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WELL-POSEDNESS AND EXPONENTIAL DECAY OF ENERGY FOR THE SOLUTION OF A WAVE EQUATION WITH NONLINEAR SOURCE AND LOCALIZED DAMPING TERMES

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ABSTRACT. We consider the wave equation with a locally damping and a nonlinear source term in a bounded domain. $y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y$, where $p > 2$. The damping is nonlinear and is effective only in a neighborhood of a suitable subset of the boundary. We show, for certain initial data and suitable conditions on g, a and p that this solution is global we use the Faedo-Galerkin method. Also we established the exponential decay of the energy when the nonlinear damping grows linearly by introducing a suitable Lyapunov functional.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, having a boundary $\Gamma = \partial\Omega$ of class C^2. We denote by ν the unit normal pointing into the exterior of Ω. We fix $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$ and we set

$$\Gamma(x^0) = \{x \in \Gamma : m(x)\nu(x) > 0\} \tag{1.1}$$

and

$$m(x) = x - x^0. \tag{1.2}$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider δ sufficiently small such that

$$\mathcal{M}_0 = \Big\{x \in \Omega : d(x, \Gamma(x^0)) < \delta\Big\} \subset \omega, \tag{1.3}$$

$$\mathcal{M}_1 = \Big\{x \in \Omega : d(x, \Gamma(x^0)) < 2\delta\Big\} \subset \omega. \tag{1.4}$$

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If $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have

$$d(x; A) = \inf_{y \in A}(|x-y|).$$

and $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \omega$.

Now consider with the following initial-boundary value problem of damped wave equation

$$\begin{cases} y_{tt} - \Delta y + a(x)g(y_t) = f(y), \quad x \in \Omega \times [0, +\infty[, \\ y = 0, \quad x \in \Gamma \times [0, \infty[, \\ y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad x \in \Omega \times [0, +\infty[, \end{cases} \tag{1.5}$$

where $f(y) = |y|^{p-2}y$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function with $g(0) = 0$ and $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded function.

In the absence of nonlinear source term (i.e., if $f = 0$), Tebou [12] has used the multipliers techniques to prove the decay estimates of global solutions for the problem (1.5) for certain initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and g having a polynomial growth near the origin. Precisely, he showed that the rate of decay of the energy is exponential or polynomial depending on exponents of the damping terms. This method is based on new integral inequality that generalizes a result of Haraux [6] and Komornik [7]. Tebou [14] studied (1.5) for a localized nonlinear strong damping. He proved that for certain initial data the global existence by using the Fadeo-Galerkin approximations and the semigroup methods, he used and also showed that the energy of the system decays exponentially by introducing a multiplier method combined with a nonlinear integral inequalities given by Martinez [9].

When $f = 0$ and the feedback term depends on the velocity in a linear way, as in the present paper, Zuazua [15] proved that the energy related to problem (1.5) decays exponentially if the damping region contains a neighbourhood of the boundary Γ or, at least, contains a neibourhood of the particular part given by (1.5).

When $g(y_t) = \operatorname{div}(a(x)\nabla y_t)$, where $a(x) = d1_\omega(x), d > 0$, Ammari et al. [2] consider the problem (1.5) without the source term $f(y)$. They obtained a logarithmic decay of energy. Their idea is to transform the resolvent problem to a transmission system to easily use the so-called Carleman estimate.

When $g(\Delta y_t) = |\Delta y_t|^{p-2}\Delta y_t$ and the source term is absent, Tebou [13] investigates the global existence of solution with initial-boundary value conditions. Meanwhile, he proved that the rate of decay of the energy is exponential or polynomial depending on exponents of the damping terms.

In the presence of the viscoelastic term Cavalcanti et al. [5] studied (1.5) in the presence of a linear localised frictional damping $(a(x)y_t)$. They obtained an exponential rate of decay by assuming that the kernel term is decaying exponentially. This work was later improved by Berrimi and Messaoudi [4] by introducing a different functional which allowed them to weaken the conditions on viscoelastic damping.

Motivated by previous works, it is interesting to investigate the global existence and decay of solutions to problem (1.5). Firstly, we show that, under suitable conditions on the functions g and a, the parameter p and certain initial data in the stable set, the existence of regular and weak solutions to problem (1.5).

After that, we establish the rate of decay of solutions by the perturbed energy method. Precisely, we show that the decay rate of energy function is exponential. In this way, we can extend the results of [14] where the authors considered (1.5) without source term and the results of [10] and [11] in the linear damping term.

This article is organized as follows. In the next section, we give some preliminaries. In Section 3, we prove the existence and uniqueness for regular and weak solutions. Then in Section 4, we are devoted to the proof of decay estimate.

2. Preliminaries

To state and prove our result, we need some assumptions.

(A1) $g:\mathbb{R}\to\mathbb{R}$ is non decreasing function of class C^1 functions such that $g(0)=0$ and

$$(\exists \tau_0,\tau_1>0)\quad \tau_0\le g'(s)\le \tau_1,\quad \text{for all } s\in\mathbb{R}.$$

(A2) The nonnegative function $a:\Omega\to[0,+\infty)$ is assumed bounded such that

$$\begin{aligned}(\exists a_0>0)\quad & a(x)\ge a_0>0,\quad \text{a.e. in } \omega,\\ & a(x)\in W^{1,\infty}(\Omega).\end{aligned}\tag{2.1}$$

(A3) Let p be a number with $2\le p<+\infty$, $n=1,2$, and $2\le p\le\frac{2n-2}{n-2}$, $n\ge 3$.

Now, we define the following functionals

$$I(y(t))=\|\nabla y(t)\|^2-\|y(t)\|_p^p,$$
$$J(y(t))=\frac{1}{2}\|\nabla y(t)\|^2-\frac{1}{p}\|y(t)\|_p^p.$$

We define the energy as

$$E(t)=\frac{1}{2}\|y_t(t)\|^2+\frac{1}{2}\|\nabla y(t)\|^2-\frac{1}{p}\|y(t)\|_p^p=\frac{1}{2}\|y_t(t)\|^2+J(y(t)),\quad \text{for all } t\ge 0.\tag{2.2}$$

The energy E is a nonincreasing function of the time variable t, and its derivative satisfies

$$E'(t)=-\int_\Omega a(x)y_t g(y_t)\,dx\le 0,\quad \text{for all } t\ge 0.\tag{2.3}$$

We can define the stable set as

$$\mathcal{W}=\Big\{y\mid y\in H_0^1(\Omega), I(y)>0\Big\}\cup\{0\}.$$

For later applications, we list up some lemmas.

Lemma 2.1 ([1])**.** *Let q be a number with $2\le q<+\infty$, $n=1,2$, or $2\le q\le 2n/(n-2)$, $n\ge 3$, then there exists a constant $C_s=C(\Omega,q)$ such that*

$$\|y\|_q\le C_s\|\nabla y\|,\quad \text{for } y\in H_0^1(\Omega).$$

Lemma 2.2 ([8]). *Let $\mathcal{Q}$ a bounded domain of $\mathbb{R}_x \times \mathbb{R}_t$, φ_m and φ functions of $L^q(\mathcal{Q})$, $1 < q < +\infty$, such that*

$$\|\varphi_m\|_{L^q(\mathcal{Q})} \leq C, \quad \varphi_m \to \varphi, \quad a.e. \text{ in } \mathcal{Q}.$$

Then,

$$\varphi_m \to \varphi \quad \text{in } L^q \text{ weak}.$$

Lemma 2.3. *Suppose that $n \geq 3$ and $p \leq \frac{2n}{n-2}$. Let $y(t)$ be a local solution on $[0, t_m]$ with the initial data $y_0 \in \mathcal{W}$ such that*

$$C_s^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} < 1.$$

Then, $y(t) \in \mathcal{W}$ for all $t \in [0, t_m]$.

Proof. We introduce

$$t^* = \inf\{t \in [0, T^*] \mid y(t) \notin \mathcal{W}\} \neq \emptyset.$$

For continuity in time of $y(t)$, $y(t) \in \mathcal{W}$ for all $0 \leq t \leq t^*$ and $y(t^*) \notin \mathcal{W}$, then we have $y(t^*) \neq 0$.

From the continuity of y and the definition of t^*

$$I(y(t^*)) = 0. \tag{2.4}$$

Hence, we get

$$J(y(t)) = \frac{p-2}{2p} \|\nabla y(t)\|^2 + \frac{1}{p} I(y(t)) \geq \frac{p-2}{2p} \|\nabla y(t)\|^2, \quad \text{on } [0, t^*]. \tag{2.5}$$

By the energy identity (2.2) and (2.5), we get

$$\|\nabla y(t)\|^2 \leq \frac{2p}{p-2} J(y(t)) \leq \frac{2p}{p-2} E(t) \leq \frac{2p}{p-2} E(0), \quad \text{on } [0, t^*]. \tag{2.6}$$

Hence, from the Sobolev-Poincaré inequality, we get

$$\begin{aligned} \|y(t)\|_p^p &\leq C_s^p \|\nabla y(t)\|^p \leq C_s^p \|\nabla y(t)\|^{p-2} \|\nabla y(t)\|^2 \\ &\leq C_s^p \left(\frac{2p}{p-2} E(0) \right)^{\frac{p-2}{2}} \|\nabla y(t)\|^2, \quad \text{on } [0, t^*]. \end{aligned} \tag{2.7}$$

As $t \to t^*$ and $\alpha < 1$, we obtain

$$\|y(t^*)\|_p^p \leq \alpha \|\nabla y(t^*)\|^2 \quad \text{and} \quad \|\nabla y(t^*)\|^2 \neq 0.$$

Then

$$\|y(t^*)\|_p^p \leq \|\nabla y(t^*)\|^2.$$

As a result, we obtain $I(y(t^*)) > 0$, which contradicts to (2.4). Thus, we conclude that $u(t) \in \mathcal{W}$, on $[0, t^*]$. This ends the proof of Lemma 2.3. $\square$

3. Well-Posedness

In this section we prove the existence of regular solutions to problem (1.5) and for this purpose we employ Galerkin method. Then, using a density argument we extend the same result to weak solutions.

Theorem 3.1. *Let $y_0 \in H^2(\Omega) \cap \mathcal{W}$, $y_1 \in H_0^1(\Omega)$. Assume that* **(A1)-(A3)** *hold. Then problem* (1.5) *admits a unique regular solution $y(x,t)$ in the class*

$$y \in L^\infty([0,\infty); H^2(\Omega) \cap \mathcal{W}), \quad y_t \in L^\infty([0,\infty); H_0^1(\Omega)), \quad y_{tt} \in L^\infty([0,\infty); L^2(\Omega)).$$

Theorem 3.2. *Let $y_0 \in \mathcal{W}$, $y_1 \in L^2(\Omega)$. Assume that* **(A1)-(A3)** *hold. Then problem* (1.5) *possesses a weak solution in the class*

$$y \in C^0([0,\infty); \mathcal{W}) \cap C^1([0,\infty); L^2(\Omega)).$$

Proof. We employ the Faedo-Galerkin approximation method to construct a global solution, let $\{w^i \mid i \in \mathbb{N}\}$ be the Hilbert basis of $L^2(\Omega)$, $H_0^1(\Omega)$ and $H^2(\Omega)$ given by

$$\begin{cases} -\Delta w^i = \lambda^i w^i, & \text{in } \Omega, \\ w^i = 0, & \text{on } \Gamma. \end{cases}$$

Set V^m the space generated by $\{w^1, w^2, \dots, w^i\}$ and we construct approximate solutions y^m, $m = 1, 2, 3, \dots$, in the form

$$y^m(t,x) = \sum_{j=1}^m c^{j,m}(t) w^j(x),$$

where $c^{j,m}$ is determined by the ordinary differential equations

$$(y_{tt}^m(t), v) - (\Delta y^m(t), v) + (a(x)g(y_t^m), v) = (|y^m|^{p-2} y^m, v), \quad \text{for all } v \in V^m, \tag{3.1}$$

on some interval $[0, t_m)$. Let y_0^m and y_1^m in V^m be such that

$$y^m(0) = y_0^m = \sum_{j=1}^m (y_0, w^j) w^j \to y_0, \quad \text{in } H^2(\Omega) \cap \mathcal{W} \text{ as } m \to +\infty, \tag{3.2}$$

$$y_t^m(0) = y_1^m = \sum_{j=1}^m (y_1, w^j) w^j \to y_1, \quad \text{in } H_0^1(\Omega) \text{ as } m \to +\infty, \tag{3.3}$$

and

$$\Delta y_0^m - a(x)g(y_1^m) + |y_0^m|^{p-2} y_0^m \to \Delta y_0 - a(x)g(y_1) + |y_0|^{p-2} y_0, \quad \text{in } L^2(\Omega) \text{ as } m \to +\infty. \tag{3.4}$$

3.1. **A priori estimates.**

3.1.1. *The first estimate.* We are going to use some a priori estimates to show that $t_m = +\infty$.

Choosing $v = 2y_t^m$ in (3.1), using Green's formula and then integrating over $(0, t)$, we find

$$\|y_t^m(t)\|^2 + 2J(y^m(t)) + 2\int_0^t \int_\Omega a(x)y_t^m(s)g(y_t^m(s))\,dx\,ds = \|y_1^m\|^2 + 2J(y_0^m),$$

for all $t \in [0, t_m)$. Using (3.2) and (3.3), we obtain

$$\|y_t^m(t)\|^2 + 2J(y^m(t)) + 2\int_0^t \int_\Omega a(x)y_t^m(s)g(y_t^m(s))\,dx\,ds \leq C_0, \tag{3.5}$$

where $J(y^m(t)) = \frac{1}{2}\|\nabla y(^m t)\|^2 - \frac{1}{p}\|y^m(t)\|_p^p$, for some C_0 independent of m. These estimates imply that the solution y^m exists globally in $[0, +\infty[$.

Estimate (3.5) yields

$$y^m \text{ is bounded in } L^\infty(0, T, \mathcal{W}), \tag{3.6}$$

$$y_t^m \text{ is bounded in } L^\infty(0, T, L^2(\Omega)), \tag{3.7}$$

$$a(x)y_t^m g(y_t^m) \text{ is bounded in } L^1(\Omega \times (0, T)). \tag{3.8}$$

We prove that $a(x)g(y_t^m(t))$ is bounded, using (**A1**) and (3.8), we have

$$\int_0^T \int_\Omega a^2(x)g^2(y_t^m)\,dx\,dt \leq \tau_1\|a\|_\infty \int_0^T \int_\Omega a(x)|y_t^m g(y_t^m)|\,dx\,dt \leq K.$$

Then

$$a(x)g(y_t^m) \text{ is bounded in } L^2(\Omega \times (0, T)). \tag{3.9}$$

3.1.2. *The second estimate.* We now proceed with further a priori estimates. In doing so, differentiating (3.1) with respect to t, we get

$$(y_{ttt}^m(t) - \Delta y_t^m(t) + a(x)y_{tt}^m g'(y_t^m(t)), v) = ((p-1)|y^m(t)|^{p-2}y_t^m(t), v).$$

Choosing $v = y_{tt}^m$, we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (|y_{tt}^m(t)|^2 + |\nabla y_t^m(t)|^2)\,dx + 2\int_\Omega a(x)|y_{tt}^m(t)|^2 g'(y_t^m(t))\,dx \\ =&2(p-1)\int_\Omega |y^m(t)|^{p-2}y_t^m(t)y_{tt}^m(t)\,dx. \end{aligned} \tag{3.10}$$

From Hölder's, Young's inequalities and (3.6), we have

(3.11)
$$
\begin{aligned}
&\left|\int_\Omega |y^m(t)|^{p-2} y_t^m(t) y_{tt}^m(t)\,dx\right|\\
\le& C(\Omega)\Big(\int_\Omega 1^{\frac{p-1}{p-2}}\,dx\Big)^{\frac{p-2}{p-1}}\Big(\int_\Omega |y^m(t)|^{2(p-1)}\,dx\Big)^{\frac{p-2}{2(p-1)}}\Big(\int_\Omega |y_t^m(t)|^{2(p-1)}\Big)^{\frac{1}{2(p-1)}}\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C_s\|\nabla y^m(t)\|^{p-2}\|\nabla y_t^m(t)\|\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C\|\nabla y_t^m(t)\|\int_\Omega |y_{tt}^m(t)|\,dx\\
\le& C(\varepsilon)\|\nabla y_t^m(t)\|^2+\varepsilon\|y_{tt}^m(t)\|^2.
\end{aligned}
$$

Integrating (3.10) over $(0,t)$ and using (3.11), we have

$$
\begin{aligned}
&\int_\Omega (|y_{tt}^m(t)|^2+|\nabla y_t^m(t)|^2)\,dx+2\tau_0\int_0^t\int_\Omega a(x)|y_{tt}^m(s)|^2\,dx\,ds\\
\le&\|y_{tt}^m(0)\|^2+\|\nabla y_1^m\|^2+C\int_0^t \|y_{tt}^m(s)\|^2+\|\nabla y_t^m(s)\|^2\,ds.
\end{aligned}\tag{3.12}
$$

We shall estimate $\|y_{tt}^m(0)\|$. To this end, choose $v=y_{tt}^m$ in (3.1) and set $t=0$ to derive

$$\|y_{tt}^m(0)\|^2=\int_\Omega y_{tt}^m(0)(\Delta y_0^m-a(x)g(y_1^m)+|y_0^m|^{p-2}y_0^m)\,dx,$$

from which, thanks to (3.4) and Cauchy-Schwarz inequality, we find $\|y_{tt}^m(0)\|\le C_1$, where C_1 is a positive constant independent of m.

We gain from (3.12) and Gronwall's lemma that

$$\|y_{tt}^m(t)\|^2+\|\nabla y_t^m(t)\|^2\le C_2,\tag{3.13}$$

for all $t\in[0,T]$, and C_2 is a positive constant independent of m. We conclude from (3.13) that

$$y_t^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)),\tag{3.14}$$

$$y_{tt}^m \text{ is bounded in } L^\infty(0,T,L^2(\Omega)).\tag{3.15}$$

3.1.3. *The third estimate.* Choosing $v=-\Delta y_t^m$ in (3.1) and then integrating over $[0,t]$ for all $t\in[0,T]$, we obtain

$$
\begin{aligned}
&\int_\Omega (|\nabla y_t^m(t)|^2+|\Delta y^m(t)|^2)\,dx-2\int_0^t\int_\Omega a(x)\Delta y_t^m g(y_t^m)\,dx\,ds\\
=&\|\nabla y_1^m\|^2+\|\Delta y_0^m\|^2-2\int_0^t\int_\Omega |y^m(s)|^{p-2}y^m(s)\Delta y_t^m(s)\,dx\,ds.
\end{aligned}\tag{3.16}
$$

Since $g(0)=0$ and $y_t^m=0$ on Γ, applying the Green formula, we obtain

$$-\int_\Omega a(x)\Delta y_t^m g(y_t^m)\,dx=\int_\Omega \nabla a(x)\nabla y_t^m g(y_t^m)\,dx+\int_\Omega a(x)|\nabla y_t^m|^2 g'(y_t^m)\,dx,$$

using **(A1)**, we obtain

$$\int_\Omega \nabla a(x)\nabla y_t^m g(y_t^m)\,dx \le C_s\tau_1\|\nabla a\|_\infty \int_\Omega |\nabla y_t^m|^2\,dx. \tag{3.17}$$

Thanks to Green's formula, Hölder's inequality, we have

$$\begin{aligned} -\int_\Omega |y^m(t)|^{p-2}y^m(t)\Delta y_t^m(t)\,dx &= \int_\Omega \nabla(|y^m(t)|^{p-2}y^m(t))\nabla y_t^m(t)\,dx \\ &\le \frac{1}{2}\|\nabla y^m(t)\|^{2(p-1)} + \frac{1}{2}\|\nabla y_t^m(t)\|^2. \end{aligned} \tag{3.18}$$

Reporting estimate (3.17) and (3.18) in (3.16), we find

$$\begin{aligned} &\|\nabla y_t^m(t)\|^2 + \|\Delta y^m(t)\|^2 + 2\tau_0\int_0^t\int_\Omega a(x)|\nabla y_t^m(s)|^2\,dx\,ds \\ \le&\|\nabla y_1^m\|^2 + \|\Delta y_0^m\|^2 + \int_0^t \|\nabla y^m(s)\|^{2(p-1)}\,ds + \left(\frac{1}{2} + C_s\tau_1\|\nabla a\|_\infty\right)\int_0^t \|\nabla y_t^m(s)\|^2\,ds. \end{aligned}$$

By Gronwall lemma, we obtain

$$\|\nabla y_t^m(t)\|^2 + \|\Delta y^m(t)\|^2 \le C_3, \tag{3.19}$$

where C_3 is a positive constant independent of m. We conclude from (3.19) that

$$y^m \text{ is bounded in } L^\infty(0,T,H^2(\Omega)). \tag{3.20}$$

Furthermore, we have from **(A3)**, Lemma (2.1) and (3.6) that

$$|y^m|^{p-2}y^m \text{ is bounded in } L^\infty(0,T,H_0^1(\Omega)). \tag{3.21}$$

3.2. **Solvability of** (1.5)**.** Applying the Dunford-Pettis theorem and the Riesz lemma we conclude from (3.6), (3.7), (3.9), (3.14), (3.15), (3.20) and (3.21), replacing the sequence y^m with a subsequence if needed, that

$$y^m \rightharpoonup y \text{ weakly star in } L^\infty(0,T,H^2(\Omega)\cap\mathcal{W}), \tag{3.22}$$

$$y_t^m \rightharpoonup y_t \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.23}$$

$$y_{tt}^m \rightharpoonup y_{tt} \text{ weakly star in } L^\infty(0,T,L^2(\Omega)), \tag{3.24}$$

$$|y^m|^{p-2}y^m \rightharpoonup \chi \text{ weakly star in } L^\infty(0,T,H_0^1(\Omega)), \tag{3.25}$$

$$a(x)g(y_t^m) \rightharpoonup \varphi \text{ weakly star in } L^2(\Omega\times(0,T)). \tag{3.26}$$

3.2.1. *Analysis of the nonlinear terms.* From (3.6), we see that

$$y^m \text{ is bounded in } L^2(0,T,H^1(\Omega)). \tag{3.27}$$

Then, we have y^m is bounded in $H^1(\mathcal{Q})$, where $\mathcal{Q} = [0,T]\times\Omega$ and the injection $H^1(\mathcal{Q}) \hookrightarrow L^2(\mathcal{Q})$ is compact, and there exists a subsequence of y^m still denoted by the same notation such that

$$y^m \to y, \quad \text{a.e. in } L^2(\mathcal{Q}) \tag{3.28}$$

$$y_t^m \to y_t, \quad \text{a.e. in } L^2(\mathcal{Q}) \tag{3.29}$$

We deduce from (3.28) that

$$|y^m|^{p-2}y^m \to |y|^{p-2}y, \quad \text{a.e. in } \mathcal{Q}.$$

From Lemma (2.2), we deduce

$$|y^m|^{p-2}y^m \rightharpoonup |y|^{p-2}y, \quad \text{weakly star in } L^\infty(0,T,H_0^1(\Omega)). \tag{3.30}$$

By (3.26) and (3.30), we obtain $\chi = |y|^{p-2}y$. It remains now to prove that

$$\int_0^T\int_\Omega a(x)g(y_t^m)v\,dx\,dt \to \int_0^T\int_\Omega a(x)g(y_t)v\,dx\,dt, \quad \text{for all } v\in L^2(0,T,L^2(\Omega)).$$

We have $a(x)g(y_t)\in L^1(\mathcal{Q})$. Since g is continuous, we deduce from (3.29), that

$$a(x)g(y_t^m) \to a(x)g(y_t, \quad \text{a.e. in } \ \mathcal{Q}. \tag{3.31}$$

$$a(x)y_t^m g(y_t^m) \to a(x)y_t g(y_t), \quad \text{a.e. in } \mathcal{Q}.$$

Using (3.8) and Fatou's Lemma, we deduce that

$$\int_0^T\int_\Omega a(x)y_t g(y_t)\,dx\,dt \le K.$$

By using Cauchy-Schwarz's inequality, we obtain

$$\int_0^T\int_\Omega |a(x)g(y_t)|\,dx\,dt \le c|\mathcal{Q}|^{\frac{1}{2}}\Big(\int_0^T\int_\Omega |a(x)g(y_t)|^2\,dx\,dt\Big)^{\frac{1}{2}} \le \widetilde{K}.$$

Let $Q\subset[0,T]\times\Omega$. We set

$$Q_1 = \Big\{(t,x)\in[0,T]\times\Omega \mid |g(y_t^m)|\le |Q|^{-1/2}\Big\}, \quad Q_2 = Q\setminus Q_1$$

and $J(r) = \inf\Big\{|s| \mid s\in\mathbb{R}, |g(s)|\ge r\Big\}$. Then, we have

$$\begin{aligned}\int_Q a(x)g(y_t^m)\ dx\ dt &= \int_{Q_1} a(x)g(y_t^m)\,dx\,dt + \int_{Q_2} a(x)g(y_t^m)\,dx\,dt\\ &\le \|a\|_\infty |Q|^{1/2} + J(|Q|^{\frac{-1}{2}})^{-1}\int_{Q_2} a(x)|y_t^m g(y_t^m)|\,dx\,dt.\end{aligned}$$

Applying (3.8), we find

$$\sup_m \int_Q a(x)g(y_t^m)\,dx\,dt \to 0, \quad \text{when } |Q|\to 0,$$

and from (3.31), we deduce thanks to Vitali's Theorem that

$$a(x)g(y_t^m) \to a(x)g(y_t), \quad \text{in } L^1([0,T]\times\Omega).$$

Hence, (3.26) yields $a(x)g(y_t) = \varphi \in L^2(\mathcal{Q})$ and

$$a(x)g(y_t^m) \rightharpoonup a(x)g(y_t), \quad \text{in } L^2(\mathcal{Q}).$$

We deduce, for all $v\in L^2([0,T]\times L^2(\Omega))$, that

$$\int_0^T\int_\Omega a(x)g(y_t^m)v\,dx\,dt \to \int_0^T\int_\Omega a(x)g(y_t)v\,dx\,dt. \tag{3.32}$$

Convergences (3.22)–(3.26), (3.30) and (3.32) permit us to pass to the limit in the (3.1). As w^j is a basis of $H^2(\Omega)$, then, for all $T>0$, for all $\theta \in D(0,T)$ and for all $v \in L^2([0,T] \times L^2(\Omega))$, after passing to the limit we obtain

$$\begin{aligned}(3.33)\qquad &\int_0^T \int_\Omega (y_{tt}(t), v(t))\theta(t)\,dt - \int_0^T (\Delta y(t), v(t))\theta(t)\,dt \\ &+ \int_0^T (a(x)(g(y_t), v(t))\theta(t)\,dt - \int_0^T (|y|^{p-2}(t)y(t), v(t))\theta(t)\,dt = 0.\end{aligned}$$

From (3.33) and taking $v \in D(0,T))$, we show that

$$y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y, \quad \text{in } D'(\Omega \times (0,T))$$

Now, since y_{tt}, $a(x)g(y_t)$, $|y|^{p-2}y \in L^2(0,\infty,L^2(\Omega))$ we have $\Delta y \in L^2(0,\infty,L^2(\Omega))$ and therefore

$$y_{tt} - \Delta y + a(x)g(y_t) = |y|^{p-2}y, \quad \text{in } L^\infty(0,\infty,L^2(\Omega))$$

3.3. **Uniqueness.** Let y_1 and y_2 be solutions to problem (1.5). Then, defining $z = y_1 - y_2$, we obtain

$$(z_{tt}, v) + (\nabla z, \nabla v) + (a(x)(g(y_{1,t}) - g(y_{2,t})), v) = (|y_1|^{p-2}y_1 - |y_2|^{p-2}y_2, v),$$

for all $v \in H_0^1(\Omega)$. Substituting $v = z_t(t)$ in the above equality and observing that g is nondecreasing, it results that

$$(3.34)\quad \frac{d}{dt}\Big\{\|z_t\|^2+\|\nabla z\|^2\Big\}+2\int_\Omega a(x)(g(y_{1,t})-g(y_{2,t}))z_t\,dx = 2\int_\Omega(|y_1|^{p-2}y_1-|y_2|^{p-2}y_2)z_t(t)\,dx.$$

It follows from the mean value theorem that

$$\begin{aligned}&\Big|y_1(x,t)|^{p-2}y_1(x,t) - |y_2(x,t)|^{p-2}y_2(x,t)\Big| \\ \le&(p-1)(|y_1(x,t)| + |y_2(x,t)|)^{p-2}|y_1(x,t) - y_2(x,t)|,\end{aligned}$$

from (3.34) and using the monotonicity of g a hence, we conclude that

$$\frac{d}{dt}\Big\{\|z_t\|^2 + \|\nabla z\|^2\Big\} \le 2(p-1)\int_\Omega (|y_1(x,t)| + |y_2(x,t)|)^{p-2}|z(t)||z_t(t)|\,dx.$$

Using analogous arguments like those used in the second estimate, we obtain

$$(3.35)\quad \frac{d}{dt}\Big\{\|z_t\|^2 + \|\nabla z\|^2\Big\} + 2\int_\Omega a(x)(g(y_{1,t}) - g(y_{2,t}))z_t\,dx \le C(\|z_t\|^2 + \|\nabla z\|^2).$$

Integrating the inequality (3.35) over $(0,t)$ and making use of Gronwall's lemma we conclude that $\|z_t\|^2 = \|\nabla z\|^2 = 0$. This concludes the first part of the proof.

3.4. **Weak solutions.** In order to obtain existence for weak solutions we use standard arguments of density. Indeed, let us assume that $\{y_0, y_1\} \in \mathcal{W} \times L^2(\Omega)$. So, let $\{y_0^\mu, y_1^\mu\} \in \mathcal{W} \times L^2(\Omega)$ be such that

$$y_0^\mu \to y_0, \quad \text{in } \mathcal{W}, \quad \text{and} \quad y_1^\mu \to y_1, \quad \text{in } L^2(\Omega). \tag{3.36}$$

Then, for each $\mu \in \mathbb{N}$ there exists y^μ regular solution of (1.5) belonging to the class of Theorem (3.1). Repeating the same arguments used in the first estimate we obtain

$$\|y_t^\mu(t)\|^2 + \|\nabla y(^\mu t)\|^2 - \frac{2}{p}\|y^\mu(t)\|_p^p + 2\int_0^t \int_\Omega a(x) y_t^\mu(s) g(y_t^\mu(s))\, dx\, ds \le C, \tag{3.37}$$

where C is a positive constant independent of μ.

Defining $z^{\mu,\sigma} = y^\mu - y^\sigma$, $\mu, \sigma \in \mathbb{N}$, where y^μ and y^σ are smooth solutions of (1.5), we obtain by the monotonicity of g that

(3.38)
$$\frac{1}{2} \cdot \frac{d}{dt}\Big\{\|z_t^{\mu,\sigma}\|^2 + \|\nabla z^{\mu,\sigma}\|^2\Big\} \le K(p)\int_\Omega (|y^\mu(x,t)| + |y^\sigma(x,t)|)^{p-2} |z^{\mu,\sigma}(t)||z_t^{\mu,\sigma}(t)|\, dx.$$

Combining (3.37) and (3.38) we obtain, after integrating over $(0,t)$ and using Gronwall's lemma, that

$$\|y_t^\mu(t) - y_t^\sigma(t)\|^2 + \|\nabla y^\mu(t) - \nabla y^\sigma(t)\|^2 \le K(p,T)(\|y_1^\mu - y_1^\sigma\|^2 + \|\nabla y_0^\mu - \nabla y_0^\sigma\|^2), \tag{3.39}$$

where $K(p,T)$ is a positive constant independent of $\mu, \sigma \in \mathbb{N}$.

From (3.36) and (3.39), we conclude that there exists a function y such that, for all $T > 0$, we have

$$y^\mu \to y \text{ strongly in } C^0(0,T,\mathcal{W}), \tag{3.40}$$

$$y_t^\mu \to y_t \text{ strongly in } C^0(0,T,L^2(\Omega)). \tag{3.41}$$

From (3.37), (3.40) and (3.41) we also have,

$$\begin{aligned} y_t^\mu &\rightharpoonup y_t \text{ weakly star in } L^2_{loc}(0,\infty,L^2(\Omega)),\\ |y^\mu|^{p-2}y^\mu &\rightharpoonup |y|^{p-2}y \text{ weakly star in } L^2_{loc}(0,\infty,L^2(\Omega)),\\ a(x)g(y_t^\mu) &\rightharpoonup a(x)g(y_t) \text{ weakly star in } L^2(\Omega\times(0,T)). \end{aligned} \tag{3.42}$$

The weak convergences from the estimate given by (3.37) and the convergences obtained in (3.40)–(3.42) are sufficient to pass to the limit in order to obtain a weak solution to problem (1.5). □

4. Stability Result

In this section, we state and prove the stability result for the energy of the problem (1.5). The stability result reads as follows.

Theorem 4.1. *Let $y_0 \in H^2(\Omega) \cap \mathcal{W}$, $y_1 \in H_0^1(\Omega)$. Assume that* **(A1)-(A3)** *hold. The energy of the unique solution of the problem* (1.5)*, given by* (2.2)*, decays exponentially to zero, there exist positive constants K and λ, independent of the initial data, with*

$$E(t) \le K E(0) e^{-\lambda t}, \quad \textit{for all}\, t \ge 0. \tag{4.1}$$

We first consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} 0 \le \psi \le 1, & \\ \psi = 1, & \text{in } \bar{\Omega}\backslash \mathcal{M}_1, \\ \psi = 0, & \text{in } \mathcal{M}_0. \end{cases} \tag{4.2}$$

For $M > 0$ and $\mu > 0$, define the perturbed energy

$$\widehat{E}(t) = M.E(t) + E^\mu(t)\rho(t), \tag{4.3}$$

where

$$\rho(t) = 2\int_\Omega y_t(h.\nabla y)\,dx + \theta \int_\Omega y_t y\,dx, \tag{4.4}$$

$$h(x) = m(x)\psi(x), \tag{4.5}$$

and $\theta \in]n-2, n[$.

Lemma 4.1. *There exist two positive constants λ_1 and λ_2 such that*

$$\lambda_1 E(t) \le \widehat{E}(t) \le \lambda_2 E(t), \quad \textit{for all}\, t \ge 0. \tag{4.6}$$

Proof. Thanks to Cauchy-Schwarz's inequality, we have

$$|\rho(t)| \le 2\mathcal{R}(x^0)\|\nabla y\|\|y_t\| + \theta\sqrt{C_s}\|\nabla y\|\|y_t\|, \tag{4.7}$$

where

$$\mathcal{R}(x^0) = \max_{x\in\bar{\Omega}} |x - x^0|. \tag{4.8}$$

From (4.7) we obtain

$$\begin{aligned} |\rho(t)| &\le (\theta\sqrt{C_s} + 2\mathcal{R}(x^0))\left\{\frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|\nabla y\|^2\right\} \\ &\le (\theta\sqrt{C_s} + 2\mathcal{R}(x^0))E(t). \end{aligned}$$

Then, for M large enough, we obtain (4.6), where $\lambda_1 = M - E^\mu(0)(\theta\sqrt{C_s} + 2\mathcal{R}(x^0))$ and $\lambda_2 = M + E^\mu(0)(\theta\sqrt{C_s} + 2\mathcal{R}(x^0))$. □

Lemma 4.2. *The functional $\rho(t)$ defined in* (4.4) *satisfies*

$$\begin{aligned} \rho'(t) = &\int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial \nu}\right)^2 d\Gamma - (n-\theta)\int_\Omega |y_t|^2\,dx - (\theta - n + 2)\int_\Omega |\nabla y|^2\,dx \\ &- \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi y_t^2\,dx + n\int_{\mathcal{M}_1}(1-\psi)y_t^2\,dx + (n-2)\int_{\mathcal{M}_1}(\psi - 1)|\nabla y|^2 dx \\ &+ \int_{\mathcal{M}_1\backslash\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx - 2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\backslash\mathcal{M}_0} m_i\frac{\partial\psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx \\ &- \theta\int_\Omega y.a(x)g(y_t)\,dx - \int_\Omega 2(h.\nabla y)a(x)g(y_t)\,dx \\ &+ 2\int_\Omega h.\nabla y|y|^{p-2}y\,dx + \theta\int_\Omega |y|^p\,dx. \end{aligned} \tag{4.9}$$

Proof. Taking the derivative of $\rho(t)$ with respect to t,

$$\begin{aligned}\rho'(t) =&2\int_\Omega y_{tt}(h\nabla y)\,dx + 2\int_\Omega y_t(h\nabla y_t)\,dx + \theta\int_\Omega y_{tt}y\,dx + \theta\int_\Omega y_t^2\,dx\\ =&2\int_\Omega y_t(h\nabla y_t)\,dx + +\theta\int_\Omega y_{tt}y\,dx + 2\int_\Omega h.\nabla y.\Delta y\,dx\\ &-2\int_\Omega h.\nabla y.a(x)g(y_t)\,dx + 2\int_\Omega h.\nabla y|y|^p y\,dx + \theta\int_\Omega |y_t|^2\,dx.\end{aligned}\tag{4.10}$$

Using (1.1)–(1.4), (4.2), (4.5) and Green formulas the first term of the right hand side of (4.10), we have

$$\begin{aligned}2\int_\Omega y_t(h\nabla y_t)dx =& -\int_\Omega \operatorname{div}(h)y_t^2\,dx\\ =& -\int_{\Omega\setminus\mathcal{M}_1}\operatorname{div}(\psi.m)y_t^2dx - \int_{\mathcal{M}_1}\operatorname{div}(\psi.m)y_t^2dx\\ =& -n\int_{\Omega\setminus\mathcal{M}_1} y_t^2dx - \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi y_t^2\,dx - n\int_{\mathcal{M}_1}\psi y_t^2dx.\end{aligned}$$

Then

$$2\int_\Omega y_t(h\nabla y_t) = -n\int_\Omega y_t^2dx + n\int_{\mathcal{M}_1}(1-\psi)y_t^2dx - \int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi y_t^2\,dx.\tag{4.11}$$

Using the first equation of (1.5) and applying the Green formula, the second term of the right hand side of (4.10), we obtain

$$\theta\int_\Omega y_{tt}y\,dx = -\theta\int_\Omega|\nabla y|^2\,dx - \theta\int_\Omega a(x)yg(y_t)\,dx + \theta\int_\Omega |y|^p\,dx.\tag{4.12}$$

We have $\frac{\partial y}{\partial x_k} = \frac{\partial y}{\partial\nu}\nu_k$, which implies

$$h\nabla y = (h.\nu)\frac{\partial y}{\partial\nu} \quad\text{and}\quad |\nabla y|^2 = \left(\frac{\partial y}{\partial\nu}\right)^2 \quad\text{on }\Gamma.$$

From the above expressions and using Green's formulas, the third term of the right hand side of (4.10) can be rewritten as follows

$$\begin{aligned}&2\int_\Omega(h\nabla y)\Delta y\,dx\\ =&2\int_\Gamma(h.\nu)|\nabla y|^2\,d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx - 2\int_\Omega h(\nabla y)\nabla(\nabla y)\,dx\\ =&2\int_\Gamma(h.\nu)\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx - \int_\Omega h\nabla(|\nabla y|^2)\,dx\\ =&\int_\Gamma(h.\nu)\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma - 2\sum_{i,k=1}^n\int_\Omega \frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx + \int_\Omega \operatorname{div}(h)|\nabla y|^2\,dx.\end{aligned}\tag{4.13}$$

So, by using (1.2), (4.2) and (4.5), the second term of (4.13) gives

$$
\begin{aligned}
(4.14)\quad & -2\sum_{i,k=1}^{n}\int_{\Omega}\frac{\partial h_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx \\
&= -2\sum_{i,k=1}^{n}\int_{\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx-2\sum_{i,k=1}^{n}\int_{\Omega\backslash\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial(m_i\psi_i)}{\partial x_k}\,dx \\
&= -2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1}\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\psi_i\frac{\partial m_i}{\partial x_k}\,dx-2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1}m_i\frac{\partial \psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx \\
&\quad -2\sum_{i,k=0}^{n}\int_{\Omega\backslash\mathcal{M}_1}\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_k}\,dx \\
&= -2\int_{\mathcal{M}_1}\psi|\nabla y|^2\,dx-2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\backslash\mathcal{M}_0}m_i\frac{\partial \psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\cdot\frac{\partial y}{\partial x_k}\,dx-2\int_{\Omega\backslash\mathcal{M}_1}|\nabla y|^2\,dx.
\end{aligned}
$$

Similarly, the third term of (4.13) can be rewritten as follows

$$
\begin{aligned}
(4.15)\quad & \int_{\Omega}(\mathrm{div}h)|\nabla y|^2\,dx=\int_{\Omega\backslash\mathcal{M}_1}\mathrm{div}(\psi m)|\nabla y|^2\,dx+\int_{\mathcal{M}_1}\mathrm{div}(\psi m)|\nabla y|^2\,dx \\
&= n\int_{\Omega\backslash\mathcal{M}_1}|\nabla y|^2\,dx+\int_{\mathcal{M}_1\backslash\mathcal{M}_0}m\nabla\psi|\nabla y|^2\,dx+n\int_{\mathcal{M}_1}\psi|\nabla y|^2\,dx.
\end{aligned}
$$

Inserting (4.14) and (4.15) in (4.13), we arrive at

$$
\begin{aligned}
(4.16)\quad 2\int_{\Omega}(h\nabla y)\Delta y\,dx &= \int_{\Gamma}(h.\nu)\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma+(n-2)\int_{\Omega}|\nabla y|^2\,dx \\
&\quad +(n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla y|^2\,dx \\
&\quad -2\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\backslash\mathcal{M}_0}m_i\frac{\partial\psi_i}{\partial x_k}\cdot\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}\,dx \\
&\quad +\int_{\mathcal{M}_1\backslash\mathcal{M}_0}m\nabla\psi|\nabla y|^2\,dx.
\end{aligned}
$$

Simple substitution of (4.11), (4.12) and (4.16) give (4.9) ends the proof of Lemma 4.2. □

Lemma 4.3. *We have*

$$
\begin{aligned}
(4.17)\quad |\rho'(t)| \le & -K_nE(t)+B\int_{\Omega}|\nabla y|^2\,dx+A\int_{\omega}|y_t|^2\,dx \\
& -\theta\int_{\Omega}a(x)yg(y_t)\,dx-2\int_{\Omega}(h\nabla y)a(x)g(y_t)\,dx \\
& +2\int_{\Omega}h.\nabla y|y|^{p-2}y\,dx+\left(\theta+\frac{K_n}{p}\right)\int_{\Omega}|y|^p dx,
\end{aligned}
$$

where

$$K_n = \min\Big\{2(n-\theta), 2(\theta-n+2)\Big\}, \quad A = \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + n$$

and

$$B = 3\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| + (n-2).$$

Proof. Next, we estimate some terms on the RHS of identity (4.9).

Taking (1.1)–(1.4), (4.2) and (4.5), we have

(4.18)
$$\int_\Gamma (h.\nu)\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma = \int_{\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma + \int_{\Gamma\setminus\Gamma(x^0)}(m.\nu)\psi\left(\frac{\partial y}{\partial\nu}\right)^2 d\Gamma \le 0,$$

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|y_t|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\omega |y_t|^2\,dx, \tag{4.19}$$

$$n\int_{\mathcal{M}_1}(1-\psi)|y_t|^2\,dx \le n\int_\omega |y_t|^2\,dx, \tag{4.20}$$

$$2\left|\sum_{i,k=0}^{n}\int_{\mathcal{M}_1\setminus\mathcal{M}_0}\frac{\partial y}{\partial x_k}\cdot\frac{\partial y}{\partial x_i}m_i\frac{\partial\psi_i}{\partial x_i}\,dx\right| \le 2\mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla y|^2\,dx, \tag{4.21}$$

$$\int_{\mathcal{M}_1\setminus\mathcal{M}_0} m\nabla\psi|\nabla y|^2\,dx \le \mathcal{R}(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)|\int_\Omega|\nabla y|^2\,dx \tag{4.22}$$

and

$$(n-2)\int_{\mathcal{M}_1}(\psi-1)|\nabla y|^2\,dx \le (n-2)\int_\Omega|\nabla y|^2\,dx. \tag{4.23}$$

Taking into account (4.18)–(4.23) into (4.9) we obtain (4.17). The proof of Lemma 4.3 is completed. ☐

Proof. (of Theorem 4.1) Taking the derivative of (4.3) with respective to t, we have

$$\widehat{E'}(t) = M\,E'(t) + \mu E'(t)E^{\mu-1}(t)\rho(t) + E^\mu(t)\rho'(t).$$

Using (2.2) and (4.17), we have

$$\begin{aligned}\widehat{E'}(t) \le{}& M\,E'(t) + C_\mu E^\mu(0)|E'(t)| - K_n.E^{\mu+1}(t)\\ &+ A\,E^\mu(t)\int_\omega|y_t|^2\,dx + B\,E^\mu(t)\int_\Omega|\nabla y|^2\,dx\\ &+ 2E^\mu(t)\int_\Omega(h\nabla y)a(x)g(y_t)\,dx - \theta E^\mu(t)\int_\Omega y\,a(x)g(y_t)\,dx\\ &+ 2E^\mu(t)\int_\Omega h\nabla y|y|^{p-2}y\,dx + \left(\theta+\frac{K_n}{p}\right)E^\mu(t)\int_\Omega|y|^p dx.\end{aligned} \tag{4.24}$$

Next, we will estimate some terms on the right-hand side of identity (4.24). Using (2.3), we get

$$\begin{aligned} A\,E^{\mu}(t)\int_{\omega}|y_t|^2\,dx \leq \frac{1}{\tau_1}\frac{A}{a_0}E^{\mu}(t)\int_{\Omega}a(x)y_t g(y_t)\,dx \leq CE^{\mu}(t)(-E'(t)) \\ \leq CE^{\mu}(0)|E'(t)|. \end{aligned} \tag{4.25}$$

By (2.2), we have

$$B.E^{\mu}(t)\int_{\Omega}|\nabla y|^2\;dx \leq BE^{\mu+1}(t). \tag{4.26}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} 2.E^{\mu}(t)\int_{\Omega}h.a(x)\nabla y g(y_t)\,dx &\leq 2\mathcal{R}(x^0)E^{\mu}(t)\|\nabla y\|\Big(\int_{\Omega}a^2(x)g^2(y_t)\,dx\Big)^{\frac{1}{2}} \\ &\leq 2c\mathcal{R}(x^0)\sqrt{\|a\|_{\infty}}E^{\mu+\frac{1}{2}}(t)\Big(\int_{\omega}a(x)y_t(t)g(y_t)\,dx\Big)^{\frac{1}{2}} \\ &\leq 2c\mathcal{R}(x^0)\sqrt{\|a\|_{\infty}}E^{\mu+\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}}. \end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned} 2.E^{\mu}(t)\int_{\Omega}h.a(x)\nabla y g(y_t)\,dx &\leq c\mathcal{R}(x^0)\|a\|_{\infty}E^{2\mu+1}(t) + c\mathcal{R}(x^0)|E'(t)| \\ &\leq c\mathcal{R}(x^0)\|a\|_{\infty}E^{\mu}(0)E^{\mu+1}(t) + c\mathcal{R}(x^0)|E'(t)| \\ &\leq \frac{K_n}{6}E^{\mu+1}(t) + c\mathcal{R}(x^0)|E'(t)|. \end{aligned} \tag{4.27}$$

Using Cauchy-Schwarz, Young's and Sobolev-Poincares inequalities, we get

$$\begin{aligned} \theta E^{\mu}(t)\int_{\Omega}y.a(x)g(y_t)\,dx &\leq \theta C_s' E^{\mu}(t)\|\nabla y\|\left(\int_{\omega}a^2(x)g^2(y_t)\,dx\right)^{\frac{1}{2}} \\ &\leq C\frac{\|a\|_{\infty}}{2}E^{\mu}(0)E^{\mu+1}(t) + C'\frac{\|a\|_{\infty}}{2}|E'(t)| \\ &\leq \frac{K_n}{6}E^{\mu+1}(t) + C'\frac{\|a\|_{\infty}}{2}|E'(t)|. \end{aligned} \tag{4.28}$$

By Cauchy-Schwarz and Young's inequalities, we find

$$\begin{aligned} 2E^{\mu}(t)\int_{\Omega}h.\nabla y|y|^{p-2}y\,dx &\leq 2.E^{\mu}(t)\mathcal{R}(x^0)\|\nabla u\|\left(\int_{\Omega}|y|^{2(p-1)}\,dx\right)^{\frac{1}{2}} \\ &\leq 2c\mathcal{R}(x^0)E^{\mu+\frac{1}{2}}(t)\|y\|_{2(p-1)}^{p-1} \\ &\leq 2c\mathcal{R}(x^0)E^{\mu+\frac{1}{2}}(t)\|\nabla y\|_2^{p-1}, \end{aligned}$$

where

$$p \leq \frac{2n-2}{n-2},$$

we obtain

$$2.E^{\mu}(t)\int_{\Omega} h\nabla y|y|^{p-2}y\,dx \le 2c\mathcal{R}(x^0)E^{\mu+\frac{1}{2}}(t)E^{\frac{p-1}{2}}(t) \le 2c\mathcal{R}(x^0)E^{\mu+\frac{p}{2}}(t) \le 2c\mathcal{R}(x^0)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0). \tag{4.29}$$

Using Sobolev-Poincaré and Young's inequalities, we get

$$\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\,dx \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\|\nabla y\|^p,$$

where

$$p \le \frac{2n}{n-2},$$

we obtain

$$\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\ dx \le 2C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)E^{\frac{K_n}{p}}(t) \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(t) \le C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0). \tag{4.30}$$

Combining (4.26), (4.29) and (4.30), we get

$$2.E^{\mu}(t)\int_{\Omega} h.\nabla y|y|^{p-2}y\,dx + B\,E^{\mu}(t)\int_{\Omega}|\nabla y|^2\ dx + \left(\theta+\frac{K_n}{p}\right)E^{\mu}(t)\int_{\Omega}|y|^p\ dx \le 2c\mathcal{R}(x^0)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0) + BE^{\mu+1}(t) + C_s^p\left(\theta+\frac{K_n}{p}\right)E^{\mu+1}(t)E^{\frac{p-2}{2}}(0) \le \frac{K_n}{6}E^{(\mu+1)}. \tag{4.31}$$

Reporting (4.25), (4.27), (4.28) and (4.31) in (4.24), we find

$$\widehat{E'}(t) \le M.E'(t) + CE^{\mu}(0)|E'(t)| + C|E'(t)| - \frac{K_n}{2}E^{\mu+1}(t).$$

Choosing $\mu = 0$ and M large enough to obtain

$$\widehat{E'}(t) \le -\frac{K_n}{2}E(t) \le -\frac{K_n}{2\lambda_1}\hat{E}(t). \tag{4.32}$$

Finally, by combining (4.6) and (4.32) we obtain (4.1), which complete the proof. □

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