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WEAVING CONTINUOUS CONTROLLED K-g-FUSION FRAMES IN HILBERT SPACES

PRASENJIT GHOSH[1] AND TAPAS K. SAMANTA[2]

ABSTRACT. We introduce the notion of weaving continuous controlled K-g-fusion frame in Hilbert space. Some characterizations of weaving continuous controlled K-g-fusion frame have been presented. We extend some of the recent results of woven K-g-fusion frame and controlled K-g-fusion frame to woven continuous controlled K-g-fusion frame. Finally, a perturbation result of woven continuous controlled K-g-fusion frame has been studied.

1. INTRODUCTION AND PRELIMINARIES

Duffin and Schaeffer [13] introduced frame for Hilbert space to study some fundamental problems in non-harmonic Fourier series. Later on, after some decades, frame theory was popularized by Daubechies et al. [11]. At present, frame theory has been widely used in signal and image processing, filter bank theory, coding and communications, system modeling and so on.

Let H be a separable Hilbert space associated with the inner product $\langle \cdot, \cdot \rangle$. Frame for Hilbert space was defined as a sequence of basis-like elements in Hilbert space. A sequence $\{f_i\}_{i=1}^{+\infty} \subset H$ is called a frame for H, if there exist positive constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{+\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.$$

The constants A and B are called lower and upper bounds, respectively.

Key words and phrases. Frame, g-fusion frame, continuous g-fusion frame, controlled frame, woven frame.

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Throughout this paper, H is considered to be a separable Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$ and $\mathbb{H}$ is the collection of all closed subspaces of H. (X, μ) denotes abstract measure space with positive measure μ. I_H is the identity operator on H. $\mathcal{B}(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2. In particular, $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H. For $S \in \mathcal{B}(H)$, we denote $\mathcal{N}(S)$ and $\mathcal{R}(S)$ for null space and range of S, respectively. Also, $P_M \in \mathcal{B}(H)$ is the orthonormal projection of H onto a closed subspace $M \subset H$. The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $R, S \in \mathcal{S}(H)$

$$R \leq S \Leftrightarrow \langle Rf, f \rangle \leq \langle Sf, f \rangle, \quad \text{for all } f \in H.$$

$\mathcal{GB}(H)$ denotes the set of all bounded linear operators which have bounded inverse. If $S, R \in \mathcal{GB}(H)$, then R^*, R^{-1} and SR also belongs to $\mathcal{GB}(H)$. An operator $U \in \mathcal{B}(H)$ is called positive if $\langle Uf, f \rangle \geq 0$ for all $f \in H$. In notation, we can write $U \geq 0$. If $V \in B(H)$ is positive then there exists a unique positive U such that $V^2 = U$. This will be denoted by $V = U^{1/2}$. Moreover, if an operator V commutes with U then V commutes with every operator in the C^*-algebra generated by U and I, specially V commutes with $U^{1/2}$. $\mathcal{GB}^+(H)$ is the set of all positive operators in $\mathcal{GB}(H)$ and T, U are invertible operators in $\mathcal{GB}(H)$. For each $m > 1$, we define $[m] = \{1, 2, \ldots, m\}$.

We present some theorems in operator theory which will be needed throughout this paper.

Theorem 1.1 (Douglas' factorization theorem [12]). *Let $S, V \in \mathcal{B}(H)$. Then the following conditions are equivalent.*

(i) $\mathcal{R}(S) \subseteq \mathcal{R}(V)$.
(ii) $SS^* \leq \lambda^2 VV^*$ *for some* $\lambda > 0$.
(iii) $S = VW$ *for some bounded linear operator* W *on* H.

Theorem 1.2 ([15]). *Let $M \subset H$ be a closed subspace and $T \in \mathcal{B}(H)$. Then $P_M T^* = P_M T^* P_{\overline{TM}}$. If T is an unitary operator (i.e., $T^*T = I_H$), then $P_{\overline{TM}} T = T P_M$.*

Theorem 1.3 ([8]). *Let H_1, H_2 be two Hilbert spaces and $U : H_1 \to H_2$ be a bounded linear operator with closed range $\mathcal{R}_U$. Then, there exists a bounded linear operator $U^\dagger : H_2 \to H_1$ such that $UU^\dagger x = x$ for all $x \in \mathcal{R}_U$.*

1.1. ***K*-*g*-fusion frame.** Construction of K-g-fusion frames and their dual were presented by Sadri and Rahimi [1] to generalize the theory of K-frame [16], fusion frame [9], and g-frame [35].

Definition 1.1 ([1]). Let $\{W_j\}_{j \in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j \in J}$ be a collection of positive weights, $\{H_j\}_{j \in J}$ be a sequence of Hilbert spaces. Suppose $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$ and $K \in \mathcal{B}(H)$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is called a K-g-fusion frame for H respect to $\{H_j\}_{j \in J}$ if there exist constants $0 < A \leq B < +\infty$

such that

$$A\|K^*f\|^2 \leq \sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j}(f)\right\|^2 \leq B\|f\|^2,$$

for all $f \in H$. The constants A and B are called the lower and upper bounds of K-g-fusion frame, respectively. If $K = I_H$ then the family is called g-fusion frame and it has been widely studied in [18–20, 31].

Define the space

$$\ell^2\left(\{H_j\}_{j\in J}\right) = \Big\{\{f_j\}_{j\in J} : f_j \in H_j, \sum_{j\in J}\|f_j\|^2 < +\infty\Big\},$$

with inner product given by

$$\langle \{f_j\}_{j\in J}, \{g_j\}_{j\in J}\rangle = \sum_{j\in J}\langle f_j, g_j\rangle_{H_j}.$$

Clearly, $\ell^2\left(\{H_j\}_{j\in J}\right)$ is a Hilbert space with the pointwise operations [1].

1.2. **Controlled K-g-fusion frame.** Controlled frame is one of the newest generalization of frame. P. Balaz et al. [6] introduced controlled frame to improve the numerical efficiency of interactive algorithms for inverting the frame operator. In recent times, several generalizations of controlled frame namely, controlled K-frame [26], controlled g-frame [27], controlled fusion frame [23], controlled g-fusion frame [34], controlled K-g-fusion frame [28] etc. have been appeared.

Definition 1.2 ([28])**.** Let $K \in \mathcal{B}(H)$ and $\{W_j\}_{j\in J}$ be a collection of closed subspaces of H and $\{v_j\}_{j\in J}$ be a collection of positive weights. Let $\{H_j\}_{j\in J}$ be a sequence of Hilbert spaces, $T, U \in \mathcal{G}\mathcal{B}(H)$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in J$. Then the family $\Lambda_{TU} = \{(W_j, \Lambda_j, v_j)\}_{j\in J}$ is a (T,U)-controlled K-g-fusion frame for H if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|K^*f\|^2 \leq \sum_{j\in J} v_j^2 \left\langle \Lambda_j P_{W_j}Uf, \Lambda_j P_{W_j}Tf\right\rangle \leq B\|f\|^2, \tag{1.1}$$

for all $f \in H$. If Λ_{TU} satisfies only the right inequality of (1.1) it is called a (T,U)-controlled g-fusion Bessel sequence in H.

Let Λ_{TU} be a (T,U)-controlled g-fusion Bessel sequence in H with a bound B. The synthesis operator $T_C : \mathcal{K}_{\Lambda_j} \to H$ is defined as

$$T_C\left(\left\{v_j\left(T^*P_{W_j}\Lambda_j^*\Lambda_jP_{W_j}U\right)^{1/2}f\right\}_{j\in J}\right) = \sum_{j\in J} v_j^2 T^*P_{W_j}\Lambda_j^*\Lambda_jP_{W_j}Uf,$$

for all $f \in H$ and the analysis operator $T_C^* : H \to \mathcal{K}_{\Lambda_j}$ is given by

$$T_C^*f = \left\{v_j\left(T^*P_{W_j}\Lambda_j^*\Lambda_jP_{W_j}U\right)^{1/2}f\right\}_{j\in J}, \quad \text{for all } f \in H,$$

where

$$\mathcal{K}_{\Lambda_j} = \left\{ \left\{ v_j \left(T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U \right)^{1/2} f \right\}_{j \in J} : f \in H \right\} \subset \ell^2 \left(\{H_j\}_{j \in J} \right).$$

The frame operator $S_C : H \to H$ is defined as follows:

$$S_C f = T_C T_C^* f = \sum_{j \in J} v_j^2 T^* P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} U f,$$

for all $f \in H$ and it is easy to verify that

$$\langle S_C f, f \rangle = \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j} U f, \Lambda_j P_{W_j} T f \right\rangle,$$

for all $f \in H$. Furthermore, if Λ_{TU} is a (T, U)-controlled K-g-fusion frame with bounds A and B, then $AKK^* \leq S_C \leq BI_H$.

1.3. **Continuous controlled g-fusion frame.** In recent times, controlled frames and their generalizations are also studied in continuous case by many researchers. P. Ghosh and T. K. Samanta studied continuous version of controlled g-fusion frame in [21].

Definition 1.3 ([21]). Let $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{GB}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda_x, v(x))\}_{x \in X}$ is called a continuous (T, U)-controlled generalized fusion frame or continuous (T, U)-controlled g-fusion frame for H with respect to (X, μ) and v, if

(i) for each $f \in H$, the mapping $x \mapsto P_{F(x)}(f)$ is measurable (i.e., is weakly measurable);

(ii) there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \int_X v^2(x) \left\langle \Lambda_x P_{F(x)} U f, \Lambda_x P_{F(x)} T f \right\rangle d\mu_x \leq B\|f\|^2, \tag{1.2}$$

for all $f \in H$, where $P_{F(x)}$ is the orthogonal projection of H onto the subspace $F(x)$. The constants A, B are called the frame bounds. If only the right inequality of (1.2) holds then Λ_{TU} is called a continuous (T, U)-controlled g-fusion Bessel family for H.

Let Λ_{TU} be a continuous (T, U)-controlled g-fusion Bessel family for H. Then the operator $S_C : H \to H$ defined by

$$\langle S_C f, g \rangle = \int_X v^2(x) \left\langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \right\rangle d\mu_x,$$

for all $f, g \in H$, is called the frame operator. If Λ_{TU} is a continuous (T, U)-controlled g-fusion frame for H, then from (1.2), we get

$$A \langle f, f \rangle \leq \langle S_C f, f \rangle \leq B \langle f, f \rangle, \quad \text{for all } f \in H.$$

The bounded linear operator $T_C : L^2(X, K) \to H$ defined by

$$\langle T_C \Phi, g \rangle = \int_X v^2(x) \left\langle T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U f, g \right\rangle d\mu_x,$$

where for all $f \in H$, $\Phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda_x^* \Lambda_x P_{F(x)} U \right)^{1/2} f \right\}_{x \in X}$ and $g \in H$, is called synthesis operator and its adjoint operator is called analysis operator.

1.4. **Weaving frame.** Woven frame is a new notion in frame theory which has been introduced by Bemrose et al. [7]. Two frames $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ for H are called woven if there exist constants $0 < A \leq B < +\infty$ such that for any subset $\sigma \subset I$ the family $\{f_i\}_{i \in \sigma} \cup \{g_i\}_{i \in \sigma^c}$ is a frame for H. This frame has been generalized for the discrete as well as the continuous case such as woven fusion frame [17], woven g-frame [24], woven g-fusion frame [25], woven K-g-fusion frame [32], continuous weaving frame [36], continuous weaving fusion frame [33], continuous weaving g-frames [3], weaving continuous K-g-frames [5], controlled weaving frames [29], continuous controlled K-g-frames [30] etc.

In this paper, woven continuous controlled K-g-fusion frame in Hilbert spaces is presented and some of their properties are going to be established. We discuss sufficient conditions for weaving continuous controlled K-g-fusion frame. Construction of woven continuous controlled K-g-fusion frame by bounded linear operator is given. At the end, we discuss a perturbation result of woven continuous controlled K-g-fusion frame.

2. Weaving Continuous Controlled K-g-Fusion Frame

In this section, we first give the continuous version of controlled K-g-fusion frame for H and then present weaving continuous controlled K-g-fusion frame for H.

Definition 2.1. Let $K \in \mathcal{B}(H)$ and $F : X \to \mathbb{H}$ be a mapping, $v : X \to \mathbb{R}^+$ be a measurable function and $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda(x) \in \mathcal{B}(F(x), K_x)$ and $T, U \in \mathcal{GB}^+(H)$. Then $\Lambda_{TU} = \{(F(x), \Lambda(x), v(x))\}_{x \in X}$ is called a continuous (T, U)-controlled K-g-fusion frame for H with respect to (X, μ) and v, if

(i) for each $f \in H$, the mapping $x \mapsto P_{F(x)}(f)$ is measurable (i.e., is weakly measurable);

(ii) there exist constants $0 < A \leq B < +\infty$ such that

$$A \|K^* f\|^2 \leq \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \leq B \|f\|^2, \tag{2.1}$$

for all $f \in H$, where $P_{F(x)}$ is the orthogonal projection of H onto the subspace $F(x)$. The constants A, B are called the frame bounds.

Now, we consider the following cases.

(i) If only the right inequality of (2.1) holds, then Λ_{TU} is called a continuous (T, U)-controlled K-g-fusion Bessel family for H.

(*ii*) If $U = I_H$, then Λ_{TU} is called a continuous (T, I_H)-controlled K-g-fusion frame for H.
(*iii*) If $T = U = I_H$, then Λ_{TU} is called a continuous K-g-fusion frame for H (for more details, refer to [4]).
(*iv*) If $K = I_H$, then Λ_{TU} is called a continuous (T, U)-controlled g-fusion frame for H.

Remark 2.1. If the measure space $X = \mathbb{N}$ and μ is the counting measure then a continuous (T, U)-controlled K-g-fusion frame will be the discrete (T, U)-controlled K-g-fusion frame.

2.0.1. *Example.* Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an standard orthonormal basis for H. Consider

$$\mathcal{B} = \left\{x \in \mathbb{R}^3 : \|x\| \leq 1\right\}.$$

Then it is a measure space equipped with the Lebesgue measure μ. Let us now consider that $\{B_1, B_2, B_3\}$ is a partition of $\mathcal{B}$ where $\mu(B_1) \geq \mu(B_2) \geq \mu(B_3) > 1$. Let $\mathbb{H} = \{W_1, W_2, W_3\}$, where $W_1 = \overline{\text{Span}}\,\{e_1, e_2\}$, $W_2 = \overline{\text{Span}}\,\{e_2, e_3\}$ and $W_3 = \overline{\text{Span}}\,\{e_1, e_3\}$. Define $F : \mathcal{B} \to \mathbb{H}$ by

$$F(x) = \begin{cases} W_1, & \text{if } x \in B_1, \\ W_2, & \text{if } x \in B_2, \\ W_3, & \text{if } x \in B_3, \end{cases}$$

and $v : \mathcal{B} \to [0, +\infty)$ by

$$v(x) = \begin{cases} 1, & \text{if } x \in B_1, \\ 2, & \text{if } x \in B_2, \\ -1, & \text{if } x \in B_3. \end{cases}$$

It is easy to verify that F and v are measurable functions. For each $x \in \mathcal{B}$, define the operators

$$\Lambda(x)(f) = \frac{1}{\sqrt{\mu(B_k)}} \langle f, e_k \rangle\, e_k,$$

$f \in H$, where k is such that $x \in \mathcal{B}_k$ and $K : H \to H$ by

$$Ke_1 = e_1, \quad Ke_2 = e_2, \quad Ke_3 = 0.$$

It is easy to verify that $K^* e_1 = e_1$, $K^* e_2 = e_2$, $K^* e_3 = 0$. Now, for any $f \in H$, we have

$$\|K^* f\|^2 = \left\|\sum_{i=1}^{3} \langle f, e_k \rangle\, K^* e_k\right\|^2 = |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2 \leq \|f\|^2.$$

Let $T\,(f_1, f_2, f_3) = (5 f_1, 4 f_2, 5 f_3)$ and $U\,(f_1, f_2, f_3) = \left(\frac{f_1}{6}, \frac{f_2}{3}, \frac{f_3}{6}\right)$ be two operators on H. Then it is easy to verify that $T, U \in \mathcal{GB}^+(H)$ and $TU = UT$. Now, for any

$f = (f_1, f_2, f_3) \in H$, we have

$$\begin{aligned}&\int_{\mathcal{B}} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ =& \sum_{i=1}^{3} \int_{\mathcal{B}_i} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ =& \frac{5}{6} f_1^2 + \frac{16}{3} f_2^2 + \frac{5}{6} f_3^2.\end{aligned}$$

This implies that

$$\frac{5}{6} \|K^* f\|^2 \leq \int_{\mathcal{B}} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \leq \frac{16}{3} \|f\|^2.$$

Thus, Λ_{TU} be a continuous (T, U)-controlled K-g-fusion frame for $\mathbb{R}^3$.

Now, we present woven continuous controlled K-g-fusion frame for H.

Definition 2.2. A family of continuous (T, U)-controlled K-g-fusion frames given by $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in X}$ for H is said to be woven continuous (T, U)-controlled K-g-fusion frame if there exist universal positive constants $0 < A \leq B < +\infty$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ is a continuous (T, U)-controlled K-g-fusion frame for H with bounds A and B.

Each family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ is called a weaving continuous (T, U)-controlled K-g-fusion frame. For abbreviation, we use W. C. C. K. G. F. F. instead of the statement of woven continuous (T, U)-controlled K-g-fusion frame.

In the following proposition, we will see that every woven continuous controlled K-g-fusion frame has a universal upper bound.

Proposition 2.1. *Suppose for each $i \in [m]$, $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x\in X}$ be a continuous (T, U)-controlled K-g-fusion Bessel family for H with bound B_i. Then for any partition $\{\sigma_i\}_{i\in[m]}$ of X, the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ is a continuous (T, U)-controlled K-g-fusion Bessel family for H.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be a arbitrary partition of X. For each $f \in H$, we have

$$\begin{aligned}&\sum_{i\in[m]} \int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\ \leq& \sum_{i\in[m]} \int_{X} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\ \leq& \left(\sum_{i\in[m]} B_i \right) \|f\|^2.\end{aligned}$$

This completes the proof. □

Next, we give a characterization of W. C. C. K. G. F. F. for H in terms of an operator.

Theorem 2.1. *Let the families given by* $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x \in X}$ *and* $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x \in X}$ *be continuous* (T, U)*-controlled* K*-g-fusion frames for* H*. The the following statements are equivalent.*

(i) Λ *and* Γ *are W. C. C. K. G. F. F. for* H*.*

(ii) *For each partition* σ *of* X*, there exist* $\alpha > 0$ *and a bounded linear operator* $\Theta_\sigma : L^2_\sigma(X, K) \to H$ *defined by*

$$\langle \Theta_\sigma \Phi, g \rangle = \int_\sigma v^2(x) \left\langle T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U f, g \right\rangle d\mu_x + \int_{\sigma^c} v^2(x) \left\langle T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U f, g \right\rangle d\mu_x,$$

$g \in H$ *such that* $\alpha K K^* \leq \Theta_\sigma \Theta_\sigma^*$*, where*

$$L^2_\sigma(X, K) = \left\{ \Phi = \phi \cup \psi : \int_X \|\Phi\|^2 d\mu < +\infty \right\},$$

where for all $f \in H$*,*

$$\phi = \left\{ v(x) \left(T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U \right)^{1/2} f \right\}_{x \in \sigma}$$

and

$$\psi = \left\{ v(x) \left(T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U \right)^{1/2} f \right\}_{x \in \sigma^c}.$$

Proof. $(i) \Rightarrow (ii)$ Suppose that A and B are the universal lower and upper bounds for Λ and Γ. Take $\Theta_\sigma = T_C^\sigma$, for every partition σ of X, where T_C^σ is the synthesis operator of

$$\{(F(x), \Lambda(x), v(x))\}_{x \in \sigma} \cup \{(G(x), \Lambda(x), v(x))\}_{x \in \sigma^c}.$$

Thus, for each $\Phi \in L^2_\sigma(X, K)$, we have

$$\begin{aligned} \langle \Theta_\sigma \Phi, g \rangle &= \langle T_C^\sigma \Phi, g \rangle \\ &= \int_\sigma v^2(x) \left\langle T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U f, g \right\rangle d\mu_x \\ &\quad + \int_{\sigma^c} v^2(x) \left\langle T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U f, g \right\rangle d\mu_x, \quad g \in H. \end{aligned}$$

Since Λ and Γ are woven, for each $f \in H$, we have

$$A \|K^* f\|^2 \leq \|(T_C^\sigma)^* f\|^2 = \|\Theta_\sigma^* f\|^2.$$

Thus, $\alpha K K^* \leq \Theta_\sigma \Theta_\sigma^*$, $\alpha = A$.

$(ii) \Rightarrow (i)$ Let σ be a partition of X and $f \in H$. Now it is easy to verify that

$$\Theta_\sigma^* f = \left\{ v(x) \left(T^* P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} U \right)^{1/2} f \right\}_{x \in \sigma}$$

$$\cup \left\{ v(x) \left(T^* P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} U \right)^{1/2} f \right\}_{x \in \sigma^c}.$$

Thus, for each $f \in H$, we have

$$\begin{aligned} \alpha \|K^* f\|^2 \leq \|\Theta_\sigma^* f\|^2 = & \int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ & + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x. \end{aligned}$$

Hence, Λ and Γ are W. C. C. K. G. F. F. for H. This completes the proof. □

In the following theorem, we will construct W. C. C. K. G. F. F. for H by using a bounded linear operator.

Theorem 2.2. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i \in [m], x \in \sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If $V \in \mathcal{B}(H)$ is invertible such that V^* commutes with T, U and V commutes with K, then $\left\{ \left(V F_i(x), \Lambda_i(x) P_{F_i(x)} V^*, v_i(x) \right) \right\}_{i \in [m], x \in \sigma_i}$ is a W. C. C. K. G. F. F. for H.*

Proof. Since $P_{F_i(x)} V^* = P_{F_i(x)} V^* P_{V F_i(x)}$ for all $x \in \sigma_i$ and $i \in [m]$, the mapping $x \mapsto P_{V F_i(x)}$ is weakly measurable. For each $f \in H$, we have

$$\begin{aligned} & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} T f \right\rangle d\mu_x \\ = & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* U f, \Lambda_i(x) P_{F_i(x)} V^* T f \right\rangle d\mu_x \\ = & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U V^* f, \Lambda_i(x) P_{F_i(x)} T V^* f \right\rangle d\mu_x \\ \leq & B \|V^* f\|^2 \leq B \|V\|^2 \|f\|^2. \end{aligned}$$

On the other hand, for each $f \in H$, we have

$$\begin{aligned} & \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} T f \right\rangle d\mu_x \\ \geq & A \|K^* V^* f\|^2 = A \|V^* K^* f\|^2 \geq A \left\| V^{-1} \right\|^{-2} \|K^* f\|^2. \end{aligned}$$

This completes the proof. □

Corollary 2.1. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i \in [m], x \in \sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If $V \in \mathcal{B}(H)$ is invertible such that V^* commutes with T, U and V commutes with K, then $\left\{ \left(V F_i(x), \Lambda_i(x) P_{F_i(x)} V^*, v_i(x) \right) \right\}_{i \in [m], x \in \sigma_i}$ is a W. C. C. VKV^*. G. F. F. for H.*

Proof. According to the proof of Theorem 2.2, universal upper bounds is $B\|V\|^2$. On the other hand, for each $f \in H$, we have

$$\frac{A}{\|V\|^2}\left\|(VKV^*)^* f\right\|^2 = \frac{A}{\|V\|^2}\left\|VK^*V^*f\right\|^2 \leq A\left\|K^*V^*f\right\|^2$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}UV^*f, \Lambda_i(x)P_{F_i(x)}TV^*f\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}Uf, \Gamma_i(x)P_{VF_i(x)}Tf\right\rangle d\mu_x,$$

where $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$. This completes the proof. □

Theorem 2.3. *Let $V \in \mathcal{B}(H)$ be invertible operator such that $V^*, (V^{-1})^*$ commutes with T and U. Suppose $\left\{\left(VF_i(x), \Lambda_i(x)P_{F_i(x)}V^*, v_i(x)\right)\right\}_{i\in[m], x\in\sigma_i}$ is a W. C. C. K. G. F. F. for H with universal bounds A and B. Then $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ be a W. C. C. $V^{-1}KV$. G. F. F. for H.*

Proof. Now, for each $f \in H$, using Theorem 1.2, and taking $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$, we have

$$\frac{A}{\|V\|^2}\left\|\left(V^{-1}KV\right)^* f\right\|^2 = \frac{A}{\|V\|^2}\left\|V^*K^*(V^{-1})^*f\right\|^2$$
$$\leq A\left\|K^*\left(V^{-1}\right)^* f\right\|^2$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}U\left(V^{-1}\right)^* f, \Gamma_i(x)P_{VF_i(x)}T\left(V^{-1}\right)^* f\right\rangle d\mu_x$$
$$\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)U\left(V^{-1}\right)^* f, \Gamma_i(x)T\left(V^{-1}\right)^* f\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)\left(V^{-1}\right)^* Uf, \Gamma_i(x)\left(V^{-1}\right)^* Tf\right\rangle d\mu_x$$
$$= \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x.$$

On the other hand, for each $f \in H$, it is easy to verify that

$$\sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq B\left\|V^{-1}\right\|^2\|f\|^2.$$

This completes the proof. □

Next, we will see that the intersection of components of a W. C. C. K. G. F. F. with a closed subspace is a W. C. C. K. G. F. F. for the smaller space.

Theorem 2.4. *Let $\{F(x), \Lambda(x), v(x)\}_{x\in X}$ and $\{G(x), \Gamma(x), w(x)\}_{x\in X}$ be W. C. C. K. G. F. F. for H and W be a closed subspace of H. Then the families given by*

$\{F(x)\cap W, \Lambda(x), v(x)\}_{x\in X}$ *and* $\{G(x)\cap W, \Gamma(x), w(x)\}_{x\in X}$ *are W. C. C. K. G. F. F. for* W.

Proof. The operators $P_{F(x)\cap W} = P_{F(x)}(P_W)$ and $P_{G(x)\cap W} = P_{G(x)}(P_W)$ are orthogonal projections of H onto $F(x)\cap W$ and $G(x)\cap W$, respectively. Let σ be a measurable subset of X. Then for each $f\in W$, we have

$$\begin{aligned}
&\int_{\sigma} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x\\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x\\
&=\int_{\sigma} v^2(x)\left\langle \Lambda(x)P_{F(x)}P_WUf, \Lambda(x)P_{F(x)}P_WTf\right\rangle d\mu_x\\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)}P_WUf, \Gamma(x)P_{G(x)}P_WTf\right\rangle d\mu_x\\
&=\int_{\sigma} v^2(x)\left\langle \Lambda(x)P_{F(x)\cap W}Uf, \Lambda(x)P_{F(x)\cap W}Tf\right\rangle d\mu_x\\
&\quad+\int_{\sigma^c} w^2(x)\left\langle \Gamma(x)P_{G(x)\cap W}Uf, \Gamma(x)P_{G(x)\cap W}Tf\right\rangle d\mu_x.
\end{aligned}$$

This completes the proof. □

The following theorem states the equivalence between W. C. C. K. G. F. F. and a bounded linear operator.

Theorem 2.5. *Let* $V\in\mathcal{B}(H)$ *be an invertible operator such that* V^* *commutes with* T, U. *Suppose* K *be a bounded linear operator on* H *which have closed range. Let* $\Lambda_{TU} = \{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ *be a W. C. C. K. G. F. F. for* H *with universal bounds* A *and* B. *Then the family given by*

$$\Delta_{TU} = \left\{\left(VF_i(x), \Lambda_i(x)P_{F_i(x)}V^*, v_i(x)\right)\right\}_{i\in[m], x\in\sigma_i}$$

is a W. C. C. K. G. F. F. for H *if and only if there exists a* $\delta > 0$ *such that for each* $f\in H$, *we have* $\|V^*f\|\geq\delta\|K^*f\|$.

Proof. Suppose that Δ_{TU} is a W. C. C. K. G. F. F. for H with bounds C and D. Then for each $f\in H$, using the Theorem 1.2, and taking $\Lambda_i(x)P_{F_i(x)}V^* = \Gamma_i(x)$, we have

$$\begin{aligned}
C\|K^*f\|^2 &\leq \sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Gamma_i(x)P_{VF_i(x)}Uf, \Gamma_i(x)P_{VF_i(x)}Tf\right\rangle d\mu_x\\
&=\sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}V^*Uf, \Lambda_i(x)P_{F_i(x)}V^*Tf\right\rangle d\mu_x\\
&=\sum_{i\in[m]}\int_X v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}UV^*f, \Lambda_i(x)P_{F_i(x)}TV^*f\right\rangle d\mu_x
\end{aligned}$$

$$\leq B \left\| V^* f \right\|^2.$$

Thus,

$$\left\| V^* f \right\| \geq \sqrt{C/B} \left\| K^* f \right\|, \quad \text{for all } f \in H.$$

Conversely, suppose $\left\| V^* f \right\| \geq \delta \left\| K^* f \right\|$ for all $f \in H$. Since K have a closed range, by Theorem 1.3, for all $f \in H$, we get

$$\left\| V^* f \right\| = \left\| \left(K^{\dagger} \right)^* K^* V^* f \right\| \leq \left\| K^{\dagger} \right\| \left\| K^* V^* f \right\|.$$

Now, for $f \in H$, we have

$$\begin{aligned}
&\sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} V^* P_{V F_i(x)} T f \right\rangle d\mu_x \\
&= \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U V^* f, \Lambda_i(x) P_{F_i(x)} T V^* f \right\rangle d\mu_x \\
&\geq A \left\| K^* V^* f \right\|^2 \geq A \left\| K^{\dagger} \right\|^{-2} \left\| V^* f \right\|^2 \geq A \delta^2 \left\| K^{\dagger} \right\|^{-2} \left\| K^* f \right\|^2.
\end{aligned}$$

This completes the proof. □

The next theorem shows that it is enough to cheek continuous weaving controlled K-g-fusion woven on smaller measurable space than the original.

Theorem 2.6. *Suppose for each $i \in [m]$, $\left\{ (F_i(x), \Lambda_i(x), v_i(x)) \right\}_{x \in X}$ be a continuous (T, U)-controlled K-g-fusion frame for H with universal bounds A_i and B_i. If there exists a measurable subset $Y \subset X$ such that the family of continuous (T, U)-controlled K-g-fusion frame $\left\{ (F_i(x), \Lambda_i(x), v_i(x)) \right\}_{i \in [m], x \in Y}$ is a W. C. C. K. G. F. F. for H with universal frame bounds A and B. Then the family given by $\left\{ (F_i(x), \Lambda_i(x), v_i(x)) \right\}_{i \in [m], \, x \in X}$ is a W. C. C. K. G. F. F. for H with universal frame bounds A and $\sum_{i \in [m]} B_i$.*

Proof. Let $\{\sigma_i\}_{i \in [m]}$ be an arbitrary partition of X. For each $f \in H$, we define $\varphi : X \to \mathbb{C}$ by

$$\varphi(x) = \sum_{i \in [m]} \chi_{\sigma_i}(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle.$$

Then φ is measurable. Now, for each $f \in H$, we have

$$\begin{aligned}
&\sum_{i \in [m]} \int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\leq \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\leq \left(\sum_{i \in [m]} B_i \right) \|f\|^2.
\end{aligned}$$

It is easy to verify that $\{\sigma_i \cap Y\}_{i\in[m]}$ is a partitions of Y. Thus, the family given by $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i\cap Y}$ is a continuous (T, U)-controlled K-g-fusion frame for H with lowest frame bound A. Therefore,

$$\begin{aligned}&\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \geq&\sum_{i\in[m]}\int_{\sigma_i\cap Y} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \geq& A\left\|K^*f\right\|^2.\end{aligned}$$

This completes the proof. □

In the following theorem, we show that it is possible to remove vectors from continuous controlled K-g-fusion frames and still be left with woven frames.

Theorem 2.7. *Let $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i}$ be a W. C. C. K. G. F. F. for H with universal bounds A and B. If there exists $0 < D < A$ and a measurable subset $Y \subset X$ and $n \in [m]$ such that for $f \in H$*

$$\sum_{i\in[m]\setminus\{n\}}\int_{X\setminus Y} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq D\left\|K^*f\right\|^2,$$

then the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in Y}$ is a W. C. C. K. G. F. F. for H with frame bounds $A - D$ and B.

Proof. Suppose that $\{\sigma_i\}_{i\in[m]}$ and $\{\gamma_i\}_{i\in[m]}$ are partitions of Y and $X\setminus Y$, respectively. For a given $f \in H$, we define $\varphi : Y \to \mathbb{C}$ by

$$\varphi(x) = \sum_{i\in[m]} \chi_{\sigma_i}(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle,$$

and $\phi : X \to \mathbb{C}$ by

$$\phi(x) = \sum_{i\in[m]} \chi_{\sigma_i\cup\gamma_i}(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle.$$

Since $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{i\in[m], x\in\sigma_i\cup\gamma_i}$ is a continuous (T, U)-controlled K-g-fusion frame for H and $\varphi = \phi|_Y$, φ and ϕ are measurable. So, for each $f \in H$, we have

$$\begin{aligned}&\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x\\ \leq&\sum_{i\in[m]}\int_{\sigma_i\cup\gamma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x \leq B\|f\|^2.\end{aligned}$$

Now, we assume that $\{\xi_i\}_{i\in[m]}$ such that $\xi_n = \theta$. Then $\{\xi_i \cup \sigma_i\}_{i\in[m]}$ is a partition of X and so for any $f \in H$, we have

$$\sum_{i\in[m]}\int_{\sigma_i} v_i^2(x)\left\langle \Lambda_i(x)P_{F_i(x)}Uf, \Lambda_i(x)P_{F_i(x)}Tf\right\rangle d\mu_x$$

$$
\begin{aligned}
&= \sum_{i \in [m] \setminus \{n\}} \Bigg[\int_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \int_{\xi_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma_n} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \Bigg] \\
&\geq \sum_{i \in [m] \setminus \{n\}} \Bigg[\int_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \int_{X \setminus Y} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma_n} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \Bigg] \\
&= \sum_{i \in [m]} \int_{\xi_i \cup \sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\quad - \sum_{i \in [m] \setminus \{n\}} \int_{X \setminus Y} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&\geq (A - D) \|K^* f\|^2 .
\end{aligned}
$$

This completes the proof. □

Proposition 2.2. *Let $K \in \mathcal{B}(H)$ be a closed range operator, $V \in \mathcal{B}(H)$ be a unitary operator and $\{(F(x), \Lambda(x), v(x))\}_{x \in X}$ be a continuous (T, U)-controlled K-g-fusion frame for H with bounds A, B. If $\|I_H - V\|^2 \left\|K^{\dagger}\right\|^2 \leq A/B$ and V commutes with T, U, then*

$$
\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x \in X}, \quad \Lambda' = \left\{ \left(V^{-1} F(x), \Lambda(x) V, v(x) \right) \right\}_{x \in X}
$$

are W. C. C. K. G. F. F. for $\mathcal{R}_K$.

Proof. Let σ be a partition of X. Since $K \in \mathcal{B}(H)$ has a closed range, for $f \in \mathcal{R}_K$, we have $\|f\|^2 \leq \left\|K^{\dagger}\right\|^2 \|K^* f\|^2$. Now, for each $f \in \mathcal{R}_K$, we have

$$
\begin{aligned}
&\int_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma^c} v^2(x) \left\langle \Lambda(x) V P_{V^{-1} F(x)} U f, \Lambda(x) V P_{V^{-1} F(x)} T f \right\rangle d\mu_x \\
&= \int_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x
\end{aligned}
$$

$$+\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}UVf, \Lambda(x)P_{F(x)}TVf\right\rangle d\mu_x$$
$$\geq \int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x$$
$$-\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}U\left(I_H - V\right)f, \Lambda(x)P_{F(x)}T\left(I_H - V\right)f\right\rangle d\mu_x$$
$$\geq A\left\|K^*f\right\|^2 - B\left\|I_H - V\right\|^2\left\|f\right\|^2$$
$$\geq A\left\|K^*f\right\|^2 - B\left\|I_H - V\right\|^2\left\|K^\dagger\right\|^2\left\|K^*f\right\|^2$$
$$=\left(A - B\left\|I_H - V\right\|^2\left\|K^\dagger\right\|^2\right)\left\|K^*f\right\|^2.$$

Hence, the families Λ and Λ' are W. C. C. K. G. F. F. for $\mathcal{R}_K$. □

Next, we will see that under some sufficient conditions sum of two continuous (T, U)-controlled K-g-fusion frames is woven with itself.

Theorem 2.8. *Let $K \in \mathcal{B}(H)$ be an invertible operator, the families given by $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x\in X}$ and $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x\in X}$ be continuous (T, U)-controlled K-g-fusion frames for H with bounds A, B and C, D, respectively. Suppose for each $x \in X$*

(i) $F(x) \subset G(x)^\perp$;
(ii) $\Lambda(x)P_{F(x)}\mathcal{R}(U) \perp \Lambda(x)P_{G(x)}\mathcal{R}(T)$;
(iii) $\Lambda(x)P_{F(x)}\mathcal{R}(T) \perp \Lambda(x)P_{G(x)}\mathcal{R}(U)$.

If for any partition σ of X, $(T^\sigma_\Gamma)^*$ is bounded below then

$$\Delta = \{(F(x) + G(x), \Lambda(x), v(x))\}_{x\in X},$$

and Λ are W. C. C. K. G. F. F. for H.

Proof. Since for each $x \in X$, $F(x) \subset G(x)^\perp$, we have $P_{F(x)+G(x)} = P_{F(x)} + P_{F(x)}$. Now, for each $x \in X$, using the given conditions (ii) and (iii), we have

$$\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x$$
$$=\int_X v^2(x)\left\langle \Lambda(x)\left(P_{F(x)} + P_{G(x)}\right)Uf, \Lambda(x)\left(P_{F(x)} + P_{G(x)}\right)Tf\right\rangle d\mu_x$$
$$=\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x$$
$$+\int_X v^2(x)\left\langle \Lambda(x)P_{G(x)}Uf, \Lambda(x)P_{G(x)}Tf\right\rangle d\mu_x \tag{2.2}$$
$$\leq (B + D)\|f\|^2.$$

On the other hand, from (2.2), we get

$$\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x \geq (A+C)\|K^*f\|^2,$$

for all $f \in H$. Thus, Δ is a continuous (T,U)-controlled K-g-fusion frame for H with bounds $(A+C)$ and $(B+D)$.

Furthermore, since K is a invertible operator and for any partition σ of X, $(T_\Gamma^\sigma)^*$ is bounded below, for each $f \in H$, there exists $M > 0$ such that

$$\|(T_\Gamma^\sigma)^* f\|^2 \geq M^2\|f\|^2 \geq \frac{M^2}{\|K\|^2}\|K^*f\|^2.$$

Now, for each $f \in H$, we have

$$\begin{aligned}
&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&=\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&\quad-\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_\sigma v^2(x)\left\langle \Lambda(x)\left(P_{F(x)}+P_{G(x)}\right)Uf, \Lambda(x)\left(P_{F(x)}+P_{G(x)}\right)Tf\right\rangle d\mu_x \\
&=\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_\sigma v^2(x)\left\langle \Lambda(x)P_{G(x)}Uf, \Lambda(x)P_{G(x)}Tf\right\rangle d\mu_x \\
&\geq A\|K^*f\|^2 + \|(T_\Gamma^\sigma)^* f\|^2 \geq \left(A+\frac{M^2}{\|K\|^2}\right)\|K^*f\|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_\sigma v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_{\sigma^c} v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
&\leq\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)+G(x)}Uf, \Lambda(x)P_{F(x)+G(x)}Tf\right\rangle d\mu_x \\
&\quad+\int_X v^2(x)\left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x
\end{aligned}$$

$$\leq (2B+D)\|f\|^2.$$

Thus, Δ and Λ are W. C. C. K. G. F. F. for H. Similarly, it can be shown that Δ and Γ are W. C. C. K. G. F. F. for H. This completes the proof. ☐

In the following theorem, we present a sufficient condition for weaving continuous controlled K-g-fusion frame in terms of positive operators associated with given continuous controlled K-g-fusion frame.

Theorem 2.9. *Let the families given by* $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x\in X}$ *and* $\Gamma = \{(G(x), \Lambda(x), v(x))\}_{x\in X}$ *be continuous* (T,U)*-controlled* K*-*g*-fusion frames for* H*. Suppose for each* $x \in X$*, the operator* $U_x : H \to H$ *defined by*

$$\langle U_x(f), g\rangle = \int_X v^2(x)\, \langle T^*\Delta(x)Uf, g\rangle \, d\mu_x,$$

$f, g \in H$*, where* $\Delta(x) = P_{G(x)}\Gamma^*(x)\Gamma(x)P_{G(x)} - P_{F(x)}\Lambda^*(x)\Lambda(x)P_{F(x)}$*, is a positive operator. Then* Λ *and* Γ *are W. C. C. K. G. F. F. for* H*.*

Proof. Let A, B and C, D be frame bounds of Λ and Γ, respectively. Take σ be any partition of X. Then for each $f \in H$, we have

$$\begin{aligned}
A\|K^*f\|^2 \leq & \int_X v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
= & \int_\sigma v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
& + \int_{\sigma^c} v^2(x) \left\langle T^*P_{F(x)}\Lambda(x)^*\Lambda(x)P_{F(x)}Uf, f\right\rangle d\mu_x \\
= & \int_\sigma v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
& - \int_{\sigma^c} v^2(x) \left\langle T^*\Delta(x)Uf, f\right\rangle d\mu_x \\
& + \int_{\sigma^c} v^2(x) \left\langle T^*P_{G(x)}\Gamma(x)^*\Gamma(x)P_{G(x)}Uf, f\right\rangle d\mu_x \\
\leq & \int_\sigma v^2(x) \left\langle \Lambda(x)P_{F(x)}Uf, \Lambda(x)P_{F(x)}Tf\right\rangle d\mu_x \\
& + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x)P_{G(x)}Uf, \Gamma(x)P_{G(x)}Tf\right\rangle d\mu_x \\
\leq & (B+D)\|f\|^2.
\end{aligned}$$

Thus, Λ and Γ are W. C. C. K. G. F. F. for H with universal bounds A and $B+D$. ☐

Theorem 2.10. *Suppose for each* $i \in [m]$*,* $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x\in X}$ *be a continuous* (T,U)*-controlled* K*-*g*-fusion frame for* H *with bounds* A_i *and* B_i*. Suppose* Y *be* *measurable subset X and there exists $N > 0$ such that for all $i, k \in [m]$ with $i \neq k$*

$$0 \leq \int_Y \langle \Gamma_{i,k} U f, \Gamma_{i,k} T f \rangle \, d\mu_x \leq N \min\{\Theta, \Omega\}, \quad f \in H,$$

where

$$\begin{aligned}
\Gamma_{i,k} =& v_i^2(x) \Lambda_i(x) P_{F_i(x)} - v_k^2(x) \Lambda_k(x) P_{F_k(x)}, \\
\Theta =& \int_Y v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x, \\
\Omega =& \int_Y v_k^2(x) \left\langle \Lambda_i(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x.
\end{aligned}$$

Then the family $\{(F_i(x), \Lambda_i(x), v_i(x))\}_{x \in X, i \in [m]}$ is W. C. C. K. G. F. F. for H with universal bounds $\frac{A}{(m-1)(N+1)+1}$ and B, where $A = \sum_{i \in [m]} A_i$ and $B = \sum_{i \in [m]} B_i$.

Proof. Let $\{\sigma_i\}_{i \in [m]}$ be a partition of X. Then for $f \in H$, we have

$$\begin{aligned}
\sum_{i \in [m]} A_i \|K^* f\|^2 \leq& \sum_{i \in [m]} \int_X v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
=& \sum_{i \in [m]} \sum_{k \in [m]} \int_{\sigma_k} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
\leq& \sum_{i \in [m]} \Bigg[\int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&+ \sum_{k \in [m], k \neq i} \int_{\sigma_k} \langle \Gamma_{i,k} U f, \Gamma_{i,k} T f \rangle \, d\mu_x \\
&+ \sum_{k \in [m], k \neq i} \int_{\sigma_k} v_k^2(x) \left\langle \Lambda_k(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x \Bigg], \\
\Gamma_{i,k} =& v_i^2(x) \Lambda_i(x) P_{F_i(x)} - v_k^2(x) \Lambda_k(x) P_{F_k(x)} \\
\leq& \sum_{i \in [m]} \Bigg[\int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
&+ \sum_{k \in [m], k \neq i} (N+1) \int_{\sigma_k} v_k^2(x) \left\langle \Lambda_k(x) P_{F_k(x)} U f, \Lambda_k(x) P_{F_k(x)} T f \right\rangle d\mu_x \Bigg], \\
=& D \sum_{i \in [m]} \int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x,
\end{aligned}$$

where $D = \{(m-1)(N+1)+1\}$. Thus, for each $f \in H$, we have

$$\begin{aligned}
\frac{A}{(m-1)(N+1)+1} \|K^* f\|^2 \leq& \sum_{i \in [m]} \int_{\sigma_i} v_i^2(x) \left\langle \Lambda_i(x) P_{F_i(x)} U f, \Lambda_i(x) P_{F_i(x)} T f \right\rangle d\mu_x \\
\leq& B \|f\|^2.
\end{aligned}$$

This completes the proof. □

3. Perturbation of Woven Continuous Controlled g-Fusion Frame

In frame theory, one of the most important problem is the stability of frame under some perturbation. P. Casazza and Chirstensen [10] have been generalized the Paley-Wiener perturbation theorem to perturbation of frame in Hilbert space. P. Ghosh and T. K. Samanta have studied perturbation of dual g-fusion frame and continuous controlled g-fusion frame in [18, 21]. In this section, we will see that under some small perturbations, continuous controlled K-g-fusion frames constitute woven continuous controlled K-g-fusion frame.

Theorem 3.1. *Let the families given by* $\Lambda = \{(F(x), \Lambda(x), v(x))\}_{x \in X}$ *and* $\Gamma = \{(G(x), \Gamma(x), v(x))\}_{x \in X}$ *be continuous* (T, U)*-controlled* K*-*g*-fusion frames for* H *with bounds* A, B *and* C, D*, respectively. Suppose that there exist non-negative constants* λ_1, λ_2 *and* μ *with* $0 < \lambda_1 < 1$*,* $\mu < (1 - \lambda_1) A - \lambda_2 B$ *such that for each* $f \in H$*, we have*

$$\begin{aligned} 0 \leq & \int_X v^2(x) \left\langle T^* \Delta(x) U f, f \right\rangle d\mu_x \\ \leq & \lambda_1 \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ & + \lambda_2 \int_X v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x + \mu \left\| K^* f \right\|^2, \end{aligned}$$

where $\Delta(x) = \left(P_{F(x)} \Lambda(x)^* \Lambda(x) P_{F(x)} - P_{G(x)} \Gamma(x)^* \Gamma(x) P_{G(x)} \right)$*. Then,* Λ *and* Γ *are W. C. C. K. G. F. F. for* H*.*

Proof. Let σ be a partition of X. Now, for each $f \in H$, we have

$$\begin{aligned} & \int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x \\ \geq & \int_\sigma v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x - \int_{\sigma^c} v^2(x) \left\langle T^* \Delta(x) U f, f \right\rangle d\mu_x \\ & + \int_{\sigma^c} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ \geq & \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x - \int_X v^2(x) \left\langle T^* \Delta(x) U f, f \right\rangle d\mu_x \\ \geq & (1 - \lambda_1) \int_X v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\ & - \lambda_2 \int_X v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x - \mu \left\| K^* f \right\|^2 \end{aligned}$$

$\geq ((1-\lambda_1) A - \lambda_2 B - \mu) \|K^* f\|^2.$

On the other hand,

$$\begin{aligned}
&\int_{\sigma} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{\sigma^c} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x \\
\leq &\int_{X} v^2(x) \left\langle \Lambda(x) P_{F(x)} U f, \Lambda(x) P_{F(x)} T f \right\rangle d\mu_x \\
&\quad + \int_{X} v^2(x) \left\langle \Gamma(x) P_{G(x)} U f, \Gamma(x) P_{G(x)} T f \right\rangle d\mu_x \\
\leq &(B+D)\|f\|^2.
\end{aligned}$$

This completes the proof. □

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References

[1] R. Ahmadi, G. Rahimlou, V. Sadri and R. Z. Farfar, *Constructions of K-g fusion frames and their duals in Hilbert spaces*, Bull. Transilv. Univ. Brasov Ser. III. Math. Comput. Sci. **13**(62) (2020), 17–32. https://doi.org/10.31926/but.mif.2020.12.61.1.2

[2] S. T. Ali, J. P. Antonie and J. P. Gazeau, *Continuous frames in Hilbert spaces*, Annals of Physics **222** (1993), 1–37. https://doi.org/10.1006/aphy.1993.1016

[3] E. Alizadeh and V. Sadri, *On continuous weaving G-frames in Hilbert spaces*, Wavelets and Linear Algebra **7**(1) (2020), 23–36. https://doi.org/10.22072/wala.2020.114423.1248

[4] E. Alizadeh, A. Rahimi, E. Osgooei and M. Rahman, *Continuous K-G-fusion frames in Hilbert spaces*, TWMS J. Pure Appl. Math. **11**(1) (2021), 44–55.

[5] E. Alizadeh and V. Sadri, *Construction of weaving continuous g-frames for operators in Hilbert spaces*, Probl. Anal. Issues Anal. **10**(2) (2021), 3–17. https://doi.org/10.15393/j3.art.2021.9310

[6] P. Balazs, J. P. Antonie and A. Grybos, *Weighted and controlled frames: Mutual relationship and first numerical properties,* Int. J. Wavelets Multiresolut. Inf. Process. **14**(1) (2010), 109–132. https://doi.org/10.1142/S0219691310003377

[7] T. Bemrose, P. G. Casazza, K. Grochenic, M. C. Lammers and R. G. Lynch, *Weaving frames*, Operators and Matrices **10**(4) (2016), 1093–1116. https://doi.org/10.7153/oam-10-61

[8] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, 2008.

[9] P. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math. **345** (2004), 87–114. https://doi.org/10.1090/conm/345/06242

[10] P. Casazza and O. Christensen, *Perturbation of operators and applications to frame theory*, J. Fourier Anal. Appl. **3** (1997), 543–557. https://doi.org/10.1007/BF02648883

[11] I. Daubechies, A. Grossmann and Y. Mayer, *Painless nonorthogonal expansions*, J. Math. Phys. **27**(5) (1986), 1271–1283. https://doi.org/10.1063/1.527388

[12] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*. Proc. Amer. Math. Soc. **17** (1966), 413–415. https://doi.org/10.1080/03081087.2017.1402859

[13] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

[14] M. H. Faroughi, A. Rahimi and R. Ahmadi, *GC-fusion frames*, Methods Funct. Anal. Topology **16**(2) (2010), 112–119.

[15] P. Gavruta, *On the duality of fusion frames*, J. Math. Anal. Appl. **333** (2007), 871–879. `https://doi.org/10.1016/j.jmaa.2006.11.052`

[16] L. Gavruta, *Frames for operator*, Appl. Comput. Harmon. Anal. **32**(1) (2012), 139–144. `https://doi.org/10.1016/j.acha.2011.07.006`

[17] S. Garg, K. L. Vashisht and G. Verma, *On weaving fusion frames for Hilbert spaces*, International Conference on Sampling Theory and Applications (SampTA) (2017), 381–385. `https://doi.org/10.1109/SAMPTA.2017.8024363`

[18] P. Ghosh and T. K. Samanta, *Stability of dual g-fusion frame in Hilbert spaces*, Methods Funct. Anal. Topology **26**(3) (2020), 227–240.

[19] P. Ghosh and T. K. Samanta, *Generalized atomic subspaces for operators in Hilbert spaces*, Math. Bohem. **147**(2) (2022), 325–345. `https://doi.org/10.21136/MB.2021.0130-20`

[20] P. Ghosh and T. K. Samanta, *Generalized fusion frame in tensor product of Hilbert spaces*, J. Indian Math. Soc. **89** (1–2) (2022), 58–71. `https://doi.org/10.18311/jims/2022/29307`

[21] P. Ghosh and T. K. Samanta, *Continuous controlled generalized fusion frames in Hilbert spaces*, J. Indian Math. Soc. (to appear).

[22] G. Kaiser, *A Friendly Guide to Wavelets*, Birkhauser, 1994.

[23] A. Khosravi and K. Musazadeh, *Controlled fusion frames*, Methods Funct. Anal. Topology **18**(3) (2012), 256–265.

[24] D. Li, J. Leng and T. Huang, *On weaving g-frames for Hilbert spaces*, Complex Anal. Oper. Theory **14**(33) (2020). `https://doi.org/10.1007/s11785-020-00991-7`

[25] M. Mohammadrezaee, M. Rashidi-Kouchi, A. Nazari and A. Oloomi, *Woven g-fusion frames in Hilbert spaces*, Sahand Communications in Mathematical Analysis **18**(3) (2021), 133–151. `https://doi.org/10.22130/scma.2021.137940.870`

[26] M. Nouri, A. Rahimi and Sh. Najafizadeh, *Controlled K-frames in Hilbert spaces,* Int. J. Anal. Appl. **4**(2) (2015), 39–50.

[27] A. Rahimi and A. Fereydooni, *Controlled g-frames and their g-multipliers in Hilbert spaces*, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **21**(2) (2013), 223–236. `https://doi.org/10.2478/auom-2013-0035`

[28] G. Rahimlou, V. Sadri and R. Ahmadi, *Construction of controlled K-g-fusion frame in Hilbert spaces*, UPB Scientific Bulletin, Series A **82**(1) (2020).

[29] R. Rezapour, A. Rahimi, E. Osgooei and H. Dehghan, *Controlled weaving frames in Hilbert spaces*, Infinite Dimensional Analysis Quantum Probability and Related Topics **22**(1) (2019), Paper ID 1950003. `https://doi.org/10.1142/S0219025719500036`

[30] R. Rezapour, A. Rahimi, E. Osgooei and H. Dehghan, *Continuous controlled K-g-frames in Hilbert spaces*, Indian J. Pure Appl. Math. **50** (2019), 863–875. `https://doi.org/10.1007/s13226-019-0359-y`

[31] V. Sadri, Gh. Rahimlou, R. Ahmadi and R. Zarghami Farfar, *Generalized fusion frames in Hilbert spaces*, Infinite Dimensional Analysis Quantum Probability and Related Topics **23**(2) (2020), Paper ID 2050015. `https://doi.org/10.1142/S0219025720500150`

[32] V. Sadri, G. Rahimlou and R. Ahmadi, *K-g-fusion woven in Hilbert spaces*, TWMS J. Pure Appl. Math. **11**(3) (2021), 947–958.

[33] V. Sadri, R. Ahmadi and G. Rahimlou, *On continuous weaving fusion frames in Hilbert spaces*, Int. J. Wavelets Multiresolut. Inf. Process. **18**(5) (2020), Paper ID 2050035, 17 pages. `https://doi.org/10.1142/S0219691320500356`

[34] H. Shakoory, R. Ahamadi, N. Behzadi and S. Nami, *(C, C')-Controlled g-fusion frames,* Iran. J. Math. Sci. **18**(1) (2023), 179–191. `https://doi.org/10.52547/ijmsi.18.1.179`

[35] W. Sun, *G-frames and G-Riesz bases*, J. Math. Anal. **322**(1) (2006), 437–452. https://doi.org/10.1016/j.jmaa.2005.09.039

[36] L. K. Vashisht and Deepshikha, *On continuous weaving frames*, Adv. Pure Appl. Math. **8**(1) (2017), 15–31. https://doi.org/10.1515/apam-2015-0077

[1]Department of Pure Mathematics,
University of Calcutta,
35, Ballygunge Circular Road, Kolkata, 700019, West Bengal, India
Email address: prasenjitpuremath@gmail.com

[2]Department of Mathematics,
Uluberia College,
Uluberia, Howrah, 711315, West Bengal, India
Email address: mumpu_tapas5@yahoo.co.in