell+2)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}+\ell\right)}P_{T,\ell}^{\left(\frac{1}{2},\frac{1}{2}\right)}(t),$$
$$V_{T,\ell}(t)=\frac{2^{2l}(\Gamma(\ell+1))^2}{\Gamma(2l+1)}P_{T,\ell}^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t),$$
$$W_{T,\ell}(t)=\frac{2^{2l}(\Gamma(\ell+1))^2}{\Gamma(2l+1)}P_{T,\ell}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t).$$

3. Function Approximation by Shifted Jacobi Polynomials

Consider the function $z(t)$ to be square integrable with respect to $\omega_T^{(\alpha,\beta)}(t)$ in $[0,T]$, then, we have (see [19, 39]):

$$z(t)=\sum_{j=0}^{\infty}a_jP_{T,j}^{(\alpha,\beta)}(t),\tag{3.1}$$

that the coefficients of the series (a_j) are gained by

$$a_j = \frac{1}{h_{T,k}^{(\alpha,\beta)}} \int_0^T \omega_T^{(\alpha,\beta)} P_{T,j}^{(\alpha,\beta)}(t) z(t) dt, \quad j = 0, 1, \dots$$

So, we will obtain the approximate solution by finite number of terms from the series in (3.1), then

$$z(t) \simeq z_M(t) = \sum_{j=0}^{M} a_j P_{T,j}^{(\alpha,\beta)}(t) = A^T \Phi_{T,M}(t), \tag{3.2}$$

where $A = [a_0, a_1, \dots, a_M]^T$ and $\Phi_{T,M}(t) = [P_{T,0}^{(\alpha,\beta)}(t), P_{T,1}^{(\alpha,\beta)}(t), \dots, P_{T,M}^{(\alpha,\beta)}(t)]^T$.

We consider that

$$S(t) = [1, t, t^2, t^3, \dots, t^M]^T.$$

By (3.2), the vector $\Phi_{T,M}(t)$ can be shown as

$$\Phi_{T,M}(t) = R_{(\alpha,\beta)} S(t), \tag{3.3}$$

that $R_{(\alpha,\beta)}$ is a square matrix of order $(M+1) \times (M+1)$, as follows

$$r_{\ell+1,k+1} = \begin{cases} (-1)^{\ell-k} \dfrac{(\alpha+\ell)!(\alpha+\beta+k+\ell)!}{(\alpha+\beta+\ell)!(\alpha+k)!(k!)(\ell-k)!T^k}, & \ell \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

for $0 \leq \ell,\, k \leq M$.

Let $M = 4$, $\alpha = \beta = 0$, then

$$R_{(0,0)} = \frac{1}{T^i} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 \\ 1 & -20 & 90 & -140 & 70 \end{bmatrix}.$$

Hence, using (3.3), we get

$$S(t) = R_{(\alpha,\beta)}^{-1} \Phi_{T,M}(t). \tag{3.4}$$

Note 3. We can caculate this matrix $R_{(\alpha,\beta)}$ for other orthogonal polynomials as well. For instance, let $M = 4$, $\beta = \frac{-1}{2}$, $\alpha = \frac{1}{2}$, then the orthogonal polynomials will be of the fourth kind shifted Chebyshev type, hence $R_{(\alpha,\beta)}$ of order 4×4 for these polynomials as follows

$$R_{(\frac{1}{2},\frac{-1}{2})} = \frac{1}{T^i} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 1 & -12 & 16 & 0 & 0 \\ -1 & 24 & -80 & 64 & 0 \\ 1 & -40 & 240 & -448 & 256 \end{bmatrix}.$$

4. Novel Shifted Jacobi Polynomials Operational Matrix($NSJOM$)

Operational matrix, which are applied in different areas of numerical analysis and to solve problems of different types and topics are of especial importance such as integral equations, DEs, integro-DEs, ordinary and partial FDEs [36, 39–41, 44–52]. In this part, we investigate the ($SJOM$) of fractional variable-order to support the numerical solution of (1.1), (1.2). Therefore, we convert the problem into the system of algebraic of equations which is solved numerically in collocation points.

At first, we deduce $D^{\zeta_\ell(t)}\Phi_{T,M}(t)$, $\ell = 1, 2, \ldots, n$, as follow.

According to the previous content, we have: $\Phi_{T,M}(t) = R_{(\alpha,\beta)}S(t)$, thus

(4.1)
$$D^{\zeta_\ell(t)}\Phi_{T,M}(t) = D^{\zeta_\ell(t)}(R_{(\alpha,\beta)}S(t)) = R_{(\alpha,\beta)}D^{\zeta_\ell(t)}[1, t, \ldots, t^M]^T, \quad \ell = 1, 2, \ldots, n.$$

Combining (2.2) and (4.1), gives:

$$\begin{aligned}
D^{\zeta_\ell(t)}\Phi_{T,M}(t) &= R_{(\alpha,\beta)}D^{\zeta_\ell(t)}(S(t)) \\
&= R_{(\alpha,\beta)}\left[0, \frac{\Gamma(2)t^{(1-\zeta_\ell(t))}}{\Gamma(2-\zeta_\ell(t))}, \cdots, \frac{\Gamma(M+1)t^{(M-\zeta_\ell(t))}}{\Gamma(M+1-\zeta_\ell(t))}\right]^T \\
&= R_{(\alpha,\beta)}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_\ell(t)}}{\Gamma(2-\zeta_\ell(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_\ell(t)}}{\Gamma(3-\zeta_\ell(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(M)t^{-\zeta_\ell(t)}}{\Gamma(M+1-\zeta_\ell(t))} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^M \end{bmatrix} \\
&= R_{(\alpha,\beta)}G_\ell(t)S(t), \quad \ell = 1, 2, \ldots, n,
\end{aligned}$$

where

$$G_\ell(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_\ell(t)}}{\Gamma(2-\zeta_\ell(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_\ell(t)}}{\Gamma(3-\zeta_\ell(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(M)t^{-\zeta_\ell(t)}}{\Gamma(M+1-\zeta_\ell(t))} \end{bmatrix}, \quad \ell = 1, 2, \ldots, n.$$

Using (3.4), we have

$$D^{\zeta_\ell(t)}\Phi_{T,M}(t) = R_{(\alpha,\beta)}G_\ell(t)R^{-1}_{(\alpha,\beta)}\Phi_{T,M}(t), \quad \ell = 1, 2, \ldots, n.$$

The operational matrix of $D^{\zeta_\ell(t)}\Phi_{T,M}(t)$, $\ell = 1, 2, \ldots, n$, is $R_{(\alpha,\beta)}G_\ell(t)R^{-1}_{(\alpha,\beta)}$.

Here, we estimate the variable-order fractional of the calculated function that obtained in (3.2) as follows

$$\begin{aligned} D^{\zeta_\ell(t)}z(t) \simeq D^{\zeta_\ell(t)}(A^T\Phi_{T,M}(t)) =& A^T D^{\zeta_\ell(t)}\Phi_{T,M}(t) \\ =& A^T R_{(\alpha,\beta)}G_\ell(t)R^{-1}_{(\alpha,\beta)}\Phi_{T,M}(t), \quad \ell = 1,2,\dots,n. \end{aligned} \tag{4.2}$$

By using (4.2), hence (1.1) turned into

$$\sum_{s=1}^{n} \beta_s(A^T R_{(\alpha,\beta)}G_s(t)R^{-1}_{(\alpha,\beta)}\Phi_{T,M}(t)) + \beta_{n+1}A^T\Phi_{T,M}(t-\tau) = f(t), \quad t \in [0,T], \tag{4.3}$$

with periodic condition

$$A^T\Phi_{T,M}(0) = A^T\Phi_{T,M}(T)$$

or anti-periodic condition as

$$A^T\Phi_{T,M}(0) = -A^T\Phi_{T,M}(T).$$

Finally, we use t_k, $k = 0,1,2,\dots,m$, where they are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$. Therefore, (4.3) converted into the following form

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s(A^T R_{(\alpha,\beta)}G_s(t_\ell)R^{-1}_{(\alpha,\beta)}\Phi_{T,M}(t_\ell)) + \alpha_{n+1}A^T\Phi_{T,M}(t_\ell-\tau) = f(t_\ell), \\ &\ell = 0,1,2,\dots,m. \end{aligned} \tag{4.4}$$

So, we can solve the system in (4.4) with the conditions mentioned numerically for determining the vector A. Therefore, the numerical solution that presented in (3.2) can be obtained.

5. Error Analysis

In this part, for estimating an upper bound for the absolute error, the Lagrange interpolation polynomials is used. Namely, by using the current technique ($NSJOM$) with error approximation and the residual correction method [53,54], an effective error estimation will be gained for the variable-order FDEs.

5.1. **Error bound.** Now, let $z(t)$ on $[0,T]$ be the smooth function and suppose that $z_M(t) \in \prod_M^{\alpha,\beta}$ is the best approximation for it. Our aim is to obtain an analytical form of norm of error for $z_M(t)$ by developing it into Jacobi polynomials. assume

$$\prod_M^{\alpha,\beta} = \text{Span}\left\{P^{(\alpha,\beta)}_{T,i}(t), i = 0,1,2,\dots,M\right\}.$$

According to concept and definition of the best approximation, we can write

$$\text{for all } v_M(t) \in \prod_M^{\alpha,\beta} \|z(t) - z_M(t)\|_\infty \le \|z(t) - v_M(t)\|_\infty.$$

Let $v_M(t)$ be the interpolating polynomials at node points t_i, $i = 0, 1, \ldots, m$ (where t_i are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$). It is clear that $v_M(t)$ satisfies in the above inequality. Then by the Lagrange interpolation polynomials formula and its error formula, have

$$z(t) - v_M(t) = \frac{z^{(M+1)}(\xi)}{(M+1)!} \prod_{j=0}^{M} (t - t_j),$$

that $0 < \xi < T$, and

$$\|z(t) - v_M(t)\|_\infty \leq \max_{0 \leq t \leq T} |z^{(M+1)}(\xi)| \frac{\| \prod_{j=0}^{M} (t - t_j)\|_\infty}{(M+1)!}.$$

Note that $z(t)$ on $[0, T]$ is smooth, therefore, there is a constant C_1, as

$$\max_{0 \leq t \leq T} |z^{(M+1)}(\xi)| \leq C_1. \tag{5.1}$$

We want to minimize the factor $\| \prod_{j=0}^{M} (t - t_j)\|_\infty$ as follows

One-to-one mapping $t = \frac{T}{2}(w + 1)$ between the interval $[-1, 1]$ and $[0, T]$ is used to deduce that [39, 55]

$$\begin{aligned} \min_{0 \leq t_i \leq T} \max_{0 \leq t \leq T} |\prod_{i=0}^{M} (t - t_i)| &= \min_{-1 \leq w_i \leq 1} \max_{-1 \leq w \leq 1} \left| \prod_{i=0}^{M} \frac{T}{2}(w - w_i) \right| \\ &= \left(\frac{T}{2}\right)^{M+1} \min_{-1 \leq w_i \leq 1} \max_{-1 \leq w \leq 1} \left| \prod_{i=0}^{M} (w - w_i) \right| \\ &= \left(\frac{T}{2}\right)^{M+1} \min_{-1 \leq w_i \leq 1} \max_{-1 \leq w \leq 1} \left| \frac{P^{(\alpha,\beta)}_{M+1}(w)}{\mu^{(\alpha,\beta)}_M} \right|, \end{aligned} \tag{5.2}$$

where $\mu^{(\alpha,\beta)}_M = \frac{\Gamma(2M+\alpha+\beta+1)}{2^M M! \Gamma(M+\alpha+\beta+1)}$ is the last factor of $P^{(\alpha,\beta)}_{M+1}(w)$ and w_j are the roots of $P^{(\alpha,\beta)}_{M+1}(w)$. It is clear that

$$\max_{-1 \leq w \leq 1} |P^{(\alpha,\beta)}_{M+1}(w)| = P^{(\alpha,\beta)}_{M+1}(1) = \frac{\Gamma(\beta + M + 2)}{\Gamma(\beta + 1)(M + 1)!}.$$

Using (5.1) and (5.2), gives the following result

$$\|z(t) - z_M(t)\|_\infty \leq C_1 \frac{\left(\frac{T}{2}\right)^{M+1} \Gamma(\beta + M + 2)}{\mu^{(\alpha,\beta)}_M ((M+1)!)^2 \Gamma(\beta + 1)}.$$

Therefore, an upper bound for absolute error between the exact and approximate solutions was stimated.

5.2. **Error function estimation.** In this subsection, we have introduced the error approximation based on the error function of residual of the proposed scheme and the approximate solution (3.2) is refined by the residual correction technique. The error approxmation of residual was used to ontain the error of some methods for different equations [41, 54, 56, 57].

At first, we mark $e_M(t) = z_M(t) - z(t)$ be the error function for the $NSJOM$ approximation $z_M(t)$ to $z(t)$, that $z(t)$ is the truaccurate solution of (1.1) or (1.2).

Therefore, $z_M(t)$ satisfies the following relation

$$\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z_M(t) + \beta_{n+1} z_M(t-\tau) = f(t) + R_M(t), \quad 0 \le t \le T, \tag{5.3}$$

with periodic condition as

$$z_M(0) = z_M(T)$$

or anti-periodic condition as

$$z_M(0) = -z_M(T),$$

where $R_M(t)$ is the residual function of (1.1) or (1.2), which is approximated by replacing the $z_M(t)$ with $z(t)$ in (1.1) or (1.2). By subtract (1.1) or (1.2) from (5.3), the error problem is constructed in the form of

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} \mathbf{e}_M(t) + \beta_{n+1} \mathbf{e}_M(t-\tau) = R_M(t), \quad 0 \le t \le T, \\ &\mathbf{e}_M(0) = \mathbf{e}_M(T) \quad \text{or} \quad \mathbf{e}_M(0) = -\mathbf{e}_M(T), \end{aligned} \tag{5.4}$$

Thus, the (5.4) can be solved like the way it was presented in the previous section and we obtain the following estimation to $\mathbf{e}_M(t)$

$$\mathbf{e}_M(t) = \sum_{s=0}^{M} d_s P_{T,i}^{(\alpha,\beta)}(t) = D^T \Phi_{T,M}(t),$$

Note that if the accurate solution of the problem (1.1) or (1.2) is unknown, then we can gain the estimation of maximum amount of absolute errors by

$$\mathbf{E}_M(t) = \max\{\mathbf{e}_M(t), 0 \le t \le T\}.$$

The above estimation of error, is influenced by the rate of expansions convergence in Jacobi polynomials. Thus, the rates of convergence in temporal discretizations, are provided by it [39, 57].

6. Numerical Experiences

In this section, several numerical examples are presented to demonstrate the applicability, efficiency, accuracy, generality of this scheme. We obtain the outcomes of the current method by Mathematica 10 software. To test our technique, we have compared in terms of absolute errors of exact solution with current method and fractional backward differential formulas ($FBDF$) which defined as: $|z_{exact}(t) - z_n(t)|$.

Gathering of the outcomes obtained via this method with the true solution of each example displays that our scheme is in the best agreement compared to other methods. As this method is easy to implement, consistent and stable, it is therefore more reliable and applicable.

Example 6.1 ([11]). Consider the below $FDDE$ for $0 < \zeta \leq 1$

$$D^{\zeta} z(t) + z(t-\tau) = \frac{\Gamma(3) z(t)^{2-\zeta}}{\Gamma(3-\zeta)} - \frac{\Gamma(2) z(t)^{1-\zeta}}{\Gamma(2-\zeta)} + (t-\tau)^2 - (t-\tau) - 1, \tag{6.1}$$

$$z(t) = t^2 - t - 1, \quad t \in [-\tau, 0],$$

$$z(0) = -z(T).$$

With anti-periodic condition. The true solution is $z(t) = t^2 - t - 1$ and $T = 2$, $0 \leq t \leq T$, $\tau = 1$, $\zeta = 0.2$.

According to the presented concepts, we approximate the solution of this example and observe that results of this scheme are in the best agreement with the accurate solution compared to method ($FBDF$). From table 1, where the absolute errors (at $t = 1$) of the exact solution with our scheme and method in [11] are recorded, we find that the numerical results which getted by our method are very close to the exact solution and we achieved an excellent estimation for the true solution by using current technique. In figure 1 compared the exact and calculated solution which acknowledges the utility, accuracy and validity of $NSJOM$ scheme. Furthermore, in figure 2 the absolute error of exact solution with our scheme for this instance has been drawn. In this instance for $M = 2$ and $M = 4$, we have $A = [-0.66667, 1, +0.66667]^T$, $A = [-0.66667, 1, +0.66667, -4.82688 \times 10^{-16}, -1.16563 \times 10^{-16}]^T$, respectively.

TABLE 1. Comparison of absolute error of true solution with scheme in [11] and current method with $\beta = 0$, $\alpha = 0$ at $t = 1.0$. for Example 6.1

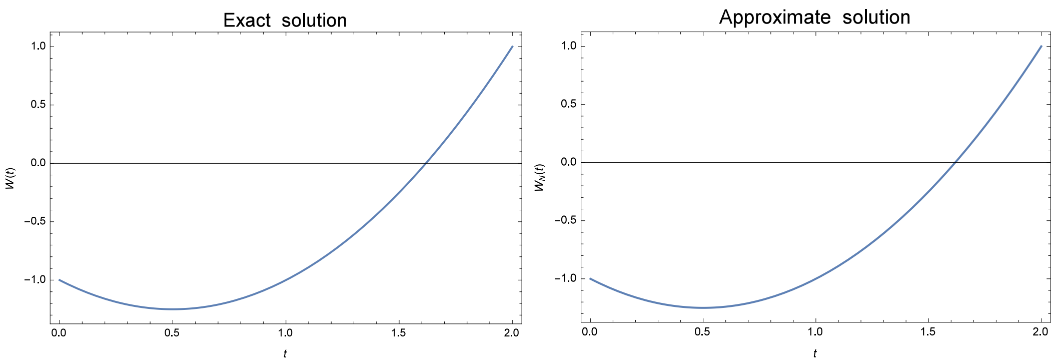
Current method	$M = 2$	4.44089×10^{-16}
	$M = 4$	6.66134×10^{-16}
Scheme in [11]	$M = 20$	1.73817×10^{-2}
	$M = 40$	1.02509×10^{-2}
	$M = 200$	2.87569×10^{-3}
	$M = 400$	1.65502×10^{-3}
	$M = 2000$	4.57442×10^{-4}
	$M = 4000$	2.62785×10^{-4}
	$M = 20000$	7.25263×10^{-5}

Example 6.2 ([11]). Consider the below $FDDE$ for $0 < \zeta_1 < \zeta_2 \leq 1$, with periodic condition

(6.2)

$$D^{\zeta_2} z(t) + D^{\zeta_1} z(t) + z(t-\tau)$$

$$= \frac{\Gamma(3) z(t)^{2-\zeta_2}}{\Gamma(3-\zeta_2)} - \frac{\Gamma(2) z(t)^{1-\zeta_2}}{\Gamma(2-\zeta_2)} + \frac{\Gamma(3) z(t)^{2-\zeta_1}}{\Gamma(3-\zeta_1)} - \frac{\Gamma(2) z(t)^{1-\zeta_1}}{\Gamma(2-\zeta_1)} + (t-\tau)^2 - (t-\tau),$$



(A) Accurate solution (B) Estimate solution

FIGURE 1. Comparison of accurate and estimate solution (z_2) of *NSJOM* scheme for Example 6.1

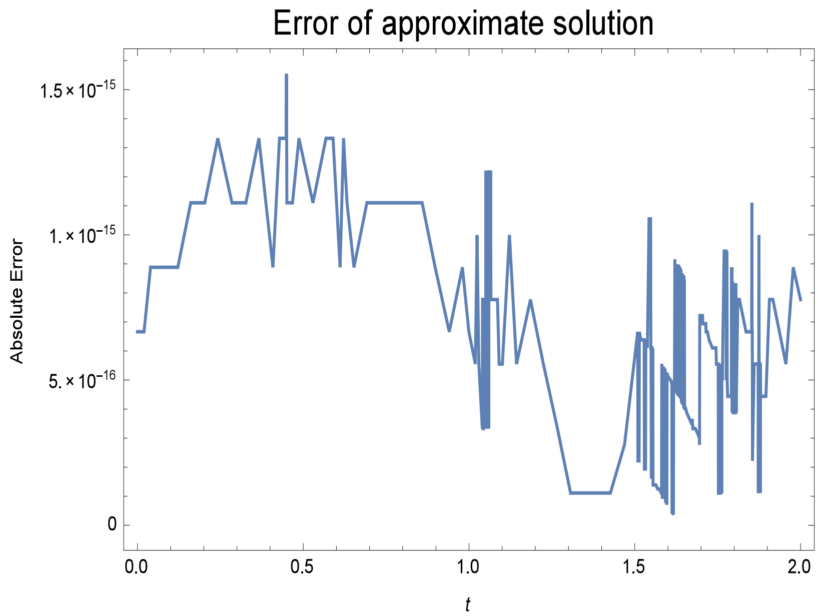


FIGURE 2. The absolute error between true and estimate solution (z_2) for Example 6.1

$$z(t) = t^2 - t, \quad t \in [-\tau, 0],$$
$$z(0) = z(T).$$

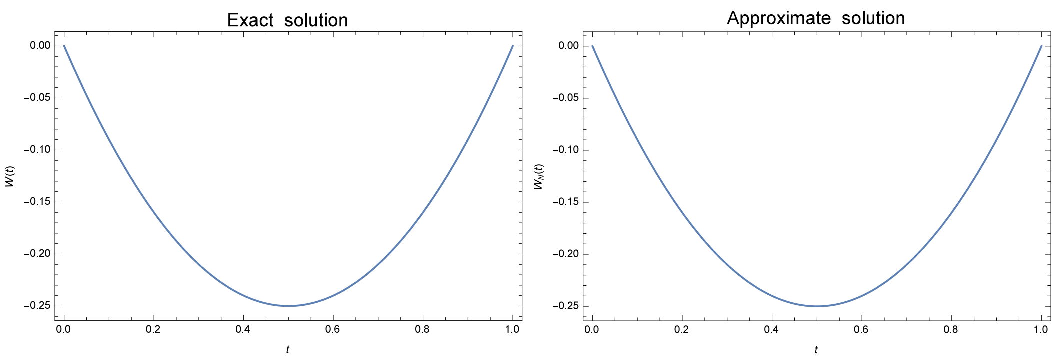
In this problem the true solution is $z(t) = t^2 - t$ and $T = 1$, $0 \le t \le T$, $\zeta_1 = 0.3$, $\zeta_2 = 0.4$, $\tau = 1$.

Using the process mentioned in Example 6.1, we get the solution of this example and compare the obtained results with *FBDF* scheme. The outcomes show that our method is much better than the mentioned method. In table 2, the absolute errors of

TABLE 2. Comparison of absolute error of true solution with method in [11] and our scheme with $\beta = 0$, $\alpha = 0$, at $t = 1.0$. for Example 6.2

Current method	$M = 2$	2.22045×10^{-16}
	$M = 4$	1.62195×10^{-15}
Scheme in [11]	$M = 20$	1.55922×10^{-3}
	$M = 40$	5.10934×10^{-4}
	$M = 200$	3.67922×10^{-5}
	$M = 400$	1.18622×10^{-5}
	$M = 2000$	8.59612×10^{-7}
	$M = 4000$	2.78025×10^{-7}

our technique and scheme in [11] (at $t = 1$) are given and compared. In figure 3 the true and caculated solution are compared and in figure 4 the absolute error of exact solution with our scheme for this instance has been shown. Not that these figures and Tables show a good agreement between accurate and approximate solution. In this problem for $M = 2$ and $M = 4$, we have $A = [-0.166667, -6.78159 \times 10^{-18}, +0.166667]^T$, $A = [-0.166667, -6.78159 \times 10^{-18}, +0.166667, -2.40920 \times 10^{-16}, -6.46958 \times 10^{-17}]^T$, respectively.



(A) Accurate solution (B) Estimate solution

FIGURE 3. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.2

Example 6.3. Consider the variable-order $FDDE$ with anti-periodic condition

$$D^{\zeta(t)}z(t) + z(t-\tau) = \frac{\Gamma(3)z(t)^{2-\zeta(t)}}{\Gamma(3-\zeta(t))} - \frac{\Gamma(2)z(t)^{1-\zeta(t)}}{\Gamma(2-\zeta(t))} + (t-\tau)^2 - (t-\tau) - 1, \tag{6.3}$$
$$z(t) = t^2 - t - 1, \quad t \in [-\tau, 0],$$
$$z(0) = -z(T).$$

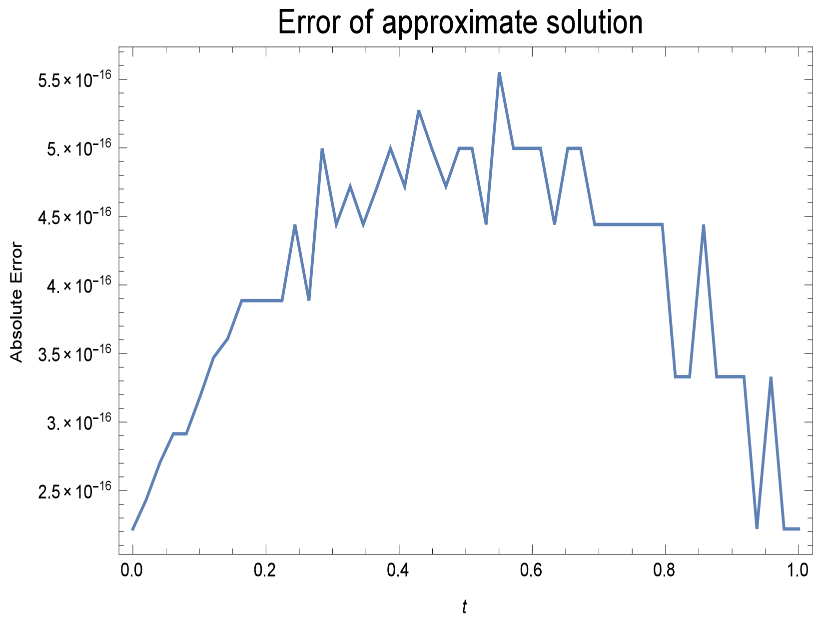


FIGURE 4. The absolute error between true and estimate solution (z_2) for Example 6.2

TABLE 3. Absolute errors of true solution and our method $(z_M(t))$ with $\beta = 0$, $\alpha = 0$ and $T = 2$ for Example 6.2

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	0	0
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0
CPU time	$0.171601s$	$3.19802s$

True solution is $z(t) = t^2 - t - 1$ and $T = 2$, $0 \leq t \leq T$, $\tau = 1$, $\zeta(t) = \frac{t}{7}$.

In this example, we estimated the solution of (6.3) for several values of α and β, by our $NSJOM$ scheme and recorded the needed consumption time (CPU time), the results related to the relative and absolute errors of this estimated solution with the exact solution in tables 3–8. Moreover, we show the absolute error with $\alpha = 1$, $\beta = 1$

TABLE 4. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0

TABLE 5. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	0	0
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0
CPU time	$0.156001s$	$9.001258s$

in Figure 6 and relative errors for various value of α and β in figures 7–9 for this instance. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 2$, have $A = [-0.66667, 1, +0.66667]^T$;
- For $\alpha = 0$, $\beta = 0$ and $M = 3$, have $A = [-0.66667, 1, +0.66667, 0]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 2$, have $A = [-0.75, 0.66667, 0.4]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 3$, have $A = [-0.75, 0.66667, 0.4, 1.94241 \times 10^{-16}]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 2$, have $A = [-0.8, 0.5, 0.26667]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 3$, have $A = [-0.8, 0.5, 0.26667, 0]^T$.

TABLE 6. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0

TABLE 7. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	4.440×10^{-16}	1.776×10^{-15}
0.2	4.440×10^{-16}	1.554×10^{-15}
0.4	2.220×10^{-16}	1.110×10^{-15}
0.6	2.220×10^{-16}	6.661×10^{-16}
0.8	2.220×10^{-16}	2.220×10^{-16}
1.0	2.220×10^{-16}	4.440×10^{-16}
1.2	0	8.881×10^{-16}
1.4	5.551×10^{-17}	1.276×10^{-15}
1.6	4.093×10^{-16}	1.366×10^{-15}
1.8	1.276×10^{-15}	1.387×10^{-15}
2.0	8.881×10^{-16}	1.776×10^{-15}
CPU time	$0s$	$0s$

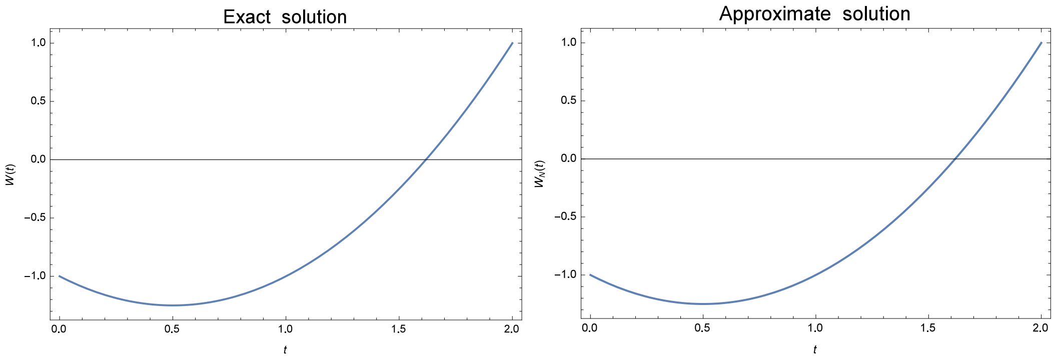
Example 6.4. Consider the following variable-order *FDDE*

(6.4)

$$
\begin{aligned}
&D^{\zeta_2} z(t) + D^{\zeta_1} z(t) + z(t-\tau) \\
=&\frac{\Gamma(3) z(t)^{2-\zeta_2}}{\Gamma(3-\zeta_2)} - \frac{\Gamma(2) z(t)^{1-\zeta_2}}{\Gamma(2-\zeta_2)} + \frac{\Gamma(3) z(t)^{2-\zeta_1}}{\Gamma(3-\zeta_1)} - \frac{\Gamma(2) z(t)^{1-\zeta_1}}{\Gamma(2-\zeta_1)} + (t-\tau)^2 - (t-\tau), \\
&z(t) = t^2 - t, \quad t \in [-\tau, 0], \\
&z(0) = z(T).
\end{aligned}
$$

TABLE 8. Relative errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	8.330×10^{-16}	1.776×10^{-15}
0.4	9.362×10^{-16}	2.004×10^{-15}
0.6	9.221×10^{-16}	2.551×10^{-15}
0.8	1.004×10^{-15}	3.276×10^{-15}
1.0	1.389×10^{-15}	4.551×10^{-15}
1.2	2.320×10^{-15}	6.351×10^{-15}
1.4	3.531×10^{-15}	7.440×10^{-15}
1.6	4.089×10^{-15}	8.241×10^{-15}
1.8	4.224×10^{-15}	8.878×10^{-15}
2.0	5.551×10^{-15}	7.983×10^{-15}



(A) Accurate solution (B) Estimate solution

FIGURE 5. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.3 ($\zeta(t) = 0.5t$).

This problem is the periodic conditions type and the true solution is $z(t) = t^2 - t$ and $0 \le t \le T$, $T = 1$, $\zeta_1(t) = \frac{t}{2}$, $\zeta_2 = \frac{t}{4}$, $\tau = 1$.

We estimated the solution of (6.4) for various values of α and β, by our $NSJOM$ scheme and presented the CPU time required for our scheme, the results related to the relative and absolute errors of this estimated solution with the exact solution in tables 9–14. Moreover, we show the absolute error with $\alpha = 1$, $\beta = 1$ in Figure 6 and relative errors for various value of α and β in figures 12–13 for this instance. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 2$, have $A = [-0.16667, 0, +0.16667]^T$;
- For $\alpha = 0$, $\beta = 0$ and $M = 4$, have $A = [-0.16667, 0, +0.16667, 0, 0]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 2$, have $A = [-0.1875, 0, 0.1]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 4$, have $A = [-0.1875, 0, 0.1, 0, 0]^T$;

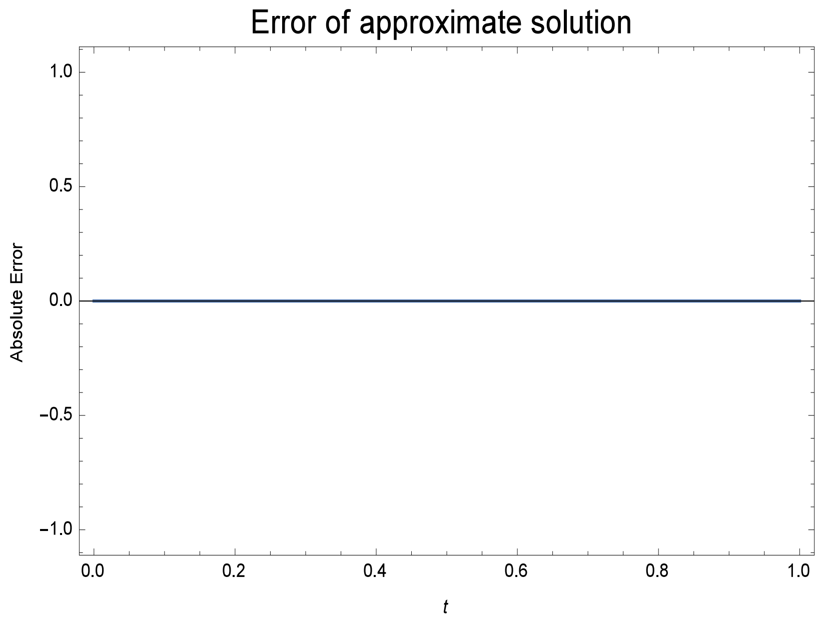


FIGURE 6. The absolute error between true and estimate solution (z_2) for Example 6.3 $(\zeta(t) = 0.5t)$

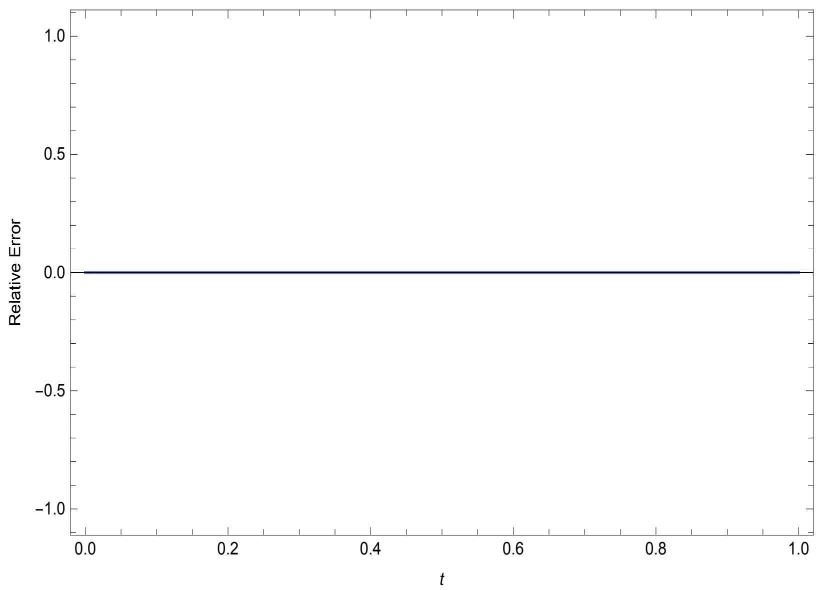


FIGURE 7. The relative error between accurate and estimate solution(z_2) with $\beta = 0$, $\alpha = 0$, at $T = 2.0$ for Example 6.3

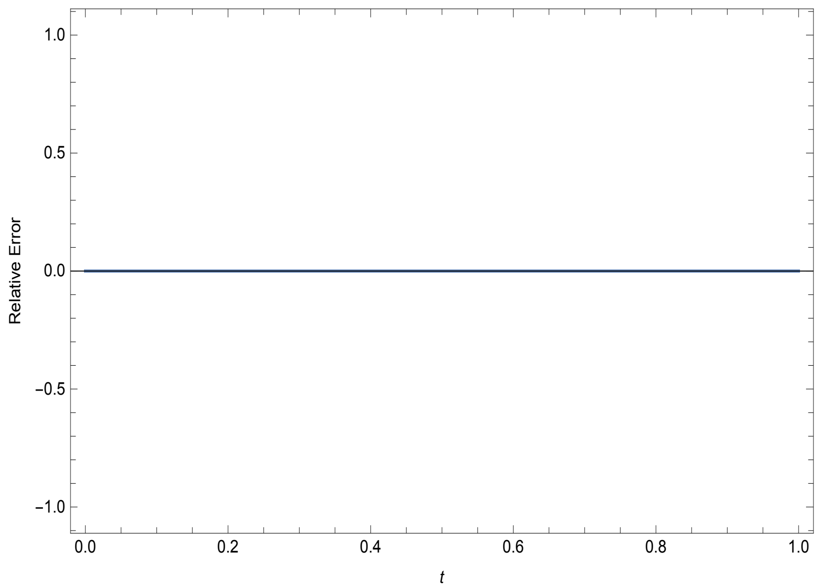


FIGURE 8. The relative error between accurate and estimate solution (z_2) with $\beta = 1$, $\alpha = 1$, at $T = 2.0$ for Example 6.3

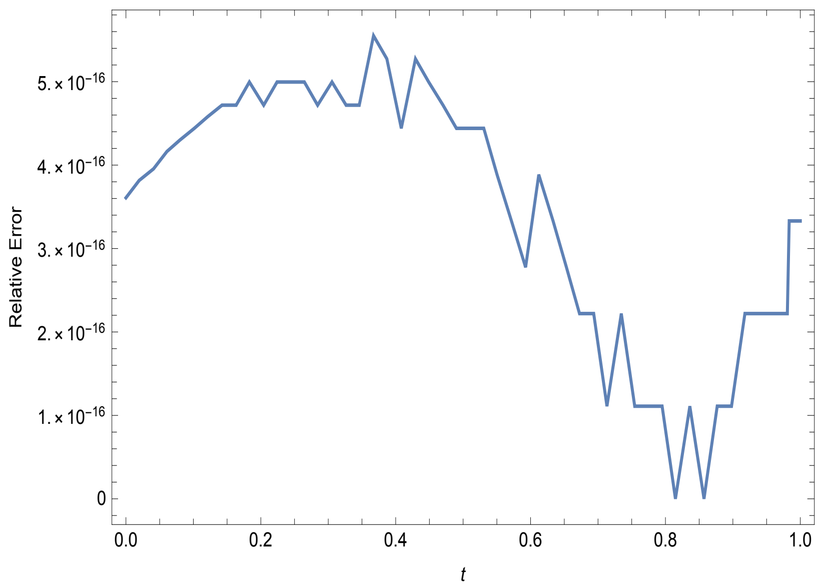


FIGURE 9. The relative error between accurate and estimate solution (z_2) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, at $T = 2.0$ for Example 6.3

TABLE 9. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.12	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0.093601s$	$4.007500s$

TABLE 10. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

- For $\alpha = 1$, $\beta = 1$ and $M = 2$, have $A = [-0.2, 0, 0.06667]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 4$, have $A = [-0.2, 0, 0.06667, 0, 0]^T$.

Example 6.5. Consider the below $FDDE$ for $0 < \zeta \leq 1$

(6.5)

$$D^{\zeta} z(t) - z(t - \tau) + z(t) = g(t),$$

$$g(t) = \frac{2\exp(t)(-1 + t)}{1 + \exp(2)} - \frac{2\exp(t - \tau)(-1 + t - \tau)}{1 + \exp(2)}$$

TABLE 11. Absolute errors of true solution and our method $(z_M(t))$ with $\beta = 1$, $\alpha = 1$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0s$	$0.062400s$

TABLE 12. Relative errors of true solution and our method $(z_M(t))$ with $\beta = 1$, $\alpha = 1$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

$$- \frac{2t^{\zeta}(-2+\zeta)(t^2 + \exp(t)t^{\tau}(-1+t+\zeta)\Gamma(2-\eta) - \exp(t)t^{\zeta}(-1+t+\zeta)\Gamma(2-\zeta,t))}{\Gamma(3-\zeta)(1+\exp(2))},$$

$$z(t) = \frac{2\exp(t)(-1+t)}{1+\exp(2)} - \frac{2\exp(2)}{1+\exp(2)} + 1, \quad t \in [-\tau, 0],$$

$$z(0) = -z(T).$$

This problem is the anti-periodic conditions type and the true solution is $z(t) = \frac{2\exp(t)(-1+t)}{1+\exp(2)} - \frac{2\exp(2)}{1+\exp(2)} + 1$ and $0 \le t \le T$, $T = 2$, $\tau = 0.01\exp(-t)$, $\zeta = 0.2$.

The solution of (6.3) for several values of α and β, by our $NSJOM$ scheme is stimated and is recorded the CPU time required for our scheme, the results related

TABLE 13. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 1$ for Example 6.4

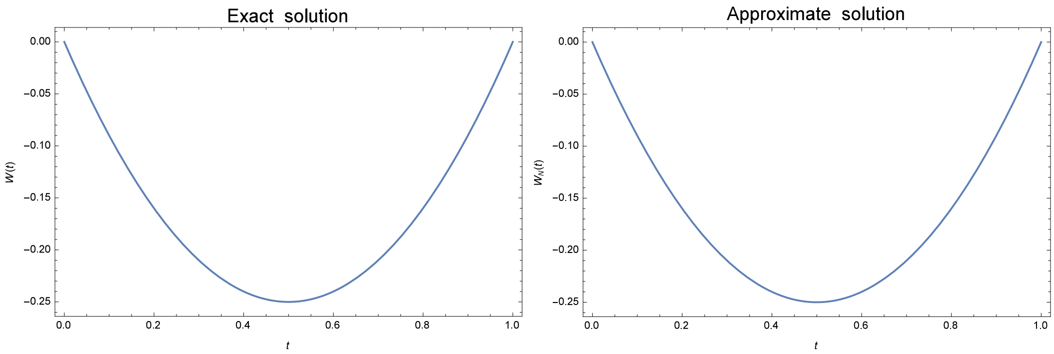
$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0.109201s$	$51.339929s$

TABLE 14. Relative errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

to the absolute and relative errors of this estimated solution with the exact solution in tables 15 and 16. In Figure 14 compared the exact and calculated solution which acknowledges the utility, accuracy and validity of *NSJOM* technique. Furthermore, in Figure 15 the absolute error of exact solution with our scheme for this instance has been drawn. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 10$, have $A = [-0.523188, 0.854347, 0.49638, 0.142, 0.0264148, 0.00361749, 3.90992 \times 10^{-4}, 3.48 \times 10^{-5}, 2.64 \times 10^{-6}, 1.72 \times 10^{-7}, 1.076 \times 10^{-8}]^T$;



(A) Accurate solution (B) Estimate solution

FIGURE 10. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.4.

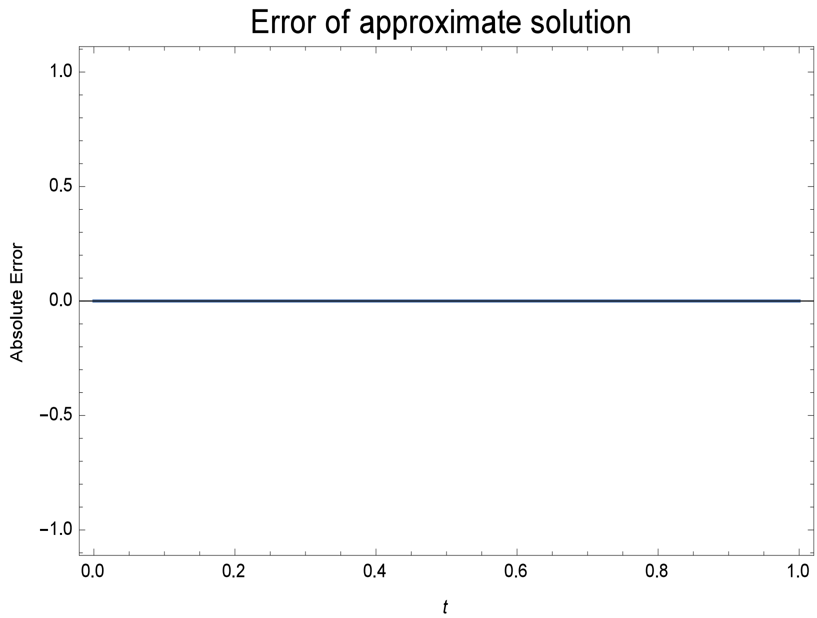


FIGURE 11. The absolute error between true and estimate solution (z_2) for Example 6.4.

- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 10$, have $A = [-0.58565, 0.54580, 0.29286, 0.08042, 0.014593, 0.0019657, 2.09955 \times 10^{-4}, 1.885 \times 10^{-5}, 1.3976 \times 10^{-6}, 9.0947 \times 10^{-8}, 5.641 \times 10^{-9}]^T$;

- For $\alpha = 1$, $\beta = 1$ and $M = 10$, have $A = [-0.622464, 0.396745, 0.192682, 0.049892, 0.008714, 0.00114289, 1.1968 \times 10^{-4}, 1.0416 \times 10^{-5}, 7.75225 \times 10^{-7}, 5.0021 \times 10^{-8}, 3.077 \times 10^{-9}]^T$.

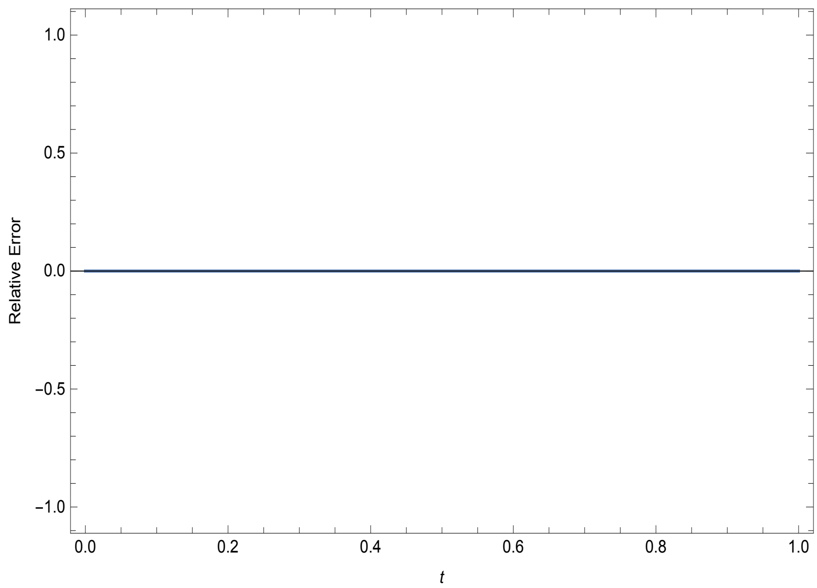


FIGURE 12. The relative errors between estimate solution (z_4) and accurate solution with $\beta = 0$, $\alpha = 0$, at $t = 1.0$. for Example 6.4.

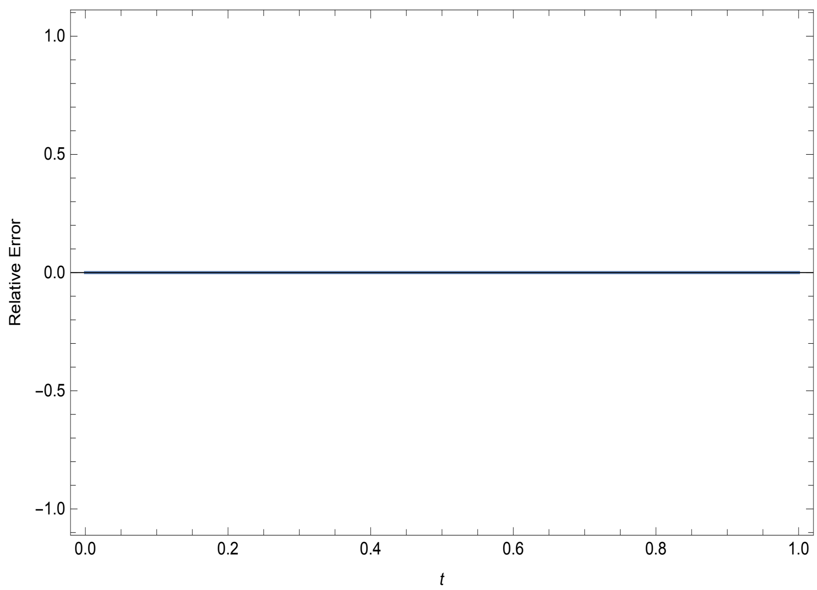


FIGURE 13. The relative error between exact and estimate solution (z_4) with $\beta = 1$, $\alpha = 1$, at $t = 1.0$. for Example 6.4.

Table 15. Absolute errors of true solution and our method ($z_M(t)$) with $M = 10$ and $T = 2$ for Example 6.5 by $NSJOM$.

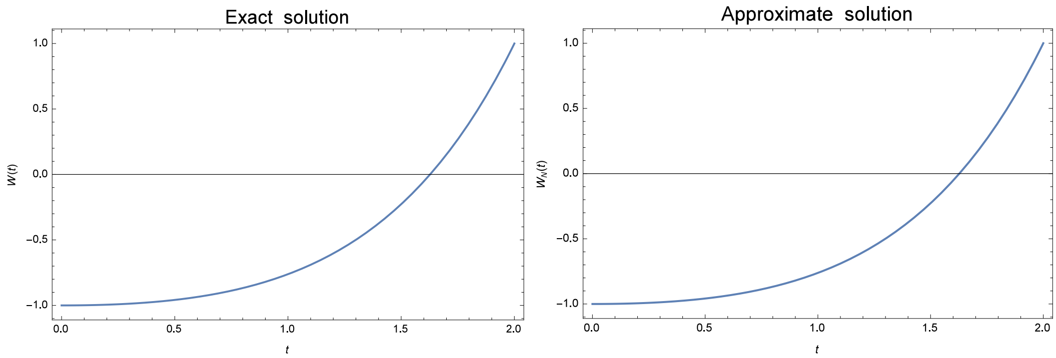
$t \in [0, T]$	$\alpha = 1$, $\beta = 1$	$\alpha = 0$, $\beta = 0$	$\alpha = 0.5$, $\beta = 0.5$
0	1.025×10^{-10}	9.479×10^{-11}	2.289×10^{-10}
0.2	3.779×10^{-9}	3.953×10^{-9}	4.087×10^{-9}
0.4	9.779×10^{-10}	9.667×10^{-10}	1.100×10^{-9}
0.6	6.661×10^{-10}	7.927×10^{-10}	9.265×10^{-10}
0.8	3.614×10^{-10}	5.863×10^{-10}	7.199×10^{-10}
1.0	6.661×10^{-10}	5.181×10^{-10}	6.502×10^{-10}
1.2	5.453×10^{-10}	4.376×10^{-10}	5.645×10^{-10}
1.4	4.185×10^{-10}	4.157×10^{-10}	5.281×10^{-10}
1.6	3.271×10^{-10}	3.433×10^{-10}	4.215×10^{-10}
1.8	6.981×10^{-10}	4.248×10^{-10}	4.307×10^{-10}
2.0	2.003×10^{-10}	9.479×10^{-11}	2.288×10^{-10}
CPU time	$1.076407s$	$1.076407s$	$1.544410s$

Table 16. Absolute errors of true solution and our method ($z_M(t)$) with $M = 15$ and $T = 2$ for Example 6.5 by $NSJOM$.

$t \in [0, T]$	$\alpha = 1$, $\beta = 1$	$\alpha = 0$, $\beta = 0$	$\alpha = 0.5$, $\beta = 0.5$
0	6.661×10^{-15}	8.881×10^{-16}	2.377×10^{-15}
0.2	7.016×10^{-14}	3.352×10^{-14}	3.907×10^{-13}
0.4	3.753×10^{-14}	1.487×10^{-14}	8.705×10^{-15}
0.6	2.775×10^{-14}	9.547×10^{-15}	8.635×10^{-13}
0.8	2.442×10^{-14}	7.549×10^{-15}	2.615×10^{-13}
1.0	2.152×10^{-14}	5.772×10^{-15}	6.163×10^{-13}
1.2	2.087×10^{-14}	5.551×10^{-15}	5.394×10^{-13}
1.4	1.909×10^{-14}	3.996×10^{-15}	9.005×10^{-146}
1.6	1.909×10^{-14}	3.556×10^{-15}	2.132×10^{-13}
1.8	2.131×10^{-14}	3.330×10^{-15}	4.916×10^{-13}
2.0	1.187×10^{-14}	1.110×10^{-15}	2.373×10^{-13}
CPU time	$2.552407s$	$2.558416s$	$3.000810s$

7. Conclusions

In this work, we have presented the (NSJOM) technique for the generalized linear variable-order $FDDE$ with anti-periodic and periodic condition by turning the main problem to an algebraic equations system that this system is solved numerically. We have shown that the presented method has good convergence, its concepts are simple and it's easy to implement. The obtained results are excellent compared to other



(A) Accurate solution (B) Estimate solution

FIGURE 14. Comparison of accurate and estimate solution (z_{15}) of *NSJOM* method for Example 6.5

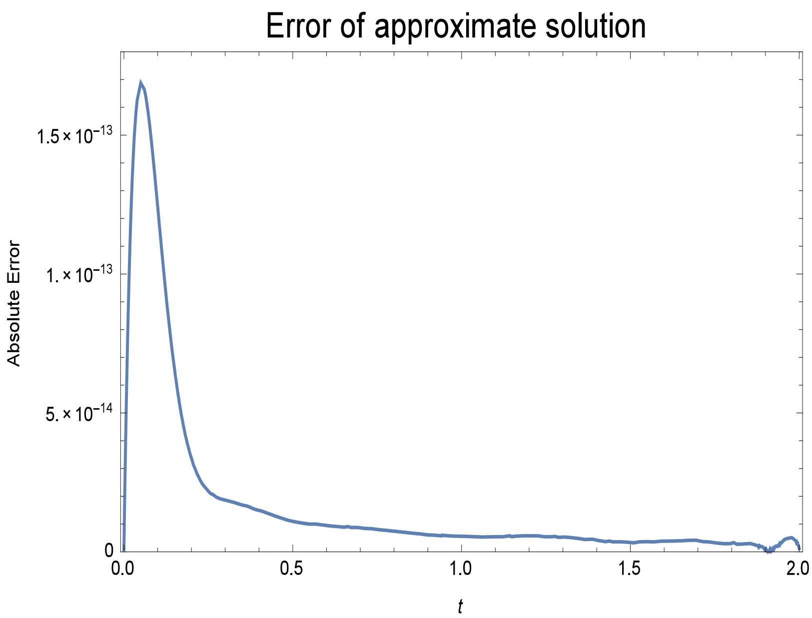


FIGURE 15. The absolute error between exact and estimate solution (z_{15}) for Example 6.5

method. Finally, the numerical results have been reported to clarify the validity and efficiency of this method.

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IMPROVED JENSEN-TYPE INEQUALITIES FOR (p, h)-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. The main goal of this article is to present multiple term refinements of the well-known Jensen's inequality for h-convex functions for a non-negative super-multiplicative and super-additive function h. For example, we show that

$$h(1-v)f(0)+h(v)f(1)\geqslant f(v)+\sum_{n=0}^{N-1}h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),$$

for the h-convex function f and certain positive summands. The significance of the obtained results is the way they extend known results from the setting of convex functions to other classes of functions.

1. Introduction and Preliminaries

Convex functions and their inequalities have played a major role in the study of various topics in Mathematics; including applied Mathematics, Mathematical Analysis, and Mathematical Physics. Recall that a function $f : I \to \mathbb{R}$ is said to be convex on the interval I if

$$f((1-v)a+vb)\leqslant(1-v)f(a)+vf(b), \tag{1.1}$$

for all $a, b \in I$ and $v \in (0, 1)$. If this inequality is reversed, then f is said to be concave.

Recent studies of the topic have investigated possible refinements of the above inequality, where adding a positive term to the left side becomes possible. This idea has been treated in [4, 10–13, 15], where not only refinements have been investigated, but reversed versions have been also discussed.

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The notion of convexity has been expanded and generalized in various ways utilizing novel and modern methods in recent years.

To motivate our work, let us recall the definitions of some special classes of functions. Let I be a p-convex subset of $\mathbb{R}$ (that means, $[(1-v)a^p+vb^p]^{\frac{1}{p}} \in I$ for all $a,b \in I$ and $v \in (0,1)$).

Definition 1.1 ([16])**.** A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to the class $PC(I)$, if

$$f\left(\left[(1-v)a^p+vb^p\right]^{\frac{1}{p}}\right) \leqslant (1-v)f(a)+vf(b), \tag{1.2}$$

for all $a,b \in I$, $p \in \mathbb{R}\backslash\{0\}$ and $v \in (0,1)$.

Definition 1.2 ([9])**.** Let $h : (0,1) \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is an h-convex function or that f belongs to the class $SX(I)$, if f is non-negative and for all $a,b \in I$ and $v \in (0,1)$

$$f((1-v)a+vb) \leqslant h(1-v)f(a)+h(v)f(b). \tag{1.3}$$

If this inequality is reversed, then f is said to be h-concave.

Definition 1.3 ([6])**.** Let $h : (0,1) \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (p,h)-convex function or that f belongs to the class $ghx(h,p,I)$, if f is non-negative and

$$f\left(\left[(1-v)a^p+vb^p\right]^{\frac{1}{p}}\right) \leqslant h(1-v)f(a)+h(v)f(b), \tag{1.4}$$

for all $a,b \in I$ and $v \in (0,1)$. Similarly, if the inequality sign in (1.4) is reversed, then f is said to be a (p,h)-concave function or belong to the class $ghv(h,p,I)$.

Definition 1.4 ([7])**.** Let $h : J \to \mathbb{R}$. If

$$h(x)h(y) \leqslant h(xy), \tag{1.5}$$

for all $x,y \in J$, then h is said to be a super-multiplicative function. If (1.5) is reversed, then h is said to be a sub-multiplicative function. If equality holds in (1.5), then h is said to be a multiplicative function.

Definition 1.5 ([7])**.** Let $h : J \to \mathbb{R}$. If for all $x,y \in J$

$$h(x)+h(y) \leqslant h(x+y), \tag{1.6}$$

then h is said to be a super-additive function. If inequality (1.6) is reversed, we say that h is a sub-additive function. If equality (1.6) holds, we say that h is an additive function.

Example 1.1. Let $h : (0,+\infty) \to (0,+\infty)$ be defined by $h(x) = x^k$. Then h is

(a) additive if $k = 1$;
(b) sub-additive if $k \in (-\infty,1)$;
(c) super-additive if $k \in (1,+\infty)$.

This latter conclusion follows from the fact that $h(x)=x^k$ is convex and $h(0)=0$, when $k>1$.

Let $h:[1,+\infty)\mapsto\mathbb{R}^+$ be defined by $h(x)=x^3-x^2+x$. We have

(a) $h(xy)-h(x)h(y)=xy(x+y)(1-x)(1-y)\geqslant 0$;

(b) $h(x+y)-h(x)-h(y)=xy(x+y+(x-1)+(y-1))\geqslant 0$.

Then h is a super-multiplicative and super-additive function.

The following theorem is the Jensen type inequality for (p,h)-convex functions.

Theorem 1.1 ([6]). *Let $v_1,\dots,v_n$ be positive real numbers, $n\geqslant 2$, such that $\sum_{k=1}^n v_k=1$. If h is a non-negative super-multiplicative function, f an (p,h)-convex function and $a_1,\dots,a_n\in I$, then*

$$f\left(\left(\sum_{k=1}^n v_k a_k^p\right)^{\frac{1}{p}}\right)\leqslant\sum_{k=1}^n h(v_k)f(a_k). \tag{1.7}$$

If h is sub-multiplicative and f an (p,h)-concave function, then inequality (1.7) *is reversed.*

The organization of the paper will be as follows. We firstly derive the refinements of Jensen-type and a variant of Jensen-type inequalities for h-convex functions. Next, we further refine our presented refinements and point out more or less direct consequences of our results for (p,h)-convex functions and its reversed, and in the last section we give the matrix versions of these inequalities studied in Section 2 and 3.

2. New Refinements of the Jensen's Inequality for h-Convex Functions

In this part of the paper, we present our main results concerning h-convex functions. The applications of these inequalities and their relations to the literature will be done in Remark 2.1. In order to do that, we start with some basic results which are important in terms of proving our main results.

We will see that our results extend the results in [1,10] to the context of h-convex functions, with the existence of multiple refining terms.

Definition 2.1. Let n be a positive integer. The sequence $(r_n(v))$ of functions on $[0,1]$ is defined by

$$\begin{aligned} r_0(v) &=\min\{v,1-v\},\\ r_n(v) &=\min\{2r_{n-1}(v),1-2r_{n-1}(v)\}. \end{aligned}$$

For all integers n, we have the following explicit formula of the function $r_n(v)$, proved by D. Choi in [3]. We also refer the reader to [12] for similar treatment.

Lemma 2.1 ([3]). *Let $\ell\geqslant 0$ and $1\leqslant k\leqslant 2^n$ be integers. If $\frac{k-1}{2^n}\leqslant v\leqslant\frac{k}{2^n}$, then*

$$r_n(v)=\begin{cases}2^n v-k+1, & \text{if } \frac{k-1}{2^n}\leqslant v\leqslant\frac{2k-1}{2^{n+1}},\\ k-2^n v, & \text{if } \frac{2k-1}{2^{n+1}}\leqslant v\leqslant\frac{k}{2^n}.\end{cases}$$

In the following lemma, we prove an essential inequality that will be needed in the sequel.

Lemma 2.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$, and f be a function defined on $[0,1]$. For a nonnegative integer N, define $\Psi_N(v)$ by*

$$\Psi_N(v) = h(1-v)f(0) + h(v)f(1) - \sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2\pi}\right)}(v), \tag{2.1}$$

where

$$\Delta_{f,h}^{(0,1)}(n,k) = h\left(\frac{1}{2}\right)\left[f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right)\right] - f\left(\frac{2k-1}{2^{n+1}}\right),$$

for $\frac{k-1}{2^N} \leqslant v \leqslant \frac{k}{2^N}$ and $k = 1, \dots, 2^N$. Then

$$\Psi_N(v) \geqslant h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right).$$

Proof. We proceed by induction on N. For $N = 1$: if $v \in [0, \frac{1}{2}]$, we have

$$\begin{aligned}
\Psi_1(v) &= h(1-v)f(0) + h(v)f(1) - h(2r_0(v))\Delta_{f,h}(0,1)\chi_{(0,1)}(v) \\
&= h(1-v)f(0) + h(v)f(1) - h(2v)\Delta_{f,h}(0,1) \\
&= \left(h(v) - h(2v)h\left(\frac{1}{2}\right)\right)f(1) + \left(h(1-v) - h(2v)h\left(\frac{1}{2}\right)\right)f(0) + h(2v)f\left(\frac{1}{2}\right) \\
&\geqslant \left(h(1-v) - h(2v)h\left(\frac{1}{2}\right)\right)f(0) + h(2v)f\left(\frac{1}{2}\right) \\
&\geqslant h(1-2v)f(0) + h(2v)f\left(\frac{1}{2}\right).
\end{aligned}$$

If $v \in [\frac{1}{2}, 1]$, then $1 - v \in [0, \frac{1}{2}]$. So, by changing v by $1-v$ and $f(v)$ by $f(1-v)$, the desired inequality for the case $v \in [\frac{1}{2}, 1]$ is obtained.

Now, assume that (2.1) holds and let $\frac{m-1}{2^{N+1}} \leqslant v \leqslant \frac{m}{2^{N+1}}$ for $m = 1, \dots, 2^{N+1}$. If $m = 2k-1$, then $\frac{k-1}{2^N} \leqslant v \leqslant \frac{2k-1}{2^{N+1}} < \frac{k}{2^N}$ and

$$\begin{aligned}
\Psi_{N+1}(v) =& \Psi_N(v) - h(2r_N(v))\Delta_{f,h}^{(0,1)}(N,k) \\
\geqslant& h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
& - h\left(2^{N+1}v - 2k + 2\right)\Delta_{f,h}^{(0,1)}(N,k) \\
=& h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
& - h\left(2^{N+1}v - 2k + 2\right) h\left(\frac{1}{2}\right)\left(f\left(\frac{k-1}{2^N}\right) + f\left(\frac{k}{2^N}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ h\left(2^{N+1}v - 2k + 2\right) f\left(\frac{2k-1}{2^{N+1}}\right) \\
\geqslant & h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
&- h\left(2^N v - k + 1\right)\left(f\left(\frac{k-1}{2^N}\right) + f\left(\frac{k}{2^N}\right)\right) \\
&+ h\left(2^{N+1}v - 2k + 2\right) f\left(\frac{2k-1}{2^{N+1}}\right) \\
=&\Big(h\left(k - 2^N v\right) - h\left(2^N v - k + 1\right)\Big) f\left(\frac{k-1}{2^N}\right) \\
&+ h\left(2^{N+1}v - 2k + 2\right) f\left(\frac{2k-1}{2^{N+1}}\right) \\
\geqslant & h\left(2k - 1 - 2^{N+1}v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^{N+1}v - 2k + 2\right) f\left(\frac{2k-1}{2^{N+1}}\right) \\
=& h\left(m - 2^{N+1}v\right) f\left(\frac{m-1}{2^{N+1}}\right) + h\left(2^{N+1}v - m + 1\right) f\left(\frac{m}{2^{N+1}}\right),
\end{aligned}
$$

by Lemma 2.1. Similarly, if $m = 2k$, then $\frac{k-1}{2^N} < \frac{2k-1}{2^{N+1}} \leqslant v \leqslant \frac{k}{2^N}$ and

$$
\begin{aligned}
\Psi_{N+1}(v) =& \Psi_N(v) - h(2r_N(v))\Delta_{f,h}^{(0,1)}(N,k) \\
\geqslant & h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
&- h\left(2k - 2^{N+1}v\right) \Delta_{f,h}^{(0,1)}(N,k) \\
=& h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
&- h\left(2k - 2^{N+1}v\right) h\left(\frac{1}{2}\right)\left(f\left(\frac{k-1}{2^N}\right) + f\left(\frac{k}{2^N}\right)\right) \\
&+ h\left(2k - 2^{N+1}v\right) f\left(\frac{2k-1}{2^{N+1}}\right) \\
\geqslant & h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right) \\
&- h\left(k - 2^N v\right)\left(f\left(\frac{k-1}{2^N}\right) + f\left(\frac{k}{2^N}\right)\right) + h\left(2k - 2^{N+1}v\right) f\left(\frac{2k-1}{2^{N+1}}\right).
\end{aligned}
$$

Thus, we have shown that

$$
\Psi_{N+1}(v) = \Big(h\left(2^N v - k + 1\right) - h\left(k - 2^N v\right)\Big) f\left(\frac{k}{2^N}\right) + h\left(2k - 2^{N+1}v\right) f\left(\frac{2k-1}{2^{N+1}}\right)
$$

$$
\begin{aligned}
&\geqslant h\left(2^{N+1}v-2k+1\right)f\left(\frac{k}{2^N}\right)+h\left(2k-2^{N+1}v\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
&=h\left(2^{N+1}v-m+1\right)f\left(\frac{m}{2^{N+1}}\right)+h\left(m-2^{N+1}v\right)f\left(\frac{m-1}{2^{N+1}}\right).
\end{aligned}
$$

This completes the proof. □

Now we show the first result concerning h-convex functions, when h is super-multiplicative and super-additive.

Theorem 2.1. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and f be an h-convex function on $[0,1]$. If N is a positive integer, then*

$$
h(1-v)f(0)+h(v)f(1)\geqslant f(v)+\sum_{n=0}^{N-1}h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v) \tag{2.2}
$$

and

$$
\begin{aligned}
h(1-v)f(0)+h(v)f(1)\leqslant& f(0)+f(1)-f(1-v)\\
&-\sum_{n=0}^{N-1}h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,2^n-k+1)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),
\end{aligned} \tag{2.3}
$$

where $v\in[0,1]$ and

$$
\Delta_{f,h}^{(0,1)}(n,k)=h\left(\frac{1}{2}\right)\left[f\left(\frac{k-1}{2^n}\right)+f\left(\frac{k}{2^n}\right)\right]-f\left(\frac{2k-1}{2^{n+1}}\right).
$$

Proof. By Lemma 2.2 and the h-convexity of the function f, we get

$$
\begin{aligned}
\Psi_N(v)&\geqslant h\left(k-2^Nv\right)f\left(\frac{k-1}{2^N}\right)+h\left(2^Nv-k+1\right)f\left(\frac{k}{2^N}\right)\\
&\geqslant f\left(\left(k-2^Nv\right)\frac{k-1}{2^N}+\left(2^Nv-k+1\right)\frac{k}{2^N}\right)\\
&=f(v).
\end{aligned}
$$

This end the proof of (2.2).

Replacing v by $1-v$ in (2.2) and noting that $r_n(v)=r_n(1-v)$, we have

$$
\begin{aligned}
&(h(1-v)+h(v))\left(f(0)+f(1)\right)-h(v)f(0)-h(1-v)f(1)\\
\leqslant&-f(1-v)+(h(1-v)+h(v))\left(f(0)+f(1)\right)\\
&-\sum_{n=0}^{N-1}h((2r_n(v)))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(1-v).
\end{aligned}
$$

So,

$$
h(1-v)f(0)+h(v)f(1)\leqslant(h(1-v)+h(v))\left(f(0)+f(1)\right)-f(1-v)
$$

$$
\begin{aligned}
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n}\frac{k}{2^n}\right)}(1-v)\\
\leqslant & h(1)(f(0)+f(1))-f(1-v)\\
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n}\frac{k}{2^n}\right)}(1-v)\\
\leqslant & (f(0)+f(1))-f(1-v)\\
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(1-v).
\end{aligned}
$$

Now, replacing k by 2^n-k+1 in the inner summation and noting that

$$
\frac{k-1}{2^n} < 1-v < \frac{k}{2^n} \quad \text{if and only if} \quad 1-\frac{k}{2^n} < v < 1-\frac{k-1}{2^n},
$$

we obtain (2.3) and the proof is completed. $\square$

Remark 2.1. Before proceeding to further results, we explain a little about Theorem 2.6. Notice that for $h(x)=x$, we recapture Theorem 2.1 in [4].

Corollary 2.1. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and f be a h-convex function on $[a,b]$. Then for all positive integers N,*

$$
h(1-v)f(a)+h(v)f(b) \geqslant f(va+(1-v)b)+\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),
$$

where $v\in[0,1]$ and

$$
\Delta_{f,h}^{(a,b)}(n,k) = h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right)+g\left(\frac{k}{2^n}\right)\right]-g\left(\frac{2k-1}{2^{n+1}}\right)
$$

and $g(t):=f((1-t)a+tb)$.

Proof. For the h-convex function f, define $g:[0,1]\to\mathbb{R}$ by $g(t):=f((1-t)a+tb)$. Then, g is h-convex on $[0,1]$. Applying Theorem 2.1 on the function g implies the result. $\square$

The following result provides a two-parameter refined version of the basic inequality for h-convex functions. We encourage the reader to see the main results in [1, 10], where this type was treated for convex functions, without any refining terms.

Theorem 2.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and let f be an h-convex function on $[a,b]$. If $0<v\leqslant\tau<1$, then for all positive integers N*

$$
\begin{aligned}
h(1-v)f(a)+h(v)f(b) \geqslant & f((1-v)a+vb)\\
&+h\left(\frac{v}{\tau}\right)\Big[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)\Big]
\end{aligned}
$$

$$+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right),$$

where

$$\Delta_{f,h}^{(a,b)}(n,k)=h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right)+g\left(\frac{k}{2^n}\right)\right]-g\left(\frac{2k-1}{2^{n+1}}\right)$$

and $g(t):=f((1-t\tau)a+t\tau b)$.

Proof. Since, h is super-multiplicative and super-additive, we have

$$\begin{aligned}
&h(1-v)f(a)+h(v)f(b)-h\left(\frac{v}{\tau}\right)[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)]\\
=&\left(h(1-v)-h\left(\frac{v}{\tau}\right)h(1-\tau)\right)f(a)+\left(h(v)-h\left(\frac{v}{\tau}\right)h(\tau)\right)f(b)\\
&+h\left(\frac{v}{\tau}\right)f((1-\tau)a+\tau b)\\
\geqslant& h\left(1-\frac{v}{\tau}\right)f(a)+h\left(\frac{v}{\tau}\right)f((1-\tau)a+\tau b)\\
\geqslant& f\left[\left(1-\frac{v}{\tau}\right)a+\left(\frac{v}{\tau}\right)((1-\tau)a+\tau b)\right]\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right)\\
=&f((1-v)a+vb)+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right).
\end{aligned}$$

This completes the proof. □

Notice that in Theorem 2.2, if we ignore the sum, we can rewrite the result in the simpler form

$$\frac{h(1-v)f(a)+h(v)f(b)-f((1-v)a+vb)}{h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)}\geqslant h\left(\frac{v}{\tau}\right).$$

This form is easier to view for comparison purpose with the main results in [1, 10]. Thus, Theorem 2.2 presents the h-convex version with multiple term refinements of the main results in these references.

On the other hand, a reverse of Theorem 2.2 can be stated as follows. We, once again, refer the reader to [1, 10] where this type of inequalities was treated in its simplest form for convex functions.

Theorem 2.3. *Let h be a non-negative multiplicative and super-additive function on $[0,+\infty)$. If f is h-convex on $[a,b]$ and $0<v\leqslant\tau<1$ then for all positive integer N*

$$\begin{aligned}
\text{(2.4)}\quad &h(1-v)f(a)+h(v)f(b)\\
&\leqslant f((1-v)a+vb)+h\left(\frac{1-v}{1-\tau}\right)\Big[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)\Big]
\end{aligned}$$

$$-\sum_{n=0}^{N-1} h\left(2\left(\frac{1-\tau}{1-v}\right) r_n\left(\frac{1-\tau}{1-v}\right)\right) \sum_{k=1}^{2^n} \Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right),$$

where

$$\Delta_{f,h}^{(a,b)}(n,k) = h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right) + g\left(\frac{k}{2^n}\right)\right] - g\left(\frac{2k-1}{2^{n+1}}\right)$$

and $g(t) := f((1-t(1-v))a + t(1-v)b)$.

Proof. Since, h is multiplicative and super-additive, we have

$$\begin{aligned}
&h(1-\tau)f(a) + h(\tau)f(b) - \frac{f(1-v)}{h\left(\frac{1-v}{1-\tau}\right)}f(a) - \frac{f(v)}{h\left(\frac{1-v}{1-\tau}\right)}f(b) + \frac{1}{h\left(\frac{1-v}{1-\tau}\right)}f((1-v)a+vb)\\
=&\left(h(1-\tau) - h(1-\tau)\right)f(a) + \left(h(\tau) - h\left(\frac{v(1-\tau)}{1-v}\right)\right)f(b)\\
&+ h\left(\frac{1-\tau}{1-v}\right)f((1-v)a+vb)\\
\geqslant& h\left(1-\frac{1-\tau}{1-v}\right)f(b) + h\left(\frac{1-\tau}{1-v}\right)f((1-v)a+vb)\\
\geqslant& f\left[\left(1-\frac{1-\tau}{1-v}\right)b + \left(\frac{1-\tau}{1-v}\right)((1-v)a+vb)\right]\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{1-\tau}{1-v}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right)\\
=& f((1-\tau)a+\tau b)\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{1-\tau}{1-v}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right).
\end{aligned}$$

Multiplying the last inequality by $h\left(\frac{1-v}{1-\tau}\right)$, the desired inequality is obtained. □

While the above results treat the values $0 \leqslant v \leqslant 1$, it has been of interest in the literature to deal with the cases $v \notin [0,1]$. We refer the reader to [2, 13] for related discussion when $v \geqslant 1$ or $v \leqslant 0$. In the following two results, this is treated for h-convex functions, where multiple-term refinements are provided.

Theorem 2.4. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, f be h-convex on $\mathbb{R}$ and $v \geqslant 0$. If N is a positive integer, then*

$$\begin{aligned}
&h(v+1)f(b) - h(v)f(a)\\
&+\sum_{n=0}^{N-1} h(v+1)h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big)\\
\leqslant& f\left((1+v)b - va\right),
\end{aligned}$$

where $g(t) := f((1-(t+v)a + t(1+v)b)$.

Proof. Notice first that for $v \geqslant 0$, one has

$$b = \frac{v}{v+1}a + \frac{1}{v+1}\left((1+v)b - va\right).$$

h-convexity of f and Theorem 2.1, implies

$$\begin{aligned} f(b) \leqslant & h\Big(\frac{v}{v+1}\Big)f(a) + h\Big(\frac{1}{v+1}\Big)f\left((1+v)b - va\right) \\ & - \sum_{n=0}^{N-1} h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big) \\ \leqslant & \frac{h(v)}{h(v+1)}f(a) + \frac{1}{h(v+1)}f\left((1+v)b - va\right) \\ & - \sum_{n=0}^{N-1} h(v+1)h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big). \end{aligned}$$

This completes the proof. □

A more straightforward form of Theorem 2.4 can be stated as follows.

Theorem 2.5. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, and $f:\mathbb{R}\to\mathbb{R}$ be h-convex. If N is a positive integer and $a<b$, then*

$$\begin{aligned} & h(1+v)f(a) - h(v)f(b) \\ & + \sum_{k=1}^{N} h(2^k v)\left[h\left(\frac{1}{2}\right)\left(f(a) + f\left(\frac{\left(2^{k-1}-1\right)a+b}{2^{k-1}}\right)\right) - f\left(\frac{\left(2^k-1\right)a+b}{2^k}\right)\right] \\ \text{(2.5)}\quad & \leqslant f((1+v)a - vb), \end{aligned}$$

where $v \geqslant 0$.

Proof. We proceed by induction on N. So, assume that f is h-convex, $a<b$ and $v \geqslant 0$. We have

$$\begin{aligned} & h(1+v)f(a) - h(v)f(b) + h(2v)\left[h\left(\frac{1}{2}\right)(f(a)+f(b)) - f\left(\frac{a+b}{2}\right)\right] \\ = & \left(h(1+v)+h(v)\right)f(a) + \left(h(2v)h\left(\frac{1}{2}\right) - h(v)\right)f(b) - h(2v)f\left(\frac{a+b}{2}\right) \\ \leqslant & h(1+2v)f(a) - h(2v)f\left(\frac{a+b}{2}\right) \\ \leqslant & f\left((1+2v)a - 2v\frac{a+b}{2}\right) \\ = & f((1+v)a - vb), \end{aligned}$$

where we have applied Theorem 2.4, with v and b replaced by $2v$ and $\frac{a+b}{2}$, respectively. We emphasize here that when $a<b$ we have $a<\frac{a+b}{2}$. Moreover, when $v\geqslant 0$ we have $2v\geqslant 0$, justifying the application of Theorem 2.4.

Now assume that, for some $N\in\mathbb{N}$, (2.5) holds whenever $a<b$ and $v\geqslant 0$. We assert the truth of the inequality for $N+1$. Observe that

$$\begin{aligned}A =&h(1+v)f(a)-h(v)f(b)\\&+\sum_{k=1}^{N+1}h(2^kv)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1\right)a+b}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b}{2^{k}}\right)\right]\\=&h(1+v)f(a)-h(v)f(b)+h(2v)\left[h\left(\frac{1}{2}\right)\Big(f(a)+f(b)\Big)-f\left(\frac{a+b}{2}\right)\right]\\&+\sum_{k=2}^{N+1}h(2^kv)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1)a+b\right.}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b}{2^{k}}\right)\right]\\=&h(1+2v)f(a)-h(2v)f\left(\frac{a+b}{2}\right)\end{aligned}$$

(2.6)

$$+\sum_{k=1}^{N+1}h(2^{k+1}v)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k}-1)a+b\right.}{2^{k}}\right)\right)-f\left(\frac{\left(2^{k+1}-1\right)a+b'}{2^{k+1}}\right)\right].$$

For simplicity, let $2v=r$, $\frac{a+b}{2}=b'$. Then (2.6) becomes

$$\begin{aligned}A =&h(1+r)f(a)-h(r)f\left(b'\right)\\&+\sum_{k=1}^{N}h(2^kr)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1\right)a+b'}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b'}{2^{k}}\right)\right]\\\leqslant&f\left((1+r)a-rb'\right)\\=&f((1+v)a-vb),\end{aligned}$$

where we have used the inductive step to obtain (2.6). Observe that when $a<b$ we have $a<b'$, which justifies the application of the inductive step. □

Other generalized external forms for h-convex functions can be stated as follows. We remark that these results extend known results for convex functions, as one can see in [14, 18].

Theorem 2.6. *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be h-convex, $b\in\mathbb{R}$ and let $\{v_k\}$ be such that $v_k>0$ for $k=1,2,\dots,n$. If $\{b_k\}\subset\mathbb{R}$, and h is a non-negative super-multiplicative function on $[0,+\infty)$, then*

$$h(1+\beta)h(a)-\sum_{k=1}^{n}h\left(v_k\right)f\left(b_k\right)\leqslant f\left((1+\beta)a-\sum_{k=1}^{n}v_kb_k\right),$$

where $\sum_{k=1}^{n}v_k=\beta$.

Proof. Notice first that for $s \geqslant 0$ and $x, y \in \mathbb{R}$ one has

$$y = \frac{s}{s+1}x + \frac{1}{s+1}\left((1+s)y - sx\right).$$

h-Convexity of f implies

$$\begin{aligned} f(y) &\leqslant h\Big(\frac{s}{s+1}\Big)f(x) + h\Big(\frac{1}{s+1}\Big)f\left((1+s)y - sx\right) \\ &\leqslant \frac{h(s)}{h(s+1)}f(x) + \frac{1}{h(s+1)}f\left((1+s)y - sx\right), \end{aligned}$$

which implies

$$h(s+1)f(y) - f\left((1+s)y - sx\right) \leqslant h(s)f(x). \tag{2.7}$$

Now, applying (2.7), we have

$$\begin{aligned} & h(1+\beta)h(a) - f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right) \\ =& h(1+\beta)h(a) - f\left((1+\beta)a - \beta\sum_{k=1}^{n}\frac{v_k}{\beta}b_k\right) \\ \leqslant& h(1+\beta)h(a) - h(1+\beta)h(a) + h(\beta)f\left(\sum_{k=1}^{n}\frac{v_k}{\beta}b_k\right) \\ \leqslant& h(\beta)\sum_{k=1}^{n} h\left(\frac{v_k}{\beta}\right)f(b_k) \\ \leqslant& \sum_{k=1}^{n} h\left(v_k\right)f(v_k). \end{aligned}$$

This completes the proof. ☐

In the following result, we present a one-term refinement of Theorem 2.6.

Theorem 2.7. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be h-convex, $b \in \mathbb{R}$ and let $\{v_k\}$ be such that $v_k > 0$ for $k = 1, 2, \ldots, n$. If $\{b_k\} \subset \mathbb{R}$, and h is a non-negative super-multiplicative and super-additive function on $[0, +\infty)$, then we have*

$$\begin{aligned} & h(1+\beta)h(a) - \sum_{k=1}^{n} h\left(v_k\right)f\left(b_k\right) \\ \leqslant& f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right) \\ & - h\left((n+1)r_0\right)\left[h\left(\frac{1}{n+1}\right)\left(f(a) + \sum_{k=1}^{n} f(b_k)\right) - f\left(\frac{a + \sum_{k=1}^{n} b_k}{n+1}\right)\right], \end{aligned}$$

where $\sum_{k=1}^{n} v_k = \beta$ and $r_0 = \min\{v_1, v_2, \ldots, v_n\}$.

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned}
I :=& h(1+\beta)f(a) - \sum_{k=1}^{n} h(v_k) f(b_k) \\
&+ h((n+1)r_0)\left[h\left(\frac{1}{n+1}\right)\left(f(a)+\sum_{k=1}^{n} f(b_k)\right) - f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right)\right] \\
\leqslant& \left(h(1+\beta)+h((n+1)r_0)h\left(\frac{1}{n+1}\right)\right) f(a) \\
&+ \sum_{k=1}^{n}\left(-h(v_k)+h(r_0)\right) f(b_k) - h((n+1)r_0) f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right) \\
\leqslant& h(1+\beta+r_0) f(a) - \sum_{k=1}^{n} h(v_k - r_0) f(b_k) - h((n+1)r_0) f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right).
\end{aligned}$$

Let $\gamma_{n+1} = (n+1)r_0$ and for $1 \leqslant k \leqslant n$, let $\gamma_k = v_k - r_0$. Further, denote $\beta + r_0$ by λ. Then

$$\sum_{k=1}^{n} \gamma_k + \gamma_{n+1} = \beta + r_0 = \lambda.$$

Therefore, we may apply Theorem 2.6, we obtain

$$\begin{aligned}
I \leqslant& f\left((1+\beta+r_0)a - \sum_{k=1}^{n}(v_k - r_0) b_k - (n+1)r_0 \frac{a+\sum_{k=1}^{n} b_k}{n+1}\right) \\
=& f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right).
\end{aligned}$$

□

3. Refinement and Reverse of Jensen's Inequality for (p,h)-Convex Function

In this part of the paper, we present our main results concerning (p,h)-convex functions. The applications of these inequalities and their relations to the literature will be done in Remark 3.1 and in the last section.

In the following result, we present a one-term refinement of Jensen's type inequality for (p,h)-convex function and its reverse.

Before we state our first result, we remind the reader of the following lemma, which was shown in [8].

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be convex, $\{x_1,\ldots,x_n\} \subset I$ and $\{p_1,\ldots,p_n\} \subset (0,1)$ be such that $\sum_{i=1}^{n} p_i = 1$. Then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) + np_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \leqslant \sum_{i=1}^{n} p_i f(x_i) \tag{3.1}$$

and

$$f\left(\sum_{i=1}^{n} p_i x_i\right) + np_{\max}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \geqslant \sum_{i=1}^{n} p_i f(x_i), \tag{3.2}$$

where $p_{\min} = \min\{p_1, \dots, p_n\}$ *and* $p_{\max} = \max\{p_1, \dots, p_n\}$.

The following result presents the (p,h)-convex version of the above lemma.

Theorem 3.1. *Let* $v_1, \dots, v_n$ *be positive real numbers,* $n \geqslant 2$*, such that* $\sum_{k=1}^n v_k = 1$*. Let* f *be a* (p,h)*-convex function, and* $x_1, \dots, x_n \in I$*.*

(a) *If* h *is a non-negative super-multiplicative and super-additive function on* $[0,1]$*, then*

$$f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right) + h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(x_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right) \leqslant \sum_{k=1}^n h(v_k) f(x_k),$$

where $r_0 = \min\{v_1, v_2, \dots, v_n\}$

(b) *If* h *is a non-negative multiplicative and super-additive function on* $[0,+\infty)$*, then*

$$f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right) + h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(x_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right) \geqslant \sum_{k=1}^n h(v_k) f(x_k),$$

where $R_0 = \max\{v_1, v_2, \dots, v_n\}$.

Proof. We prove the first inequality. Since, h is a super-multiplicative and super-additive function, we have

$$\begin{aligned}
&\sum_{k=1}^n h(v_k) f(x_k) - h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(x_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right)\\
=&\sum_{k=1}^n \left(h(v_k) - h(nr_0)h\left(\frac{1}{n}\right)\right) f(x_k) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\\
\geqslant&\sum_{k=1}^n h(v_k - r_0) f(x_k) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\\
\geqslant& f\left[\left(\sum_{k=1}^n (v_k - r_0) x_k^p + nr_0\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right]\\
=& f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right),
\end{aligned}$$

where the last inequality follows by the Jensen's inequality for the (p,h)-convex function f. The second inequality is equivalent to the following inequality

$$\sum_{k=1}^n \left(\frac{h(nR_0)h\left(\frac{1}{n}\right) - h(v_k)}{h(nR_0)}\right) f(x_k) + f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{1/p}\right)\frac{1}{k(nR_0)} \geqslant f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{1/p}\right).$$

Since, h is a multiplicative and super-additive function, we have

$$\sum_{k=1}^n \left(\frac{h(nR_0)h\left(\frac{1}{n}\right) - h(v_k)}{h(nR_0)}\right) f(x_k) + \frac{1}{h(nR_0)} f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{1/p}\right)$$

$$\begin{aligned}
&\geqslant \sum_{k=1}^{n}\left(\frac{h\left(R_0\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right) f\left(x_k\right)+\frac{1}{h\left(nR_0\right)} f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right)\\
&\geqslant \sum_{k=1}^{n}\left(\frac{h\left(R_0-v_k\right)}{h\left(nR_0\right)}\right) f\left(x_k\right)+\frac{1}{h\left(nR_0\right)} f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right)\\
&\geqslant \sum_{k=1}^{n} h\left(\frac{R_0-v_k}{nR_0}\right) f\left(x_k\right)+h\left(\frac{1}{nR_0}\right) f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right)\\
&\geqslant f\left(\left(\sum_{k=1}^{n} \frac{R_0-v_k}{nR_0} x_k^p+\frac{1}{nR_0}\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right)\\
&= f\left(\sum_{k=1}^{n}\left(\frac{R_0-v_k}{nR_0}+\frac{v_k}{nR_0}\right) x_k^p\right)^{1/p}\\
&= f\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{1/p}\right),
\end{aligned}$$

where the last inequality follows by the Jensen's inequality for the (p,h)-convex function f. □

For further generalisation of Theorem 3.1, we need the following lemma.

Lemma 3.2 ([5]). *Let ϕ be a strictly increasing convex function defined on an interval I. If x, y, z and w are points in I such that $z-w \leqslant x-y$, where $w \leqslant z \leqslant x$ and $y \leqslant x$, then*

$$0 \leqslant \phi(z)-\phi(w) \leqslant \phi(x)-\phi(y).$$

This lemma will be simply used to prove the following generalization of Theorem 3.1.

Theorem 3.2. *Let $v_1, \ldots, v_n$ be positive real numbers, $n \geqslant 2$, such that $\sum_{k=1}^{n} v_k=1$. Let f be a (p,h)-convex function, $x_1, \ldots, x_n \in I$ and $\lambda \geqslant 1$.*

(a) *If h is a non-negative super-multiplicative and super-additive function on $[0,1]$, then*

$$\begin{aligned}
&f^{\lambda}\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right)+h^{\lambda}(nr_0)\left(\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right)^{\lambda}-f^{\lambda}\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)\right)\\
&\leqslant\left(\sum_{k=1}^{n} h\left(v_k\right) f\left(x_k\right)\right)^{\lambda}, \qquad (3.3)
\end{aligned}$$

where $r_0=\min\{v_1, v_2, \ldots, v_n\}$

(b) *If h is a non-negative multiplicative and super-additive function on $[0,+\infty)$, then*

$$\left(\sum_{k=1}^{n} h\left(v_k\right) f\left(x_k\right)\right)^{\lambda}$$

$$\leqslant f^{\lambda}\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right) + h^{\lambda}(nR_0)\left(\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right)^{\lambda} - f^{\lambda}\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)\right),$$

where $R_0 = \max\{v_1, v_2, \ldots, v_n\}$.

Proof. Let

$$x = \sum_{k=1}^{n} h(v_k) f(x_k), \quad y = f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right),$$
$$z = h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right), \quad w = h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)$$

and

$$z' = h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right), \quad w' = h(nR_0) f\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right).$$

Then based on Theorem 3.1, we have

$$z - w \leqslant x - y \leqslant z' - w'.$$

The first and the second inequalities in Theorem 3.2 follow directly by applying Lemma 3.2, with $\phi(x) = x^{\lambda}$, where $\lambda \geqslant 1$ to the inequalities $z - w \leqslant x - y$, with $w \leqslant z \leqslant x$, $y \leqslant x$ and $x - y \leqslant z' - w'$ with $y \leqslant x \leqslant z'$, $w' \leqslant z'$, respectively. This completes the proof. ☐

Remark 3.1. Before proceeding to further results, we explain a little about Theorem 3.2. Notice that if we take $f(x) = e^x$, $h(x) = x$ and $x_i = \ln a_i$ for $a_i > 0$ we recapture Theorems 2.2 and 2.4 in [17].

4. Refinement and Reverse of Jensen's Inequality for (p, h) Operator Convex Function

Let $\mathbf{M}_\ell$ be the algebra of all complex matrices of order $\ell \times \ell$. A matrix $A \in \mathbf{M}_\ell$ is called Hermitian if $A = A^*$, where A^* is the adjoint of A. The notation $A \geqslant 0$ $(A > 0)$ is used to mean that A is positive semi-definite (positive definite). If A and B are Hermitian and $A - B$ is positive semi-definite, then we write $A \geqslant B$.

In this section, we extend some results from the context of real functions and real numbers to that of matrices. In the following we suppose that $I \subset \mathbb{R}^+$ and $p > 0$.

Definition 4.1. Let $h : [0, 1] \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is operator (p, h)-convex or that f belongs to the class *opgx*(h, p, I), if

$$f\left([(1-v)A^p + vB^p]^{\frac{1}{p}}\right) \leqslant h(1-v) f(A) + h(v) f(B), \tag{4.1}$$

for all $A, B \in \mathbf{M}_\ell^+$ with $\sigma(A), \sigma(B) \subset I$, and $v \in (0, 1)$. Similarly, if the inequality sign in (4.1) is reversed, then f is said to be a (p, h)-concave function or belong to the class *ghv*(h, p, I).

The matrix version of Jensen type inequality for operator (p,h)-convex functions is as follows.

Theorem 4.1. *Let h be a non-negative super-multiplicative function on $[0,1]$ and assume $f \in opgx(p,h,I)$. For $k=1,\dots,n$, let A_k be a positive semi-definite matrix with spectrum in I and let $v_1,\dots,v_n$ be positive real numbers, such that $\sum_{k=1}^n v_k = 1$. Then*

$$f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right) \leqslant \sum_{k=1}^n h(v_k) f(A_k).$$

Theorem 4.2. *Let $f \in opgx(p,h,I)$, $A_1,\dots,A_n$ be positive semi-definite matrices in $\mathbf{M}_\ell$ with spectra in I and $v_1,\dots,v_n$ be positive real numbers, such that $\sum_{k=1}^n v_k = 1$.*

(a) *If h is a non-negative super-multiplicative and super-additive function on $[0,1]$, then*

$$\begin{aligned} &f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right) + h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(A_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right)\\ &\leqslant \sum_{k=1}^n h(v_k) f(A_k), \end{aligned}$$

where $r_0 = \min\{v_1, v_2, \dots, v_n\}$

(b) *If h is a non-negative multiplicative and super-additive function on $[0,+\infty)$, then*

$$\begin{aligned} \sum_{k=1}^n h(v_k) f(A_k) \leqslant &f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right)\\ &+ h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(A_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right), \end{aligned}$$

where $R_0 = \max\{v_1, v_2, \dots, v_n\}$.

Proof. We prove the first inequality. Since, h is a super-multiplicative and super-additive function, we have

$$\begin{aligned} &\sum_{k=1}^n h(v_k) f(A_k) - h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f(A_k) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right)\\ &=\sum_{k=1}^n \left(h(v_k) - h(nr_0)h\left(\frac{1}{n}\right)\right) f(A_k) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\\ &\geqslant \sum_{k=1}^n h(v_k - r_0) f(A_k) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\\ &\geqslant f\left[\left(\sum_{k=1}^n (v_k - r_0) A_k^p + nr_0\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)\right)^{\frac{1}{p}}\right] \end{aligned}$$

$$=f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{\frac{1}{p}}\right),$$

where the last inequality follows by the Jensen's inequality for the (p,h) operator convex function f. This proves the first desired inequality.

To prove the second desired inequality, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\frac{h\left(nR_0\right)h\left(\frac{1}{n}\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}\left(\frac{h\left(R_0\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}\left(\frac{h\left(R_0-v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}h\left(\frac{R_0-v_k}{nR_0}\right)f\left(A_k\right)+h\left(\frac{1}{nR_0}\right)f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&f\left(\left(\sum_{k=1}^{n}\frac{R_0-v_k}{nR_0}A_k^p+\frac{1}{nR_0}\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
=&f\left(\sum_{k=1}^{n}\left(\frac{R_0-v_k}{nR_0}+\frac{v_k}{nR_0}\right)A_k^p\right)^{1/p}\\
=&f\left(\left(\sum_{k=1}^{n}\frac{1}{n}A_k^p\right)^{1/p}\right),
\end{aligned}$$

where the last inequality follows by the Jensen's inequality for the (p,h) operator convex function f. □

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DIRECT LIMIT FF (m,n)-ARY HYPERMODULES

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Abstract. The purpose of this paper is the study of direct limit in the category of (m,n)-ary hypermodules over (m,n)-hyperring R. In this regards, we introduce and study $R_{(m,n)} - Hmod$, the category of $R_{(m,n)} - Hmod$, and direct limit in this category. In particular, we study a direct limit of morphisms, direct systems of kernels, and cokernels. Finally, we investigate the relationship between the functor home and direct limit and prove that the functor *hom* preserves direct limit in category $R_{(m,n)} - Hmod$.

1. Introduction

The notion of n-ary groups (also called n-group or multi-ary group) is a generalization of groups. An n-ary group (G, f) is a pair of a set G and a map $f : G\times\cdots\times G \to G$, which is called an n-ary operation. The earliest work on these structures was done in 1904 by Krasner [23] and in 1928 by Dörnet [20]. Such n-ary groups have many applications in computer science, coding theory, topology, combinatorics, and quantum physic (for more details see [16–19, 30, 31]). One of the applications in algebraic hyperstructures theory was defined by Marty [28]. Many researchers developed this theory of view point of theory and application(for more see [5, 11, 12, 14, 15, 36]).

Ameri et al. [3] introduced and studied the notion of hyperalgebraic, a framework to formulate algebraic hyperstructures in a general manner, also R. Ameri and I. G. Rosenberg [2]. Davvaz and Vougiouklis [15] studied n-ary hypergroups. After that, a generalization of it, such as (m,n)-hyperrings and (m,n)-hypermodules were introduced and studied in different contexts(some of the studies can be found in [4, 6, 7, 9, 24–26, 29]). On the other hand, fundamental relations, as the smallest equivalence

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relation on an algebraic hyperstructure such as a hypergroup, hyperring, hypermodules or in general a hyperalgebra such that its quotient is a group, ring, module, or algebra respectively, play an important role to study the theory of algebraic hyperstructure. In fact, the fundamental relation on an algebraic hyperstructure induces a functor from a category of algebraic hyperstructures such as a category of hypergroups and hypermodules to its related classical algebraic structure such as the category of group and modules. R. Ameri in [1] introduced and studied the category of hypergroups and hypermodules. Recently, various kinds of categories of hyperstructures have been studied in numerous papers(for instance see [1, 21, 22, 27, 32–35]). In this paper, we follow [21] and introduce and study direct limit in the category of (m, n)-hypermodules. This work is a generalization of the paper A. Asadi, R. Ameri, Direct Limit of Krasner (m, n)-Hyperrings [8], with more details of categorical properties related to direct limit. In Section 2, we give some basic preliminaries about (m, n)-rings and (m, n)-hypermodules. In Section 3, we introduce a direct system of (m, n)-ary hypermodules and use it to introduce direct limit in category (m, n)-hypermodules. In Section 4, the properties of direct limit of a direct system of (m, n)-ary hypermodules are investigated. In Section 5, the direct limit of morphisms is studied and some basic properties of the are obtained. In section 6, direct systems of kernels and cokernels of a direct system of (m, n)-ary hypermodules are studied. Finally, in section 7, the behavior of direct limits under home representable functors is studied, and it is shown that these functors preserve limits.

2. Preliminaries

In this section, we give some definitions and results of n-array hyperstructures which we need in what follows.

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \to P^*(H)$ is called an n-ary *hyperoperation,* where $P^*(H)$ is the set of all nonempty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid.*

We shall use the following abbreviated notation.

The sequence $x_i, x_{i+1}, \dots, x_j$ will be denoted by x_i^j. For $j < i, x_i^j$ is the empty set. Using this notation, $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$ will be written as $f\left(x_1^i, y_{i+1}^j, z_{j+1}^n\right)$. In the case when $y_{i+1} = \cdots = y_j = y$ the last expression will be written $f\left(x_1^i, y_{(j-i)}, z_{j+1}^n\right)$.

If f is an n-array hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t-array hyperoperation f_l is given by

$$f_l\left(x_1^{l(n-1)+1}\right) = \underbrace{f\Big(f\Big(\dots, f(f}_{l}(x_1^n\Big), x_{n+1}^{2n-1}\Big), \dots,\Big), x_{(l-1)(n-1)+1}^{l(n-1)+1}\Big).$$

For nonempty subsets $A_1, A_2, \dots, A_n$ of H, define

$$f(A_1^n) = f(A_1, A_2, \dots, A_n) = \bigcup\{f\left(x_1^n\right) \mid x_i \in A_i, i = 1, 2, \dots, n\}.$$

An n-array hyperoperation f is called *associative* if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, \ldots, x_{n-1} \in H$. An n-array hypergroupoid with the associative n-array hyperoperation is called an *n-ary semihypergroup.*

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution, $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an n-ary quasihypergroup. If (H, f) is an n-ary semihypergroup and n-array quasihypergroup, then (H, f) is called an n-ary hypergroup. An n-ary hypergroupoid (H, f) is commutative if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$, then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

Definition 2.1 ([15])**.** Let (H, f) be an n-array hypergroup and B be a non-empty subset of H. B is called an n-ary subhypergroup of (H, f), if $f(x_1^n) \subseteq B$ for all $x_1^n \in B$, and the equation $b \in f\left(b_1^{i-1}, x_i, b_{i+1}^n\right)$ has a solution, $x_i \in B$ for every $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

Definition 2.2 ([15])**.** Let (H, f) be a commutative n-ary hypergroup. (H, f) is called canonical n-ary hypergroup if the following statements are satisfied:

(1) there exists unique $e \in H$, such that for every $x \in H$, $f(x, \underbrace{e, \ldots, e}_{(n-1)}) = x$;

(2) for all $x \in H$ there exists unique $x^{-1} \in H$, such that $e \in f(x, x^{-1} \underbrace{e, \ldots, e}_{(n-2)})$;

(3) if $x \in f(x_1^n)$, then for all i, we have $x_i \in f\left(x, x^{-1}, \ldots, x_{i-1}^{-1}, x_{i+1}^{-1}, \ldots, x_n^{-1}\right)$.

We say that e is the scaler identity of (H, f) and x^{-1} is the inverse of x. Notice the inverse of e is e.

Definition 2.3 ([29])**.** A (Krasner) (m, n)-hyperring is algebraic hyperstructure (R, h, k) which satisfies the following axioms:

(1) (R, h) is a canonical m-ary hypergroup;

(2) (R, k) is an n-ary semigroup;

(3) the n-ary operation k is distributive to the m-array hyperoperation h, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$,

$$k\left(a_1^{i-1}, h(x_1^m), a_{i+1}^n\right) = h\left(k(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, k(a_1^{i-1}, x_m, a_{i+1}^n)\right);$$

(4) 0 is a zero element (absorbing element), of the n-ary operation k, i.e., for $x_2^n \in R$ we have $k(0, x_2^n) = k(x_2, 0, x_3^n) = \cdots = k(x_2^n, 0)$.

A nonempty subset S of R is called a subhyperring of R if (R, h, k) is a Krasner (m, n)-hyperring. Let I be a non-empty subset of R. We say that I is a hyperideal of (R, h, k) if (I, h) is a canonical m-ary hypergroup of (R, h) and $k(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$, for every $x_1^n \in R$ and $1 \leq i \leq n$.

Definition 2.4. Let M be a nonempty set. Then (M, f, g) is an (m, n)-hypermodule over an (m, n)-hyperring (R, h, k), if (M, f) is an m-ary hypergroup and the map $g : \underbrace{R \times \cdots \times R}_{n-1} \times M \to P^*(M)$ satisfies the following conditions:

(i) $g\left(r_1^{n-1}, f(x_1^m)\right) = f\left(g(r_1^{n-1}, x_1), \ldots, g(r_1^{n-1}, x_m)\right)$;

(ii) $g\left(r_1^{i-1}, h(s_1^m), r_{i+1}^{n-1}, x\right) = f\left(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x\right)$;

(iii) $g\left(r_1^{i-1}, k(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x\right) = g\left(r_1^{n-1}, g(r_m^{n+m-2}, x)\right)$;

(iv) $0 \in g\left(r_1^{i-1}, 0, r_{i+1}^{n-1}, x\right)$.

If g is an n-ary hyperoperation, $S_1, \ldots, S_{n-1}$ are subsets of R and $M_1 \subseteq M$, we set

$$g(S_1^{n-1}, M_1) = \bigcup\{g(r_1^{n-1}, x) \mid r_i \in S_i, i = 1, \ldots, n-1, x \in M_1\}.$$

If $n = m = 2$ then an (m, n)-ary hypermodule M is hypermodule.

Let (M, f, g) be an (m, n)-hypermodule over an (m, n)-hyperring (R, h, k). A non-empty subset N of M is called an (m, n)-ary sub-hypermodule of M if (N, f) is m-array subhypergroup of (M, f) and $g(R^{(n-1)}, N) \in P^*(N)$.

Definition 2.5. A canonical (m, n)-hypermodule (M, f, g) is an (m, n)-hypermodule with a canonical m-array hypergroup (M, f) over a Krasner (m, n)-hyperring (R, h, k).

A Krasner (m, n)-hyperring (R, h, k) is *commutative* if (R, k) is a commutative n-ary semigroup. Also, we say that (R, h, k) is a *scaler identity* if there exists an element 1_R, such that $x = k(x, 1_R^{(n-1)})$ for all $x \in R$. Later on, let (R, h, k) be a commutative Krasner (m, n)-hyperring with a scaler identity 1_R. For all $r_1^{n-1} \in R$ and $x \in M$ we have

$$g(r_1^{n-1}, 0_M) = \{0_M\}, \quad g(0_R^{n-1}, x) = \{0_M\} \quad \text{and} \quad g(1_R^{n-1}, x) = \{x\}.$$

Moreover, let $g(r_1^{i-1}, -r_i, r_{i+1}^{n-1}, x) = -g(r_1, \ldots, r_{n-1}, x) = g(r_1^{n-1}, -x)$.

Definition 2.6 ([29])**.** Let (M_1, f_1, g_1) and (M_2, f_2, g_2) be two (m, n)-hypermodules over an (m, n)-hyperring (R, h, k). We say that $\phi : M_1 \to M_2$ is a homomorphism of (m, n)-hypermodules if for all x_1^m, x of M_1 and $r_1^{n-1} \in R$: $\phi\left(f_1(x_1, \ldots, x_m)\right) = f_2\left(\phi(x_1), \ldots, \phi(x_m)\right)$, $\phi\left(g_1(r_1^{n-1}, x)\right) = g_2\left(r_1^{n-1}, \phi(x)\right)$.

If in the above definition we consider a map $\phi : M_1 \to P^*(M_2)$, then we obtain a *multivalued homomorphism,* shortly we write m-homomorphism.

Example 2.1. We shall provide an example of an m-homomorphism. Let A and B be two canonical hypergroup as Tables 1 and 2.

Define $0 * x = 0$ and $1 * x = x$ for all $x \in A, B$. Then, it is easy to check that $(A, +, *)$ is a Krasner hyperring, and A and B are also A-hypermodule with the external multiplication $*$. Let $\varphi : B \to A$ with $\varphi(1) = \varphi(-1) = 1$ and $\varphi(0) = 0$. Clearly, φ is an m-homomorphism.

+	0	1
0	0	1
1	1	$\{o, 1\}$

TABLE 1. $(A, +)$

$+'$	0	1	-1
0	0	1	-1
1	1	1	$\{o, 1\text{,-}1\}$
-1	-1	$\{o, 1\text{,-}1\}$	-1

TABLE 2. $(B, +')$

Definition 2.7 ([9])**.** A linear combination of family $A = \{x_i \mid i \in I\}$ of elements of M is a sum of the form $f\left(g(r_{11}^{1(n-1)}, x_1), \ldots, g(r_{11}^{l(n-1)}, x_l), o^{(m-l)}\right)$ with $l \leq m$ and if $l > m, l = t(m-1)+1$, a linear combination of A is the form of

$$\underbrace{f\Big(f\Big(\ldots, f\Big(f}_{t}\left(g(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{m1}^{m(n-1)}, x_m\Big)\Big), g\Big(r_{(m+1)1}^{(m+1)(n-1)}, x_{m+1}\Big), \ldots,\right.$$

$$g\Big(r_{(2m-1)1}^{(2m-1)(n-1)}, x_{2m-1}\Big), \ldots\Big), g\Big(r_{((l-1)(m-1)+2)1}^{((l-1)(m-1)+2))(n-1)}, \ldots, g\Big(r_{(l(m-1)+1)1}^{(l(m-1)+1)(n-1)}\Big)\Big)\Big),$$

where $r_{ij} \in R$ and set $\{r_{ij}, r_{ij} \neq 0\}$ is finite.

A linear combination of family $\{x_i \mid i \in I\}$ of elements of M is a sum of the form

$$\Big\{f\Big(g\Big(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{l1}^{l(n-1)}, x_l\Big)\Big) \mid x_i, i \in I\Big\}$$

is linear dependent if there exists a linear combination

$$f\Big(g\Big(r_{11}^{1(n-1)}, x_1\Big), \ldots, g\Big(r_{l1}^{l(n-1)}, x_l\Big)\Big)$$

containing 0, without being all r_{ij} equal to 0. Otherwise, $\{x_i \mid i \in I\}$ is called linear independent.

Definition 2.8 ([9])**.** A subset X of M generates M if every element of M belongs to linear combination of elements from X.

Definition 2.9 ([21])**.** The category $R_{(m,n)} - Hmod$ of (m, n)-ary hypermodules defined as follows:

(i) the objects of $R_{(m,n)} - Hmod$ are (m, n)-hypermodules,
(ii) for the objects M and K, the set of all morphisms from M to K is defined as follows:

$$Hom_R(M, K) = \{f \mid f : M \to P^*(K) \text{ is an m-homomorphism}\};$$

(iii) the composition gf of morphisms $f : M \to P^*(K)$ and $g : K \to P^*(L)$ defined as follows:

$$gf : H \to P^*(K), \quad gf(x) = \bigcup_{t \in f(x)} g(t);$$

(iv) for any object H, the morphism $1_H : H \to P^*(H)$, defined by $1_H(x) = \{x\}$, is the identity morphism.

Remark 2.1. Consider a category whose objects are all (m,n)-hypermodules and whose morphisms are all R-homomorphisms denoted by $R_{(m,n)} - hmod$. The class of all R-homomorphisms from A into B is denoted by $hom_R(A,B)$. In addition, $R_{s(m,n)} - hmod$ is the category of all (m,n)-hypermodules whose morphisms are all strong R-homomorphisms. The class of all strong R-homomorphisms from A into B is denoted by $hom_{R_S}(A,B)$. It is easy to observe that $R_{s(m,n)} - hmod$ is a subcategory of $R_{(m,n)} - hmod$.

Definition 2.10 ([21])**.** Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules. We define a hyperoperation on $\prod_{i\in I} M_i$ as follows:

$$F\{a_{i1}^{im}\} = \left(\{t_i\} \mid t_i \in f_i(a_{i1}^{im}), \{a_{i1}^{im}\} \in \prod_{i\in I} M_i\right).$$

For $r \in R$ and $a_i \in \prod_{i\in I} M_i$, define

$$G\left(r_1^{(n-1)}\{a_i\}_{(i\in I)}\right) = \left\{g_i\left(r_1^{(n-1)}, a_i\right)\right\}_{i\in I}.$$

then $\prod_{i\in I} M_i$, together with m-array hyperoperation F and n-array operation G is called *direct hyper product* $\{M_i \mid i \in I\}$.

Theorem 2.1 ([21])**.** *Let $\{M_i \mid i \in I\}$ be a family of (m,n)-hypermodules, and $\{\phi_i : M \to p^*(M_i) \mid i \in I\}$ be a family of m-homomorphisms. Then there exists a unique m-homomorphism*

$$\left(\left\{\phi : M \to p^*\left(\prod_{i\in I} M_i\right)\right)\right\}$$

such that, $\Pi_i \phi = \phi_i$ for all $i \in I$, and this property determines $\prod_{i\in I} M_i$ uniquely up to isomorphism. In other words, $\prod_{i\in I} M_i$ is a product in the category of $R_{(m,n)} - Hmod$.

Definition 2.11 ([21])**.** The direct hypersum of a family $\{M_i \mid i \in I\}$ of (m,n)-hypermodules, denoted by $\coprod_{i\in I} M_i$ is the set of all $\{a_i\}_{i\in I}$, where a_i can be non-zero only for a finite number of indices.

Proposition 2.1 ([21])**.** *If $\{M_i \mid i \in I\}$ is a family of (m,n)-hypermodules, then*

(i) $\coprod_{i\in I} M_i$ *is an (m,n)-hypermodule.*

(ii) *for each* $k \in I$, *the map* $\ell_k : M_k \to \coprod_{i\in I} M_i$, *given by* $l_k(a) = \{a_i\}_{i\in I}$, *where* $a_i = 0$, *for* $i \neq k$, *and* $a_k = a$, *is* m*-homomorphism.*

(iii) *for each* $i \in I$, $\ell_i(M_i)$ *is a subhypermodule of* $\coprod_{i\in I} M_i$. *The map* ℓ_k *is called the canonical injection.*

Theorem 2.2 ([21])**.** *Let* $\{M_i \mid i \in I\}$ *be a family of* (m,n)*-hypermodules and* $\{\phi_i : M_i \to M \mid i \in I\}$ *be a family of* m*-homomorphisms of* (m,n)*-hypermodules. Then, there is a unique* m*-homomorphism* $\phi : \coprod_{i\in I} M_i \to M$ *such that* $\phi\ell_i = \phi_i$, *for all* $i \in I$ *and this property determines* $\coprod_{i\in I} M_i$ *uniquely up to isomorphism. In the other words* $\coprod_{i\in I} M_i$ *is a coproduct in the category of* $R_{(m,n)} - Hmod$.

Remark 2.2. In the following sections of this paper, we consider the category of all (m,n)-hypermodules over a (m,n)-hyperring R, in the sense of Canonical (m,n)-hypermodules over Krasner (m,n)-hyperring R with a scaler identity. We denote this category by $R_{(m,n)} - Khmod$. Hence the objects of $R_{(m,n)} - Khmod$ are Canonical (m,n)-hypermodules over Krasner (m,n)-herringbone.

3. The Direct Limit

Definition 3.1 ([37])**.** Let (A,Λ) be a quasi-ordered directed(to the right) set, i.e. for the two elements $i, j \in \Lambda$ there exists (at least one) $k \in \Lambda$ with $i \leq K$ and $j \leq K$.

A direct system of (m,n)-ary hypermodules $(M_i, \phi_{ij})_\Lambda$ consists of

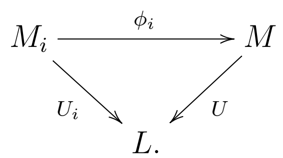
(1) a family of (m,n)-ary hypermodules $(M_i)_\Lambda)$ and
(2) a family of morphisms $\phi_{ij} : M_i \to M_j$ for all pairs (i,j) with $i \leq j$, satisfying

$$\phi_{ii} = id_{M_i} \quad \text{and} \quad \phi_{jk}\phi_{ij} = \phi_{ik}, \quad \text{for } i \leq j \leq k.$$

A direct system of morphisms from $(M_i, \phi_{ij})_\Lambda$ into an $R - (m,n)$-hypermodules L is a a family of morphisms $\{U_i : M_i \to L\}$ with $U_j\phi_{ij} = U_i$ whenever $i \leq j$.

Definition 3.2. Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R - (m,n)$-hypermodules and M an $R - (m,n)$-hypermodule.

A direct system of morphisms $\{\phi_i : M_i \to M\}_\Lambda$ is said to be a direct limit of $(M_i, \phi_{ij})_\Lambda$ if, for every direct system of morphisms $\{U_i : M_i \to L\}_\Lambda, L \in R_{(m,n)} - hmod$, there is a unique morphism $U : M \to L$ which makes the following diagram commutative for every $i \in \Lambda$



If $\{\phi'_i : M_i \to M'\}_\Lambda$ is another direct limit of $(M_i, \phi_{ij})_\Lambda$, then by definition there is an isomorphism $H : M_i \to M'$ with $H\phi_i = \phi'_i$ for $i \in \Lambda$. Hence M is uniquely determined up to isomorphism.

We write $M = \varinjlim M_i$ and $(\phi_i, \varinjlim M_i)$ for the direct limit.

Example 3.1. A collection of subsets M_i of a set M can be partially ordered by inclusion. If the collection is directed, its direct limit is the union $\cup M_i$. The same is true for a directed collection of subgroups of a given group.

Theorem 3.1. *Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules. For every pair $i \leq j$ we put $M_{i,j} = M_i$ and obtain with canonical embedding ℓ_i the following mappings:*

$$M_{i,j} \xrightarrow{\phi_{ij}} M_j \xrightarrow{\ell_j} \coprod_\Lambda M_k,$$

$$M_{i,j} \xrightarrow{id_{M_i}} M_i \xrightarrow{\ell_i} \coprod_\Lambda M_k.$$

The difference yields morphisms $F\{-\ell_i, \ell_j\phi_{ij}, o^{(m-2)}\} : M_{i,j} \longrightarrow \coprod_\Lambda M_k$ and with the coproduct we obtain a morphism $\phi : \coprod_{i\leq j} M_{i,j} \longrightarrow \coprod_\Lambda M_k$.

$Cok\phi$ together with the morphisms

$$\phi_i = Cok\phi\ell_i : M_i \longrightarrow \coprod_\Lambda M_k \longrightarrow Cok\phi$$

form a direct limit of $(M_i, \phi_{ij})_\Lambda$.

Proof. Let $\{U_i : M_i \to L\}_\Lambda$ be a direct limit of morphisms and $\overline{U} : \coprod_\Lambda M_k \longrightarrow L$ with $\overline{U}\ell_k = U_k$. We have $0 \in \overline{U}(F(-\ell_i, \ell_j\phi_{ij}, o^{(m-2)})) = f_l(-U_i, U_j\phi_{ij}, o^{(m-2)})$ for $i \leq j$. Hence, $\overline{U}\phi = 0$ and the diagram

$$\begin{array}{ccccc} \coprod_{i\leq j} M_{ij} & \xrightarrow{\phi} & \coprod_\Lambda M_k & \longrightarrow & Coke\phi \\ & & \downarrow{\scriptstyle \overline{U}} & & \\ & & L & & \end{array}$$

can be extended to a commutative diagram by a unique $U : Coke\phi \to L$ (definition of cokernel). □

Remark 3.1 ([37]). Regarding the quasi-ordered set Λ as a (directed) category, a directed system of (m,n)-hypermodules corresponds to a functor $\phi : \Lambda \to R_{(m,n)} - hmod$. Then direct system of morphisms is functorial morphisms between ϕ and constant functor $\Lambda \to R_{(m,n)} - hmod$. Then the direct limit is called the colimit of the functor ϕ. Instead of Λ, more general categories can serve as source and Instead of $R_{(m,n)} - hmod$, other categories may be used as target.

4. Properties of the Direct Limit

Theorem 4.1. *Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules with direct limit $(\phi_i, \varinjlim M_i)$.*

(1) *For* $m_j \in M_j, j \in \Lambda$, *we have* $0 \in \phi_j(m_j)$ *if and only if, for some* $k \geq j$, $0 \in \phi_{jk}(m_j)$.
(2) *For* $m, n \in \varinjlim M_i$, *there exist* $k \in \Lambda$ *and elements* $m_k, n_k \in M_k$ *with* $m \in \phi_k(m_k)$ *and* $n \in \phi_k(n_k)$.
(3) *If* N *is a finitely generated submodules of* $\varinjlim M_i$, *then there exist* $k \in \Lambda$ *with* $N \subset \phi_k(m_k)(= \operatorname{Im} \phi_k)$.
(4) $\varinjlim M_i = \bigcup_{\Lambda} \operatorname{Im} \phi_i$.

Proof. (1) If $0 \in \phi_{jk}(m_j)$, then also $0 \in \phi_j(m_j) = \phi_k \phi_{jk} m_j$.

Assume on the other hand $0 \in \phi_j(m_j)$, i.e., within Theorem 3.1

$$\ell_j m_j \in \operatorname{Im} F, \quad \ell_j m_j = \sum_{(i,l)\in E} f(-\ell_i, \ell_i \phi_{il}, 0^{(m-2)}) m_{il}, \quad m_{il} \in M_{i,l},$$

where E is a finite set of pairs $i \leq l$.

Choose any $k \in \Lambda$ bigger than all the indices occurring in E and $j \leq k$.

For $i \leq k$ the $\phi_{ik} : M_i \to M_k$ yield a morphism $\psi_k : \coprod_{i \leq k} M_i \to M_k$ with $\psi_k \ell_i = \phi_{ik}$ and

$$\begin{aligned} \phi_{jk} m_j =& \psi_k \ell_j m_j = \sum_E f\left(\psi_k \ell_l \phi_{il}, -\ell_i \psi_k, 0^{(m-2)}\right) m_{il} \\ =& \sum_E f\left(\phi_{lk} \phi_{il}, -\phi_{ik}, 0^{(m-2)}\right) m_{il} \ni 0. \end{aligned}$$

(2) For $m \in \varinjlim M_i$, let $(m_{i_1}, \ldots, m_{i_r})$ be a preimage of m in $\coprod_\lambda M_k$ (under Coke F). For $k \geq i_1, \ldots, i_r$ we get

$$m \in f_i\left(\phi_{i_1}(m_{i_1}), \ldots, \phi_{i_r}(m_{i_r})\right) = \phi_k\left(f_i(\phi_{i_1 k}(m_{i_1}), \ldots, \phi_{i_r k}(m_{i-r}))\right).$$

For $m, n \in \varinjlim M_i$, and $k, l \in \Lambda, m_k \in M_k, n_l \in M_l$ with $m \in \phi_k(m_k), n \in \phi_l(n_l)$, we choose $s \geq k, s \geq l$ to obtain $m \in \phi_s(\phi_{ks}(m_k)), n \in \phi_s(\phi_{ls}(n_l))$.

(3), (4) are consequences of (2). □

5. Direct Limit of Morphisms

Theorem 5.1. *Let* $(M_i, \phi_{ij})_\Lambda$ *and* $(N_i, \psi_{ij})_\Lambda$ *be two direct systems of* $R-(m,n)$*-hypermodules over the same set* Λ *and* $(\phi_i, \varinjlim M_(i)$, *resp.* $(\psi_i, \varinjlim N_i)$ *their direct limits.*

For any family of morphisms $\{u_i : M_i \to N_i\}_\Lambda$, *with* $\phi_{ij} u_j = \psi_{ij} u_i$ *for all indices* $i \leq j$, *there is unique morphism*

$$u : \varinjlim M_i \to \varinjlim N_i,$$

such that, for every $i \in \Lambda$, the following diagram is commutative

$$\begin{array}{ccc} M_i & \xrightarrow{u_i} & N_i \\ \phi_i \downarrow & & \downarrow \psi_i \\ \varinjlim M_i & \xrightarrow{u} & \varinjlim N_i \end{array}.$$

If all the u_i are monic (epic), then u is monic (epic).
Notation: $u = \varinjlim u_i$.

Proof. The mappings $\{\psi_i u_i : M_i \to \varinjlim N_i\}_\Lambda$ form a direct system of morphisms since for $i \leq j$ we get $\psi_j u_j = \psi_j \psi_{ij} u_i = \psi_i u_i$. Hence the existence of u follows from the defining property of the direct limit.

Consider $m \in \varinjlim M_i$ with $0 \in u(m)$. By (4.1), there exist $k \in \Lambda$ and $m_k \in M_k$ with $m \in \phi_k(m_k)$ and hence $0 \in u(\phi_k(m_k)) = \psi_k(u_k(m_k))$. Now there exists $l \geq K$ whith $0 \in \psi_{lk}(u_k(m_k)) = u_l(\phi_{kl}(m_k))$. If u_l is monic, then $\phi_{kl}(m_k) = 0$ and also $m \in \phi_k(m_k) = 0$. Consequently, if all $\{u_i\}_\Lambda$ are monic, then u is monic.

For $n \in \varinjlim N_i$ By (4.1), there exist $k \in \Lambda$ and $n_k \in N_k$ with $n \in \psi_k(n_k)$. If u_k is surjective, then $n_k \in u_k(m_k)$ for some $m_k \in M_k$ and $n \in \psi_k(u_k(m_k)) = u(\phi_k(m_k))$. If all the $\{u_i\}_\Lambda$ are surjective, then u is surjective. □

6. Direct Systems of Kernels and Cokernels

Using Theorem 5.1, we obtain, for $i \leq j$, commutative diagrams

$$\begin{array}{ccccccc} Ke\ u_i & \longrightarrow & M_i & \xrightarrow{u_i} & N_i & \longrightarrow & Coke\ u_i \\ & & \downarrow & & \downarrow & & \\ Ke\ u_j & \longrightarrow & M_j & \xrightarrow{u_j} & N_j & \longrightarrow & Coke\ u_j \end{array}$$

which can be extended by $k_{ij} : Ke\ u_i \to Ke\ u_j$ and $h_{ij} : coke\ u_i \to coke\ u_j$ to commutative diagrams.

$(Ke\ u_i, k_{ij})_\Lambda$ and $(coke\ u_i, h_{ij})_\Lambda$ also form direct system of (m,n)-hypermodules.

Theorem 6.1. *Consider direct systems of $R-(m,n)$-hypermodules*

$$(L_i, \phi_{ij})_\Lambda, (M_i, \psi_{ij})_\Lambda, (N_i, \mu_{ij})_\Lambda,$$

with direct limits $(\phi_i, \varinjlim L_i)$, $(\psi_i, \varinjlim M_i)$, $(\mu_i, \varinjlim N_i)$ and families of morphism $\{u_i\}_\Lambda$, $\{v_i\}_\Lambda$, which make the following diagrams commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_i & \xrightarrow{u_i} & M_i & \xrightarrow{v_i} & N_i & \longrightarrow & 0 \\ & & \downarrow \varphi_{ij} & & \downarrow \psi_{ij} & & \downarrow \mu_{ij} & & \\ 0 & \longrightarrow & L_j & \xrightarrow{u_j} & M_j & \xrightarrow{v_j} & N_j & \longrightarrow & 0. \end{array}$$

Then, $U = \varinjlim u_i$ *and* $V = \varinjlim v_i$*, the following sequence is also exact:*

$$0 \longrightarrow \varinjlim L_i \xrightarrow{U} \varinjlim M_i \xrightarrow{V} \varinjlim N_i \longrightarrow 0\ .$$

Proof. It has already been shown in (5.1) that U is monic and V is epic. $\operatorname{Im} U \subseteq Ke\, V$ is obvious. Consider $m \in Ke\ V$. There exist $k \in \Lambda$ and $m_k \in M_k$ with $m \in \psi_k(m_k)$ and $0 \in V(m) \in V(\psi_k(m_k)) = \mu_k(V_k(m_k))$.

Now by (4.1), we can find an $s \in \Lambda$ with $0 \in V_s(\psi_{ks}(m_k)) = \psi_{ks}(V_{ks}(m_k))$.

This implies $\psi_{ks}(m_k) = u_s l_s$ for some $l_s \in L_s$ and

$$U(\varphi_s(l_s)) = \psi_s(u_s(l_s)) = \psi_s(\psi_{ks}(m_s)) = \psi_k(m_k) \ni m.$$

Consequently, $m \in \operatorname{Im} U$ and $\operatorname{Im} U = Ke\, V$. □

Theorem 6.2. *Let* M *be an* $R-(m,n)$*-hypermodule,* Λ *a set, and* $\{M_i\}_\Lambda$ *a family of subhypermodules of* M *directed with respect to inclusion and with* $\bigcup_\Lambda M_i = M$*, then* $\varinjlim M_i = M$*.*

Proof. Defining $i \leq j$ if $M_i \subset M_j$ for $i, j \in \Lambda$, the set Λ becomes quasi-ordered and directed. With the inclusion $\varphi_{ij} : M_i \to M_j$ for $i \leq j$, the family $\{M_i, \varphi_{ij}\}_\Lambda$ is a direct system of (m,n)-hypermodules and $\varinjlim M_i = M$.

In particular, every (m,n)-hypermodule is a direct limit of its finitely generated subhypermodules. □

7. Home-Functor and Direct Limit

Let $(M_i, \phi_{ij})_\Lambda$ be a direct system of $R-(m,n)$-hypermodules with direct limit $(\phi_i, \varinjlim M_i)$ and K an $R-(m,n)$-hypermodule. with the assignments, for $i \leq j$,

$$h_{ij} := hom(k, \phi_{ij}) : hom(k, M_i) \to hom(k, M_j), \quad \alpha_i \mapsto \phi_{ij}\alpha_i,$$

we obtain a direct system of $\mathbb{Z}-(m,n)$-hypermodules $(hom(k, M_i), h_{ij})_\Lambda$ with direct limit $(h_i, \varinjlim hom(k, M_i))$ and the assignment

$$u_i := hom(k, \phi_i) : hom(k, M_i) \to hom(k, \varinjlim M_i), \quad \alpha_i \mapsto \phi_i\alpha_i,$$

defines a direct system of $\mathbb{Z}$-morphisms ($\mathbb{Z}$ is as an (m,n)-hyperring) and hence a $\mathbb{Z}$-morphism

$$\Phi_K := \varinjlim u_i : \varinjlim hom(k, M_i) \to hom(k, \varinjlim M_i).$$

These $\mathbb{Z}$-morphisms may be regarded as $End(K)$-morphisms.

Theorem 7.1. *If* K *is a finitely generated* $R-(m,n)$*-hypermodule, then* Φ_K *is monic.*

Proof. Consider $\alpha \in Ke\, \Phi_K$. There exist $i \in \Lambda$ and $\alpha_i \in hom(K, M_i)$ with $\alpha \in h_i(\alpha_i)$ and $0 \in \varphi_i(\alpha_i)$, Since $\alpha_i(K) \subset Ke\varphi_i$ is a finitely generated (m,n)-subhypermodule of M_i, There exists $i \leq j \in \Lambda$ with $0 \in \varphi_{ij}(\alpha_i(K))$ (by (4.1)). This implies $h_{ij}(\alpha_i) = \varphi_{ij}(\alpha_i) = 0$ and $h_i(\alpha_i) = 0$ in $hom(k, \varinjlim M_i)$. □

Theorem 7.2. *An $R-(m,n)$-hypermodule K is finitely generated if and only if*

$$\Phi_K : \varinjlim hom(k, M_i) \to hom(k, \varinjlim M_i)$$

is an isomorphism for every direct system $(M_i, \psi_{ij})_\Lambda$ of (m,n)-hypermodules with ψ_{ij} monomorphisms.

Proof. Let K be finitely generated. By (7.1), Φ_K is monic. With the φ_{ij} monic, the φ_i are monic. For every $\alpha \in hom(k, \varinjlim M_i)$, the image $\alpha(K)$ is finitely generated. By (4.1), $\alpha(K) \subset \varphi_k(M_k)$ for some $k \in \Lambda.$, with $\varphi_k^{-1} : \psi_k(M_k)$ we get $\varphi_k^{-1}\alpha \in hom(k, M_k)$ and $\Phi_k h_k(\varphi_k^{-1}\alpha) = \varphi_k(\varphi_k^{-1}\alpha) \ni \alpha$, i.e., Φ_k is surjective.

On the other hand. Assume Φ_k is an isomorphisms for the direct system $(K_i, \varphi_{ij})_\Lambda$ of the finitely generated (m,n)-subhypermodules $K_i \subset K$, i.e.,

$$\varinjlim hom(K, K_i) \simeq hom(K, \varinjlim K_i) \simeq hom(K, K).$$

By (4.1), there exist $j \in \Lambda$ and $\alpha_j \in hom(K, K_j)$ with $\alpha_j\varphi_j = id_K$, i.e., $K = \alpha_j\varphi_j\alpha_j K = \varphi_j K_j$. Hence, K is finitely generated. □

8. Conclusions and Future Works

In this paper, the category of (m,n)-hypermodules introduced and studied, especially the subclass of canonical (m,n)-hypermodules was investigated. Also, direct limit in category (m,n)-hypermodules was introduced and its basic properties has been discussed. In this regards, the relationship between direct limit and functor home in this category was investigated. The paper provided a good introduction to study the category of (m,n)- hypermodules as a generalization of category of (m,n)-modules as well as hypermodules. At the end, the paper provide a good introduction to study the homology of (m,n)-hypermodules, as well as hyperstructures in general.

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GENERALIZATION OF LUPAŞ-KANTOROVICH OPERATORS CONNECTED WITH PÓLYA DISTRIBUTION

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Abstract. The motive of this paper is to introduce the generalization of Lupaş-Kantorovich operators connected with Pólya distribution and establish the rate of convergence in terms of modulus of continuity. Furthermore, a Voronovskaja type asymptotic formula for these operators is studied. In the end, few numerical examples with graphical representation are added to depict the effect of convergence of the operators.

1. Introduction

About a decade ago, Gurdek et al. [20] defined the Baskakov operators for functions of two variables and analysed the approximation degree and differential properties of these operators. Agrawal et al. [8, 9] considered the bivariate form of the Lupaş Durrmeyer operators with Pólya distribution which was considered by Gupta and Rassias in [19]. In 2010, Gadjiev and Gorbanalizadeh [15] constructed the two dimensional extension of Bernstein–Stancu type polynomials and investigated the degree of convergence of these polynomials. The Kantrovich variants of various operators have been intensively studied in [1–4, 11, 16] and [23]. Very recently, Agrawal et al. [7] discussed the approximation features of the Kantorovich modification of the operators proposed by Stancu [26] and introduced their bivariate extension. For more related work, we suggest the readers (see [5, 14, 17, 18, 22, 24, 25, 27]). Inspired by the above work, we now introduce the bivariate form of the operators defined in [6] and given as:

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$$\left(\widetilde{Q}_n^{(1/n)}f\right)(x) = (1+n)\sum_{j=0}^{n}\widetilde{q}_{n,j}^{(1/n)}(x)\int_{I_{j,n}} f(\kappa)d\kappa, \quad x\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{1.1}$$

where

$$\widetilde{q}_{n,j}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{j}\left(\frac{2x(n+1)-1}{2}\right)_j\left(\frac{2n(1-x)-2x+1}{2}\right)_{n-j},$$

$I_{j,n} = \left[\frac{j}{n+1},\frac{j+1}{n+1}\right]$ and $(nx)_j = \prod_{i=0}^{j-1}(nx+i)$.

These operators preserve the linear functions along with the constants. In [6], the authors have provided moments and established some direct results for the operators defined by (1.1).

2. Preliminary Results

Let J be the interval $[\frac{1}{4},\frac{3}{4}]$. Then on $J^2 = J\times J$, The space of continuous functions with real values is denoted by $C(J^2)$. The norm for this space is $||g||_{C(J^2)} = \sup_{(x,y)\in J^2}|g(x,y)|$.

For $f\in C(J^2)$ and $(x,y)\in J^2$, we define

$$\begin{aligned}\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}f\right)(x,y) =&(1+n_1)(1+n_2)\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\widetilde{q}_{n_1,n_2,j_1,j_2}^{(1/n_1,1/n_2)}(x,y)\\ &\times\int_{I_{j_1,n_1}}\int_{I_{j_2,n_2}} f(u,v)dudv,\end{aligned}$$

where

$$\begin{aligned}\widetilde{q}_{n_1,n_2,j_1,j_2}^{(1/n_1,1/n_2)}(x,y) =&\frac{2(n_1!)}{(2n_1)!}\frac{2(n_2!)}{(2n_2)!}\binom{n_1}{j_1}\binom{n_2}{j_2}\left(\frac{2x(n_1+1)-1}{2}\right)_{j_1}\\ &\times\left(\frac{2n_1(1-x)-2x+1}{2}\right)_{n_1-j_1}\left(\frac{2y(n_2+1)-1}{2}\right)_{j_2}\\ &\times\left(\frac{2n_2(1-y)-2y+1}{2}\right)_{n_2-j_2}.\end{aligned}$$

The following lemmas are helpful in determining the key outcomes.

Lemma 2.1 ([6]). *For $x\in[\frac{1}{4},\frac{3}{4}]$ and $n=1,2,3,\dots,$ we have*

$$\begin{aligned}\left(\widetilde{Q}_n^{(1/n)}e_0\right)(x) =&1, \quad \left(\widetilde{Q}_n^{(1/n)}e_1\right)(x) = x,\\ \left(\widetilde{Q}_n^{(1/n)}e_2\right)(x) =&\frac{1}{12(1+n)^3}\Big\{12n^3x^2+12n^2x(x+2)+n\left(-12x^2+48x-11\right)\\ &-12x^2+24x-5\Big\}.\end{aligned}$$

Lemma 2.2 ([6]). *For $x \in [\frac{1}{4}, \frac{3}{4}]$ and $n = 1, 2, 3, \ldots,$ we have*

$$\begin{aligned}\Big(\widetilde{Q}_n^{(1/n)}(e_1 - xe_0)\Big)(x) =&0,\\ \Big(\widetilde{Q}_n^{(1/n)}(e_1 - xe_0)^2\Big)(x) =&\frac{1}{12(1+n)^3}\Big\{ - 24n^2(x-1)x\\ &+ n\left(-48x^2 + 48x - 11\right) - 24x^2 + 24x - 5\Big\}.\end{aligned}$$

Lemma 2.3. *If we denote $e_{ij} = x^i y^j$, where $i, j = 0, 1, 2$ and $i + j \leq 2$, then*

$$\begin{aligned}\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}e_{00}\Big)(x,y) =&1, \quad \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}e_{10}\Big)(x,y) = x,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}e_{01}\Big)(x,y) =&y,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}e_{20}\Big)(x,y) =&\frac{1}{12(1+n_1)^3}\Big\{12n_1^3x^2 + 12n_1^2x(x+2)\\ &+ n_1\left(-12x^2 + 48x - 11\right) - 12x^2 + 24x - 5\Big\},\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}e_{02}\Big)(x,y) =&\frac{1}{12(1+n_2)^3}\Big\{12n_2^3y^2 + 12n_2^2y(y+2)\\ &+ n_2\left(-12y^2 + 48y - 11\right) - 12y^2 + 24y - 5\Big\}.\end{aligned}$$

Lemma 2.4. *The following result holds:*

$$\begin{aligned}\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)\Big)(x) =&0, \quad (\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y))(x) = 0,\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)^2\Big)(x,y) =&\frac{1}{12(1+n_1)^3}\Big\{ - 24n_1^2(x-1)x\\ &+ n_1\left(-48x^2 + 48x - 11\right) - 24x^2 + 24x - 5\Big\},\\ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y)^2\Big)(x,y) =&\frac{1}{12(1+n_2)^3}\Big\{ - 24n_2^2(y-1)y\\ &+ n_2\left(-48y^2 + 48y - 11\right) - 24y^2 + 24y - 5\Big\}\\ &= O\left(\frac{1}{n}\right), \quad \text{when } n \to +\infty.\end{aligned}$$

Also,

$$\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(u-x)^4\Big)(x,y) = O\left(\frac{1}{n^2}\right), \quad \text{when } n \to +\infty$$

and

$$\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}(v-y)^4\Big)(x,y) = O\left(\frac{1}{n^2}\right), \quad \text{when } n \to +\infty.$$

3. Rate of Convergence

For $f \in C(J^2)$, the full modulus of continuity with respect to x and y is given as

$$\overline{\omega}(f,h) = \max\Big\{|f(x_1,y_1) - f(x_2,y_2)| : (x_1,y_1) \text{ and } (x_2,y_2) \in J^2\Big\}, \quad h > 0,$$

with the condition that

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \leq h.$$

And the partial moduli of continuity is given as

$$\omega_1(f,h) = \max\Big\{|f(x,y_1) - f(x,y_2)| : (x,y_1) \text{ and } (x,y_2) \in J^2 \text{ with } |y_1-y_2| \leq h\Big\}$$

and

$$\omega_2(f,h) = \max\Big\{|f(x_1,y) - f(x_2,y)| : (x_1,y) \text{ and } (x_2,y) \in J^2 \ \text{ with } |x_1-x_2| \leq h\Big\},$$

respectively.

They meet the well-known features of the usual modulus of continuity, as defined in [10]. Various results related to the partial moduli of continuity have been studied by researchers (for instance, one may refer [21]).

Theorem 3.1. *If $f \in C(J^2)$, the operators $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f$ converge uniformly to f on J^2.*

Proof. Clearly,

$$\lim_{n_1\to+\infty, n_2\to+\infty} \widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} e_{ij} = e_{ij},$$

for (i,j) taking the values $(0,0)$, $(1,0)$ and $(1,1)$, and

$$\lim_{n_1\to+\infty, n_2\to+\infty} \widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (e_{02}+e_{20}) = e_{02}+e_{20}.$$

Thus, on applying [12, Theorem 2.1], we obtain the desired result. □

Theorem 3.2. *For $f \in C(J^2)$ and $\zeta,\eta \in J^2$, we have*

$$\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\Big)(\zeta,\eta) - f(\zeta,\eta)\Big| \leq 2\Big\{\omega_1\Big(f, \frac{1}{\sqrt{n_1+1}}\Big) + \omega_2\Big(f, \frac{1}{\sqrt{n_2+1}}\Big)\Big\}.$$

Proof. From the property of partial moduli of continuity, we get

$$\begin{aligned}
\Big|\Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\Big)(\zeta,\eta) - f(\zeta,\eta)\Big| \leq& \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,v) - f(\zeta,\eta)|\Big)(\zeta,\eta)\\
\leq& \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,v) - f(u,\eta)|\Big)(\zeta,\eta)\\
&+ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,\eta) - f(\zeta,\eta)|\Big)(\zeta,\eta)\\
\leq& \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} \omega_2(f,|v-\eta|)\Big)(\zeta,\eta)\\
&+ \Big(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} \omega_1(f,|u-\zeta|)\Big)(\zeta,\eta)\\
\leq& \Big(1 + h_{n_2}^{-1}(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|v-\eta|)(\eta)\Big)\,\omega_2(f,h_{n_2})\\
&+ \Big(1 + h_{n_1}^{-1}(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}|u-\zeta|)(\zeta)\Big)\,\omega_1(f,h_{n_1}),
\end{aligned}$$

where $h_{n_1}, h_{n_2} > 0$.

Making use of Cauchy-Schwarz inequality, we may write

$$\left|\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2} f\right)(\zeta,\eta) - f(\zeta,\eta)\right| \leq \left(1 + h_{n_2}^{-1}\sqrt{\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}(v-\eta)^2\right)(\eta)}\right)\omega_2(f,h_{n_2}) + \left(1 + h_{n_1}^{-1}\sqrt{\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}(u-\zeta)^2\right)(\zeta)}\right)\omega_1(f,h_{n_1}).$$

Thus, by choosing $h_{n_1} = \frac{1}{\sqrt{n_1+1}}$ and $h_{n_2} = \frac{1}{\sqrt{n_2+1}}$, we reach the required result. □

Theorem 3.3. *For $f \in C(J^2)$ and $\zeta, \eta \in J^2$, we have*

$$\left|\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2} f\right)(\zeta,\eta) - f(\zeta,\eta)\right| \leq ||f_\zeta||\sqrt{\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}(u-\zeta)^2\right)(\zeta,\eta)} + ||f_\eta||\sqrt{\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}(v-\eta)^2\right)(\zeta,\eta)}.$$

Proof. If $(\zeta,\eta) \in J^2$, then

$$f(u,v) - f(\zeta,\eta) = \int_\zeta^u f_s(s,v)ds + \int_\eta^v f_t(\zeta,t)dt.$$

Applying the operators $\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}$ on both sides of the sides of above inequality, we get

$$\left|\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2} f\right)(u,v) - f(\zeta,\eta)\right| \leq \left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}\int_\zeta^u f_s(s,v)ds\right)(\zeta,\eta) + \left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}\int_\eta^v f_t(\zeta,t)dt\right)(\zeta,\eta),$$

as

$$\left|\int_\zeta^u f_s(s,v)ds\right| \leq ||f_\zeta|| \cdot |u-\zeta|$$

and

$$\left|\int_\eta^v f_t(\zeta,t)dt\right| \leq ||f_\eta|| \cdot |v-\eta|,$$

therefore,

$$\left|\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2} f\right)(u,v) - f(\zeta,\eta)\right| \leq ||f_\zeta||\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}|u-\zeta|\right)(\zeta,\eta) + ||f_\eta||\left(\widetilde{Q}^{(1/n_1,1/n_2)}_{n_1,n_2}|v-\eta|\right)(\zeta,\eta).$$

We got the desired conclusion by using the Cauchy-Schwarz inequality. □

For $(u,v),(\zeta,\eta) \in J^2$, we define the Lipschitz class (as defined in [13]), $\mathrm{Lip}_K\,\alpha$, as follows:

$$\mathrm{Lip}_K\,\alpha = \left\{f \in C(J^2) : |f(u,v) - f(\zeta,\eta)| \leq K\left\{(u-\zeta)^2 + (v-\eta)^2\right\}^{\frac{\alpha}{2}};\ \alpha \in (0,1]\right\}.$$

Theorem 3.4. *If $f \in \mathrm{Lip}_K\,\alpha$, then the following conclusion is correct:*

$$\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\right)(\zeta,\eta) - f(\zeta,\eta)\right| \leq K\left\{\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (u-\zeta)^2\right)(\zeta,\eta) + \left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (v-\eta)^2\right)(\zeta,\eta)\right\}^{\frac{\alpha}{2}}.$$

Proof. If $f \in \mathrm{Lip}_K\,\alpha$, then we may write

$$\begin{aligned}\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\right)(\zeta,\eta) - f(\zeta,\eta)\right| &\leq \left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} |f(u,v) - f(\zeta,\eta)|\right)(\zeta,\eta) \\ &\leq K\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} \left\{|u-\zeta|^2 + |v-\eta|^2\right\}^{\frac{\alpha}{2}}\right)(\zeta,\eta).\end{aligned}$$

Using the Hölder's inequality and $v_1 = \frac{2}{\alpha}$ and $w_1 = \frac{2}{2-\alpha}$, we obtain

$$\left|\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} f\right)(\zeta,\eta) - f(\zeta,\eta)\right| \leq K\left\{\left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (u-\zeta)^2\right)(\zeta,\eta) + \left(\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)} (v-\eta)^2\right)(\zeta,\eta)\right\}^{\frac{\alpha}{2}}.$$

Hence, the required result follows. □

4. Voronovskaja–type Theorem

Let $C^2(J^2)$ be the space containing the functions f that have the property $f \in C(J^2)$ and $f^{(i,j)} \in C(J^2)$, $0 \leq i + j \leq 2$.

Here,

$$f^{(i,j)} = \left\{\frac{\partial^i f}{\partial \zeta^i}, \frac{\partial^i f}{\partial \eta^i} : \ i = 1,2\right\}, \quad \zeta,\eta \in J^2.$$

The space $C^2(J^2)$ is equipped with the norm

$$||f||_{C^2(J^2)} = ||f||_{C(J^2)} + \left\|\frac{\partial f}{\partial \zeta}\right\|_{C(J^2)} + \left\|\frac{\partial f}{\partial \eta}\right\|_{C(J^2)} + \left\|\frac{\partial^2 f}{\partial \zeta^2}\right\|_{C(J^2)} + \left\|\frac{\partial^2 f}{\partial \eta^2}\right\|_{C(J^2)}.$$

Theorem 4.1. *Let $f \in C^2(J^2)$, then*

$$\lim_{n\to+\infty} n\left\{\left(\widetilde{Q}_{n,n}^{(1/n,1/n)} f\right)(\zeta,\eta) - f(\zeta,\eta)\right\} = \zeta(\zeta-1) f_{\zeta\zeta}(\zeta,\eta) + \eta(\eta-1) f_{\eta\eta}(\zeta,\eta).$$

Proof. Let $(\zeta,\eta),(u,v) \in J^2$. Applying Taylor's expansion, we get

$$\begin{aligned}f(u,v) =& f(\zeta,\eta) + f_\eta(\zeta,\eta)(v-\eta) + f_\zeta(\zeta,\eta)(u-\zeta) + \frac{1}{2}\Big\{f_{\eta\eta}(\zeta,\eta)(v-\eta)^2 \\ &+ f_{\zeta\zeta}(\zeta,\eta)(u-\zeta)^2 + 2f_{\zeta\eta}(\zeta,\eta)(u-\zeta)(v-\eta)\Big\} \\ &+ \xi(u,v)\Big\{(u-\zeta)^2 + (v-\eta)^2\Big\},\end{aligned}$$

where $\xi(u,v)$ vanishes as $(u,v) \to (\zeta,\eta)$.

By the linearity of $\widetilde{Q}_{n,n}^{(1/n,1/n)}$, we have

$$\begin{aligned}(\widetilde{Q}_{n,n}^{(1/n,1/n)}f)(u,v) =& f(\zeta,\eta) + f_\eta(\zeta,\eta)(\widetilde{Q}_n^{(1/n)}(v-\eta))(\eta) + f_\zeta(\zeta,\eta)(\widetilde{Q}_n^{(1/n)}(u-\zeta))(\zeta)\\ &+ \frac{1}{2}\Big\{ f_{\eta\eta}(\widetilde{Q}_n^{(1/n)}(v-\eta)^2)(\eta) + f_{\zeta\zeta}(\widetilde{Q}_n^{(1/n)}(u-\zeta)^2)(\zeta)\\ &+ 2f_{\zeta\eta}(\zeta,\eta)(\widetilde{Q}_n^{(1/n)}(u-\zeta))(\zeta)(\widetilde{Q}_n^{(1/n)}(v-\eta))(\eta)\Big\}\\ &+ \widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}.\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}&\left|\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}\right|\\ \leq& \Big\{\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)\Big\}^{\frac{1}{2}}\Big\{\Big(\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big((u-\zeta)^2+(v-\eta)^2\Big)^2\Big)(\zeta,\eta)\Big\}^{\frac{1}{2}}\\ \leq& \sqrt{2}\Big\{\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)\Big\}^{\frac{1}{2}}\Big\{(\widetilde{Q}_{n,n}^{(1/n,1/n)}(u-\zeta)^4)(\zeta)+(\widetilde{Q}_{n,n}^{(1/n,1/n)}(v-\eta)^4)(\eta)\Big\}^{\frac{1}{2}}.\end{aligned}$$

In view of Theorem 3.1, we have

$$\lim_{n\to+\infty}\widetilde{Q}_{n,n}^{(1/n,1/n)}\xi^2(u,v)(\zeta,\eta)=0.$$

Using Lemma 2.4, we may write

$$\lim_{n\to+\infty} n\widetilde{Q}_{n,n}^{(1/n,1/n)}\Big\{\xi(u,v)\Big((u-\zeta)^2+(v-\eta)^2\Big)\Big\}(\zeta,\eta)=0.$$

Finally, on using the values from Lemma 2.4, the proof of the theorem follows. □

5. Graphical Analysis

For validating the convergence results obtained in the above sections, we provide few numerical examples involving illustrative graphics.

Example 5.1. For $f(x,y)=x^2-x+y^2-y$, we show the convergence of $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}$ to $f(x,y)=x^2-x+y^2-y$ for $n_1=n_2=50$ and $n_1=n_2=200$ in Figure 1 and Figure 2, respectively.

Example 5.2. For $f(x,y)=-\sqrt{7}\,(x^2+2xy-2x+y^2-2y+1)+x^2-10xy$, we show the convergence of $\widetilde{Q}_{n_1,n_2}^{(1/n_1,1/n_2)}$ to $f(x,y)$ for $n_1=n_2=5$ and $n_1=n_2=50$ in Figure 3 and Figure 4, respectively.

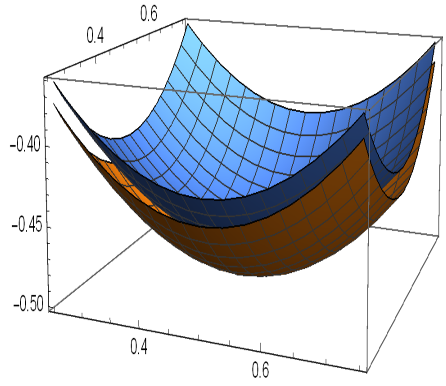

FIGURE 1. Graphs of $\widetilde{Q}_{50,50}^{(1/50,1/50)}$ (blue) and $f(x,y) = x^2 - x + y^2 - y$ (yellow).

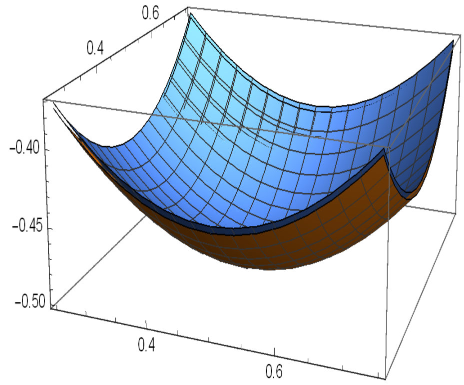

FIGURE 2. Graphs of $\widetilde{Q}_{200,200}^{(1/200,1/200)}$ (blue) and $f(x,y) = x^2 - x + y^2 - y$ (yellow).

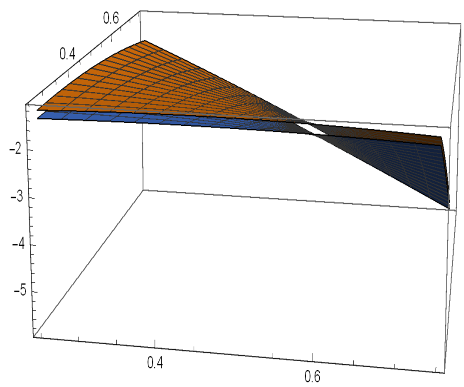

FIGURE 3. Graphs of $\widetilde{Q}_{5,5}^{(1/5,1/5)}$ (blue) and $f(x,y) = -\sqrt{7}\,(x^2 + 2xy - 2x + y^2 - 2y + 1) + x^2 - 10xy$ (yellow).

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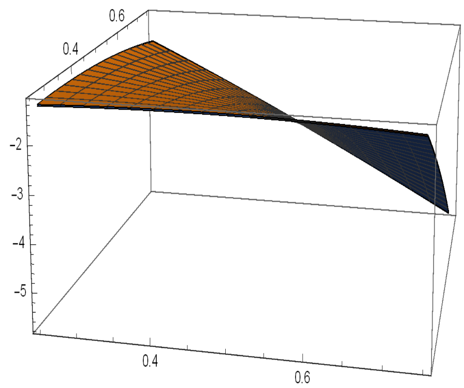

FIGURE 4. Graphs of $\widetilde{Q}_{50,50}^{(1/50,1/50)}$ (blue) and $f(x,y) = -\sqrt{7}\,(x^2+2xy-2x+y^2-2y+1)+x^2-10xy$ (yellow).

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WEAVING CONTINUOUS CONTROLLED K-g-FUSION FRAMES IN HILBERT SPACES

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ABSTRACT. We introduce the notion of weaving continuous controlled K-g-fusion frame in Hilbert space. Some characterizations of weaving continuous controlled K-g-fusion frame have been presented. We extend some of the recent results of woven K-g-fusion frame and controlled K-g-fusion frame to woven continuous controlled K-g-fusion frame. Finally, a perturbation result of woven continuous controlled K-g-fusion frame has been studied.

1. INTRODUCTION AND PRELIMINARIES

Duffin and Schaeffer [13] introduced frame for Hilbert space to study some fundamental problems in non-harmonic Fourier series. Later on, after some decades, frame theory was popularized by Daubechies et al. [11]. At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on.

Let H be a separable Hilbert space associated with the inner product $\langle \cdot, \cdot \rangle$. Frame for Hilbert space was defined as a sequence of basis-like elements in Hilbert space. A sequence $\{f_i\}_{i=1}^{+\infty} \subset H$ is called a frame for H, if there exist positive constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{+\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The constants A and B are called lower and upper bounds, respectively.

Key words and phrases. Frame, g-fusion frame, continuous g-fusion frame, controlled frame, woven frame.

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Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\mathbb{H}$ is the collection of all closed subspaces of H. (X, μ) denotes abstract measure space with positive measure μ. I_H is the identity operator on H. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2. In particular, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H. For $S \in \mathcal{B}(H)$, we denote $\mathcal{N}(S)$ and $\mathcal{R}(S)$ for null space and range of S, respectively. Also, $P_M \in \mathcal{B}(H)$ is the orthonormal projection of H onto a closed subspace $M \subset H$. The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $R, S \in \mathcal{S}(H)$

$$R \leq S \Leftrightarrow \langle Rf, f \rangle \leq \langle Sf, f \rangle, \quad \text{for all } f \in H.$$

$\mathcal{GB}(H)$ denotes the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{GB}(H)$, then R^*, R^{-1} and SR also belongs to $\mathcal{GB}(H)$. An operator $U \in \mathcal{B}(H)$ is called positive if $\langle Uf, f \rangle \geq 0$ for all $f \in H$. In notation, we can write $U \geq 0$. If $V \in B(H)$ is positive then there exists a unique positive U such that $V^2 = U$. This will be denoted by $V = U^{1/2}$. Moreover, if an operator V commutes with U then V commutes with every operator in the C^*-algebra generated by U and I, specially V commutes with $U^{1/2}$. $\mathcal{GB}^+(H)$ is the set of all positive operators in $\mathcal{GB}(H)$ and T, U are invertible operators in $\mathcal{GB}(H)$. For each $m > 1$, we define $[m] = \{1, 2, \ldots, m\}$.

We present some theorems in operator theory which will be needed throughout this paper.

Theorem 1.1 (Douglas' factorization theorem [12])**.** *Let $S, V \in \mathcal{B}(H)$. Then the following conditions are equivalent.*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(V)$.
(ii) $SS^* \leq \lambda^2 VV^*$ *for some* $\lambda > 0$.
(iii) $S = VW$ *for some bounded linear operator* W *on* H.

Theorem 1.2 ([15])**.** *Let $M \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_M T^* = P_M T^* P_{\overline{TM}}$. If T is an unitary operator (i.e., $T^* T = I_H$), then $P_{\overline{TM}} T = T P_M$.*

Theorem 1.3 ([8])**.** *Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_U$. Then, there exists a bounded linear operator $U^\dagger : H_2 \to H_1$ such that $UU^\dagger x = x$ for all $x \in \mathcal{R}_U$.*

1.1. ***K*-*g*-fusion frame.** Construction of K-g-fusion frames and their dual were presented by Sadri and Rahimi [1] to generalize the theory of K-frame [16], fusion frame [9], and g-frame [35].

Definition 1.1 ([1])**.** Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights, $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces. Suppose $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$ and $K \in \mathcal{B}(H)$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a K-g-fusion frame for H respect to $\{H_j\}_{j \in J}$ if there exist constants $0 < A \leq B < +\infty$

such that

$$A\left\|K^{*}f\right\|^{2}\leq\sum_{j\in J}v_{j}^{2}\left\|\Lambda_{j}P_{W_{j}}(f)\right\|^{2}\leq B\left\|f\right\|^{2},$$

for all $f\in H$. The constants A and B are called the lower and upper bounds of K-g-fusion frame, respectively. If $K=I_{H}$ then the family is called g-fusion frame and it has been widely studied in [18–20, 31].

Define the space

$$\ell^{2}\left(\{H_{j}\}_{j\in J}\right)=\left\{\{f_{j}\}_{j\in J}:f_{j}\in H_{j},\sum_{j\in J}\left\|f_{j}\right\|^{2}<+\infty\right\},$$

with inner product given by

$$\langle\{f_{j}\}_{j\in J},\{g_{j}\}_{j\in J}\rangle=\sum_{j\in J}\langle f_{j},g_{j}\rangle_{H_{j}}.$$

Clearly, $\ell^{2}\left(\{H_{j}\}_{j\in J}\right)$ is a Hilbert space with the pointwise operations [1].

1.2. **Controlled K-g-fusion frame.** Controlled frame is one of the newest generalization of frame. P. Balaz et al. [6] introduced controlled frame to improve the numerical efficiency of interactive algorithms for inverting the frame operator. In recent times, several generalizations of controlled frame namely, controlled K-frame [26], controlled g-frame [27], controlled fusion frame [23], controlled g-fusion frame [34], controlled K-g-fusion frame [28] etc. have been appeared.

Definition 1.2 ([28])**.** Let $K\in\mathcal{B}(H)$ and $\{W_{j}\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_{j}\}_{j\in J}$ be a collection of positive weights. Let $\{H_{j}\}_{j\in J}$ be a sequence of Hilbert spaces, $T,U\in\mathcal{G}\mathcal{B}(H)$ and $\Lambda_{j}\in\mathcal{B}(H,H_{j})$ for each $j\in J$. Then the family $\Lambda_{TU}=\{(W_{j},\Lambda_{j},v_{j})\}_{j\in J}$ is a (T,U)-controlled K-g-fusion frame for H if there exist constants $0<A\leq B<+\infty$ such that

$$A\|K^{*}f\|^{2}\leq\sum_{j\in J}v_{j}^{2}\left\langle\Lambda_{j}P_{W_{j}}Uf,\Lambda_{j}P_{W_{j}}Tf\right\rangle\leq B\|f\|^{2},\tag{1.1}$$

for all $f\in H$. If Λ_{TU} satisfies only the right inequality of (1.1) it is called a (T,U)-controlled g-fusion Bessel sequence in H.

Let Λ_{TU} be a (T,U)-controlled g-fusion Bessel sequence in H with a bound B. The synthesis operator $T_{C}:\mathcal{K}_{\Lambda_{j}}\to H$ is defined as

$$T_{C}\left(\left\{v_{j}\left(T^{*}P_{W_{j}}\Lambda_{j}^{*}\Lambda_{j}P_{W_{j}}U\right)^{1/2}f\right\}_{j\in J}\right)=\sum_{j\in J}v_{j}^{2}T^{*}P_{W_{j}}\Lambda_{j}^{*}\Lambda_{j}P_{W_{j}}Uf,$$

for all $f\in H$ and the analysis operator $T_{C}^{*}:H\to\mathcal{K}_{\Lambda_{j}}$ is given by

$$T_{C}^{*}f=\left\{v_{j}\left(T^{*}P_{W_{j}}\Lambda_{j}^{*}\Lambda_{j}P_{W_{j}}U\right)^{1/2}f\right\}_{j\in J},\quad\text{for all } f\in H,$$

where

$$\mathcal{K}_{\Lambda_j} = \left\{ \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} : f \in H \right\} \subset \ell^2 \left(\{H_j\}_{j \in J} \right).$$

The frame operator $S_C : H \to H$ is defined as follows:

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f,$$

for all $f \in H$ and it is easy to verify that

$$\langle S_C f, f \rangle = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle,$$

for all $f \in H$. Furthermore, if Λ_{TU} is a (T, U)-controlled K-g-fusion frame with bounds A and B, then $AKK^* \leq S_C \leq BI_H$.

1.3. **Continuous controlled g-fusion frame.** In recent times, controlled frames and their generalizations are also studied in continuous case by many researchers. P. Ghosh and T. K. Samanta studied continuous version of controlled g-fusion frame in [21].

Definition 1.3 ([21])**.** Let $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{GB}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a continuous (T, U)-controlled generalized fusion frame or continuous (T, U)-controlled g-fusion frame for H with respect to (X, μ) and v, if

(i) for each $f \in H$, the mapping $x \mapsto P_{F(x)}(f)$ is measurable (i.e., is weakly measurable);

(ii) there exist constants $0 < A \leq B < +\infty$ such that

$$A \|f\|^2 \leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B \|f\|^2, \tag{1.2}$$

for all $f \in H$, where $P_{F(x)}$ is the orthogonal projection of H onto the subspace $F(x)$. The constants A, B are called the frame bounds. If only the right inequality of (1.2) holds then Λ_{TU} is called a continuous (T, U)-controlled g-fusion Bessel family for H.

Let Λ_{TU} be a continuous (T, U)-controlled g-fusion Bessel family for H. Then the operator $S_C : H \to H$ defined by

$$\langle S_C f, g \rangle = \int_X v^2(x) \left\langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \right\rangle d\mu_x,$$

for all $f, g \in H$, is called the frame operator. If Λ_{TU} is a continuous (T, U)-controlled g-fusion frame for H, then from (1.2), we get

$$A \langle f, f \rangle \leq \langle S_C f, f \rangle \leq B \langle f, f \rangle, \quad \text{for all } f \in H.$$

The bounded linear operator $T_C : L^2(X, K) \to H$ defined by

$$\langle T_C \Phi, g \rangle = \int_X v^2(x) \left\langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \right\rangle d\mu_x,$$

where for all $f \in H$, $\Phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U \right)^{1/2} f \right\}_{x \in X}$ and $g \in H$, is called synthesis operator and its adjoint operator is called analysis operator.

1.4. **Weaving frame.** Woven frame is a new notion in frame theory which has been introduced by Bemrose et al. [7]. Two frames $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ for H are called woven if there exist constants $0 < A \le B < +\infty$ such that for any subset $\sigma \subset I$ the family $\{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c}$ is a frame for H. This frame has been generalized for the discrete as well as the continuous case such as woven fusion frame [17], woven g-frame [24], woven g-fusion frame [25], woven K-g-fusion frame [32], continuous weaving frame [36], continuous weaving fusion frame [33], continuous weaving g-frames [3], weaving continuous K-g-frames [5], controlled weaving frames [29], continuous controlled K-g-frames [30] etc.

In this paper, woven continuous controlled K-g-fusion frame in Hilbert spaces is presented and some of their properties are going to be established. We discuss sufficient conditions for weaving continuous controlled K-g-fusion frame. Construction of woven continuous controlled K-g-fusion frame by bounded linear operator is given. At the end, we discuss a perturbation result of woven continuous controlled K-g-fusion frame.

2. Weaving Continuous Controlled K-g-Fusion Frame

In this section, we first give the continuous version of controlled K-g-fusion frame for H and then present weaving continuous controlled K-g-fusion frame for H.

Definition 2.1. Let $K \in \mathcal{B}(H)$ and $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda(x) \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{GB}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda(x), v(x))\}_{x \in X}$ is called a continuous (T, U)-controlled K-g-fusion frame for H with respect to (X, μ) and v, if

(i) for each $f \in H$, the mapping $x \mapsto P_{F(x)}(f)$ is measurable (i.e., is weakly measurable);

(ii) there exist constants $0 < A \le B < +\infty$ such that

$$A \|K^* f\|^2 \le \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \le B \|f\|^2, \tag{2.1}$$

for all $f \in H$, where $P_{F(x)}$ is the orthogonal projection of H onto the subspace $F(x)$. The constants A, B are called the frame bounds.

Now, we consider the following cases.

(i) If only the right inequality of (2.1) holds, then Λ_{TU} is called a continuous (T, U)-controlled K-g-fusion Bessel family for H.

(ii) If $U = I_H$, then Λ_{TU} is called a continuous (T, I_H)-controlled K-g-fusion frame for H.
(iii) If $T = U = I_H$, then Λ_{TU} is called a continuous K-g-fusion frame for H (for more details, refer to [4]).
(iv) If $K = I_H$, then Λ_{TU} is called a continuous (T, U)-controlled g-fusion frame for H.

Remark 2.1. If the measure space $X = \mathbb{N}$ and μ is the counting measure then a continuous (T, U)-controlled K-g-fusion frame will be the discrete (T, U)-controlled K-g-fusion frame.

2.0.1. *Example.* Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an standard orthonormal basis for H. Consider

$$\mathcal{B} = \left\{x \in \mathbb{R}^3 : \|x\| \leq 1\right\}.$$

Then it is a measure space equipped with the Lebesgue measure μ. Let us now consider that $\{B_1, B_2, B_3\}$ is a partition of $\mathcal{B}$ where $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 1$. Let $\mathbb{H} = \{W_1, W_2, W_3\}$, where $W_1 = \overline{\text{Span}}\,\{e_1, e_2\}$, $W_2 = \overline{\text{Span}}\,\{e_2, e_3\}$ and $W_3 = \overline{\text{Span}}\,\{e_1, e_3\}$. Define $F : \mathcal{B} \to \mathbb{H}$ by

$$F(x) = \begin{cases} W_1, & \text{if } x \in B_1, \\ W_2, & \text{if } x \in B_2, \\ W_3, & \text{if } x \in B_3, \end{cases}$$

and $v : \mathcal{B} \to [0, +\infty)$ by

$$v(x) = \begin{cases} 1, & \text{if } x \in B_1, \\ 2, & \text{if } x \in B_2, \\ -1, & \text{if } x \in B_3. \end{cases}$$

It is easy to verify that F and v are measurable functions. For each $x \in \mathcal{B}$, define the operators

$$\Lambda(x)(f) = \frac{1}{\sqrt{\mu(B_k)}} \langle f, e_k \rangle\, e_k,$$

$f \in H$, where k is such that $x \in \mathcal{B}_k$ and $K : H \to H$ by

$$Ke_1 = e_1, \quad Ke_2 = e_2, \quad Ke_3 = 0.$$

It is easy to verify that $K^* e_1 = e_1$, $K^* e_2 = e_2$, $K^* e_3 = 0$. Now, for any $f \in H$, we have

$$\|K^* f\|^2 = \left\| \sum_{i=1}^{3} \langle f, e_k \rangle\, K^* e_k \right\|^2 = |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2 \leq \|f\|^2.$$

Let $T\,(f_1, f_2, f_3) = (5f_1, 4f_2, 5f_3)$ and $U\,(f_1, f_2, f_3) = \left(\frac{f_1}{6}, \frac{f_2}{3}, \frac{f_3}{6}\right)$ be two operators on H. Then it is easy to verify that $T, U \in \mathcal{GB}^+(H)$ and $TU = UT$. Now, for any

$f=(f_1,f_2,f_3)\in H$, we have

$$\begin{aligned}&\int\limits_{\mathcal{B}} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\ =&\sum_{i=1}^{3}\int\limits_{\mathcal{B}_i} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\ =&\frac{5}{6}f_1^2+\frac{16}{3}f_2^2+\frac{5}{6}f_3^2.\end{aligned}$$

This implies that

$$\frac{5}{6}\left\|K^*f\right\|^2\leq \int\limits_{\mathcal{B}} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\leq \frac{16}{3}\|f\|^2.$$

Thus, Λ_{TU} be a continuous (T,U)-controlled K-g-fusion frame for $\mathbb{R}^3$.

Now, we present woven continuous controlled K-g-fusion frame for H.

Definition 2.2. A family of continuous (T,U)-controlled K-g-fusion frames given by $\{(F_i(x),\Lambda_i(x),v_i(x))\}_{i\in[m],x\in X}$ for H is said to be woven continuous (T,U)-controlled K-g-fusion frame if there exist universal positive constants $0<A\leq B<+\infty$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\{(F_i(x),\Lambda_i(x),v_i(x))\}_{i\in[m],x\in\sigma_i}$ is a continuous (T,U)-controlled K-g-fusion frame for H with bounds A and B.

Each family $\{(F_i(x),\Lambda_i(x),v_i(x))\}_{i\in[m],x\in\sigma_i}$ is called a weaving continuous (T,U)-controlled K-g-fusion frame. For abbreviation, we use W. C. C. K. G. F. F. instead of the statement of woven continuous (T,U)-controlled K-g-fusion frame.

In the following proposition, we will see that every woven continuous controlled K-g-fusion frame has a universal upper bound.

Proposition 2.1. *Suppose for each $i\in[m]$, $\{(F_i(x),\Lambda_i(x),v_i(x))\}_{x\in X}$ be a continuous (T,U)-controlled K-g-fusion Bessel family for H with bound B_i. Then for any partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\{(F_i(x),\Lambda_i(x),v_i(x))\}_{i\in[m],x\in\sigma_i}$ is a continuous (T,U)-controlled K-g-fusion Bessel family for H.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be a arbitrary partition of X. For each $f\in H$, we have

$$\begin{aligned}&\sum_{i\in[m]}\int\limits_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \leq&\sum_{i\in[m]}\int\limits_{X} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \leq&\left(\sum_{i\in[m]}B_i\right)\|f\|^2.\end{aligned}$$

This completes the proof. ∎

Next, we give a characterization of W. C. C. K. G. F. F. for H in terms of an operator.

Theorem 2.1. *Let the families given by* $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x \in X}$ *and* $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x \in X}$ *be continuous* (T, U)*-controlled* K*-g-fusion frames for* H*. The the following statements are equivalent.*

(i) *Λ and Γ are W. C. C. K. G. F. F. for H.*

(ii) *For each partition σ of X, there exist $\alpha > 0$ and a bounded linear operator $\Theta_\sigma : L^2_\sigma(X, K) \to H$ defined by*

$$\langle \Theta_\sigma \Phi, g \rangle = \int\limits_\sigma v^2(x) \left\langle T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U f, g \right\rangle d\mu_x + \int\limits_{\sigma^c} v^2(x) \left\langle T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U f, g \right\rangle d\mu_x,$$

$g \in H$ such that $\alpha K K^* \leq \Theta_\sigma \Theta_\sigma^*$, where

$$L^2_\sigma(X, K) = \left\{ \Phi = \phi \cup \psi : \int\limits_X \|\Phi\|^2 d\mu < +\infty \right\},$$

where for all $f \in H$,

$$\phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U \right)^{1/2} f \right\}_{x \in \sigma}$$

and

$$\psi = \left\{ v(x) \left(T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U \right)^{1/2} f \right\}_{x \in \sigma^c}.$$

Proof. $(i) \Rightarrow (ii)$ Suppose that A and B are the universal lower and upper bounds for Λ and Γ. Take $\Theta_\sigma = T_C^\sigma$, for every partition σ of X, where T_C^σ is the synthesis operator of

$$\{(F(x), \Lambda(x), v(x))\}_{x \in \sigma} \cup \{(G(x), \Lambda(x), v(x))\}_{x \in \sigma^c}.$$

Thus, for each $\Phi \in L^2_\sigma(X, K)$, we have

$$\begin{aligned} \langle \Theta_\sigma \Phi, g \rangle &= \langle T_C^\sigma \Phi, g \rangle \\ &= \int\limits_\sigma v^2(x) \left\langle T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U f, g \right\rangle d\mu_x \\ &\quad + \int\limits_{\sigma^c} v^2(x) \left\langle T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U f, g \right\rangle d\mu_x, \quad g \in H. \end{aligned}$$

Since Λ and Γ are woven, for each $f \in H$, we have

$$A \|K^* f\|^2 \leq \|(T_C^\sigma)^* f\|^2 = \|\Theta_\sigma^* f\|^2.$$

Thus, $\alpha K K^* \leq \Theta_\sigma \Theta_\sigma^*$, $\alpha = A$.

$(ii) \Rightarrow (i)$ Let σ be a partition of X and $f \in H$. Now it is easy to verify that

$$\Theta_\sigma^* f = \left\{ v(x) \left(T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U \right)^{1/2} f \right\}_{x \in \sigma}$$

$$\cup \left\{ v(x) \left(T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U \right)^{1/2} f \right\}_{x \in \sigma^c}.$$

Thus, for each $f \in H$, we have

$$\begin{aligned}\alpha \|K^* f\|^2 \leq \|\Theta_\sigma^* f\|^2 = & \int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ & + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x.\end{aligned}$$

Hence, Λ and Γ are W. C. C. K. G. F. F. for H. This completes the proof. □

In the following theorem, we will construct W. C. C. K. G. F. F. for H by using a bounded linear operator.

Theorem 2.2. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i \in [m], x \in \sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If $V \in \mathcal{B}(H)$ is invertible such that V^* commutes with T, U and V commutes with K, then $\left\{ \left(V F_i(x), \Lambda_i(x) P_{F_i(x)} V^*, v_i(x) \right) \right\}_{i \in [m], x \in \sigma_i}$ is a W. C. C. K. G. F. F. for H.*

Proof. Since $P_{F_i(x)} V^* = P_{F_i(x)} V^* P_{V F_i(x)}$ for all $x \in \sigma_i$ and $i \in [m]$, the mapping $x \mapsto P_{V F_i(x)}$ is weakly measurable. For each $f \in H$, we have

$$\begin{aligned}& \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} T f \right\rangle d\mu_x \\ = & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* U f, \Lambda_i(x) P_{F_i(x)} V^* T f \right\rangle d\mu_x \\ = & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U V^* f, \Lambda_i(x) P_{F_i(x)} T V^* f \right\rangle d\mu_x \\ \leq & B \|V^* f\|^2 \leq B \|V\|^2 \|f\|^2.\end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned}& \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} T f \right\rangle d\mu_x \\ \geq & A \|K^* V^* f\|^2 = A \|V^* K^* f\|^2 \geq A \left\| V^{-1} \right\|^{-2} \|K^* f\|^2.\end{aligned}$$

This completes the proof. □

Corollary 2.1. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i \in [m], x \in \sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If $V \in \mathcal{B}(H)$ is invertible such that V^* commutes with T, U and V commutes with K, then $\left\{ \left(V F_i(x), \Lambda_i(x) P_{F_i(x)} V^*, v_i(x) \right) \right\}_{i \in [m], x \in \sigma_i}$ is a W. C. C. $V K V^*$. G. F. F. for H.*

Proof. According to the proof of Theorem 2.2, universal upper bounds is $B\|V\|^2$. On the other hand, for each $f \in H$, we have

$$\frac{A}{\|V\|^2}\left\|(VKV^*)^*f\right\|^2 = \frac{A}{\|V\|^2}\left\|VK^*V^*f\right\|^2 \leq A\left\|K^*V^*f\right\|^2$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}UV^*f, \Lambda_i(x)P_{F_i(x)}TV^*f\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}Uf, \Gamma_i(x)P_{VF_i(x)}Tf\right\rangle d\mu_x,$$

where $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$. This completes the proof. □

Theorem 2.3. *Let $V \in \mathcal{B}(H)$ be invertible operator such that $V^*, (V^{-1})^*$ commutes with T and U. Suppose $\left\{\left(VF_i(x), \Lambda_i(x)P_{F_i(x)}V^*, v_i(x)\right)\right\}_{i\in[m], x\in\sigma_i}$ is a W. C. C. K. G. F. F. for H with universal bounds A and B. Then $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ be a W. C. C. $V^{-1}KV$. G. F. F. for H.*

Proof. Now, for each $f \in H$, using Theorem 1.2, and taking $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$, we have

$$\frac{A}{\|V\|^2}\left\|\left(V^{-1}KV\right)^*f\right\|^2 = \frac{A}{\|V\|^2}\left\|V^*K^*(V^{-1})^*f\right\|^2$$
$$\leq A\left\|K^*\left(V^{-1}\right)^*f\right\|^2$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}U\left(V^{-1}\right)^*f, \Gamma_i(x)P_{VF_i(x)}T\left(V^{-1}\right)^*f\right\rangle d\mu_x$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)U\left(V^{-1}\right)^*f, \Gamma_i(x)T\left(V^{-1}\right)^*f\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)\left(V^{-1}\right)^*Uf, \Gamma_i(x)\left(V^{-1}\right)^*Tf\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x.$$

On the other hand, for each $f \in H$, it is easy to verify that

$$\sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq B\left\|V^{-1}\right\|^2\|f\|^2.$$

This completes the proof. □

Next, we will see that the intersection of components of a W. C. C. K. G. F. F. with a closed subspace is a W. C. C. K. G. F. F. for the smaller space.

Theorem 2.4. *Let $\{F(x), \Lambda(x), v(x)\}_{x\in X}$ and $\{G(x), \Gamma(x), w(x)\}_{x\in X}$ be W. C. C. K. G. F. F. for H and W be a closed subspace of H. Then the families given by*

$\{F(x)\cap W, \Lambda(x), v(x)\}_{x\in X}$ *and* $\{G(x)\cap W, \Gamma(x), w(x)\}_{x\in X}$ *are W. C. C. K. G. F. F. for* W.

Proof. The operators $P_{F(x)\cap W} = P_{F(x)}(P_W)$ and $P_{G(x)\cap W} = P_{G(x)}(P_W)$ are orthogonal projections of H onto $F(x)\cap W$ and $G(x)\cap W$, respectively. Let σ be a measurable subset of X. Then for each $f \in W$, we have

$$\begin{aligned}
&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x \\
&=\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}P_WUf, \Lambda(x)P_{F(x)}P_WTf\right\rangle d\mu_x \\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)}P_WUf, \Gamma(x)P_{G(x)}P_WTf\right\rangle d\mu_x \\
&=\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)\cap W}Uf, \Lambda(x)P_{F(x)\cap W}Tf\right\rangle d\mu_x \\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)\cap W}Uf, \Gamma(x)P_{G(x)\cap W}Tf\right\rangle d\mu_x.
\end{aligned}$$

This completes the proof. □

The following theorem states the equivalence between W. C. C. K. G. F. F. and a bounded linear operator.

Theorem 2.5. *Let $V \in \mathcal{B}(H)$ be an invertible operator such that V^* commutes with T, U. Suppose K be a bounded linear operator on H which have closed range. Let $\Lambda_{TU} = \{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. Then the family given by*

$$\Delta_{TU} = \left\{\left(VF_i(x), \Lambda_i(x)P_{F_i(x)}V^*, v_i(x)\right)\right\}_{i\in[m], x\in\sigma_i}$$

is a W. C. C. K. G. F. F. for H if and only if there exists a $\delta > 0$ such that for each $f \in H$, we have $\|V^*f\| \geq \delta\,\|K^*f\|$.

Proof. Suppose that Δ_{TU} is a W. C. C. K. G. F. F. for H with bounds C and D. Then for each $f \in H$, using the Theorem 1.2, and taking $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$, we have

$$\begin{aligned}
C\,\|K^*f\|^2 &\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}Uf, \Gamma_i(x)P_{VF_i(x)}Tf\right\rangle d\mu_x \\
&= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}V^*Uf, \Lambda_i(x)P_{F_i(x)}V^*Tf\right\rangle d\mu_x \\
&= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}UV^*f, \Lambda_i(x)P_{F_i(x)}TV^*f\right\rangle d\mu_x
\end{aligned}$$

$$\leq B\left\|V^{*}f\right\|^{2}.$$

Thus,

$$\left\|V^{*}f\right\| \geq \sqrt{C/B}\left\|K^{*}f\right\|, \quad \text{for all } f \in H.$$

Conversely, suppose $\|V^{*}f\| \geq \delta\|K^{*}f\|$ for all $f \in H$. Since K have a closed range, by Theorem 1.3, for all $f \in H$, we get

$$\left\|V^{*}f\right\| = \left\|\left(K^{\dagger}\right)^{*}K^{*}V^{*}f\right\| \leq \left\|K^{\dagger}\right\|\left\|K^{*}V^{*}f\right\|.$$

Now, for $f \in H$, we have

$$\begin{aligned}
&\sum_{i\in[m]}\int_{X} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}V^{*}P_{VF_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}V^{*}P_{VF_i(x)}Tf\right\rangle d\mu_x \\
=&\sum_{i\in[m]}\int_{X} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}UV^{*}f, \Lambda_i(x)P_{F_i(x)}TV^{*}f\right\rangle d\mu_x \\
\geq& A\left\|K^{*}V^{*}f\right\|^2 \geq A\left\|K^{\dagger}\right\|^{-2}\left\|V^{*}f\right\|^2 \geq A\delta^2\left\|K^{\dagger}\right\|^{-2}\left\|K^{*}f\right\|^2.
\end{aligned}$$

This completes the proof. □

The next theorem shows that it is enough to cheek continuous weaving controlled K-g-fusion woven on smaller measurable space than the original.

Theorem 2.6. *Suppose for each $i \in [m]$, $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x\in X}$ be a continuous (T, U)-controlled K-g-fusion frame for H with universal bounds A_i and B_i. If there exists a measurable subset $Y \subset X$ such that the family of continuous (T, U)-controlled K-g-fusion frame $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in Y}$ is a W. C. C. K. G. F. F. for H with universal frame bounds A and B. Then the family given by $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in X}$ is a W. C. C. K. G. F. F. for H with universal frame bounds A and $\sum_{i\in[m]} B_i$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be an arbitrary partition of X. For each $f \in H$, we define $\varphi : X \to \mathbb{C}$ by

$$\varphi(x) = \sum_{i\in[m]} \chi_{\sigma_i}(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle.$$

Then φ is measurable. Now, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \\
\leq&\sum_{i\in[m]}\int_{X} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \\
\leq&\left(\sum_{i\in[m]} B_i\right)\|f\|^2.
\end{aligned}$$

It is easy to verify that $\{\sigma_i \cap Y\}_{i\in[m]}$ is a partitions of Y. Thus, the family given by $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m],x\in\sigma_i\cap Y}$ is a continuous (T, U)-controlled K-g-fusion frame for H with lowest frame bound A. Therefore,

$$\begin{aligned}&\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \geq&\sum_{i\in[m]}\int_{\sigma_i\cap Y} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \geq& A\left\|K^*f\right\|^2.\end{aligned}$$

This completes the proof. □

In the following theorem, we show that it is possible to remove vectors from continuous controlled K-g-fusion frames and still be left with woven frames.

Theorem 2.7. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m],x\in\sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If there exists $0 < D < A$ and a measurable subset $Y \subset X$ and $n \in [m]$ such that for $f \in H$*

$$\sum_{i\in[m]\setminus\{n\}}\int_{X\setminus Y} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq D\left\|K^*f\right\|^2,$$

then the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m],x\in Y}$ is a W. C. C. K. G. F. F. for H with frame bounds $A - D$ and B.

Proof. Suppose that $\{\sigma_i\}_{i\in[m]}$ and $\{\gamma_i\}_{i\in[m]}$ are partitions of Y and $X\setminus Y$, respectively. For a given $f \in H$, we define $\varphi : Y \to \mathbb{C}$ by

$$\varphi(x) = \sum_{i\in[m]}\chi_{\sigma_i}(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle,$$

and $\phi : X \to \mathbb{C}$ by

$$\phi(x) = \sum_{i\in[m]}\chi_{\sigma_i\cup\gamma_i}(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle.$$

Since $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m],x\in\sigma_i\cup\gamma_i}$ is a continuous (T, U)-controlled K-g-fusion frame for H and $\varphi = \phi|_Y$, φ and ϕ are measurable. So, for each $f \in H$, we have

$$\begin{aligned}&\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \leq&\sum_{i\in[m]}\int_{\sigma_i\cup\gamma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq B\|f\|^2.\end{aligned}$$

Now, we assume that $\{\xi_i\}_{i\in[m]}$ such that $\xi_n = \theta$. Then $\{\xi_i \cup \sigma_i\}_{i\in[m]}$ is a partition of X and so for any $f \in H$, we have

$$\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x$$

$$
\begin{aligned}
&= \sum_{i \in [m] \setminus \{n\}} \Bigg[\int\limits_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \int\limits_{\xi_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad + \int\limits_{\sigma_n} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \Bigg] \\
&\geq \sum_{i \in [m] \setminus \{n\}} \Bigg[\int\limits_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \int\limits_{X \setminus Y} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad + \int\limits_{\sigma_n} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \Bigg] \\
&= \sum_{i \in [m]} \int\limits_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \sum_{i \in [m] \setminus \{n\}} \int\limits_{X \setminus Y} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\geq (A - D) \left\| K^* f \right\|^2 .
\end{aligned}
$$

This completes the proof. □

Proposition 2.2. *Let $K \in \mathcal{B}(H)$ be a closed range operator, $V \in \mathcal{B}(H)$ be a unitary operator and $\{(F(x), \Lambda(x), v(x))\}_{x \in X}$ be a continuous (T, U)-controlled K-g-fusion frame for H with bounds A, B. If $\|I_H - V\|^2 \left\| K^\dagger \right\|^2 \leq A/B$ and V commutes with T, U, then*

$$
\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x \in X}, \quad \Lambda' = \left\{ \left(V^{-1} F(x), \Lambda(x) V, v(x) \right) \right\}_{x \in X}
$$

are W. C. C. K. G. F. F. for $\mathcal{R}_K$.

Proof. Let σ be a partition of X. Since $K \in \mathcal{B}(H)$ has a closed range, for $f \in \mathcal{R}_K$, we have $\|f\|^2 \leq \left\| K^\dagger \right\|^2 \|K^* f\|^2$. Now, for each $f \in \mathcal{R}_K$, we have

$$
\begin{aligned}
&\int\limits_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int\limits_{\sigma^c} v^2(x) \left\langle \Lambda(x) V P_{V^{-1} F(x)} U f, \Lambda(x) V P_{V^{-1} F(x)} T f \right\rangle d\mu_x \\
&= \int\limits_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}UVf, \Lambda(x)P_{F(x)}TVf\right\rangle d\mu_x\\
\geq &\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&-\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}U\left(I_H-V\right)f, \Lambda(x)P_{F(x)}T\left(I_H-V\right)f\right\rangle d\mu_x\\
&\geq A\left\|K^*f\right\|^2 - B\left\|I_H-V\right\|^2\left\|f\right\|^2\\
\geq &A\left\|K^*f\right\|^2 - B\left\|I_H-V\right\|^2\left\|K^\dagger\right\|^2\left\|K^*f\right\|^2\\
= &\left(A - B\left\|I_H-V\right\|^2\left\|K^\dagger\right\|^2\right)\left\|K^*f\right\|^2.
\end{aligned}
$$

Hence, the families Λ and Λ' are W. C. C. K. G. F. F. for $\mathcal{R}_K$. □

Next, we will see that under some sufficient conditions sum of two continuous (T,U)-controlled K-g-fusion frames is woven with itself.

Theorem 2.8. *Let $K \in \mathcal{B}(H)$ be an invertible operator, the families given by $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x\in X}$ and $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x\in X}$ be continuous (T,U)-controlled K-g-fusion frames for H with bounds A, B and C, D, respectively. Suppose for each $x \in X$*

(i) $F(x) \subset G(x)^\perp$;
(ii) $\Lambda(x)P_{F(x)}\mathcal{R}(U) \perp \Lambda(x)P_{G(x)}\mathcal{R}(T)$;
(iii) $\Lambda(x)P_{F(x)}\mathcal{R}(T) \perp \Lambda(x)P_{G(x)}\mathcal{R}(U)$.

If for any partition σ of X, $(T^\sigma_\Gamma)^*$ is bounded below then

$$\Delta = \{(F(x)+G(x), \Lambda(x), v(x))\}_{x\in X},$$

and Λ are W. C. C. K. G. F. F. for H.

Proof. Since for each $x \in X$, $F(x) \subset G(x)^\perp$, we have $P_{F(x)+G(x)} = P_{F(x)} + P_{F(x)}$. Now, for each $x \in X$, using the given conditions (ii) and (iii), we have

$$
\begin{aligned}
&\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x\\
= &\int_X v^2(x)\left\langle \Lambda(x)\left(P_{F(x)}+P_{G(x)}\right)Uf, \Lambda(x)\left(P_{F(x)}+P_{G(x)}\right)Tf\right\rangle d\mu_x\\
= &\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&+\int_X v^2(x)\left\langle \Lambda(x)P_{G(x)}Uf, \Lambda(x)P_{G(x)}Tf\right\rangle d\mu_x \qquad (2.2)\\
\leq &(B+D)\|f\|^2.
\end{aligned}
$$

On the other hand, from (2.2), we get

$$\int_X v^2(x) \left\langle \Lambda(x) P_{F(x)+G(x)} U f, \Lambda(x) P_{F(x)+G(x)} T f \right\rangle d\mu_x \geq (A + C) \|K^* f\|^2,$$

for all $f \in H$. Thus, Δ is a continuous (T, U)-controlled K-g-fusion frame for H with bounds $(A + C)$ and $(B + D)$.

Furthermore, since K is a invertible operator and for any partition σ of X, $(T_\Gamma^\sigma)^*$ is bounded below, for each $f \in H$, there exists $M > 0$ such that

$$\|(T_\Gamma^\sigma)^* f\|^2 \geq M^2 \|f\|^2 \geq \frac{M^2}{\|K\|^2} \|K^* f\|^2.$$

Now, for each $f \in H$, we have

$$\begin{aligned}
&\int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)+G(x)} U f, \Lambda(x) P_{F(x)+G(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma^c} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&= \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad - \int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_\sigma v^2(x) \left\langle \Lambda(x) \left(P_{F(x)} + P_{G(x)} \right) U f, \Lambda(x) \left(P_{F(x)} + P_{G(x)} \right) T f \right\rangle d\mu_x \\
&= \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_\sigma v^2(x) \left\langle \Lambda(x) P_{G(x)} U f, \Lambda(x) P_{G(x)} T f \right\rangle d\mu_x \\
&\geq A \|K^* f\|^2 + \|(T_\Gamma^\sigma)^* f\|^2 \geq \left(A + \frac{M^2}{\|K\|^2} \right) \|K^* f\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)+G(x)} U f, \Lambda(x) P_{F(x)+G(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma^c} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\leq \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)+G(x)} U f, \Lambda(x) P_{F(x)+G(x)} T f \right\rangle d\mu_x \\
&\quad + \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x
\end{aligned}$$

$$\leq (2B+D)\|f\|^2.$$

Thus, Δ and Λ are W. C. C. K. G. F. F. for H. Similarly, it can be shown that Δ and Γ are W. C. C. K. G. F. F. for H. This completes the proof. □

In the following theorem, we present a sufficient condition for weaving continuous controlled K-g-fusion frame in terms of positive operators associated with given continuous controlled K-g-fusion frame.

Theorem 2.9. *Let the families given by $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x\in X}$ and $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x\in X}$ be continuous (T,U)-controlled K-g-fusion frames for H. Suppose for each $x \in X$, the operator $U_x : H \to H$ defined by*

$$\langle U_x(f), g\rangle = \int_X v^2(x)\, \langle T^*\Delta(x)Uf, g\rangle\, d\mu_x,$$

$f, g \in H$, where $\Delta(x) = P_{G(x)}\Gamma^*(x)\Gamma(x)P_{G(x)} - P_{F(x)}\Lambda^*(x)\Lambda(x)P_{F(x)}$, is a positive operator. Then Λ and Γ are W. C. C. K. G. F. F. for H.

Proof. Let A, B and C, D be frame bounds of Λ and Γ, respectively. Take σ be any partition of X. Then for each $f \in H$, we have

$$\begin{aligned}
A\,\|K^*f\|^2 \leq &\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
=&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&+\int_{\sigma^c} v^2(x)\left\langle T^*P_{F(x)}\Lambda(x)^*\Lambda(x)P_{F(x)}Uf, f\right\rangle d\mu_x\\
=&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&-\int_{\sigma^c} v^2(x)\,\langle T^*\Delta(x)Uf, f\rangle\, d\mu_x\\
&+\int_{\sigma^c} v^2(x)\left\langle T^*P_{G(x)}\Gamma(x)^*\Gamma(x)P_{G(x)}Uf, f\right\rangle d\mu_x\\
\leq&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&+\int_{\sigma^c} v^2(x)\left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x\\
\leq&(B+D)\|f\|^2.
\end{aligned}$$

Thus, Λ and Γ are W. C. C. K. G. F. F. for H with universal bounds A and $B+D$. □

Theorem 2.10. *Suppose for each $i \in [m]$, $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x\in X}$ be a continuous (T,U)-controlled K-g-fusion frame for H with bounds A_i and B_i. Suppose Y be* *measurable subset X and there exists $N > 0$ such that for all $i, k \in [m]$ with $i \neq k$*

$$0 \leq \int\limits_Y \langle \Gamma_{i,k} U f, \Gamma_{i,k} T f \rangle \, d\mu_x \leq N \min\{\Theta, \Omega\}, \quad f \in H,$$

where

$$\begin{aligned}
\Gamma_{i,k} =& v_i^2(x) \Lambda_i(x) P_{F_i(x)} - v_k^2(x) \Lambda_k(x) P_{F_k(x)}, \\
\Theta =& \int\limits_Y v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x, \\
\Omega =& \int\limits_Y v_k^2(x) \left\langle \Lambda_i(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x.
\end{aligned}$$

Then the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x \in X, i \in [m]}$ is W. C. C. K. G. F. F. for H with universal bounds $\frac{A}{(m-1)(N+1)+1}$ and B, where $A = \sum_{i \in [m]} A_i$ and $B = \sum_{i \in [m]} B_i$.

Proof. Let $\{\sigma_i\}_{i \in [m]}$ be a partition of X. Then for $f \in H$, we have

$$\begin{aligned}
\sum_{i \in [m]} A_i \|K^* f\|^2 \leq& \sum_{i \in [m]} \int\limits_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
=& \sum_{i \in [m]} \sum_{k \in [m]} \int\limits_{\sigma_k} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
\leq& \sum_{i \in [m]} \left[\int\limits_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \right. \\
& + \sum_{k \in [m], k \neq i} \int\limits_{\sigma_k} \langle \Gamma_{i,k} U f, \Gamma_{i,k} T f \rangle \, d\mu_x \\
& \left. + \sum_{k \in [m], k \neq i} \int\limits_{\sigma_k} v_k^2(x) \left\langle \Lambda_k(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x \right], \\
\Gamma_{i,k} =& v_i^2(x) \Lambda_i(x) P_{F_i(x)} - v_k^2(x) \Lambda_k(x) P_{F_k(x)} \\
\leq& \sum_{i \in [m]} \left[\int\limits_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \right. \\
& \left. + \sum_{k \in [m], k \neq i} (N+1) \int\limits_{\sigma_k} v_k^2(x) \left\langle \Lambda_k(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x \right], \\
=& D \sum_{i \in [m]} \int\limits_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x,
\end{aligned}$$

where $D = \{(m-1)(N+1)+1\}$. Thus, for each $f \in H$, we have

$$\begin{aligned}
\frac{A}{(m-1)(N+1)+1} \|K^* f\|^2 \leq& \sum_{i \in [m]} \int\limits_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
\leq& B \|f\|^2.
\end{aligned}$$

This completes the proof. □

3. Perturbation of Woven Continuous Controlled g-Fusion Frame

In frame theory, one of the most important problem is the stability of frame under some perturbation. P. Casazza and Chirstensen [10] have been generalized the Paley-Wiener perturbation theorem to perturbation of frame in Hilbert space. P. Ghosh and T. K. Samanta have studied perturbation of dual g-fusion frame and continuous controlled g-fusion frame in [18,21]. In this section, we will see that under some small perturbations, continuous controlled K-g-fusion frames constitute woven continuous controlled K-g-fusion frame.

Theorem 3.1. *Let the families given by* $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x\in X}$ *and* $\Gamma = \{(G(x), \Gamma(x), v(x))\}_{x\in X}$ *be continuous* (T, U)*-controlled* K*-*g*-fusion frames for* H *with bounds* A, B *and* C, D*, respectively. Suppose that there exist non-negative constants* λ_1, λ_2 *and* μ *with* $0 < \lambda_1 < 1$*,* $\mu < (1-\lambda_1) A - \lambda_2 B$ *such that for each* $f \in H$*, we have*

$$\begin{aligned}0 \leq &\int_X v^2(x) \left\langle T^*\Delta(x)Uf, f\right\rangle d\mu_x \\ \leq &\lambda_1 \int_X v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\ &+ \lambda_2 \int_X v^2(x) \left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x + \mu \left\|K^*f\right\|^2,\end{aligned}$$

where $\Delta(x) = \left(P_{F(x)}\Lambda(x)^*\Lambda(x)P_{F(x)} - P_{G(x)}\Gamma(x)^*\Gamma(x)P_{G(x)}\right)$*. Then,* Λ *and* Γ *are W. C. C. K. G. F. F. for* H*.*

Proof. Let σ be a partition of X. Now, for each $f \in H$, we have

$$\begin{aligned}&\int_\sigma v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x \\ \geq &\int_\sigma v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x - \int_{\sigma^c} v^2(x) \left\langle T^*\Delta(x)Uf, f\right\rangle d\mu_x \\ &+ \int_{\sigma^c} v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\ \geq &\int_X v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x - \int_X v^2(x) \left\langle T^*\Delta(x)Uf, f\right\rangle d\mu_x \\ \geq &(1-\lambda_1) \int_X v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\ &- \lambda_2 \int_X v^2(x) \left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x - \mu \left\|K^*f\right\|^2\end{aligned}$$

$\geq \left(\left(1-\lambda_1\right) A - \lambda_2 B - \mu \right) \left\| K^* f \right\|^2.$

On the other hand,

$$
\begin{aligned}
&\int_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x \\
\leq &\int_{X} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{X} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x \\
\leq &(B+D) \|f\|^2.
\end{aligned}
$$

This completes the proof. □

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FUZZY ALMOST HYPERIDEALS AND FUZZY ALMOST QUASI-HYPERIDEALS IN SEMIHYPERGROUPS

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Abstract. Studying fuzzy hyperideals is necessary for comprehending semihypergroups. The idea of fuzzy hyperideals is expanded upon by several concepts. The notion of almost fuzzy hyperideals is one of them. In this article, we first define the notions of fuzzy almost hyperideals and fuzzy almost quasi-hyperideals in semihypergroups. We investigate the fundamental characteristics of fuzzy almost hyperideals and fuzzy quasi-hyperideals. Additionally, we establish the connection between fuzzy (resp., quasi-) hyperideals and almost (resp., quasi-) hyperideals.

1. Introduction and Preliminaries

The idea of almost left (resp., right, two-sided) ideals plays a crucial role in characterizing semigroups that do not contain any proper left (resp., right, two-sided) ideals. Grošek and Satko [6,7] took on this issue for the first time. Bogdanović [1] considered a similar problem for almost bi-ideals in semigroups the following year. Researchers have studied a variety of almost ideals in semigroups and applied the concept of fuzzy sets, introduced by Zadeh [20], to several kinds of almost ideals (see [2,10,14,18]).

At the 8th International Congress of Scandinavian Mathematicians, Marty [11] introduced the concept of algebraic hyperstructures. Semihypergroups are a generalization of semigroups in that each product of two elements is a nonempty set rather than an element. This generalization of semihypergroups is applicable in many scientific disciplines, including biology (see [13]). Almost hyperideals, introduced by Suebsung et al. [17], were the ones that were first proposed the idea of almost ideals for

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semihypergroups. They looked into some of the essential properties of almost hyperideals. The concept of almost quasi-hyperideals in semihypergroups was defined, and their characteristics were given by Suebsung et al. [19] in 2021. Later, Muangdoo et al. [12] investigated a semihypergroup analog of the problem considered by Bogdanović. They introduced the idea of almost bi-hyperideals and fuzzy almost bi-hyperideals in semihypergroups. There were several significant studies and linkages made between these ideas.

We note that Suebsung et al. [17, 19] only considered the notion of almost (resp., quasi-) hyperideals into account in their studies. It is intriguing to consider whether we can use the concept of fuzzy sets in these kinds of analyses. In fact, in semihypergroups, we introduce the idea of fuzzy almost (resp., quasi-) hyperideals. There are given some essential properties of such introductory notions. Fuzzy almost (resp., quasi-) hyperideals and other kinds of fuzzy almost ideals have relationships. Additionally, the characteristic function is used to describe the relationship between fuzzy almost (resp., quasi-) hyperideals and almost (resp., quasi-) hyperideals.

2. Preliminaries

In this section we give some brief concepts and results, which will be helpful in next sections. Firstly, the concept of semihypergroups will be recalled as follows.

Let $\mathcal{H}$ be a non-empty set and $\mathcal{P}^*(\mathcal{H}) := \mathcal{P}(\mathcal{H}) \setminus \{\varnothing\}$ denotes the set of all non-empty subsets of $\mathcal{H}$. The map $\circ\colon \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is called the *hyperoperation* or the *join operation* on the set $\mathcal{H}$. A couple $(\mathcal{H}, \circ)$ is called a *hypergroupoid* if $\circ$ is a hyperoperation on $\mathcal{H}$. For $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a hypergroupoid $\mathcal{H}$, we will denote

$$\mathcal{A} \circ \mathcal{B} = \bigcup_{a\in\mathcal{A}, b\in\mathcal{B}} a \circ b, \quad a \circ \mathcal{A} = \{a\} \circ \mathcal{A} \quad \text{and} \quad a \circ \mathcal{B} = \{a\} \circ \mathcal{B}.$$

A hypergroupoid $(\mathcal{H}, \circ)$ is called a *semihypergroup* if for every $x, y, z \in \mathcal{H}$ we have $(x \circ y) \circ z = x \circ (y \circ z)$. Throughout this paper, we simply denote a semihypergroup $(\mathcal{H}, \circ)$ by $\mathcal{H}$, and $\mathcal{H}$ is understood to be a semihypergroup. A *subsemihypergroup* $\mathcal{Q}$ of $\mathcal{H}$ is a non-empty subset of $\mathcal{H}$ such that $\mathcal{Q} \circ \mathcal{Q} \subseteq \mathcal{Q}$. A *left* (*resp., right*) *hyperideal* $\mathcal{Q}$ of $\mathcal{H}$ if $\mathcal{H} \circ \mathcal{Q} \subseteq \mathcal{Q}$ (resp., $\mathcal{Q} \circ \mathcal{H} \subseteq \mathcal{Q}$). By a *hyperideal* $\mathcal{Q}$ of $\mathcal{H}$, we mean a non-empty set of $\mathcal{H}$ which is both a left and a right hyperideal of $\mathcal{H}$. A subsemihypergroup $\mathcal{Q}$ of $\mathcal{H}$ is called a *quasi-ideal* of $\mathcal{H}$ if $\mathcal{Q} \circ \mathcal{H} \cap \mathcal{H} \circ \mathcal{Q} \subseteq \mathcal{Q}$. In [5], the readers can find more information about the many types of hyperideals in semihypergroups From now on, we write $\mathcal{A}\mathcal{B}$ instead of $\mathcal{A} \circ \mathcal{B}$, for any nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$.

A non-empty subset $\mathcal{Q}$ of $\mathcal{H}$ is said to be:

(1) an *almost ideal* [17] of $\mathcal{H}$ if $h_1\mathcal{Q} \cap \mathcal{Q} \neq \emptyset$ and $\mathcal{Q}h_2 \cap \mathcal{Q} \neq \emptyset$ for all $h_1, h_2 \in \mathcal{H}$;
(2) an *almost quasi-hyperideal* [19] of $\mathcal{H}$ if $(h\mathcal{Q} \cap \mathcal{Q}h) \cap \mathcal{Q} \neq \emptyset$ for all $h \in \mathcal{H}$.

Example 2.1. Let $\mathcal{H} = \{a, b, c, d\}$. Define a hyperoperation $\circ$ on $\mathcal{H}$ by the following table:

$\circ$	a	b	c	d
a	a	$\{a,b\}$	$\{a,c\}$	H
b	b	b	$\{b,d\}$	$\{b,d\}$
c	c	$\{c,d\}$	c	$\{c,d\}$
d	d	d	d	d

Then $\mathcal{H}$ is a semihypergroup (see [8]). We can carefully calculate that $\{a,b,d\}$ is an almost hyperideal of $\mathcal{H}$ but it is not a hyperideal of $\mathcal{H}$. Furthermore, $\{a,d\}$ is an almost quasi-hyperideal of $\mathcal{H}$ but it is not a quasi-hyperideal of $\mathcal{H}$.

The above example illustrates the difference between (resp., quasi-) hyperideals and almost (resp., quasi-) hyperideals in semihypergroups. Now, we recall the concept of fuzzy sets.

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, where $\mathcal{F}$ is a nonempty indexed set, we define

$$\bigvee_{i\in\mathcal{F}} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{F}} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We observe that if $\mathcal{F}$ is finite, then

$$\bigvee_{i\in\mathcal{F}} h_i := \max_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{F}} h_i := \min_{i\in\mathcal{F}}\{h_i\}.$$

Let $\mathcal{T}$ be a non-empty set. We call a mapping $\eta\colon \mathcal{T} \to [0,1]$ a *fuzzy set* of $\mathcal{T}$ (see [20]). For any non-empty subset A of $\mathcal{T}$, the *characteristic function* λ_A of A in $\mathcal{T}$ is a fuzzy set of $\mathcal{T}$ defined by $\lambda_A(x) := 1$ if $x \in A$ and $\lambda_A(x) := 0$ if $x \notin A$ for all $x \in \mathcal{T}$. For any $\alpha \in [0,1]$ can be regarded as a fuzzy set of $\mathcal{T}$ by assigning $\alpha(x) := \alpha$ for all $x \in \mathcal{T}$.

For any two fuzzy sets η and ν of a non-empty set $\mathcal{T}$, define the symbol as follows:

(1) $\eta \subseteq \nu \Leftrightarrow \eta(h) \leq \nu(h)$ for all $h \in \mathcal{T}$;
(2) $\eta = \nu \Leftrightarrow \eta \subseteq \nu$ and $\nu \subseteq \eta$;
(3) $(\eta \cap \nu)(h) = \min\{\eta(h), \nu(h)\} = \eta(h) \wedge \nu(h)$ for all $h \in \mathcal{T}$;
(4) $(\eta \cup \nu)(h) = \max\{\eta(h), \nu(h)\} = \eta(h) \vee \nu(h)$ for all $h \in \mathcal{T}$;

We note here that the symbol $\eta \supseteq \nu$, we mean $\nu \subseteq \eta$.

The concept of semihypergroups can be studied in terms of fuzzy sets by the following setting. Let η and ν be fuzzy sets of $\mathcal{H}$. Define the product $\eta \circ \nu$ by

$$(\eta \circ \nu)(h) = \begin{cases} \bigvee_{h=h_1h_2} \{\eta(h_1) \wedge \nu(h_2)\}, & \text{if } h = h_1h_2 \text{ for some } h_1, h_2 \in \mathcal{H}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $h \in \mathcal{H}$.

By the above definition, one can prove the following important result.

Lemma 2.1 ([12]). *Let $\mathcal{K}$ and $\mathcal{L}$ be non-empty subsets of $\mathcal{H}$. Then the following holds:*

(1) *$\mathcal{K} \subseteq \mathcal{L}$ if and only if $\lambda_{\mathcal{K}} \subseteq \lambda_{\mathcal{L}}$;*
(2) *$\lambda_{\mathcal{K}} \cap \lambda_{\mathcal{L}} = \lambda_{\mathcal{K}\cap\mathcal{L}}$;*
(3) *$\lambda_{\mathcal{K}} \circ \lambda_{\mathcal{L}} = \lambda_{\mathcal{K}\mathcal{L}}$.*

Definition 2.1 ([15])**.** Let $u \in \mathcal{H}$ and $t \in (0,1]$. A fuzzy set u_t of $\mathcal{H}$ defined by

$$u_t(x) := \begin{cases} t, & \text{if } u = x, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{H}$, is called a *fuzzy point* of $\mathcal{H}$.

We observe that for any characteristic function of a singleton set of $\mathcal{H}$ can be regarded as a fuzzy point of $\mathcal{H}$. That is, for any $a \in \mathcal{H}$, we have $\lambda_{\{a\}} = a_1$.

3. On Fuzzy Almost (resp., quasi-) Hyperideals

The concepts of fuzzy almost hyperideals and fuzzy quasi-hyperideals in semihypergroups are defined in this section. This section will demonstrate how these notions are distinct from fuzzy hyperideals and fuzzy quasi-hyperideals in semihypergroups. The properties of the notions we defined are investigated.

Definition 3.1. A fuzzy set η of $\mathcal{H}$ is said to be:

(1) a *fuzzy almost left* (*resp., right*) *hyperideal* of $\mathcal{H}$ if for any fuzzy point h_t of $\mathcal{H}$ there exists $x \in \mathcal{H}$ such that $(\eta \circ h_t)(x) \wedge \eta(x) \neq 0$ (resp., $(h_t \circ \eta)(x) \wedge \eta(x) \neq 0$);
(2) a *fuzzy almost* (*two-sided*) *hyperideal* of $\mathcal{H}$ if it is both a fuzzy left almost hyperideal and a fuzzy right almost hyperideal of $\mathcal{H}$.

Example 3.1. Let $\mathcal{H} = \{a, b, c, u, v\}$. Define a hyperoperation $\circ$ on $\mathcal{H}$ by the following table:

$\circ$	a	b	c	u	v
a	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
b	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
c	a	a	$\{a,b,c\}$	a	$\{a,b,c\}$
u	$\{a,b,u\}$	$\{a,b,u\}$	$\mathcal{H}$	$\{a,b,u\}$	$\mathcal{H}$
v	$\{a,b,u\}$	$\{a,b,u\}$	$\mathcal{H}$	$\{a,b,u\}$	$\mathcal{H}$

Then $\mathcal{H}$ is a semihypergroup (see [5]). We define a fuzzy set η of $\mathcal{H}$ by

$$\eta(a) = 0, \quad \eta(b) = 0, \quad \eta(c) = 0.6, \quad \eta(u) = 0.4 \quad \text{and} \quad \eta(v) = 0.$$

We can see that for any $t \in (0,1]$:

(1) $(a_t \circ \eta)(c) \wedge \eta(c) \neq 0$ and $(\eta \circ a_t)(u) \wedge \eta(u) \neq 0$;
(2) $(b_t \circ \eta)(c) \wedge \eta(c) \neq 0$ and $(\eta \circ b_t)(u) \wedge \eta(u) \neq 0$;
(3) $(c_t \circ \eta)(c) \wedge \eta(c) \neq 0$ and $(\eta \circ c_t)(c) \wedge \eta(c) \neq 0$;
(4) $(u_t \circ \eta)(u) \wedge \eta(u) \neq 0$ and $(\eta \circ u_t)(u) \wedge \eta(u) \neq 0$;
(5) $(v_t \circ \eta)(c) \wedge \eta(c) \neq 0$ and $(\eta \circ v_t)(c) \wedge \eta(c) \neq 0$.

Therefore, η is a fuzzy almost hyperideal of $\mathcal{H}$. Since $(1 \circ \eta)(a) = 0.6 > 0 = \eta(a)$, we have that η is not a fuzzy left hyperideal of $\mathcal{H}$. That is, η is not a fuzzy hyperideal of $\mathcal{H}$.

It is not difficult to verify that any fuzzy left (resp., right, two-sided) hyperideal is a fuzzy almost left (resp., right, two-sided) hyperideal. In addition, Example 3.1 illustrates that a fuzzy almost hyperideal may not be a fuzzy hyperideal. This example demonstrates how fuzzy hyperideals in semihypergroups are generalized by the concept of fuzzy almost hyperideals. We refer the readers to [3, 4] for more information about fuzzy left (resp., right, two-sided) hyperideals.

Definition 3.2. A fuzzy set η on $\mathcal{H}$ is said to be a *fuzzy almost quasi-hyperideal* of $\mathcal{H}$ if for any fuzzy point h_t of $\mathcal{H}$ there exists $x \in \mathcal{H}$ such that $(\eta \circ h_t)(x) \wedge (h_t \circ \eta)(x) \wedge \eta(x) \neq 0$.

Example 3.2. Let $\mathcal{H} = \{a, b, c, d\}$. Define a hyperoperation $\circ$ on $\mathcal{H}$ by the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a, b\}$
d	a	a	$\{a, b\}$	$\{a, b, c\}$

Then $\mathcal{H}$ is a semihypergroup (see [5]). We define a fuzzy set η of $\mathcal{H}$ by

$$\eta(a) = 0.7, \quad \eta(b) = 0, \quad \eta(c) = 0.2 \quad \text{and} \quad \eta(d) = 0.4.$$

We can see that there exists $a \in \mathcal{H}$ such that $(h_t \circ \eta)(a) \wedge (\eta \circ h_t)(a) \wedge \eta(a) \neq 0$ for all fuzzy point h_t of $\mathcal{H}$. Therefore, η is a fuzzy almost quasi-hyperideal of $\mathcal{H}$. Since $(1 \circ \eta)(b) \wedge (\eta \circ 1)(b) = 0.4 > 0 = \eta(b)$, we have that η is not a fuzzy quasi-hyperideal of $\mathcal{H}$.

We can observe that any fuzzy quasi-hyperideal of semihypergroups is a fuzzy almost fuzzy quasi-hyperideal. We can see from the preceding example that the converse does not hold. For further detail on fuzzy quasi-hyperideals, we recommend readers to [16].

Remark 3.1. Examples 3.1 and 3.2 indicate how fuzzy almost (resp., quasi-) hyperideals extend on the idea of fuzzy (resp., quasi-) hyperideals. Verifying a relationship between fuzzy almost hyperideals and fuzzy almost quasi-hyperideals is not complicated. In semihypergroups, any fuzzy almost quasi-hyperideal is also a fuzzy almost hyperideal. Example 3.1 illustrates how these concepts differ from one another. Indeed, for any $t \in (0, 1]$, we have $(a_t \circ \eta)(x) \wedge (\eta \circ a_t)(x) \wedge \eta(x) = 0$ for all $x \in \mathcal{H}$.

In the following paper, we focus only on fuzzy almost hyperideals and fuzzy almost quasi-hyperideals in semihypergroups. However, the verification of our subsequent results is limited to fuzzy almost hyperideals since each fuzzy most quasi-hyperideal is a fuzzy almost hyperideal. The following result is required to examine the features of fuzzy almost (resp., quasi-) hyperideals in semihypergroups.

Lemma 3.1. *Let η, ν and θ be fuzzy sets of $\mathcal{H}$. We have that if $\eta \subseteq \nu$, then $\eta \circ \theta \subseteq \nu \circ \theta$ and $\theta \circ \eta \subseteq \theta \circ \nu$.*

Proof. We illustrate only that $\eta \circ \theta \subseteq \nu \circ \theta$. For verifying that $\theta \circ \eta \subseteq \theta \circ \nu$, it can be done similarly. Assume that $\eta \subseteq \nu$. Let $x \in \mathcal{H}$. If there is no $u, v \in \mathcal{H}$ such that $x \in uv$, then $(\eta \circ \theta)(x) \leq (\nu \circ \theta)(x)$. On the other hand, we have that

$$(\eta \circ \theta)(x) = \bigvee_{x \in uv} \{\eta(u) \wedge \theta(v)\} \leq \bigvee_{x \in uv} \{\nu(u) \wedge \theta(v)\} = (\nu \circ \theta)(x).$$

Therefore, we obtain our claim. $\square$

Here is our initial significant conclusion. When determining if a fuzzy set is a fuzzy almost (resp., quasi-) hyperideal, we do not always need to check with the definition. The result examines whether there is a fuzzy almost (resp., quasi-) hyperideal less than it, in which case it is also a fuzzy almost (resp., quasi-) hyperideal.

Theorem 3.1. *Let η and ν be fuzzy sets of $\mathcal{H}$. We have that if η is a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$ such that $\eta \subseteq \nu$, then ν is a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. Suppose that η is a fuzzy almost hyperideal of $\mathcal{H}$ such that $\eta \subseteq \nu$. By Lemma 3.1 and the definition of fuzzy almost hyperideal of $\mathcal{H}$, we obtain that there exist $x, y \in \mathcal{H}$ such that

$$0 \neq (h_t \circ \eta)(x) \wedge \eta(x) \leq (h_t \circ \nu)(x) \wedge \nu(x)$$

and

$$0 \neq (\eta \circ h'_{t'})(x) \wedge \eta(x) \leq (\eta \circ h'_{t'})(x) \wedge \nu(x),$$

for any fuzzy points h_t and $h'_{t'}$ of $\mathcal{H}$. This shows that ν is a fuzzy almost hyperideal of $\mathcal{H}$. For illustrating that ν is a fuzzy almost quasi-hyperideal of $\mathcal{H}$ can be done similarly. $\square$

By Theorem 3.1, we obtain the following consequence immediately.

Corollary 3.1. *Let η be a fuzzy set of $\mathcal{H}$ and ν be a fuzzy almost (quasi-) hyperideal of $\mathcal{H}$. Then $\eta \cup \nu$ is a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. By Theorem 3.1 and the fact that $\eta \subseteq \eta \cup \nu$, we obtain our claim. $\square$

The following example shows the contrast of Corollary 3.1.

Example 3.3. Let $\mathcal{H} = \{a, b, c\}$. Define a hyperoperation $\circ$ on $\mathcal{H}$ by the following table:

$\circ$	a	b	c
a	$\{a\}$	$\{b, c\}$	$\{c\}$
b	$\{b, c\}$	$\{b, c\}$	$\{c\}$
c	$\{b, c\}$	$\{b, c\}$	$\{c\}$

Then $\mathcal{H}$ is a semihypergroup. Define fuzzy sets η and ν of $\mathcal{H}$ by

$$\eta(a) = 0, \quad \eta(b) = 0, \quad \eta(c) = 0.1, \quad \nu(a) = 0, \quad \nu(b) = 0.6 \quad \text{and} \quad \nu(c) = 0.$$

We can carefully calculate that η and ν are fuzzy almost hyperideals of $\mathcal{H}$, but $\eta \cap \nu$ is not a fuzzy almost hyperideal of $\mathcal{H}$. Similarly, we can show that η and ν are fuzzy almost quasi-hyperideals of $\mathcal{H}$, but $\eta \cap \nu$ is not a fuzzy almost quasi-hyperideal of $\mathcal{H}$.

In the next couple results, we study relationships between almost (resp., quasi-) hyperideals and fuzzy almost (resp., quasi-) hyperideals in semihypergroups. Firstly, we represent almost (resp., quasi-) hyperideals in terms of fuzzy almost (resp., quasi-) hyperideals.

Theorem 3.2. *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is an almost (resp., quasi-) hyperideal of $\mathcal{H}$;*
(2) *$\lambda_{\mathcal{Q}}$ is a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{Q}$ is an almost hyperideal of $\mathcal{H}$. Let h_t be a fuzzy point of $\mathcal{H}$. By our assumption, we have that $h\mathcal{Q} \cap \mathcal{Q} \neq \emptyset$. This means that there exists $x \in \mathcal{Q}$ such that $x \in hq_1$ for some $q_1 \in \mathcal{Q}$. Therefore,

$$(h_t \circ \lambda_{\mathcal{Q}})(x) = \bigvee_{x \in uv} \{h_t(u) \wedge \lambda_{\mathcal{Q}}(v)\} = 1.$$

Similarly, we have that $\mathcal{Q}h \cap \mathcal{Q} \neq \emptyset$. This means that there exists $y \in \mathcal{Q}$ such that $y \in q_2h$ for some $q_2 \in \mathcal{Q}$. Therefore,

$$(\lambda_{\mathcal{Q}} \circ h_t)(x) = \bigvee_{x \in uv} \{\lambda_{\mathcal{Q}}(u) \wedge h_t(v)\} = 1.$$

This shows that $\lambda_{\mathcal{Q}}$ is a fuzzy almost hyperideal of $\mathcal{H}$.

(2) $\Rightarrow$ (1). Assume that $\lambda_{\mathcal{Q}}$ is a fuzzy almost hyperideal of $\mathcal{H}$. Let $h, h' \in \mathcal{H}$. By our presumption, for any $t, t' \in (0, 1]$ there exist $x, y \in \mathcal{H}$ such that

$$(h_t \circ \lambda_{\mathcal{Q}})(x) \wedge \lambda_{\mathcal{Q}}(x) \neq 0 \tag{3.1}$$

and

$$(\lambda_{\mathcal{Q}} \circ h'_{t'})(y) \wedge \lambda_{\mathcal{Q}}(y) \neq 0. \tag{3.2}$$

By (3.1), we have that $x \in hu$ for some $u \in \mathcal{Q}$ and $x \in \mathcal{Q}$. That is, $x \in h\mathcal{Q} \cap \mathcal{Q}$, so $h\mathcal{Q} \cap \mathcal{Q} \neq \emptyset$. On the other hand, by (3.2), we also conclude that $\mathcal{Q}h' \cap \mathcal{Q} \neq \emptyset$. Therefore, $\mathcal{Q}$ is an almost hyperideal of $\mathcal{H}$.

In showing that $\mathcal{Q}$ is an almost quasi-hyperideal if and only if $\lambda_{\mathcal{Q}}$ is a fuzzy quasi-hyperideal can be completed in a similar way. □

In order to describe fuzzy almost (resp., quasi-) hyperideals using almost (resp., quasi-) hyperideals, we need the following notion. Let η be a fuzzy set of $\mathcal{H}$. The *support* of η, denoted by $\operatorname{supp}(\eta)$, is defined to be the set $\{h \in \mathcal{H} \mid \eta(h) \neq 0\}$.

Theorem 3.3. *Let η be a fuzzy set of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *η is a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$;*
(2) *$\operatorname{supp}(\eta)$ is an almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2). Assume that η is a fuzzy almost hyperideal of $\mathcal{H}$. Let $h \in \mathcal{H}$. Then there exists $x \in \mathcal{H}$ such that $(\eta \circ h_t)(x) \wedge \eta(x) \neq 0$, where $t \in (0,1]$. Hence, $(\eta \circ h_t)(x) \neq 0$ and $\eta(x) \neq 0$. That is, $x = uh$ for some $u \in \mathcal{H}$ with $\eta(u) \neq 0$, and $\eta(x) \neq 0$. Thus, $x = uh \subseteq \text{supp}(\eta)h$ and $x \in \text{supp}(\eta)$. This means that $\text{supp}(\eta)h \cap \text{supp}(\eta) \neq \emptyset$. By similar arguments, we have that $h' \text{supp}(\eta) \cap \text{supp}(\eta) \neq \emptyset$ for any $h' \in \mathcal{H}$. This shows that $\text{supp}(\eta)$ is an almost hyperideal of $\mathcal{H}$.

(2) $\Rightarrow$ (1). Assume that $\text{supp}(\eta)$ is an almost hyperideal of $\mathcal{H}$. Let h_t be a fuzzy point of $\mathcal{H}$. By Theorem 3.2, $\lambda_{\text{supp}(\eta)}$ is a fuzzy almost hyperideal of $\mathcal{H}$. Then, we have that there exists $x \in \mathcal{H}$ such that

$$(h_t \circ \lambda_{\text{supp}(\eta)})(x) \wedge \lambda_{\text{supp}(\eta)}(x) \neq 0.$$

This implies that $x = hu$ for some $u \in \text{supp}(\eta)$ and $x \in \text{supp}(\eta)$. Thus, we have that

$$(h_t \circ \eta)(x) \wedge \eta(x) \neq 0.$$

Similarly, for any fuzzy point $h'_{t'}$ of $\mathcal{H}$, we have that there exists $y \in \mathcal{H}$ such that $(\eta \circ h'_{t'})(y) \wedge \eta(y) \neq 0$. Altogether, η is a fuzzy almost hyperideal of $\mathcal{H}$.

Illustrating that η is a fuzzy almost quasi-hyperideal of $\mathcal{H}$ if and only if $\text{supp}(\eta)$ is an almost quasi-hyperideal of $\mathcal{H}$ can be done similarly. □

The existence of proper almost (resp., quasi-) hyperideals in semihypergroups can be described using fuzzy almost (resp., quasi-) hyperideals by the following consequence.

Corollary 3.2. *The following statements are equivalent:*

(1) $\mathcal{H}$ *has no proper almost* (*resp., quasi-*) *hyperideal;*
(2) $\text{supp}(\eta) = \mathcal{H}$ *for every fuzzy almost* (*resp., quasi-*) *hyperideal* η *of* $\mathcal{H}$*.*

4. Minimality and Maximality of Fuzzy Almost (resp., quasi-) Hyperideals

We define the minimalities of almost (resp., quasi-) hyperideals and fuzzy almost (resp., quasi-) hyperideals in semihypergroups. The relationship between minimal almost (resp., quasi-) hyperideals and minimal fuzzy almost (resp., quasi-) hyperideals is investigated.

Definition 4.1. An almost (resp., quasi-) hyperideal $\mathcal{Q}$ of $\mathcal{H}$ is said to be *minimal* if for any almost (resp., quasi-) hyperideal $\mathcal{M}$ of $\mathcal{H}$, we have $\mathcal{M} = \mathcal{Q}$ whenever $\mathcal{M} \subseteq \mathcal{Q}$.

Definition 4.2. A fuzzy almost (resp., quasi-) hyperideal η of $\mathcal{H}$ is said to be *minimal* if for any fuzzy almost (resp., quasi-) hyperideal ν of $\mathcal{H}$, we have $\text{supp}(\nu) = \text{supp}(\eta)$ whenever $\nu \subseteq \eta$.

Example 4.1. (a) By Example 3.1, we see that $\{a\}$ and $\{u\}$ are minimal almost hyperideals of $\mathcal{H}$. Moreover, for any $t \in (0,1]$, a fuzzy set η of $\mathcal{H}$ defined by $\eta(x) = 0$ if $x \in \{a,b,v\}$ and $\eta(x) = t$ if $x \in \{c,u\}$, is a minimal fuzzy almost hyperideal of $\mathcal{H}$.

(b) By Example 3.2, we see that $\{a\}$ is a minimal almost quasi-hyperideal of $\mathcal{H}$. Moreover, for any $t \in (0,1]$, a fuzzy set η of $\mathcal{H}$ defined by $\eta(x) = t$ if $x = a$ and $\eta(x) = 0$ if $x \in \{b, c, d\}$, is a minimal fuzzy almost quasi-hyperideal of $\mathcal{H}$.

Minimal almost (resp., quasi-) hyperideals are represented using fuzzy almost (resp., quasi-) hyperideals as follows.

Theorem 4.1. *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is a minimal almost (resp., quasi-) hyperideal of $\mathcal{H}$;*
(2) *$\lambda_{\mathcal{Q}}$ is a minimal fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2). Assume that $\mathcal{Q}$ is a minimal almost hyperideal of $\mathcal{H}$. By Theorem 3.2, $\lambda_{\mathcal{Q}}$ is a fuzzy almost hyperideal of $\mathcal{H}$. Let ν be a fuzzy almost hyperideal of $\mathcal{H}$ such that $\nu \subseteq \lambda_{\mathcal{Q}}$. Now, we know, by Theorem 3.3, that $\text{supp}(\nu)$ is an almost hyperideal of $\mathcal{H}$. Since $\text{supp}(\nu) \subseteq \text{supp}(\lambda_{\mathcal{Q}}) = \mathcal{Q}$, by the minimality of $\mathcal{Q}$, we have $\text{supp}(\nu) = \text{supp}(\lambda_{\mathcal{Q}})$. This shows that $\text{supp}(\lambda_{\mathcal{Q}})$ is a minimal fuzzy almost hyperideal of $\mathcal{H}$.

(2) $\Rightarrow$ (1). Assume that $\lambda_{\mathcal{Q}}$ is a minimal fuzzy almost hyperideal of $\mathcal{H}$. By Theorem 3.2, $\mathcal{Q}$ is an almost hyperideal of $\mathcal{H}$. Let $\mathcal{M}$ be an almost hyperideal of $\mathcal{H}$ such that $\mathcal{M} \subseteq \mathcal{Q}$. Then, by Lemma 2.1 and Theorem 3.2, $\lambda_{\mathcal{M}}$ is a fuzzy almost hyperideal of $\mathcal{H}$ such that $\lambda_{\mathcal{M}} \subseteq \lambda_{\mathcal{Q}}$. This implies that $\text{supp}(\lambda_{\mathcal{M}}) \subseteq \text{supp}(\lambda_{\mathcal{Q}})$. By the minimality of $\lambda_{\mathcal{Q}}$, we have $\text{supp}(\lambda_{\mathcal{M}}) = \text{supp}(\lambda_{\mathcal{Q}})$. That is, $\mathcal{M} = \mathcal{Q}$. Therefore, $\mathcal{Q}$ is minimal.

We can demonstrate that $\mathcal{Q}$ is a minimal almost quasi-hyperideal if and only if $\lambda_{\mathcal{Q}}$ is a minimal fuzzy almost quasi-hyperideal by the same technique. □

Next, we define the maximalists of almost (resp., quasi-) hyperideals and fuzzy almost (resp., quasi-) hyperideals in semihypergroups. The relationship between maximal almost (resp., quasi-) hyperideals and maximal fuzzy almost (resp., quasi-) hyperideals is investigated.

Definition 4.3. An almost (resp., quasi-) hyperideal $\mathcal{M}$ of $\mathcal{H}$ is said to be *maximal* if for all almost (resp., quasi-) hyperideal $\mathcal{L}$ of $\mathcal{H}$ such that $\mathcal{M} \subseteq \mathcal{L}$ implies $\mathcal{M} = \mathcal{L}$.

Definition 4.4. A fuzzy almost (resp., quasi-) hyperideal η of $\mathcal{H}$ is said to be *maximal* if for all fuzzy almost (resp., quasi-) hyperideal ν of $\mathcal{H}$ such that $\eta \subseteq \nu$ implies $\text{supp}(\eta) = \text{supp}(\nu)$.

Maximal almost (resp., quasi-) hyperideals are represented using fuzzy almost (resp., quasi-) hyperideals as follows.

Theorem 4.2. *Let $\mathcal{M}$ be a non-empty subset of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{M}$ is a maximal almost (resp., quasi-) hyperideal of $\mathcal{H}$;*
(2) *$\lambda_{\mathcal{M}}$ is a maximal fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. (1) ⇒ (2). Assume that $\mathcal{M}$ is a maximal almost hyperideal of $\mathcal{H}$. By Theorem 3.2, $\lambda_{\mathcal{M}}$ is a fuzzy almost hyperideal of $\mathcal{H}$. Let ν be a fuzzy almost hyperideal of $\mathcal{H}$ such that $\lambda_{\mathcal{M}} \subseteq \nu$. Now, we know, by Theorem 3.3, that $\operatorname{supp}(\nu)$ is an almost hyperideal of $\mathcal{H}$. Since $\operatorname{supp}(\lambda_{\mathcal{M}}) \subseteq \operatorname{supp}(\nu) = \mathcal{M}$, by the maximality of $\mathcal{M}$, we have $\operatorname{supp}(\nu) = \operatorname{supp}(\lambda_{\mathcal{M}})$. This shows that $\operatorname{supp}(\lambda_{\mathcal{M}})$ is a maximal fuzzy almost hyperideal of $\mathcal{H}$.

(2) ⇒ (1). Assume that $\lambda_{\mathcal{M}}$ is a maximal fuzzy almost hyperideal of $\mathcal{H}$. By Theorem 3.2, $\mathcal{M}$ is an almost hyperideal of $\mathcal{H}$. Let $\mathcal{L}$ be an almost hyperideal of $\mathcal{H}$ such that $\mathcal{M} \subseteq \mathcal{L}$. Then, by Lemma 2.1 and Theorem 3.2, $\lambda_{\mathcal{L}}$ is a fuzzy almost hyperideal of $\mathcal{H}$ such that $\lambda_{\mathcal{M}} \subseteq \lambda_{\mathcal{L}}$. This implies that $\operatorname{supp}(\lambda_{\mathcal{M}}) \subseteq \operatorname{supp}(\lambda_{\mathcal{L}})$. By the of $\lambda_{\mathcal{M}}$, we have $\operatorname{supp}(\lambda_{\mathcal{M}}) = \operatorname{supp}(\lambda_{\mathcal{L}})$. That is, $\mathcal{M} = \mathcal{L}$. Therefore, $\mathcal{M}$ is maximal.

We can demonstrate that $\mathcal{M}$ is a maximal almost quasi-hyperideal if and only if $\lambda_{\mathcal{M}}$ is a maximal fuzzy almost quasi-hyperideal by the same technique. □

5. Prime of (fuzzy) almost (resp., quasi-) hyperideals

We introduce various notions of prime almost (resp., quasi-) hyperideals and prime fuzzy almost (resp., quasi-) hyperideals in semihypergroups. Their fundamental related property is provided.

First of all the primes of almost (reps., quasi-) hyperideals are defined.

Definition 5.1. Let $\mathcal{Q}$ be an almost (resp., quasi-) hyperideal of $\mathcal{H}$. Then $\mathcal{Q}$ is said to be:

(1) *prime* if for any almost (resp., quasi-) hyperideals $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{H}$, we have $\mathcal{M} \subseteq \mathcal{Q}$ or $\mathcal{L} \subseteq \mathcal{Q}$ whenever $\mathcal{M}\mathcal{L} \subseteq \mathcal{Q}$;
(2) *semiprime* if for any almost (resp., quasi-) hyperideal $\mathcal{M}$ of $\mathcal{H}$, we have $\mathcal{M} \subseteq \mathcal{Q}$ whenever $\mathcal{M}^2 \subseteq \mathcal{Q}$;
(3) *strongly prime* if for any almost (resp., quasi-) hyperideals $\mathcal{M}$ and $\mathcal{L}$ of $\mathcal{H}$, we have $\mathcal{M} \subseteq \mathcal{Q}$ or $\mathcal{L} \subseteq \mathcal{Q}$ whenever $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{Q}$.

The following definition, we provide the primes of fuzzy almost (resp., quasi-) hyperideals.

Definition 5.2. Let η be a fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$. Then η is said to be:

(1) *prime* if for any two fuzzy almost hyperideals ν and ϑ of $\mathcal{H}$, we have $\nu \subseteq \eta$ or $\vartheta \subseteq \eta$ whenever $\nu \circ \vartheta \subseteq \eta$;
(2) *semiprime* if for any fuzzy almost (resp., quasi-) hyperideal ν of $\mathcal{H}$, we have $\nu \subseteq \eta$ whenever $\nu \circ \nu \subseteq \eta$;
(3) *strongly prime* if for any two fuzzy almost (resp., quasi-) hyperideals ν and ϑ of $\mathcal{H}$, we have $\nu \subseteq \eta$ or $\vartheta \subseteq \eta$ whenever $(\nu \circ \vartheta) \cap (\vartheta \circ \nu) \subseteq \eta$.

It is clear that every fuzzy strongly prime almost (resp., quasi-) hyperideal is a fuzzy prime almost (resp., quasi-) hyperideal, and every fuzzy prime almost (resp., quasi-) hyperideal is a fuzzy semiprime almost (resp., quasi-) hyperideal.

A necessary auxiliary result should be presented without proof before we can start our theorem.

Lemma 5.1. *Let η and ν be fuzzy sets of $\mathcal{H}$. Then the following statements hold:*

(a) $\mathrm{supp}(\eta) \cap \mathrm{supp}(\nu) \subseteq \mathrm{supp}(\eta \cap \nu)$;

(b) $\mathrm{supp}(\eta)\, \mathrm{supp}(\nu) \subseteq \mathrm{supp}(\eta \circ \nu)$.

Theorem 5.1. *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{H}$. Then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is a strongly prime (resp., prime, semiprime) almost (resp., quasi-) hyperideal of $\mathcal{H}$;*

(2) *$\lambda_{\mathcal{Q}}$ is a strongly prime (resp., prime, semiprime) fuzzy almost (resp., quasi-) hyperideal of $\mathcal{H}$.*

Proof. (1) ⇒ (2). Assume that $\mathcal{Q}$ is an almost hyperideal of $\mathcal{H}$. Then, by Theorem 3.2, $\lambda_{\mathcal{Q}}$ is a fuzzy almost hyperideals of $\mathcal{H}$. Let η and ν be fuzzy almost hyperideals of $\mathcal{H}$ such that $(\eta \circ \nu) \cap (\nu \circ \eta) \subseteq \lambda_{\mathcal{Q}}$. By Lemma 5.1, we have that

$$\begin{aligned} \mathrm{supp}(\eta)\, \mathrm{supp}(\nu) \cap \mathrm{supp}(\nu)\, \mathrm{supp}(\eta) &\subseteq \mathrm{supp}(\eta \circ \nu) \cap \mathrm{supp}(\nu \circ \eta) \\ &\subseteq \mathrm{supp}((\eta \circ \nu) \cap (\nu \circ \eta)) \subseteq \mathrm{supp}(\lambda_{\mathcal{Q}}). \end{aligned}$$

By Theorem 3.3, we have $\mathrm{supp}(\eta)$ and $\mathrm{supp}(\nu)$ are almost hyperideals of $\mathcal{H}$. Thus, by our presumption, we have $\mathrm{supp}(\eta) \subseteq \mathrm{supp}(\lambda_{\mathcal{Q}})$ or $\mathrm{supp}(\nu) \subseteq \mathrm{supp}(\lambda_{\mathcal{Q}})$. This implies that $\eta \subseteq \lambda_{\mathcal{Q}}$ or $\nu \subseteq \lambda_{\mathcal{Q}}$. Therefore, $\lambda_{\mathcal{Q}}$ is strongly prime.

(2) ⇒ (1). Assume that $\lambda_{\mathcal{Q}}$ is a strongly prime fuzzy almost hyperideal of $\mathcal{H}$. Then, by Theorem 3.2, $\mathcal{Q}$ is an almost hyperideal of $\mathcal{H}$. Let $\mathcal{L}$ and $\mathcal{M}$ be almost hyperideals of $\mathcal{H}$ such that $\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M} \subseteq \mathcal{Q}$. By Lemma 2.1 and 5.1, we have that

$$(\lambda_{\mathcal{M}} \circ \lambda_{\mathcal{L}}) \cap (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{M}}) = \lambda_{\mathcal{M}\mathcal{L}} \cap \lambda_{\mathcal{L}\mathcal{M}} = \lambda_{\mathcal{M}\mathcal{L} \cap \mathcal{L}\mathcal{M}} \subseteq \lambda_{\mathcal{Q}}.$$

By Theorem 3.2, we have $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{L}}$ are fuzzy almost hyperideals of $\mathcal{H}$. Thus, by our assumption, we have $\lambda_{\mathcal{M}} \subseteq \lambda_{\mathcal{Q}}$ or $\lambda_{\mathcal{L}} \subseteq \lambda_{\mathcal{Q}}$. According to Lemma 2.1, it implies that $\mathcal{M} \subseteq \mathcal{Q}$ or $\mathcal{L} \subseteq \mathcal{Q}$. This shows that $\mathcal{Q}$ is strongly prime.

Using a similar methodology, we can show the connection between prime almost hyperideals and prime fuzzy almost hyperideals. We may demonstrate this for the semiprime property by applying $\mathcal{M} = \mathcal{L}$ in the proof. Since the hyperideality and fuzzy hyperideality do not act in the proof, we do not present the evidence of almost quasi-hyperideals and fuzzy almost quasi-hyperideals. □

6. Conclusion

We introduce concepts that we introduce in this study, fuzzy almost hyperideals and fuzzy almost quasi-hyperideals in semihypergroups. We investigate the properties of fuzzy almost (resp., quasi-) hyperideals. Additionally, we establish the connection between almost (resp., quasi-) hyperideals and fuzzy almost (resp., quasi-) hyperideals. Investigated are the minimality, maximality and primes properties of the concepts we defined. Future research will expand this study to include some fuzzy set generalizations.

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EXISTENCE OF SOLUTIONS FOR INHOMOGENEOUS BIHARMONIC PROBLEM INVOLVING CRITICAL HARDY-SOBOLEV EXPONENTS

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Abstract. This paper is devoted to the study of biharmonic problems. More precisely, we consider the following inhomogeneous problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda\left(\frac{u}{|x|^{4-\alpha}}\right) + f(x), & x \in \Omega,\\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 5$, under sufficient conditions on the data and the considered parameters, we prove the existence and multiplicity of solutions, by virtue of Ekeland's Variational Principle and the Mountain Pass Lemma.

1. Introduction

In this paper, we consider the following inhomogeneous problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda\left(\frac{u}{|x|^{4-\alpha}}\right) + f(x), & x \in \Omega,\\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 5$, containing 0 in its interior, $0 < \mu < \overline{\mu} := \frac{N^2(N-4)^2}{16}$, $\lambda > 0$, $0 \leq s,\ \alpha < 4$, $\alpha \neq 0$, $f \in H^{-2}(\Omega)$ ($H^{-2}(\Omega)$ denotes the dual space of the Sobolev space $H_0^2(\Omega)$), Δ^2 is the biharmonic operator and $2^*(s) = \frac{2(N-s)}{N-4}$ is the Sobolev critical exponent.

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The nonlinearity has a critical growth imposed by the critical exponent of Sobolev and the singular potentials, which causes a loss of compactness of the considered problem, consequently the classical methods cannot be applied directly, which make the study hard and more difficult.

We quote here some realized problems: The regular case in our problem, i.e., $\mu = \lambda = s = 0$ has been studied by Deng et al. [5]. By using Ekeland's Variational Principle [6] and the Mountain Pass Lemma [1], they proved the existence of multiple solutions for $f \neq 0$ satisfying a suitable assumption.

For $s = \lambda = 0$ and $f \equiv 0$, D'Ambrosio and Jannelli in [2], proved that there exists radial solutions U_μ positive, symmetric, decreasing and solve

$$\Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, u(x) > 0.$$

In [7], Kang and Xu studied the following problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda |u|^{q-2}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $0 \leq s < 4$ and $2 \leq q < 2^* = \frac{2N}{N-4}$. By variational arguments the existence of nontrivial solutions of the problem is established.

In what follows, we state our main results for which we consider the following hypothesis

$$0 < \inf\left\{ C_N (T(u))^{\frac{N-2s+4}{8-2s}} - \int_\Omega f u dx : \ u \in H_0^2(\Omega), \int_\Omega \left(\frac{|u|^{2^*(s)}}{|x|^s}\right) dx = 1 \right\}, \tag{1.2}$$

where

$$C_N = \left(\frac{8-2s}{N-4}\right)\left(\frac{N-4}{N-2s+4}\right)^{\frac{N-2s+4}{8-2s}}$$

and

$$T(u) = \int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right) - \lambda\left(\frac{u^2}{|x|^{4-\alpha}}\right)\right) dx.$$

Theorem 1.1. i) *Let $\mu \in]0, \overline{\mu}[$, $\lambda \in]0, \lambda_1[$ and f satisfying the condition* (1.2)*, then the problem* (1.1) *has at least a solution.*

ii) *There exists $\widehat{\mu} \in]0, \overline{\mu}[$ such that, for $\mu \in]0, \widehat{\mu}[$, $\lambda \in]0, \lambda_1[$ and f satisfying the condition* (1.2)*, then* (1.1) *has at least two solutions, if*

1) $0 < \alpha \leq \frac{1}{2}$ *for* $N \geq 5$;
2) $\frac{1}{2} < \alpha < 4$ *for* $5 \leq N < 12$.

The positive constants λ_1 and $\widehat{\mu}$ will be given later.

This paper is organized as follows. In the forthcoming section, we give some preliminaries and technical lemmas used in our work. In section 3 we give a detailed proof of Theorem 1.1.

2. Preliminary Results

2.1. **Definitions and notations.** Throughout this article, $\|\cdot\|_{-}$ denotes the norm of the Sobolev $H^{-2}(\Omega)$, $o_n(1)$ is any quantity which tends to zero as n goes to infinity and $\mathcal{O}(\varepsilon^s)$ verifies $|\frac{\mathcal{O}(\varepsilon^s)}{\varepsilon^s}| \leq C$, where C is a positive constant.

Problem (1.1) is related to the following Rellich inequality [8]

$$\int_\Omega \frac{u^2}{|x|^4} dx \leq \frac{1}{\overline{\mu}} \int_\Omega |\Delta u|^2 dx, \quad \text{for all } u \in H_0^2(\Omega), \tag{2.1}$$

where $H_0^2(\Omega)$ is the completetion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega (|\Delta u|^2 dx)^{\frac{1}{2}}$. Then the following best constant is defined

$$A_{\mu,s}(\Omega) := \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \frac{\int_\Omega \left(|\Delta u|^2 - \mu \frac{u^2}{|x|^4} \right) dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}}, \quad \text{for } 0 < \mu < \overline{\mu}. \tag{2.2}$$

Note that it is well known that $A_{\mu,s}(\Omega)$ is independent of any $\Omega \subset \mathbb{R}^N$ and that is not obtained except in the case with $\Omega = \mathbb{R}^N$. Moreover, the minimizers of $A_{\mu,s}(\Omega)$ have been investigated by [7]. Thus, we will simply denote $A_{\mu,s}(\Omega) = A_{\mu,s}(\mathbb{R}^N) = A_{\mu,s}$.

The authors in [2, 7] proved that $A_{\mu,s}$ is attained in $\mathbb{R}^N$ by the functions

$$\left\{ y_\varepsilon(x) = \varepsilon^{\frac{4-N}{2}} U_\mu \left(\frac{x}{\varepsilon} \right) : \varepsilon > 0 \right\},$$

and achieved

$$\int_\Omega \left(|\Delta y_\varepsilon(x)|^2 - \mu \left(\frac{|y_\varepsilon(x)|^2}{|x|^4} \right) \right) dx = \int_\Omega \left(\frac{|y_\varepsilon(x)|^{2^*(s)}}{|x|^s} \right) dx = A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)},$$

such as U_μ satisfies for $\mu \in]0, \overline{\mu}[$:

(a) $\lim_{\rho \to 0} \rho^{a(\mu)} U_\mu(\rho) = k_1$, $\lim_{\rho \to 0} \rho^{a(\mu)+1} U'_\mu(\rho) = k_3$;

(b) $\lim_{\rho \to +\infty} \rho^{b(\mu)} U_\mu(\rho) = k_2$, $\lim_{\rho \to +\infty} \rho^{b(\mu)+1} U'_\mu(\rho) = k_4$,

where $k_i \in \mathbb{R}$, $i = 1, \ldots, 4$ and $b(\mu) = (\frac{N-4}{2})(2 - \theta(\frac{\mu}{\overline{\mu}}))$, $a(\mu) = (\frac{N-4}{2})\theta(\frac{\mu}{\overline{\mu}})$, $\theta : [0,1] \to [0,1]$ is given by

$$\theta(t) = 1 - \frac{\sqrt{(N-2)^2 + 4 - \sqrt{16(N-2)^2 + t(N-4)^2 N^2}}}{N-4}.$$

Let us define $\vartheta : [0,1] \to [0,1]$ as follows:

$$\vartheta(t) = \frac{t(t-2)((N-4)t+4)((N-4)t - 2N + 4)}{N^2}.$$

Let us put

$$\varsigma_\alpha = \frac{1}{16}(N-4-\alpha)(N-4+\alpha)(N^2 - \alpha^2),$$

$$\zeta_s = \frac{(N-4)^2(s-4)}{(N+4)^4}\Big[N^2s^3 - (2N^3+4N^2)s^2 + (N^4+10N^3-20N^2+64N-64)s - 6N^4 + 20N^3 - 64N^2 + 64N\Big],$$

and set $\widehat{\mu} = \min(\varsigma_\alpha, \zeta_s)$.

Remark 2.1. (a) θ is continuous and strictly increasing.
(b) ϑ is an increasing homeomorphism and its inverse is θ.

In this paper, we use $H_0^2(\Omega)$ to denote the completetion of $C_0^\infty(\Omega)$ with respect to the norm,

$$\|u\|^2 := \int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right) dx.$$

By (2.1), this norm is equivalent to the usual norm $(\int_\Omega |\Delta u|^2 dx)^{\frac{1}{2}}$.

Let $u \in H_0^2(\Omega)$ be a weak solution of (1.1) if for all $\varphi \in H_0^2(\Omega)$,

$$\int_\Omega \Delta u \Delta\varphi - \int_\Omega \left(\frac{\mu}{|x|^4}\right) u\varphi dx - \int_\Omega \left(\frac{|u|^{2^*(s)-2}}{|x|^s}\right) u\varphi dx - \int_\Omega \left(\frac{\lambda}{|x|^{4-\alpha}}\right) u\varphi dx - \int_\Omega fu\varphi dx = 0.$$

It is true that the weak solutions of Problem (1.1) are equivalent to the nonzero critical points of the energy functional associated to (1.1) given by the following expression:

$$I(u) = \frac{1}{2}T(u) - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx - \int_\Omega fu dx, \quad \text{for all } u \in H_0^2(\Omega).$$

Definition 2.1. A functional $I \in C^1(H_0^2(\Omega); \mathbb{R})$ satisfies the Palais-Smale condition at level c, ($(PS)_c$ for short), if any sequence $(u_n) \subset H_0^2(\Omega)$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } H^{-2}(\Omega),$$

contains a strongly convergent subsequence.

2.2. **Eigenvalue problem.** Due to the Rellich inequality, the operator $Lu := \Delta^2 u - \mu\frac{u}{|x|^4}$ is definite on $H_0^2(\Omega)$. Moreover, the following eigenvalue problem with Hardy potentials and singular coefficient

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \lambda\left(\frac{u}{|x|^{4-\alpha}}\right), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha < 4$, $\lambda \in \mathbb{R}$, has the first eigenvalue λ_1 given by:

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right) dx}{\int_\Omega \frac{u^2}{|x|^{4-\alpha}} dx}.$$

Since the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega, |x|^{\alpha-4})$ is compact, by choosing a minimizing sequence, we easily infer that λ_1 can be obtained in $H_0^2(\Omega)$ and $\lambda_1 > 0$.

2.3. **Nehari manifold.** As the energy functional I is well defined in $H_0^2(\Omega)$ and belongs to $C^1(H_0^2(\Omega),\mathbb{R})$ and is not bounded from below on $H_0^2(\Omega)$, we consider it on the Nehari manifold

$$\mathcal{N} := \{u \in H_0^2(\Omega) : \langle I'(u), u\rangle = 0\}.$$

It is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. We can split $\mathcal{N}$ for:

$$\mathcal{N}^+ :=\{u \in \mathcal{N} : \langle I''(u), u\rangle > 0\},$$
$$\mathcal{N}^- :=\{u \in \mathcal{N} : \langle I''(u), u\rangle < 0\}$$

and

$$\mathcal{N}^0 := \{u \in \mathcal{N} : \langle I''(u), u\rangle = 0\}.$$

Denote $\inf\limits_{\mathcal{N}} I = c_0$.

2.4. **Some technical lemmas.**

Lemma 2.1. *If* $\mu \in]0,\overline{\mu}[$, $\alpha > 0$ *and* $0 < \lambda < \lambda_1$, *then*

$$\inf\left\{(T(u))^{\frac{1}{2}} : \int\limits_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}dx = 1\right\} = M > 0.$$

Proof. We know that

$$\lambda_1\int\limits_{\Omega}\frac{u^2}{|x|^{4-\alpha}}dx \leq \int\limits_{\Omega}\left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right)dx,$$

we deduce that

$$T(u) \geq \left(1-\frac{\lambda}{\lambda_1}\right)\int\limits_{\Omega}\left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right)dx.$$

Thus, by Rellich inequality, we get

$$\left(1-\frac{\lambda}{\lambda_1}\right)\left(1-\frac{\mu}{\overline{\mu}}\right)\int\limits_{\Omega}|\Delta u|^2dx \leq T(u) \leq \int\limits_{\Omega}|\Delta u|^2dx.$$

Then $(T(u))^{\frac{1}{2}} \geq \sqrt{K}S > 0$ for all $u \in H_0^2(\Omega)$ such that $\int\limits_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}dx = 1$. Here $S = \inf\limits_{u\in H_0^2(\Omega)\setminus\{0\}}\frac{\int_{\Omega}|\Delta u|^2dx.}{\int_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}dx}$ and $K = (1-\frac{\lambda}{\lambda_1})(1-\frac{\mu}{\overline{\mu}})$. We immediately have that $M > 0$. □

Lemma 2.2. *Let* $f \neq 0$ *satisfying condition* (1.2). *Then* $\mathcal{N}^0 = \emptyset$.

Proof. Suppose that $\mathcal{N}^0 \neq \emptyset$, then for $u \in \mathcal{N}^0$ we have

$$T(u) = (2^*(s)-1)\int\limits_{\Omega}\frac{|u|^{2^*(s)}}{|x|^s}dx.$$

Thus,

$$0=\langle I''(u),u\rangle = T(u)-\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx-\int_\Omega fudx = (2^*(s)-2)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx-\int_\Omega fudx. \tag{2.3}$$

From (1.2) and (2.3), we obtain

$$\begin{aligned} 0 &< C_N(T(u))^{\frac{N-2s+4}{8-2s}} - \int_\Omega fudx \\ &= (2^*(s)-1)\left[\left(\frac{T(u)}{(2^*-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx}\right)^{\frac{N-2s+4}{8-2s}} - 1\right]\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx \\ &= 0, \end{aligned}$$

which yields a contradiction. □

Lemma 2.3. *Let $f \neq 0$ satisfying* (1.2). *For every $u \in H_0^2(\Omega)$, $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \mathcal{N}^-$. In particular,*

$$t^+ > \left[\frac{T(u)}{(2^*(s)-1)\left(\frac{N-2s+4}{8-2s}\right)}\right]^{\frac{N-2s+4}{8-2s}} = t_{\max}(u) \quad \text{and} \quad I(t^+u) = \max_{t\geq t_{\max}} I(tu).$$

Moreover, if $\int_\Omega fudx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-u \in \mathcal{N}^+$, $t^- < t_{\max}(u)$ and $I(t^-u) = \min_{0\leq t\leq t_{\max}} I(tu)$.

Proof. The lemma is proved in the same way as in [5]. □

Lemma 2.4. *Let $f \neq 0$ satisfying* (1.2). *For each $u \in \mathcal{N} \setminus \{0\}$, there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in H_0^2(\Omega) \setminus \{0\}$, $\|w\| < \varepsilon$, satisfying the following there conditions:*

$$t(0) = 1,$$

$$t(w)(u-w) \in \mathcal{N}, \quad \text{for all } \|w\| < \varepsilon,$$

$$\langle t'(0), v\rangle = \frac{\int_\Omega [2\Delta u\Delta v - 2\left(\frac{\mu}{|x|^4} + \frac{\lambda}{|x|^{4-\alpha}}\right)uv - 2^*(s)\frac{|u|^{2^*(s)-2}}{|x|^s}uv - fv]dx}{T(u)-(2^*(s)-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx}. \tag{2.4}$$

Proof. Define the map $F : \mathbb{R} \times H_0^2(\Omega) \to \mathbb{R}$,

$$F(t,w) = sT(u-w) - t^{2^*(s)-1}\int_\Omega \frac{|u-\omega|^{2^*(s)}}{|x|^s}dx - \int_\Omega (u-w)fdx.$$

Since $F(1,0) = 0$, $\frac{\partial F}{\partial t}(1,0) = T(u)-(2^*(s)-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx \neq 0$, applying the implicit function theorem at the point $(1,0)$, we can get the result of this lemma. □

In the following lemma, we prove that $\mathcal{N}^-$ is closed and disconnects $H_0^2(\Omega)$ in exactly two connected components E_1 and E_2.

$$E_1 = \left\{ u \in H_0^2(\Omega) : u = 0 \text{ or } \|u\| < t^+ \left(\frac{u}{\|u\|} \right) \right\}$$

and

$$E_2 = \left\{ u \in H_0^2(\Omega) \backslash \{0\} : \ \|u\| > t^+ \left(\frac{u}{\|u\|} \right) \right\}.$$

Lemma 2.5. *Assume that condition* (1.2) *is satisfied, then*

(a) $\mathcal{N}^-$ *is closed;*
(b) $H_0^2 \backslash \mathcal{N}^- = E_1 \cup E_2$*;*
(c) $\mathcal{N}^+ \subset E_1$.

Proof. Let $(u_n) \subset \mathcal{N}^-$ and $w = \lim\limits_{n\to+\infty} u_n$, then $w \in \mathcal{N}$. Assume by contradiction that $w \notin \mathcal{N}^-$, then

$$T(u_n) - (2^*(s) - 1) \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx < 0, \tag{2.5}$$

$T(w) - (2^*(s) - 1) \int_\Omega \frac{|w|^{2^*(s)}}{|x|^s} dx = 0$. So, $w \in \mathcal{N}^0$ this implies that $w = 0$. From (2.5) and Lemma 2.1, we get $KS^2 < (2^*(s) - 1) \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx$, so $KS^2 < (2^*(s) - 1) \int_\Omega \frac{|w|^{2^*(s)}}{|x|^s} dx$, which yields to a contradiction.

Let $u \in \mathcal{N}^-$ and $v = \frac{u}{\|u\|}$, then $t^+(u) = 1$, and there exists a unique $t^+(v)$ such that $t^+(v)v \in \mathcal{N}^-$. As $t^+(v)v = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|}u \in \mathcal{N}^-$, then $t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = t^+(u) = 1$. Thus, if $u \in H_0^2(\Omega)$ and $t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} \neq 1$, then $u \notin \mathcal{N}^-$ and $H_0^2(\Omega) = E_1 \cup E_2$.

Let $u \in \mathcal{N}^+$. Then $t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = t^-(u) = 1$. Since $t^+(u) > t^-(u)$, it follows that $t^+(u) = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} > 1$. So, $\|u\| < t^+\left(\frac{u}{\|u\|}\right)$, and we conclude that $\mathcal{N}^+ \subset E_1$. ☐

Let the cut-off function $\varphi(x) = \varphi(|x|) \in C_0^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ in $B(0, R)$ and $\varphi(x) = 1$ in $B(0, \frac{R}{2})$. Set $u_\varepsilon = \varphi(x) y_\varepsilon(x)$, the following asymptotic properties hold.

Proposition 2.1. *Suppose that* $N \ge 5$, $\mu \in]0, \overline{\mu}[$. *Then*

(1) $\int_\Omega \left(|\Delta u_\varepsilon|^2 - \mu \left(\frac{|u_\varepsilon|^2}{|x|^4} \right) \right) dx = A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} + \mathcal{O}(\varepsilon^{2b(\mu)-N+4})$;
(2) $\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx = A_{\mu,s}^{\left(\frac{N-4}{4-s}\right)} + \mathcal{O}(\varepsilon^{2^*(s)b(\mu)-N+s})$;
(3) $\int_\Omega |x|^{\alpha-4} |u_\varepsilon|^2 dx = \mathcal{O}(\varepsilon^\alpha)$;
(4) $\int_\Omega \frac{|u_\varepsilon|^{2^*(s)-1} u_0}{|x|^s} dx = \varepsilon^{\frac{N-4}{2}} u_0(0) E + \mathcal{O}(\varepsilon^{\frac{N-4}{2}})$, *where* $E = \int_{\mathbb{R}^N} \frac{U_\mu^{2^*(s)-1}(x)}{|x|^s} dx$ *and* $\mu < \zeta_s$.

Proof. For the estimates (1), (2) one can see in [7], we only verify (3) and (4). Take $R>0$ small enough such that $B(0,\frac{R}{2})\subset\Omega$

$$\begin{aligned}\int\limits_{\Omega}|x|^{\alpha-4}u_\varepsilon^2dx &= \int\limits_{\Omega\setminus B(0,\frac{R}{2})}|x|^{\alpha-4}u_\varepsilon^2dx+\int\limits_{B(0,\frac{R}{2})}|x|^{\alpha-4}u_\varepsilon^2dx\\ &=\mathcal{O}(\varepsilon^{4-N+2b(\mu)})+\omega_N\int_0^{\frac{R}{2}}\rho^{\alpha-4}y_\varepsilon^2(\rho)\rho^{N-1}d\rho\\ &=\mathcal{O}(\varepsilon^{4-N+2b(\mu)})+\omega_N\varepsilon^{4-N}\int_0^{\frac{R}{2}}\rho^{\alpha-4-N-1}U_\mu^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1}d\rho\\ &=\mathcal{O}(\varepsilon^{\alpha}),\end{aligned}$$

because

$$\begin{aligned}\int\limits_{\Omega\setminus B(0,\frac{R}{2})}|x|^{\alpha-4}u_\varepsilon^2dx &\le \omega_N\int_{\frac{R}{2}}^{R}\rho^{\alpha-4}y_\varepsilon^2(\rho)\rho^{N-1}d\rho\\ &=\omega_N\varepsilon^{4-N}\int_{\frac{R}{2}}^{R}\rho^{\alpha-4}U_\varepsilon^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1}d\rho\\ &=\mathcal{O}(\varepsilon^{4-N+2b(\mu)})\end{aligned}$$

and

$$\omega_N\varepsilon^{4-N}\int_0^{\frac{R}{2}}\rho^{\alpha-4+N-1}U_\mu^2\left(\frac{\rho}{\varepsilon}\right)d\rho=\omega_N\varepsilon^{\alpha}\int_0^{\frac{R}{2\varepsilon}}\rho^{\alpha-4+N-1-2b(\mu)}d\rho.$$

Since $\alpha-4+N-1-2b(\mu)<-1$, we get that

$$\omega_N\varepsilon^{4-N}\int_0^{\frac{R}{2}}\rho^{\alpha-4-N-1}U_\mu^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1}d\rho=K\varepsilon^{\alpha}.$$

It follows from $\int\limits_{\Omega\setminus B(0,\frac{R}{2})}|x|^{\alpha-4}u_\varepsilon^2dx=\mathcal{O}(\varepsilon^{4-N+2b(\mu)})$ and $0<\alpha<2b(\mu)+4-N$, that

$$\int\limits_{\Omega}|x|^{\alpha-4}u_\varepsilon^2dx=\mathcal{O}(\varepsilon^{\alpha}),$$

$$\begin{aligned}\int\limits_{\Omega}|x|^{-s}u_\varepsilon^{2^*(s)-1}u_0(x)dx &=\varepsilon^{\frac{N-4}{2}}\int\limits_{\mathbb{R}^N}|y|^{-s}[\varphi^{2^*(s)-1}(\varepsilon y)-1]U_\varepsilon^{2^*(s)-1}(y)u_0(\varepsilon y)dy\\ &\quad+\varepsilon^{\frac{N-4}{2}}\int\limits_{\mathbb{R}^N}|y|^{-s}U_\varepsilon^{2^*(s)-1}(y)[u_0(\varepsilon y)-u_0(0)]dy\\ &\quad+\varepsilon^{\frac{N-4}{2}}\int\limits_{\mathbb{R}^N}|y|^{-s}U_\varepsilon^{2^*(s)-1}(y)dy\\ &=\mathcal{O}\left(\varepsilon^{\frac{N-4}{2}}\right)+\varepsilon^{\frac{N-4}{2}}u_0(0)E,\end{aligned}$$

where

$$
\begin{aligned}
E=&\int\limits_{\mathbb{R}^N}\frac{U_\mu^{2^*(s)-1}(x)}{|x|^s}dx=\omega_N\int\limits_0^{+\infty}U_\mu^{2^*(s)-1}(r)r^{N-s-1}dr\\
\leq& C_1\int\limits_0^{R}r^{N-s-1-(2^*(s)-1)a(\mu)}dr+\omega_N\int\limits_R^{M}U_\mu^{2^*(s)-1}(r)r^{N-s-1}dr\\
&+C_2\int\limits_M^{+\infty}r^{N-s-1-(2^*(s)-1)b(\mu)}dr.
\end{aligned}
$$

Let $N-s-(2^*(s)-1)a(\mu)-1>-1$ and $N-s-(2^*(s)-1)b(\mu)-1<-1$, thus $\mu<\zeta_s$. □

3. Proof of Theorem 1.1

The current section contains two subsections. In the first subsection we consider $0<\lambda<\lambda_1$ and $0<\mu<\overline{\mu}$, in the second subsection, we take $0<\lambda<\lambda_1$ and $0<\mu<\widehat{\mu}$.

3.1. **Existence of solution in $\mathcal{N}^+$.**

Using Ekeland's variational principl, we prove the existence of a solution in $\mathcal{N}^+$.

Proposition 3.1. *Let f satisfying* (1.2). *Then $c_0=\inf\limits_{u\in\mathcal{N}}I(u)$ is achieved at a point $u_0\in\mathcal{N}^+$, which is a critical point and even a local minimum for I.*

Proof. We start by showing that I is bounded from below in $\mathcal{N}$. Indeed, for $u\in\mathcal{N}$ we have:

$$T(u)-\int\limits_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx-\int\limits_\Omega fudx=0.$$

Thus,

$$
\begin{aligned}
I(u)&=\frac{1}{2}T(u)-\frac{1}{2^*(s)}\int\limits_\Omega\frac{|u|^{2^*(s)}}{|x|^s}dx-\int\limits_\Omega fudx\\
&=\left(\frac{4-s}{2(N-s)}\right)T(u)-\left(\frac{N+4-2s}{2(N-s)}\right)\int\limits_\Omega fudx\\
&\geq-\frac{(N+4-2s)^2}{8(N-s)(4-s)}\|f\|_-^2.
\end{aligned}
$$

In particular,

$$c_0\geq-\frac{(N+4-2s)^2}{8(N-s)(4-s)}\|f\|_-^2.$$

From Lemma 2.3, we can get $t_0=t_0(v)$ such that $t_0v\in\mathcal{N}$ and $I(t_0v)>0$. Moreover,

$$I(t_0v)=\frac{1}{2}t_0^2T(v)-\frac{t_0^{2^*(s)}}{2^*(s)}\int\limits_\Omega\frac{|v|^{2^*(s)}}{|x|^s}dx-t_0\int\limits_\Omega fvdx$$

$$= -\frac{1}{2}t_0^2 T(v) + \left(1 - \frac{1}{2^*(s)}\right) t_0^{2^*(s)} \int_\Omega \frac{|v|^{2^*(s)}}{|x|^s} dx$$
$$< -\frac{4-s}{2(N-s)} t_0^2 T(v) < 0.$$

Hence,

$$c_0 \leq I(t_0 v) < 0. \tag{3.1}$$

Applying the Ekeland's variational principle to the minimization problem (1.1), we can get a minimizing sequence $(u_n) \subset \mathcal{N}^+$ satisfying :

(i) $I(u_n) < c_0 + \frac{1}{n}$;
(ii) $I(u_n) \leq I(w) + \frac{1}{n}\|w - u_n\|$, for all $w \in \mathcal{N}$.

By taking n large enough, we get from (3.1):

$$I(u_n) = \frac{4-s}{2(N-s)} T(u_n) - \frac{N+4-2s}{2(N-s)} \int_\Omega f u_n dx < c_0 + \frac{1}{n} \leq -\frac{4-s}{2(N-s)} t_0^2 T(u_n).$$

This implies that

$$\int_\Omega f u_n dx \geq \frac{(4-s)t_0^2}{N+4-2s} T(u_n), \tag{3.2}$$

consequently, $u_n \neq 0$ and we have:

$$\frac{4-s}{N+4-2s} \cdot \frac{t_0^2}{\|f\|_-} T(u_n) \leq \|u_n\| \leq \frac{N+4-2s}{(4-s)\rho} \|f\|_-, \tag{3.3}$$

where the constant $\rho > 0$ verifies:

$$T(u) \geq \rho \|u\|^2. \tag{3.4}$$

Next we shall prove that $\|I'(u_n)\| \to 0$ as $n \to +\infty$. Hence, let us assume $\|I'(u_n)\| > 0$ for n large enough. By Applying Lemma 2.4, with $u = u_n$ and $w = \sigma\left(\frac{I'(u_n)}{\|I'(u_n)\|}\right)$, $\sigma > 0$, we can find some $t_n(\sigma) = t\sigma\left(\frac{I'(u_n)}{\|I'(u_n)\|}\right)$ such that

$$w_\sigma = t_n(\sigma)\left[u_n - \sigma \frac{I'(u_n)}{\|I'(u_n)\|}\right] \in \mathcal{N}.$$

By condition (ii), we obtain:

$$\frac{1}{n}\|w - u_n)\| \geq I(u_n) - I(w_\sigma)$$
$$= (1 - t_n(\sigma))\langle I'(w_\sigma), u_n\rangle + \sigma t_n(\sigma)\left\langle I'(w_\sigma), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle + o_n(\sigma).$$

Dividing by σ and passing to the limit as σ goes to zero we derive that:

$$\frac{1}{n}(1 + |t_n'(0)|\, \|u_n\|) \geq -t_n'(0)\langle I'(u_n), u_n\rangle + \|I'(u_n)\| = \|I'(u_n)\|,$$

where $t'_n(0) = \langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle$. So, we conclude that

$$\|I'(u_n)\| \leq \frac{C}{n}(1 + |t'_n(0)|), \quad C > 0.$$

The proof will be completed once we have shown that $|t'_n(0)|$ uniformly bounded with respect to n. From (2.4) and the estimate (3.3), we get:

$$|t'_n(0)| \leq \frac{C_1}{\left|T(u_n) - (2^*(s)-1)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx\right|},$$

C_1 is a suitable constant. Hence, we must prove that $|T(u_n) - (2^*(s)-1)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called (u_n), we have

$$\left|T(u_n) - (2^*(s)-1)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx\right| = o_n(1). \tag{3.5}$$

According to (3.3) and (3.5), there exists a constant $C_2 > 0$ such that

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx \geq C_2.$$

In addition, from (3.5) and by the fact that $u_n \in \mathcal{N}$, we get

$$\int_\Omega f u_n dx = (2^*(s)-2)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx + o_n(1).$$

This together with (1.2) imply that

$$0 < (2^*(s)-2)\left[\left(\frac{T(u_n)}{(2^*(s)-1)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx}\right)^{\frac{2^*(s)-1}{2^*(s)-2}} - 1\right] = o_n(1),$$

which is clearly impossible.

In conclusion,

$$I'(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.6}$$

Let $u_0 \in H_0^2(\Omega)$ be the weak limit in $H_0^2(\Omega)$ of (u_n). From (3.2) we derive that $\int_\Omega f u_0 > 0$, and from (3.6) that $\langle I'(u_0), w\rangle = 0$, for all $w \in H_0^2(\Omega)$, i.e., u_0 is a weak solution for (1.1). In fact, $u_0 \in \mathcal{N}$ and $c_0 \leq I(u_0) \leq \lim_{n\to+\infty} I(u_n) = c_0$. So, we deduce that $u_n \to v$ strongly in $H_0^2(\Omega)$ and $I(u_0) = c_0 = \inf_{u\in\mathcal{N}} I(u)$. Moreover, $u_0 \in \mathcal{N}^+$. So u_0 is a local minimum for I. □

3.2. **Existence of solution in $\mathcal{N}^-$.** In this subsection, for proof of the existence of a solution in $\mathcal{N}^-$, we shall find the range of c where I verifies the $(PS)_c$ condition.

Lemma 3.1. *Let (u_n) be any sequence of $H_0^2(\Omega)$ satisfying the following conditions:*

(a) $I(u_n) \to c$ *with* $c < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}$;

(b) $\|I'(u_n)\| \to 0$ *as* $n \to +\infty$.

Then (u_n) has a strongly convergent subsequence.

Proof. We have $I(u_n) = c + o_n(1)$ and

$$\langle I'(u_n), u_n\rangle = T(u_n) - \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx - \int_\Omega f u_n dx + o_n(1). \tag{3.7}$$

Then

$$\frac{4-s}{2(N-s)} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx + o_n(1) = c + \frac{1}{2}\int_\Omega f u_n dx - \frac{1}{2}\langle I'(u_n), u_n\rangle + \mathcal{O}(1).$$

By using Hölder inquality, we get

$$\frac{4-s}{2(N-s)} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx \le c + \frac{1}{2}\|f\|_- \|u_n\| + \frac{1}{2}\|I'(u_n)\|_- \|u_n\|. \tag{3.8}$$

From (3.4), (3.7) and (3.8), we have for all $\varepsilon > 0$:

$$\begin{aligned}\rho\|u_n\| \le T(u_n) &\le \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx + \int_\Omega f u_n dx + \langle I'(u_n), u_n\rangle \\ &\le \frac{2(N-s)}{4-s} c + \frac{N+4-2s}{4-s} (\|f\|_- + \|I'(u_n)\|_-)\|u_n\| + \varepsilon\|u_n\|.\end{aligned}$$

So, $T(u_n)$ is uniformly bounded. For a subsequence of (u_n), we can get a $u \in H_0^2(\Omega)$ such that $u_n \rightharpoonup u$. So, from (b), we obtain that

$$\langle I'(u), w\rangle = 0, \quad \text{for all } w \in H_0^2(\Omega).$$

Then u is a weak solution for (1.1). In particular $u \ne 0$, $u \in \mathcal{N}$ and $I(u) \ge c_0$. We have:

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ weakly in } H_0^2(\Omega), \\ &u_n \rightharpoonup u \ \text{ weakly in } L^2(\Omega, |x|^{-4}) \text{ and } L^{2^*(s)}(\Omega, |x|^{-s}), \\ &u_n \to u \ \text{ strongly in } L^2(\Omega, |x|^{\alpha-4}), \\ &u_n \to u \ \text{ strongly in } L^q(\Omega) \text{ for all } 1 \le q < 2^*(s). \end{aligned}$$

Let $u_n = u + v_n$. So, $v_n \rightharpoonup 0$ in $H_0^2(\Omega)$. As in Brezis-Lieb Lemma (see [4]), we conclude that

$$c + o_n(1) = I(u) + I(v_n) + \int_\Omega f v_n dx \tag{3.9}$$

and

$$o_n(1) = I'(v_n) + \int_\Omega f v_n dx.$$

Without loss of generality, as $n \to +\infty$ we may assume that

$$T(v_n) \to l, \quad \int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx \to l.$$

From (2.2) we obtain

$$l \geq A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

By (3.9), we deduce that $I(u) = c - \frac{4-s}{2(N-s)} l \leq c - \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} < c_0$, which contradicts the fact that $c_0 = \inf I$. Hence, $l = 0$ and $u_n \to u$ strongly in $H_0^2(\Omega)$ as $n \to +\infty$. □

Lemma 3.2. *Let* $f \neq 0$ *satisfying* (1.2) *and if* $0 < \alpha \leq \frac{1}{2}$ *for* $N \geq 5$ *or* $\frac{1}{2} < \alpha < 4$ *for* $5 \leq N < 12$, *then for all* $t > 0$, *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$I(u_0 + tu_\varepsilon) < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}. \tag{3.10}$$

Proof. We infer from [3] that:

$$\begin{aligned}\int_\Omega \frac{|u_0 + tu_\varepsilon|^{2^*(s)}}{|x|^s} dx =& \int_\Omega \frac{|u_0|^{2^*(s)}}{|x|^s} dx + t^{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx \\ &+ 2^*(s) t \int_\Omega \frac{|u_0|^{2^*(s)-2} u_0 u_\varepsilon}{|x|^s} dx + 2^*(s) t^{2^*(s)-1} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)-1} u_0}{|x|^s} dx \\ &+ \mathcal{O}\left(\varepsilon^{2b(\mu)+4-N}\right).\end{aligned}$$

Since $u_0 \in \mathcal{N}$ is a solution of (1.1) and from Proposition 2.1, we obtain:

$$\begin{aligned}I(u_0 + tu_\varepsilon) =& I(u_0) + \frac{t^2}{2} T(u_\varepsilon) - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx \\ &- \frac{1}{2^*(s)} \int_\Omega \frac{|u_0 + tu_\varepsilon|^{2^*(s)} - |u_0|^{2^*(s)} - |tu_\varepsilon|^{2^*(s)} - 2^*(s)|u_0|^{2^*(s)-2} u_0 t u_\varepsilon}{|x|^s} dx \\ =& I(u_0) + \frac{t^2}{2} T(u_\varepsilon) - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx - t^{2^*(s)-1} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)-1} u_0}{|x|^s} dx \\ &- \mathcal{O}(\varepsilon^{2b(\mu)+4-N}) \\ =& I(u_0) + \frac{t^2}{2} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} - \frac{t^{2^*(s)}}{2^*(s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} - t^{2^*(s)-1} \varepsilon^{\frac{N-4}{2}} u_0(0) E \\ &+ \mathcal{O}\left(\varepsilon^{2^*(s)b(\mu)-N+s}\right) - \mathcal{O}\left(\varepsilon^{\alpha}\right) + o_n\left(\varepsilon^{\frac{N-4}{2}}\right) + \mathcal{O}\left(\varepsilon^{2b(\mu)-N+4}\right).\end{aligned}$$

Define

$$g(t)=\frac{t^2}{2}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-\frac{t^{2^*(s)}}{2^*(s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t^{2^*(s)-1}\varepsilon^{\frac{N-4}{2}}u_0(0)E,\quad t>0,$$

and assume that $g(t)$ achieves its maximum at $t_0>0$. Since

$$t_0A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t_0^{2^*(s)-1}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}=(2^*(s)-1)t_0^{2^*(s)-2}\varepsilon^{\frac{N-4}{2}}u_0(0)E,$$

necessarily $0<t_0<1$ and $t_0\to 1$ as $\varepsilon\to 0$.

Note that $t\to\frac{t^2}{2}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-\frac{t^{2^*(s)}}{2^*s}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}$ rises monotonically on $[0,1]$, so,

$$\begin{aligned}I(u_0+tu_\varepsilon)<&c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t^{2^*-1}\varepsilon^{\frac{N-4}{2}}u_0(0)E+\mathcal{O}\left(\varepsilon^{2^*(s)b(\mu)-N+s}\right)\\&-\mathcal{O}\left(\varepsilon^{\alpha}\right)+o_n\left(\varepsilon^{\frac{N-4}{2}}\right)+\mathcal{O}\left(\varepsilon^{2b(\mu)+4-N}\right).\end{aligned}$$

We distinguish the following two cases.

Case 1. When $2^*(s)b(\mu)-N>2b(\mu)+4-N>\frac{N-4}{2}\geq\alpha$ if $5\leq N$, we have $0<\mu\leq\varsigma_\alpha$ and $0<\alpha\leq\frac{1}{2}$, then, for $\mu\in]0,\widehat{\mu}[$, we obtain:

$$I(u_0+tu_\varepsilon)<c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Case 2. When $2^*(s)b(\mu)-N>2b(\mu)+4-N>\alpha>\frac{N-4}{2}$ if $5\leq N<12$, we have $0<\mu<\varsigma_{\frac{N-4}{2}}$ and $\frac{1}{2}\leq\alpha<4$, then, for $\mu\in]0,\widehat{\mu}[$, we obtain:

$$I(u_0+tu_\varepsilon)<c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.\qquad\square$$

Finally, it remains to show the following proposition.

Proposition 3.2. *Suppose that f verifies conditions of Lemma* 3.2. *Then I has a minimizer $u_1\in\mathcal{N}^-$ such that $c_1=I(u_1)$. Moreover, u_1 is a solution of Problem* (1.1).

Proof. Let $(v_n)\subset\mathcal{N}^-$ such that

$$I(v_n)\to c_1\quad\text{and}\quad I'(v_n)\to 0,\quad\text{in } H^{-2}(\Omega).$$

For $u\in H_0^2(\Omega)$ such that $\|u\|=1$. By Lemma 2.3, there exists a unique $t^+(u)>0$ such that $t^+(u)u\in\mathcal{N}^-$ and $I(t^+(u)u)=\max_{s\geq t_{\max}}I(su)$. According to Lemma 2.5, we have $u_0\in E_1$, we can choose a constant c', which satisfies $0<t^+(u)\leq c'$, for all $\|u\|=1$, we claim that

$$u_0+t_0u_\varepsilon\in E_2,\tag{3.11}$$

where $t_0=\left(\frac{|c'^2-\|u_0\|^2|}{\|u_\varepsilon\|}\right)^{\frac{1}{2}}+1$. In fact, a direct computation shows that:

$$\begin{aligned}\|u_0+t_0u_\varepsilon\|^2=&\|u_0\|^2+t_0^2\|u_\varepsilon\|^2+2t_0\int_\Omega\left(\Delta u_0\Delta u_\varepsilon-\mu\frac{u_0u_\varepsilon}{|x|^4}\right)dx\\=&\|u_0\|^2+t_0^2\|u_\varepsilon\|^2+o_n(1)\end{aligned}$$

$$>c'^2 \geq \left[t^+\left(\frac{u_0+t_0u_\varepsilon}{\|u_0+t_0u_\varepsilon\|}\right)\right]^2,$$

for $\varepsilon > 0$ small enough. Thus, claim (3.11) holds. We fix $\varepsilon > 0$ such that both (3.10) and (3.11) hold by the choice of t_0. We set

$$\Gamma = \{\gamma \in C([0;1] : H_0^2(\Omega)) : \gamma(0) = u_0, \gamma(1) = u_0 + t_0u_\varepsilon\},$$

and take $h(t) = u_0 + tt_0u_\varepsilon$, which belongs to Γ. From Lemma 3.1, we conclude that:

$$c = \inf_{h\in\Gamma}\max_{t\in[0;1]} I(h(t)) < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Since every $h \in \Gamma$ intersects $\mathcal{N}^-$, we get that:

$$c_1 = \inf_{\mathcal{N}^-} I \leq c < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Using Lemma 3.2, we deduce that v_n converges strongly to u_1 in $H_0^2(\Omega)$. Thus, $u_1 \in \mathcal{N}^-$ and $c_1 = I(u_1)$. Then $I'(u_1) = 0$, and thus u_1 is a solution of Problem (1.1). We conclude that Problem (1.1) admits also a solution in $\mathcal{N}^-$. □

Proof of Theorem 1.1. By Porpositions 3.1, 3.2 and as $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$ we deduce that the problem (1.1) admits two solutions u_0 and u_1 with $u_0 \neq u_1$. □

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KRAGUJEVAC JOURNAL OF MATHEMATICS

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