Kragujevac Journal of Mathematics
Volume 50(1) (2026), Pages 39–69.

A NOVEL SHIFTED JACOBI OPERATIONAL MATRIX METHOD FOR LINEAR MULTI-TERMS DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL VARIABLE-ORDER WITH PERIODIC AND ANTI-PERIODIC CONDITIONS

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Abstract. This paper investigates the generalized linear multi-terms delay fractional differential equation of variable order with periodic and anti-periodic conditions. In this work, a novel shifted Jacobi operational matrix technique is applied to solve a class of these equations, so that the original problem becomes a system of algebraic equations that can be solved by numerical methods. The proposed technique is successfully applied to the aforementioned problem. Sufficient and complete numerical tests are presented to demonstrate the accuracy, generality, efficiency of presented technique and the flexibility of this scheme. The numerical results of this method are compared with other existing methods such as fractional backward differential formulas ($FBDF$). Comparing the outcomes of these schemes, as well as comparing the current technique ($NSJOM$) with the exact solution, demonstrates the efficiency and validity of this method. It should be noted that the implementation of current method is considered very easy and general for many numerical techniques. Furthermore, the error and its bound are estimated.

1. Introduction

In the last three decades, analysis and applications of fractional calculus have been the fastest growing active area of research. Currently, it has become an important tool because of its vast applications in different scientific fields for example, physics, chemistry, blood circulation phenomena, electrodynamics, biophysics, capacitor theory,

Key words and phrases. Periodic and anti-periodic conditions, shifted Jacobi operational matrix technique, Caputo differential operator, multi-terms delay differential equations, fractional variable-order.

2020 *Mathematics Subject Classification.* Primary: 65M99. Secondary: 34A08, 46E22, 65F25.

DOI 10.46793/KgJMat2601.039K

Received: February 21, 2021.

Accepted: May 16, 2023.

Complex environment, polymer rheology, experimental data fitting, dynamic systems, etc. (see [4–7, 11] and references therein). The increasing development of efficient and suitable methods with high accuracy to solve FDEs has caused the interest of many researchers to increase in this field. There are many important and popular methods for estimating of numerical solution of FDEs which can be implied to both linear and nonlinear FDEs, namely fractional linear multi-steps methods and convolution quadrature are presented by Lubich [12]. Galeone and Garrappa presented Fractional Adams-Molton methods for FDEs [13]. Trapezoidal methods to solve FDEs is proposed via Garrappa in [15]. The numerical solution to solve linear multi-term FDEs: systems of equations have presented by Edwards et al. [16]. Ford and Diethelm have suggested the multi-order FDE and their numerical solution in [17] and the numerical analysis for distributed-order DEs is given by these authors [18] and etc.

Incorporating the delay into FDEs creates new perspectives, especially in the field of bioengineering[10],because the realization of dynamics occurring in biological tissues is improved in bioengineering by fractional derivatives [8, 10].

In mathematical sciences, the DDEs are a kind of DEs in that the derivative of an unknown function at a definite time is presented in terms of the values of the function at prior times. The DDEs are also called time-delay systems, systems of deed-time or systems of aftereffect, differential-difference type equations, hereditary systems, deviating arguments equations [21].

Fractional DDEs differ from the ordinary type in which the derivative at any time depends on the solution (and when the equations are neutral then related to the derivative) at previous times. Many real-world happenings can be modeled as the $FDDE$s [11]. The $FDDE$s have many usages in different scientific areas by modeling different problems like electro dynamics, economy, biology, finance, control, physics, chemistry and etc. [21–27].

In the past years, numerical solution of the $FDDE$s analyzed and approximated by Margado et al. in [28]. Cermak et al. in [29] examined the stability areas of systems of $FDDE$s. Lazarovic and Spansic in [30] analyzed the stability for systems of $FDDE$s by means of Gr$\ddot{u}$nwalds approach. A New Predictor-Corrector method ($NPCM$) and new iteration technique have proposed in [31, 32], to numerically solve FDEs. A predictor-corrector method for solving nonlinear $FDDE$s in [14] have peresented via Bhalekar and Daftardar-Gejji. In [9], the algorithm of Adams-Bashforth-moulton which was peresented in [6, 20, 33], is proposed for solving the $FDDE$s. A new technique to solve nonlinear $FDDE$s have presented by Varsha et al. [10]. The Reproducing kernel Hilbert Space method to solve nonlinear $FDDE$s have employed via Ghasemi et al. [21]. In have [8] authors provided a new numerical method for solving FDDEs and Khodabandehlo et al. in [1–3] have proposed a $NSJOM$ technique for nonlinear variable-order $FDDE$s.

Furthermore, the spectral techniques that depend on an orthogonal polynomials set, are applied to solve the FDEs. The classical Jacobi polynomials are one of the most famous, which are as follows:

$$P_n^{(\alpha,\beta)}(t), \quad \beta > -1, \alpha > -1,\ n \geq 0.$$

These polynomials have been used widely in mathematical analysis and practical applications owning to they have the benefits of getting the numeric solutions in parameters β and α. Then, the systematic study of Jacobi polynomials with general indexes α and β will be useful and obviously, this case, in addition to extending the time interval $t \in [0, I]$, can be considered as one of the goals and novelties of this version [19]. Moreover, in recent years interest of researchers has increased in this area (area of variable FDEs) [34–38].

In this paper, generalize the orthogonal polynomials in the base of solution is the our goal. In fact, we present a $NSJOM$ method for the fractional derivatives to solve a class of linear multi-terms variable $FDDE$s with periodic condition which as follow:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z(t) \ + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = z_T, \end{aligned} \tag{1.1}$$

where $z_T = z(T)$. Also, the linear multi-terms variable $FDDE$s with anti-periodic condition is:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z(t) \ + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = -z_T, \end{aligned} \tag{1.2}$$

where $\beta_s \in \mathbb{R}$, $s = 1, 2, \ldots, n+1$, $\beta_{n+1} \neq 0$, $0 < T$ and D^{ζ_s}, $s = 1, 2, \ldots, n$, are the Caputo's derivative of variable-order fractional.

Note 1. If $\zeta_s(t)$, $s = 1, 2, \ldots, n$, are constants, then (1.1) and (1.2) will be as follow:

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s} z(t) + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = z_T, \end{aligned}$$

and

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s} z(t) + \beta_{n+1} z(t-\tau) = f(t), \quad 0 \leq t \leq T, \\ &z(t) = k(t), \quad t \in [-\tau, 0], \\ &z(0) = -z_T. \end{aligned}$$

Also note that: we can use many polynomials such as Gegenbauer, Legendre, Fibonacci, all Chebyshev, Lucas, Vieta-Lucas polynomials, and etc. in our novel suggestion scheme.

The numerical outcomes gained for the mentioned equation in this paper reveal that the current technique has high efficiency and accuracy. By comparing numerical results getted via this technique with other available methods, and focusing on them, we find out that the suggested method capable of solving the variable-order $FDDE$, playing role of a powerful effective and practical numerical technique.

2. Fundamentals and Preliminaries

In this section, some of the mos basic fractional calculus theory properties will be mentioned. Then, some important features of Jacobi polynomials, that are relevant for the development of the proposed technique, will be presented [39, 42, 43].

Definition 2.1. The left and right-sided Caputo fractional derivatives of order ζ, $q-1<\zeta\leq q$, are determined as

$$D_{-}^{\zeta}z(t)=\frac{(-1)^{q}}{\Gamma(q-\zeta)}\int_{t}^{T}\frac{z'(s)}{(s-t)^{\zeta-q+1}}ds,$$

$$D_{+}^{\zeta}z(t)=\frac{1}{\Gamma(q-\zeta)}\int_{0}^{t}\frac{z'(s)}{(t-s)^{\zeta-q+1}}ds.$$

that

$$D_{+}^{\zeta}t^{m}=\begin{cases}0, & \text{for } m\in\mathbb{N}_0 \text{ and } m<\lceil\zeta\rceil,\\ \dfrac{\Gamma(m+1)}{\Gamma(m-\zeta+1)}t^{m-\zeta}, & \text{for } m\in\mathbb{N}_0 \text{ and } m>\lceil\zeta\rceil.\end{cases}$$

and

$$D_{-}^{\zeta}(T-t)^{m}=\begin{cases}0, & \text{for } m\in\mathbb{N}_0 \text{ and } m<\lceil\zeta\rceil,\\ \dfrac{(-1)^{m}\Gamma(m+1)}{\Gamma(m-\zeta+1)}(T-t)^{m-\eta}, & \text{for } m\in\mathbb{N}_0 \text{ and } m>\lceil\zeta\rceil.\end{cases}$$

where $\lceil\cdot\rceil$ is the ceiling function and $N_0=\{0,1,2,\dots\}$. And for constants δ and γ, we will have $D_{\pm}^{\zeta}(\delta\psi(t)\ +\gamma\eta(t))=\delta D_{\pm}^{\zeta}(\psi(t))+\gamma D_{\pm}^{\zeta}(\eta(t))$.

Definition 2.2. The Caputo derivative with fractional variable-order $\zeta(t)$ for $z(t)\in C^{m}[0,T]$ is as follows [35, 40]:

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(s)}{(t-s)^{\zeta(t)}}ds+\frac{z(0^+)-z(0^-)}{\Gamma(1-\zeta(t))}t^{-\zeta(t)}. \tag{2.1}$$

At the starting point and for $0<\zeta(t)<1$, we have:

$$D^{\zeta(t)}z(t)=\frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(s)}{(t-s)^{\zeta(t)}}ds.$$

Also, for constants a and b we have $D^{\zeta(t)}(a\,z_1(t)+b\,z_2(t))=a\,D^{\zeta(t)}z_1(t)+b\,D^{\zeta(t)}z_2(t)$. Using (2.1), then: $D^{\zeta(t)}K=0$, K is a constant.

On the other hand

$$D^{\zeta(t)}t^m = \begin{cases} 0, & \text{for } m=0, \\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\zeta(t))}t^{m-\zeta(t)}, & \text{for } m=1,2,\dots \end{cases} \tag{2.2}$$

2.1. **Shifted Jacobi polynomials and their properties.** Suppose $P_n^{(\alpha,\beta)}(s)$, $\beta > -1$, $\alpha > -1$ as the n-th degree Jacobi orthogonal polynomial in $-1 \le s \le 1$.

As any classical orthogonal polynomials, $P_n^{(\alpha,\beta)}(s)$ form an orthogonal system with respect to weight function $\omega^{(\alpha,\beta)}(s) = (1-s)^\alpha(1+s)^\beta$, namely [39]:

$$\int_{-1}^{1} P_\ell^{(\alpha,\beta)}(s)P_k^{\alpha,\beta)}(s)\omega^{(\alpha,\beta)}ds = h_k^{(\alpha,\beta)}\delta_{\ell,k},$$

where

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\beta+\alpha+1)},$$

$\delta_{\ell,k}$ is the Kronecker function and

$$P_\ell^{(\alpha,\beta)}(s) = \sum_{j=0}^{\ell} \frac{\Gamma(\alpha+\ell+1)\Gamma(\alpha+\ell+1+\beta+j)}{\Gamma(\alpha+\beta+\ell+1)\Gamma(\alpha+1+j)\Gamma(j+1)\Gamma(\ell-j+1)}\left(\frac{s-1}{2}\right)^j, \tag{2.3}$$

is the analytical form of the ℓ-th order Jacobi polynomial [19]. The polynomials given in (2.3) can be obtained as follow:

$$y_{1,\ell}^{\alpha,\beta}P_\ell^{(\alpha,\beta)}(s) \;= y_{2,\ell}^{\alpha,\beta}P_{\ell-1}^{(\alpha,\beta)}(s) - y_{3,\ell}^{\alpha,\beta}P_{\ell-2}^{(\alpha,\beta)}(s), \quad \ell = 2,3,\dots,$$

where

$$\begin{aligned} y_{1,\ell}^{\alpha,\beta} &= 2l(\alpha+\ell+\beta)(\alpha+2l-2+\beta), \\ y_{2,\ell}^{\alpha,\beta} &= (\alpha+2l-1+\beta)(\alpha^2-\beta^2+(\alpha+2l+\beta)(\alpha+2l+\beta-2)s), \\ y_{3,\ell}^{\alpha,\beta} &= 2(\alpha+\ell-1)(\beta+\ell-1)(\alpha+2l+\beta). \end{aligned}$$

That start values as follow

$$P_0^{(\alpha,\beta)}(s) = 1 \quad \text{and} \quad P_1^{(\alpha,\beta)}(s) = \frac{1}{2}[(\alpha+\beta+2)s + (\alpha-\beta)].$$

In order to use the polynomial of (2.3) on the interval $0 \le t \le T$, we need to change the variable $s = \left(\frac{2t}{T}-1\right)$. Therefore, the shifted Jacobi orthogonal polynomials $P_j^{(\alpha,\beta)}\left(\frac{2t}{T}-1\right)$ which marked by $P_{T,j}^{(\alpha,\beta)}(t)$ will be constructed . Then $P_{T,j}^{(\alpha,\beta)}(t)$ form an orthogonal system with $\omega_T^{(\alpha,\beta)}(t) = t^\beta(T-t)^\alpha$ as the weight function for $0 \le t \le T$ with the following orthogonal feature:

$$\int_0^T P_{T,\ell}^{(\alpha,\beta)}(t)P_{T,k}^{(\alpha,\beta)}(t)\omega_T^{(\alpha,\beta)}dt = h_{T,k}^{(\alpha,\beta)}\delta_{\ell,k},$$

where

$$h_{T,k}^{(\alpha,\beta)} = \left(\frac{T}{2}\right)^{\alpha+\beta+1} h_k^{(\alpha,\beta)}.$$

Also,

$$P_{T,\ell}^{(\alpha,\beta)}(t)=\sum_{j=0}^{\ell}(-1)^{\ell+j}\frac{\Gamma(\beta+1+\ell)\Gamma(\alpha+j+\ell+1+\beta)}{\Gamma(\alpha+\ell+1+\beta)\Gamma(\beta+1+j)\Gamma(j+1)\Gamma(\ell-j+1)T^j}t^j$$
$$=\sum_{j=0}^{\ell}\frac{\Gamma(\ell+1+\alpha)\Gamma(\alpha+j+\ell+\beta+1)}{\Gamma(\alpha+\ell+1+\beta)\Gamma(\alpha+1+j)\Gamma(j+1)\Gamma(\ell-j+1)T^j}(T-t)^j$$

is the analytical form of the ℓ-th order Shifted Jacobi polynomial [19] and we have

$$P_{T,\ell}^{(\alpha,\beta)}(0)=(-1)^{\ell}\frac{\Gamma(\beta+\ell+1)}{\Gamma(\beta+1)\Gamma(j+1)},$$
$$P_{T,\ell}^{(\alpha,\beta)}(T)=\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\Gamma(j+1)},$$

in the endpoint values.

Note 2. The Jacobi's shifted orthogonal polynomials constitute infinite number of orthogonal polynomials such as the shifted Chebyshev polynomials of the first, second, third and fourth kinds $T_{T,\ell}(t)$, $U_{T,\ell}(t)$, $V_{T,\ell}(t)$ and $W_{T,\ell}(t)$, respectively; the shifted Gegenbauer polynomials $G_{T,\ell}^{(\alpha,\beta)}(t)$ and the shifted Legendre polynomials $\ell_{T,\ell}(t)$. These polynomials, which are all orthogonal, are related to $P_{T,\ell}^{(\alpha,\beta)}(t)$ as follow:

$$\ell_{T,\ell}(t)=P_{T,\ell}^{(0,0)}(t),$$
$$G_{T,\ell}^{(\alpha,\beta)}(t)=\frac{\Gamma(\ell+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}+\ell\right)}P_{T,\ell}^{\left(\alpha-\frac{1}{2},\beta-\frac{1}{2}\right)}(t),$$
$$T_{T,\ell}(t)=\frac{\Gamma(\ell+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\ell\right)}P_{T,\ell}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t),$$
$$U_{T,\ell}(t)=\frac{\Gamma(\ell+2)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}+\ell\right)}P_{T,\ell}^{\left(\frac{1}{2},\frac{1}{2}\right)}(t),$$
$$V_{T,\ell}(t)=\frac{2^{2l}(\Gamma(\ell+1))^2}{\Gamma(2l+1)}P_{T,\ell}^{\left(-\frac{1}{2},\frac{1}{2}\right)}(t),$$
$$W_{T,\ell}(t)=\frac{2^{2l}(\Gamma(\ell+1))^2}{\Gamma(2l+1)}P_{T,\ell}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(t).$$

3. Function Approximation by Shifted Jacobi Polynomials

Consider the function $z(t)$ to be square integrable with respect to $\omega_T^{(\alpha,\beta)}(t)$ in $[0,T]$, then, we have (see [19,39]):

$$z(t)=\sum_{j=0}^{\infty}a_jP_{T,j}^{(\alpha,\beta)}(t), \tag{3.1}$$

that the coefficients of the series (a_j) are gained by

$$a_j = \frac{1}{h_{T,k}^{(\alpha,\beta)}} \int_0^T \omega_T^{(\alpha,\beta)} P_{T,j}^{(\alpha,\beta)}(t) z(t) dt, \quad j = 0, 1, \dots$$

So, we will obtain the approximate solution by finite number of terms from the series in (3.1), then

$$z(t) \simeq z_M(t) = \sum_{j=0}^{M} a_j P_{T,j}^{(\alpha,\beta)}(t) = A^T \Phi_{T,M}(t), \tag{3.2}$$

where $A = [a_0, a_1, \dots, a_M]^T$ and $\Phi_{T,M}(t) = [P_{T,0}^{(\alpha,\beta)}(t), P_{T,1}^{(\alpha,\beta)}(t), \dots, P_{T,M}^{(\alpha,\beta)}(t)]^T$.

We consider that

$$S(t) = [1, t, t^2, t^3, \dots, t^M]^T.$$

By (3.2), the vector $\Phi_{T,M}(t)$ can be shown as

$$\Phi_{T,M}(t) = R_{(\alpha,\beta)} S(t), \tag{3.3}$$

that $R_{(\alpha,\beta)}$ is a square matrix of order $(M+1) \times (M+1)$, as follows

$$r_{\ell+1,k+1} = \begin{cases} (-1)^{\ell-k} \dfrac{(\alpha+\ell)!(\alpha+\beta+k+\ell)!}{(\alpha+\beta+\ell)!(\alpha+k)!(k!)(\ell-k)!T^k}, & \ell \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

for $0 \leq \ell,\ k \leq M$.

Let $M = 4$, $\alpha = \beta = 0$, then

$$R_{(0,0)} = \frac{1}{T^i} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 \\ 1 & -20 & 90 & -140 & 70 \end{bmatrix}.$$

Hence, using (3.3), we get

$$S(t) = R_{(\alpha,\beta)}^{-1} \Phi_{T,M}(t). \tag{3.4}$$

Note 3. We can caculate this matrix $R_{(\alpha,\beta)}$ for other orthogonal polynomials as well. For instance, let $M = 4$, $\beta = \frac{-1}{2}$, $\alpha = \frac{1}{2}$, then the orthogonal polynomials will be of the fourth kind shifted Chebyshev type, hence $R_{(\alpha,\beta)}$ of order 4×4 for these polynomials as follows

$$R_{(\frac{1}{2},\frac{-1}{2})} = \frac{1}{T^i} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 1 & -12 & 16 & 0 & 0 \\ -1 & 24 & -80 & 64 & 0 \\ 1 & -40 & 240 & -448 & 256 \end{bmatrix}.$$

4. Novel Shifted Jacobi Polynomials Operational Matrix($NSJOM$)

Operational matrix, which are applied in different areas of numerical analysis and to solve problems of different types and topics are of especial importance such as integral equations, DEs, integro-DEs, ordinary and partial FDEs [36, 39–41, 44–52]. In this part, we investigate the ($SJOM$) of fractional variable-order to support the numerical solution of (1.1), (1.2). Therefore, we convert the problem into the system of algebraic of equations which is solved numerically in collocation points.

At first, we deduce $D^{\zeta_\ell(t)}\Phi_{T,M}(t)$, $\ell = 1, 2, \dots, n$, as follow.

According to the previous content, we have: $\Phi_{T,M}(t) = R_{(\alpha,\beta)}S(t)$, thus

$$D^{\zeta_\ell(t)}\Phi_{T,M}(t) = D^{\zeta_\ell(t)}(R_{(\alpha,\beta)}S(t)) = R_{(\alpha,\beta)}D^{\zeta_\ell(t)}[1, t, \dots, t^M]^T, \quad \ell = 1, 2, \dots, n. \tag{4.1}$$

Combining (2.2) and (4.1), gives:

$$\begin{aligned}
D^{\zeta_\ell(t)}\Phi_{T,M}(t) &= R_{(\alpha,\beta)}D^{\zeta_\ell(t)}(S(t)) \\
&= R_{(\alpha,\beta)}\left[0, \frac{\Gamma(2)t^{(1-\zeta_\ell(t))}}{\Gamma(2-\zeta_\ell(t))}, \dots, \frac{\Gamma(M+1)t^{(M-\zeta_\ell(t))}}{\Gamma(M+1-\zeta_\ell(t))}\right]^T \\
&= R_{(\alpha,\beta)}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_\ell(t)}}{\Gamma(2-\zeta_\ell(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_\ell(t)}}{\Gamma(3-\zeta_\ell(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(M)t^{-\zeta_\ell(t)}}{\Gamma(M+1-\zeta_\ell(t))} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^M \end{bmatrix} \\
&= R_{(\alpha,\beta)}G_\ell(t)S(t), \quad \ell = 1, 2, \dots, n,
\end{aligned}$$

where

$$G_\ell(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\zeta_\ell(t)}}{\Gamma(2-\zeta_\ell(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\zeta_\ell(t)}}{\Gamma(3-\zeta_\ell(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(M)t^{-\zeta_\ell(t)}}{\Gamma(M+1-\zeta_\ell(t))} \end{bmatrix}, \quad \ell = 1, 2, \dots, n.$$

Using (3.4), we have

$$D^{\zeta_\ell(t)}\Phi_{T,M}(t) = R_{(\alpha,\beta)}G_\ell(t)R^{-1}_{(\alpha,\beta)}\Phi_{T,M}(t), \quad \ell = 1, 2, \dots, n.$$

The operational matrix of $D^{\zeta_\ell(t)}\Phi_{T,M}(t)$, $\ell = 1, 2, \dots, n$, is $R_{(\alpha,\beta)}G_\ell(t)R^{-1}_{(\alpha,\beta)}$.

Here, we estimate the variable-order fractional of the calculated function that obtained in (3.2) as follows

$$\begin{aligned} D^{\zeta_\ell(t)} z(t) \simeq D^{\zeta_\ell(t)}(A^T \Phi_{T,M}(t)) =& A^T D^{\zeta_\ell(t)} \Phi_{T,M}(t) \\ =& A^T R_{(\alpha,\beta)} G_\ell(t) R^{-1}_{(\alpha,\beta)} \Phi_{T,M}(t), \quad \ell = 1, 2, \ldots, n. \end{aligned} \tag{4.2}$$

By using (4.2), hence (1.1) turned into

$$\sum_{s=1}^{n} \beta_s (A^T R_{(\alpha,\beta)} G_s(t) R^{-1}_{(\alpha,\beta)} \Phi_{T,M}(t)) + \beta_{n+1} A^T \Phi_{T,M}(t - \tau) = f(t), \quad t \in [0, T], \tag{4.3}$$

with periodic condition

$$A^T \Phi_{T,M}(0) = A^T \Phi_{T,M}(T)$$

or anti-periodic condition as

$$A^T \Phi_{T,M}(0) = -A^T \Phi_{T,M}(T).$$

Finally, we use t_k, $k = 0, 1, 2, \ldots, m$, where they are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$. Therefore, (4.3) converted into the following form

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s (A^T R_{(\alpha,\beta)} G_s(t_\ell) R^{-1}_{(\alpha,\beta)} \Phi_{T,M}(t_\ell)) + \alpha_{n+1} A^T \Phi_{T,M}(t_\ell - \tau) = f(t_\ell), \\ &\ell = 0, 1, 2, \ldots, m. \end{aligned} \tag{4.4}$$

So, we can solve the system in (4.4) with the conditions mentioned numerically for determining the vector A. Therefore, the numerical solution that presented in (3.2) can be obtained.

5. Error Analysis

In this part, for estimating an upper bound for the absolute error, the Lagrange interpolation polynomials is used. Namely, by using the current technique ($NSJOM$) with error approximation and the residual correction method [53,54], an effective error estimation will be gained for the variable-order FDEs.

5.1. **Error bound.** Now, let $z(t)$ on $[0, T]$ be the smooth function and suppose that $z_M(t) \in \prod_M^{\alpha,\beta}$ is the best approximation for it. Our aim is to obtain an analytical form of norm of error for $z_M(t)$ by developing it into Jacobi polynomials. assume

$$\prod_M^{\alpha,\beta} = \text{Span}\left\{ P^{(\alpha,\beta)}_{T,i}(t), i = 0, 1, 2, \ldots, M \right\}.$$

According to concept and definition of the best approximation, we can write

$$\text{for all } v_M(t) \in \prod_M^{\alpha,\beta} \|z(t) - z_M(t)\|_\infty \leq \|z(t) - v_M(t)\|_\infty.$$

Let $v_M(t)$ be the interpolating polynomials at node points t_i, $i = 0, 1, \ldots, m$ (where t_i are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$). It is clear that $v_M(t)$ satisfies in the above inequality. Then by the Lagrange interpolation polynomials formula and its error formula, have

$$z(t) - v_M(t) = \frac{z^{(M+1)}(\xi)}{(M+1)!}\prod_{j=0}^{M}(t - t_j),$$

that $0 < \xi < T$, and

$$\|z(t) - v_M(t)\|_\infty \leq \max_{0\leq t\leq T}|z^{(M+1)}(\xi)|\frac{\|\prod_{j=0}^{M}(t-t_j)\|_\infty}{(M+1)!}.$$

Note that $z(t)$ on $[0, T]$ is smooth, therefore, there is a constant C_1, as

$$\max_{0\leq t\leq T}|z^{(M+1)}(\xi)| \leq C_1. \tag{5.1}$$

We want to minimize the factor $\|\prod_{j=0}^{M}(t - t_j)\|_\infty$ as follows

One-to-one mapping $t = \frac{T}{2}(w + 1)$ between the interval $[-1, 1]$ and $[0, T]$ is used to deduce that [39, 55]

$$\begin{aligned}\min_{0\leq t_i\leq T}\max_{0\leq t\leq T}|\prod_{i=0}^{M}(t-t_i)| &= \min_{-1\leq w_i\leq 1}\max_{-1\leq w\leq 1}\left|\prod_{i=0}^{M}\frac{T}{2}(w-w_i)\right| \\ &= \left(\frac{T}{2}\right)^{M+1}\min_{-1\leq w_i\leq 1}\max_{-1\leq w\leq 1}\left|\prod_{i=0}^{M}(w-w_i)\right| \\ &= \left(\frac{T}{2}\right)^{M+1}\min_{-1\leq w_i\leq 1}\max_{-1\leq w\leq 1}\left|\frac{P^{(\alpha,\beta)}_{M+1}(w)}{\mu^{(\alpha,\beta)}_M}\right|,\end{aligned} \tag{5.2}$$

where $\mu^{(\alpha,\beta)}_M = \frac{\Gamma(2M+\alpha+\beta+1)}{2^M M!\Gamma(M+\alpha+\beta+1)}$ is the last factor of $P^{(\alpha,\beta)}_{M+1}(w)$ and w_j are the roots of $P^{(\alpha,\beta)}_{M+1}(w)$. It is clear that

$$\max_{-1\leq w\leq 1}|P^{(\alpha,\beta)}_{M+1}(w)| = P^{(\alpha,\beta)}_{M+1}(1) = \frac{\Gamma(\beta+M+2)}{\Gamma(\beta+1)(M+1)!}.$$

Using (5.1) and (5.2), gives the following result

$$\|z(t) - z_M(t)\|_\infty \leq C_1\frac{\left(\frac{T}{2}\right)^{M+1}\Gamma(\beta+M+2)}{\mu^{(\alpha,\beta)}_M((M+1)!)^2\Gamma(\beta+1)}.$$

Therefore, an upper bound for absolute error between the exact and approximate solutions was stimated.

5.2. **Error function estimation.** In this subsection, we have introduced the error approximation based on the error function of residual of the proposed scheme and the approximate solution (3.2) is refined by the residual correction technique. The error approxmation of residual was used to ontain the error of some methods for different equations [41, 54, 56, 57].

At first, we mark $e_M(t) = z_M(t) - z(t)$ be the error function for the $NSJOM$ approximation $z_M(t)$ to $z(t)$, that $z(t)$ is the truaccurate solution of (1.1) or (1.2).

Therefore, $z_M(t)$ satisfies the following relation

$$\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} z_M(t) + \beta_{n+1} z_M(t-\tau) = f(t) + R_M(t), \quad 0 \le t \le T, \tag{5.3}$$

with periodic condition as

$$z_M(0) = z_M(T)$$

or anti-periodic condition as

$$z_M(0) = -z_M(T),$$

where $R_M(t)$ is the residual function of (1.1) or (1.2), which is approximated by replacing the $z_M(t)$ with $z(t)$ in (1.1) or (1.2). By subtract (1.1) or (1.2) from (5.3), the error problem is constructed in the form of

$$\begin{aligned} &\sum_{s=1}^{n} \beta_s D^{\zeta_s(t)} \mathbf{e}_M(t) + \beta_{n+1} \mathbf{e}_M(t-\tau) = R_M(t), \quad 0 \le t \le T, \\ &\mathbf{e}_M(0) = \mathbf{e}_M(T) \quad \text{or} \quad \mathbf{e}_M(0) = -\mathbf{e}_M(T), \end{aligned} \tag{5.4}$$

Thus, the (5.4) can be solved like the way it was presented in the previous section and we obtain the following estimation to $\mathbf{e}_M(t)$

$$\mathbf{e}_M(t) = \sum_{s=0}^{M} d_s P_{T,i}^{(\alpha,\beta)}(t) = D^T \Phi_{T,M}(t),$$

Note that if the accurate solution of the problem (1.1) or (1.2) is unknown, then we can gain the estimation of maximum amount of absolute errors by

$$\mathbf{E}_M(t) = \max\{\mathbf{e}_M(t), 0 \le t \le T\}.$$

The above estimation of error, is influenced by the rate of expansions convergence in Jacobi polynomials. Thus, the rates of convergence in temporal discretizations, are provided by it [39, 57].

6. Numerical Experiences

In this section, several numerical examples are presented to demonstrate the applicability, efficiency, accuracy, generality of this scheme. We obtain the outcomes of the current method by Mathematica 10 software. To test our technique, we have compared in terms of absolute errors of exact solution with current method and fractional backward differential formulas ($FBDF$) which defined as: $|z_{exact}(t) - z_n(t)|$.

Gathering of the outcomes obtained via this method with the true solution of each example displays that our scheme is in the best agreement compared to other methods. As this method is easy to implement, consistent and stable, it is therefore more reliable and applicable.

Example 6.1 ([11]). Consider the below $FDDE$ for $0 < \zeta \leq 1$

$$D^{\zeta}z(t) + z(t-\tau) = \frac{\Gamma(3)z(t)^{2-\zeta}}{\Gamma(3-\zeta)} - \frac{\Gamma(2)z(t)^{1-\zeta}}{\Gamma(2-\zeta)} + (t-\tau)^2 - (t-\tau) - 1, \tag{6.1}$$
$$z(t) = t^2 - t - 1, \quad t \in [-\tau, 0],$$
$$z(0) = -z(T).$$

With anti-periodic condition. The true solution is $z(t) = t^2 - t - 1$ and $T = 2$, $0 \leq t \leq T$, $\tau = 1$, $\zeta = 0.2$.

According to the presented concepts, we approximate the solution of this example and observe that results of this scheme are in the best agreement with the accurate solution compared to method ($FBDF$). From table 1, where the absolute errors (at $t = 1$) of the exact solution with our scheme and method in [11] are recorded, we find that the numerical results which getted by our method are very close to the exact solution and we achieved an excellent estimation for the true solution by using current technique. In figure 1 compared the exact and calculated solution which acknowledges the utility, accuracy and validity of $NSJOM$ scheme. Furthermore, in figure 2 the absolute error of exact solution with our scheme for this instance has been drawn. In this instance for $M = 2$ and $M = 4$, we have $A = [-0.66667, 1, +0.66667]^T$, $A = [-0.66667, 1, +0.66667, -4.82688 \times 10^{-16}, -1.16563 \times 10^{-16}]^T$, respectively.

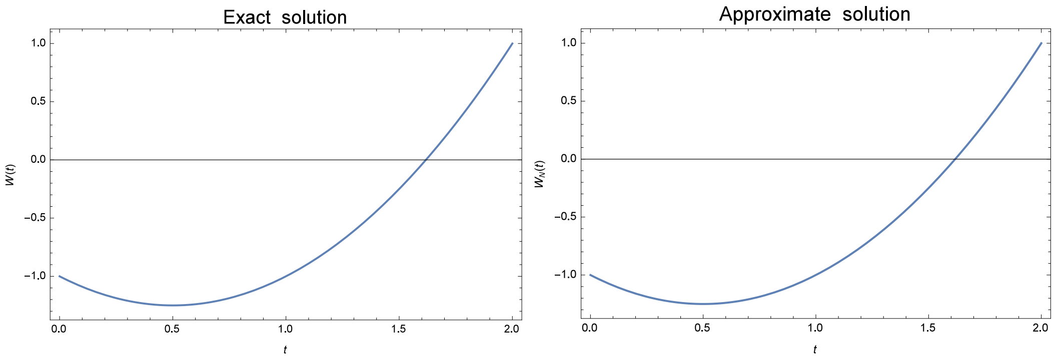
TABLE 1. Comparison of absolute error of true solution with scheme in [11] and current method with $\beta = 0$, $\alpha = 0$ at $t = 1.0$. for Example 6.1

Current method	$M = 2$	4.44089×10^{-16}
	$M = 4$	6.66134×10^{-16}
Scheme in [11]	$M = 20$	1.73817×10^{-2}
	$M = 40$	1.02509×10^{-2}
	$M = 200$	2.87569×10^{-3}
	$M = 400$	1.65502×10^{-3}
	$M = 2000$	4.57442×10^{-4}
	$M = 4000$	2.62785×10^{-4}
	$M = 20000$	7.25263×10^{-5}

Example 6.2 ([11]). Consider the below $FDDE$ for $0 < \zeta_1 < \zeta_2 \leq 1$, with periodic condition

(6.2)

$$\begin{aligned} &D^{\zeta_2}z(t) + D^{\zeta_1}z(t) + z(t-\tau) \\ =&\frac{\Gamma(3)z(t)^{2-\zeta_2}}{\Gamma(3-\zeta_2)} - \frac{\Gamma(2)z(t)^{1-\zeta_2}}{\Gamma(2-\zeta_2)} + \frac{\Gamma(3)z(t)^{2-\zeta_1}}{\Gamma(3-\zeta_1)} - \frac{\Gamma(2)z(t)^{1-\zeta_1}}{\Gamma(2-\zeta_1)} + (t-\tau)^2 - (t-\tau), \end{aligned}$$



(A) Accurate solution (B) Estimate solution

FIGURE 1. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.1

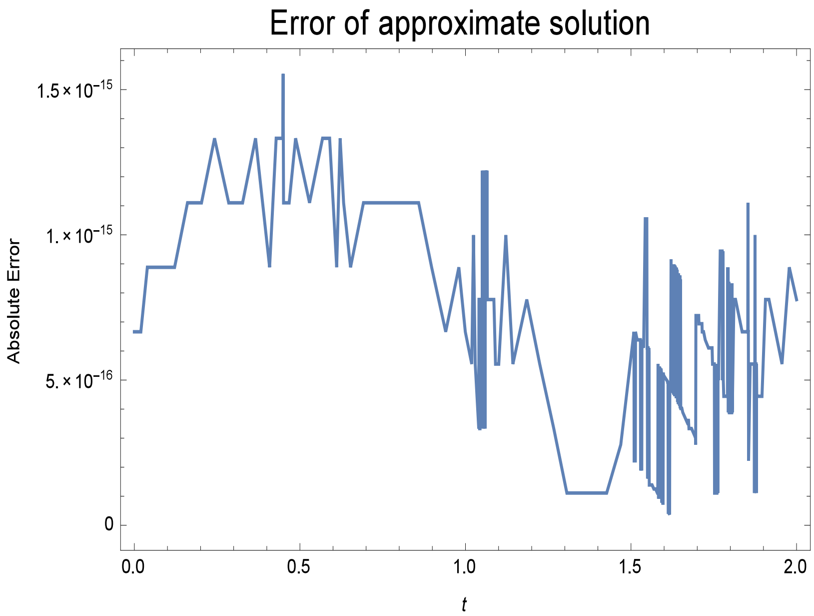


FIGURE 2. The absolute error between true and estimate solution (z_2) for Example 6.1

$$z(t) = t^2 - t, \quad t \in [-\tau, 0],$$
$$z(0) = z(T).$$

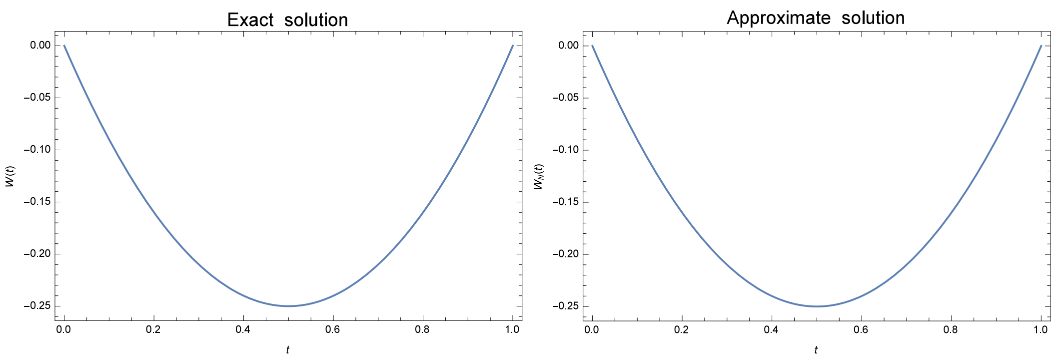
In this problem the true solution is $z(t) = t^2 - t$ and $T = 1$, $0 \le t \le T$, $\zeta_1 = 0.3$, $\zeta_2 = 0.4$, $\tau = 1$.

Using the process mentioned in Example 6.1, we get the solution of this example and compare the obtained results with $FBDF$ scheme. The outcomes show that our method is much better than the mentioned method. In table 2, the absolute errors of

TABLE 2. Comparison of absolute error of true solution with method in [11] and our scheme with $\beta = 0$, $\alpha = 0$, at $t = 1.0$. for Example 6.2

Current method	$M = 2$	2.22045×10^{-16}
	$M = 4$	1.62195×10^{-15}
Scheme in [11]	$M = 20$	1.55922×10^{-3}
	$M = 40$	5.10934×10^{-4}
	$M = 200$	3.67922×10^{-5}
	$M = 400$	1.18622×10^{-5}
	$M = 2000$	8.59612×10^{-7}
	$M = 4000$	2.78025×10^{-7}

our technique and scheme in [11] (at $t = 1$) are given and compared. In figure 3 the true and caculated solution are compared and in figure 4 the absolute error of exact solution with our scheme for this instance has been shown. Not that these figures and Tables show a good agreement between accurate and approximate solution. In this problem for $M = 2$ and $M = 4$, we have $A = [-0.166667, -6.78159 \times 10^{-18}, +0.166667]^T$, $A = [-0.166667, -6.78159 \times 10^{-18}, +0.166667, -2.40920 \times 10^{-16}, -6.46958 \times 10^{-17}]^T$, respectively.



(A) Accurate solution (B) Estimate solution

FIGURE 3. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.2

Example 6.3. Consider the variable-order $FDDE$ with anti-periodic condition

$$D^{\zeta(t)} z(t) + z(t-\tau) = \frac{\Gamma(3) z(t)^{2-\zeta(t)}}{\Gamma(3-\zeta(t))} - \frac{\Gamma(2) z(t)^{1-\zeta(t)}}{\Gamma(2-\zeta(t))} + (t-\tau)^2 - (t-\tau) - 1, \tag{6.3}$$
$$z(t) = t^2 - t - 1, \quad t \in [-\tau, 0],$$
$$z(0) = -z(T).$$

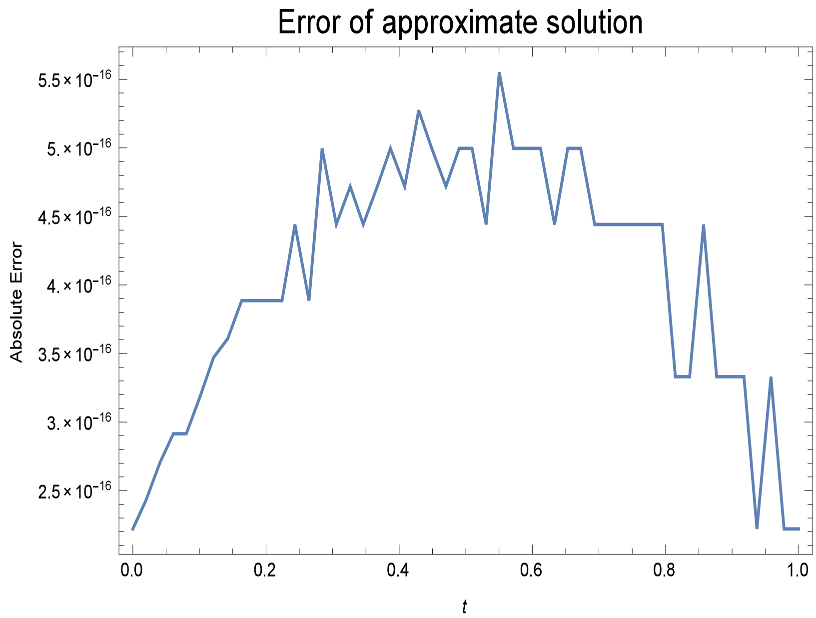


FIGURE 4. The absolute error between true and estimate solution (z_2) for Example 6.2

TABLE 3. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 2$ for Example 6.2

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	0	0
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0
CPU time	$0.171601s$	$3.19802s$

True solution is $z(t) = t^2 - t - 1$ and $T = 2$, $0 \le t \le T$, $\tau = 1$, $\zeta(t) = \frac{t}{7}$.

In this example, we estimated the solution of (6.3) for several values of α and β, by our $NSJOM$ scheme and recorded the needed consumption time (CPU time), the results related to the relative and absolute errors of this estimated solution with the exact solution in tables 3–8. Moreover, we show the absolute error with $\alpha = 1$, $\beta = 1$

TABLE 4. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0

TABLE 5. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	0	0
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0
CPU time	$0.156001s$	$9.001258s$

in Figure 6 and relative errors for various value of α and β in figures 7–9 for this instance. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 2$, have $A = [-0.66667, 1, +0.66667]^T$;
- For $\alpha = 0$, $\beta = 0$ and $M = 3$, have $A = [-0.66667, 1, +0.66667, 0]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 2$, have $A = [-0.75, 0.66667, 0.4]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 3$, have $A = [-0.75, 0.66667, 0.4, 1.94241 \times 10^{-16}]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 2$, have $A = [-0.8, 0.5, 0.26667]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 3$, have $A = [-0.8, 0.5, 0.26667, 0]^T$.

TABLE 6. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	0	0
0.4	0	0
0.6	0	0
0.8	0	0
1.0	0	0
1.2	0	0
1.4	0	0
1.6	0	0
1.8	0	0
2.0	0	0

TABLE 7. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0	4.440×10^{-16}	1.776×10^{-15}
0.2	4.440×10^{-16}	1.554×10^{-15}
0.4	2.220×10^{-16}	1.110×10^{-15}
0.6	2.220×10^{-16}	6.661×10^{-16}
0.8	2.220×10^{-16}	2.220×10^{-16}
1.0	2.220×10^{-16}	4.440×10^{-16}
1.2	0	8.881×10^{-16}
1.4	5.551×10^{-17}	1.276×10^{-15}
1.6	4.093×10^{-16}	1.366×10^{-15}
1.8	1.276×10^{-15}	1.387×10^{-15}
2.0	8.881×10^{-16}	1.776×10^{-15}
CPU time	$0s$	$0s$

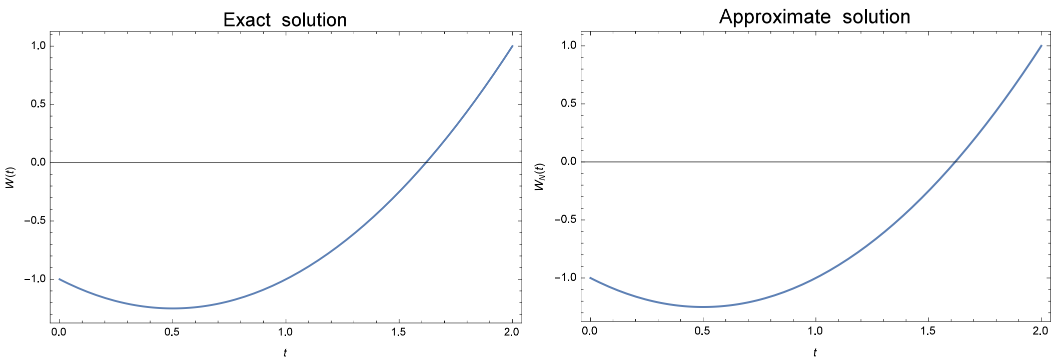
Example 6.4. Consider the following variable-order *FDDE*

(6.4)

$$
\begin{aligned}
&D^{\zeta_2} z(t) + D^{\zeta_1} z(t) + z(t-\tau) \\
=&\frac{\Gamma(3) z(t)^{2-\zeta_2}}{\Gamma(3-\zeta_2)} - \frac{\Gamma(2) z(t)^{1-\zeta_2}}{\Gamma(2-\zeta_2)} + \frac{\Gamma(3) z(t)^{2-\zeta_1}}{\Gamma(3-\zeta_1)} - \frac{\Gamma(2) z(t)^{1-\zeta_1}}{\Gamma(2-\zeta_1)} + (t-\tau)^2 - (t-\tau), \\
&z(t) = t^2 - t, \quad t \in [-\tau, 0], \\
&z(0) = z(T).
\end{aligned}
$$

TABLE 8. Relative errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 2$ for Example 6.3

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 3$
0.2	8.330×10^{-16}	1.776×10^{-15}
0.4	9.362×10^{-16}	2.004×10^{-15}
0.6	9.221×10^{-16}	2.551×10^{-15}
0.8	1.004×10^{-15}	3.276×10^{-15}
1.0	1.389×10^{-15}	4.551×10^{-15}
1.2	2.320×10^{-15}	6.351×10^{-15}
1.4	3.531×10^{-15}	7.440×10^{-15}
1.6	4.089×10^{-15}	8.241×10^{-15}
1.8	4.224×10^{-15}	8.878×10^{-15}
2.0	5.551×10^{-15}	7.983×10^{-15}



(A) Accurate solution (B) Estimate solution

FIGURE 5. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.3 ($\zeta(t) = 0.5t$).

This problem is the periodic conditions type and the true solution is $z(t) = t^2 - t$ and $0 \le t \le T$, $T = 1$, $\zeta_1(t) = \frac{t}{2}$, $\zeta_2 = \frac{t}{4}$, $\tau = 1$.

We estimated the solution of (6.4) for various values of α and β, by our $NSJOM$ scheme and presented the CPU time required for our scheme, the results related to the relative and absolute errors of this estimated solution with the exact solution in tables 9–14. Moreover, we show the absolute error with $\alpha = 1$, $\beta = 1$ in Figure 6 and relative errors for various value of α and β in figures 12–13 for this instance. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 2$, have $A = [-0.16667, 0, +0.16667]^T$;
- For $\alpha = 0$, $\beta = 0$ and $M = 4$, have $A = [-0.16667, 0, +0.16667, 0, 0]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 2$, have $A = [-0.1875, 0, 0.1]^T$;
- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 4$, have $A = [-0.1875, 0, 0.1, 0, 0]^T$;

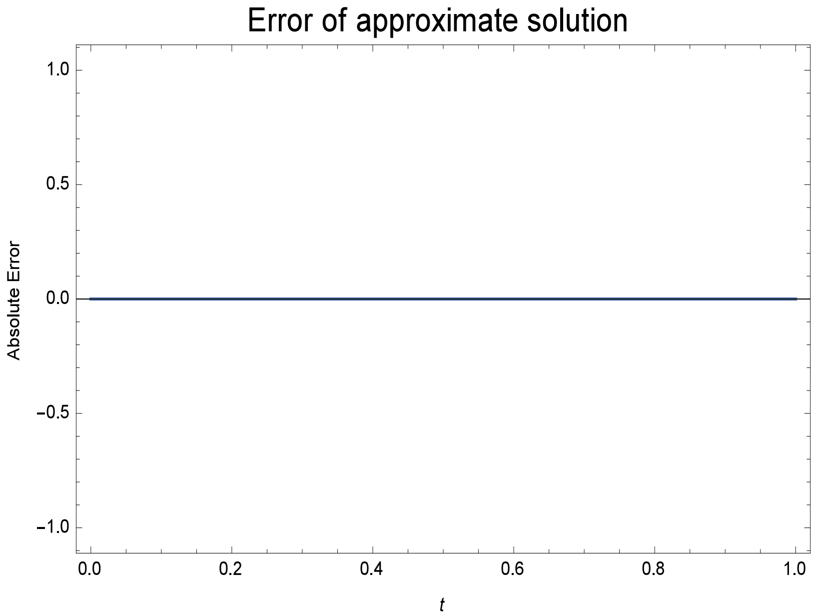


FIGURE 6. The absolute error between true and estimate solution (z_2) for Example 6.3 ($\zeta(t) = 0.5t$)

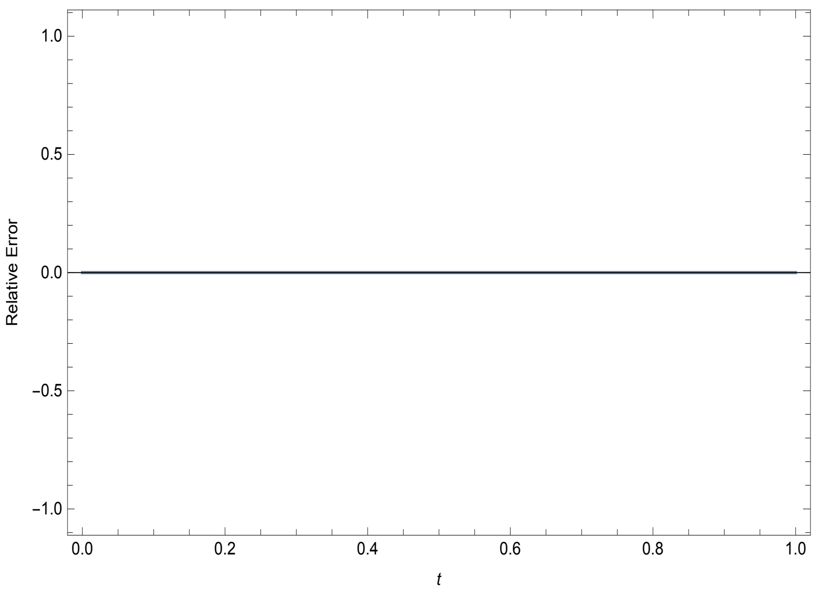


FIGURE 7. The relative error between accurate and estimate solution(z_2) with $\beta = 0$, $\alpha = 0$, at $T = 2.0$ for Example 6.3

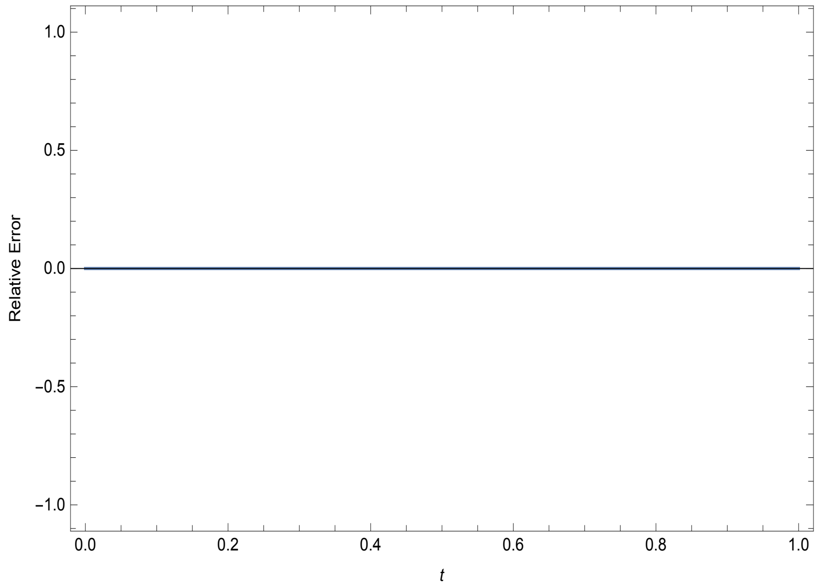


FIGURE 8. The relative error between accurate and estimate solution (z_2) with $\beta = 1$, $\alpha = 1$, at $T = 2.0$ for Example 6.3

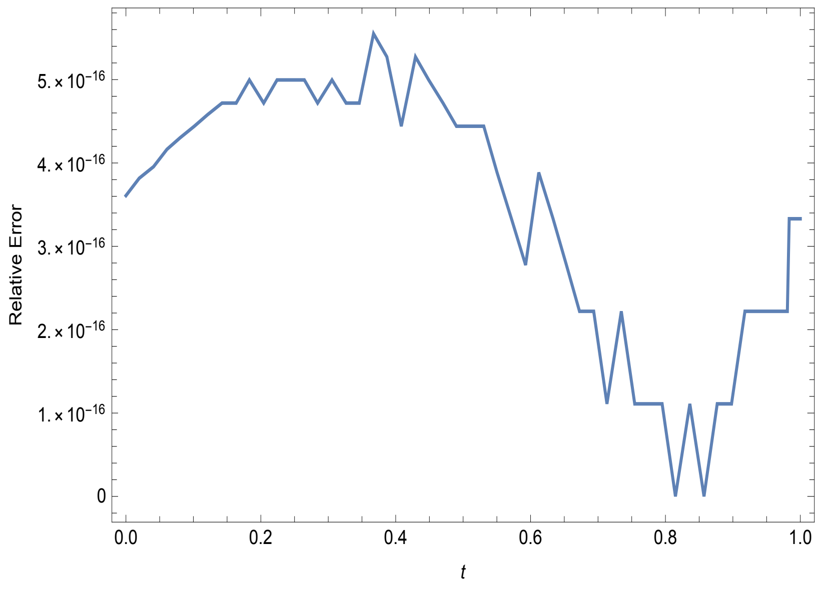


FIGURE 9. The relative error between accurate and estimate solution (z_2) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, at $T = 2.0$ for Example 6.3

TABLE 9. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.12	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0.093601s$	$4.007500s$

TABLE 10. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 0$, $\alpha = 0$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

- For $\alpha = 1$, $\beta = 1$ and $M = 2$, have $A = [-0.2, 0, 0.06667]^T$;
- For $\alpha = 1$, $\beta = 1$ and $M = 4$, have $A = [-0.2, 0, 0.06667, 0, 0]^T$.

Example 6.5. Consider the below $FDDE$ for $0 < \zeta \leq 1$

(6.5)

$$D^{\zeta} z(t) - z(t - \tau) + z(t) = g(t),$$

$$g(t) = \frac{2\exp(t)(-1 + t)}{1 + \exp(2)} - \frac{2\exp(t - \tau)(-1 + t - \tau)}{1 + \exp(2)}$$

TABLE 11. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0s$	$0.062400s$

TABLE 12. Relative errors of true solution and our method ($z_M(t)$) with $\beta = 1$, $\alpha = 1$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

$$- \frac{2t^{\zeta}(-2+\zeta)(t^2 + \exp(t)t^{\tau}(-1+t+\zeta)\Gamma(2-\eta) - \exp(t)t^{\zeta}(-1+t+\zeta)\Gamma(2-\zeta, t))}{\Gamma(3-\zeta)(1+\exp(2))},$$

$$z(t) = \frac{2\exp(t)(-1+t)}{1+\exp(2)} - \frac{2\exp(2)}{1+\exp(2)} + 1, \quad t \in [-\tau, 0],$$

$$z(0) = -z(T).$$

This problem is the anti-periodic conditions type and the true solution is $z(t) = \frac{2\exp(t)(-1+t)}{1+\exp(2)} - \frac{2\exp(2)}{1+\exp(2)} + 1$ and $0 \le t \le T$, $T = 2$, $\tau = 0.01\exp(-t)$, $\zeta = 0.2$.

The solution of (6.3) for several values of α and β, by our $NSJOM$ scheme is stimated and is recorded the CPU time required for our scheme, the results related

TABLE 13. Absolute errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 1$ for Example 6.4

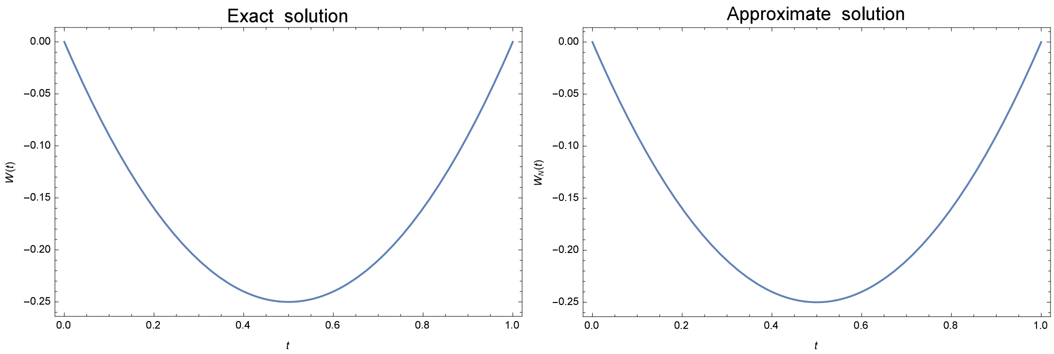
$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0
CPU time	$0.109201s$	$51.339929s$

TABLE 14. Relative errors of true solution and our method ($z_M(t)$) with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $T = 1$ for Example 6.4

$t \in [0, T]$	Current method, $M = 2$	Current method, $M = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

to the absolute and relative errors of this estimated solution with the exact solution in tables 15 and 16. In Figure 14 compared the exact and calculated solution which acknowledges the utility, accuracy and validity of $NSJOM$ technique. Furthermore, in Figure 15 the absolute error of exact solution with our scheme for this instance has been drawn. In this instance, we have:

- For $\alpha = 0$, $\beta = 0$ and $M = 10$, have $A = [-0.523188, 0.854347, 0.49638, 0.142, 0.0264148, 0.00361749, 3.90992 \times 10^{-4}, 3.48 \times 10^{-5}, 2.64 \times 10^{-6}, 1.72 \times 10^{-7}, 1.076 \times 10^{-8}]^T$;



(A) Accurate solution (B) Estimate solution

FIGURE 10. Comparison of accurate and estimate solution (z_2) of $NSJOM$ scheme for Example 6.4.

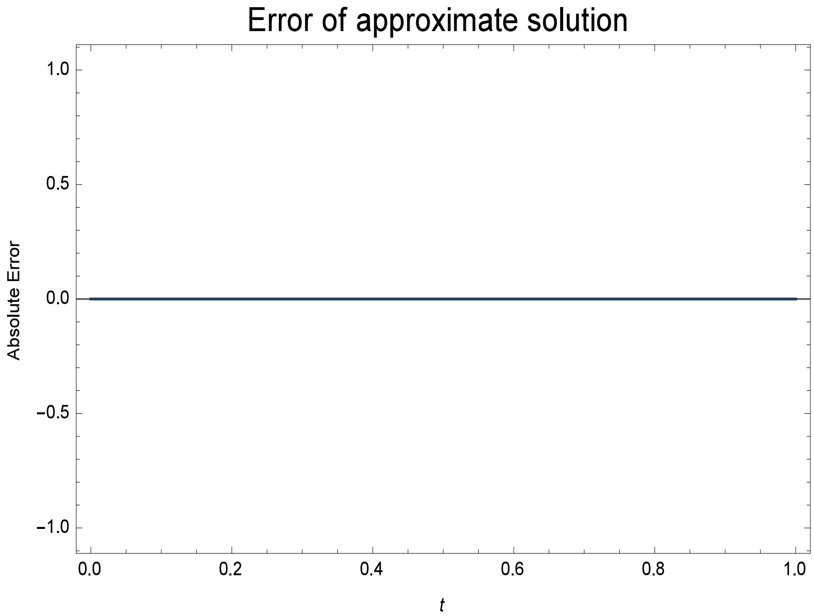


FIGURE 11. The absolute error between true and estimate solution (z_2) for Example 6.4.

- For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $M = 10$, have $A = [-0.58565, 0.54580, 0.29286, 0.08042, 0.014593, 0.0019657, 2.09955 \times 10^{-4}, 1.885 \times 10^{-5}, 1.3976 \times 10^{-6}, 9.0947 \times 10^{-8}, 5.641 \times 10^{-9}]^T$;

- For $\alpha = 1$, $\beta = 1$ and $M = 10$, have $A = [-0.622464, 0.396745, 0.192682, 0.049892, 0.008714, 0.00114289, 1.1968 \times 10^{-4}, 1.0416 \times 10^{-5}, 7.75225 \times 10^{-7}, 5.0021 \times 10^{-8}, 3.077 \times 10^{-9}]^T$.

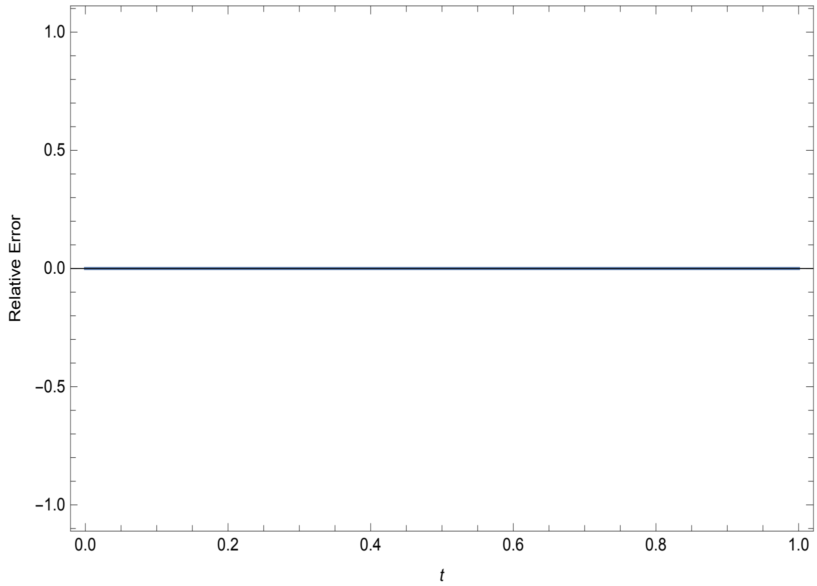


FIGURE 12. The relative errors between estimate solution (z_4) and accurate solution with $\beta = 0$, $\alpha = 0$, at $t = 1.0$. for Example 6.4.

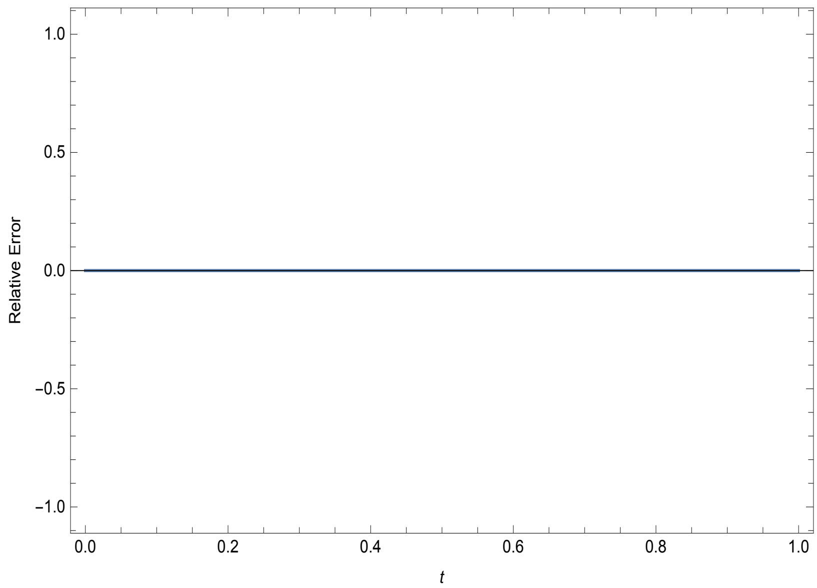


FIGURE 13. The relative error between exact and estimate solution (z_4) with $\beta = 1$, $\alpha = 1$, at $t = 1.0$. for Example 6.4.

TABLE 15. Absolute errors of true solution and our method ($z_M(t)$) with $M = 10$ and $T = 2$ for Example 6.5 by $NSJOM$.

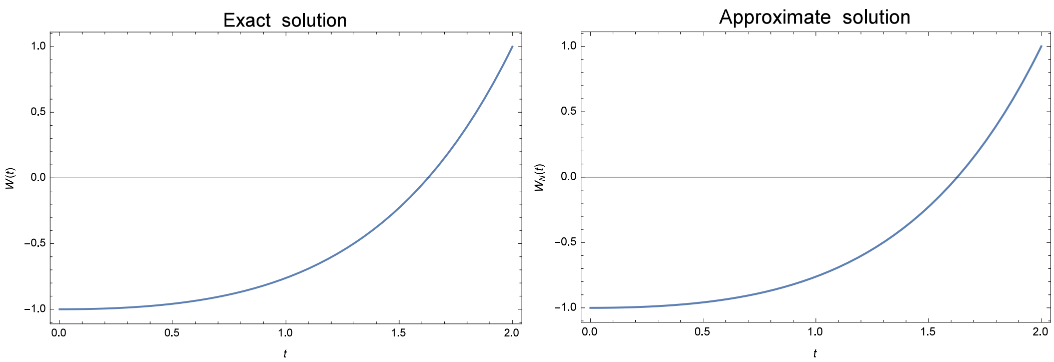
$t \in [0, T]$	$\alpha = 1,\ \beta = 1$	$\alpha = 0,\ \beta = 0$	$\alpha = 0.5,\ \beta = 0.5$
0	1.025×10^{-10}	9.479×10^{-11}	2.289×10^{-10}
0.2	3.779×10^{-9}	3.953×10^{-9}	4.087×10^{-9}
0.4	9.779×10^{-10}	9.667×10^{-10}	1.100×10^{-9}
0.6	6.661×10^{-10}	7.927×10^{-10}	9.265×10^{-10}
0.8	3.614×10^{-10}	5.863×10^{-10}	7.199×10^{-10}
1.0	6.661×10^{-10}	5.181×10^{-10}	6.502×10^{-10}
1.2	5.453×10^{-10}	4.376×10^{-10}	5.645×10^{-10}
1.4	4.185×10^{-10}	4.157×10^{-10}	5.281×10^{-10}
1.6	3.271×10^{-10}	3.433×10^{-10}	4.215×10^{-10}
1.8	6.981×10^{-10}	4.248×10^{-10}	4.307×10^{-10}
2.0	2.003×10^{-10}	9.479×10^{-11}	2.288×10^{-10}
CPU time	$1.076407s$	$1.076407s$	$1.544410s$

TABLE 16. Absolute errors of true solution and our method ($z_M(t)$) with $M = 15$ and $T = 2$ for Example 6.5 by $NSJOM$.

$t \in [0, T]$	$\alpha = 1,\ \beta = 1$	$\alpha = 0,\ \beta = 0$	$\alpha = 0.5,\ \beta = 0.5$
0	6.661×10^{-15}	8.881×10^{-16}	2.377×10^{-15}
0.2	7.016×10^{-14}	3.352×10^{-14}	3.907×10^{-13}
0.4	3.753×10^{-14}	1.487×10^{-14}	8.705×10^{-15}
0.6	2.775×10^{-14}	9.547×10^{-15}	8.635×10^{-13}
0.8	2.442×10^{-14}	7.549×10^{-15}	2.615×10^{-13}
1.0	2.152×10^{-14}	5.772×10^{-15}	6.163×10^{-13}
1.2	2.087×10^{-14}	5.551×10^{-15}	5.394×10^{-13}
1.4	1.909×10^{-14}	3.996×10^{-15}	9.005×10^{-146}
1.6	1.909×10^{-14}	3.556×10^{-15}	2.132×10^{-13}
1.8	2.131×10^{-14}	3.330×10^{-15}	4.916×10^{-13}
2.0	1.187×10^{-14}	1.110×10^{-15}	2.373×10^{-13}
CPU time	$2.552407s$	$2.558416s$	$3.000810s$

7. CONCLUSIONS

In this work, we have presented the (NSJOM) technique for the generalized linear variable-order $FDDE$ with anti-periodic and periodic condition by turning the main problem to an algebraic equations system that this system is solved numerically. We have shown that the presented method has good convergence, its concepts are simple and it's easy to implement. The obtained results are excellent compared to other



(A) Accurate solution (B) Estimate solution

FIGURE 14. Comparison of accurate and estimate solution (z_{15}) of $NSJOM$ method for Example 6.5

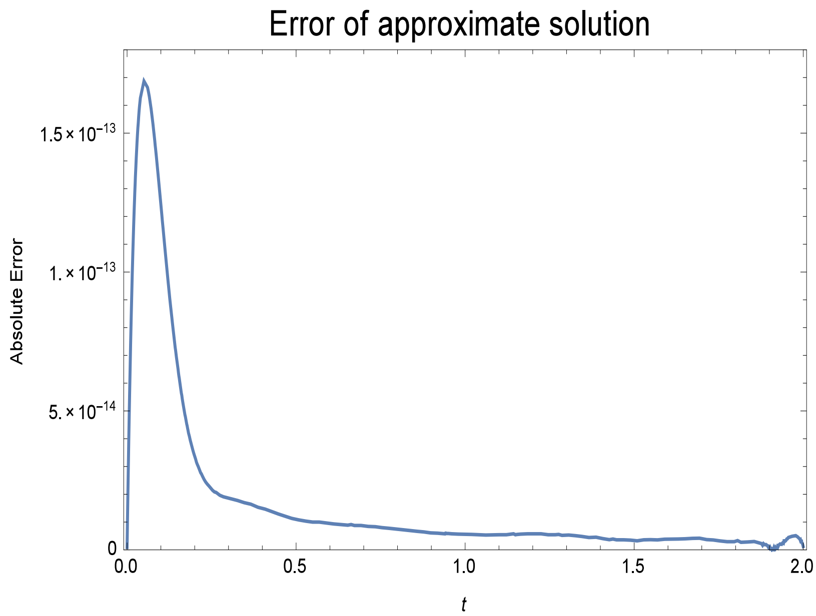


FIGURE 15. The absolute error between exact and estimate solution (z_{15}) for Example 6.5

method. Finally, the numerical results have been reported to clarify the validity and efficiency of this method.

Acknowledgements. We are grateful to the anonymous reviewers for their helpful comments, which undoubtedly led to the definite improvement in the paper.

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