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EXISTENCE OF SOLUTIONS FOR INHOMOGENEOUS BIHARMONIC PROBLEM INVOLVING CRITICAL HARDY-SOBOLEV EXPONENTS

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Abstract. This paper is devoted to the study of biharmonic problems. More precisely, we consider the following inhomogeneous problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda\left(\frac{u}{|x|^{4-\alpha}}\right) + f(x), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 5$, under sufficient conditions on the data and the considered parameters, we prove the existence and multiplicity of solutions, by virtue of Ekeland's Variational Principle and the Mountain Pass Lemma.

1. Introduction

In this paper, we consider the following inhomogeneous problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda\left(\frac{u}{|x|^{4-\alpha}}\right) + f(x), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 5$, containing 0 in its interior, $0 < \mu < \overline{\mu} := \frac{N^2(N-4)^2}{16}$, $\lambda > 0$, $0 \leq s,\ \alpha < 4$, $\alpha \neq 0$, $f \in H^{-2}(\Omega)$ ($H^{-2}(\Omega)$ denotes the dual space of the Sobolev space $H_0^2(\Omega)$), Δ^2 is the biharmonic operator and $2^*(s) = \frac{2(N-s)}{N-4}$ is the Sobolev critical exponent.

Key words and phrases. Palais-Smale condition, Ekeland's variational principle, critical Hardy-Sobolev exponent, singularity, biharmonic problem.

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The nonlinearity has a critical growth imposed by the critical exponent of Sobolev and the singular potentials, which causes a loss of compactness of the considered problem, consequently the classical methods cannot be applied directly, which make the study hard and more difficult.

We quote here some realized problems: The regular case in our problem, i.e., $\mu = \lambda = s = 0$ has been studied by Deng et al. [5]. By using Ekeland's Variational Principle [6] and the Mountain Pass Lemma [1], they proved the existence of multiple solutions for $f \neq 0$ satisfying a suitable assumption.

For $s = \lambda = 0$ and $f \equiv 0$, D'Ambrosio and Jannelli in [2], proved that there exists radial solutions U_μ positive, symmetric, decreasing and solve

$$\Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, u(x) > 0.$$

In [7], Kang and Xu studied the following problem

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda|u|^{q-2}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $0 \leq s < 4$ and $2 \leq q < 2^* = \frac{2N}{N-4}$. By variational arguments the existence of nontrivial solutions of the problem is established.

In what follows, we state our main results for which we consider the following hypothesis

$$0 < \inf\left\{C_N(T(u))^{\frac{N-2s+4}{8-2s}} - \int_\Omega fu dx : \ u \in H_0^2(\Omega), \int_\Omega \left(\frac{|u|^{2^*(s)}}{|x|^s}\right) dx = 1\right\}, \tag{1.2}$$

where

$$C_N = \left(\frac{8-2s}{N-4}\right)\left(\frac{N-4}{N-2s+4}\right)^{\frac{N-2s+4}{8-2s}}$$

and

$$T(u) = \int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right) - \lambda\left(\frac{u^2}{|x|^{4-\alpha}}\right)\right) dx.$$

Theorem 1.1. i) *Let* $\mu \in]0, \overline{\mu}[$, $\lambda \in]0, \lambda_1[$ *and* f *satisfying the condition* (1.2)*, then the problem* (1.1) *has at least a solution.*

ii) *There exists* $\widehat{\mu} \in]0, \overline{\mu}[$ *such that, for* $\mu \in]0, \widehat{\mu}[$, $\lambda \in]0, \lambda_1[$ *and* f *satisfying the condition* (1.2)*, then* (1.1) *has at least two solutions, if*

1) $0 < \alpha \leq \frac{1}{2}$ *for* $N \geq 5$;
2) $\frac{1}{2} < \alpha < 4$ *for* $5 \leq N < 12$.

The positive constants λ_1 and $\widehat{\mu}$ will be given later.

This paper is organized as follows. In the forthcoming section, we give some preliminaries and technical lemmas used in our work. In section 3 we give a detailed proof of Theorem 1.1.

2. Preliminary Results

2.1. **Definitions and notations.** Throughout this article, $\|\cdot\|_-$ denotes the norm of the Sobolev $H^{-2}(\Omega)$, $o_n(1)$ is any quantity which tends to zero as n goes to infinity and $\mathcal{O}(\varepsilon^s)$ verifies $|\frac{\mathcal{O}(\varepsilon^s)}{\varepsilon^s}| \leq C$, where C is a positive constant.

Problem (1.1) is related to the following Rellich inequality [8]

$$\int_\Omega \frac{u^2}{|x|^4}dx \leq \frac{1}{\overline{\mu}}\int_\Omega |\Delta u|^2 dx, \quad \text{for all } u \in H_0^2(\Omega), \tag{2.1}$$

where $H_0^2(\Omega)$ is the completetion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega(|\Delta u|^2 dx)^{\frac{1}{2}}$. Then the following best constant is defined

$$A_{\mu,s}(\Omega) := \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\Delta u|^2 - \mu\frac{u^2}{|x|^4}\right)dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}}}, \quad \text{for } 0 < \mu < \overline{\mu}. \tag{2.2}$$

Note that it is well known that $A_{\mu,s}(\Omega)$ is independent of any $\Omega \subset \mathbb{R}^N$ and that is not obtained except in the case with $\Omega = \mathbb{R}^N$. Moreover, the minimizers of $A_{\mu,s}(\Omega)$ have been investigated by [7]. Thus, we will simply denote $A_{\mu,s}(\Omega) = A_{\mu,s}(\mathbb{R}^N) = A_{\mu,s}$.

The authors in [2, 7] proved that $A_{\mu,s}$ is attained in $\mathbb{R}^N$ by the functions

$$\left\{y_\varepsilon(x) = \varepsilon^{\frac{4-N}{2}}U_\mu\left(\frac{x}{\varepsilon}\right) : \varepsilon > 0\right\},$$

and achieved

$$\int_\Omega \left(|\Delta y_\varepsilon(x)|^2 - \mu\left(\frac{|y_\varepsilon(x)|^2}{|x|^4}\right)\right)dx = \int_\Omega \left(\frac{|y_\varepsilon(x)|^{2^*(s)}}{|x|^s}\right)dx = A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)},$$

such as U_μ satisfies for $\mu \in]0, \overline{\mu}[$:

(a) $\lim_{\rho\to 0}\rho^{a(\mu)}U_\mu(\rho) = k_1,\ \lim_{\rho\to 0}\rho^{a(\mu)+1}U'_\mu(\rho) = k_3$;

(b) $\lim_{\rho\to +\infty}\rho^{b(\mu)}U_\mu(\rho) = k_2,\ \lim_{\rho\to +\infty}\rho^{b(\mu)+1}U'_\mu(\rho) = k_4$,

where $k_i \in \mathbb{R}$, $i = 1, \dots, 4$ and $b(\mu) = (\frac{N-4}{2})(2 - \theta(\frac{\mu}{\overline{\mu}}))$, $a(\mu) = (\frac{N-4}{2})\theta(\frac{\mu}{\overline{\mu}})$, $\theta : [0,1] \to [0,1]$ is given by

$$\theta(t) = 1 - \frac{\sqrt{(N-2)^2 + 4 - \sqrt{16(N-2)^2 + t(N-4)^2N^2}}}{N-4}.$$

Let us define $\vartheta : [0,1] \to [0,1]$ as follows:

$$\vartheta(t) = \frac{t(t-2)((N-4)t + 4)((N-4)t - 2N + 4)}{N^2}.$$

Let us put

$$\varsigma_\alpha = \frac{1}{16}(N - 4 - \alpha)(N - 4 + \alpha)(N^2 - \alpha^2),$$

$$\zeta_s = \frac{(N-4)^2(s-4)}{(N+4)^4}\Big[N^2s^3 - (2N^3+4N^2)s^2 + (N^4+10N^3-20N^2+64N-64)s - 6N^4 + 20N^3 - 64N^2 + 64N\Big],$$

and set $\widehat{\mu} = \min(\varsigma_\alpha, \zeta_s)$.

Remark 2.1. (a) θ is continuous and strictly increasing.
(b) ϑ is an increasing homeomorphism and its inverse is θ.

In this paper, we use $H_0^2(\Omega)$ to denote the completetion of $C_0^\infty(\Omega)$ with respect to the norm,

$$\|u\|^2 := \int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right) dx.$$

By (2.1), this norm is equivalent to the usual norm $(\int_\Omega |\Delta u|^2 dx)^{\frac{1}{2}}$.

Let $u \in H_0^2(\Omega)$ be a weak solution of (1.1) if for all $\varphi \in H_0^2(\Omega)$,

$$\int_\Omega \Delta u \Delta\varphi - \int_\Omega \left(\frac{\mu}{|x|^4}\right) u\varphi dx - \int_\Omega \left(\frac{|u|^{2^*(s)-2}}{|x|^s}\right) u\varphi dx - \int_\Omega \left(\frac{\lambda}{|x|^{4-\alpha}}\right) u\varphi dx - \int_\Omega fu\varphi dx = 0.$$

It is true that the weak solutions of Problem (1.1) are equivalent to the nonzero critical points of the energy functional associated to (1.1) given by the following expression:

$$I(u) = \frac{1}{2}T(u) - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx - \int_\Omega fu dx, \quad \text{for all } u \in H_0^2(\Omega).$$

Definition 2.1. A functional $I \in C^1(H_0^2(\Omega);\mathbb{R})$ satisfies the Palais-Smale condition at level c, ($(PS)_c$ for short), if any sequence $(u_n) \subset H_0^2(\Omega)$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } H^{-2}(\Omega),$$

contains a strongly convergent subsequence.

2.2. **Eigenvalue problem.** Due to the Rellich inequality, the operator $Lu := \Delta^2 u - \mu\frac{u}{|x|^4}$ is definite on $H_0^2(\Omega)$. Moreover, the following eigenvalue problem with Hardy potentials and singular coefficient

$$\begin{cases} \Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \lambda\left(\frac{u}{|x|^{4-\alpha}}\right), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha < 4$, $\lambda \in \mathbb{R}$, has the first eigenvalue λ_1 given by:

$$\lambda_1 = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right) dx}{\int_\Omega \frac{u^2}{|x|^{4-\alpha}} dx}.$$

Since the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega, |x|^{\alpha-4})$ is compact, by choosing a minimizing sequence, we easily infer that λ_1 can be obtained in $H_0^2(\Omega)$ and $\lambda_1 > 0$.

2.3. **Nehari manifold.** As the energy functional I is well defined in $H_0^2(\Omega)$ and belongs to $C^1(H_0^2(\Omega),\mathbb{R})$ and is not bounded from below on $H_0^2(\Omega)$, we consider it on the Nehari manifold

$$\mathcal{N} := \{u \in H_0^2(\Omega) : \langle I'(u), u\rangle = 0\}.$$

It is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. We can split $\mathcal{N}$ for:

$$\begin{aligned}\mathcal{N}^+ :=&\{u \in \mathcal{N} : \langle I''(u), u\rangle > 0\},\\ \mathcal{N}^- :=&\{u \in \mathcal{N} : \langle I''(u), u\rangle < 0\}\end{aligned}$$

and

$$\mathcal{N}^0 := \{u \in \mathcal{N} : \langle I''(u), u\rangle = 0\}.$$

Denote $\inf\limits_{\mathcal{N}} I = c_0$.

2.4. **Some technical lemmas.**

Lemma 2.1. *If* $\mu \in]0, \overline{\mu}[$, $\alpha > 0$ *and* $0 < \lambda < \lambda_1$, *then*

$$\inf\left\{(T(u))^{\frac{1}{2}} : \int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1\right\} = M > 0.$$

Proof. We know that

$$\lambda_1 \int\limits_\Omega \frac{u^2}{|x|^{4-\alpha}}dx \leq \int\limits_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right)dx,$$

we deduce that

$$T(u) \geq \left(1 - \frac{\lambda}{\lambda_1}\right)\int\limits_\Omega \left(|\Delta u|^2 - \mu\left(\frac{u^2}{|x|^4}\right)\right)dx.$$

Thus, by Rellich inequality, we get

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\left(1 - \frac{\mu}{\overline{\mu}}\right)\int\limits_\Omega |\Delta u|^2 dx \leq T(u) \leq \int\limits_\Omega |\Delta u|^2 dx.$$

Then $(T(u))^{\frac{1}{2}} \geq \sqrt{K}S > 0$ for all $u \in H_0^2(\Omega)$ such that $\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx = 1$. Here $S = \inf\limits_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega |\Delta u|^2 dx.}{\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx}$ and $K = (1 - \frac{\lambda}{\lambda_1})(1 - \frac{\mu}{\overline{\mu}})$. We immediately have that $M > 0$. $\square$

Lemma 2.2. *Let* $f \neq 0$ *satisfying condition* (1.2). *Then* $\mathcal{N}^0 = \emptyset$.

Proof. Suppose that $\mathcal{N}^0 \neq \emptyset$, then for $u \in \mathcal{N}^0$ we have

$$T(u) = (2^*(s) - 1)\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx.$$

Thus,

$$0=\langle I''(u),u\rangle = T(u)-\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx-\int_\Omega fudx = (2^*(s)-2)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx-\int_\Omega fudx. \tag{2.3}$$

From (1.2) and (2.3), we obtain

$$\begin{aligned} 0 &< C_N(T(u))^{\frac{N-2s+4}{8-2s}} - \int_\Omega fudx \\ &= (2^*(s)-1)\left[\left(\frac{T(u)}{(2^*-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx}\right)^{\frac{N-2s+4}{8-2s}} - 1\right]\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx \\ &= 0, \end{aligned}$$

which yields a contradiction. □

Lemma 2.3. *Let $f \neq 0$ satisfying (1.2). For every $u \in H_0^2(\Omega)$, $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \mathcal{N}^-$. In particular,*

$$t^+ > \left[\frac{T(u)}{(2^*(s)-1)\left(\frac{N-2s+4}{8-2s}\right)}\right]^{\frac{N-2s+4}{8-2s}} = t_{\max}(u) \quad \text{and} \quad I(t^+u) = \max_{t\geq t_{\max}} I(tu).$$

Moreover, if $\int_\Omega fudx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-u \in \mathcal{N}^+$, $t^- < t_{\max}(u)$ and $I(t^-u) = \min_{0\leq t\leq t_{\max}} I(tu)$.

Proof. The lemma is proved in the same way as in [5]. □

Lemma 2.4. *Let $f \neq 0$ satisfying (1.2). For each $u \in \mathcal{N}\setminus\{0\}$, there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in H_0^2(\Omega)\setminus\{0\}$, $\|w\| < \varepsilon$, satisfying the following there conditions:*

$$t(0) = 1,$$

$$t(w)(u-w) \in \mathcal{N}, \quad \text{for all } \|w\| < \varepsilon,$$

$$\langle t'(0), v\rangle = \frac{\int_\Omega [2\Delta u\Delta v - 2\left(\frac{\mu}{|x|^4}+\frac{\lambda}{|x|^{4-\alpha}}\right)uv - 2^*(s)\frac{|u|^{2^*(s)-2}}{|x|^s}uv - fv]dx}{T(u)-(2^*(s)-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx}. \tag{2.4}$$

Proof. Define the map $F : \mathbb{R}\times H_0^2(\Omega) \to \mathbb{R}$,

$$F(t,w) = sT(u-w) - t^{2^*(s)-1}\int_\Omega \frac{|u-\omega|^{2^*(s)}}{|x|^s}dx - \int_\Omega (u-w)fdx.$$

Since $F(1,0) = 0$, $\frac{\partial F}{\partial t}(1,0) = T(u) - (2^*(s)-1)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}dx \neq 0$, applying the implicit function theorem at the point $(1,0)$, we can get the result of this lemma. □

In the following lemma, we prove that $\mathcal{N}^-$ is closed and disconnects $H_0^2(\Omega)$ in exactly two connected components E_1 and E_2.

$$E_1 = \left\{ u \in H_0^2(\Omega) : u = 0 \text{ or } \|u\| < t^+\left(\frac{u}{\|u\|}\right)\right\}$$

and

$$E_2 = \left\{ u \in H_0^2(\Omega) \backslash \{0\} : \ \|u\| > t^+\left(\frac{u}{\|u\|}\right)\right\}.$$

Lemma 2.5. *Assume that condition* (1.2) *is satisfied, then*

(a) $\mathcal{N}^-$ *is closed;*
(b) $H_0^2 \backslash \mathcal{N}^- = E_1 \cup E_2$*;*
(c) $\mathcal{N}^+ \subset E_1$.

Proof. Let $(u_n) \subset \mathcal{N}^-$ and $w = \lim\limits_{n\to+\infty} u_n$, then $w \in \mathcal{N}$. Assume by contradiction that $w \notin \mathcal{N}^-$, then

$$T(u_n) - (2^*(s)-1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx < 0, \tag{2.5}$$

$T(w) - (2^*(s)-1)\int\limits_\Omega \frac{|w|^{2^*(s)}}{|x|^s}dx = 0$. So, $w \in \mathcal{N}^0$ this implies that $w = 0$. From (2.5) and Lemma 2.1, we get $KS^2 < (2^*(s)-1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx$, so $KS^2 < (2^*(s)-1)\int\limits_\Omega \frac{|w|^{2^*(s)}}{|x|^s}dx$, which yields to a contradiction.

Let $u \in \mathcal{N}^-$ and $v = \frac{u}{\|u\|}$, then $t^+(u) = 1$, and there exists a unique $t^+(v)$ such that $t^+(v)v \in \mathcal{N}^-$. As $t^+(v)v = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|}u \in \mathcal{N}^-$, then $t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = t^+(u) = 1$. Thus, if $u \in H_0^2(\Omega)$ and $t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} \neq 1$, then $u \notin \mathcal{N}^-$ and $H_0^2(\Omega) = E_1 \cup E_2$.

Let $u \in \mathcal{N}^+$. Then $t^-\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} = t^-(u) = 1$. Since $t^+(u) > t^-(u)$, it follows that $t^+(u) = t^+\left(\frac{u}{\|u\|}\right)\frac{1}{\|u\|} > 1$. So, $\|u\| < t^+\left(\frac{u}{\|u\|}\right)$, and we conclude that $\mathcal{N}^+ \subset E_1$. ☐

Let the cut-off function $\varphi(x) = \varphi(|x|) \in C_0^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ in $B(0,R)$ and $\varphi(x) = 1$ in $B(0,\frac{R}{2})$. Set $u_\varepsilon = \varphi(x)y_\varepsilon(x)$, the following asymptotic properties hold.

Proposition 2.1. *Suppose that* $N \ge 5$, $\mu \in]0, \overline{\mu}[$. *Then*

(1) $\int\limits_\Omega \left(|\Delta u_\varepsilon|^2 - \mu\left(\frac{|u_\varepsilon|^2}{|x|^4}\right)\right)dx = A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} + \mathcal{O}(\varepsilon^{2b(\mu)-N+4})$;
(2) $\int\limits_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}dx = A_{\mu,s}^{\left(\frac{N-4}{4-s}\right)} + \mathcal{O}(\varepsilon^{2^*(s)b(\mu)-N+s})$;
(3) $\int\limits_\Omega |x|^{\alpha-4}|u_\varepsilon|^2 dx = \mathcal{O}(\varepsilon^\alpha)$;
(4) $\int\limits_\Omega \frac{|u_\varepsilon|^{2^*(s)-1}u_0}{|x|^s}dx = \varepsilon^{\frac{N-4}{2}}u_0(0)E + \mathcal{O}(\varepsilon^{\frac{N-4}{2}})$, *where* $E = \int\limits_{\mathbb{R}^N} \frac{U_\mu^{2^*(s)-1}(x)}{|x|^s}dx$ *and* $\mu < \zeta_s$.

Proof. For the estimates (1), (2) one can see in [7], we only verify (3) and (4). Take $R > 0$ small enough such that $B(0, \frac{R}{2}) \subset \Omega$

$$\begin{aligned}\int_\Omega |x|^{\alpha-4}u_\varepsilon^2 dx &= \int_{\Omega\setminus B(0,\frac{R}{2})} |x|^{\alpha-4}u_\varepsilon^2 dx + \int_{B(0,\frac{R}{2})} |x|^{\alpha-4}u_\varepsilon^2 dx \\ &= \mathcal{O}(\varepsilon^{4-N+2b(\mu)}) + \omega_N \int_0^{\frac{R}{2}} \rho^{\alpha-4} y_\varepsilon^2(\rho)\rho^{N-1} d\rho \\ &= \mathcal{O}(\varepsilon^{4-N+2b(\mu)}) + \omega_N \varepsilon^{4-N} \int_0^{\frac{R}{2}} \rho^{\alpha-4-N-1} U_\mu^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1} d\rho \\ &= \mathcal{O}(\varepsilon^{\alpha}),\end{aligned}$$

because

$$\begin{aligned}\int_{\Omega\setminus B(0,\frac{R}{2})} |x|^{\alpha-4}u_\varepsilon^2 dx &\le \omega_N \int_{\frac{R}{2}}^{R} \rho^{\alpha-4} y_\varepsilon^2(\rho)\rho^{N-1} d\rho \\ &= \omega_N \varepsilon^{4-N} \int_{\frac{R}{2}}^{R} \rho^{\alpha-4} U_\varepsilon^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1} d\rho \\ &= \mathcal{O}(\varepsilon^{4-N+2b(\mu)})\end{aligned}$$

and

$$\omega_N \varepsilon^{4-N} \int_0^{\frac{R}{2}} \rho^{\alpha-4+N-1} U_\mu^2\left(\frac{\rho}{\varepsilon}\right) d\rho = \omega_N \varepsilon^{\alpha} \int_0^{\frac{R}{2\varepsilon}} \rho^{\alpha-4+N-1-2b(\mu)} d\rho.$$

Since $\alpha - 4 + N - 1 - 2b(\mu) < -1$, we get that

$$\omega_N \varepsilon^{4-N} \int_0^{\frac{R}{2}} \rho^{\alpha-4-N-1} U_\mu^2\left(\frac{\rho}{\varepsilon}\right)\rho^{N-1} d\rho = K\varepsilon^{\alpha}.$$

It follows from $\int_{\Omega\setminus B(0,\frac{R}{2})} |x|^{\alpha-4}u_\varepsilon^2 dx = \mathcal{O}(\varepsilon^{4-N+2b(\mu)})$ and $0 < \alpha < 2b(\mu) + 4 - N$, that

$$\int_\Omega |x|^{\alpha-4}u_\varepsilon^2 dx = \mathcal{O}(\varepsilon^{\alpha}),$$

$$\begin{aligned}\int_\Omega |x|^{-s}u_\varepsilon^{2^*(s)-1}u_0(x) dx =& \varepsilon^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |y|^{-s}[\varphi^{2^*(s)-1}(\varepsilon y) - 1] U_\varepsilon^{2^*(s)-1}(y) u_0(\varepsilon y) dy \\ &+ \varepsilon^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |y|^{-s} U_\varepsilon^{2^*(s)-1}(y)[u_0(\varepsilon y) - u_0(0)] dy \\ &+ \varepsilon^{\frac{N-4}{2}} \int_{\mathbb{R}^N} |y|^{-s} U_\varepsilon^{2^*(s)-1}(y) dy \\ =& \mathcal{O}\left(\varepsilon^{\frac{N-4}{2}}\right) + \varepsilon^{\frac{N-4}{2}} u_0(0) E,\end{aligned}$$

where

$$\begin{aligned}
E =& \int\limits_{\mathbb{R}^N} \frac{U_\mu^{2^*(s)-1}(x)}{|x|^s} dx = \omega_N \int\limits_0^{+\infty} U_\mu^{2^*(s)-1}(r) r^{N-s-1} dr \\
\leq& C_1 \int\limits_0^{R} r^{N-s-1-(2^*(s)-1)a(\mu)} dr + \omega_N \int\limits_R^{M} U_\mu^{2^*(s)-1}(r) r^{N-s-1} dr \\
&+ C_2 \int\limits_M^{+\infty} r^{N-s-1-(2^*(s)-1)b(\mu)} dr.
\end{aligned}$$

Let $N - s - (2^*(s) - 1)a(\mu) - 1 > -1$ and $N - s - (2^*(s) - 1)b(\mu) - 1 < -1$, thus $\mu < \zeta_s$. □

3. Proof of Theorem 1.1

The current section contains two subsections. In the first subsection we consider $0 < \lambda < \lambda_1$ and $0 < \mu < \overline{\mu}$, in the second subsection, we take $0 < \lambda < \lambda_1$ and $0 < \mu < \widehat{\mu}$.

3.1. Existence of solution in $\mathcal{N}^+$.

Using Ekeland's variational principl, we prove the existence of a solution in $\mathcal{N}^+$.

Proposition 3.1. *Let f satisfying* (1.2)*. Then $c_0 = \inf\limits_{u\in\mathcal{N}} I(u)$ is achieved at a point $u_0 \in \mathcal{N}^+$, which is a critical point and even a local minimum for I.*

Proof. We start by showing that I is bounded from below in $\mathcal{N}$. Indeed, for $u \in \mathcal{N}$ we have:

$$T(u) - \int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx - \int\limits_\Omega fu dx = 0.$$

Thus,

$$\begin{aligned}
I(u) &= \frac{1}{2} T(u) - \frac{1}{2^*(s)} \int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx - \int\limits_\Omega fu dx \\
&= \left(\frac{4-s}{2(N-s)} \right) T(u) - \left(\frac{N+4-2s}{2(N-s)} \right) \int\limits_\Omega fu dx \\
&\geq -\frac{(N+4-2s)^2}{8(N-s)(4-s)} \|f\|_-^2.
\end{aligned}$$

In particular,

$$c_0 \geq -\frac{(N+4-2s)^2}{8(N-s)(4-s)} \|f\|_-^2.$$

From Lemma 2.3, we can get $t_0 = t_0(v)$ such that $t_0 v \in \mathcal{N}$ and $I(t_0 v) > 0$. Moreover,

$$I(t_0 v) = \frac{1}{2} t_0^2 T(v) - \frac{t_0^{2^*(s)}}{2^*(s)} \int\limits_\Omega \frac{|v|^{2^*(s)}}{|x|^s} dx - t_0 \int\limits_\Omega fv dx$$

$$= -\frac{1}{2}t_0^2 T(v) + \left(1 - \frac{1}{2^*(s)}\right) t_0^{2^*(s)} \int_\Omega \frac{|v|^{2^*(s)}}{|x|^s} dx$$
$$< -\frac{4-s}{2(N-s)} t_0^2 T(v) < 0.$$

Hence,

$$c_0 \leq I(t_0 v) < 0. \tag{3.1}$$

Applying the Ekeland's variational principle to the minimization problem (1.1), we can get a minimizing sequence $(u_n) \subset \mathcal{N}^+$ satisfying :

(i) $I(u_n) < c_0 + \frac{1}{n}$;
(ii) $I(u_n) \leq I(w) + \frac{1}{n}\|w - u_n\|$, for all $w \in \mathcal{N}$.

By taking n large enough, we get from (3.1):

$$I(u_n) = \frac{4-s}{2(N-s)} T(u_n) - \frac{N+4-2s}{2(N-s)} \int_\Omega f u_n dx < c_0 + \frac{1}{n} \leq -\frac{4-s}{2(N-s)} t_0^2 T(u_n).$$

This implies that

$$\int_\Omega f u_n dx \geq \frac{(4-s)t_0^2}{N+4-2s} T(u_n), \tag{3.2}$$

consequently, $u_n \neq 0$ and we have:

$$\frac{4-s}{N+4-2s} \cdot \frac{t_0^2}{\|f\|_-} T(u_n) \leq \|u_n\| \leq \frac{N+4-2s}{(4-s)\rho} \|f\|_-, \tag{3.3}$$

where the constant $\rho > 0$ verifies:

$$T(u) \geq \rho \|u\|^2. \tag{3.4}$$

Next we shall prove that $\|I'(u_n)\| \to 0$ as $n \to +\infty$. Hence, let us assume $\|I'(u_n)\| > 0$ for n large enough. By Applying Lemma 2.4, with $u = u_n$ and $w = \sigma\left(\frac{I'(u_n)}{\|I'(u_n)\|}\right)$, $\sigma > 0$, we can find some $t_n(\sigma) = t\sigma\left(\frac{I'(u_n)}{\|I'(u_n)\|}\right)$ such that

$$w_\sigma = t_n(\sigma)\left[u_n - \sigma \frac{I'(u_n)}{\|I'(u_n)\|}\right] \in \mathcal{N}.$$

By condition (ii), we obtain:

$$\frac{1}{n}\|w - u_n)\| \geq I(u_n) - I(w_\sigma)$$
$$= (1 - t_n(\sigma))\langle I'(w_\sigma), u_n\rangle + \sigma t_n(\sigma)\left\langle I'(w_\sigma), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle + o_n(\sigma).$$

Dividing by σ and passing to the limit as σ goes to zero we derive that:

$$\frac{1}{n}(1 + |t_n'(0)|\ \|u_n\|) \geq -t_n'(0)\langle I'(u_n), u_n\rangle + \|I'(u_n)\| = \|I'(u_n)\|,$$

where $t_n'(0) = \langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle$. So, we conclude that

$$\|I'(u_n)\| \le \frac{C}{n}(1 + |t_n'(0)|), \quad C > 0.$$

The proof will be completed once we have shown that $|t_n'(0)|$ uniformly bounded with respect to n. From (2.4) and the estimate (3.3), we get:

$$|t_n'(0)| \le \frac{C_1}{\left|T(u_n) - (2^*(s) - 1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx\right|},$$

C_1 is a suitable constant. Hence, we must prove that $|T(u_n) - (2^*(s) - 1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called (u_n), we have

$$\left|T(u_n) - (2^*(s) - 1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx\right| = o_n(1). \tag{3.5}$$

According to (3.3) and (3.5), there exists a constant $C_2 > 0$ such that

$$\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx \ge C_2.$$

In addition, from (3.5) and by the fact that $u_n \in \mathcal{N}$, we get

$$\int\limits_\Omega f u_n dx = (2^*(s) - 2)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx + o_n(1).$$

This together with (1.2) imply that

$$0 < (2^*(s) - 2)\left[\left(\frac{T(u_n)}{(2^*(s) - 1)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}dx}\right)^{\frac{2^*(s)-1}{2^*(s)-2}} - 1\right] = o_n(1),$$

which is clearly impossible.

In conclusion,

$$I'(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.6}$$

Let $u_0 \in H_0^2(\Omega)$ be the weak limit in $H_0^2(\Omega)$ of (u_n). From (3.2) we derive that $\int\limits_\Omega f u_0 > 0$, and from (3.6) that $\langle I'(u_0), w\rangle = 0$, for all $w \in H_0^2(\Omega)$, i.e., u_0 is a weak solution for (1.1). In fact, $u_0 \in \mathcal{N}$ and $c_0 \le I(u_0) \le \lim\limits_{n\to+\infty} I(u_n) = c_0$. So, we deduce that $u_n \to v$ strongly in $H_0^2(\Omega)$ and $I(u_0) = c_0 = \inf\limits_{u\in\mathcal{N}} I(u)$. Moreover, $u_0 \in \mathcal{N}^+$. So u_0 is a local minimum for I. □

3.2. **Existence of solution in $\mathcal{N}^-$.** In this subsection, for proof of the existence of a solution in $\mathcal{N}^-$, we shall find the range of c where I verifies the $(PS)_c$ condition.

Lemma 3.1. *Let (u_n) be any sequence of $H_0^2(\Omega)$ satisfying the following conditions:*

(a) $I(u_n) \to c$ *with* $c < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}$;

(b) $\|I'(u_n)\| \to 0$ *as* $n \to +\infty$.

Then (u_n) has a strongly convergent subsequence.

Proof. We have $I(u_n) = c + o_n(1)$ and

$$\langle I'(u_n), u_n\rangle = T(u_n) - \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx - \int_\Omega f u_n dx + o_n(1). \tag{3.7}$$

Then

$$\frac{4-s}{2(N-s)} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx + o_n(1) = c + \frac{1}{2}\int_\Omega f u_n dx - \frac{1}{2}\langle I'(u_n), u_n\rangle + \mathcal{O}(1).$$

By using Hölder inquality, we get

$$\frac{4-s}{2(N-s)} \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx \le c + \frac{1}{2}\|f\|_- \|u_n\| + \frac{1}{2}\|I'(u_n)\|_- \|u_n\|. \tag{3.8}$$

From (3.4), (3.7) and (3.8), we have for all $\varepsilon > 0$:

$$\begin{aligned} \rho\|u_n\| \le T(u_n) &\le \int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} dx + \int_\Omega f u_n dx + \langle I'(u_n), u_n\rangle \\ &\le \frac{2(N-s)}{4-s} c + \frac{N+4-2s}{4-s} (\|f\|_- + \|I'(u_n)\|_-)\|u_n\| + \varepsilon\|u_n\|. \end{aligned}$$

So, $T(u_n)$ is uniformly bounded. For a subsequence of (u_n), we can get a $u \in H_0^2(\Omega)$ such that $u_n \rightharpoonup u$. So, from (b), we obtain that

$$\langle I'(u), w\rangle = 0, \quad \text{for all } w \in H_0^2(\Omega).$$

Then u is a weak solution for (1.1). In particular $u \neq 0$, $u \in \mathcal{N}$ and $I(u) \ge c_0$. We have:

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ weakly in } H_0^2(\Omega), \\ &u_n \rightharpoonup u \ \text{ weakly in } L^2(\Omega, |x|^{-4}) \text{ and } L^{2^*(s)}(\Omega, |x|^{-s}), \\ &u_n \to u \ \text{ strongly in } L^2(\Omega, |x|^{\alpha-4}), \\ &u_n \to u \ \text{ strongly in } L^q(\Omega) \text{ for all } 1 \le q < 2^*(s). \end{aligned}$$

Let $u_n = u + v_n$. So, $v_n \rightharpoonup 0$ in $H_0^2(\Omega)$. As in Brezis-Lieb Lemma (see [4]), we conclude that

$$c + o_n(1) = I(u) + I(v_n) + \int_\Omega f v_n dx \tag{3.9}$$

and

$$o_n(1) = I'(v_n) + \int_\Omega f v_n dx.$$

Without loss of generality, as $n \to +\infty$ we may assume that

$$T(v_n) \to l, \quad \int_\Omega \frac{|v_n|^{2^*(s)}}{|x|^s} dx \to l.$$

From (2.2) we obtain

$$l \geq A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

By (3.9), we deduce that $I(u) = c - \frac{4-s}{2(N-s)} l \leq c - \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} < c_0$, which contradicts the fact that $c_0 = \inf I$. Hence, $l = 0$ and $u_n \to u$ strongly in $H_0^2(\Omega)$ as $n \to +\infty$. □

Lemma 3.2. *Let $f \neq 0$ satisfying (1.2) and if $0 < \alpha \leq \frac{1}{2}$ for $N \geq 5$ or $\frac{1}{2} < \alpha < 4$ for $5 \leq N < 12$, then for all $t > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$*

$$I(u_0 + tu_\varepsilon) < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}. \tag{3.10}$$

Proof. We infer from [3] that:

$$\begin{aligned}\int_\Omega \frac{|u_0 + tu_\varepsilon|^{2^*(s)}}{|x|^s} dx =& \int_\Omega \frac{|u_0|^{2^*(s)}}{|x|^s} dx + t^{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx \\ &+ 2^*(s) t \int_\Omega \frac{|u_0|^{2^*(s)-2} u_0 u_\varepsilon}{|x|^s} dx + 2^*(s) t^{2^*(s)-1} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)-1} u_0}{|x|^s} dx \\ &+ \mathcal{O}\left(\varepsilon^{2b(\mu)+4-N}\right).\end{aligned}$$

Since $u_0 \in \mathcal{N}$ is a solution of (1.1) and from Proposition 2.1, we obtain:

$$\begin{aligned}I(u_0 + tu_\varepsilon) =& I(u_0) + \frac{t^2}{2} T(u_\varepsilon) - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx \\ &- \frac{1}{2^*(s)} \int_\Omega \frac{|u_0 + tu_\varepsilon|^{2^*(s)} - |u_0|^{2^*(s)} - |tu_\varepsilon|^{2^*(s)} - 2^*(s)|u_0|^{2^*(s)-2} u_0 t u_\varepsilon}{|x|^s} dx \\ =& I(u_0) + \frac{t^2}{2} T(u_\varepsilon) - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|x|^s} dx - t^{2^*(s)-1} \int_\Omega \frac{|u_\varepsilon|^{2^*(s)-1} u_0}{|x|^s} dx \\ &- \mathcal{O}(\varepsilon^{2b(\mu)+4-N}) \\ =& I(u_0) + \frac{t^2}{2} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} - \frac{t^{2^*(s)}}{2^*(s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)} - t^{2^*(s)-1} \varepsilon^{\frac{N-4}{2}} u_0(0) E \\ &+ \mathcal{O}\left(\varepsilon^{2^*(s)b(\mu)-N+s}\right) - \mathcal{O}\left(\varepsilon^{\alpha}\right) + o_n\left(\varepsilon^{\frac{N-4}{2}}\right) + \mathcal{O}\left(\varepsilon^{2b(\mu)-N+4}\right).\end{aligned}$$

Define

$$g(t)=\frac{t^2}{2}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-\frac{t^{2^*(s)}}{2^*(s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t^{2^*(s)-1}\varepsilon^{\frac{N-4}{2}}u_0(0)E,\quad t>0,$$

and assume that $g(t)$ achieves its maximum at $t_0>0$. Since

$$t_0A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t_0^{2^*(s)-1}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}=(2^*(s)-1)t_0^{2^*(s)-2}\varepsilon^{\frac{N-4}{2}}u_0(0)E,$$

necessarily $0<t_0<1$ and $t_0\to 1$ as $\varepsilon\to 0$.

Note that $t\to\frac{t^2}{2}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-\frac{t^{2^*(s)}}{2^*s}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}$ rises monotonically on $[0,1]$, so,

$$\begin{aligned}I(u_0+tu_\varepsilon)<&c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}-t^{2^*-1}\varepsilon^{\frac{N-4}{2}}u_0(0)E+\mathcal{O}\left(\varepsilon^{2^*(s)b(\mu)-N+s}\right)\\&-\mathcal{O}\left(\varepsilon^{\alpha}\right)+o_n\left(\varepsilon^{\frac{N-4}{2}}\right)+\mathcal{O}\left(\varepsilon^{2b(\mu)+4-N}\right).\end{aligned}$$

We distinguish the following two cases.

Case 1. When $2^*(s)b(\mu)-N>2b(\mu)+4-N>\frac{N-4}{2}\geq\alpha$ if $5\leq N$, we have $0<\mu\leq\varsigma_\alpha$ and $0<\alpha\leq\frac{1}{2}$, then, for $\mu\in]0,\widehat{\mu}[$, we obtain:

$$I(u_0+tu_\varepsilon)<c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Case 2. When $2^*(s)b(\mu)-N>2b(\mu)+4-N>\alpha>\frac{N-4}{2}$ if $5\leq N<12$, we have $0<\mu<\varsigma_{\frac{N-4}{2}}$ and $\frac{1}{2}\leq\alpha<4$, then, for $\mu\in]0,\widehat{\mu}[$, we obtain:

$$I(u_0+tu_\varepsilon)<c_0+\frac{4-s}{2(N-s)}A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.\qquad\square$$

Finally, it remains to show the following proposition.

Proposition 3.2. *Suppose that f verifies conditions of Lemma* 3.2. *Then I has a minimizer $u_1\in\mathcal{N}^-$ such that $c_1=I(u_1)$. Moreover, u_1 is a solution of Problem* (1.1).

Proof. Let $(v_n)\subset\mathcal{N}^-$ such that

$$I(v_n)\to c_1\quad\text{and}\quad I'(v_n)\to 0,\quad\text{in } H^{-2}(\Omega).$$

For $u\in H_0^2(\Omega)$ such that $\|u\|=1$. By Lemma 2.3, there exists a unique $t^+(u)>0$ such that $t^+(u)u\in\mathcal{N}^-$ and $I(t^+(u)u)=\max_{s\geq t_{\max}}I(su)$. According to Lemma 2.5, we have $u_0\in E_1$, we can choose a constant c', which satisfies $0<t^+(u)\leq c'$, for all $\|u\|=1$, we claim that

$$u_0+t_0u_\varepsilon\in E_2,\tag{3.11}$$

where $t_0=\left(\frac{|c'^2-\|u_0\|^2|}{\|u_\varepsilon\|}\right)^{\frac{1}{2}}+1$. In fact, a direct computation shows that:

$$\begin{aligned}\|u_0+t_0u_\varepsilon\|^2=&\|u_0\|^2+t_0^2\|u_\varepsilon\|^2+2t_0\int_\Omega\left(\Delta u_0\Delta u_\varepsilon-\mu\frac{u_0u_\varepsilon}{|x|^4}\right)dx\\=&\|u_0\|^2+t_0^2\|u_\varepsilon\|^2+o_n(1)\end{aligned}$$

$$>c'^2 \geq \left[t^+ \left(\frac{u_0 + t_0 u_\varepsilon}{\|u_0 + t_0 u_\varepsilon\|} \right) \right]^2,$$

for $\varepsilon > 0$ small enough. Thus, claim (3.11) holds. We fix $\varepsilon > 0$ such that both (3.10) and (3.11) hold by the choice of t_0. We set

$$\Gamma = \{\gamma \in C([0;1] : H_0^2(\Omega)) : \gamma(0) = u_0, \gamma(1) = u_0 + t_0 u_\varepsilon\},$$

and take $h(t) = u_0 + tt_0 u_\varepsilon$, which belongs to Γ. From Lemma 3.1, we conclude that:

$$c = \inf_{h \in \Gamma} \max_{t \in [0;1]} I(h(t)) < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Since every $h \in \Gamma$ intersects $\mathcal{N}^-$, we get that:

$$c_1 = \inf_{\mathcal{N}^-} I \leq c < c_0 + \frac{4-s}{2(N-s)} A_{\mu,s}^{\left(\frac{N-s}{4-s}\right)}.$$

Using Lemma 3.2, we deduce that v_n converges strongly to u_1 in $H_0^2(\Omega)$. Thus, $u_1 \in \mathcal{N}^-$ and $c_1 = I(u_1)$. Then $I'(u_1) = 0$, and thus u_1 is a solution of Problem (1.1). We conclude that Problem (1.1) admits also a solution in $\mathcal{N}^-$. ☐

Proof of Theorem 1.1. By Porpositions 3.1, 3.2 and as $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$ we deduce that the problem (1.1) admits two solutions u_0 and u_1 with $u_0 \neq u_1$. ☐

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