Kragujevac Journal of Mathematics
Volume 50(1) (2026), Pages 25–38.

MULTIPLE POSITIVE SOLUTIONS OF DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEMS

N. S. GOPAL[1,2] AND JAGAN MOHAN JONNALAGADD[2,*]

Abstract. In this work, we deal with the following two-point non-linear Dirichlet boundary value problem for a finite nabla fractional difference equation:

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) = f(u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0. \end{cases}$$

Here a, $b \in \mathbb{R}$ with $b-a \in \mathbb{N}_3$, $1 < \alpha < 2$, $f : \mathbb{R} \to \mathbb{R}^{+} \cup \{0\}$ is a continuous function, and $\nabla^{\alpha}_{\rho(a)}$ denotes the α^{th} order Riemann-Liouville nabla difference operator. First, we construct an associated Green's function and obtain some of its properties. Under suitable conditions on the non-linear part of the difference equation, we deduce some results for at least two and at least three positive solutions of the considered problem. For this purpose, we use a few prominent conical shell fixed point theorems.

1. Introduction

In the year 1695, "L'Hospital inquires Leibniz on the differential operator $\frac{d^n}{dt^n}$: What if the order will be $\frac{1}{2}$? To which Leibniz replied: It will lead to a paradox from which one day useful consequences will be drawn". This question gave birth to a branch of mathematics that we know today as fractional calculus. Although it started around the same time as differential calculus, most of the early developments of fractional calculus were confined to the basement for a long time. Today fractional calculus has been successfully used for mathematical modelling in medical sciences, computational biology, economics, physics and several areas of engineering. For further applications

Key words and phrases. Nabla fractional difference, boundary value problem, Dirichlet boundary conditions, Green's function, cone, fixed point, positive solution.

2020 *Mathematics Subject Classification.* Primary: 34K37, 39A12, 39A60.

DOI 10.46793/KgJMat2601.025G

Received: February 02, 2022.

Accepted: May 16, 2023.

and historical literature, we refer here to a few classical texts on fractional calculus [34, 36, 37] and [31].

On the other hand, discrete fractional calculus deals with arbitrary order differences and sums defined on a discrete domain in either a forward (delta) or backward (nabla) sense. The theory of discrete fractional calculus is relatively new, with the most notable works done in the past decade. The notions of the nabla fractional difference and sum can be traced back to the work [14] and [35]. In this line, Atici and Eloe [?] developed the nabla fractional Riemann-Liouville difference operator, initiated the study of nabla fractional initial value problem and established the exponential law, product rule and nabla Laplace transform. Following their works, the contributions of several mathematicians have made the theory of discrete fractional calculus a fruitful field of research in science and engineering. We refer here to a recent monograph [12] and the references therein.

The study of boundary value problems has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the fractional side, there is a sudden growth in interest for the development of nabla fractional boundary value problems. Many authors have studied nabla fractional boundary value problems recently. To name a few, [2, 16] and [19] worked with self-adjoint Caputo nabla boundary value problem. Brackins [9] studied a particular class of self-adjoint Riemann-Liouville nabla boundary value problem and derived the Green's function associated with it along with a few of its properties. Gholami et al. [17] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla boundary value problem with Dirichlet boundary conditions. Jonnalagadda [13, 20, 21, 23–25] analysed some qualitative properties of two-point non-linear Riemann-Liouville nabla boundary value problem associated with various types of boundary conditions.

There has been an increasing interest in multiple fixed-point theorems and their applications to boundary value problems for differential equations and finite difference equations. Interest in triple solutions was born from the Leggett–Williams multiple fixed-point theorem [33]. Following this, two triple fixed-point theorems by Avery [5], and Avery and Peterson [7] have been developed and applied to specific boundary value problems for ordinary differential equations as well as for their discrete analogues [3, 7]. Also, Avery and Henderson [6] have established twin fixed-point theorem by dual application of Krasnosel'skii fixed-point theorem. The applications of the above fixed-point theorems in discrete fractional calculus are scarce. To the best of our knowledge, there has been no progress in this line, in the domain of nabla fractional calculus.

Our purpose of this article is to establish sufficient conditions for the existence of multiple positive solutions of the following standard two-point non-linear nabla fractional boundary value problem with Dirichlet boundary conditions

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) = f(u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$, with $b-a \in \mathbb{N}_3, 1 < \alpha < 2$ and $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, using conical shell fixed-point theorems such as Leggett-Williams [33] and Avery-Henderson [6].

The present article is organized as follows. Section 2 contains a few preliminaries on nabla fractional calculus. In Sections 3, we present sufficient conditions on three and two positive solutions of (1.1) using fixed-point theorems by Leggett-Williams [33] and Avery-Henderson [6], respectively, on a suitable cone.

2. Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^+$, respectively. We use the following notations, definitions and known results of nabla fractional calculus [12]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}_a^b$, where $b-a \in \mathbb{Z}^+$, are defined by

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\}, \quad \mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}.$$

Definition 2.2. We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$, by

$$\rho(t) = t-1, \quad t \in \mathbb{N}_{a+1}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\Big(\nabla u\Big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\Big(\nabla^N u\Big)(t) = \Big(\nabla\Big(\nabla^{N-1} u\Big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.3 ([12])**.** Let $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$. The generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.4 (See [12])**.** Let $t,\ a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided the right-hand side exists.

We observe the following properties of the nabla fractional Taylor monomials.

Lemma 2.1 ([19, 24])**.** *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold.*

(a) *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$ and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.*
(b) *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.*
(c) *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.*
(d) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.*
(e) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a non-decreasing function of t.*

(f) *If* $t \in \mathbb{N}_s$ *and* $\mu > 0$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* t.
(g) *If* $0 < v \le \mu$, *then* $H_v(t, a) \le H_\mu(t, a)$, *for each fixed* $t \in \mathbb{N}_a$.

Definition 2.5 ([12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 ([12])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \le N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by

$$\left(\nabla_a^{\nu} u\right)(t) = \left(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

Now, we write the expression for the Green's function corresponding to (1.1) and state a few properties of the same, which will be used later.

Theorem 2.1 ([9, 17, 25])**.** *Let* $1 < \alpha < 2$ *and* $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$. *The equivalent form of* (1.1) *is given by*

$$u(t) = \sum_{s=a+2}^{b} G(t, s) f(u(s)), \quad t \in \mathbb{N}_a^b, \tag{2.1}$$

where the Green's function is given by

$$G(t, s) = \begin{cases} G_1(t, s) = \frac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)), & t \in \mathbb{N}_a^{s-1}, \\ G_2(t, s) = \frac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)) - H_{\alpha-1}(t, \rho(s)), & t \in \mathbb{N}_s^{b}. \end{cases} \tag{2.2}$$

Theorem 2.2 ([9, 17, 25])**.** *The Green's function* $G(t, s)$ *defined in* (2.2) *satisfies the following properties:*

(a) $G(a, s) = G(b, s) = 0$, *for all* $s \in \mathbb{N}_{a+1}^b$;
(b) $G(t, a + 1) = 0$, *for all* $t \in \mathbb{N}_a^b$;
(c) $G(t, s) > 0$, *for all* $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$;
(d) $\max_{t \in \mathbb{N}_{a+1}^{b-1}} G(t, s) = G(s - 1, s)$, *for all* $s \in \mathbb{N}_{a+2}^b$;
(e) $\sum_{s=a+1}^{b} G(t, s) \le \lambda$, *for all* $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, *where*

$$\lambda = \left(\frac{b - a - 1}{\alpha \Gamma(\alpha + 1)}\right) \left(\frac{(\alpha - 1)(b - a) + 1}{\alpha}\right)^{\overline{\alpha - 1}}. \tag{2.3}$$

3. Multiple Positive Solutions

In this section, we establish sufficient conditions on the existence of at least two and three positive solutions of (1.1) using Avery-Henderson [6] and Leggett-Williams [33] fixed-point theorems respectively, on a suitable cone, by suitably constructing the growth conditions on the non-linear part of the boundary value problem.

Definition 3.1 ([1])**.** Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed non-empty convex set $K \subset \mathcal{B}$ is called a cone provided,

(i) $eu + iv \in K$, for all $u, v \in K$ and all $e, i \geq 0$;
(ii) $u \in K$ and $-u \in K$ implies $u = 0$.

Definition 3.2 ([28])**.** An operator $T : \mathcal{B} \to \mathcal{B}$ is called completely continuous, if it is continuous and maps bounded sets into pre-compact sets.

Definition 3.3 ([1])**.** A functional α_1 is said to be a non-negative continuous concave functional on a cone K of a real Banach space $\mathcal{B}$, if $\alpha_1 : K \to [0, +\infty)$ is continuous and

$$\alpha_1(tx + (1-t)y) \geq t\alpha_1(x) + (1-t)\alpha_1(y),$$

for all $x, y \in K$ and $t \in [0, 1]$.

The following theorems which are useful for the main results has appeared in [13] and the same has been proved here for the completeness of the article.

Lemma 3.1. *Let a, b be two real numbers such that $0 < a < b$ and $1 < \alpha < 2$. Then $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. It is enough to show that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$. Consider

$$\begin{aligned}
&\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right)\\
=&\frac{-(b-s)^{\overline{\alpha-1}}(\alpha-1)(a-\rho(s))^{\overline{\alpha-2}} + (a-s)^{\overline{\alpha-1}}(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}\\
=&\frac{(\alpha-1)\left((a-s)(a-\rho(s))^{\overline{\alpha-2}}(b-\rho(s))^{\overline{\alpha-2}} - (b-s)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}\right)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}\\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(-b+s+a-s)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}\\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(a-b)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}.
\end{aligned}$$

Since $b > a$, it follows from Lemma 2.1 that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$. The proof is complete. □

Lemma 3.2. *There exits a number $\gamma \in (0, 1)$, such that*

$$\min_{t \in \mathbb{N}_c^d} G(t, s) \geq \gamma \max_{t \in \mathbb{N}_a^b} G(t, s) = \gamma G(s-1, s), \tag{3.1}$$

where $c, d \in \mathbb{N}_{a+1}^{b-1}$, such that $c = a + \left\lceil \frac{b-a+1}{4} \right\rceil$ and $d = a + 3\left\lfloor \frac{b-a+1}{4} \right\rfloor$.

Proof. We make use Definition 2.4 and properties of Taylor monomials and Green's function from Lemma 2.1 and Theorem 2.2, respectively.

Consider, for $s \in \mathbb{N}_{a+2}^{b}$,

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s > t, \\ \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(t-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \leq t. \end{cases}$$

Now, for $s > t$ and $c \leq t \leq d$, $G_1(t,s)$ is an increasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_1(t,s) = G_1(c,s) = \frac{(c-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)}.$$

For $t > s$ and $c \leq t \leq d$, $G_2(t,s)$ is a decreasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_2(t,s) = G_2(d,s) = \frac{(d-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)} - \frac{(d-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Thus,

$$\begin{aligned} \min_{t\in\mathbb{N}_c^d} G(t,s) &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{c}, \\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s \in \mathbb{N}_{c+1}^{d-1}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_{d}^{b}, \end{cases} \\ &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{r}, \\ G_1(c,s), & \text{for } s \in \mathbb{N}_{r}^{b}, \end{cases} \end{aligned}$$

where $c < r < d$. Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{a+2}^{r}, \\ \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{r}^{b}. \end{cases}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma(s) \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.2}$$

where

$$\gamma(s) = \min\left\{\frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}\right\}.$$

For $s \in \mathbb{N}_{r}^{b}$, denote by

$$\gamma_1(s) = \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} \geq \frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}.$$

Similarly, for $s \in \mathbb{N}_{a+2}^{r}$, we take

$$\gamma_2(s) = \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}\right).$$

By Lemma 3.1, we see that $\frac{(d-s+1)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}$ is a decreasing function for $s \in \mathbb{N}_{a+2}^{r}$. Then

$$\begin{aligned}\gamma_2(s) &\geq \frac{1}{(s-a-1)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right) \\ &> \frac{1}{(d-a)^{\overline{\alpha-1}}}\left((d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}\right).\end{aligned}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.3}$$

where

$$\gamma = \min\left\{\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}\right\}. \tag{3.4}$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{a+2}^{b}$, implying that $\gamma > 0$. It would be suffice to prove that one of the terms $\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}$, $1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}$ is less than 1. It follows from Lemma 2.1 that

$$\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} < 1.$$

Therefore, we conclude that $\gamma \in (0,1)$. The proof is complete. □

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector of vector space $\mathbb{R}^{b-a+1}$. Denote by

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ is a Banach space equipped with the maximum norm, i.e.,

$$\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|.$$

Define the operator $T : \mathcal{B} \to \mathcal{B}$ by

$$(Tu)(t) = \sum_{s=a+2}^{b} G(t,s) f(u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.5}$$

Since T is defined on a discrete finite domain, it is trivially completely continuous. We also observe from (2.1) and (3.5), that u is a fixed point of T, if and only if u is a solution of (1.1).

Define the cone

$$K = \left\{u \in \mathcal{B} \mid u(t) \geq 0, \text{ for } t \in \mathbb{N}_a^b \text{ and } \min_{t\in\mathbb{N}_c^d} u(t) \geq \gamma\|u\|\right\}.$$

First, we show that $T : K \to K$. Let $u \in K$. Clearly, $(Tu)(t) \geq 0$, for $t \in \mathbb{N}_a^b$. Consider

$$\begin{aligned}\min_{t\in\mathbb{N}_c^d} (Tu)(t) &= \min_{t\in\mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)\\ &\geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d} (G(t,s))\, f(u(s)) \geq \sum_{s=a+2}^{b} \gamma \max_{t\in\mathbb{N}_a^b} (G(t,s))\, f(u(s))\\ &\geq \gamma \max_{t\in\mathbb{N}_a^b} \left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)\\ &= \gamma\|Tu\|.\end{aligned}$$

Thus, we have $T : K \to K$. Take

$$D = \sum_{s=a+2}^{b} G(s-1,s). \tag{3.6}$$

We define the following sets

$$\begin{aligned}K_{c'} &= \{u \in K \mid \|u\| < c'\},\\ K_{\alpha_2}(a',b') &= \{u \in K \mid a' \leq \alpha_2(u), \|u\| \leq b'\},\end{aligned}$$

where $\alpha_2 : K \to [0,+\infty)$ is a non-negative continuous concave functional. We state here the Leggett-Williams fixed-point theorem as follows. The proof of the same can be found in [33] and applications can be found in [3,8]. Also, we would like to refer here to a paper by Kwong [30], which talks about the geometrical view of the Leggett-Williams fixed point theorem.

Theorem 3.1. *Let $T : \bar{K_{c'}} \to \bar{K_{c'}}$ be completely continuous and α_2 be a non-negative continuous concave functional on K, such that $\alpha_2(x) \leq \|u\|$, for all $u \in \bar{K_{c'}}$. Suppose there exist $0 < d' < a' < b' \leq c'$, such that*

(a) *$\{u \in K_{\alpha_2}(a',b') : \alpha_2(u) > a'\} \neq \emptyset$ and $\alpha_2(Tu) > a'$, for $u \in K_{\alpha_2}(a',b')$;*
(b) *$\|Tu\| < d'$, for $\|u\| \leq d'$;*
(c) *$\alpha_2(Tu) > a'$, for $u \in K_{\alpha_2}(a',c')$ with $\|Tu\| > b'$.*

Then, T has at least three fixed points u_1, u_2, u_3 satisfying

$$\begin{aligned}\|u_1\| &< d', \quad a' < \alpha_2(u_2),\\ \|u_3\| &> d' \quad \textit{and} \quad \alpha_2(u_3) < a'.\end{aligned}$$

We introduce here the growth conditions on the non-linear function f, in line with [3].

Theorem 3.2. *Suppose there exist numbers a', b', d', where $0 < d' < a' < \gamma b' < b'$, such that f satisfies the following*

(a) *$f(u) > \frac{a'}{\gamma D}$, if $u \in [a',b']$;*
(b) *$f(u) < \frac{d'}{D}$, if $u \in [0,d']$;*

(c) *There exists* c' *such that* $c' > b'$ *and if* $u \in [0, c']$ *then* $f(u) < \frac{c'}{D}$.

Then, the boundary value problem (1.1) *has at least three positive solutions.*

Proof. Define a non-negative continuous concave functional $\alpha_2 : K \to [0, \infty)$ with $\alpha_2(u) \leq \|u\|$, for all $u \in K$, by

$$\alpha_2(u) = \min_{t \in \mathbb{N}_c^d} u(t).$$

Claim 1. If there exists a positive number r such that $u \in [0, r]$ implies $f(u) < \frac{r}{D}$, then $T : K_r \to K_r$. Suppose that $u \in K_r$. Then,

$$\begin{aligned}
\|Tu\| &= \max_{t \in \mathbb{N}_a^b} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\
&\leq \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} [G(t,s)] \, f(u(s)) \\
&= \sum_{s=a+2}^{b} G(s-1,s) f(u(s)) \\
&< \frac{r}{D} \sum_{s=a+2}^{b} G(s-1,s) = r.
\end{aligned}$$

Thus, $T : K_r \to K_r$. Hence, we have that if condition (c) holds, then there exists a number c' such that $c' > b'$ and $T : K_{c'} \to K_{c'}$. Note that with $r = d'$ and using condition (b), we get that $T : K_{d'} \to K_{d'}$.

Claim 2. $\{u \in K_{\alpha_2}(a', b') \mid \alpha_2(u) > a'\} \neq \emptyset$ and $\alpha_2(Tu) > a'$ for $u \in K_{\alpha_2}(a', b')$.

Since $u = \frac{a'+b'}{2} \in \{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\}$, it is non-empty. Let $u \in K_{\alpha_2}(a', b')$. By using condition (a), we have

$$\begin{aligned}
\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\
&\geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} [G(t,s)] \, f(u(s)) \geq \gamma \sum_{s=a+2}^{b} G(s-1,s) f(u(s)) \\
&> a'.
\end{aligned}$$

Thus, if $u \in K_{\alpha_2}(a', b')$, then $\alpha_2(Tu) > a'$.

Claim 3. If $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$, then $\alpha_2(Tu) > a'$. Suppose $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$. Then,

$$\begin{aligned}
\alpha_2(Tu) &= \min_{t \in \mathbb{N}_c^d} \left(\sum_{s=a+2}^{b} G(t,s) f(u(s)) \right) \\
&\geq \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} (G(t,s)) \, f(u(s)) \geq \gamma \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} (G(t,s)) \, f(u(s))
\end{aligned}$$

$$\begin{aligned}
&\geq \gamma \max_{t\in\mathbb{N}_c^d}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)\\
&= \gamma\|Tu\|\\
&> \gamma b' > a'.
\end{aligned}$$

Thus, $\alpha_2(Ax) > a'$. Hence, all the hypothesis of Theorem 3.1 are satisfied. Therefore, the boundary value problem (1.1) has at least three positive solutions. □

It has been observed that the flexibility of suitable choice of functionals over norms is the main advantage of Avery-type fixed-point theorems over Leggett–Williams fixed-point theorem [7, 30]. We define here the following subset of K for a positive number q:

$$K(\theta,q) = \{u \in K \mid \theta(u) < q\},$$

and the set $\partial K(\theta,q) = \{u \in K : \theta(u) = q\}$, where θ is a non-negative continuous functional on K.

The following is a twin fixed point theorem by Avery and Henderson [6].

Theorem 3.3. *Let K be a cone in a real Banach space $\mathcal{B}$. Let α_1 and γ_1 be increasing, non-negative continuous functionals on K. Let θ be a non-negative continuous functional on K with $\theta(0) = 0$ such that for some positive constants r and M,*

$$\alpha_1(u) \leq \theta(u) \leq \gamma_1(u) \quad \textit{and} \quad \|u\| \leq M\alpha_1(u),$$

for all $u \in \overline{K(\alpha_1,r)}$. Assume that there exist two positive numbers p and q with $p < q < r$, such that

$$\theta(ku) \leq k\theta(u), \quad \textit{for } 0 \leq k \leq 1 \textit{ and } u \in \partial K(\theta,q).$$

Suppose there exist a completely continuous operator $T : \overline{K(\alpha_1,r)} \to K$, satisfying

(a) *$\alpha_1(Tu) > r$, for all $u \in \partial K(\alpha_1,r)$;*
(b) *$\theta(Tu) < q$, for all $u \in \partial K(\theta,q)$;*
(c) *$K(\gamma_1,p) \neq \emptyset$ and $\gamma_1(Tu) > p$, for all $u \in \partial K(\gamma_1,p)$.*

Then, T has at least two fixed points u_1 and u_2 belonging to $\overline{K(\alpha_1,r)}$, such that

$$p < \gamma_1(u_1), \quad \textit{with } \theta(u_1) < q,$$

and

$$q < \theta(u_2), \quad \textit{with } \alpha_1(u_2) < r.$$

We introduce growth conditions on the non-linear function f here in line with [10]. Set $l = b - a + 1$.

Theorem 3.4. *Suppose that there exist positive constants p, q and r, such that $p < q < r$ and assume that function f satisfies the following conditions:*

(a) *$f(u) > \frac{r}{\gamma l G(s-1,s)}$, for all $u \in [r, \frac{r}{\gamma}]$;*
(b) *$f(u) < \frac{q}{l G(s-1,s)}$, for all $u \in [q, \frac{q}{\gamma}]$;*
(c) *$f(u) > \frac{p}{l G(s-1,s)}$, for all $u \in [\gamma p, p]$.*

Then, the operator T has at least two fixed points, u_1 and u_2, such that

$$p < \gamma_1(u_1), \quad \text{with } \theta(u_1) < q,$$

and

$$q < \theta(u_2), \quad \text{with } \alpha_1(u_2) < r.$$

Proof. We need to verify that the completely continuous operator T satisfies the hypothesis of Theorem 3.3. Denote by

$$\alpha_1(u) = \min_{t\in\mathbb{N}_c^d} u(t), \quad \theta(u) = \max_{t\in\mathbb{N}_c^d} u(t), \quad \gamma_1(u) = \|u\|.$$

For all $u \in K$, we have $\alpha_1(u) \leq \theta(u) \leq \gamma_1(u)$. Let $u \in K$. Then,

$$\alpha_1(u) = \min_{t\in\mathbb{N}_c^d} u(t) \geq \gamma \max_{t\in\mathbb{N}_a^b} u(t) = \gamma\gamma_1(u) = \gamma\|u\|.$$

Hence, for all $k \geq 0$ and $u \in K$, we have

$$\theta(ku) = \max_{t\in\mathbb{N}_c^d}(ku(t)) = k \max_{t\in\mathbb{N}_c^d} u(t) = k\theta(u).$$

Claim 1. If $u \in \partial K(\alpha_1, r)$, then $\alpha_1(Tu) > r$. Let $u \in \partial K(\alpha_1, r)$, i.e., $\min_{t\in\mathbb{N}_c^d} u(t) = r$. Then, $\alpha_1(u) = r \geq \gamma\|u\|$, implying that

$$r \leq \|u\| \leq \frac{r}{\gamma}, \quad \text{for } u \in \partial K(\alpha_1, r).$$

Using condition (a), we have

$$\begin{aligned}
\alpha_1(Tu) &= \min_{t\in\mathbb{N}_c^d}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right) \\
&\geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}(G(t,s))\, f(u(s)) \geq \gamma \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}(G(t,s))\, f(u(s)) \\
&> \gamma \frac{r}{\gamma l G(s-1,s)} \max_{t\in\mathbb{N}_a^b}(G(t,s))\, l \\
&= r.
\end{aligned}$$

Thus, condition (a) of Theorem 3.3 is satisfied.

Claim 2. If $u \in \partial K(\theta, q)$, then $\theta(Tu) < q$. Let $u \in \partial K(\theta, q)$, i.e., $\max_{t\in\mathbb{N}_c^d} u(t) = q$. We have

$$\theta(u) = q \geq \alpha_1(u) \geq \gamma\|u\| \quad \text{and} \quad \|u\| \geq \theta(u) = q,$$

implying that

$$q \leq \|u\| \leq \frac{q}{\gamma}, \quad \text{for } u \in \partial K(\theta, q).$$

Using condition (b), for $u \in \partial K(\theta, q)$, we have

$$\theta(Tu) = \max_{t\in\mathbb{N}_c^d}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)$$

$$
\begin{aligned}
&\leq \max_{t\in\mathbb{N}_a^b}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right) \leq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right) f(u(s))\\
&< \frac{q}{lG(s-1,s)} \max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right) l\\
&= q.
\end{aligned}
$$

Thus, condition (b) of Theorem 3.3 is satisfied. Now, since $K(\gamma_1, p) = \{u \in K \mid \|u\| < p\} \neq \emptyset$, we observe that $p \geq \gamma_1(u) \geq \alpha_1(u) \geq \gamma p$, for $u \in \partial K(\gamma_1, p)$. Using condition (c), we have

$$
\begin{aligned}
\gamma_1(Tu) &= \max_{t\in\mathbb{N}_a^b}\left(\sum_{s=a+2}^{b} G(t,s)f(u(s))\right)\\
&\geq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right) f(u(s))\\
&> \frac{p}{lG(s-1,s)} \max_{t\in\mathbb{N}_a^b}\left(G(t,s)\right) l\\
&= p.
\end{aligned}
$$

Thus, all the conditions of Theorem 3.3 are satisfied. Hence, T has at least two fixed points. The proof is complete. □

Conclusion

In the present article, we have established sufficient conditions for the existence of multiple positive solutions of the standard two-point non-linear nabla fractional boundary value problem with Dirichlet boundary conditions using fixed-point theorems such as Leggett–Williams and Avery–Henderson on a suitable constructed cone. To the best of our knowledge use of above conical shell fixed point theorem in nabla fractional calculus is unknown.

Acknowledgements. Authors N. S. Gopal acknowledges the review and editorial board for their suggestions and also the financial support received through the Senior Research Fellowship [09/1026(0028)/2019-EMR-I] from CSIR-HRDG New Delhi, Government of India.

References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics **141**, Cambridge University Press, Cambridge, 2001.

[2] K. Ahrendt, L. De Wolf, L. Mazurowski, K. Mitchell, T. Rolling and D. Veconi, *Initial and boundary value problems for the Caputo fractional self-adjoint difference equations*, Enlightenment in Pure Appl. Math. **2**(1) (2016).

[3] D. Anderson, R. Avery and A. C. Peterson, *Three positive solutions to a discrete focal boundary value problem. Positive solutions of non-linear problems*, J. Comput. Appl. Math. **88**(1) (1998), 103–118. http://dx.doi.org/10.1016/S0377-0427(97)00201-X

[4] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ. Special Edition I (2009), 12 pages. http://dx.doi.org/10.14232/ejqtde.2009.4.3
[5] R. I. Avery, *A generalization of the Leggett-Williams fixed point theorem*, Math. Sci. Res. Hot-Line **3**(7) (1999), 9–14.
[6] R. I. Avery, C. J. Chyan and J. Henderson, *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations*, Comput. Math. Appl. **42**(3–5) (2001), 695–704. http://dx.doi.org/10.1016/S0898-1221(01)00188-2
[7] R. I. Avery and A. C. Peterson, *Three positive fixed points of non-linear operators on ordered Banach spaces*, Comput. Math. Appl. **42**(3–5) (2001), 313–322. http://dx.doi.org/10.1016/S0898-1221(01)00156-0
[8] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311**(2) (2005) 495–505. https://doi.org/10.1016/j.jmaa.2005.02.052
[9] A. Brackins, *Boundary value problems of nabla fractional difference equations*, Thesis (Ph.D.), The University of Nebraska - Lincoln, 2014, 92 pages.
[10] A. Cabada and N. Dimitrov, *Existence of solutions of nth-order non-linear difference equations with general boundary conditions*, Acta Math. Sci. Ser. B (Engl. Ed.) **40**(1) (2020), 226–236. http://dx.doi.org/10.1007/s10473-020-0115-y
[11] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Application*, Birkhauser, Boston, MA 2001.
[12] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Springer, Cambridge, 2015.
[13] N. S. Gopal and J. M. Jonnalagadda, *Existence and uniqueness of solutions to a nabla fractional difference equation with dual nonlocal boundary conditions*, Foundations **2** (2022), 151–166. http://dx.doi.org/10.3390/foundations2010009
[14] H. L. Gray and N. Fan Zhang, *On a new definition of the fractional difference*, Math. Comp. **50**(182) (1988), 513–529.
[15] D. J. Guo and V. Lakshmikantham, *Non-linear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering **5**, Academic Press, Inc., Boston, MA, 1988.
[16] J. St. Goar, *A Caputo boundary value problem in Nabla fractional calculus*, Thesis (Ph.D.), University of Nebraska - Lincoln, 2016, 112 pages.
[17] Y. Gholami and K. Ghanbari, *Coupled systems of fractional* ∇*-difference boundary value problems*, Differ. Equ. Appl. **8**(4) (2016), 459–470.
[18] Y. He, M. Suna and C. Hou, *Multiple positive solutions of non-linear boundary value problem for finite fractional difference*, Abstr. Appl. Anal. (2014), Article ID 147975, 12 pages. http://dx.doi.org/10.1155/2014/147975
[19] A. Ikram, *Lyapunov inequalities for nabla Caputo boundary value problems.* J. Difference Equ. Appl. **25**(6) (2019), 757–775. https://doi.org/10.48550/arXiv.1907.08847
[20] J. M. Jonnalagadda and N. S. Gopal, *Green's function for a discrete fractional boundary value problem*, Differ. Equ. Appl. **14**(2) (2022), 163–178. http://dx.doi.org/10.7153/dea-2022-14-10
[21] J. M. Jonnalagadda, *An ordering on Green's function and a Lyapunov-type inequality for a family of nabla fractional boundary value problems*, Fract. Differ. Calc. **9**(1) (2019), 109–124.http://dx.doi.org/10.7153/fdc-2019-09-08
[22] J. M. Jonnalagadda, *On-Hilfer type nabla difference equations*, Int. J. Differ. Equ. **15**(1) (2020), 91–107.
[23] J. M. Jonnalagadda, *Lyapunov-type inequalities for discrete Riemann-Liouville fractional boundary value problems*, Int. J. Difference Equ. **13**(2) (2018), 85–103.
[24] J. M. Jonnalagadda, *On a nabla fractional boundary value problem with general boundary conditions*, AIMS Math. **5**(1) (2020), 204–215. https://doi.org/10.3934/math.2020012

[25] J. M. Jonnalagadda, *On two-point Riemann-Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13**(2) (2018), 141–166.
[26] M. S. Keener and C. C. Travis, *Positive cones and focal points for a class of nth-order differential equations*, Trans. Amer. Math. Soc. **237** (1978), 331–351.
[27] K. Deimling, *Non-linear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[28] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989.
[29] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Translated from the Russian by Richard E. Flaherty, Ltd. Groningen, 1964, 381 pages.
[30] K. M. Kwong, *On Krasnoselskii's cone fixed point theorem*, Fixed Point Theory Appl. (2008), Article ID 164537, 18 pages. https://doi.org/10.1155/2008/164537
[31] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies **204**, Elsevier Science B.V., Amsterdam, 2006.
[32] M. G.Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Trans. (1950), 128 pages.
[33] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of non-linear operators on ordered Banach spaces*, Indiana Univ. Math. J. **28**(4) (1979), 673–688.
[34] K. S. Miller and B. Ross, *Fractional Difference Calculus. Univalent Functions, Fractional Calculus, and their Applications*, Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1989.
[35] K. S. Miller and B. Ross, *Univalent functions, fractional calculus, and their applications*, Papers from the Symposium held at Nihon University, New York, 1989, 404 pages.
[36] I. Podlubny, *Fractional Differential Equations. An introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Mathematics in Science and Engineering **198**, Academic Press, Inc., San Diego, CA, 1999.
[37] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Translated from the 1987 Russian, Gordon and Breach Science Publishers, Yverdon, 1993.

[1]Presidency College,
Kempapura, Bangalore - 560024, Karnataka, India.
Email address: nsgopal94@gmail.com

[2]Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078, Telangana, India.

[*]Corresponding Author
Email address: j.jaganmohan@hotmail.com