KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 50(1) (2026), PAGES 71–89.

IMPROVED JENSEN-TYPE INEQUALITIES FOR (p,h)-CONVEX FUNCTIONS WITH APPLICATIONS

MOHAMED AMINE IGHACHANE[1], LAKHLIFA SADEK[2], AND MOHAMMAD SABABHEH[3]

ABSTRACT. The main goal of this article is to present multiple term refinements of the well-known Jensen's inequality for h-convex functions for a non-negative super-multiplicative and super-additive function h. For example, we show that

$$h(1-v)f(0)+h(v)f(1)\geqslant f(v)+\sum_{n=0}^{N-1}h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),$$

for the h-convex function f and certain positive summands. The significance of the obtained results is the way they extend known results from the setting of convex functions to other classes of functions.

1. INTRODUCTION AND PRELIMINARIES

Convex functions and their inequalities have played a major role in the study of various topics in Mathematics; including applied Mathematics, Mathematical Analysis, and Mathematical Physics. Recall that a function $f:I\to\mathbb{R}$ is said to be convex on the interval I if

$$f((1-v)a+vb)\leqslant(1-v)f(a)+vf(b), \tag{1.1}$$

for all $a,b\in I$ and $v\in(0,1)$. If this inequality is reversed, then f is said to be concave.

Recent studies of the topic have investigated possible refinements of the above inequality, where adding a positive term to the left side becomes possible. This idea has been treated in [4, 10–13, 15], where not only refinements have been investigated, but reversed versions have been also discussed.

Key words and phrases. (p,h)-convex function, operator (p,h)-convex function, Jensen's inequality.
2020 *Mathematics Subject Classification.* Primary: 15A39, 15B48, 26D15, 15A60.
DOI 10.46793/KgJMat2601.071I
Received: February 17, 2023.
Accepted: May 17, 2023.

The notion of convexity has been expanded and generalized in various ways utilizing novel and modern methods in recent years.

To motivate our work, let us recall the definitions of some special classes of functions. Let I be a p-convex subset of $\mathbb{R}$ (that means, $[(1-v)a^p + vb^p]^{\frac{1}{p}} \in I$ for all $a, b \in I$ and $v \in (0,1)$).

Definition 1.1 ([16])**.** A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to the class $PC(I)$, if

$$f\left(\left[(1-v)a^p + vb^p\right]^{\frac{1}{p}}\right) \leqslant (1-v)f(a) + vf(b), \tag{1.2}$$

for all $a, b \in I$, $p \in \mathbb{R}\backslash\{0\}$ and $v \in (0,1)$.

Definition 1.2 ([9])**.** Let $h : (0,1) \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is an h-convex function or that f belongs to the class $SX(I)$, if f is non-negative and for all $a, b \in I$ and $v \in (0,1)$

$$f((1-v)a + vb) \leqslant h(1-v)f(a) + h(v)f(b). \tag{1.3}$$

If this inequality is reversed, then f is said to be h-concave.

Definition 1.3 ([6])**.** Let $h : (0,1) \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is a (p,h)-convex function or that f belongs to the class $ghx(h,p,I)$, if f is non-negative and

$$f\left(\left[(1-v)a^p + vb^p\right]^{\frac{1}{p}}\right) \leqslant h(1-v)f(a) + h(v)f(b), \tag{1.4}$$

for all $a, b \in I$ and $v \in (0,1)$. Similarly, if the inequality sign in (1.4) is reversed, then f is said to be a (p,h)-concave function or belong to the class $ghv(h,p,I)$.

Definition 1.4 ([7])**.** Let $h : J \to \mathbb{R}$. If

$$h(x)h(y) \leqslant h(xy), \tag{1.5}$$

for all $x, y \in J$, then h is said to be a super-multiplicative function. If (1.5) is reversed, then h is said to be a sub-multiplicative function. If equality holds in (1.5), then h is said to be a multiplicative function.

Definition 1.5 ([7])**.** Let $h : J \to \mathbb{R}$. If for all $x, y \in J$

$$h(x) + h(y) \leqslant h(x+y), \tag{1.6}$$

then h is said to be a super-additive function. If inequality (1.6) is reversed, we say that h is a sub-additive function. If equality (1.6) holds, we say that h is an additive function.

Example 1.1. Let $h : (0,+\infty) \to (0,+\infty)$ be defined by $h(x) = x^k$. Then h is

(a) additive if $k = 1$;
(b) sub-additive if $k \in (-\infty, 1)$;
(c) super-additive if $k \in (1, +\infty)$.

This latter conclusion follows from the fact that $h(x) = x^k$ is convex and $h(0) = 0$, when $k > 1$.

Let $h : [1, +\infty) \mapsto \mathbb{R}^+$ be defined by $h(x) = x^3 - x^2 + x$. We have

(a) $h(xy) - h(x)h(y) = xy(x + y)(1 - x)(1 - y) \geqslant 0$;

(b) $h(x + y) - h(x) - h(y) = xy(x + y + (x - 1) + (y - 1)) \geqslant 0$.

Then h is a super-multiplicative and super-additive function.

The following theorem is the Jensen type inequality for (p, h)-convex functions.

Theorem 1.1 ([6]). *Let $v_1, \dots, v_n$ be positive real numbers, $n \geqslant 2$, such that $\sum_{k=1}^{n} v_k = 1$. If h is a non-negative super-multiplicative function, f an (p, h)-convex function and $a_1, \dots, a_n \in I$, then*

$$f\left(\left(\sum_{k=1}^{n} v_k a_k^p\right)^{\frac{1}{p}}\right) \leqslant \sum_{k=1}^{n} h\left(v_k\right) f\left(a_k\right). \tag{1.7}$$

If h is sub-multiplicative and f an (p, h)-concave function, then inequality (1.7) *is reversed.*

The organization of the paper will be as follows. We firstly derive the refinements of Jensen-type and a variant of Jensen-type inequalities for h-convex functions. Next, we further refine our presented refinements and point out more or less direct consequences of our results for (p, h)-convex functions and its reversed, and in the last section we give the matrix versions of these inequalities studied in Section 2 and 3.

2. New Refinements of the Jensen's Inequality for h-Convex Functions

In this part of the paper, we present our main results concerning h-convex functions. The applications of these inequalities and their relations to the literature will be done in Remark 2.1. In order to do that, we start with some basic results which are important in terms of proving our main results.

We will see that our results extend the results in [1, 10] to the context of h-convex functions, with the existence of multiple refining terms.

Definition 2.1. Let n be a positive integer. The sequence $(r_n(v))$ of functions on $[0, 1]$ is defined by

$$\begin{aligned} r_0(v) &= \min\{v, 1 - v\}, \\ r_n(v) &= \min\{2r_{n-1}(v), 1 - 2r_{n-1}(v)\}. \end{aligned}$$

For all integers n, we have the following explicit formula of the function $r_n(v)$, proved by D. Choi in [3]. We also refer the reader to [12] for similar treatment.

Lemma 2.1 ([3]). *Let $\ell \geqslant 0$ and $1 \leqslant k \leqslant 2^n$ be integers. If $\frac{k-1}{2^n} \leqslant v \leqslant \frac{k}{2^n}$, then*

$$r_n(v) = \begin{cases} 2^n v - k + 1, & \text{if } \frac{k-1}{2^n} \leqslant v \leqslant \frac{2k-1}{2^{n+1}}, \\ k - 2^n v, & \text{if } \frac{2k-1}{2^{n+1}} \leqslant v \leqslant \frac{k}{2^n}. \end{cases}$$

In the following lemma, we prove an essential inequality that will be needed in the sequel.

Lemma 2.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$, and f be a function defined on $[0,1]$. For a nonnegative integer N, define $\Psi_N(v)$ by*

$$\Psi_N(v) = h(1-v)f(0) + h(v)f(1) - \sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2\pi}\right)}(v), \tag{2.1}$$

where

$$\Delta_{f,h}^{(0,1)}(n,k) = h\left(\frac{1}{2}\right)\left[f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right)\right] - f\left(\frac{2k-1}{2^{n+1}}\right),$$

for $\frac{k-1}{2^N} \leqslant v \leqslant \frac{k}{2^N}$ and $k = 1,\dots,2^N$. Then

$$\Psi_N(v) \geqslant h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right).$$

Proof. We proceed by induction on N. For $N = 1$: if $v \in [0, \frac{1}{2}]$, we have

$$\begin{aligned}
\Psi_1(v) &= h(1-v)f(0) + h(v)f(1) - h(2r_0(v))\Delta_{f,h}(0,1)\chi_{(0,1)}(v)\\
&= h(1-v)f(0) + h(v)f(1) - h(2v)\Delta_{f,h}(0,1)\\
&= \left(h(v) - h(2v)h\left(\frac{1}{2}\right)\right)f(1) + \left(h(1-v) - h(2v)h\left(\frac{1}{2}\right)\right)f(0) + h(2v)f\left(\frac{1}{2}\right)\\
&\geqslant \left(h(1-v) - h(2v)h\left(\frac{1}{2}\right)\right)f(0) + h(2v)f\left(\frac{1}{2}\right)\\
&\geqslant h(1-2v)f(0) + h(2v)f\left(\frac{1}{2}\right).
\end{aligned}$$

If $v \in [\frac{1}{2}, 1]$, then $1 - v \in [0, \frac{1}{2}]$. So, by changing v by $1 - v$ and $f(v)$ by $f(1-v)$, the desired inequality for the case $v \in [\frac{1}{2}, 1]$ is obtained.

Now, assume that (2.1) holds and let $\frac{m-1}{2^{N+1}} \leqslant v \leqslant \frac{m}{2^{N+1}}$ for $m = 1,\dots,2^{N+1}$. If $m = 2k-1$, then $\frac{k-1}{2^N} \leqslant v \leqslant \frac{2k-1}{2^{N+1}} < \frac{k}{2^N}$ and

$$\begin{aligned}
\Psi_{N+1}(v) =& \Psi_N(v) - h(2r_N(v))\Delta_{f,h}^{(0,1)}(N,k)\\
\geqslant& h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right)\\
&- h\left(2^{N+1}v - 2k + 2\right)\Delta_{f,h}^{(0,1)}(N,k)\\
=& h\left(k - 2^N v\right) f\left(\frac{k-1}{2^N}\right) + h\left(2^N v - k + 1\right) f\left(\frac{k}{2^N}\right)\\
&- h\left(2^{N+1}v - 2k + 2\right) h\left(\frac{1}{2}\right)\left(f\left(\frac{k-1}{2^N}\right) + f\left(\frac{k}{2^N}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&+h\left(2^{N+1}v-2k+2\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
\geqslant &h\left(k-2^{N}v\right)f\left(\frac{k-1}{2^{N}}\right)+h\left(2^{N}v-k+1\right)f\left(\frac{k}{2^{N}}\right)\\
&-h\left(2^{N}v-k+1\right)\left(f\left(\frac{k-1}{2^{N}}\right)+f\left(\frac{k}{2^{N}}\right)\right)\\
&+h\left(2^{N+1}v-2k+2\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
=&\Big(h\left(k-2^{N}v\right)-h\left(2^{N}v-k+1\right)\Big)f\left(\frac{k-1}{2^{N}}\right)\\
&+h\left(2^{N+1}v-2k+2\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
\geqslant &h\left(2k-1-2^{N+1}v\right)f\left(\frac{k-1}{2^{N}}\right)+h\left(2^{N+1}v-2k+2\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
=&h\left(m-2^{N+1}v\right)f\left(\frac{m-1}{2^{N+1}}\right)+h\left(2^{N+1}v-m+1\right)f\left(\frac{m}{2^{N+1}}\right),
\end{aligned}$$

by Lemma 2.1. Similarly, if $m=2k$, then $\frac{k-1}{2^N}<\frac{2k-1}{2^{N+1}}\leqslant v\leqslant\frac{k}{2^N}$ and

$$\begin{aligned}
\Psi_{N+1}(v)=&\Psi_N(v)-h(2r_N(v))\Delta_{f,h}^{(0,1)}(N,k)\\
\geqslant &h\left(k-2^{N}v\right)f\left(\frac{k-1}{2^{N}}\right)+h\left(2^{N}v-k+1\right)f\left(\frac{k}{2^{N}}\right)\\
&-h\left(2k-2^{N+1}v\right)\Delta_{f,h}^{(0,1)}(N,k)\\
=&h\left(k-2^{N}v\right)f\left(\frac{k-1}{2^{N}}\right)+h\left(2^{N}v-k+1\right)f\left(\frac{k}{2^{N}}\right)\\
&-h\left(2k-2^{N+1}v\right)h\left(\frac{1}{2}\right)\left(f\left(\frac{k-1}{2^{N}}\right)+f\left(\frac{k}{2^{N}}\right)\right)\\
&+h\left(2k-2^{N+1}v\right)f\left(\frac{2k-1}{2^{N+1}}\right)\\
\geqslant &h\left(k-2^{N}v\right)f\left(\frac{k-1}{2^{N}}\right)+h\left(2^{N}v-k+1\right)f\left(\frac{k}{2^{N}}\right)\\
&-h\left(k-2^{N}v\right)\left(f\left(\frac{k-1}{2^{N}}\right)+f\left(\frac{k}{2^{N}}\right)\right)+h\left(2k-2^{N+1}v\right)f\left(\frac{2k-1}{2^{N+1}}\right).
\end{aligned}$$

Thus, we have shown that

$$\Psi_{N+1}(v)=\Big(h\left(2^{N}v-k+1\right)-h\left(k-2^{N}v\right)\Big)f\left(\frac{k}{2^{N}}\right)+h\left(2k-2^{N+1}v\right)f\left(\frac{2k-1}{2^{N+1}}\right)$$

$$\begin{aligned}
&\geqslant h\left(2^{N+1}v-2k+1\right) f\left(\frac{k}{2^N}\right)+h\left(2k-2^{N+1}v\right) f\left(\frac{2k-1}{2^{N+1}}\right)\\
&= h\left(2^{N+1}v-m+1\right) f\left(\frac{m}{2^{N+1}}\right)+h\left(m-2^{N+1}v\right) f\left(\frac{m-1}{2^{N+1}}\right).
\end{aligned}$$

This completes the proof. □

Now we show the first result concerning h-convex functions, when h is super-multiplicative and super-additive.

Theorem 2.1. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and f be an h-convex function on $[0,1]$. If N is a positive integer, then*

$$h(1-v)f(0)+h(v)f(1)\geqslant f(v)+\sum_{n=0}^{N-1} h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v) \tag{2.2}$$

and

$$\begin{aligned}
h(1-v)f(0)+h(v)f(1) \leqslant& f(0)+f(1)-f(1-v)\\
&-\sum_{n=0}^{N-1} h(2r_n(v))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,2^n-k+1)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),
\end{aligned} \tag{2.3}$$

where $v\in[0,1]$ and

$$\Delta_{f,h}^{(0,1)}(n,k)=h\left(\frac{1}{2}\right)\left[f\left(\frac{k-1}{2^n}\right)+f\left(\frac{k}{2^n}\right)\right]-f\left(\frac{2k-1}{2^{n+1}}\right).$$

Proof. By Lemma 2.2 and the h-convexity of the function f, we get

$$\begin{aligned}
\Psi_N(v) &\geqslant h\left(k-2^Nv\right) f\left(\frac{k-1}{2^N}\right)+h\left(2^Nv-k+1\right) f\left(\frac{k}{2^N}\right)\\
&\geqslant f\left(\left(k-2^Nv\right)\frac{k-1}{2^N}+\left(2^Nv-k+1\right)\frac{k}{2^N}\right)\\
&= f(v).
\end{aligned}$$

This end the proof of (2.2).

Replacing v by $1-v$ in (2.2) and noting that $r_n(v)=r_n(1-v)$, we have

$$\begin{aligned}
&(h(1-v)+h(v))\left(f(0)+f(1)\right)-h(v)f(0)-h(1-v)f(1)\\
\leqslant& -f(1-v)+(h(1-v)+h(v))\left(f(0)+f(1)\right)\\
&-\sum_{n=0}^{N-1} h((2r_n(v)))\sum_{k=1}^{2^n}\Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(1-v).
\end{aligned}$$

So,

$$h(1-v)f(0)+h(v)f(1)\leqslant(h(1-v)+h(v))\left(f(0)+f(1)\right)-f(1-v)$$

$$\begin{aligned}
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n}\frac{k}{2^n}\right)}(1-v)\\
\leqslant & h(1)(f(0)+f(1))-f(1-v)\\
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n}\frac{k}{2^n}\right)}(1-v)\\
\leqslant & (f(0)+f(1))-f(1-v)\\
&-\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(0,1)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(1-v).
\end{aligned}$$

Now, replacing k by 2^n-k+1 in the inner summation and noting that

$$\frac{k-1}{2^n} < 1-v < \frac{k}{2^n} \quad \text{if and only if} \quad 1-\frac{k}{2^n} < v < 1-\frac{k-1}{2^n},$$

we obtain (2.3) and the proof is completed. □

Remark 2.1. Before proceeding to further results, we explain a little about Theorem 2.6. Notice that for $h(x)=x$, we recapture Theorem 2.1 in [4].

Corollary 2.1. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and f be a h-convex function on $[a,b]$. Then for all positive integers N,*

$$h(1-v)f(a)+h(v)f(b) \geqslant f(va+(1-v)b)+\sum_{n=0}^{N-1} h(2r_n(v)) \sum_{k=1}^{2^n} \Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v),$$

where $v \in [0,1]$ and

$$\Delta_{f,h}^{(a,b)}(n,k) = h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right)+g\left(\frac{k}{2^n}\right)\right]-g\left(\frac{2k-1}{2^{n+1}}\right)$$

and $g(t):=f((1-t)a+tb)$.

Proof. For the h-convex function f, define $g:[0,1]\to\mathbb{R}$ by $g(t):=f((1-t)a+tb)$. Then, g is h-convex on $[0,1]$. Applying Theorem 2.1 on the function g implies the result. □

The following result provides a two-parameter refined version of the basic inequality for h-convex functions. We encourage the reader to see the main results in [1, 10], where this type was treated for convex functions, without any refining terms.

Theorem 2.2. *Let h be a non-negative super-multiplicative and super-additive function on $[0,1]$ and let f be an h-convex function on $[a,b]$. If $0 < v \leqslant \tau < 1$, then for all positive integers N*

$$\begin{aligned}
h(1-v)f(a)+h(v)f(b) \geqslant & f((1-v)a+vb)\\
&+h\left(\frac{v}{\tau}\right)\Big[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)\Big]
\end{aligned}$$

$$+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right),$$

where

$$\Delta_{f,h}^{(a,b)}(n,k) = h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right)+g\left(\frac{k}{2^n}\right)\right]-g\left(\frac{2k-1}{2^{n+1}}\right)$$

and $g(t) := f((1-t\tau)a+t\tau b)$.

Proof. Since, h is super-multiplicative and super-additive, we have

$$\begin{aligned}
&h(1-v)f(a)+h(v)f(b)-h\left(\frac{v}{\tau}\right)\left[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)\right]\\
=&\left(h(1-v)-h\left(\frac{v}{\tau}\right)h(1-\tau)\right)f(a)+\left(h(v)-h\left(\frac{v}{\tau}\right)h(\tau)\right)f(b)\\
&+h\left(\frac{v}{\tau}\right)f((1-\tau)a+\tau b)\\
\geqslant&h\left(1-\frac{v}{\tau}\right)f(a)+h\left(\frac{v}{\tau}\right)f((1-\tau)a+\tau b)\\
\geqslant&f\left[\left(1-\frac{v}{\tau}\right)a+\left(\frac{v}{\tau}\right)((1-\tau)a+\tau b)\right]\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right)\\
=&f((1-v)a+vb)+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{v}{\tau}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{v}{\tau}\right).
\end{aligned}$$

This completes the proof. □

Notice that in Theorem 2.2, if we ignore the sum, we can rewrite the result in the simpler form

$$\frac{h(1-v)f(a)+h(v)f(b)-f((1-v)a+vb)}{h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)}\geqslant h\left(\frac{v}{\tau}\right).$$

This form is easier to view for comparison purpose with the main results in [1, 10]. Thus, Theorem 2.2 presents the h-convex version with multiple term refinements of the main results in these references.

On the other hand, a reverse of Theorem 2.2 can be stated as follows. We, once again, refer the reader to [1, 10] where this type of inequalities was treated in its simplest form for convex functions.

Theorem 2.3. *Let h be a non-negative multiplicative and super-additive function on $[0,+\infty)$. If f is h-convex on $[a,b]$ and $0<v\leqslant\tau<1$ then for all positive integer N*

$$\begin{aligned}
&h(1-v)f(a)+h(v)f(b)\\
&\leqslant f((1-v)a+vb)+h\left(\frac{1-v}{1-\tau}\right)\left[h(1-\tau)f(a)+h(\tau)f(b)-f((1-\tau)a+\tau b)\right]
\end{aligned}\tag{2.4}$$

$$-\sum_{n=0}^{N-1} h\left(2\left(\frac{1-\tau}{1-v}\right) r_n\left(\frac{1-\tau}{1-v}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right),$$

where

$$\Delta_{f,h}^{(a,b)}(n,k)=h\left(\frac{1}{2}\right)\left[g\left(\frac{k-1}{2^n}\right)+g\left(\frac{k}{2^n}\right)\right]-g\left(\frac{2k-1}{2^{n+1}}\right)$$

and $g(t):=f((1-t(1-v))a+t(1-v)b)$.

Proof. Since, h is multiplicative and super-additive, we have

$$\begin{aligned}
&h(1-\tau)f(a)+h(\tau)f(b)-\frac{f(1-v)}{h\left(\frac{1-v}{1-\tau}\right)}f(a)-\frac{f(v)}{h\left(\frac{1-v}{1-\tau}\right)}f(b)+\frac{1}{h\left(\frac{1-v}{1-\tau}\right)}f((1-v)a+vb)\\
=&\left(h(1-\tau)-h(1-\tau)\right)f(a)+\left(h(\tau)-h\left(\frac{v(1-\tau)}{1-v}\right)\right)f(b)\\
&+h\left(\frac{1-\tau}{1-v}\right)f((1-v)a+vb)\\
\geqslant&h\left(1-\frac{1-\tau}{1-v}\right)f(b)+h\left(\frac{1-\tau}{1-v}\right)f((1-v)a+vb)\\
\geqslant&f\left[\left(1-\frac{1-\tau}{1-v}\right)b+\left(\frac{1-\tau}{1-v}\right)((1-v)a+vb)\right]\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{1-\tau}{1-v}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right)\\
=&f((1-\tau)a+\tau b)\\
&+\sum_{n=0}^{N-1} h\left(2r_n\left(\frac{1-\tau}{1-v}\right)\right)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{1-\tau}{1-v}\right).
\end{aligned}$$

Multiplying the last inequality by $h\left(\frac{1-v}{1-\tau}\right)$, the desired inequality is obtained. □

While the above results treat the values $0\leqslant v\leqslant 1$, it has been of interest in the literature to deal with the cases $v\notin[0,1]$. We refer the reader to [2, 13] for related discussion when $v\geqslant 1$ or $v\leqslant 0$. In the following two results, this is treated for h-convex functions, where multiple-term refinements are provided.

Theorem 2.4. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, f be h-convex on $\mathbb{R}$ and $v\geqslant 0$. If N is a positive integer, then*

$$\begin{aligned}
&h(v+1)f(b)-h(v)f(a)\\
&+\sum_{n=0}^{N-1} h(v+1)h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big)\\
\leqslant&f\left((1+v)b-va\right),
\end{aligned}$$

where $g(t):=f((1-(t+v)a+t(1+v)b)$.

Proof. Notice first that for $v \geqslant 0$, one has

$$b = \frac{v}{v+1}a + \frac{1}{v+1}\left((1+v)b - va\right).$$

h-convexity of f and Theorem 2.1, implies

$$\begin{aligned} f(b) \leqslant & h\Big(\frac{v}{v+1}\Big)f(a) + h\Big(\frac{1}{v+1}\Big)f\left((1+v)b - va\right) \\ & - \sum_{n=0}^{N-1} h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big) \\ \leqslant & \frac{h(v)}{h(v+1)}f(a) + \frac{1}{h(v+1)}f\left((1+v)b - va\right) \\ & - \sum_{n=0}^{N-1} h(v+1)h\Big(2r_n\Big(\frac{1}{v+1}\Big)\Big)\sum_{k=1}^{2^n}\Delta_{f,h}^{(a,b)}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\Big(\frac{1}{v+1}\Big). \end{aligned}$$

This completes the proof. □

A more straightforward form of Theorem 2.4 can be stated as follows.

Theorem 2.5. *Let h be a non-negative super-multiplicative and super-additive function on $[0,+\infty)$, and $f:\mathbb{R}\to\mathbb{R}$ be h-convex. If N is a positive integer and $a<b$, then*

$$\begin{aligned} & h(1+v)f(a) - h(v)f(b) \\ & + \sum_{k=1}^{N} h(2^k v)\left[h\left(\frac{1}{2}\right)\left(f(a) + f\left(\frac{\left(2^{k-1}-1\right)a+b}{2^{k-1}}\right)\right) - f\left(\frac{\left(2^{k}-1\right)a+b}{2^{k}}\right)\right] \\ \text{(2.5)}\quad & \leqslant f((1+v)a - vb), \end{aligned}$$

where $v \geqslant 0$.

Proof. We proceed by induction on N. So, assume that f is h-convex, $a<b$ and $v \geqslant 0$. We have

$$\begin{aligned} & h(1+v)f(a) - h(v)f(b) + h(2v)\left[h\left(\frac{1}{2}\right)(f(a)+f(b)) - f\left(\frac{a+b}{2}\right)\right] \\ = & \left(h(1+v)+h(v)\right)f(a) + \left(h(2v)h\left(\frac{1}{2}\right) - h(v)\right)f(b) - h(2v)f\left(\frac{a+b}{2}\right) \\ \leqslant & h(1+2v)f(a) - h(2v)f\left(\frac{a+b}{2}\right) \\ \leqslant & f\left((1+2v)a - 2v\frac{a+b}{2}\right) \\ = & f((1+v)a - vb), \end{aligned}$$

where we have applied Theorem 2.4, with v and b replaced by $2v$ and $\frac{a+b}{2}$, respectively. We emphasize here that when $a<b$ we have $a<\frac{a+b}{2}$. Moreover, when $v\geqslant 0$ we have $2v\geqslant 0$, justifying the application of Theorem 2.4.

Now assume that, for some $N\in\mathbb{N}$, (2.5) holds whenever $a<b$ and $v\geqslant 0$. We assert the truth of the inequality for $N+1$. Observe that

$$\begin{aligned}
A =& h(1+v)f(a)-h(v)f(b)\\
&+\sum_{k=1}^{N+1} h(2^k v)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1\right)a+b}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b}{2^{k}}\right)\right]\\
=&h(1+v)f(a)-h(v)f(b)+h(2v)\left[h\left(\frac{1}{2}\right)\Big(f(a)+f(b)\Big)-f\left(\frac{a+b}{2}\right)\right]\\
&+\sum_{k=2}^{N+1} h(2^k v)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1)a+b\right.}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b}{2^{k}}\right)\right]\\
=&h(1+2v)f(a)-h(2v)f\left(\frac{a+b}{2}\right)\\
&+\sum_{k=1}^{N+1} h(2^{k+1} v)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{(2^{k}-1)a+b}{2^{k}}\right)\right)-f\left(\frac{\left(2^{k+1}-1\right)a+b'}{2^{k+1}}\right)\right]. \qquad (2.6)
\end{aligned}$$

For simplicity, let $2v=r$, $\frac{a+b}{2}=b'$. Then (2.6) becomes

$$\begin{aligned}
A =& h(1+r)f(a)-h(r)f\left(b'\right)\\
&+\sum_{k=1}^{N} h(2^k r)\left[h\left(\frac{1}{2}\right)\left(f(a)+f\left(\frac{\left(2^{k-1}-1\right)a+b'}{2^{k-1}}\right)\right)-f\left(\frac{\left(2^{k}-1\right)a+b'}{2^{k}}\right)\right]\\
\leqslant& f\left((1+r)a-rb'\right)\\
=&f((1+v)a-vb),
\end{aligned}$$

where we have used the inductive step to obtain (2.6). Observe that when $a<b$ we have $a<b'$, which justifies the application of the inductive step. $\square$

Other generalized external forms for h-convex functions can be stated as follows. We remark that these results extend known results for convex functions, as one can see in [14, 18].

Theorem 2.6. *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be h-convex, $b\in\mathbb{R}$ and let $\{v_k\}$ be such that $v_k>0$ for $k=1,2,\dots,n$. If $\{b_k\}\subset\mathbb{R}$, and h is a non-negative super-multiplicative function on $[0,+\infty)$, then*

$$h(1+\beta)h(a)-\sum_{k=1}^{n} h\left(v_k\right)f\left(b_k\right)\leqslant f\left((1+\beta)a-\sum_{k=1}^{n} v_k b_k\right),$$

where $\sum_{k=1}^{n} v_k=\beta$.

Proof. Notice first that for $s \geqslant 0$ and $x, y \in \mathbb{R}$ one has

$$y = \frac{s}{s+1}x + \frac{1}{s+1}\left((1+s)y - sx\right).$$

h-Convexity of f implies

$$\begin{aligned} f(y) &\leqslant h\Big(\frac{s}{s+1}\Big)f(x) + h\Big(\frac{1}{s+1}\Big)f\left((1+s)y - sx\right) \\ &\leqslant \frac{h(s)}{h(s+1)}f(x) + \frac{1}{h(s+1)}f\left((1+s)y - sx\right), \end{aligned}$$

which implies

$$h(s+1)f(y) - f\left((1+s)y - sx\right) \leqslant h(s)f(x). \tag{2.7}$$

Now, applying (2.7), we have

$$\begin{aligned} &h(1+\beta)h(a) - f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right) \\ =&h(1+\beta)h(a) - f\left((1+\beta)a - \beta\sum_{k=1}^{n}\frac{v_k}{\beta}b_k\right) \\ \leqslant&h(1+\beta)h(a) - h(1+\beta)h(a) + h(\beta)f\left(\sum_{k=1}^{n}\frac{v_k}{\beta}b_k\right) \\ \leqslant&h(\beta)\sum_{k=1}^{n}h\left(\frac{v_k}{\beta}\right)f(b_k) \\ \leqslant&\sum_{k=1}^{n}h\left(v_k\right)f(v_k). \end{aligned}$$

This completes the proof. ∎

In the following result, we present a one-term refinement of Theorem 2.6.

Theorem 2.7. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be h-convex, $b \in \mathbb{R}$ and let $\{v_k\}$ be such that $v_k > 0$ for $k = 1, 2, \ldots, n$. If $\{b_k\} \subset \mathbb{R}$, and h is a non-negative super-multiplicative and super-additive function on $[0, +\infty)$, then we have*

$$\begin{aligned} &h(1+\beta)h(a) - \sum_{k=1}^{n} h\left(v_k\right)f\left(b_k\right) \\ \leqslant&f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right) \\ &- h\left((n+1)r_0\right)\left[h\left(\frac{1}{n+1}\right)\left(f(a) + \sum_{k=1}^{n} f(b_k)\right) - f\left(\frac{a + \sum_{k=1}^{n} b_k}{n+1}\right)\right], \end{aligned}$$

where $\sum_{k=1}^{n} v_k = \beta$ and $r_0 = \min\{v_1, v_2, \ldots, v_n\}$.

Proof. Since h is super-multiplicative and super-additive, we have

$$\begin{aligned}
I :=& h(1+\beta)f(a) - \sum_{k=1}^{n} h(v_k) f(b_k) \\
&+ h((n+1)r_0)\left[h\left(\frac{1}{n+1}\right)\left(f(a) + \sum_{k=1}^{n} f(b_k)\right) - f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right)\right] \\
\leqslant& \left(h(1+\beta) + h((n+1)r_0)h\left(\frac{1}{n+1}\right)\right)f(a) \\
&+ \sum_{k=1}^{n}(-h(v_k)+h(r_0))f(b_k) - h((n+1)r_0)f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right) \\
\leqslant& h(1+\beta+r_0)f(a) - \sum_{k=1}^{n} h(v_k - r_0) f(b_k) - h((n+1)r_0) f\left(\frac{a+\sum_{k=1}^{n} b_k}{n+1}\right).
\end{aligned}$$

Let $\gamma_{n+1} = (n+1)r_0$ and for $1 \leqslant k \leqslant n$, let $\gamma_k = v_k - r_0$. Further, denote $\beta + r_0$ by λ. Then

$$\sum_{k=1}^{n} \gamma_k + \gamma_{n+1} = \beta + r_0 = \lambda.$$

Therefore, we may apply Theorem 2.6, we obtain

$$\begin{aligned}
I \leqslant& f\left((1+\beta+r_0)a - \sum_{k=1}^{n}(v_k - r_0)b_k - (n+1)r_0 \frac{a+\sum_{k=1}^{n} b_k}{n+1}\right) \\
=& f\left((1+\beta)a - \sum_{k=1}^{n} v_k b_k\right).
\end{aligned}$$

□

3. Refinement and Reverse of Jensen's Inequality for (p,h)-Convex Function

In this part of the paper, we present our main results concerning (p,h)-convex functions. The applications of these inequalities and their relations to the literature will be done in Remark 3.1 and in the last section.

In the following result, we present a one-term refinement of Jensen's type inequality for (p,h)-convex function and its reverse.

Before we state our first result, we remind the reader of the following lemma, which was shown in [8].

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be convex, $\{x_1, \dots, x_n\} \subset I$ and $\{p_1, \dots, p_n\} \subset (0,1)$ be such that $\sum_{i=1}^{n} p_i = 1$. Then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) + np_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \leqslant \sum_{i=1}^{n} p_i f(x_i) \tag{3.1}$$

and

$$f\left(\sum_{i=1}^{n} p_i x_i\right) + np_{\max}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \geqslant \sum_{i=1}^{n} p_i f(x_i), \tag{3.2}$$

where $p_{\min} = \min\{p_1, \dots, p_n\}$ *and* $p_{\max} = \max\{p_1, \dots, p_n\}$.

The following result presents the (p,h)-convex version of the above lemma.

Theorem 3.1. *Let* $v_1, \dots, v_n$ *be positive real numbers,* $n \geqslant 2$*, such that* $\sum_{k=1}^n v_k = 1$*. Let* f *be a* (p,h)*-convex function, and* $x_1, \dots, x_n \in I$*.*

(a) *If* h *is a non-negative super-multiplicative and super-additive function on* $[0,1]$*, then*

$$f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right)+h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(x_k\right)-f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right)\leqslant\sum_{k=1}^n h\left(v_k\right)f\left(x_k\right),$$

where $r_0 = \min\{v_1, v_2, \dots, v_n\}$

(b) *If* h *is a non-negative multiplicative and super-additive function on* $[0,+\infty)$*, then*

$$f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right)+h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(x_k\right)-f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right)\geqslant\sum_{k=1}^n h\left(v_k\right)f\left(x_k\right),$$

where $R_0 = \max\{v_1, v_2, \dots, v_n\}$.

Proof. We prove the first inequality. Since, h is a super-multiplicative and super-additive function, we have

$$\begin{aligned}
&\sum_{k=1}^n h\left(v_k\right)f\left(x_k\right)-h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(x_k\right)-f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\right)\\
=&\sum_{k=1}^n\left(h\left(v_k\right)-h(nr_0)h\left(\frac{1}{n}\right)\right)f\left(x_k\right)+h(nr_0)f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\\
\geqslant&\sum_{k=1}^n h\left(v_k-r_0\right)f\left(x_k\right)+h(nr_0)f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right)\\
\geqslant&f\left[\left(\sum_{k=1}^n\left(v_k-r_0\right)x_k^p+nr_0\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{\frac{1}{p}}\right]\\
=&f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{\frac{1}{p}}\right),
\end{aligned}$$

where the last inequality follows by the Jensen's inequality for the (p,h)-convex function f. The second inequality is equivalent to the following inequality

$$\sum_{k=1}^n\left(\frac{h\left(nR_0\right)h\left(\frac{1}{n}\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f(x_k)+f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{1/p}\right)\frac{1}{k\left(nR_0\right)}\geqslant f\left(\left(\frac{1}{n}\sum_{k=1}^n x_k^p\right)^{1/p}\right).$$

Since, h is a multiplicative and super-additive function, we have

$$\sum_{k=1}^n\left(\frac{h\left(nR_0\right)h\left(\frac{1}{n}\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f\left(x_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^n v_k x_k^p\right)^{1/p}\right)$$

$$
\begin{aligned}
&\geqslant \sum_{k=1}^{n}\left(\frac{h\left(R_0\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right) f\left(x_k\right)+\frac{1}{h\left(nR_0\right)} f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right) \\
&\geqslant \sum_{k=1}^{n}\left(\frac{h\left(R_0-v_k\right)}{h\left(nR_0\right)}\right) f\left(x_k\right)+\frac{1}{h\left(nR_0\right)} f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right) \\
&\geqslant \sum_{k=1}^{n} h\left(\frac{R_0-v_k}{nR_0}\right) f\left(x_k\right)+h\left(\frac{1}{nR_0}\right) f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right) \\
&\geqslant f\left(\left(\sum_{k=1}^{n} \frac{R_0-v_k}{nR_0} x_k^p+\frac{1}{nR_0} \sum_{k=1}^{n} v_k x_k^p\right)^{1/p}\right) \\
&= f\left(\sum_{k=1}^{n}\left(\frac{R_0-v_k}{nR_0}+\frac{v_k}{nR_0}\right) x_k^p\right)^{1/p} \\
&= f\left(\left(\frac{1}{n} \sum_{k=1}^{n} x_k^p\right)^{1/p}\right),
\end{aligned}
$$

where the last inequality follows by the Jensen's inequality for the (p,h)-convex function f. □

For further generalisation of Theorem 3.1, we need the following lemma.

Lemma 3.2 ([5]). *Let ϕ be a strictly increasing convex function defined on an interval I. If x, y, z and w are points in I such that $z-w \leqslant x-y$, where $w \leqslant z \leqslant x$ and $y \leqslant x$, then*

$$0 \leqslant \phi(z)-\phi(w) \leqslant \phi(x)-\phi(y).$$

This lemma will be simply used to prove the following generalization of Theorem 3.1.

Theorem 3.2. *Let $v_1, \ldots, v_n$ be positive real numbers, $n \geqslant 2$, such that $\sum_{k=1}^{n} v_k=1$. Let f be a (p,h)-convex function, $x_1, \ldots, x_n \in I$ and $\lambda \geqslant 1$.*

(a) *If h is a non-negative super-multiplicative and super-additive function on $[0,1]$, then*

$$
\begin{aligned}
& f^{\lambda}\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right)+h^{\lambda}(nr_0)\left(\left(h\left(\frac{1}{n}\right) \sum_{k=1}^{n} f\left(x_k\right)\right)^{\lambda}-f^{\lambda}\left(\left(\frac{1}{n} \sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)\right) \\
& \leqslant\left(\sum_{k=1}^{n} h\left(v_k\right) f\left(x_k\right)\right)^{\lambda}, \qquad (3.3)
\end{aligned}
$$

where $r_0=\min\{v_1, v_2, \ldots, v_n\}$

(b) *If h is a non-negative multiplicative and super-additive function on $[0,+\infty)$, then*

$$\left(\sum_{k=1}^{n} h\left(v_k\right) f\left(x_k\right)\right)^{\lambda}$$

$$\leqslant f^{\lambda}\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right) + h^{\lambda}(nR_0)\left(\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right)^{\lambda} - f^{\lambda}\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)\right),$$

where $R_0 = \max\{v_1, v_2, \ldots, v_n\}$.

Proof. Let

$$x = \sum_{k=1}^{n} h(v_k) f(x_k), \quad y = f\left(\left(\sum_{k=1}^{n} v_k x_k^p\right)^{\frac{1}{p}}\right),$$
$$z = h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right), \quad w = h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right)$$

and

$$z' = h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^{n} f\left(x_k\right)\right), \quad w' = h(nR_0) f\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}}\right).$$

Then based on Theorem 3.1, we have

$$z - w \leqslant x - y \leqslant z' - w'.$$

The first and the second inequalities in Theorem 3.2 follow directly by applying Lemma 3.2, with $\phi(x) = x^{\lambda}$, where $\lambda \geqslant 1$ to the inequalities $z - w \leqslant x - y$, with $w \leqslant z \leqslant x$, $y \leqslant x$ and $x - y \leqslant z' - w'$ with $y \leqslant x \leqslant z'$, $w' \leqslant z'$, respectively. This completes the proof. □

Remark 3.1. Before proceeding to further results, we explain a little about Theorem 3.2. Notice that if we take $f(x) = e^x$, $h(x) = x$ and $x_i = \ln a_i$ for $a_i > 0$ we recapture Theorems 2.2 and 2.4 in [17].

4. Refinement and Reverse of Jensen's Inequality for (p, h) Operator Convex Function

Let $\mathbf{M}_\ell$ be the algebra of all complex matrices of order $\ell \times \ell$. A matrix $A \in \mathbf{M}_\ell$ is called Hermitian if $A = A^*$, where A^* is the adjoint of A. The notation $A \geqslant 0$ ($A > 0$) is used to mean that A is positive semi-definite (positive definite). If A and B are Hermitian and $A - B$ is positive semi-definite, then we write $A \geqslant B$.

In this section, we extend some results from the context of real functions and real numbers to that of matrices. In the following we suppose that $I \subset \mathbb{R}^+$ and $p > 0$.

Definition 4.1. Let $h : [0, 1] \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : I \to \mathbb{R}$ is operator (p, h)-convex or that f belongs to the class $opgx(h, p, I)$, if

$$f\left(\left[(1 - v)A^p + vB^p\right]^{\frac{1}{p}}\right) \leqslant h(1 - v) f(A) + h(v) f(B), \tag{4.1}$$

for all $A, B \in \mathbf{M}_\ell^+$ with $\sigma(A), \sigma(B) \subset I$, and $v \in (0, 1)$. Similarly, if the inequality sign in (4.1) is reversed, then f is said to be a (p, h)-concave function or belong to the class $ghv(h, p, I)$.

The matrix version of Jensen type inequality for operator (p,h)-convex functions is as follows.

Theorem 4.1. *Let h be a non-negative super-multiplicative function on $[0,1]$ and assume $f \in opgx(p,h,I)$. For $k = 1,\dots,n$, let A_k be a positive semi-definite matrix with spectrum in I and let $v_1,\dots,v_n$ be positive real numbers, such that $\sum_{k=1}^n v_k = 1$. Then*

$$f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right) \leqslant \sum_{k=1}^n h\left(v_k\right) f\left(A_k\right).$$

Theorem 4.2. *Let $f \in opgx(p,h,I)$, $A_1,\dots,A_n$ be positive semi-definite matrices in $\mathbf{M}_\ell$ with spectra in I and $v_1,\dots,v_n$ be positive real numbers, such that $\sum_{k=1}^n v_k = 1$.*

(a) *If h is a non-negative super-multiplicative and super-additive function on $[0,1]$, then*

$$\begin{aligned} &f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right) + h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(A_k\right) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right) \\ &\leqslant \sum_{k=1}^n h\left(v_k\right) f\left(A_k\right), \end{aligned}$$

where $r_0 = \min\{v_1, v_2, \dots, v_n\}$

(b) *If h is a non-negative multiplicative and super-additive function on $[0,+\infty)$, then*

$$\begin{aligned} \sum_{k=1}^n h\left(v_k\right) f\left(A_k\right) \leqslant & f\left(\left(\sum_{k=1}^n v_k A_k^p\right)^{\frac{1}{p}}\right) \\ & + h(nR_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(A_k\right) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right), \end{aligned}$$

where $R_0 = \max\{v_1, v_2, \dots, v_n\}$.

Proof. We prove the first inequality. Since, h is a super-multiplicative and super-additive function, we have

$$\begin{aligned} &\sum_{k=1}^n h\left(v_k\right) f\left(A_k\right) - h(nr_0)\left(h\left(\frac{1}{n}\right)\sum_{k=1}^n f\left(A_k\right) - f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right)\right) \\ &= \sum_{k=1}^n \left(h\left(v_k\right) - h(nr_0)h\left(\frac{1}{n}\right)\right) f\left(A_k\right) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right) \\ &\geqslant \sum_{k=1}^n h\left(v_k - r_0\right) f\left(A_k\right) + h(nr_0) f\left(\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)^{\frac{1}{p}}\right) \\ &\geqslant f\left[\left(\sum_{k=1}^n \left(v_k - r_0\right) A_k^p + nr_0\left(\frac{1}{n}\sum_{k=1}^n A_k^p\right)\right)^{\frac{1}{p}}\right] \end{aligned}$$

$$=f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{\frac{1}{p}}\right),$$

where the last inequality follows by the Jensen's inequality for the (p,h) operator convex function f. This proves the first desired inequality.

To prove the second desired inequality, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\frac{h\left(nR_0\right)h\left(\frac{1}{n}\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}\left(\frac{h\left(R_0\right)-h\left(v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}\left(\frac{h\left(R_0-v_k\right)}{h\left(nR_0\right)}\right)f\left(A_k\right)+\frac{1}{h\left(nR_0\right)}f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&\sum_{k=1}^{n}h\left(\frac{R_0-v_k}{nR_0}\right)f\left(A_k\right)+h\left(\frac{1}{nR_0}\right)f\left(\left(\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
\geqslant&f\left(\left(\sum_{k=1}^{n}\frac{R_0-v_k}{nR_0}A_k^p+\frac{1}{nR_0}\sum_{k=1}^{n} v_k A_k^p\right)^{1/p}\right)\\
=&f\left(\sum_{k=1}^{n}\left(\frac{R_0-v_k}{nR_0}+\frac{v_k}{nR_0}\right)A_k^p\right)^{1/p}\\
=&f\left(\left(\sum_{k=1}^{n}\frac{1}{n}A_k^p\right)^{1/p}\right),
\end{aligned}$$

where the last inequality follows by the Jensen's inequality for the (p,h) operator convex function f. □

References

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, *Young-type inequalities and their matrix analogues*, Linear Multilinear Algebra **63** (2015), 622–635. https://doi.org/10.1080/03081087.2014.891588

[2] M. Bakherad and M. Moslehian, *Reverses and variations of the Heinz inequality*, Linear Multilinear Algebra **63**(10) (2015), 1972–1980. https://doi.org/10.1080/03081087.2014.880433

[3] D. Choi, *Multiple-term refinements of Young type inequalities,* J. Math. (2016), Article ID 4346712. https://doi.org/10.1155/2016/4346712

[4] D. Choi, M. Krnić and J. Pecarić, *Improved Jensen-type inequalities via linear interpolation and applications,* J. Math. Inequal. **11**(2) (2017), 301–322. https://dx.doi.org/10.7153/jmi-11-27

[5] Y. Al-Manasrah and F. Kittaneh, *Further generalization refinements and reverses of the Young and Heinz inequalities,* Results Math. **19** (2016), 757–768. https://doi.org/10.1007/s00025-016-0611-2

[6] Z. B. Fang and R. Shi, *On the (p,h)-convex function and some integral inequalities,* J. Inequal. Appl. **2014** (2014), Article ID 14. https://doi.org/10.1186/1029-242X-2014-45

[7] X. Jin, B. Jin, J. Ruan and X. Ma, *Some characterization of h-convex functions,* J. Math. Inequal. **16**(2) (2022), 751–764. https://dx.doi.org/10.7153/jmi-2022-16-53
[8] F. Mitroi, *About the precision in Jensen-Steffensen inequality*, An. Univ. Craiova Ser. Mat. Inform. **37**(4) (2010), 73–84. https://doi.org/10.52846/ami.v37i4.367
[9] S. Varošanec, *h-Convexity,* Math. Anal. Appl. **326** (2007), 303–311. https://doi.org/10.1016/j.jmaa.2006.02.086
[10] M. Sababheh, *Convexity and matrix means,* Linear Algebra Appl. **506** (2016), 588–602. https://doi.org/10.1016/j.laa.2016.06.027
[11] M. Sababheh, *Log and harmonically log-convex functions related to matrix norms,* Oper. Matrices **10**(2) (2016), 453–465. https://dx.doi.org/10.7153/oam-10-26
[12] M. Sababheh, *Means refinements via convexity*, Mediterr. J. Math. **14** (2017), Article ID 125. https://doi.org/10.1007/s00009-017-0924-8
[13] M. Sababheh, *Convex functions and means of matrices*, Math. Inequal. Appl. **20**(1) (2017), 29–47. https://dx.doi.org/10.7153/mia-20-03
[14] M. Sababheh, *Extrapolation of convex functions*, Filomat **32**(1) (2018), 127–139. https://doi.org/10.2298/FIL1801127S
[15] M. Sababheh, *Interpolated inequalities for unitarily invariant norms*, Linear Algebra Appl. **475** (2015), 240–250. https://doi.org/10.1016/j.laa.2015.02.026
[16] K. S. Zhang and J. P. Wan, *p-Convex functions and their properties,* Pure Appl. Math. **23**(1) (2007), 130–133.
[17] X. T. Dinh, H. Q. Duong and H. N. Nguyen, *Two new extensions of the weighted arithmetic-geometric mean inequality via weak sub-majorization,* Indian J. Pure Appl. Math. **53** (2022), 1122–1127. https://doi.org/10.1007/s13226-022-00223-y
[18] P. Vasić and J. Pečarić, *On the Jensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634/677** (1979), 50–54. https://www.jstor.org/stable/43668091

Sciences and Technologies Team (ESTE),
Higher School of Education and Training of El Jadida,
Chouaib Doukkali University, El Jadida, Morocco
Email address: mohamedamineighachane@gmail.com

Chouaib Doukkali University,
El Jadida, Morocco
Email address: lakhlifasadek@gmail.com

[3]Princess Sumaya University for Technology,
Amman, Jordan
Email address: sababheh@yahoo.com