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NEW GENERALIZED BIPARAMETRIC FUBINI-TYPE POLYNOMIALS OF LEVEL-m

WILLIAM RAMÍREZ[1,2], DANIEL BEDOYA[3], CLEMENTE CESARANO[2], ALEJANDRO URIELES[4], AND SHAHID AHMAD WANI[5]

Abstract. This paper introduces two new biparametric families of generalized Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x,y,a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x,y,a)$. We give some algebraic and differential properties, as well as relationships between these classes of polynomials with other polynomials and numbers. In addition, we introduce the new generalized biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$. As a result, we derive the product formula for $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$ and provide some factorizations of biparametric Fubini-type polynomials matrix $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$, which involve the generalized Pascal matrix.

1. Introduction

This section presents some generating functions that are well-known Fubini numbers and Fubini polynomials, which have many applications in several fields of mathematics, physics, and engineering. The Fubini numbers $w_g(n)$, were studied, and are given by the following generating function (see, [3, p. 11, Eq. (1)]):

$$\frac{1}{2-e^z}=\sum_{n=0}^{+\infty} w_g(n)\frac{z^n}{n!},\quad |z|<\ln 2,$$

where $w_g(0)=1$.

Key words and phrases. Generalized parametric Fubini-type polynomials, Fubini polynomials, Apostol-Bernoulli polynomials of higher order, Apostol-Genocchi polynomials of higher order, Stirling numbers of the second kind, generalized Pascal matrix.

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On the other hand, the Fubini-type numbers were defined and are given by the following generating function (see, [10, p. 397, Eq. (10.21)]):

$$\frac{e^z-1}{2-e^z}=\sum_{n=0}^{+\infty} w_M(n)\frac{z^n}{n!}, \quad |z|<\ln 2.$$

Also, a new family of numbers a_n was defined, which are obtained by modifying the numbers $w_g(n)$, give by (see, [4, p. 1609, Eq. (14)]):

$$\frac{2}{(2-e^z)^2}=\sum_{n=0}^{+\infty} a_n\frac{z^n}{n!}, \quad |z|<\ln 2.$$

Recently, the generalized Fubini-type polynomial $a_n^{(\alpha)}(x)$ of order α, was defined by means of the following generating function (see, [4, p. 1611, Eq. (18)]):

$$\left(\frac{2}{(2-e^z)^2}\right)^{\alpha} e^{xz}=\sum_{n=0}^{+\infty} a_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad |z|<\ln 2,$$

where $\alpha\in\mathbb{N}_0$. Observe that $a_n^{(\alpha)}(0)=a_n^{(\alpha)}$ denotes the Fubini-type numbers of order α.

Two new families of biparametric Fubini-type polynomials were defined, which are given by the following generating function (see, [14, Eq. (2.2) and Eq. (2.3)]):

$$\left(\frac{2}{(2-e^z)^2}\right)^{\alpha} e^{xz}\cos(yz)=\sum_{n=0}^{+\infty} a_n^{(c,\alpha)}(x,y)\frac{z^n}{n!}, \tag{1.1}$$

$$\left(\frac{2}{(2-e^z)^2}\right)^{\alpha} e^{xz}\sin(yz)=\sum_{n=0}^{+\infty} a_n^{(s,\alpha)}(x,y)\frac{z^n}{n!}. \tag{1.2}$$

Another important family in this area of study is the Genocchi polynomials $G_n(x)$, a sequence of polynomials related to a generalization of the Genocchi numbers. These polynomials arise in number theory and are related to the Riemann zeta function and Fourier series. Their generating function is given (see, [12, Eq. (2)]):

$$\sum_{n=0}^{+\infty}\frac{G_n(x)}{n!}t^n=\frac{2te^t}{e^t+1}e^{xt}, \quad |t|<\pi.$$

In the past, some authors have introduced and studied new families of biparametric polynomials in which they demonstrate interesting properties and presented useful applications in this area of mathematics (see, [1,5,7,8,13,14]). One of the applications introduced by a large number of authors in various contexts is the matrix approach of several families of polynomials [6,16,17]. This paper will introduce two new classes of biparametric Fubini-type polynomials of level-m (see, [4,11,14,17]). Finally, we will show the relationships between these polynomials and other polynomials and numbers, as well as the generalized biparametric Fubini-type polynomials matrices.

This paper will be organized as follow. Section 2 contains the definitions and some auxiliary results. Section 3 will define the generalized biparametric Fubini-type of level-m polynomials and proved some algebraic and differential properties of them, as well as their relationships with other families of polynomials and the Stirling

numbers of the second kind. Part 4 will introduce the generalized biparametric Fubini-type polynomial matrix derived a product formula for it. In Section 5 we will give some factorizations for such a matrix, which involves summation matrices and the generalized Pascal matrix of the first kind, respectively. Finally, the conclusion of the paper will be presented.

2. Previous Definitions and Notations

Throughout this paper, we use the following standard notions by: $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural, nonnegative integer, real and complex numbers, respectively. Furthermore, we denote the symbol of Pochhammer by

$$(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1),$$

where $(\lambda)_0 = 1$ and $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. For the complex logarithm, we consider the principal branch. All matrices are in $M_{n+1}(\mathbb{K})$, the set of all $(n+1)\times(n+1)$ matrices over the field $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$. Also, for i, j any nonnegative integers we adopt the following convention $\binom{i}{j} = 0$, whenever $j > i$. Aditionally, we take that $0^0 = 1$ and $0^n = 0$ if $n \in \mathbb{N}$.

For real parameters x and y the Taylor series representation in $z = 0$ of the following functions $e^{xz}\cos(yz)$ and $e^{xz}\sin(yz)$ are given by (see, [14]):

$$\begin{aligned} F_c(z;x;y) =& e^{xz}\cos(yz) = \sum_{k=0}^{+\infty} C_k(x,y)\frac{z^k}{k!}, \\ F_s(z;x;y) =& e^{xz}\sin(yz) = \sum_{k=0}^{+\infty} S_k(x,y)\frac{z^k}{k!}, \end{aligned}$$

where the expressions $C_k(x,y)$ and $S_k(x,y)$ are given by:

$$\begin{aligned} C_k(x;y) =& \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k}{2j} x^{k-2j}y^{2j}, \\ S_k(x;y) =& \sum_{j=0}^{\left[\frac{k-1}{2}\right]} (-1)^j \binom{k}{2j+1} x^{k-2j-1}y^{2j+1}. \end{aligned}$$

Proposition 2.1. *For $n \in \mathbb{N}$, let $\{a_n^{(c,\alpha)}(x,y)\}_{n\geq 0}$ and $\{a_n^{(s,\alpha)}(x,y)\}_{n\geq 0}$ be the sequences of Fubini-type biparametric polynomials of order α, respectively. Then, the following statements hold (see, [4, 14]).*

(a) *Special values. For $\alpha, \beta \in \mathbb{N}$,*

$$a_n^{(\alpha+\beta)} = \sum_{k=0}^{n} \binom{n}{k} a_k^{(\alpha)} a_{n-k}^{(\beta)}.$$

(b) *Summation formulas:*

$$\begin{aligned}
a_n^{(c,\alpha)}(x+y,y) &= \sum_{k=0}^{n} \binom{n}{k} a_k^{(c,\alpha)}(x,y) y^{n-k}, \quad \alpha \in \mathbb{N}_0, \\
a_n^{(s,\alpha)}(x+y,y) &= \sum_{k=0}^{n} \binom{n}{k} a_k^{(s,\alpha)}(x,y) y^{n-k}, \quad \alpha \in \mathbb{N}_0, \\
a_n^{(\alpha)}(x+1) &= \sum_{k=0}^{n} \binom{n}{k} a_k^{(\alpha)}(x), \\
\frac{\partial^m}{\partial x^m} a_n^{(\alpha)}(x) &= m! \binom{n}{m} a_{n-m}^{(\alpha)}(x), \quad n, m \in \mathbb{N}, n \geqslant m, \\
\int_0^1 a_{n-m}^{(\alpha)}(x)dx &= \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{a_k^{(\alpha)}}{n+1-k-m}.
\end{aligned}$$

For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, the Stirling numbers of second kind $S(n,k)$ are defined by means to of the following expansion (see, [2, p. 207]):

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k).$$

The Jacobi polynomial of the degree n and order (α, β), with $\alpha, \beta > -1$, $P_n^{(\alpha,\beta)}(x)$ may be defined through Rodrigues' formula (see, [11, p. 395]):

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dx^n} \left\{ (1-x)^{n+\alpha}(1+x)^{n+\alpha} \right\}.$$

The relationship between the n-th monomial x^n and the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be written as

$$x^n = n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}} P_k^{(\alpha,\beta)}(1-2x). \tag{2.1}$$

Proposition 2.2. *For $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$, $\{G_n(x)\}_{n\geq 0}$ and $\{\mathscr{E}_n(x;\lambda)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level-m, Genocchi polynomials and Apostol-Euler polynomials, respectively. Then, we have the relationships:*

(a) [15, Eq. (4)]

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x), \tag{2.2}$$

(b) [12, Eq. (8)]

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} G_k(x) + G_{n+1}(x) \right], \tag{2.3}$$

(c) [9, Eq. (32)]

$$x^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} \mathscr{E}_k(x;\lambda) + \mathscr{E}_n(x;\lambda)\right]. \tag{2.4}$$

Let x be any nonzero real number. The generalized Pascal matrix of first kind $P[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [17, Definition 1]):

$$p_{i,j}(x) := \begin{cases} \binom{i}{j}(x)^{i-j}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Let the $(n+1)\times(n+1)$ Fibonacci matrix be $\mathscr{F} = [f_{i,j}]$, $i,j = 0,1,2,\ldots,n$, whose entries are given by (see, [18, Eq. (16)]):

$$f_{i,j} := \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0, \\ 0, & \text{if } i-j+1 < 0, \end{cases}$$

where F_n is the n-th Fibonacci number. Also, they gave the inverse of $\mathscr{F}$ as follows. If $\mathscr{F}^{-1} = [f'_{i,j}]$, where $i,j = 0,1,2,\ldots,n$, then

$$f'_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i = j+1, i = j+2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Let $\{L_n\}_{n\geq 1}$ be the Lucas numbers sequence, i.e., $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$. The Lucas matrix $\mathscr{L} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [19, Eq. (2)]):

$$l_{i,j} := \begin{cases} L_{i-j+1}, & \text{if } i-j \geq 0, \\ 0, & \text{if } i-j < 0, \end{cases}$$

where L_n is the n-th Lucas number.

3. Generalized Biparametric Fubini Type Polynomials of Level-m

In this section, we introduce two new biparametric family of Fubini type polynomials of level-m and prove some algebraic and differential properties of them.

Definition 3.1. For $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$, $a \in \mathbb{R}^+$ and $z \in \mathbb{C}\setminus\{0\}$, the generalized biparametric Fubini-type polynomials in the variable x and y, parameter a order α and level-m, are defined through the following generating functions

$$\left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} + 1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz) = \sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!}, \qquad |z| < \ln 2, \tag{3.1}$$

$$\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\sin(yz)=\sum_{n=0}^{+\infty}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!},\quad |z|<\ln 2. \tag{3.2}$$

For $x=0$, $y=0$ in (3.1) and $x=0$, $y=\frac{\pi}{2z}$ in (3.2) we obtain the generalized biparametric Fubini-type numbers of parameter a order α and level-m

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(a):=\mathscr{A}_{n,c}^{[m-1,\alpha]}(0;0;a),$$
$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(a):=\mathscr{A}_{n,s}^{[m-1,\alpha]}(0;\frac{\pi}{2z};a).$$

By comparing Definition 3.1 with (1.1) and (1.2) we have

$$\mathscr{A}_{n,c}^{[0,\alpha]}(x;y;e)=a_n^{(c,\alpha)}(x,y),$$
$$\mathscr{A}_{n,s}^{[0,\alpha]}(x;y;e)=a_n^{(s,\alpha)}(x,y).$$

Example 3.1. Using (3.3) for any $m=2$ and $a=2$, the first generalized biparametric Fubini-type numbers, order α and level-m are:

$$\begin{aligned}
\mathscr{A}_{0,c}^{[1,\alpha]}(0;0;2)&=2^{\alpha},\\
\mathscr{A}_{1,c}^{[1,\alpha]}(0;0;2)&=2^{\alpha}\left(\frac{42272111787129327}{18014398509481984}\alpha\right),\\
\mathscr{A}_{2,c}^{[1,\alpha]}(0;0;2)&=2^{\alpha}\left(\frac{1786931434943558184171171423472929}{3245185536584267267831560205762 56}\right)\alpha^2\\
&\quad+2^{\alpha}\left(\frac{1259095235339315236735153314706143}{3245185536584267267831560205762 56}\right)\alpha.
\end{aligned}$$

Example 3.2. Using (3.3) for any $m=2$ and $a=2$, the first generalized biparametric Fubini-type polynomials in the variable x and y, order α and level-m are:

$$\begin{aligned}
\mathscr{A}_{0,c}^{[1,\alpha]}(x;y;2)&=2^{\alpha},\\
\mathscr{A}_{1,c}^{[1,\alpha]}(x;y;2)&=2^{\alpha}\left(\frac{42272111787129327}{18014398509481984}\alpha+x\right),\\
\mathscr{A}_{2,c}^{[1,\alpha]}(x;y;2)&=2^{\alpha}\left(\frac{1786931434943558184171171423472929}{3245185536584267267831560205762 56}\right)\alpha^2\\
&\quad+2^{\alpha}\left(\frac{1523013335141436710453162369089536}{3245185536584267267831560205762 56}\right)\alpha x\\
&\quad+2^{\alpha}\left(\frac{1259095235339315236735153314706143}{3245185536584267267831560205762 56}\right)\alpha\\
&\quad+2^{\alpha}\left(x^2-y^2\right).
\end{aligned}$$

Example 3.3. Using (3.4) for any $m=4$, and $a=3$, the first generalized biparametric Fubini-type polynomials in the variable x and y, order α and level-m are:

$$\mathscr{A}_{0,s}^{[3,\alpha]}(x;y;3)=2^{\alpha}y,$$

$$
\begin{aligned}
\mathscr{A}_{1,s}^{[3,\alpha]}(x;y;3) &= 2^{\alpha}\left(\frac{11481054201676165}{4503599627370496}\alpha y + xy\right),\\
\mathscr{A}_{2,s}^{[3,\alpha]}(x;y;3) &= 2^{\alpha}\left(\frac{39544381674547776728728658732167 5}{6084722881095501127184175385804 8}\right)\alpha^2 y\\
&\quad + 2^{\alpha}\left(\frac{31023642854693546475776800456704 0}{6084722881095501127184175385804 8}\right)\alpha xy\\
&\quad + 2^{\alpha}\left(\frac{26174879013924692794648460905197 9}{6084722881095501127184175385804 8}\right)\alpha y\\
&\quad + 2^{\alpha}\left(x^2 y - \frac{20282409603651670423947251286016}{6084722881095501127184175385804 8}y^3\right).
\end{aligned}
$$

Theorem 3.1. *For $m \in \mathbb{N}$, let $\{\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\}_{n\geq 0}$ and $\{\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\}_{n\geq 0}$ be the sequence of generalized biparametric Fubini-type polynomials, with parameter $a \in \mathbb{R}^+$, order α and level-m. Then, the following statements hold.*

(a) *For every $\alpha = 0$ and $n \in \mathbb{N}_0$*

$$
\mathscr{A}_{n,c}^{[m-1,0]}(x;y;a) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \tag{3.3}
$$

$$
\mathscr{A}_{n,s}^{[m-1,0]}(x;y;a) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \tag{3.4}
$$

(b) *For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{N}_0$, we have the relationship*

$$
\begin{aligned}
\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a) &= \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(a) C_k(x,y)\\
&= \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k}\binom{k}{2j} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(a) x^{k-2j} y^{2j},\\
\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a) &= \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(a) S_k(x,y)\\
&= \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{n}{k}\binom{k}{2j+1} (-1)^j \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(a) x^{k-2j-1} y^{2j+1}.
\end{aligned}
$$

(c) *Differential relationships. For $m \in \mathbb{N}$ and $n, k \in \mathbb{N}_0$, with $0 \leq k \leq n$, we have*

$$
\begin{aligned}
\frac{\partial^k}{\partial x^k}\left[\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\right] &= \frac{n!}{(n-k)!}\mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a),\\
\frac{\partial^k}{\partial x^k}\left[\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a))\right] &= \frac{n!}{(n-k)!}\mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a).
\end{aligned}
$$

(d) *Integral formulas. For $m \in \mathbb{N}$, is fulfilled*

$$\int_{x_0}^{x_1} \mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\,dx = \frac{\left[\mathscr{A}_{n+1,c}^{[m-1,\alpha]}(x_1;y;a) - \mathscr{A}_{n+1,c}^{[m-1,\alpha]}(x_0;y;a)\right]}{n+1},$$
$$\int_{x_0}^{x_1} \mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\,dx = \frac{\left[\mathscr{A}_{n+1,s}^{[m-1,\alpha]}(x_1;y;a) - \mathscr{A}_{n+1,s}^{[m-1,\alpha]}(x_0;y;a)\right]}{n+1}.$$

(e) *Addition theorem of the argument*

$$\mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,c}^{[m-1,\beta]}(y;0,a), \tag{3.5}$$

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k}\binom{k}{2j} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a)y^k, \tag{3.6}$$

$$C_n(x+y,y) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k,c}^{[m-1,-\alpha]}(y;0;a), \tag{3.7}$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha+\beta]}(x+y;y;a) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{k,s}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,s}^{[m-1,\beta]}\left(y;\frac{\pi}{2z},a\right),$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \binom{n}{k}\binom{k}{2j+1} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a)y^k,$$

$$S_n(x+y,y) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k,s}^{[m-1,-\alpha]}\left(y;\frac{\pi}{2z};a\right).$$

Proof. For (3.5), from Definition 3.1, we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a)\frac{z^n}{n!} &= \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} + 1}{(2-e^z)^2}\right]^{(\alpha+\beta)} e^{(x+y)z}\cos(yz) \\
&= \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} + 1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz) \\
&\quad \times \left[\frac{\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} + 1}{(2-e^z)^2}\right]^{\beta} e^{yz}\cos(0) \\
&= \sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!} \sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\beta]}(y;0;a)\frac{z^n}{n!} \\
&= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,c}^{[m-1,\beta]}(y;0;a)\frac{z^n}{n!}.
\end{aligned}$$

□

Proof. We give proof of (3.7)

$$\begin{aligned}
\sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a)\frac{z^n}{n!} &= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{(\alpha+\beta)} e^{(x+y)z}\cos(yz)\\
&= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz)\\
&\quad\times\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\beta} e^{yz}\cos(0)\\
&= \sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!}\sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,-\alpha]}(y;0;a)\frac{z^n}{n!}\\
&= e^{(x+y)z}\cos(yz)\\
&= \sum_{n=0}^{+\infty} C_n(x+y,y)\frac{z^n}{n!}.
\end{aligned}$$

□

From (2.1) and Proposition 2.2, we have deduced some algebraic relationship connecting the polynomials $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, with other families of polynomials.

Theorem 3.2. *For $m\in\mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, are related with the Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$, by means of the following identity:*

$$\begin{aligned}
(3.8)\quad \mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{n-k}(-1)^v\binom{n}{k}\binom{k}{2j}\binom{n-k+\alpha}{n-k-v}\\
&\quad\times\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k-1}}\mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)P_v^{(\alpha,\beta)}(1-2y),\\
(3.9)\quad \mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{n-k}(-1)^v\binom{n}{k}\binom{k}{2j+1}\binom{n-k+\alpha}{n-k-v}\\
&\quad\times\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k-1}}\mathscr{A}_{k,s}^{[m-1,\alpha]}(x;y;a)P_v^{(\alpha,\beta)}(1-2y).
\end{aligned}$$

Proof. By replacing (2.1) into the right-hand side of (3.6), we obtain

$$\begin{aligned}
\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{n}{k}\binom{k}{2j}\mathscr{A}_k^{[m-1,\alpha]}(x;y;a)(n-k)!\\
&\quad\times\sum_{v=0}^{n-k}(-1)^v\binom{n-k+\alpha}{n-k-v}\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k+1}}P_v^{(\alpha,\beta)}(1-2y)
\end{aligned}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{n-k} \binom{n}{k} \binom{k}{2j} \binom{n-k+\alpha}{n-k-v} (-1)^v (n-k)! \times \frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k+1}} \mathscr{A}_{k,c}^{[m-1,\alpha]}(x,y;a) P_v^{(\alpha,\beta)}(1-2y).$$

Therefore, (3.8) holds. The proof of (3.9) is similar. □

Theorem 3.3. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$, and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, are related with the generalized Bernoulli polynomials of level-m $B_n^{[m-1]}(x)$, by means of the following identity:*

(3.10)

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k} \frac{v!}{(v+m)!} \binom{n}{k} \binom{k}{2j} \binom{k}{v} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) B_{k-v}^{[m-1]}(y),$$

(3.11)

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k} \frac{v!}{(v+m)!} \binom{n}{k} \binom{k}{2j} \binom{k}{v} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) B_{k-v}^{[m-1]}(y).$$

Proof. By substituting (2.2) into the right-hand side of (3.6) and making the corresponding modifications, we obtain (3.10). The proof of (3.11) is similar. □

Theorem 3.4. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y,a)$and $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y,a)$, are related with the Genocchi polynomials $G_n(x)$, by means of*

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j} \binom{k+1}{v} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) G_v(y)$$

$$+ \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) G_{v+1}(y), \tag{3.12}$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j+1} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) G_{v+1}(y)$$

$$+ \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j+1} \binom{k+1}{v} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) G_v(y). \tag{3.13}$$

Proof. By substituting (2.3) into the right-hand side of (3.6), we obtain the proof (3.12). The proof of (3.13) is similar. □

Theorem 3.5. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m and $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$, and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$are related with the Apostol-Euler*

polynomials $\mathscr{E}_n(x;\lambda)$, by means of the following identity:

(3.14)

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j}\binom{k}{v}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_v(y;\lambda) + \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_k(y;\lambda),$$

(3.15)

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j+1}\binom{k}{v}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_v(y;\lambda) + \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j+1}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_k(y;\lambda).$$

Proof. By substituting (2.4) into the right-hand side of (3.6), it suffices to follow the proof given in Theorem 3.2, making the corresponding modifications. The proof of (3.14) and (3.15) sre similar. □

Proposition 3.1. *For $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$, $a \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$, we have*

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\lceil\frac{k}{2}\rceil}\sum_{v=0}^{k}v!\binom{n}{k}\binom{k}{2j}\binom{x}{v}\mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x,y;a)S(k,v),$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\lceil\frac{k-1}{2}\rceil}\sum_{v=0}^{k}v!\binom{n}{k}\binom{k}{2j+1}\binom{x}{v}\mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x,y;a)S(k,v).$$

4. The Generalized Biparametric Fubini-type Polynomials Matrix

Inspired by [17], in which the authors introduce the generalized Euler polynomial matrix, in this section we addressed the generalized biparametric Fubini-type polynomials matrix and we will show you some of their properties.

Definition 4.1. The generalized $(n+1)\times(n+1)$ biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ with $m \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and a positive real numbers are defined by

$$\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y;a) = \begin{cases} \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathscr{D}_{i,j,s}^{[m-1,\alpha]}(x;y;a) = \begin{cases} \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Since, $\mathscr{A}_{n,c}^{[m-1,0]}(x;0;a) = x^n$ and $\mathscr{A}_{n,s}^{[m-1,0]}\left(x;\frac{\pi}{2z};a\right) = x^n$, we have

$$\begin{aligned}\mathscr{D}_c^{[m-1,0]}(x;0;a) =& P[x],\\ \mathscr{D}_s^{[m-1,0]}\left(x;\frac{\pi}{2z};a\right) =& P[x].\end{aligned}$$

Theorem 4.1. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ *satisfy the following product formula*

$$\begin{aligned}\mathscr{D}_c^{[m-1,\alpha+\beta]}(x+y;y;a) =& \mathscr{D}_c^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_c^{[m-1,\beta]}(y;0;a)\\ =& \mathscr{D}_c^{[m-1,\beta]}(x;y;a)\,\mathscr{D}_c^{[m-1,\alpha]}(y;0;a)\\ =& \mathscr{D}_c^{[m-1,\alpha]}(y;0;a)\,\mathscr{D}_c^{[m-1,\beta]}(x;y;a),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathscr{D}_s^{[m-1,\alpha+\beta]}(x+y;y;a) =& \mathscr{D}_s^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_s^{[m-1,\beta]}\left(y;\frac{\pi}{2z};a\right)\\ =& \mathscr{D}_s^{[m-1,\beta]}(x;y;a)\,\mathscr{D}_s^{[m-1,\alpha]}\left(y;\frac{\pi}{2z};a\right)\\ =& \mathscr{D}_s^{[m-1,\alpha]}\left(y;\frac{\pi}{2z};a\right)\,\mathscr{D}_s^{[m-1,\beta]}(x;y;a).\end{aligned} \tag{4.2}$$

Proof. Let $V_{i,j,c}^{[m-1,\alpha,\beta]}(a)(x,y)$ be the (i,j)-th entry of the matrix product $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_c^{[m-1,\beta]}(y;0;a)$. Then, by the addition formula (3.5) we have

$$\begin{aligned}V_{i,j,c}^{[m-1,\alpha,\beta]}(a)(x,y) =& \sum_{k=0}^{n}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =& \sum_{k=j}^{i}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =& \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =& \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathscr{A}_{i-j-k,c}^{[m-1,\alpha]}(x;y,a)\mathscr{A}_{k,c}^{[m-1,\beta]}(y;0;a)\\ =& \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha+\beta]}(x+y;y;a),\end{aligned}$$

which implies (4.1). The second and third equalities of the theorem and (4.2), can be derived in a similar way. □

Corollary 4.1. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ *satisfy the following relation*

$$\begin{aligned}\mathscr{D}_c^{[m-1,\alpha]}(x+y;y;a) &= \mathscr{D}_c^{[m-1,\alpha]}(x;y;a)P[y] = P[x]\mathscr{D}_c^{[m-1,\alpha]}(y;y;a)\\ &= \mathscr{D}_c^{[m-1,\alpha]}(y;y;a)P[x],\\ \mathscr{D}_s^{[m-1,\alpha]}(x+y;y;a) &= \mathscr{D}_s^{[m-1,\alpha]}(x;y;a)P[y] = P[x]\mathscr{D}_s^{[m-1,\alpha]}(y;y;a)\end{aligned}$$

$$= \mathscr{D}_s^{[m-1,\alpha]}(y;y;a)P[x],$$
$$\mathscr{D}_c^{[m-1]}(x+y;y) = P[x]\mathscr{D}_c^{[m-1]}(y;y;a) = P[y]\mathscr{D}_c^{[m-1]}(x;y,a).$$

Example 4.1. Using (3.3), for $m=1$, $\alpha=1$ the first four polynomials $\mathscr{A}_{k,c}^{[1-1,\alpha]}(x;y;a)$, $k=0,1,2,3$ are

$$\begin{aligned}
\mathscr{A}_{0,c}^{[1-1,1]}(x;y,a) &= 2,\\
\mathscr{A}_{1,c}^{[1-1,1]}(x;y;a) &= 2x+4,\\
\mathscr{A}_{2,c}^{[1-1,1]}(x;y;a) &= 2x^2+8x+16-2y^2,\\
\mathscr{A}_{3,c}^{[1-1,1]}(x;y;a) &= 2x^3+12x^2+48x-12y^2-6xy^2+88.
\end{aligned}$$

Hence, for $n=3$ we have

$$\mathscr{D}_c^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 & 0\\ 2x+4 & 2 & 0 & 0\\ 2x^2+8x+16-2y^2 & 4x+8 & 2 & 0\\ 2x^3+12x^2+48x-12y^2-6xy+88 & 6x^2+24x+48-6y^2 & 6x+12 & 2 \end{bmatrix}.$$

Example 4.2. Using (3.4), for $m=1$, $\alpha=1$ the first four polynomials $\mathscr{A}_{k,s}^{[1-1,\alpha]}(x;y;a)$, $k=0,1,2,3$ are:

$$\begin{aligned}
\mathscr{A}_{0,s}^{[1-1,1]}(x;y,a) &= 2y,\\
\mathscr{A}_{1,s}^{[1-1,1]}(x;y;a) &= 2xy+4y,\\
\mathscr{A}_{2,s}^{[1-1,1]}(x;y;a) &= 2x^2y+8x+16y-\frac{2}{3}y^2,\\
\mathscr{A}_{3,s}^{[1-1,1]}(x;y;a) &= 2yx^3+12x^2y+48xy-4y^3-2xy^3+88y.
\end{aligned}$$

Hence, for $n=3$, we have

$$\mathscr{D}_s^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2y & 0 & 0 & 0\\ 2xy+4y & 2y & 0 & 0\\ 2x^2y+8x+16y-\frac{2}{3}y^2 & 4xy+8y & 2y & 0\\ 2yx^3+12x^2y+48xy-4y^3-2xy^3+88y & 6x^2y+24x+48y-2y^2 & 6xy+12y & 2y \end{bmatrix}.$$

5. Matrix Factorization of the New Family of Biparametric Fubini-type Polynomials of Level-m Trough the Fibonacci and Lucas Matrices

For $m\in\mathbb{N}$, $a\in\mathbb{R}^+$, $\alpha\in\mathbb{N}_0$ and $0\le i,j\le n$, let $\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a)$ be the matrices whose entries are defined as follows. Inspired by the ideas of [17], the entries of our auxiliary matrices are (see, [17, Eq. (4.39)]):

$$\begin{aligned}
\tilde{r}_{i,j,c}^{[m-1,\alpha]}(x;y;a) =& \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - \binom{i-1}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\
&- \binom{i-2}{j}\mathscr{A}_{i-j-2,c}^{[m-1,\alpha]}(x;y;a),
\end{aligned}$$

$$\tilde{r}_{i,j,s}^{[m-1,\alpha]}(x;y;a) = \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - \binom{i-1}{j}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a) - \binom{i-2}{j}\mathscr{A}_{i-j-2,s}^{[m-1,\alpha]}(x;y;a).$$

On the other hand, $\mathscr{J}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{J}_s^{[m-1,\alpha]}(x;y;a)$ are the matrices whose entries are given by (see, [17, Eq. (4.40)])

$$\tilde{s}_{i,j,c}^{[m-1,\alpha]}(x;y;a) = \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - \binom{i}{j+1}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a) - \binom{i}{j+2}\mathscr{A}_{i-j-2,c}^{[m-1,\alpha]}(x;y;a),$$

$$\tilde{s}_{i,j,s}^{[m-1,\alpha]}(x;y;a) = \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - \binom{i}{j+1}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a) - \binom{i}{j+2}\mathscr{A}_{i-j-2,s}^{[m-1,\alpha]}(x;y;a).$$

Using the definitions of $\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a)$, $\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a)$, $\mathscr{J}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{J}_s^{[m-1,\alpha]}(x;y;a)$, it is observed that

$$\begin{aligned}
\tilde{r}_{0,0,c}^{[m-1,\alpha]}(x;y;a) &= \tilde{r}_{1,1,c}^{[m-1,\alpha]}(x;y;a) = \tilde{s}_{0,0,c}^{[m-1,\alpha]}(x;y;a)\\
&= \tilde{s}_{1,1,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{0,c}^{[m-1,\alpha]}(x,y,a),\\
\tilde{r}_{0,j,c}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{0,j,c}^{[m-1,\alpha]}(x;y;a) = 0, \quad j \geq 1,\\
\tilde{r}_{1,0,c}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{1,0,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{1,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a),\\
\tilde{r}_{1,j,c}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{1,j,c}^{[m-1,\alpha]}(x;y,a) = 0, \quad j \geq 2,\\
\tilde{r}_{i,0,c}^{[m-1,\alpha]}(x;y;a) &= \mathscr{A}_{i,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-1,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,c}^{[m-1,\alpha]}(x;y;a), \quad i \geq 2,\\
\tilde{s}_{i,0,c}^{[m-1,\alpha]}(x;y;a) &= \mathscr{A}_{i,c}^{[m-1,\alpha]}(x;y;a) - 2\mathscr{A}_{i-1,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,c}^{[m-1,\alpha]}(x;y;a), \quad i \geq 2,\\
\tilde{r}_{0,0,s}^{[m-1,\alpha]}(x;y;a) &= \tilde{r}_{1,1,s}^{[m-1,\alpha]}(x;y;a) = \tilde{s}_{0,0,s}^{[m-1,\alpha]}(x;y;a)\\
&= \tilde{s}_{1,1,s}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{0,s}^{[m-1,\alpha]}(x,y,a),\\
\tilde{r}_{0,j,s}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{0,j,s}^{[m-1,\alpha]}(x;y;a) = 0, \quad j \geq 1,\\
\tilde{r}_{1,0,s}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{1,0,s}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{1,s}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{0,s}^{[m-1,\alpha]}(x;y;a),\\
\tilde{r}_{1,j,s}^{[m-1,\alpha]}(x;y;a) &= \tilde{s}_{1,j,s}^{[m-1,\alpha]}(x;y,a) = 0, \quad j \geq 2,\\
\tilde{r}_{i,0,s}^{[m-1,\alpha]}(x;y;a) &= \mathscr{A}_{i,s}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-1,s}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,s}^{[m-1,\alpha]}(x;y;a), \quad i \geq 2,\\
\tilde{s}_{i,0,s}^{[m-1,\alpha]}(x;y;a) &= \mathscr{A}_{i,s}^{[m-1,\alpha]}(x;y;a) - 2\mathscr{A}_{i-1,s}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,s}^{[m-1,\alpha]}(x;y;a), \quad i \geq 2.
\end{aligned}$$

For $m \in \mathbb{N}$, $a \in \mathbb{R}^+$, $\alpha \in \mathbb{N}_0$, $0 \leq i, j \leq n$, let $\mathscr{L}_{1,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{L}_{1,s}^{[m-1,\alpha]}(x;y;a)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,1,c}^{[m-1,\alpha]}(x;y;a) &= \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - 3\binom{i-j}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\ &\quad + 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\alpha]}(x;y;a),\\ \hat{l}_{i,j,1,s}^{[m-1,\alpha]}(x;y;a) &= \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - 3\binom{i-j}{j}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a)\\ &\quad + 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}\mathscr{A}_{k-j,s}^{[m-1,\alpha]}(x;y;a).\end{aligned}$$

Similarly, let $\mathscr{L}_{2,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{L}_{2,s}^{[m-1,\alpha]}(x;y;a)$, $(n+1)\times(n+1)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,2,c}^{[m-1,\alpha]}(x;y;a) &= \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - 3\binom{i}{j+1}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\ &\quad + 5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a),\\ \hat{l}_{i,j,2,s}^{[m-1,\alpha]}(x;y;a) &= \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - 3\binom{i}{j+1}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a)\\ &\quad + 5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k}\mathscr{A}_{i-k,s}^{[m-1,\alpha]}(x;y;a).\end{aligned}$$

Next we will show, factorizations of the matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$, involving the Fibonacci and Lucas matrices, respectively.

Theorem 5.1. *The generalized biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ can be factored in terms of the Fibonacci matrix $\mathscr{F}$ as follows:*

$$\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) = \mathscr{F}\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a), \tag{5.1}$$

$$\mathscr{D}_s^{[m-1,\alpha]}(x;y;a) = \mathscr{F}\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a), \tag{5.2}$$

$$\mathscr{D}_c^{[m-1,\alpha]}(x;y,a) = \mathscr{J}_c^{[m-1,\alpha]}(x;y;a)\mathscr{F}, \tag{5.3}$$

$$\mathscr{D}_s^{[m-1,\alpha]}(x;y,a) = \mathscr{J}_s^{[m-1,\alpha]}(x;y;a)\mathscr{F}. \tag{5.4}$$

Proof. It suffices to prove that $\mathscr{F}^{-1}\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) = \mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a)$. Recall that $\mathscr{F}^{-1} = f'_{i,j}$, where $f'_{i,j}$ is given in equation (2.5). Since $f'_{0,j} = 0$ for $j \geq 1$, we have:

$$f'_{0,0}\mathscr{D}_{0,0,c}^{[m-1,\alpha]}(x;y,a) = \mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a)$$

and

$$\tilde{r}_{0,0,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a) = \sum_{k=0}^{n} f'_{0,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y;a).$$

Since $\mathscr{D}_{0,j,c}^{[m-1,\alpha]}(x;y,a) = 0$ and $f'_{0,j} = 0$ for $j \geq 1$, we obtain:

$$\sum_{k=0}^{n} f'_{0,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) = f'_{0,0}\mathscr{D}_{0,j,c}^{[m-1,\alpha]}(x;y,a) = \tilde{r}_{0,j,c}^{[m-1,\alpha]}(x;y;a), \quad \text{for } j \geq 1.$$

Next, for $f'_{1,0} = -1$, $f'_{1,1} = 1$, and $f'_{1,j} = 0$ for $j \geq 2$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{1,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& f'_{1,0}\mathscr{D}_{0,0,c}^{[m-1,\alpha]}(x;y,a) + f'_{1,1}\mathscr{D}_{1,0,c}^{[m-1,\alpha]}(x;y,a),\\
\sum_{k=0}^{n} f'_{1,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& -\mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a) + \mathscr{A}_{1,c}^{[m-1,\alpha]}(x;y;a) = \tilde{r}_{1,0,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

From equation (2.5), for $i = 2, 3, \ldots, n$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& \mathscr{A}_{i,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-1,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,c}^{[m-1,\alpha]}(x;y;a)\\
=& \tilde{r}_{i,0,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

Now, for $i \geq 2$ and $j \geq 1$, using equation (2.5) and the definition of $\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y,a)$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& f'_{i,i}\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y,a) + f'_{i,i-1}\mathscr{D}_{i-1,j,c}^{[m-1,\alpha]}(x;y,a)\\
&+ f'_{i,i-2}\mathscr{D}_{i-2,j,c}^{[m-1,\alpha]}(x;y,a),
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - \binom{i-1}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\
&- \binom{i-2}{j}\mathscr{A}_{i-j-2,c}^{[m-1,\alpha]}(x;y;a),\\
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& \tilde{r}_{i,j,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

Therefore, we conclude that

$$\mathscr{F}^{-1}\mathscr{D}_{c}^{[m-1,\alpha]}(x;y;a) = \mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a).$$

The proof is complete. □

The proofs of (5.2), (5.3) and (5.4) follow an analogous path.

The expressions (5.1) (5.2), (5.3) and (5.4) allow us to deduce the following identity:

$$\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{F}^{-1}\,\mathscr{J}_{c}^{[m-1,\alpha]}(x;y,a)\,\mathscr{F},$$

$$\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a) = \mathscr{F}^{-1}\,\mathscr{J}_{s}^{[m-1,\alpha]}(x;y,a)\,\mathscr{F}.$$

An analogous reasoning used in the proof of Theorem 5.1, allows us to prove the results below.

Example 5.1. For $m=1$ and $\alpha=1$, the matrices, for $n=2$, $K_{Fub,c}^{[m-1,1]}(x;y,a)$ and $\mathscr{F}$ are

$$\mathbb{K}_{Fub,c}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+2 & 2 & 0 \\ 2x^2+6x+10-2y^2 & 4x+6 & 2 \end{bmatrix},$$

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathscr{F}\mathbb{K}_{Fub,c}^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}$$

and

$$\mathscr{D}_{c}^{[m-1,1]}(x,y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.$$

This is a particular case of the statement (5.1) of Theorem 5.1.

Example 5.2. For $m=1$ and $\alpha=1$, the matrices, for $n=2$, $\mathscr{J}^{[m-1,1]}(x;y;a)$ and $\mathscr{F}$ are

$$\mathscr{J}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+2 & 2 & 0 \\ 2x^2+4x+6-2y^2 & 4x+6 & 2 \end{bmatrix},$$

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathscr{J}^{[m-1,1]}(x;y;a)\mathscr{F} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}$$

and

$$\mathscr{D}_{c}^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.$$

This is a particular case of the statement (5.3) of Theorem 5.1.

Theorem 5.2. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_{c}^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_{s}^{[m-1,\alpha]}(x;y;a)$ *can be factored in terms of the Lucas matrix*

$\mathscr{L}$ of the following form:

$$
\begin{aligned}
\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) =& \mathscr{L}\,\mathscr{L}_{1,c}^{[m-1,\alpha]}(x;y;a),\\
\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) =& \mathscr{L}_{2,c}^{[m-1,\alpha]}(x;y;a)\mathscr{L},\\
\mathscr{D}_s^{[m-1,\alpha]}(x;y;a) =& \mathscr{L}\,\mathscr{L}_{1,s}^{[m-1,\alpha]}(x;y;a),\\
\mathscr{D}_s^{[m-1,\alpha]}(x;y;a) =& \mathscr{L}_{2,s}^{[m-1,\alpha]}(x;y;a)\mathscr{L}.
\end{aligned}
\tag{5.5}
$$

Example 5.3. For $m=1$ and $\alpha=1$, the matrices, for $n=2$ $\mathscr{L}_{1,c}^{[m-1,1]}(x;y;a)$ and $\mathscr{L}$ are

$$
\mathscr{L}_{1,c}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0\\ 2x-2 & 2 & 0\\ 2x^2+2x+14-2y^2 & 4x+2 & 2 \end{bmatrix},
$$

$$
\mathscr{L} = \begin{bmatrix} 1 & 0 & 0\\ 3 & 1 & 0\\ 4 & 3 & 1 \end{bmatrix},
$$

$$
\mathscr{L}\,\mathscr{L}_{1,c}^{[m-1,1]}(x;y;a) = \begin{bmatrix} 2 & 0 & 0\\ 2x+4 & 2 & 0\\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}
$$

and

$$
\mathscr{D}_c^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0\\ 2x+4 & 2 & 0\\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.
$$

This is a particular case of the statement (5.5) of Theorem 5.2.

6. Conclusion

In this paper, we have introduced two new biparametric families of generalized Fubini-type polynomials of level-m, and explored their key algebraic and differential properties. By leveraging these properties, we have derived several formulas and identities that contribute to the understanding of these polynomials. Additionally, we have presented a new matrix representation for the generalized biparametric Fubini-type polynomials, offering a novel approach to their application. Given the broad range of applications of special polynomials in mathematics, physics, and engineering, the results of this study have significant potential for further development in these fields. However, future research may focus on exploring the specific advantages and limitations of these new polynomials in practical applications.

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A GENETIC ALGORITHM MODEL IMPROVED WITH ZECKENDORF REPRESENTATIONS FOR PREVENTIVE MAINTENANCE SCHEDULING PROBLEM

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Abstract. This study introduces a novel Genetic Algorithm (GA) model enhanced with Zeckendorf representation and Fibonacci number-based encoding to optimize preventive maintenance scheduling (PMS) problems. Conventional maintenance scheduling methods, based on random or linear encoding techniques, often fail to optimize maintenance processes effectively.

Therefore, the proposed model aims to systematically plan maintenance periods and minimize production interruptions by encoding maintenance intervals using Zeckendorf representation.

By optimizing maintenance processes, the proposed model enhances system production continuity. Experimental analyses indicate that the proposed model enhances existing production capacity and facilitate a more balanced management of maintenance operations. The electricity and water production capacities increased by 11% and 10%, respectively, while the reserve capacity improved by 9% for electricity and 17% for water.

These results show that the proposed method is a new optimization strategy for maintenance planning by enhancing the applicability of GA in preventive maintenance scheduling problems. Optimizing maintenance scheduling with Zeckendorf representation enables systematic and balanced execution of maintenance operations, thereby ensuring more efficient planning in industrial facility maintenance processes.

Key words and phrases. Fibonacci numbers, Zeckendorf representations, genetic algorithm, preventive maintenance.

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1. Introduction

Industrial facilities, power plants, and other critical infrastructures emphasize maintenance strategies due to their need for operational continuity. Maintenance processes not only extend the lifespan of equipment but also play a critical role in ensuring operational continuity and preventing sudden failures. In this context, Preventive Maintenance (PM) is a strategy that involves the planned and periodic maintenance of machines or systems. PM aims to inspect equipment at specific intervals, perform maintenance or repairs, and minimize unexpected failures and interruptions [7].

When compared to other maintenance strategies such as reactive maintenance (running to failure) and predictive maintenance (forecasting using sensor data), PM stands out for its planned and systematic nature. Preventive Maintenance Scheduling (PMS), on the other hand, involves planning when and how the specified maintenance activities will be carried out. Effective implementation of PMS optimizes maintenance processes, reduces operational costs, increases productivity, and ensures system reliability [1]. Traditional PMS approaches often employ deterministic methods, leading to the development of linear and nonlinear optimization models. Research utilizing Mixed Integer Linear Programming (MILP) and heuristic methods has been published [5,13]. Studies have been conducted to optimize maintenance planning for power generation units, minimize maintenance costs, and create appropriate maintenance schedules [16].

However, in large-scale and complex maintenance planning problems, traditional methods are often computationally intensive and time-consuming in achieving optimal solutions. Therefore, heuristic and meta-heuristic algorithms, such as Genetic Algorithm (GA), Tabu Search (TS), and Particle Swarm Optimization (PSO), are widely preferred. In recent years, hybrid models combining these approaches have been introduced, improving solution quality and the efficiency of the optimization process [2,3].

The rapid growth of computational optimization has led to the development of numerous algorithms aimed at solving complex scheduling and resource allocation problems. Among these, Genetic Algorithms (GAs) have emerged as versatile and powerful tools for addressing various optimization challenges due to their ability to explore large search spaces and locate near-optimal solutions [3,18]. Traditional GAs typically rely on binary or floating-point representations to encode solutions, but these methods often struggle to balance diversity and convergence.

The methods employed to solve PMS problems remain open to further improvement. Enhancing the solution quality and success of proposed methods is both feasible and necessary. Motivated by this, the study proposes a novel preventive maintenance scheduling method by incorporating Fibonacci-based Zeckendorf and k-Zeckendorf representations into a hybrid GA-TS approach. Fibonacci-based representations aim to improve system performance by distributing maintenance durations and intervals

more systematically and evenly. The effectiveness of the proposed approach is analyzed in terms of the supply-demand balance in electricity and water production.

This study introduces a novel enhancement to the standard GA framework by integrating Zeckendorf representations, a numerical encoding technique based on Fibonacci numbers. Unlike traditional representations, Zeckendorf encoding leverages the unique properties of Fibonacci numbers to provide a compact, efficient, and inherently diverse representation of solutions. This method has been applied to the preventive maintenance scheduling problem, a critical optimization challenge faced by industries such as cogeneration plants that rely on continuous operations. The scheduling problem requires balancing equipment availability, resource utilization, and demand fulfillment while minimizing operational disruptions.

The significance of this study lies in the potential to improve the performance of genetic algorithms through a structured yet adaptable coding schema. The proposed method not only achieves better optimization results by reducing redundancy and increasing the diversity of the search space, but also provides a new perspective on how numerical representations can affect algorithmic efficiency.

In the following sections of this paper, existing PMS approaches in the literature are discussed, the details of the proposed method are explained, and the performance of the method is evaluated based on experimental results.

2. Mathematical Model

This section explains how the proposed genetic algorithm approach has been developed with a mathematical approach. For this purpose, detailed explanations of Fibonacci base and Zeckendorf representations are provided, and the computation processes are demonstrated.

In number theory, there exist various alternative methods for representing a number A. These representations often depend on the properties of A and the specific mathematical framework being employed. Such methods can provide deeper insights into the structure and behavior of numbers, offering tools for analyzing their relationships and applications. Among these representations, one of the most notable is the representation using Fibonacci numbers, commonly referred to as the *Fibonacci representation*. It is well established [12] that the Fibonacci sequence, denoted by $\{F_n\}$, is defined recursively as follows:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{if } n > 1,$$

with initial values $F_0 = 0$ and $F_1 = 1$. Excluding the initial values $F_0 = 0$ and $F_1 = 1$ of the Fibonacci sequence, the subsequent 12 Fibonacci numbers and their corresponding indices are summarized as follows in Table 1.

For example, every positive integer A can be written as

$$A = F_{n_1} + F_{n_2} + \cdots + F_{n_r}, \quad n_1 > n_2 > \cdots > n_r \geq 2,$$

which is called a *Fibonacci representation* of A.

TABLE 1. The F_n Fibonacci numbers for $2 \leq n \leq 13$

n	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	2	3	5	8	13	21	34	55	89	144	233

The most widely recognized method for representing non-negative integers using only the digits $\{0, 1\}$ is the binary numeral system. However, alternative methods exist for representing integers using just $\{0, 1\}$, some of which are less immediately intuitive. One notable example is *Zeckendorf's Fibonacci representation*.

Zeckendorf's theorem is a well-known result that states every positive integer can be uniquely expressed as a sum of distinct Fibonacci numbers, provided that the indices of the Fibonacci numbers satisfy the condition $n_j - n_{j-1} \geq 2$, $j = 1, 2, \ldots, r$, $n_r \geq 2$. This unique representation is referred to as *Zeckendorf's Fibonacci representation* [10, 11]. The Zeckendorf theorem offers an alternative to the binary numeral system, providing a unique representation of integers that has practical applications in areas such as data transmission and compression [17].

Zeckendorf's theorem states that every positive integer A can be uniquely expressed as the sum of one or more distinct Fibonacci numbers. This representation ensures that the sum excludes any two consecutive Fibonacci numbers [10]. Zeckendorf's Fibonacci encoding utilizes this unique representation. Specifically, any positive integer $A = (\ldots d_3 d_2 d_1)_{\text{fib}}$ can be expressed as

$$A = \sum_{i=1}^{n} d_i F_{n_i}$$

where F_{n_i}, $n_i \geq 2$ is the n_i^{th} Fibonacci number, $n_i - n_{i-1} \geq 2$, $d_i \in \{0, 1\}$, and $d_n = 1$. This encoding is referred to as *Zeckendorf's Fibonacci coding* and is denoted as $A = (d_i)_{\text{fib}}$, $i = 1, 2, \ldots$

For $A \leq 233$, a 12-bit $\{0, 1\}$ array is generated in Zeckendorf's Fibonacci representation. For instance, the representation of $A = 123$ is given by the sum $89 + 34$. In Zeckendorf's Fibonacci coding, this can be written as $123 = (1010000000)_{\text{fib}}$.

A number representation system is generally most effective when it provides a unique representation for every integer. Zeckendorf's Fibonacci encoding ensures uniqueness by prohibiting consecutive 1's in the representation. However, if this condition is relaxed, while every number can still be expressed as a sum of Fibonacci numbers, some numbers will have multiple possible sums, leading to non-unique representations.

It can be observed that the number $A = 123$ can be expressed in 7 different ways using various combinations of Fibonacci numbers. To address this non-uniqueness, let us slightly modify Zeckendorf's Fibonacci encoding to establish a new $\{0, 1\}$-based representation that ensures a single, unique form for each number.

In [4], a Zeckendorf-Wythoff sequence is introduced, which organizes positive integers into columns based on their Zeckendorf representations. The n^{th} column of the Zeckendorf-Wythoff sequence consists of integers A, listed in ascending order, whose

Zeckendorf representations terminate with F_{n+1}. The Zeckendorf-Wythoff sequence is defined such that each row corresponds to a Fibonacci sequence. If the largest Fibonacci number in the Zeckendorf representation of A is F_n, then A begins with F_n.

Each row of the sequence consists of A integers, all having Zeckendorf representations of the same structure. In particular, for any two consecutive numbers in the row, the indices of the Fibonacci numbers F_n corresponding to the digits in their Zeckendorf representations differ by exactly one [9, 10].

Let $R(A)$ represent the number of ways the non-negative integer A can be expressed as the sum of distinct Fibonacci numbers. It is given that for integers A in the odd-numbered columns of a row, the successive values of $R(A)$ form an arithmetic progression, and $R(A-1)$ remains constant. Moreover, if there exists an A such that $F_n < A < F_{n+1} - 1$ in a column of the Zeckendorf-Wythoff sequence, then $A + F_{n+k}$, $k \geq 2$, also belongs to the same column. These properties of Zeckendorf representations and Wythoff pairs are utilized to determine $R(A)$ [10, 11].

The Zeckendorf-Wythoff sequence is presented in Table 2, arranged into 14 columns, as it corresponds to the use of 8, 12, and 14-bit arrays.

TABLE 2. Zeckendorf-Wythoff sequence

$w_{(i,j)}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$w_{(1,j)}$	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$w_{(2,j)}$	4	7	11	18	29	47	76	123	199	322	521	843	1364	2207
$w_{(3,j)}$	6	10	16	26	42	68	110	178	288	466	754	1220	1974	3194

Table 3 provides the number of distinct Fibonacci representations for each number A, organized according to the columns in Table 2.

TABLE 3. $R(A)$ values for numbers A given in Table 2

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$R(w_{(1,j)})$	1	1	2	2	3	3	4	4	5	5	6	6	7	7
$R(w_{(2,j)})$	1	1	3	3	5	5	7	7	9	9	11	11	13	13
$R(w_{(3,j)})$	2	2	4	4	6	6	8	8	10	10	12	12	14	14

The number of representations of A as sums of distinct Fibonacci numbers can be derived from the Zeckendorf representation of A. Specifically, if the Zeckendorf representation of A ends with F_n, where $n \geq 2$, then a constant q satisfies:

$$R(A) = R(A-1)R(F_n) - q, \quad 0 \leq q \leq R(A-1).$$

Here, $R(F_n) = \lfloor n/2 \rfloor$ and $R(F_n - 1) = 1$ for $n \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to a real number x.

Table 3 clearly demonstrates that the different Fibonacci representations of the numbers A in each row increase to the right, following the pattern outlined in Table 2.

Furthermore, this increase occurs at a faster rate in the lower rows. This indicates that smaller numbers in the lower rows can possess a greater number of representations. For instance, the number A can have six different representations if $A = 144$, seven different representations if $A = 123$, and eight different representations if $A = 110$.

Although $R(A)$ indicates the number of different representations of the number A, detailed information about these representations can only be explicitly expressed using the Zeckendorf representation. An alternative canonical representation can be developed to express any given number as a sum of Fibonacci numbers, while permitting the inclusion of consecutive Fibonacci numbers within the same sum. This approach is facilitated by adopting a base system where Fibonacci numbers act as placeholders, thereby allowing adjacent terms. The validity of this method stems from the property that the sum of any two consecutive Fibonacci numbers equals the next Fibonacci number. Consequently, any representation of the form $(\dots 100 \dots)_{\text{fib}}$ can be systematically replaced with $(\dots 011 \dots)_{\text{fib}}$. In this study, each instance of $(\dots 100 \dots)_{\text{fib}}$ was transformed into $(\dots 011 \dots)_{\text{fib}}$. This alternative representation not only serves as a canonical form but also provides a more compact encoding compared to Zeckendorf's Fibonacci representation.

To verify that every number can be represented in this system, one can write down the k-Zeckendorf representations of several numbers using 12 bits. By constructing such representations, it becomes evident how numbers can be uniquely expressed following the rules of the k-Zeckendorf system. For clarity, a table can be provided to illustrate these representations. Table 4 includes examples of numbers along with their corresponding k-Zeckendorf representations in 12-bit format.

TABLE 4. Representations of k-Zeckendorf using Zeckendorf's Fibonacci coding

N	Zeckendorf's code	1-Zeckendorf's code	...	6-Zeckendorf's code	7-Zeckendorf's code	8-Zeckendorf's code
68	100100000	100011000	...	011010110	-	-
123	1010000000	1001100000	...	0111011000	0111010110	-
178	10010000000	10001100000	...	01101100000	01101011000	01101010110
233	100000000000	011000000000	...	010101010110	-	-

The least number of Fibonacci numbers used in a representation corresponds to the number of 1's in Zeckendorf's Fibonacci coding. This is because Zeckendorf's theorem guarantees the use of the minimal number of Fibonacci numbers in any valid representation, which is why it is also referred to as the minimal Fibonacci representation. Consequently, any representation of the form $(...100...)_{\text{fib}}$, where a 1 is followed by two or more 0's, can be systematically replaced with $(..011..)_{\text{fib}}$ by redistributing the terms according to the Zeckendorf rules. This replacement not only adheres to the uniqueness property of Zeckendorf's representation but also ensures the representation remains minimal by avoiding consecutive Fibonacci numbers and maintaining the fewest possible 1's in the coding.

3. Genetic Algorithms for Preventive Maintenance Scheduling

Genetic algorithms (GA) are a powerful computational method inspired by natural selection and biological evolution processes. This technique is considered one of the most common applications of evolutionary computing methods used to solve complex search and optimization problems. The concept of genetic algorithms was first introduced by John Holland [8] and was further developed by his students and colleagues in the following years, making it applicable to a wide range of optimization problems.

The fundamental philosophy of GA is to evolve the population by selecting solutions with high fitness values and applying genetic operators (such as crossover, mutation, etc.) [14]. This approach enables effective exploration of the vast solution space while reducing the risk of getting stuck in local minima. It is particularly advantageous in terms of flexibility and applicability for combinatorial and constrained optimization problems.

The Preventive Maintenance Scheduling problem involves optimizing maintenance activities to ensure the efficient operation of critical systems while minimizing downtime and costs [15]. Given the combinatorial nature of PMS, exact methods become computationally infeasible for large-scale problems. Consequently, heuristic and metaheuristic approaches, particularly GAs, have garnered significant attention in the literature due to their ability to provide near-optimal solutions within reasonable computation timeframes.

GAs are population-based optimization techniques inspired by natural selection and evolutionary principles. In the context of PMS, GAs encode maintenance schedules as chromosomes that evolve over multiple generations through selection, crossover, and mutation operators. The fitness function evaluates each schedule based on criteria such as total energy production, water supply, maintenance costs, and demand-supply balance.

The application of GA to PMS problems follows specific fundamental steps [6]. First, maintenance schedules are encoded as chromosomes. In this encoding process, each gene represents the maintenance status of a unit in a specific week; that is, the unit is either operational or undergoing maintenance during that week. This transformation allows maintenance processes to be structured in a format that can be processed by the genetic algorithm.

Next, an initial population is generated. The initial population is typically produced randomly; however, certain heuristic methods suitable for the problem's nature can also be employed. This process is crucial to enable the genetic algorithm to search a wide solution set.

In each generation, the chromosomes created are evaluated based on a fitness function. The fitness value measures how well the maintenance schedules meet energy and water demands and whether they minimize production interruptions caused by

maintenance. More suitable maintenance plans create a more balanced production-consumption relationship by improving the system's operational efficiency.

One of the key components of genetic algorithms is the selection process, which ensures that individuals with high fitness are chosen to reproduce in the next generation. Among the most commonly used selection methods are roulette wheel selection and tournament selection, both of which aim to pass on high-performing chromosomes through generations.

The crossover process is performed by exchanging genetic information between two selected parent chromosomes. This operation enables the creation of potentially better maintenance schedules and enhances genetic diversity, facilitating the exploration of a broader solution space.

The mutation process is applied to prevent premature convergence of the genetic algorithm and to allow further exploration of the solution space. By introducing small changes in randomly selected chromosomes, genetic diversity is preserved, and alternative solutions can be discovered.

Finally, the algorithm terminates when a predefined number of generations is reached or when the fitness function converges to a certain level. At this point, the best chromosome obtained is considered the final solution for maintenance scheduling.

4. New Models And Solutions

In this study, a novel GA model based on Zeckendorf representation and Fibonacci number decomposition is proposed to enhance the efficiency of preventive maintenance scheduling (PMS). The primary objective of the proposed approach is to optimize the scheduling of preventive maintenance activities for electricity generation and water desalination units while minimizing deviations between production and demand. Unlike traditional GA models that use direct encoding schemes, this study employs k-Zeckendorf representation to structure maintenance schedules more efficiently, thereby improving the search process and convergence speed.

The proposed model integrates three key components: (1) initial population generation using k-Zeckendorf representation, (2) a customized fitness function based on production-demand deviation, and (3) genetic operations (selection, crossover, and mutation) adapted to k-Zeckendorf-based scheduling.

Maintenance schedules for turbines and distillers are initialized using k-Zeckendorf representation, which ensures that each maintenance period is uniquely encoded in terms of Fibonacci numbers. This encoding reduces redundancy in schedule representation and enables the algorithm to explore feasible solutions more efficiently. Maintenance initiation weeks are selected based on k-Zeckendorf-decomposed values, ensuring that units undergo preventive maintenance without clustering within the same periods.

In preventive maintenance scheduling, electricity and water production are affected by maintenance periods, leading to fluctuations in meeting demand. Therefore, optimal maintenance scheduling must ensure appropriate allocation of production capacities. The fitness function evaluates the deviation between available electricity and water production and demand values, defined as follows:

$$F = \sum_{t=1}^{T} \left[w_e |P_e(t) - D_e(t)| + w_w |P_w(t) - D_w(t)| \right],$$

where F represents the fitness function, T denotes the total number of weeks, P_e and D_e are the electricity production and demand, respectively, P_w and D_w are the water production and demand, respectively, w_e represents the weight coefficient for the electricity production-demand deviation, and w_w denotes the weight coefficient for the water production-demand deviation.

Electricity production may decrease due to maintenance on turbines. The total electricity production per week is calculated as follows:

$$P_e(t) = \sum_{i=1}^{N_T} \left(A_i(t) \cdot C_{T,i} \right),$$

where N_T is the total number of turbines, $A_i(t)$ is a binary variable indicating whether the turbine is under maintenance (1 = operational, 0 = under maintenance), and $C_{T,i}$ denotes the capacity (MW) of the i-th turbine. Similarly, water production decreases due to maintenance on distillers. The weekly water production is calculated as follows:

$$P_w(t) = \sum_{j=1}^{N_D} \left(B_j(t) \cdot C_{D,j} \right),$$

where N_D is the total number of distillers, $B_j(t)$ is a binary variable indicating whether the distiller is under maintenance (1 = operational, 0 = under maintenance), and $C_{D,j}$ represents the capacity (MIGD) of the j-th distiller.

Thus, the fitness function penalizes large production deficits, especially during critical demand periods, guiding the genetic algorithm towards solutions that optimize unit maintenance intervals while maintaining supply-demand balance.

The fitness function evaluates the difference between the available electricity and water production and the corresponding demand values. It is defined as follows:

The proposed GA model is applied to a PMS problem that includes eight turbines and sixteen distillers operating under predefined demand constraints. Each unit undergoes scheduled maintenance for a predetermined period-four weeks for turbines and five weeks for distillers. The numerical values for these units are presented in Table 5, derived from literature to enable a fair and effective comparison [3].

The GA iterates over 100 generations with a population size of 50, dynamically adjusting schedules based on varying fitness values.

To evaluate the effectiveness of the proposed approach, weekly electricity and water production levels are analyzed in comparison with demand values. Additionally,

Table 5. Numerical Values of Units

Unit Type	Number of Units	Maintenance Duration (Weeks)	Production Capacity
Turbines	8	4	47,040 MW per unit
Distillers	16	5	50.4 MIGD (12 units) 40.2 MIGD (4 units)
Boilers	8	5	Support Only (No Direct Production)

total annual production, demand, and production surplus are calculated to assess system performance. The resulting schedules demonstrate that maintenance activities are well-distributed, production shortages are prevented, and resource utilization is optimized.

5. Experimental Results

This section evaluates the performance of the proposed genetic algorithm and k-Zeckendorf representation-based PMS model. The production and consumption values of electricity and water, total production surplus, and fitness function values are analyzed.

5.1. **Experimental Setup.** The proposed genetic algorithm and k-Zeckendorf representation based preventive maintenance scheduling model in this study was implemented on a computer with an Intel Core i7-12700H processor and 16 GB of RAM. The algorithm was implemented using Python 3.8 programming language, and libraries such as NumPy and Matplotlib were utilized.

To assess the model's performance, the optimization process was executed with a population of 50 individuals over 100 generations. The core parameters of the genetic algorithm are presented in Table 6.

Table 6. Key parameters of the proposed model

Parameter	Value	Description
Population Size	50	Number of individuals evaluated simultaneously
Number of Generations	100	Evolution cycles in the solution space
Crossover Rate	0.8	Probability of creating new generation individuals
Mutation Rate	0.1	Probability of applying random mutation
Selection Method	Tournament Selection	Method for selecting individuals for crossover
Encoding Type	k-Zeckendorf	Maintenance scheduling encoded using Zeckendorf representation

Preventive maintenance schedules were generated for both electricity generation units (turbines) and water treatment units (distillers). Each turbine was assigned a maintenance period of four weeks, while each distiller was assigned five weeks. Using Zeckendorf representation, maintenance intervals were systematically planned in Fibonacci-based sequences, ensuring a balanced distribution of maintenance operations across the system.

5.2. **Results and Analysis.** This section analyzes the experimental results of the proposed genetic algorithm and the k-Zeckendorf representation-based Preventive Maintenance Scheduling model. The study evaluates key performance metrics, including electricity and water production, supply-demand balance, total fitness function value, and convergence analysis. Additionally, the proposed method is compared with the conventional GA to assess the impact of Zeckendorf-based encoding on the optimization process.

To evaluate the proposed model's ability to optimize maintenance scheduling, the weekly electricity and water production values were compared with demand values. Figure 1 compares electricity production values against demand and also displays the surplus production amount.

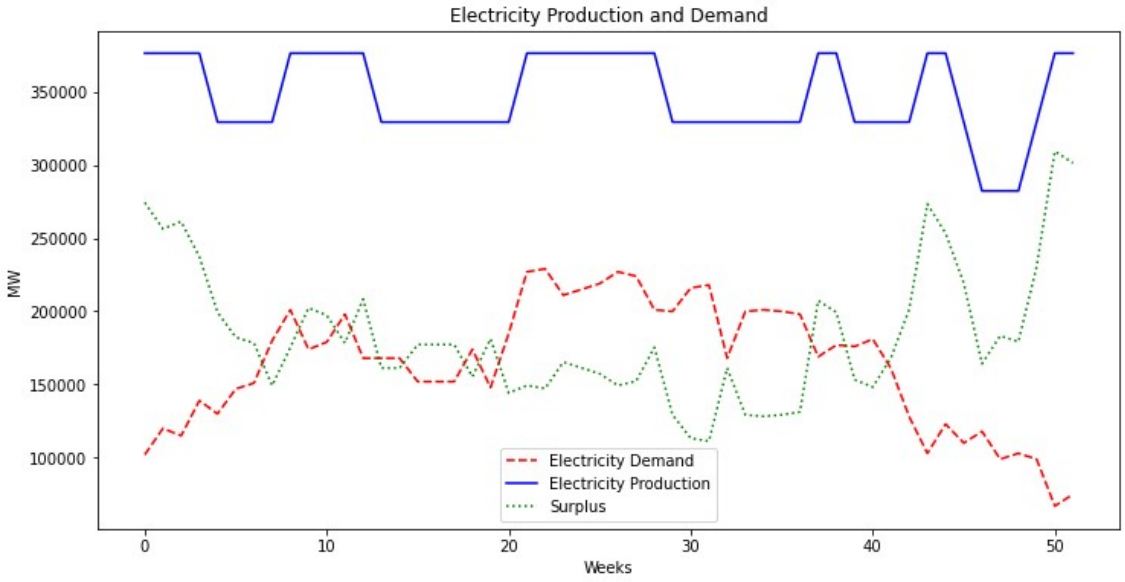


FIGURE 1. Comparison of electricity production values with demand

As seen in Figure 1, electricity production consistently meets demand; however, short-term production declines occur during maintenance periods. Nevertheless, encoding maintenance scheduling using k-Zeckendorf representation ensures a balanced distribution of maintenance periods across the system, preventing these declines from causing disruptions.

Similarly, Figure 2 presents the relationship between water production and demand. Due to the proposed maintenance planning approach, fluctuations in water production have been largely controlled outside critical maintenance periods, ensuring that demand is met throughout the year. Production deficits during maintenance processes have not led to any service interruptions.

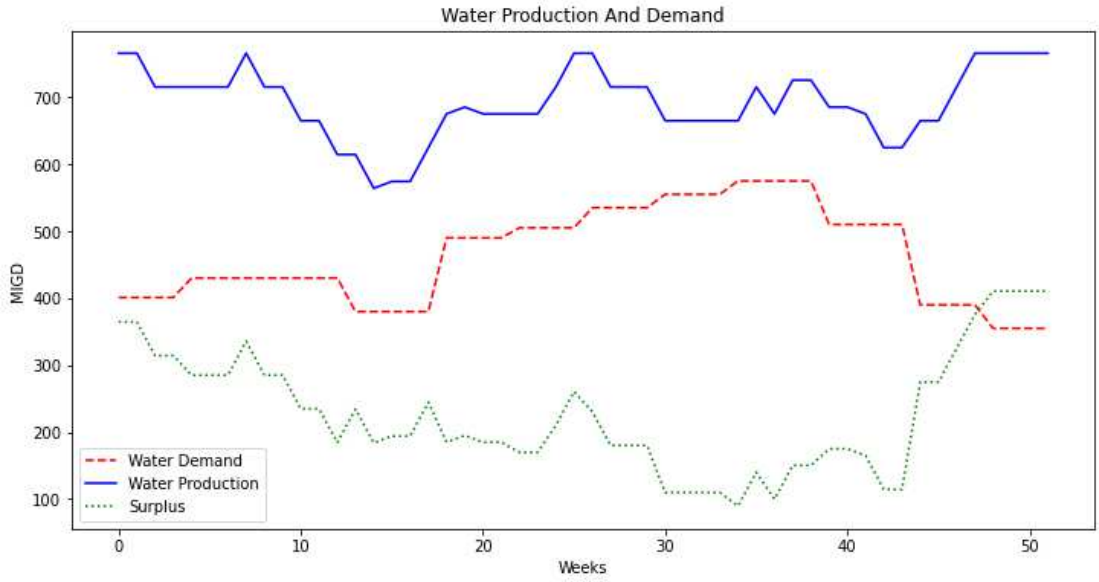


FIGURE 2. Comparison of Water Production with Demand

When the results of the proposed method are compared with studies that solve the same problem using genetic algorithms, it is observed that the proposed approach achieves superior outcomes. Table 7 provides a comparative evaluation of the results obtained by the model.

TABLE 7. Comparative Results of the Proposed Model

Models	GA+TS [3]	Proposed Model	Improvement Rate
Available Electricity (MW)	17,687,040.00	19,568,640.00	11%
Electricity Demand (MW)	8,590,000.00	8,590,000.00	-
Surplus (MW)	9,097,040.00	10,978,640.00	21%
Reserve Percentage	51%	56%	9%
Available Water (MIGD)	36,321.60	39,811.20	10%
Water Demand (MIGD)	24,130.00	24,130.00	-
Surplus (MIGD)	12,191.60	15,681.20	29%
Reserve Percentage	34%	39%	17%

The results presented in Table 7 demonstrate that the proposed *k-Zeckendorf representation*-based genetic algorithm model provides significant performance improvements compared to the conventional Hybrid GA-TS model. According to the comparisons made in terms of electricity and water production, the proposed model was observed to increase existing electricity generation by 11% and water production by 10%.

Notably, in terms of reserve capacity-i.e., the system's production capacity beyond demand-an improvement of 9% in electricity and 17% in water was achieved. This indicates that the operational flexibility of the facility has increased due to a more balanced distribution of maintenance processes and more effective management of production interruptions.

Furthermore, the 21% increase in surplus electricity production and the 29% increase in surplus water production reveal that the system possesses greater backup capacity and that maintenance processes have a reduced impact on critical production periods. This finding confirms that the proposed model minimizes production interruptions and responds more effectively to demand fluctuations.

The findings indicate that the *k-Zeckendorf representation*-based maintenance scheduling enhances system production continuity by structuring maintenance planning more efficiently. In particular, Zeckendorf encoding, which optimizes the search space within the evolutionary process of genetic algorithms, has enabled systematic rather than random scheduling of maintenance periods. Consequently, this has increased reserve capacity and strengthened production reliability.

Based on these results, it can be concluded that the proposed model offers a feasible, efficient, and high-performance solution for PMS problems.

6. Conclusion

This study proposed a genetic algorithm model incorporating Zeckendorf representation and Fibonacci-based encoding to address preventive maintenance scheduling problems. Traditional methods relying on random or linear encoding approaches often struggle to systematically plan maintenance periods, leading to production interruptions. By optimizing maintenance scheduling using Zeckendorf encoding, the proposed model effectively minimized production disruptions and improved the supply-demand balance.

Experimental results demonstrated that the proposed model increased electricity production capacity by 11% and water production capacity by 10%. Reserve capacity improved by 9% for electricity and 17% for water, reinforcing production flexibility.

The optimization process achieved faster convergence than conventional methods, reducing solution time. This study enhances the applicability of genetic algorithms for PMS problems, improving their effectiveness in maintenance scheduling. The integration of Zeckendorf representation with genetic algorithms has proven to be a more balanced and effective solution, particularly for large-scale maintenance planning problems. Future research will focus on applying this model to different energy plants,

production facilities, and infrastructure systems. Further development will explore hybrid approaches and other metaheuristic optimization techniques to enhance model performance.

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ψ-HILFER-ATANGANA-BALEANU FRACTIONAL INTEGRAL OPERATORS AND SOME FRACTIONAL INTEGRAL INEQUALITIES

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Abstract. We present new fractional integral operators known as the left- and right-sided ψ-Hilfer-Atangana-Baleanu fractional integral operators. These fractional integrals generalize the Atangana-Baleanu and the ABK fractional integral operators. As a practical application, we derive a generalization of the Hermite-Hadamard inequality for s-convex functions in the second sense, utilizing the newly defined integral operators. Additionally, we established several Hermite-Hadamard type fractional integral inequalities for functions whose derivatives in absolute value raised to some nonnegative powers are s-convex in the second sense, using the ψ-Hilfer-Atangana-Baleanu fractional integral operator. We also present some specific cases of our main results.

1. Introduction

Fractional inequalities have a significant impact in the research of various fields of mathematics, including mathematical analysis, probability theory, differential equations, and many more. These inequalities involve fractional derivatives or fractional integrals of functions, which generalize the concept of differentiation and integration to non-integer orders. Fractional inequalities are not only theoretical tools but also have practical applications in fields such as physics, engineering, and biology, where phenomena are often described by equations involving fractional derivatives. They are essential for understanding the behavior of solutions to fractional differential equations

Key words and phrases. Hermite-Hadamard inequality, Atangana-Baleanu fractional integral operators, fractional integral inequalities, convex functions, Hölder's inequality, power mean inequality, s-convex functions.

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and for establishing the conditions under which such solutions exist and are unique. Due to their potential applications, many researchers have studied and established several fractional inequalities involving various fractional derivatives and fractional integrals over the years. Most of these inequalities are generalizations, extensions or refinements of the classical inequalities within the framework of fractional calculus. It is well known that convex functions have a significant impact in the research of inequalities. In the last few decades, many researchers have investigated several inequalities involving convex functions and their numerous generalizations by employing different fractional derivatives and/or fractional integral operators. For example, in [33], Tunc et al. introduced the generalized fractional integrals of a function with respect to another function and established the Hermite-Hadamard inequality as well as some Hermite-Hadamard type inequalities involving the newly introduced fractional integral operators for convex functions. In [29], Rahman et al. established some new fractional integral inequalities for a certain class of n ($n \in \mathbb{N}$) positive continuous functions and also for decreasing functions by utilizing the fractional proportional integral operators. Rashid et al. in [30] defined a new fractional integral operator, which is known as generalized proportional fractional (GPF) integral in the sense of another function, and proved several fractional inequalities involving the newly defined GPF-integral with respect to another function. By using some generalized fractional integrals, Du et al. [9] derived some Bullen-type inequalities where the first derivative of the functions under consideration are either Lipschitzian, bounded or generalized (s, m)-preinvex. In [10], Du and Peng established some Hermite-Hadamard type inequalities for multiplicatively convex functions via multiplicative Riemann-Liouville fractional integrals. In [32], Saleh et al. investigated some new Newton-type inequalities for functions whose first derivatives are α-convex by utilizing the Riemann-Liouville fractional integrals. Recently, Kermausuor and Nwaeze [21] introduced some new fractional integral operators called the k-Atangana-Baleanu fractional integral operators and established a generalization of the Hermite-Hadamard inequality as well as several other fractional integral inequalities for functions whose derivatives are bounded and functions whose derivative in absolute value is convex. For further details on the results mentioned above and other related findings, we encourage interested readers to consult the papers listed in [2–4, 6, 9, 11, 14–18, 20, 21, 23–25, 27, 29–36] and the other related references cited in this paper.

Motivated by the ongoing research on fractional inequalities and their applications, this paper seeks to introduce new generalized fractional integral operators called the left- and right-sided ψ-Hilfer-Atanga-Baleanu (ψ-HAB) fractional integral operators. We will subsequently establish Hermite-Hadamard type inequalities for functions that are s-convex in the second sense and functions whose derivative in absolute value to certain powers are s-convex in the second sense, all using the newly defined fractional integral operators. In the next section, we present some preliminary definitions and results, followed by the main results in Section 3 and a conclusion in Section 4.

2. Preliminaries

We highlight some basic concepts and results in the literature that will be useful to our work. We start with the definitions of some fractional integrals.

Definition 2.1 ([13, 26, 28])**.** The left- and right-sided Riemann-Liouville fractional integrals of a real-valued function $\mathcal{G}$ of order $\omega > 0$ are given by

$$I^{\omega}_{d_1^+}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{d_1}^{r}(r-z)^{\omega-1}\mathcal{G}(z)dz$$

and

$$I^{\omega}_{d_2^-}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{r}^{d_2}(z-r)^{\omega-1}\mathcal{G}(z)dz,$$

where Γ is the gamma function given by

$$\Gamma(\omega) := \int_0^{\infty} z^{\omega-1}e^{-z}dz, \quad \mathrm{Re}(\omega) > 0.$$

The following definition is an important generalization of the Riemann-Liouville fractional integrals known as the ψ-Riemann-Liouville fractional integrals.

Definition 2.2 ([22])**.** Let (d_1, d_2), $-\infty \leq d_1 < d_2 \leq +\infty$, be a finite or infinite interval of the real line $\mathbb{R}$ and $\omega > 0$. Also, let ψ be an increasing and positive function on $(d_1, d_2]$, having a continuous derivative ψ' on (d_1, d_2). The left and right-sided Riemann-Liouville fractional integrals of a function $\mathcal{G}$ with respect to the function ψ on $[d_1, d_2]$ are defined by

$$I^{\omega;\psi}_{d_1^+}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{d_1}^{r}(\psi(r)-\psi(z))^{\omega-1}\psi'(z)\mathcal{G}(z)dz$$

and

$$I^{\omega;\psi}_{d_2^-}\mathcal{G}(r) = \frac{1}{\Gamma(\omega)}\int_{r}^{d_2}(\psi(z)-\psi(r))^{\omega-1}\psi'(z)\mathcal{G}(z)dz.$$

The left- and right-sided Atangana-Baleanu fractional integrals are defined as follows.

Definition 2.3 ([1,5])**.** The left- and right-sided Atangana-Baleanu fractional integral of a real-valued function $\mathcal{G}$ of order $\omega \in (0, 1)$ are defined by

$${}^{AB}I^{\omega}_{d_1^+}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^{r}(r-z)^{\omega-1}\mathcal{G}(z)dz$$

and

$${}^{AB}I^{\omega}_{d_2^-}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2}(z-r)^{\omega-1}\mathcal{G}(z)dz,$$

where $B(\omega) > 0$ is a normalization function and satisfies the property $B(0) = B(1) = 1$.

In [19], Kashuri introduced the following generalization of the Atangana-Baleanu fractional integral operators called the ABK-fractional integral operators.

Definition 2.4. The left- and right-sided ABK-fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega \in (0,1)$ are given by

$$ {}^{ABK\rho}_{d_1^+}I_r^{\omega}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\rho^{1-\omega}\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^{r}\frac{z^{\rho-1}}{(r^{\rho}-z^{\rho})^{1-\omega}}\mathcal{G}(z)dz, \quad r > d_1, $$

and

$$ {}^{ABK\rho}_{d_2^-}I_r^{\omega}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\rho^{1-\omega}\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2}\frac{z^{\rho-1}}{(z^{\rho}-r^{\rho})^{1-\omega}}\mathcal{G}(z)dz, \quad r < d_2, $$

where $\rho > 0$ and $B(\omega) > 0$ satisfies the property $B(0) = B(1) = 1$.

Recently, Kermausuor and Nwaeze also introduced another generalization of the Atangana-Baleanu fractional integrals called the k-Atangana-Baleanu fractional integrals as defined below.

Definition 2.5 ([21])**.** The left- and right-sided k-Atangana-Baleanu fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega > 0$, are defined as

$$ {}^{AB}{}_{k}I_{d_1^+}^{\omega}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{kB(\omega)\Gamma_k(\omega)}\int_{d_1}^{r}(r-z)^{\frac{\omega}{k}-1}\mathcal{G}(z)dz, \quad r > d_1, $$

and

$$ {}^{AB}{}_{k}I_{d_2^-}^{\omega}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{kB(\omega)\Gamma_k(\omega)}\int_{r}^{d_2}(z-r)^{\frac{\omega}{k}-1}\mathcal{G}(z)dz, \quad r < d_2, $$

where $k > 0$ and $B(\omega) > 0$ satisfies the property $B(0) = B(1) = 1$.

As mentioned in the introduction, several fractional integral inequalities have been established by utilizing the aforementioned fractional integrals. For instance, by utilizing the Atangana-Baleanu fractional integral operators, Fernandez and Mohammed [12] established the following generalization of the Hermite-Hadamard inequality.

Theorem 2.1. *If $\mathcal{G} : [d_1, d_2] \to \mathbb{R}$ is convex on $[d_1, d_2]$ and $\mathcal{G} \in L_1([d_1, d_2])$, then the inequalities*

$$ \begin{aligned} \mathcal{G}\left(\frac{d_1+d_2}{2}\right) \leq & \frac{B(\omega)\Gamma(\omega)}{2[(d_2-d_1)^{\omega}+(1-\omega)\Gamma(\omega)]}\left[{}^{AB}I_{d_1^+}^{\omega}\mathcal{G}(d_2) + {}^{AB}I_{d_2^-}^{\omega}\mathcal{G}(d_1)\right] \\ \leq & \frac{\mathcal{G}(d_1)+\mathcal{G}(d_2)}{2} \end{aligned} $$

hold for $\omega \in (0,1)$.

Similarly, Kashuri presented the following generalization of the Hermite-Hadamard inequality by utilizing the ABK-fractional integrals.

Theorem 2.2. *Let $\omega \in (0,1)$ and $\rho > 0$. Let $\mathcal{G} : [d_1^\rho, d_2^\rho] \to \mathbb{R}$ with $0 \leq d_1 < d_2$ be a convex function. Then, the following inequalities for the ABK-fractional integrals hold:*

$$\begin{aligned}
&\frac{2(d_2^\rho - d_1^\rho)^\omega}{B(\omega)\Gamma(\omega+1)\rho^{2-\rho}}\mathcal{G}\left(\frac{d_1^\rho + d_2^\rho}{2}\right) + \frac{1-\omega}{B(\omega)}\left[\mathcal{G}(d_1^\rho) + \mathcal{G}(d_2^\rho)\right] \\
&\leq {}^{ABK\rho}_{\quad d_1^+}I^\omega_{d_2^\rho}\mathcal{G}(d_2^\rho) + {}^{ABK\rho}_{\quad d_2^-}I^\omega_{d_1^\rho}\mathcal{G}(d_1^\rho) \\
&\leq \left(\frac{(d_2^\rho - d_1^\rho)^\omega + \rho(1-\omega)\Gamma(\omega)}{\rho B(\omega)\Gamma(\omega)}\right)\left[\mathcal{G}(d_1^\rho) + \mathcal{G}(d_2^\rho)\right].
\end{aligned}$$

To end this section, we present the definition of s-convex functions in the second sense.

Definition 2.6 ([7])**.** Let $s \in (0,1]$. A function $\mathcal{G} : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex in the second sense on I, if

$$\mathcal{G}(r\delta_1 + (1-r)\delta_2) \leq r^s\mathcal{G}(\delta_1) + (1-r)^s\mathcal{G}(\delta_2),$$

for all $\delta_1, \delta_2 \in I$ and $r \in [0,1]$.

Remark 2.1. If $s = 1$ in Definition 2.6, then we have the classical definition of a convex function.

3. Main Results

We commence by defining the new fractional integrals that generalize the Atangana-Baleanu fractional integrals.

Definition 3.1. Let $d_1, d_2 \in \mathbb{R}$ with $d_1 < d_2$ and let ψ be a strictly increasing and positive function on $[d_1, d_2]$, having a continuous derivative ψ' on (d_1, d_2). The left- and right-sided ψ-Hilfer-Atangana-Baleanu (HAB) fractional integral operators of a real-valued function $\mathcal{G}$ of order $\omega > 0$, are defined as

$${}^{HAB}_{\quad\psi}I^\omega_{d_1^+}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{d_1}^{r}(\psi(r)-\psi(z))^{\omega-1}\psi'(z)\mathcal{G}(z)dz, \quad r > d_1,$$

and

$${}^{HAB}_{\quad\psi}I^\omega_{d_2^-}\mathcal{G}(r) = \frac{1-\omega}{B(\omega)}\mathcal{G}(r) + \frac{\omega}{B(\omega)\Gamma(\omega)}\int_{r}^{d_2}(\psi(z)-\psi(r))^{\omega-1}\psi'(z)\mathcal{G}(z)dz, \quad r < d_2,$$

where $B(\omega) > 0$ is a normalization function and satisfies the property $B(0) = B(1) = 1$.

Remark 3.1. If $\psi(t) = t$ in Definition 3.1, then we have the Atangana-Baleanu fractional integral operators, and if $\psi(t) = \frac{t^\rho}{\rho}$, $\rho > 0$, then we have the ABK-fractional integral operators.

Next, we investigate the Hermite-Hadamard inequality by utilizing the ψ-Hilfer-Atangana-Baleanu fractional integral operators. In what follows, we assume that ψ meets the requirements of Definition 3.1.

Lemma 3.1. *For $\mu < \nu$, we have*

(3.1)
$$\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr = \frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^\omega}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\mu)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\nu)\right]$$

and

(3.2)
$$\int_0^1 r^{\omega-1}\mathcal{G}(r\nu+(1-r)\mu)dr = \frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^\omega}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\nu)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\mu)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\mu)\right].$$

Proof. By using change of variables with $x = r\mu + (1-r)\nu$, we have

$$\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr = \frac{1}{(\nu-\mu)^\omega}\int_\mu^\nu (\nu-x)^{\omega-1}\mathcal{G}(x)dx. \tag{3.3}$$

Since ψ is strictly increasing on $[\mu,\nu]$ and hence has an inverse, we deduce that for any $x \in [\mu,\nu]$, $x = \psi(z)$ for some $z \in [\mu,\nu]$. So using change of variables again with $x = \psi(z)$ in (3.3) and Definition 3.1, we have

$$\begin{aligned}
\int_0^1 r^{\omega-1}\mathcal{G}(r\mu+(1-r)\nu)dr =& \frac{1}{(\nu-\mu)^\omega}\int_\mu^\nu (\nu-x)^{\omega-1}\mathcal{G}(x)dx \\
=& \frac{1}{(\nu-\mu)^\omega}\int_{\psi^{-1}(\mu)}^{\psi^{-1}(\nu)} (\psi(\psi^{-1}(\nu))-\psi(z))^{\omega-1}\psi'(z)\mathcal{G}(\psi(z))dz \\
=& \frac{B(\omega)\Gamma(\omega)}{\omega(\nu-\mu)^\omega}\Bigg[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\mu)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu)) \\
& - \frac{1-\omega}{B(\omega)}(\mathcal{G}\circ\psi)(\psi^{-1}(\nu))\Bigg].
\end{aligned}$$

This proves the identity in (3.1). The identity in (3.2) is obtained by using a similar argument. □

Theorem 3.1. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta,\epsilon] \to \mathbb{R}$ be an s-convex function in the second sense. Then, the inequalities*

$$\begin{aligned}
& \frac{2^s(\epsilon-\delta)^\omega}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)] \\
\le & {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \\
\le & \left[\frac{\omega(\epsilon-\delta)^\omega}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right) + \frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)] \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
& \left[\frac{2^s(\epsilon-\delta)^\omega}{B(\omega)\Gamma(\omega)} + \frac{2^s(1-\omega)}{B(\omega)}\right]\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) \\
\le & {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))
\end{aligned}$$

$$\leq \left[\frac{\omega(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+s}+\mathbb{B}(\omega,s+1)\right)+\frac{1-\omega}{B(\omega)}\right][\mathcal{G}(\delta)+\mathcal{G}(\epsilon)] \tag{3.5}$$

hold for $\omega \in (0,1)$*, where* $\mathbb{B}(\cdot,\cdot)$ *is the beta function defined by*

$$\mathbb{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Proof. Under the condition of s-convexity in the second sense of $\mathcal{G}$ on $[\delta,\epsilon]$, it follows that for any $p_1,p_2\in[\delta,\epsilon]$

$$\mathcal{G}\left(\frac{p_1+p_2}{2}\right)\leq\frac{\mathcal{G}(p_1)+\mathcal{G}(p_2)}{2^s}. \tag{3.6}$$

If we take $p_1=r\epsilon+(1-r)\delta$ and $p_2=(1-r)\epsilon+r\delta$ for $r\in[0,1]$ in (3.6), then we get

$$2^s\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\leq\mathcal{G}(r\epsilon+(1-r)\delta)+\mathcal{G}((1-r)\epsilon+r\delta). \tag{3.7}$$

Multiply (3.7) by $\frac{\omega}{B(\omega)\Gamma(\omega)}r^{\omega-1}$ and integrate with respect to r over $[0,1]$ to obtain

$$\begin{aligned}\frac{2^s}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\leq&\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}(r\epsilon+(1-r)\delta)dr\\&+\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^1 r^{\omega-1}\mathcal{G}((1-r)\epsilon+r\delta)dr.\end{aligned} \tag{3.8}$$

Using Lemma 3.1 and the inequality in (3.8), we get

$$\begin{aligned}&\frac{2^s(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]\\&\leq{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)).\end{aligned} \tag{3.9}$$

This proves the first inequality in (3.4). Now, since $\mathcal{G}$ is s-convex in the second sense, we have $\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\leq\frac{\mathcal{G}(\epsilon)+\mathcal{G}(\delta)}{2^s}$. Hence, from (3.9), we have

$$\begin{aligned}&\left[\frac{2^s(\epsilon-\delta)^{\omega}}{B(\omega)\Gamma(\omega)}+\frac{2^s(1-\omega)}{B(\omega)}\right]\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)\\&\leq{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)).\end{aligned}$$

This establishes the first inequality in (3.5). To establish the second inequality in both (3.4) and (3.5), we note that, if $\mathcal{G}$ is s-convex in the second sense, then for $r\in[0,1]$, it follows that

$$\mathcal{G}(r\epsilon+(1-r)\delta)\leq r^s\mathcal{G}(\epsilon)+(1-r)^s\mathcal{G}(\delta) \tag{3.10}$$

and

$$\mathcal{G}((1-r)\epsilon+r\delta)\leq(1-r)^s\mathcal{G}(\epsilon)+r^s\mathcal{G}(\delta). \tag{3.11}$$

Adding (3.10) and (3.11), we obtain

$$\mathcal{G}(r\epsilon+(1-r)\delta)+\mathcal{G}((1-r)\epsilon+r\delta)\leq(r^s+(1-r)^s)[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)]. \tag{3.12}$$

Multiply (3.12) by $\frac{\omega}{B(\omega)\Gamma(\omega)} r^{\omega-1}$ and integrate with respect to r over $[0,1]$ to get

$$\begin{aligned}
&\frac{\omega}{B(\omega)\Gamma(\omega)} \int_0^1 r^{\omega-1} \mathcal{G}(r\epsilon + (1-r)\delta) dr + \frac{\omega}{B(\omega)\Gamma(\omega)} \int_0^1 r^{\omega-1} \mathcal{G}((1-r)\epsilon + r\delta) dr \\
\leq &\frac{\omega}{B(\omega)\Gamma(\omega)} [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \int_0^1 r^{\omega-1} (r^s + (1-r)^s) dr \\
= &\frac{\omega}{B(\omega)\Gamma(\omega)} [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \left(\frac{1}{\omega + s} + \mathbb{B}(\omega, s+1) \right). \qquad (3.13)
\end{aligned}$$

Using Lemma 3.1, we deduce from (3.13) that

$$\begin{aligned}
&{}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\delta)^+} (\mathcal{G} \circ \psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\epsilon)^-} (\mathcal{G} \circ \psi)(\psi^{-1}(\delta)) \\
\leq &\frac{\omega(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \left(\frac{1}{\omega + s} + B(\omega, s+1) \right) + \frac{1-\omega}{B(\omega)} [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \\
= &\left[\frac{\omega(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} \left(\frac{1}{\omega + s} + \mathbb{B}(\omega, s+1) \right) + \frac{1-\omega}{B(\omega)} \right] [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)]. \qquad (3.14)
\end{aligned}$$

This proves the second inequality in both (3.4) and (3.5). □

Corollary 3.1. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta, \epsilon] \to \mathbb{R}$ be a convex function. Then, the inequalities*

$$\begin{aligned}
&\frac{2(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} \mathcal{G}\left(\frac{\epsilon + \delta}{2} \right) + \frac{1-\omega}{B(\omega)} [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \\
\leq &{}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\delta)^+} (\mathcal{G} \circ \psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\epsilon)^-} (\mathcal{G} \circ \psi)(\psi^{-1}(\delta)) \\
\leq &\left[\frac{(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} + \frac{1-\omega}{B(\omega)} \right] [\mathcal{G}(\gamma) + \mathcal{G}(\omega)] \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\frac{2(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} + \frac{2(1-\omega)}{B(\omega)} \right] \mathcal{G}\left(\frac{\epsilon + \delta}{2} \right) \\
\leq &{}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\delta)^+} (\mathcal{G} \circ \psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi} I^{\omega}_{\psi^{-1}(\epsilon)^-} (\mathcal{G} \circ \psi)(\psi^{-1}(\delta)) \\
\leq &\left[\frac{(\epsilon - \delta)^{\omega}}{B(\omega)\Gamma(\omega)} + \frac{1-\omega}{B(\omega)} \right] [\mathcal{G}(\delta) + \mathcal{G}(\epsilon)] \qquad (3.16)
\end{aligned}$$

hold for $\omega \in (0,1)$.

Proof. Take $s = 1$ in Theorem 3.1. □

Remark 3.2. If we take $\psi(t) = t$ in Corollary 3.1, then the inequality (3.16) reduces to the inequality in Theorem 2.1 and if we take $\psi(t) = \frac{t^{\rho}}{\rho}$, $\rho > 0$, in Corollary 3.1, then the inequality (3.15) reduces to the inequality in Theorem 2.2 with some slight modification to the endpoints of the interval.

Below, we introduce some new inequalities of the midpoint and trapezoidal types for functions whose derivatives, in absolute value raised to certain nonnegative powers, are s-convex in the second sense, using the ψ-Hilfer-Atangana-Baleanu fractional integrals. To accomplish this, we first establish the following key identities involving these fractional integrals.

Lemma 3.2. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta, \omega] \to \mathbb{R}$ be a function. If $\mathcal{G}$ is differentiable and $\mathcal{G}' \in L_1([\delta, \epsilon])$, then the equality*

$$\begin{aligned}
&\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\left[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\right] \\
&-\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right] \\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1[(1-r)^{\omega}-r^{\omega}]\mathcal{G}'(r\delta+(1-r)\epsilon)dr \qquad (3.17)
\end{aligned}$$

holds for all $\omega > 0$.

Proof. Utilizing the technique of integration by parts, change of variables and Lemma 3.1, we get

$$\begin{aligned}
&\int_0^1(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{\omega}{\epsilon-\delta}\int_0^1(1-r)^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{\omega}{\epsilon-\delta}\int_0^1 t^{\omega-1}\mathcal{G}(t\epsilon+(1-t)\delta)dt \\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_0^1(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{\epsilon-\delta}\mathcal{G}(\epsilon)-\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right]. \qquad (3.18)
\end{aligned}$$

Using a similar reasoning, we can conclude that

$$\begin{aligned}
&\int_0^1 r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&-\frac{1}{\epsilon-\delta}\mathcal{G}(\delta)+\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))-\frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right]. \qquad (3.19)
\end{aligned}$$

The identity in (3.17) can be achieved by multiplying the identities in (3.18) and (3.19) by $\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}$, and then subtracting the resulting equations. □

Lemma 3.3. *Let $\delta, \epsilon \in \mathbb{R}$ with $\delta < \epsilon$ and $\mathcal{G} : [\delta, \epsilon] \to \mathbb{R}$ be a function. If $\mathcal{G}$ is differentiable and $\mathcal{G}' \in L_1([\delta, \epsilon])$, then the equality*

$$\begin{aligned}
&\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right] \\
&- \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right) \\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr - \int_{1/2}^{1}(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr\right] \qquad (3.20)
\end{aligned}$$

holds for all $\omega > 0$.

Proof. Utilizing the technique of integration by parts, we get

$$\begin{aligned}
&\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&\frac{r^{\omega}}{\delta-\epsilon}\mathcal{G}(r\delta+(1-r)\epsilon)\Big|_0^{1/2} - \frac{\omega}{(\delta-\epsilon)}\int_0^{1/2} r^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{2^{\omega}(\delta-\epsilon)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) - \frac{\omega}{(\delta-\epsilon)}\int_0^{1/2} r^{\omega-1}\mathcal{G}(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{2^{\omega}(\delta-\epsilon)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) + \frac{\omega}{(\epsilon-\delta)^{\omega+1}}\int_{\frac{\delta+\epsilon}{2}}^{\epsilon}(\epsilon-z)^{\omega-1}\mathcal{G}(z)dz \\
=&-\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) \\
&+\frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right].
\end{aligned}$$

So,

$$\begin{aligned}
&\int_0^{1/2} r^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&-\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) + \frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\epsilon)\right]. \qquad (3.21)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{1/2}^{1}(1-r)^{\omega}\mathcal{G}'(r\delta+(1-r)\epsilon)dr \\
=&\frac{1}{2^{\omega}(\epsilon-\delta)}\mathcal{G}\left(\frac{\delta+\epsilon}{2}\right) - \frac{B(\omega)\Gamma(\omega)}{(\epsilon-\delta)^{\omega+1}}\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) - \frac{1-\omega}{B(\omega)}\mathcal{G}(\delta)\right]. \qquad (3.22)
\end{aligned}$$

The identity in (3.20) can be achieved by multiplying the identities in (3.21) and (3.22) by $\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}$, and then taking the difference between the resulting equations. □

Now, employing Lemma 3.2 and Lemma 3.3, we derive the following Hermite-Hadamard type inequalities.

Theorem 3.2. *If $\mathcal{G}$ meets the conditions of Lemma 3.2 and $|\mathcal{G}'|$ is s-convex in the second sense, then the inequality*

$$\begin{aligned}
&\left|\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\omega)+\mathcal{G}(\epsilon)\Big]\right.\\
&\left.-\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}\Big[|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big]}{B(\omega)\Gamma(\omega)}\Bigg(\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\\
&+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}\Bigg)
\end{aligned}\tag{3.23}$$

holds for $\omega\in(0,1)$, where $\mathbb{B}_z(\cdot,\cdot)$ is the incomplete beta function stated as

$$\mathbb{B}_z(x,y)=\int_0^z t^{x-1}(1-t)^{y-1}dt.\tag{3.24}$$

Proof. Utilizing the identity in (3.17) and the s-convexity in the second sense of $|\mathcal{G}'|$, we derive

$$\begin{aligned}
&\left|\left(\frac{(\epsilon-\delta)^{\omega}+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right.\\
&\left.-\left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\int_0^1\Big|(1-r)^{\omega}-r^{\omega}\Big|\Big(r^{s}|\mathcal{G}'(\delta)|+(1-r)^{s}|\mathcal{G}'(\epsilon)|\Big)dr
\end{aligned}\tag{3.25}$$

Now, we observed that for $r\in[0,1]$,

$$(1-r)^{\omega}-r^{\omega}\begin{cases}\geq 0, & \text{for } r\in[0,1/2],\\ <0, & \text{for } r\in(1/2,1],\end{cases}$$

and hence

$$\begin{aligned}
\int_0^1|(1-r)^{\omega}-r^{\omega}|r^{s}dr=&\int_0^{1/2}((1-r)^{\omega}-r^{\omega})r^{s}dr+\int_{1/2}^1(r^{\omega}-(1-r)^{\omega})r^{s}dr\\
=&\int_0^{1/2}((1-r)^{\omega}r^{s}-r^{\omega+s})dr+\int_0^{1/2}((1-t)^{\omega+s}-t^{\omega}(1-t)^{s})dt\\
=&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\frac{1}{2^{\omega+s+1}(\omega+s+1)}\\
&+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s+1}(\omega+s+1)}-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\\
=&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)
\end{aligned}$$

$$+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}. \tag{3.26}$$

Similarly,

$$\begin{aligned}\int_0^1|(1-r)^\omega-r^\omega|(1-r)^s dr =&\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\\ &+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}.\end{aligned} \tag{3.27}$$

The desired inequality in (3.23) is obtained by substituting (3.26) and (3.27) in (3.25). □

Corollary 3.2. *If $\mathcal{G}$ meets the requirements of Lemma* 3.2 *and $|\mathcal{G}'|$ is convex, then the following inequality holds:*

$$\begin{aligned}&\left|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right.\\ &\left.-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\right|\\ \le&\frac{(\epsilon-\delta)^{\omega+1}\Big[|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big]}{B(\omega)\Gamma(\omega)}\left(\frac{1}{\omega+1}-\frac{1}{2^\omega(\omega+1)}\right),\end{aligned} \tag{3.28}$$

for $\omega\in(0,1)$.

Proof. Take $s=1$ in Theorem 3.2. □

Theorem 3.3. *If $\mathcal{G}$ meets the requirements of Lemma* 3.2 *and $|\mathcal{G}'|^q$, $q>1$, is s-convex in the second sense, then inequality*

$$\begin{aligned}&\left|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right.\\ &\left.-\Big[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\Big]\right|\\ \le&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|^q+|\mathcal{G}'(\epsilon)|^q\Big)^{\frac{1}{q}}}{B(\omega)\Gamma(\omega)}\left(\frac{2}{\omega+1}-\frac{2}{2^\omega(\omega+1)}\right)^{1-\frac{1}{q}}\\ &\times\left(\mathbb{B}_{\frac{1}{2}}(s+1,\omega+1)-\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\right.\\ &\left.+\frac{1}{\omega+s+1}-\frac{1}{2^{\omega+s}(\omega+s+1)}\right)^{\frac{1}{q}}\end{aligned} \tag{3.29}$$

holds for $\omega\in(0,1)$, where $\mathbb{B}_z(\cdot,\cdot)$ is the incomplete beta function defined in (3.24).

Proof. Utilizing the identity in (3.17) and the power-mean inequality, we derive

$$\left|\left(\frac{(\epsilon-\delta)^\omega+(1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)}\right)\Big[\mathcal{G}(\omega)+\mathcal{G}(\epsilon)\Big]\right.$$

$$
\begin{aligned}
&- \left[{}^{HAB}{}_{\psi}I^{\epsilon}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \right] \Bigg| \\
\leq & \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)} \left(\int_0^1 |(1-r)^{\omega} - r^{\omega}| dr \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 |(1-r)^{\omega} - r^{\omega}| |\mathcal{G}'(r\delta + (1-r)\epsilon)|^q dr \right)^{\frac{1}{q}}. \qquad (3.30)
\end{aligned}
$$

Using the s-convexity of $|\mathcal{G}'|^q$ and a reasoning similar to the one in the proof of Theorem 3.3, we can conclude that

$$
\begin{aligned}
&\int_0^1 |(1-r)^{\omega} - r^{\omega}| |\mathcal{G}'(r\delta + (1-r)\epsilon)|^q dr \\
\leq & \Big(|\mathcal{G}'(\delta)|^q + \mathcal{G}'(\epsilon)|^q \Big) \Bigg(\mathbb{B}_{\frac{1}{2}}(s+1, \omega+1) - \mathbb{B}_{\frac{1}{2}}(\omega+1, s+1) \\
& + \frac{1}{\omega+s+1} - \frac{1}{2^{\omega+s}(\omega+s+1)} \Bigg). \qquad (3.31)
\end{aligned}
$$

In addition,

$$
\int_0^1 \Big| (1-r)^{\omega} - r^{\omega} \Big| dr = 2 \left(\frac{1}{\omega+1} - \frac{1}{2^{\omega}(\omega+1)} \right). \qquad (3.32)
$$

The desired inequality in (3.29) is obtained by substituting (3.31) and (3.32) in (3.30). □

Corollary 3.3. *If $\mathcal{G}$ meets the requirements of Lemma 3.2 and $|\mathcal{G}'|^q$, $q > 1$, is convex, then the inequality*

$$
\begin{aligned}
&\left| \left(\frac{(\epsilon-\delta)^{\omega} + (1-\omega)\Gamma(\omega)}{B(\omega)\Gamma(\omega)} \right) \Big[\mathcal{G}(\omega) + \mathcal{G}(\epsilon) \Big] \right. \\
&\left. - \left[{}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\delta)^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{\psi^{-1}(\epsilon)^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \right] \right| \\
\leq & \frac{2^{1-\frac{1}{q}}(\epsilon-\delta)^{\omega+1} \Big(|\mathcal{G}'(\delta)|^q + \mathcal{G}'(\epsilon)|^q \Big)^{\frac{1}{q}}}{B(\omega)\Gamma(\omega)} \left(\frac{1}{\omega+1} - \frac{1}{2^{\omega}(\omega+1)} \right) \qquad (3.33)
\end{aligned}
$$

holds for $\omega \in (0,1)$.

Proof. Take $s = 1$ in Theorem 3.3. □

Theorem 3.4. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|$ is s-convex in the second sense, then the inequality*

$$
\begin{aligned}
&\left| \left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta)) \right] \right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)} \mathcal{G}\left(\frac{\epsilon+\delta}{2} \right) + \frac{1-\omega}{B(\omega)} \Big[\mathcal{G}(\delta) + \mathcal{G}(\epsilon) \Big] \right) \right|
\end{aligned}
$$

$$\leq \frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+s+1}(\omega+s+1)}+\mathbb{B}_{\frac{1}{2}}\left(\omega+1,s+1\right)\right) \tag{3.34}$$

holds for $\omega \in (0,1)$, where $\mathbb{B}_z(\cdot,\cdot)$ is the incomplete beta function defined in (3.24).

Proof. Using Lemma 3.3 and the s-convexity in the second sense of $|\mathcal{G}'|$, we obtain

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right.\\
&\left.-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right)\right|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\int_0^{1/2} r^{\omega+s}|\mathcal{G}'(\delta)|+(1-r)^s r^{\omega}|\mathcal{G}'(\epsilon)|dr\right.\\
&\left.+\int_{1/2}^{1}(1-r)^{\omega}r^s|\mathcal{G}'(\delta)|+(1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|dr\right]\\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[|\mathcal{G}'(\delta)|\left(\int_0^{1/2} r^{\omega+s}dr+\int_{1/2}^{1}(1-r)^{\omega}r^s dr\right)\right.\\
&\left.+|\mathcal{G}'(\epsilon)|\left(\int_0^{1/2}(1-r)^s r^{\omega}dr+\int_{1/2}^{1}(1-r)^{\omega+s}dr\right)\right]\\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[|\mathcal{G}'(\delta)|\left(\int_0^{1/2} r^{\omega+s}dr+\int_0^{1/2} t^{\omega}(1-t)^s dt\right)\right.\\
&\left.+|\mathcal{G}'(\epsilon)|\left(\int_0^{1/2}(1-r)^s r^{\omega}dr+\int_0^{1/2} t^{\omega+s}dt\right)\right]\\
=&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+s+1}(\omega+s+1)}+\mathbb{B}_{\frac{1}{2}}\left(\omega+1,s+1\right)\right).
\end{aligned}$$

□

Corollary 3.4. *If $\mathcal{G}$ meets the requirements of Lemma* 3.3 *and $|\mathcal{G}'|$ is convex, then the inequality*

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right.\\
&\left.-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\Big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\Big]\right)\right|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}\Big(|\mathcal{G}'(\delta)|+|\mathcal{G}'(\epsilon)|\Big)}{2^{\omega+1}\left(\omega+1\right)B(\omega)\Gamma(\omega)}
\end{aligned} \tag{3.35}$$

holds for $\omega \in (0,1)$.

Proof. Take $s=1$ in Theorem 3.4. □

Theorem 3.5. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|^q, q > 1$, is s-convex in the second sense, then the following inequality holds:*

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right| \\
\leq &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{\omega+s+1}(\omega+s+1)} + \mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\delta)|^q + \frac{|\mathcal{G}'(\epsilon)|^q}{2^{\omega+s+1}(\omega+s+1)}\right)^{\frac{1}{q}}\right], \qquad (3.36)
\end{aligned}$$

for $\omega \in (0,1)$, where $\mathbb{B}_z(\cdot,\cdot)$ is the incomplete beta function defined in (3.24).

Proof. Utilizing Lemma 3.3, the power-mean inequality and the s-convexity in the second sense of $|\mathcal{G}'|^q$, we derive

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right| \\
\leq &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_0^{1/2} r^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left(r^{\omega+s}|\mathcal{G}'(\delta)|^q + (1-r)^s r^{\omega}|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{1/2}^1 (1-r)^{\omega}dr\right)^{1-\frac{1}{q}}\left(\int_{1/2}^1\left((1-r)^{\omega}r^s|\mathcal{G}'(\delta)|^q + (1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\right] \\
= &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{1/2}\left(r^{\omega+s}|\mathcal{G}'(\delta)|^q + (1-r)^s r^{\omega}|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_{1/2}^1\left((1-r)^{\omega}r^s|\mathcal{G}'(\delta)|^q + (1-r)^{\omega+s}|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\right] \\
= &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}}\left[\left(\frac{\mathcal{G}'(\delta)|^q}{2^{\omega+s+1}(\omega+s+1)} + \mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\mathbb{B}_{\frac{1}{2}}(\omega+1,s+1)\,|\mathcal{G}'(\delta)|^q + \frac{\mathcal{G}'(\epsilon)|^q}{2^{\omega+s+1}(\omega+s+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Corollary 3.5. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|^q, q > 1$, is convex, then the inequality*

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right| \\
\leq &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega+1}(\omega+1)}\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{\omega+2}(\omega+2)} + \left(\frac{1}{2^{\omega+1}(\omega+1)} - \frac{1}{2^{\omega+2}(\omega+2)}\right)|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\left(\frac{1}{2^{\omega+1}(\omega+1)} - \frac{1}{2^{\omega+2}(\omega+2)}\right)|\mathcal{G}'(\delta)|^q + \frac{|\mathcal{G}'(\epsilon)|^q}{2^{\omega+2}(\omega+2)}\right)^{\frac{1}{q}}\right] \qquad (3.37)
\end{aligned}$$

holds for $\omega \in (0,1)$.

Proof. Take $s = 1$ in Theorem 3.5. □

Theorem 3.6. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|^q, q > 1$, is s-convex in the second sense, then the inequality*

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right| \\
&\leq \frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{G}'(\delta)|^q}{2^{s+1}(s+1)} + \left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\epsilon)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\left(\frac{1}{s+1} - \frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\delta)|^q + \frac{|\mathcal{G}'(\epsilon)|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right] \qquad (3.38)
\end{aligned}$$

holds for $\omega \in (0,1)$ and $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 3.3, the Hölder's inequality and the s-convexity in the second sense of $|\mathcal{G}'|^q$, we have

$$\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^+}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon)) + {}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^-}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right. \\
&\left. - \left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right) + \frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right| \\
\leq &\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left[\left(\int_0^{1/2} r^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left(r^s|\mathcal{G}'(\delta)|^q + (1-r)^s|\mathcal{G}'(\epsilon)|^q\right)dr\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{1/2}^{1}(1-r)^{\omega p}dr\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left(r^{s}|\mathcal{G}'(\delta)|^{q}+(1-r)^{s}|\mathcal{G}'(\epsilon)|^{q}\right)dr\right)^{\frac{1}{q}}\Bigg]\\
=&\frac{(\epsilon-\delta)^{\omega+1}}{B(\omega)\Gamma(\omega)}\left(\frac{1}{2^{\omega p+1}(\omega p+1)}\right)^{\frac{1}{p}}\Bigg[\left(\frac{|\mathcal{G}'(\delta)|^{q}}{2^{s+1}(s+1)}+\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\epsilon)|^{q}\right)^{\frac{1}{q}}\\
&+\left(\left(\frac{1}{s+1}-\frac{1}{2^{s+1}(s+1)}\right)|\mathcal{G}'(\delta)|^{q}+\frac{|\mathcal{G}'(\epsilon)|^{q}}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

□

Corollary 3.6. *If $\mathcal{G}$ meets the requirements of Lemma 3.3 and $|\mathcal{G}'|^{q}$, $q>1$, is convex, then the inequality*

$$
\begin{aligned}
&\left|\left[{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{+}}(\mathcal{G}\circ\psi)(\psi^{-1}(\epsilon))+{}^{HAB}{}_{\psi}I^{\omega}_{(\psi^{-1}(\frac{\delta+\epsilon}{2}))^{-}}(\mathcal{G}\circ\psi)(\psi^{-1}(\delta))\right]\right.\\
&\left.-\left(\frac{(\epsilon-\delta)^{\omega}}{2^{\omega-1}B(\omega)\Gamma(\omega)}\mathcal{G}\left(\frac{\epsilon+\delta}{2}\right)+\frac{1-\omega}{B(\omega)}\big[\mathcal{G}(\delta)+\mathcal{G}(\epsilon)\big]\right)\right|\\
\leq&\frac{(\epsilon-\delta)^{\omega+1}}{2^{\omega+3}B(\omega)\Gamma(\omega)}\left(\frac{4}{\omega p+1}\right)^{\frac{1}{p}}\Bigg[\left(|\mathcal{G}'(\delta)|^{q}+3|\mathcal{G}'(\epsilon)|^{q}\right)^{\frac{1}{q}}\\
&+\left(3|\mathcal{G}'(\delta)|^{q}+|\mathcal{G}'(\epsilon)|^{q}\right)^{\frac{1}{q}}\Bigg] \qquad (3.39)
\end{aligned}
$$

holds for $\omega\in(0,1)$ and $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Take $s=1$ Theorem 3.6. □

4. Conclusion

In this study, we first introduced the definitions of some new fractional integral operators referred to as ψ-Hilfer-Atangana-Baleanu (ψ-HAB) fractional integral operators. As indicated in Remark 3.1, these newly defined fractional integrals are generalizations of the Atangana-Baleanu and ABK fractional integrals. Next, we established a generalized version of the Hermite-Hadamard inequality for functions that are s-convex in the second sense via the ψ-HAB fractional integral operators. Finally, we investigated some fractional integral inequalities of the Hermite-Hadamard type involving the ψ-HAB fractional integrals for functions whose derivatives in absolute value to some nonnegative powers are s-convex in the second sense. Some particular cases of the main result were presented.

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NONLINEAR LEFT BI-SKEW LIE TYPE DERIVATIONS ON $*$-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital $*$-algebra over $\mathbb{C}$ (the field of complex number). For any $\alpha, \beta \in \mathcal{A}$, define $\alpha \circ \beta = \alpha^*\beta - \beta^*\alpha$. In this article, it is shown that a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfies $\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n)$ for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation. As applications, we apply our main result to various special classes of unital $*$-algebras, such as prime $*$-algebras, factor von Neumann algebras and von Neumann algebra with no central summands of type I_1.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra on the complex field $\mathbb{C}$. A map $\vartheta : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\vartheta(\alpha + \beta) = \vartheta(\alpha) + \vartheta(\beta)$ and $\vartheta(\alpha\beta) = \vartheta(\alpha)\beta + \alpha\vartheta(\beta)$ hold for all $\alpha, \beta \in \mathcal{A}$. An algebra with an involution '$*$' is called $*$-algebra, then an additive derivation $\vartheta : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation, if $\vartheta(\alpha^*) = \vartheta(\alpha)^*$ hold for all $\alpha \in \mathcal{A}$. For $\alpha, \beta \in \mathcal{A}$, describe the skew Lie product and bi-skew Lie product of α and β by $[\alpha, \beta]_* = \alpha\beta - \beta\alpha^*$ and $[\alpha, \beta]_\triangledown = \alpha\beta^* - \beta\alpha^*$, respectively. A map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) is said to be nonlinear skew Lie derivation (resp. nonlinear skew Lie triple derivation) if $\vartheta([\alpha, \beta]_*) = [\vartheta(\alpha), \beta]_* + [\alpha, \vartheta(\beta)]_*$ (resp. $\vartheta([[\alpha, \beta]_*, \gamma]_*) = [[\vartheta(\alpha), \beta]_*, \gamma]_* + [\alpha, \vartheta(\beta)]_*, \gamma]_* + [[\alpha, \beta]_*, \vartheta(\gamma)]_*$), holds for all $\alpha, \beta, \gamma \in \mathcal{A}$. Analogously, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called nonlinear bi-skew Lie derivation (resp. nonlinear bi-skew Lie triple derivation) if $\vartheta([\alpha, \beta]_\triangledown) = [\vartheta(\alpha), \beta]_\triangledown + [\alpha, \vartheta(\beta)]_\triangledown$ (resp. $\vartheta([[\alpha, \beta]_\triangledown, \gamma]_\triangledown) = [[\vartheta(\alpha), \beta]_\triangledown, \gamma]_\triangledown + [[\alpha, \vartheta(\beta)]_\triangledown, \gamma]_\triangledown + [[\alpha, \beta]_\triangledown, \vartheta(\gamma)]_\triangledown$) holds for

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all $\alpha, \beta, \gamma \in \mathcal{A}$. Many mathematicians have spent the last decade to the studying mappings involving different products on various types of rings and algebras. Yu et al. in [15], proved that every $*$-Lie derivation on factor von Neumann algebras is an additive $*$-derivation. Li et al. [9] discussed the skew Lie triple derivation between factor von Neumann algebras and showed that it is an additive $*$-derivation. Kong et al. [7], extended the results of [15] to prime $*$-rings. Similarly, skew Jordan derivations, skew Jordan triple derivations and skew Jordan-type derivations have been studied under certain conditions on various types of algebras such as factor von Neumann algebras, von Neumann algebras and $*$-algebras (see [8, 10, 14, 16, 17]).

For any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ and integer $n \geq 2$, define a sequence of polynomials as follows: $P_1(\lambda_1) = \lambda_1, P_2(\lambda_1, \lambda_2) = \lambda_1 \circ \lambda_2 = \lambda_1^* \lambda_2 - \lambda_2^* \lambda_1$ and $P_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) = P_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \circ \lambda_n$. The polynomial $P_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n)$ is called the left bi-skew Lie n-product of elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now, we define a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is said to be nonlinear left bi-skew Lie n-derivation if

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n)$$

holds for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$. Obviously, a nonlinear left bi-skew Lie 2-derivation is a nonlinear left bi-skew Lie derivation and a nonlinear left bi-skew Lie 3-derivation is a nonlinear left bi-skew Lie triple derivation. Nonlinear left bi-skew Lie 2-derivations, nonlinear left bi-skew Lie 3-derivations and nonlinear left bi-skew Lie n-derivations are collectively known as nonlinear left bi-skew Lie-type derivations. In 2018, Lin [11], studied nonlinear $*$-Lie type derivations on standard operator algebras and proved that every nonlinear $*$-Lie n-derivation on a standard operator algebra is an additive $*$-derivation. In the same year Lin [12], proved that every nonlinear $*$-Lie n-derivation on von Neumann algebras is an additive $*$-derivation. Recently Madni et al. [13], proved that if $\mathcal{A}$ is a unital $*$-algebra and $\vartheta : \mathcal{A} \to \mathcal{A}$ is a skew Lie n-derivation, then ϑ is additive. Moreover, if $\vartheta(\frac{iI}{2})$ is self-adjoint, then ϑ is a $*$-derivation. Kong et al. in [5], proved that if $\mathcal{A}$ is a factor von Neumann algebra and a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfy $\vartheta([\alpha, \beta]_\triangledown) = [\vartheta(\alpha), \beta]_\triangledown + [\alpha, \vartheta(\beta)]_\triangledown$ if and only if ϑ is an additive $*$-derivation. Khan [4], proved that every nonlinear bi-skew Lie triple derivation on factor von Neumann algebra is an additive $*$-derivation. Ashraf et al. [1], generalized the result of [4] and proved that every nonlinear bi-skew Lie n-derivation on factor von Neumann algebra is an additive $*$-derivation.

Inspired by the above-mentioned works, our focus will be on providing a description of nonlinear left bi-skew Lie n-derivations on $*$-algebras. More accurately, we show that, under mild assumptions, every nonlinear left bi-skew Lie n-derivation on a unital $*$-algebra is an additive $*$-derivation.

2. Main Results

The main result of this article is as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unity I and containing a non-trivial projection P_1. Write $P_2 = I - P_1$ and suppose that $\mathcal{A}$ satisfies*

$$(\spadesuit) \qquad X\mathcal{A}P_k = 0 \Rightarrow X = 0 \quad (k = 1, 2),$$

where $X \in \mathcal{A}$. Then a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) satisfies

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

Proof. Assume $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then, by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly, any $\Pi \in \mathcal{A}$ can be written as $\Pi = \Pi_{11} + \Pi_{12} + \Pi_{21} + \Pi_{22}$, where $\Pi_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. Let $\mathcal{A}^+ = \{A \in \mathcal{A} : A^* = A\}$ and $\mathcal{A}^- = \{A \in \mathcal{A} : A^* = -A\}$, $\mathcal{A}_{12}^- = \{P_1AP_2 + P_2AP_1 : A \in \mathcal{A}^-\}$, $\mathcal{A}_{kk}^- = P_k\mathcal{A}^-P_k$, $k = 1, 2$. Thus, for every $A \in \mathcal{A}^-$, we have $A = A_{11} + A_{12} + A_{22}$, where $A_{11} \in \mathcal{A}_{11}^-, A_{12} \in \mathcal{A}_{12}^-$ and $A_{22} \in \mathcal{A}_{22}^-$. Obviously, we only need to prove the necessity part. To complete the proof of the theorem stated above, several lemmas are required. These lemmas are presented as follows. □

Lemma 2.1. $\vartheta(0) = 0$.

Proof. It is obvious that

$$\begin{aligned}\vartheta(0) &= \vartheta(P_n(0, 0, \ldots, 0)) \\ &= P_n(\vartheta(0), 0, \ldots, 0) + P_n(0, \vartheta(0), \ldots, 0) + \cdots + P_n(0, 0, \ldots, \vartheta(0)) \\ &= 0.\end{aligned}$$

□

Lemma 2.2. $\vartheta(H)^* = -\vartheta(H)$ *for any $H \in \mathcal{A}^-$.*

Proof. We have the relation $P_n\Big(H, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) = H$ and

$$\begin{aligned}\vartheta(H) =& \vartheta\Big(P_n\Big(H, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big) \\ =& P_n\Big(\vartheta(H), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(H, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) \\ &+ \cdots + P_n\Big(H, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big) \\ =& \frac{I}{2}\big(\vartheta(H) - \vartheta(H)^*\big) + \Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) \\ &+ \cdots + \Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) \\ =& \frac{I}{2}\big(\vartheta(H) - \vartheta(H)^*\big) + (n-1)\Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big).\end{aligned}$$

On solving further we get

$$-\vartheta(H)^* = \frac{I}{2}\big(\vartheta(H) - \vartheta(H)^*\big) + (n-1)\Big(-\vartheta\Big(-\frac{I}{2}\Big)^* H - H\vartheta\Big(-\frac{I}{2}\Big)\Big) = \vartheta(H). \square$$

Lemma 2.3. *For any $H_{11} \in \mathcal{A}_{11}^-, H_{12} \in \mathcal{A}_{12}^-$ and $H_{22} \in \mathcal{A}_{22}^-$, we have*

$$\vartheta(H_{11} + H_{12}) = \vartheta(H_{11}) + \vartheta(H_{12})$$

and

$$\vartheta(H_{12} + H_{22}) = \vartheta(H_{12}) + \vartheta(H_{22}).$$

Proof. Let $T = \vartheta(H_{11} + H_{12}) - \vartheta(H_{11}) - \vartheta(H_{12})$. We need to demonstrate this $T = 0$. Utilizing the reality $P_n(P_2, H_{11}, -I, -I, \ldots, -I) = 0$ and Lemma 2.1, we have

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \ldots, -I))\\
=&\vartheta(P_n(P_2, H_{11}, -I, -I, \ldots, -I)) + \vartheta(P_n(P_2, H_{12}, -I, -I, \ldots, -I))\\
=&P_n(\vartheta(P_2), H_{11}, -I, -I, \ldots, -I) + P_n(P_2, \vartheta(H_{11}), -I, -I, \ldots, -I)\\
&+ P_n(P_2, H_{11}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{11}, -I, -I, \ldots, \vartheta(-I))\\
&+ P_n(\vartheta(P_2), H_{12}, -I, \ldots, -I) + P_n(P_2, \vartheta(H_{12}), -I, -I, \ldots, -I)\\
&+ P_n(P_2, H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{12}, -I, -I, \ldots, \vartheta(-I))\\
=&P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \ldots, -I)\\
&+ P_n(P_2, (\vartheta(H_{11}) + \vartheta(H_{12})), -I, -I, \ldots, -I)\\
&+ P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \ldots, -I)\\
&+ \cdots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \ldots, -I))\\
=&P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \ldots, -I) + P_n(P_2, (\vartheta(H_{11} + H_{12})), -I, -I, \ldots, -I)\\
&+ P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \ldots, -I)\\
&+ \cdots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

On equating the above two expressions, we obtain that $P_n(P_2, T, -I, \ldots, -I) = 0$. This gives us $T_{12} = T_{21} = T_{22} = 0$.

Invoking the fact $P_n(P_2 - P_1, H_{12}, -I, \ldots, -I) = 0$ and using Lemma

$$\begin{aligned}
&\vartheta(P_n((P_2 - P_1), H_{11} + H_{12}, -I, \ldots, -I))\\
=&\vartheta((P_n(P_2 - P_1), H_{11}, -I, \ldots, -I)) + \vartheta(P_n((P_2 - P_1), H_{12}, -I, \ldots, -I))\\
=&P_n(\vartheta(P_2 - P_1), H_{11}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}), -I, \ldots, -I)\\
&+ P_n((P_2 - P_1), H_{11}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1), H_{11}, -I, \ldots, \vartheta(-I))\\
&+ P_n(\vartheta(P_2 - P_1), H_{12}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{12}), -I, \ldots, -I)\\
&+ P_n((P_2 - P_1), H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1), H_{12}, -I, \ldots, \vartheta(-I))\\
=&P_n(\vartheta(P_2 - P_1), H_{11} + H_{12}, -I, \ldots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}) + \vartheta(H_{12}),\\
&- I, \ldots, -I) + P_n((P_2 - P_1), H_{11} + H_{12}, \vartheta(-I), \ldots, -I) + \cdots + P_n((P_2 - P_1),\\
&H_{11} + H_{12}, -I, \ldots, \vartheta(-I)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\vartheta(P_n((P_2-P_1),H_{11}+H_{12},-I,\dots,-I))\\
=&P_n(\vartheta(P_2-P_1),H_{11}+H_{12},-I,\dots,-I)+P_n((P_2-P_1),\vartheta(H_{11}+H_{12}),-I,\dots,-I)\\
&+P_n((P_2-P_1),H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n((P_2-P_1),H_{11}+H_{12},\\
&-I,\dots,\vartheta(-I)).
\end{aligned}$$

When comparing the two expressions above, we obtain that

$$P_n((P_2-P_1),T,-I,\dots,-I)=0.$$

This further implies that $T_{11}=0$. Hence $T=0$, that is,

$$\vartheta(H_{11}+H_{12})=\vartheta(H_{11})+\vartheta(H_{12}).$$

Similarly, we can show that $\vartheta(H_{12}+H_{22})=\vartheta(H_{12})+\vartheta(H_{22})$. □

Lemma 2.4. *For any $H_{11}\in\mathcal{A}_{11}^{-}, H_{12}\in\mathcal{A}_{12}^{-}$ and $H_{22}\in\mathcal{A}_{22}^{-}$, we have*

$$\vartheta(H_{11}+H_{12}+H_{22})=\vartheta(H_{11})+\vartheta(H_{12})+\vartheta(H_{22}).$$

Proof. Let $T=\vartheta(H_{11}+H_{12}+H_{22})-\vartheta(H_{11})-\vartheta(H_{12})-\vartheta(H_{22})$. We show that $T=0$. Using the fact that $P_n(P_1,H_{22},-I,\dots,-I)=0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}
&\vartheta(P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,-I))\\
=&\vartheta(P_n(P_1,H_{11}+H_{12},-I,\dots,-I))+\vartheta(P_n(P_1,H_{22},-I,\dots,-I))\\
=&P_n((\vartheta(P_1),H_{11}+H_{12},-I,\dots,-I))+P_n(P_1,\vartheta(H_{11}+H_{12}),-I,\dots,-I)\\
&+P_n(P_1,H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{11}+H_{12},-I,\dots,\vartheta(-I))\\
&+P_n(\vartheta(P_1),H_{22},-I,\dots,-I)+P_n(P_1,\vartheta(H_{22}),-I,\dots,-I)\\
&+P_n(P_1,H_{22},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{22},-I,\dots,\vartheta(-I))\\
=&P_n((\vartheta(P_1),H_{11}+H_{12},-I,\dots,-I))+P_n(P_1,\vartheta(H_{11})+\vartheta(H_{12}),-I,\dots,-I)\\
&+P_n(P_1,H_{11}+H_{12},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{11}+H_{12},-I,\dots,\vartheta(-I))\\
&+P_n(\vartheta(P_1),H_{22},-I,\dots,-I)+P_n(P_1,\vartheta(H_{22}),-I,\dots,-I)\\
&+P_n(P_1,H_{22},\vartheta(-I),\dots,-I)+\cdots+P_n(P_1,H_{22},-I,\dots,\vartheta(-I))\\
=&P_n((\vartheta(P_1),H_{11}+H_{12}+H_{22},-I,\dots,-I))\\
&+P_n(P_1,\vartheta(H_{11})+\vartheta(H_{12})+\vartheta(H_{22}),-I,\dots,-I)\\
&+P_n(P_1,H_{11}+H_{12}+H_{22},\vartheta(-I),\dots,-I)\\
&+\cdots+P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,\vartheta(-I)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\vartheta(P_n(P_1,H_{11}+H_{12}+H_{22},-I,\dots,-I))\\
=&P_n((\vartheta(P_1),H_{11}+H_{12}+H_{22},-I,\dots,-I))\\
&+P_n(P_1,\vartheta(H_{11}+H_{12}+H_{22}),-I,\dots,-I)
\end{aligned}$$

$$
\begin{aligned}
&+ P_n(P_1, H_{11} + H_{12} + H_{22}, \vartheta(-I), \ldots, -I) \\
&+ \cdots + P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \ldots, \vartheta(-I)).
\end{aligned}
$$

On solving the above two expressions for $\vartheta(P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \ldots, -I))$, we obtain that $P_n(P_1, T, -I, \ldots, -I) = 0$ which futher implies that $T_{11} = T_{12} = T_{21} = 0$. Using the fact that $P_n(P_2, H_{11}, -I, \ldots, -I) = 0$ and Lemmas 2.1 and 2.3, we find that

$$
\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \ldots, -I)) \\
=&\vartheta(P_n(P_2, H_{11}, -I, \ldots, -I)) + \vartheta(P_n(P_2, H_{12} + H_{22}, -I, \ldots, -I)) \\
=&P_n((\vartheta(P_2), H_{11}, -I, \ldots, -I)) + P_n(P_2, \vartheta(H_{11}), -I, \ldots, -I) \\
&+ P_n(P_2, H_{11}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{11}, -I, \ldots, \vartheta(-I)) \\
&+ P_n(\vartheta(P_2), H_{12} + H_{22}, -I, \ldots, -I) + P_n(P_2, \vartheta(H_{12}) + \vartheta(H_{22}), -I, \ldots, -I) \\
&+ P_n(P_2, H_{12} + H_{21}, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_2, H_{12} + H_{21}, -I, \ldots, \vartheta(-I)) \\
=&P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \ldots, -I)) \\
&+ P_n(P_2, \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{22}), -I, \ldots, -I) \\
&+ P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \ldots, -I) \\
&+ \cdots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \ldots, \vartheta(-I)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \ldots, -I)) \\
=&P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \ldots, -I)) \\
&+ P_n((P_2, \vartheta(H_{11}) + H_{12} + H_{22}), -I, \ldots, -I) \\
&+ P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \ldots, -I) \\
&+ \cdots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \ldots, \vartheta(-I)).
\end{aligned}
$$

On comparing the above two expressions, we obtain that $P_n(P_2, T, -I, \ldots, -I) = 0$ which further implies that $T_{22} = 0$. Thus, $T = 0$, that is,

$$\vartheta(H_{11} + H_{12} + H_{21} + H_{22}) = \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{21}) + \vartheta(H_{22}). \qquad \square$$

Lemma 2.5. *For any $H_{12}, H_{12}^{'} \in \mathcal{A}_{12}^{-}$ we have*

$$\vartheta(H_{12} + H_{12}^{'}) = \vartheta(H_{12}) + \vartheta(H_{12}^{'}).$$

Proof. Take $H_{12} = A_{12} - A_{12}^{*}$ and $H_{12}^{'} = B_{12} - B_{12}^{*}$ for $A_{12}, B_{12} \in \mathcal{A}_{12}$. Using the facts $P_n(iP_1 + iA_{12} + iA_{12}^{*}, iP_2 + iB_{12} + iB_{12}^{*}, -\frac{I}{2}, \ldots, -\frac{I}{2}) = H_{12} + H_{12}^{'} + H_{12}(H_{12}^{'})^{*} - H_{12}^{'}H_{12}^{*}$, where $H_{12}(H_{12}^{'})^{*} - H_{12}^{'}H_{12}^{*} = (A_{12}^{*}B_{12} - B_{12}^{*}A_{12}) + (A_{12}B_{12}^{*} - B_{12}A_{12}^{*})$ and making use of Lemma 2.4, we have

$$
\begin{aligned}
&\vartheta(H_{12} + H_{12}^{'}) + \vartheta(H_{12}(H_{12}^{'})^{*} - H_{12}^{'}H_{12}^{*}) \\
=&\vartheta\Big(H_{12} + H_{12}^{'} + H_{12}(H_{12}^{'})^{*} - H_{12}^{'}H_{12}^{*}\Big)
\end{aligned}
$$

$$
\begin{aligned}
=&\vartheta\Big(P_n\Big(iP_1+iA_{12}+iA_{12}^*, iP_2+iB_{12}+iB_{12}^*, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(iP_1)+\vartheta(iA_{12}+iA_{12}^*), iP_2+iB_{12}+iB_{12}^*, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\\
&+P_n\Big(iP_1+iA_{12}+iA_{12}^*, \vartheta(iP_2)+\vartheta(iB_{12}+iB_{12}^*), -\frac{I}{2},\dots,-\frac{I}{2}\Big)\\
&+P_n\Big(iP_1+iA_{12}+iA_{12}^*, iP_2+iB_{12}+iB_{12}^*, \vartheta\Big(-\frac{I}{2}\Big),\dots,-\frac{I}{2}\Big)\\
&+\cdots+P_n\Big(iP_1+iA_{12}+iA_{12}^*, iP_2+iB_{12}+iB_{12}^*, -\frac{I}{2},\dots,\vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&\vartheta\Big(P_n\Big(iP_1, iP_2, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)+\vartheta\Big(P_n(iP_1, iB_{12}+iB_{12}^*, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
&+\vartheta\Big(P_n\Big(iA_{12}+iA_{12}^*, iP_2, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
&+\vartheta\Big(P_n\Big(iA_{12}+iA_{12}^*, iB_{12}+iB_{12}^*, -\frac{I}{2},\dots,-\frac{I}{2}\Big)\Big)\\
=&\vartheta(H_{12})+\vartheta(H_{12}^{'})+\vartheta(H_{12}(H_{12}^{'})^*-H_{12}^{'}H_{12}^*).
\end{aligned}
$$

Hence, $\vartheta(H_{12}+H_{12}^{'})=\vartheta(H_{12})+\vartheta(H_{12}^{'})$ for any $H_{12}, H_{12}^{'}\in\mathcal{A}_{12}^{-}$. □

Lemma 2.6. *For any $H_{ii}, H_{ii}^{'}\in\mathcal{A}_{ii}$ for $i=1,2$, we have*

$$\vartheta(H_{11}+H_{11}^{'})=\vartheta(H_{11})+\vartheta(H_{11}^{'}) \quad \textit{and} \quad \vartheta(H_{22}+H_{22}^{'})=\vartheta(H_{22})+\vartheta(H_{22}^{'}).$$

Proof. Let $T=\vartheta(H_{11}+H_{11}^{'})-\vartheta(H_{11})-\vartheta(H_{11}^{'})$, we show that $T=0$. Using the fact that $P_n(P_2, H_{11}, -I\dots,-I)=P_n(P_2, H_{11}^{'}, -I,\dots,-I)=0$ and making use of Lemma 2.1, we obtain

$$
\begin{aligned}
&\vartheta(P_n(P_2, H_{11}+H_{11}^{'}, -I\dots,-I))\\
=&\vartheta(P_n(P_2, H_{11}, -I\dots,-I))+\vartheta(P_n(P_2, H_{11}^{'}, -I\dots,-I))\\
=&P_n(\vartheta(P_2), H_{11}, -I,\dots,-I)+P_n(P_2, \vartheta(H_{11}), -I,\dots,-I)\\
&+P_n(P_2, H_{11}, \vartheta(-I),\dots,-I)+\cdots+P_n(P_2, H_{11}, -I,\dots,\vartheta(-I))\\
&+P_n(\vartheta(P_2), H_{11}^{'}, -I,\dots,-I)+P_n(P_2, \vartheta(H_{11}^{'}), -I,\dots,-I)\\
&+P_n(P_2, H_{11}^{'}, \vartheta(-I),\dots,-I)+\cdots+P_n(P_2, H_{11}^{'}, -I,\dots,\vartheta(-I))\\
=&P_n(\vartheta(P_2), H_{11}+H_{11}^{'}, -I,\dots,I)+P_n(P_2, \vartheta(H_{11})+\vartheta(H_{11}^{'}), -I,\dots,-I)\\
&+P_n(P_2, H_{11}+H_{11}^{'}, \vartheta(-I),\dots,-I)+\cdots+P_n(P_2, H_{11}+H_{11}^{'}, -I,\dots,\vartheta(-I)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\vartheta(P_n(P_2, H_{11}+H_{11}^{'}, -I\dots,-I))\\
=&P_n(\vartheta(P_2), H_{11}+H_{11}^{'}, -I,\dots,I)+P_n(P_2, \vartheta(H_{11}+H_{11}^{'}), -I,\dots,-I)\\
&+P_n(P_2, H_{11}+H_{11}^{'}, \vartheta(-I),\dots,-I)+\cdots+P_n(P_2, H_{11}+H_{11}^{'}, -I,\dots,\vartheta(-I)).
\end{aligned}
$$

Comparing the above two expressions, we find that $P_n(P_2, T, -I, \ldots, -I) = 0$, which gives us $T_{12} = T_{21} = T_{22} = 0$.

Next, we show that $T_{11} = 0$. Let $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$ and it is easy to observe that $P_n(X, H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}), P_n(X, H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}) \in \mathcal{A}_{12}$. Thus, using Lemma 2.5, we find that

$$\begin{aligned}
&\vartheta\Big(P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&\vartheta\Big(P_n\Big(X, H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big) + \vartheta\Big(P_n\Big(X, H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(X, H_{11}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
&+ P_n\Big(\vartheta(X), H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(X, H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11}) + \vartheta(H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11} + H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big)\\
&+ \cdots + P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\vartheta\Big(P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)\\
=&P_n\Big(\vartheta(X), H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(X, \vartheta(H_{11} + H_{11}^{'}), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(X, H_{11} + H_{11}^{'}, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big)\\
&+ \cdots + P_n\Big(X, H_{11} + H_{11}^{'}, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big).
\end{aligned}$$

From the last two expressions, we get $P_n(X, T, -\frac{I}{2}, \ldots, -\frac{I}{2}) = 0$, on solving further, we obtain $-XT + TX = 0$. Multiplying by P_1 and P_2 from left and right, we obtain $P_1TP_1AP_2 = 0$ for all $A \in \mathcal{A}$. Application of the condition (♠) yields $T_{11} = 0$. Hence, $T = 0$, that is, $\vartheta(H_{11} + H_{11}^{'}) = \vartheta(H_{11}) + \vartheta(H_{11}^{'})$. Symmetrically, we can prove that $\vartheta(H_{22} + H_{22}^{'}) = \vartheta(H_{22}) + \vartheta(H_{22}^{'})$. □

Lemma 2.7. *ϑ is additive on $\mathcal{A}^-$.*

Proof. For any $H, K \in \mathcal{A}^-$, we have $H = H_{11} + H_{12} + H_{22}$ and $K = K_{11} + K_{12} + K_{22}$. With the help of Lemmas 2.4, 2.5 and 2.6, we obtain

$$\begin{aligned}\vartheta(H+K) =&\vartheta(H_{11}+H_{12}+H_{22}+K_{11}+K_{12}+K_{22})\\ =&\vartheta(H_{11}+K_{11})+\vartheta(H_{12}+K_{12})+\vartheta(H_{22}+K_{22})\\ =&\vartheta(H_{11})+\vartheta(K_{11})+\vartheta(H_{12})+\vartheta(K_{12})+\vartheta(H_{22})+\vartheta(K_{22})\\ =&\vartheta(H_{11}+H_{12}+H_{22})+\vartheta(K_{11}+K_{12}+K_{22})\\ =&\vartheta(H)+\vartheta(K).\end{aligned}$$

□

Lemma 2.8. $\vartheta(-I) = 0$.

Proof. Using Lemmas 2.2 and 2.7, it follows that $2^{n-1}V = P_n(V, -I, -I, \ldots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned}2^{n-1}\vartheta(V) =&\vartheta(P_n(V,-I,-I,\ldots,-I)).\\ =&P_n(\vartheta(V),-I,-I,\ldots,-I)+P_n(V,\vartheta(-I),-I,\ldots,I)\\ &+P_n(V,-I,\vartheta(-I),\ldots,-I)+\cdots+P_n(V,-I,-I,\ldots,\vartheta(-I))\\ =&2^{n-1}\vartheta(V)+2^{n-2}(n-1)\{-A\vartheta(-I)-\vartheta(-I)^*V\}.\end{aligned}$$

On simplifying further, we obtain

$$0 = V\vartheta(-I) + \vartheta(-I)^*V.$$

Taking $V = iI$ in the above expression, we obtain $\vartheta(-I) = -\vartheta(-I)^*$, which implies $\vartheta(-I) \in \mathcal{A}^-$. We have now $V\vartheta(-I) = \vartheta(-I)V$ for all $V \in \mathcal{A}^-$. For every $F \in \mathcal{A}$, we have $F = F_1 + iF_2$, where $F_1 = \frac{F-F^*}{2} \in \mathcal{A}^-$ and $F_2 = \frac{F+F^*}{2i} \in \mathcal{A}^-$. Thus, we get $F\vartheta(-I) = \vartheta(-I)F$ for all $F \in \mathcal{A}$, this further implies that

$$\vartheta(-I) \in Z(\mathcal{A}) \cap \mathcal{A}^-. \tag{2.1}$$

Using Lemmas 2.1 and 2.2, it follows that $0 = P_n(V, -iI, -I, \ldots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned}0 =&\vartheta(P_n(V,-iI,-I,\ldots,-I))\\ =&P_n(\vartheta(V),-iI,-I,\ldots,-I)+P_n(V,\vartheta(-I),-I,\ldots,I)\\ &+P_n(V,-I,\vartheta(-iI),\ldots,-I)+\cdots+P_n(V,-iI,-I,\ldots,\vartheta(-I))\\ =&2^{n-2}(-V\vartheta(-iI)-\vartheta(-iI)^*V).\end{aligned}$$

By following similar steps as done earlier, we get

$$\vartheta(-iI) \in Z(\mathcal{A}) \cap \mathcal{A}^-. \tag{2.2}$$

Assume $\vartheta(P_1) = L_1 + iL_2$, where $L_1, L_2 \in \mathcal{A}^+$, with the help of equation (2.1) and fact $P_n(-I, P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}) = 0$, we get

$$0 = \vartheta\Big(P_n\Big(-I, P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\Big)$$

$$\begin{aligned}
=&P_n\Big(\vartheta(-I), P_1, -\frac{I}{2}, \ldots, -\frac{I}{2}\Big) + P_n\Big(-I, \vartheta(P_1), -\frac{I}{2}, \ldots, -\frac{I}{2}\Big)\\
&+ P_n\Big(-I, P_1, \vartheta\Big(-\frac{I}{2}\Big), \ldots, -\frac{I}{2}\Big) + \cdots + P_n\Big(-I, P_1, -\frac{I}{2}, \ldots, \vartheta\Big(-\frac{I}{2}\Big)\Big)\\
=&-2\vartheta(-I)P_1 - 2iL_2.
\end{aligned}$$

Thus, we obtain

$$\vartheta(P_1) = L_1 - \vartheta(-I)P_1. \tag{2.3}$$

With the help of equations (2.2) and (2.3) and fact $P_n(-iI, P_1, -I, \ldots, -I) = 2^{n-1}iP_1$, we obtain

$$\begin{aligned}
2^{n-1}\vartheta(iP_1) =&\vartheta(P_n(-iI, P_1, -I, \ldots, -I))\\
=&P_n(\vartheta(-iI), P_1, -I, \ldots, -I) + P_n(-iI, \vartheta(P_1), -I, \ldots, -I)\\
&+ P_n(-iI, P_1, \vartheta(-I), \ldots, -I) + \cdots + P_n(-iI, P_1, -I, \ldots, \vartheta(-I))\\
=&-2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}iL_1.
\end{aligned}$$

On solving further, we obtain

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + iL_1. \tag{2.4}$$

Applying (2.3) and (2.4), we get

$$\begin{aligned}
2^{n-1}\vartheta(iP_1) =&\vartheta(P_n(P_1, iP_1, -I, \ldots, -I))\\
=&P_n(\vartheta(P_1), iP_1, -I, \ldots, -I) + P_n(P_1, \vartheta(iP_1), -I, \ldots, -I)\\
&+ P_n(P_1, iP_1, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_1, iP_1, -I, \ldots, \vartheta(-I))\\
=&-2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}i(P_1L_1 + L_1P_1).
\end{aligned}$$

Therefore, we get

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1 + L_1P_1). \tag{2.5}$$

Comparing (2.4) and (2.5), we obtain $L_1 = P_1L_1 + L_1P_1$, implies $P_1L_1P_1 = P_2L_1P_2 = 0$. Invoking the last relation in (2.3), we obtain

$$\vartheta(P_1) = P_1L_1P_2 + P_2L_1P_1 - \vartheta(-I)P_1. \tag{2.6}$$

Taking $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$, then $X \in \mathcal{A}^-$. Invoking Lemmas 2.2 and 2.7, we possess

$$\begin{aligned}
2^{n-2}\vartheta(X) =&\vartheta(P_n(P_1, X, -I, \ldots, -I))\\
=&P_n(\vartheta(P_1), X, -I, \ldots, -I) + P_n(P_1, \vartheta(X), -I, \ldots, -I)\\
&+ P_n(P_1, X, \vartheta(-I), \ldots, -I) + \cdots + P_n(P_1, X, -I, \ldots, \vartheta(-I))\\
=&2^{n-2}(\vartheta(P_1)^*X + X\vartheta(P_1)) + 2^{n-2}(P_1\vartheta(X) + \vartheta(X)P_1).
\end{aligned}$$

Accordingly, we get

$$\vartheta(X) = \vartheta(P_1)^*X + X\vartheta(P_1) + P_1\vartheta(X) + \vartheta(X)P_1. \tag{2.7}$$

Multiplying (2.7) by P_1 and P_2 from left and right, respectively, and applying (2.6), we obtain $\vartheta(-I)A_{12}$. Implementation of the condition (♠) gives us $\vartheta(-I)P_1 = 0$, employing (2.1) and condition (♠) in the last relation, we get $\vartheta(-I)P_2 = 0$, finally we obtain $\vartheta(-I) = 0$. □

Lemma 2.9. *For any $S, S_1, S_2 \in \mathcal{A}^+$*
(1) $\vartheta(S)^* = \vartheta(S)$;
(2) $\vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S$;
(3) ϑ *is additive on* $\mathcal{A}^+$;
(4) $\vartheta(S_1 + iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$.

Proof. (1) Using the fact that $P_n(-I, S, -I, \ldots, -I) = 0$ and Lemma 2.8, where $S \in \mathcal{A}^+$, we obtain

$$0 = \vartheta(P_n(-I, S, -I, \ldots, -I)) = P_n(-I, \vartheta(S), -I, \ldots, -I) = 2^{n-2}(\vartheta(S)^* - \vartheta(S)),$$

which allows us $\vartheta(S)^* = \vartheta(S)$.

(2) Using the fact that $P_n(-iI, S, -I \ldots, -I) = 2^{n-1}iS$, together with Lemmas 2.7, 2.8 and equation (2.2), where $S \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}\vartheta(iS) =& \vartheta(P_n(-iI, S, -I \ldots, -I)) \\ =& P_n(\vartheta(-iI), S, -I, \ldots, -I) + P_n(-iI, \vartheta(S), -I, \ldots, -I) \\ =& 2^{n-1}i(\vartheta(S) - \vartheta(-iI)S), \end{aligned}$$

which gives us

$$\vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S. \tag{2.8}$$

(3) For any $S_1, S_2 \in \mathcal{A}^+$, invoking Lemma 2.7 and (2.8), we have

$$\vartheta(iS_1 + iS_2) = \vartheta(iS_1) + \vartheta(iS_2) = i\vartheta(S_1) - \vartheta(-iI)S_1 + i\vartheta(S_2) - \vartheta(-iI)S_2.$$

On the other hand, we have $\vartheta(i(S_1+S_2)) = i\vartheta(S_1+S_2) - \vartheta(-iI)(S_1+S_2)$. Comparing the above two expressions for $\vartheta(i(S_1 + S_2))$, we obtain $\vartheta(S_1 + S_2) = \vartheta(S_1) + \vartheta(S_2)$.

(4) Using the fact that $P_n(S_1 + iS_2, -I, -I, \ldots, -I) = 2^{n-1}iS_2$, together with Lemmas 2.7, 2.8 and (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) =& 2^{n-1}\vartheta(iS_2) = \vartheta(P_n(S_1 + iS_2, -I, -I, \ldots, -I)) \\ =& P_n(\vartheta(S_1 + iS_2), -I, -I, \ldots, -I) \\ =& 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*). \end{aligned}$$

Thus, we have

$$2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) = 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*). \tag{2.9}$$

Invoking the fact that $P_n(-iI, S_1 + iS_2, -I, \ldots, -I) = 2^{n-1}iS_1$ and using Lemmas 2.7, 2.8 and (2.2), (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_1) - \vartheta(-iI)S_1) =& 2^{n-1}\vartheta(iS_1) \\ =& \vartheta(P_n(-iI, S_1 + iS_2, -I, \ldots, -I)) \end{aligned}$$

$$\begin{aligned}&=P_n(\vartheta(-iI), S_1+iS_2, -I, \dots, -I)\\&\quad + P_n(-iI, \vartheta(S_1+iS_2), -I, \dots, -I)\\&= -2^{n-1}\vartheta(-iI)S_1 + 2^{n-2}i(\vartheta(S_1+iS_2)+\vartheta(S_1+iS_2)^*).\end{aligned}$$

On solving further, we obtain

$$2^{n-1}\vartheta(S_1) = 2^{n-2}(\vartheta(S_1+iS_2)+\vartheta(S_1+iS_2)^*). \tag{2.10}$$

By adding (2.9) and (2.10), we get $\vartheta(S_1+iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$. □

Lemma 2.10. *For all $A \in \mathcal{A}$*
(1) $\vartheta(A^*) = \vartheta(A)^*$;
(2) ϑ *is additive on* $\mathcal{A}$.

Proof. (1) For $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$ and using Lemma 2.9 and (2.2), we obtain

$$\begin{aligned}\vartheta(A)^* &= \vartheta(A_1+iA_2)^* = (\vartheta(A_1)+i\vartheta(A_2)-\vartheta(-iI)A_2)^*\\&= \vartheta(A_1) - i\vartheta(A_2) + \vartheta(-iI)A_2 = \vartheta(A_1+i(-A_2)) = \vartheta(A_1-iA_2) = \vartheta(A^*).\end{aligned}$$

(2) For $A, B \in \mathcal{A}$, then $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1, A_2, B_1, B_2 \in \mathcal{A}^+$. Using Lemma 2.9, we obtain

$$\begin{aligned}\vartheta(A+B) &= \vartheta(A_1+iA_2+B_1+iB_2)\\&= \vartheta((A_1+B_1)+i(A_2+B_2))\\&= \vartheta(A_1+B_1)+i\vartheta(A_2+B_2)-\vartheta(-iI)(A_2+B_2)\\&= \vartheta(A_1)+\vartheta(B_1)+i\vartheta(A_2)+i\vartheta(B_2)-\vartheta(-iI)A_2-\vartheta(-iI)(B_2)\\&= (\vartheta(A_1)+i\vartheta(A_2)-\vartheta(-iI)A_2)+(\vartheta(B_1)+i\vartheta(B_2)-\vartheta(-iI)B_2)\\&= \vartheta(A)+\vartheta(B).\end{aligned}$$ □

Lemma 2.11. *For all $A \in \mathcal{A}$*
(1) $\vartheta(-iI) = 0$;
(2) $\vartheta(iA) = i\vartheta(A)$.

Proof. (1) Equation (2.4) implies

$$\vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1P_2 + P_2L_1P_1). \tag{2.11}$$

Observe that $P_n(iP_1, (A_{12}-A_{12}^*), -I, \dots, -I) = -2^{n-2}i(A_{12}+A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned}-2^{n-2}\vartheta(i(A_{12}+A_{12}^*)) &= \vartheta(P_n(iP_1, (A_{12}-A_{12}^*), -I, \dots, -I))\\&= P_n(\vartheta(iP_1), (A_{12}-A_{12}^*), -I, \dots, -I)\\&\quad + P_n(iP_1, \vartheta(A_{12}-A_{12}^*), -I, \dots, -I).\end{aligned}$$

Using (2.11), in the above expression, we obtain

$$-2^{n-2}\vartheta(i(A_{12}+A_{12}^*)) = 2^{n-2}\Big(\vartheta(-iI)A_{12} - iP_2L_1A_{12} + iP_1L_1A_{12}^* + iA_{12}L_1P_1$$

$$+ A_{12}^*\vartheta(-iI) - iA_{12}^*L_1P_2 - iP_1\vartheta(A_{12}) + iP_1\vartheta(A_{12})^* - i\vartheta(A_{12})^*P_1 + i\vartheta(A_{12})P_1\Big).$$

On the other hand, we have

$$-2^{n-2}\vartheta(i(A_{12}+A_{12}^*)) = -2^{n-2}i\vartheta(A_{12}+A_{12}^*) + \vartheta(-iI)(A_{12}+A_{12}^*).$$

Comparing the above two expressions, we obtain

$$\begin{aligned}&-i\vartheta(A_{12}) - i\vartheta(A_{12})^* + \vartheta(-iI)A_{12} + \vartheta(-iI)A_{12}^*\\ =&\vartheta(-iI)A_{12} - iP_2L_1A_{12} + iP_1L_1A_{12}^* + iA_{12}L_1P_1 + A_{12}^*\vartheta(-iI)\\ &- iA_{12}^*L_1P_2 - iP_1\vartheta(A_{12}) + iP_1\vartheta(A_{12})^* - i\vartheta(A_{12})^*P_1 + i\vartheta(A_{12})P_1.\end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, we obtain

$$P_1\vartheta(A_{12})^*P_2 = 0. \tag{2.12}$$

Alternatively observed that $P_n(iP_1, i(A_{12}+A_{12}^*), -I, \ldots, -I) = 2^{n-2}(A_{12}-A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned}2^{n-2}\vartheta(A_{12}-A_{12}^*) =&\vartheta(P_n(iP_1, i(A_{12}+A_{12}^*), -I, \ldots, -I))\\ =&P_n(\vartheta(iP_1), i(A_{12}+A_{12}^*), -I, \ldots, -I)\\ &+ P_n(iP_1, \vartheta(i(A_{12}+A_{12}^*)), -I, \ldots, -I).\end{aligned}$$

Using (2.11) in the above expression, we obtain

$$\begin{aligned}\vartheta(A_{12}-A_{12}^*) =&i\vartheta(-iI)A_{12} + P_2L_1A_{12} + P_1L_1A_{12}^* - iA_{12}L_1P_1\\ &- iA_{12}^*\vartheta(-iI) - iA_{12}^*L_1P_2 + P_1\vartheta(A_{12}) + P_1\vartheta(A_{12})^*\\ &+ iP_1\vartheta(-iI)A_{12} - \vartheta(A_{12})P_1 - \vartheta(A_{12})^*P_1 - i\vartheta(-iI)A_{12}^*.\end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, and applying (2.12), we obtain $\vartheta(-iI)A_{12} = 0$. Implementation of the condition (♠) gives us $\vartheta(-iI)P_1 = 0$, employing (2.2) and condition (♠) in $\vartheta(-iI)A_{12} = 0$, we get $\vartheta(-I)P_2 = 0$, hence we obtain $\vartheta(-iI) = 0$.

(2) Applying the fact $\vartheta(-iI) = 0$ in the Lemma 2.9 (2), we obtain $\vartheta(iS) = i\vartheta(S)$ for all $S \in \mathcal{A}^+$. Let any $A \in \mathcal{A}$ then $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$

$$\begin{aligned}\vartheta(iA) =&\vartheta(iA_1 - A_2) = \vartheta(iA_1) + \vartheta(-A_2) = i\vartheta(A_1) - \vartheta(A_2)\\ =&i(\vartheta(A_1) + i\vartheta(A_2)) = i\vartheta(A_1 + iA_2) = i\vartheta(A).\end{aligned}$$

□

Lemma 2.12. *$\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$, for all $A, B \in \mathcal{A}$.*

Proof. For any $A, B \in \mathcal{A}$, applying the fact

$$P_n(A, iB, -I, \ldots, -I) = 2^{n-2}i(A^*B + B^*A),$$

and using Lemmas 2.8, 2.10 (2) and 2.11 (2), we obtain

$$2^{n-2}i\vartheta(A^*B + B^*A) = \vartheta(P_n(A, iB, -I, \ldots, -I))$$

$$\begin{aligned}&=P_n(\vartheta(A), iB, -I, \ldots, -I) + P_n(A, \vartheta(iB), -I, \ldots, -I)\\&=2^{n-2}i(\vartheta(A)^*B + B^*\vartheta(A)) + 2^{n-2}i(A^*\vartheta(B) + \vartheta(B)^*A).\end{aligned}$$

Hence, we obtain

$$\vartheta(A^*B + B^*A) = \vartheta(A)^*B + B^*\vartheta(A) + A^*\vartheta(B) + \vartheta(B)^*A. \tag{2.13}$$

Implementing the fact $P_n(A, B, -I, \ldots, -I) = 2^{n-2}(A^*B - B^*A)$ and using Lemmas 2.8 and 2.10 (2), we obtain

$$\begin{aligned}2^{n-2}\vartheta(A^*B - B^*A) &= \vartheta(P_n(A, B, -I, \ldots, -I))\\&= P_n(\vartheta(A), B, -I, \ldots, -I) + P_n(A, \vartheta(B), -I, \ldots, -I)\\&= 2^{n-2}(\vartheta(A)^*B - B^*\vartheta(A)) + 2^{n-2}(A^*\vartheta(B) - \vartheta(B)^*A).\end{aligned}$$

Therefore, we obtain

$$\vartheta(A^*B - B^*A) = \vartheta(A)^*B - B^*\vartheta(A) + A^*\vartheta(B) - \vartheta(B)^*A. \tag{2.14}$$

Adding equations (2.13) and (2.14) and adopting Lemma 2.10 (1), we obtain

$$\vartheta(A^*B) = \vartheta(A^*)B + A^*\vartheta(B). \qquad \square$$

Accordingly, we obtain $\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$.

Invoking Lemmas 2.10 and 2.12, we obtain that ϑ is an additive $*$-derivation on $\mathcal{A}$. Hence, the proof of Theorem 2.1, is completed.

3. Corollaries

Recall that an algebra $\mathcal{A}$ is prime if for any $\alpha, \beta \in \mathcal{A}$, $\alpha\mathcal{A}\beta = \{0\}$ implies that either $\alpha = 0$ or $\beta = 0$. It is simple to check that every prime $*$-algebra satisfies (♠). Hence, as a direct significance of Theorem 2.1, we have the following result.

Corollary 3.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection. Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ (need not be linear) satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

It is commonly known that a factor von Neumann algebra is prime, thus, it always satisfies (♠). Thus, due to the quick aftereffects of Corollary 3.1, we obtain the subsequent result.

Corollary 3.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with* $\dim(\mathcal{A}) \geq 2$. *Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (♠) (see [3, 8] for details). Therefore, applying Theorem 2.1, we have the following result.

Corollary 3.3. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. Then, a map $\vartheta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \ldots, \lambda_n)) = \sum_{i=1}^{n} P_n(\lambda_1, \ldots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \ldots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$-derivation.

4. Conclusion

In this paper, we have studied nonlinear left bi-skew Lie n-derivations on arbitrary $*$-algebras. In fact, it is shown that, under certain assumptions, every nonlinear left bi-skew Lie n-derivation on a unital $*$-algebra is an additive $*$-derivation.

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SELECTION PRINCIPLES IN TOPOLOGICAL VECTOR SPACES

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Abstract. This paper investigates bounded, M-bounded, H-bounded, R-bounded topological vector spaces. The most important properties and characteristics of these classes of topological vector spaces are established.

1. Introduction

Recently the theory of selection principles has been intensively developing in topological spaces, in topological groups, in uniform spaces etc.

Significant contributions to the theory of selection principles of choice were made in [1, 3, 6–13] and others.

A good overview of the very extensive literature on selection principles is contained in [3, 10].

Lj.D.R. Kočinac [3] found uniform analogues of the most important properties of the selection principles: uniformly Menger spaces, uniformly Hurewicz spaces, uniformly Rothberger space etc. These properties are considered as types of totally bounded uniform spaces, for example, the uniformly Menger space occupies an intermediate place between totally bounded and ω-bounded spaces. To each selection property of a uniform structure defined above can correspond game.

In this paper we study important properties of the bounded, M-bounded, H-bounded and R-bounded topological vector spaces.

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2. Preliminaries and Denotations

For covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \emptyset\}$, $H \subset X$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha* \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e., for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$.

Recall that a uniform space (X, U) is called:

(a) *totally bounded*, if each $\alpha \in U$ has a finite set $H \subset X$ such that $\alpha(H) = X$ [2];
(b) *precompact*, if the uniformity U has a base consisting of finite covers [2];
(c) *ω-bounded*, if the uniformity U has a base consisting of countable cover [2, 3];
(d) *uniformly locally compact*, if the uniformity of U contains a uniform cover consisting is compact sets [2];
(e) has the *uniform Menger property*, if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, β_n is a finite subset of α_n and $\bigcup_{n\in\mathbb{N}} \beta_n$ is a cover of X [3];
(f) has the *uniform Hurewicz property* if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(\beta_n \mid n \in \mathbb{N})$ such that each β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \bigcup \beta_n$ for all but finitely many n [3];
(g) has the *uniform Rothberger property* if for each sequence $(\alpha_n \mid n \in \mathbb{N}) \subset U$ there is a sequence $(A_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $A_n \in \alpha_n$ and $\bigcup_{n\in\mathbb{N}} A_n = X$ [3].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). The mapping f is called *precompact*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [2].

For the uniformity U by τ_U we denote the topology generated by the uniformity.

A topological vector space L is said to be

(a) *M-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there exists a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of L such that $L = \bigcup_{n\in\mathbb{N}} A_n + U_n$;
(b) *H-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there exists a sequence $(A_n \mid n \in \mathbb{N})$ of finite subsets of L such that each $x \in L$ belongs to all but finitely many $A_n + U_n$;
(c) *R-bounded* if for each sequence $(U_n \mid n \in \mathbb{N})$ of 0-neighborhoods in L there is a sequence $(A_n \mid n \in \mathbb{N})$ of elements of L such that $L = \bigcup_{n\in\mathbb{N}}(x_n + U_n)$.

In other words topological vector space L is said to be M-bounded (H-bounded, R-bounded), if (L, U) a uniformly Menger (uniformly Hurewicz, uniformly Rothberger) space with respect to the uniformity U.

3. Results

Let L be a topological vector space (TVS).

Proposition 3.1. *Any totally bounded TVS L is M-bounded.*

Proof. Let L be a TVS. We will prove that it is M-bounded. Let $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniformly cover, where W_n is 0-neighborhood in L. Then, for any 0-neighborhood W_n there exist a 0-neighborhood W'_n, such that $W'_n + W'_n \subset W_n$. Clearly, $\beta_{W'_n} * \succ \alpha_{W_n}$. Therefore, for each $n \in \mathbb{N}$ there exists a finite set $M_n \subset L$, such that $\beta_{W'_n}(M_n) = L$. Put $\beta_{W'_n}(M_n) = \bigcup_{i=1}^{k} \{\beta_{W'_n}(x_i) : x_i \in M_n\}$. For each $\beta_{W'_i}(x)$ choose $W_n + x_i \in \alpha_{W_n}$, such that $\beta_{W'_n}(x_i) \subset W_n + x_i$, $i = 1, 2, \ldots, k$. Let $\alpha^0_{W_n} = (W_n + x_i \mid i = 1, 2, \ldots, k)$. Since for each $n \in \mathbb{N}$ the finite family $\alpha^0_{W_n}$ is a cover, then $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = L$. Therefore, TVS L is M-bounded. □

Corollary 3.1. *Any compact TVS L is M-bounded.*

The space of real numbers $\mathbb{R}$ is M-bounded TVS, but it is not totally bounded, i.e., not compact.

Proposition 3.2. *Any M-bounded TVS L is ω-bounded.*

Proof. Let L be a M-bounded TVS. We will prove that it is ω-bounded. It suffices to show that the uniform space (L, U) is a ω-bounded. Let $\alpha_W \in U$ be an arbitrary uniform cover, where W_n is a 0-neighborhood in L. For each $n \in \mathbb{N}$ put $\alpha_{W_n} = \alpha_W$. Then, $(\alpha_{W_n} \mid n \in \mathbb{N})$ is a sequence of uniform covers. Therefore, there exists a sequence of finite subfamilies $(\alpha^0_{W_n} \mid n \in \mathbb{N})$, such that for each $n \in \mathbb{N}$ we have $\alpha^0_{W_n} \subset \alpha_{W_n}$ and $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = X$. Since for each $n \in \mathbb{N}$ the family $\alpha^0_{W_n}$ is finite, then the family $\alpha^0_W = (\alpha_{W_1}, \alpha_{W_2}, \ldots, \alpha_{W_n}, \ldots)$ is subcover of α_W. Therefore, TVS L is ω-bounded. □

Theorem 3.1. *If a TVS L' is a linearly continuous image of a M-bounded (H-bounded, R-bounded) TVS L, then L' is also M-bounded (H-bounded, R-bounded) TVS.*

Proof. Let us consider only the M-bounded case and the remaining cases proceed similarly. Let $f : L \to L'$ be an linearly continuous mapping of the M-bounded TVS L to the TVS L'. Since every linearly continuous mapping is uniformly continuous, the mapping $f : (L, U) \to (L', U')$ is uniformly continuous. Let $(\beta_{V_n} \mid n \in \mathbb{N})$ be an arbitrary sequence of elements of U', where V_n is 0-neighborhoods in L' and put $f^{-1}\beta_{V_n} = \alpha_{f^{-1}V_n}$, $n \in \mathbb{N}$. It is easily seen that we have: if V_n is 0-neighborhoods in L then $f^{-1}V_n$ is a 0-neighborhoods in L. Since (L, U) is uniformly Menger, then there exists a sequence of finite subfamilies $(\alpha^0_{f^{-1}V_n} \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $\alpha^0_{f^{-1}V_n} \subset \alpha_{f^{-1}V_n}$ and $\bigcup_{n\in\mathbb{N}} \alpha^0_{f^{-1}V_n} = L$. For each $n \in \mathbb{N}$ put $f\alpha^0_{f^{-1}V_n} = \beta^0_{V_n}$. Then, $f(\bigcup_{n\in\mathbb{N}} \alpha^0_{f^{-1}V_n}) = \bigcup_{n\in\mathbb{N}} f\alpha^0_{f^{-1}V_n} = \bigcup_{n\in\mathbb{N}} \beta^0_{V_n} = L'$, $n \in \mathbb{N}$. This means that (L', U') has indeed the uniform Menger property. Therefore, L' is M-bounded. □

A TVS L is called σ-totally bounded, if it can be represented as the union of countably many totally bounded subspaces.

Proposition 3.3. *Any σ-totally bounded TVS is H-bounded.*

Proof. Let L be a σ-totally bounded TVS. We proof that L is H-bounded. Let $(\alpha_{V_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers, where V_n is 0-neighborhoods in L and $L = \bigcup L_{n n\in\mathbb{N}}$, where L_n is totally bounded. For each $n \in \mathbb{N}$ put $\alpha_{V_n^{L_n}} = \alpha_{V_n} \wedge \{L_n\}$, $V_n^{L_n} = V_n \bigcap L_n$. Then, $(\alpha_{V_n^{L_n}} \mid n \in \mathbb{N}) \subset U_{L_n}$ is sequence uniform covers of the space (L_n, U_{L_n}) and $V_n^{L_n}$ is 0-neighborhood in L_n. Let $W_n^{L_n}$ be 0-neighborhood such that $W_n^{L_n} + W_n^{L_n} \subset V_n^{L_n}$. Then, $\beta_{W_n^{L_n}} * \succ \alpha_{V_n^{L_n}}$. For each $n \in \mathbb{N}$ choose a finite subset $M_{L_n} \subset L_n$ such that $\beta_{W_n^{L_n}}(M_{L_n}) = \bigcup\{\beta_{W_n^{L_n}}(x_1), \beta_{W_n^{L_n}}(x_2), \ldots, \beta_{W_n^{L_n}}(x_m)\} = L_n$, where $x_i \in M_{L_n}$, $i = 1, 2, \ldots, m$. Therefore, there is $V_n^{L_n} + x_i$ from $\alpha_{V_n^{L_n}}$ such that $\beta_{W_n^{L_n}}(x_i) \subset V_n^{L_n} + x_i$, $i = 1, 2, \ldots, m$. For each $n \in \mathbb{N}$ put $\alpha^0_{V_n^{L_n}} = (V_n^{L_n} + x_i \mid i = 1, 2, \ldots, m)$. It is easy to see that each $\alpha^0_{V_n^{L_n}}$ is finite subfamily of the cover $\alpha_{V_n^{L_n}}$ and each $x \in L$ is an element of $\bigcup \alpha^0_{V_n^{L_n}}$ for all but finitely many n. Thus, L is H-bounded. □

Corollary 3.2. *Any σ-totally bounded TVS is M-bounded.*

Theorem 3.2. *Any countable discrete TVS is a R-bounded.*

Proof. Let L be a countable discrete TVS and $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be a sequence of uniform covers. Since the TVS L is a countable discrete space, there exists a base $B = (\alpha_W)$ consisting of a countable cover $\alpha_W = (\{0\} + x_n \mid x_n \in L)$ and for each $n \in N$, $\alpha_W \succ \alpha_{W_n}$. Then, we choose one $\{0\} + x_n \in \alpha_W$ such that $\{0\} + x_n \subset W_n + x_n$ and $\bigcup_{n\in\mathbb{N}}(\{0\} + x_n) = L$. Hence, TVS L is a R-bounded. □

Theorem 3.3. *For a locally compact TVS L the following are equivalent:*
(1) *L is M-bounded;*
(2) *L is ω-bounded.*

Proof. (1) $\Rightarrow$ (2) It is evident. So, we have prove only (2) $\Rightarrow$ (1). Let $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be a sequence uniform covers in U. We apply (2) to the countable uniform cover $\beta_W = (W + x_n \mid n \in \mathbb{N})$ consisting of compact sets. For each $n \in \mathbb{N}$ there exists a finite subfamily $\alpha^0_{W_n} \subset \alpha_{W_n}$ such that $W + x_n \subset \bigcup \alpha^0_{W_n}$. It is easy to see that $\bigcup_{n\in\mathbb{N}} \alpha^0_{W_n} = L$. Consequently, L is a M-bounded. □

Proposition 3.4. *Any locally compact TVS L is strongly paracompact.*

Proof. Let L be a locally compact TVS and W be a compact 0-neighborhood. Then, $\alpha_W = \{W + x : x \in L\}$ is a uniform cover consisting of compact subsets, i.e., a uniform space (L, U) is uniformly locally compact. Since (L, U) is strongly uniformly paracompact [16]. Then, TVS L is locally compact and paracompact, i.e., strongly paracompact. □

Corollary 3.3. *Any locally compact TVS L is complete.*

Proposition 3.5. *The completion of M-bounded (H-bounded, R-bounded) TVS is a M-bounded (H-bounded, R-bounded).*

Proof. Let's consider only the R-bounded case and the remaining cases proceed similarly. Let $\tilde{L}$ be a completion of the R-bounded space L and $(\tilde{\alpha}_{W_n} \mid n \in \mathbb{N}) \subset \tilde{U}$ be an arbitrary sequence. For each $n \in \mathbb{N}$, $\alpha_{W_n} = \tilde{\alpha}_{W_n} \wedge \{L\}$. Then, $(\alpha_{W_n} \mid n \in \mathbb{N})$ is a sequence uniform covers in U. Since L is a R-bounded space, there exists a sequence $(W_n + x \mid n \in \mathbb{N})$ of such that for any $n \in \mathbb{N}$, $W_n + x \in \alpha_{W_n}$ and $\bigcup_{n\in\mathbb{N}}(W_n + x)$ is a cover of the space L. It is easy to see that $\bigcup_{n\in\mathbb{N}}(\tilde{W}_n + \tilde{x}) = \tilde{L}$. Thus, the completion $\tilde{L}$ is R-bounded. □

Let $f : L \to L'$ be a linear continuous mapping of a TVS L onto a TVS L'. The mapping f is called precompact, if the mapping $f : (L, U) \to (L', U')$ of a uniform space (L, U) to uniform space (L', U') is precompact.

Theorem 3.4. *Let $f : L \to L'$ be a precompact mapping. If TVS L' is H-bounded, then a TVS L is also H-bounded.*

Proof. Let $f : L \to L'$ be a precompact mapping of a TVS L onto a TVS L' and $(\alpha_{W_n} \mid n \in \mathbb{N}) \subset U$ be an arbitrary sequence of uniform covers. Then, for any $n \in \mathbb{N}$ there exists a finite cover $\gamma_{W'_n} \in U$ and $\beta_{V_n} \in U'$, such that $f^{-1}\beta_{V_n} \wedge \gamma_{W'_n} \succ \alpha_{W_n}$. Apply to the sequence $(\beta_{V_n} \mid n \in \mathbb{N}) \subset U'$ the fact that the space (L', U') has a uniformly Hurewicz property we find a sequence $(\beta^0_{V_n} \mid n \in \mathbb{N})$ of finite subfamilies such that for each $y \in L'$ we have $y \in \bigcup \beta^0_{V_n}$ for all but finitely many n. Note, that for any $n \in \mathbb{N}$ the family $f^{-1}\beta^0_{V_n} \wedge \gamma_{W'_n}$ is finite and $\bigcup\{f^{-1}\beta^0_{V_n} \wedge \gamma_{W'_n}\} = \bigcup f^{-1}\beta^0_{V_n}$. Next, for any $f^{-1}(V_n + y_i) \bigcap W'_n + x_i \in f^{-1}\beta_0 \wedge \gamma_n$ choose $W_n + x_i \in \alpha_{W_n}$, such that $f^{-1}(V_n + y_i) \bigcap W'_n + x_i \subset W_n + x_i$. Put $\alpha^0_{W_n} = \{W_n + x_1, W_n + x_2, \ldots, W_n + x_i\}$, $f(x_i) = y_i$. It is easy to see that for each $x \in L$ we have $x \in \bigcup \alpha^0_{W_n}$, $f(x) = y$ for all but finitely many n. Therefore, L is H-bounded. □

As it is well known, to each selection principle for topological spaces it is naturally associated the corresponding game and often selection principles can be characterized game-theoretically. In TVS case to each selection principle one can assign also the corresponding game. The game MLG associated to the M-bounded property is defined in the following way. Two players, ONE and TWO, play a round for each positive integer. In the n-th round ONE chooses a 0-neighborhood U_n and TWO responds choosing a finite subset A_n. TWO wins a play $U_1, A_1; U_2, A_2, \ldots$, if $X = \bigcup_{n\in\mathbb{N}} A_n + U_n$, and otherwise ONE wins. The games HLG and RLG associated to the H-bounded and R-bounded properties are defined in the following way.

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LIMIT POINT ANALYSIS OF THE BROWDER SPECTRUM FOR OPERATOR MATRICES

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Abstract. In this paper, we investigate the limit point set of the Browder spectrum for upper triangular operator matrices $\mathcal{M}$ on Banach spaces. Utilizing the robust tools and comprehensive framework of local spectral theory, we offer a detailed analysis of this spectral feature. We establish that the relationship between the accumulation points of the Browder spectrum $\sigma_{Br}(\mathcal{M})$ of $\mathcal{M}$ and those of its diagonal entries is encapsulated by the equation:

$$\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \cup \mathcal{W}_{\operatorname{Acc}\sigma_{Br}} = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i),$$

where $\mathcal{W}_{\operatorname{Acc}\sigma_{Br}}$ denotes a specific union of "holes" in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$, comprising subsets within the intersection $\bigcap_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$. Furthermore, we delineate sufficient conditions under which the limit points of the Browder spectrum for a 3×3 upper triangular block operator matrix are precisely characterized as the union of the limit points of the spectra of its diagonal entries. Our findings significantly advance the understanding of the spectral properties of operator matrices, offering crucial insights into their structure within the context of local spectral theory. Moreover, this work extends and refines the results of A. Tajmouati et al., as presented in [17], contributing to the ongoing development and enhancement of operator matrix theory.

1. Introduction

Browder operators and their associated derivative sets serve as generalizations of Fredholm and semi-Fredholm operators, as well as their derivative sets, within

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the framework of linear operators. This area has been extensively explored in the literature, leading to numerous significant advancements. For a deeper understanding, readers are encouraged to consult the references [4, 9].

In recent years, operator matrix theory has gained substantial attention in both pure and applied mathematics. As a result, numerous researchers have focused on the spectral properties of 2×2 upper triangular block operator matrices, including works by [1–7, 10, 17, 18]. This area of study is closely tied to the exploration of Browder operators and the development of local spectral theory.

In the field of mathematical physics, many linear evolution equations can be reformulated as a Cauchy problem governed by 3×3 block operator matrices. For example, such reformulations are common in equations arising from linear thermo-viscoelasticity [14] and fluid dynamics [15]. Our motivation for this paper is to investigate why the results presented by A. Tajmouati et al. [17] remain valid for 3×3 upper triangular operator matrices derived from systems of linear evolution equations.

Accordingly, after considering on the direct sum of Banach spaces, the following bounded 3×3 upper triangular operator matrix

$$\mathcal{M} := \begin{pmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

This raises an important question.

Under what conditions on the entries of the operator matrices $\mathcal{M}$ can we obtain a relation between the limit point set for the Browder spectrum of $\mathcal{M}$ and the limit point set for the Browder spectrum of the operator entries A_i, $1 \leq i \leq 3$, in the form:

$$\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)?$$

To address this question, we propose leveraging the results of A. Tajmouati et al. in [17] after reducing the matrix form $\mathcal{M}$ into the the form of block 2×2 operator matrix

$$\mathcal{M} := \begin{pmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & A_3 \end{pmatrix},$$

where $\mathcal{X} := \begin{pmatrix} A_1 & B_1 \\ 0 & A_2 \end{pmatrix}$, $\mathcal{Y} := \begin{pmatrix} C_1 \\ B_2 \end{pmatrix}$ and $\mathcal{Z} := \begin{pmatrix} 0 & 0 \end{pmatrix}$.

While this approach might appear somewhat contrived and limited in scope, it is critical to refine our analysis to achieve more comprehensive and significant results. To achieve the best of this paper, our main objective in this work is to extend the results obtained by A. Tajmouati et al. in [17] to the case of the 3×3 block operator matrix. Specifically, we found some sufficient conditions on the entries of the operator matrices $\mathcal{M}$ to assure our interest and to prove an improvement in the theory of operators matrices and a continuation and an amelioration of the results of A. Tajmouati et al. in [17].

The paper's remaining sections are arranged as follows. In Section 2, we collect certain notes and required terminology throughout the paper and we remember some classic definitions introducing Browder's operator theory. The purpose of the work presented in Section 3 is dedicated to investigate the limit point set for Browder spectrum of a bounded 3×3 upper triangular operator matrix $\mathcal{M}$. After checking that the assertion $\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$ is always true, we investigate the local spectral theory to prove that

$$\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \cup \left[\left[\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)\right] \cap \mathrm{S}_{\mathrm{vep}}(A_3)\right] = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$$

for all bounded linear operators B_1, B_2 and C_1. In addition, we impose sufficient conditions on A_i, for $i = \{1, 2, 3\}$ under which the following equality $\operatorname{Acc}\sigma_{Br}(\mathcal{M}) = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$ is guaranteed. Furthermore, we prove that the passage from $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ to $\bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$ can be presented as well:

$$\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \cup \mathcal{W}_{\operatorname{Acc}\sigma_{Br}} = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i),$$

where $\mathcal{W}_{\operatorname{Acc}\sigma_{Br}}$ is the union of certain holes in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$, that occur to be subsets of $\bigcap_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$. Finally, we give sufficient conditions to reassure our desired equality.

2. Preliminaries

In this section, we revisit several well-known definitions and mathematical tools that are essential for the subsequent discussions. Throughout this paper, E denotes an infinite-dimensional complex Banach space, and $T \in \mathcal{L}(E)$, where $\mathcal{L}(E)$ represents the set of bounded linear operators on E. We will use the following notations and concepts:

- T^* : the adjoint of T;
- $\mathcal{N}(T)$: the null space of T;
- $\mathcal{R}(T)$: the range of T;
- $\alpha(T) = \dim \mathcal{N}(T)$: the dimension of the null space of T;
- $\beta(T = \operatorname{codim}\mathcal{R}(T))$: the codimension of the range of T;
- $\sigma(T)$: the spectrum of T.

Let us consider Υ a given subset of $\mathbb{C}$. The following notations will be clarified:

- $\operatorname{Acc}\Upsilon$: the set of all points of accumulation of Υ;
- $\operatorname{Iso}\Upsilon$: the set of all isolated points of Υ;
- Υ^c : the complement of Υ;
- $\partial\Upsilon$: the boundary of Υ;
- $\overline{\Upsilon}$: the closure of Υ;
- $\eta\Upsilon$: the polynomially convex hull of Υ.

Before passing to introduce the notion of Browder operators and their derivative sets, the local spectral theory of linear operators on a Banach space is discussed in the

following basic results, which will be utilized in the rest of this work. For additional details, we may refer to [12, 13].

Definition 2.1. Let $T \in \mathcal{L}(E)$.

T is said to have the single-valued extension property (SVEP for short) at $\tau \in \mathbb{C}$ if for every open neighborhood $\mathfrak{F}_\tau$ of τ, the constant function $g \equiv 0$ is the only analytic solution of the equation:

$$(\lambda I - T)g(\lambda) = 0, \quad \text{for all } \lambda \in \mathfrak{F}_\tau.$$

We denote by $\mathrm{S}_{\mathrm{vep}}(T)$, the open set of $\tau \in \mathbb{C}$ where T fails to have SVEP at τ.

Remark 2.1. We would like to note also that T has SVEP if $\mathrm{S}_{\mathrm{vep}}(T) = \emptyset$.

Now, we introduce an important class of operators which involves the concept of Fredholm theory.

Definition 2.2. Let E be a Banach space, we define the following.

(i) The set of upper semi-Fredholm operators on E as:

$$\Phi_+(E) = \{T \in \mathcal{L}(E) : \alpha(T) < +\infty \text{ and } \mathcal{R}(T) \text{ is closed in } E\}.$$

(ii) The set lower semi-Fredholm operators on E as:

$$\Phi_-(E) = \{T \in \mathcal{L}(E) : \beta(T) < +\infty \text{ and } \mathcal{R}(T) \text{ is closed in } E\}.$$

Accordingly to Definition 2.2, the set of Fredholm operators on E is defined by:

$$\begin{aligned}\Phi(E) :=& \Phi_+(E) \cap \Phi_-(E)\\ :=& \{T \in \mathcal{L}(E) : \beta(T) < +\infty,\ \mathcal{R}(T) \text{ is closed in } E \text{ and } \beta(T) < +\infty\}.\end{aligned}$$

Before moving to introduce some sets of Browder operators, the following quantities will be omitted for $T \in \mathcal{L}(E)$:

- the ascent of T, denoted by $a(T)$ and defined as:

$$a(T) := \inf\{q \in \mathbb{N} : \mathcal{N}(T^q) = \mathcal{N}(T^{q+1})\};$$

- the descent of T, denoted by $d(T)$ and defined as:

$$d(T) := \inf\{q \in \mathbb{N} : \mathcal{R}(T^q) = \mathcal{R}(T^{q+1})\}.$$

Before proceeding further, we introduce the following definition.

Definition 2.3. **(i)** Set of upper semi-Browder operators on E is defined as:

$$Br_+(E) := \{T \in \mathcal{L}(E) : T \in \Phi_+(E) :\ a(T) < +\infty\}.$$

(ii) Set of lower semi-Browder operators on E is defined as:

$$Br_-(E) := \{T \in \mathcal{L}(E) : T \in \Phi_-(E) :\ d(T) < +\infty\}.$$

(iii) Set of Browder operators on E is defined by:

$$\begin{aligned}Br(E) :=& Br_-(E) \cap Br_+(E)\\ :=& \{T \in \mathcal{L}(E) : T \in \Phi(E), a(T) < +\infty \text{ and } d(T) < +\infty\}.\end{aligned}$$

Continuing in this context, we introduce the definitions of the upper semi-Browder spectrum, the lower semi-Browder spectrum, and the Browder spectrum.

Definition 2.4. Let $T \in \mathcal{L}(E)$. Then, we define the following spectra:

$$\sigma_{Br^+}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br_+(E)\} - \text{upper semi-Browder spectrum of } T,$$
$$\sigma_{Br^-}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br_-(E)\} - \text{lower semi-Browder spectrum of } T,$$
$$\sigma_{Br}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br(E)\} - \text{Browder spectrum of } T.$$

Remark 2.2. **(i)** It is important to point out that $\sigma_{Br^+}(T) = \sigma_{Br^-}(T^*)$ and $\sigma_{Br^-}(T) = \sigma_{Br^+}(T^*)$, where $T^* \in \mathcal{L}(E^*)$ and $\sigma_{Br}(T) = \sigma_{Br^+}(T) \cup \sigma_{Br^-}(T)$. This proves that $\sigma_{Br}(T) = \sigma_{Br}(T^*)$.

(ii) On the basis of Theorem 3.52 in [4], we have:

$$\sigma_{Br}(T) = \sigma_{Br^+}(T) \cup \mathrm{S}_{\mathrm{vep}}(T^*) = \sigma_{Br^-}(T) \cup \mathrm{S}_{\mathrm{vep}}(T).$$

At the moment, we introduce the following upper triangular block 3×3 operator matrix defined on the direct sum of Banach spaces $E \oplus F \oplus G$ as well:

$$\mathcal{M} := \begin{pmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

Each operator entry of such kind of operator matrix is bounded and act on their corresponding spaces as:

$$A_1 : E \to E, \quad A_2 : F \to F, \quad A_3 : G \to G,$$
$$B_1 : F \to E, \quad B_2 : G \to F\ , \quad C_1 : G \to E.$$

Considering the case of infinite dimensions, it is widely acknowledged that the inclusion, $\sigma(\mathcal{M}) \subset \bigcup_{i=1}^{3} \sigma(A_i)$, can be strict. Many mathematicians have drawn the attention to study the defect set $\left(\bigcup_{i=1}^{3} \tilde{\sigma}(A_i)\right) \setminus \tilde{\sigma}(\mathcal{M})$, where $\tilde{\sigma}$ goes through various types of spectra.

The subsequent analysis requires a crucial conclusion that stems from the following theorem, whose formulation is inspired by the framework developed in [9].

Theorem 2.1. *Let* $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$. *Then,*

$$\sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)\Big] \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big] = \bigcup_{i=1}^{3} \sigma_{Br}(A_i),$$

for every $B_1 \in \mathcal{L}(F, E)$, $B_2 \in \mathcal{L}(G, F)$ *and* $C_1 \in \mathcal{L}(G, E)$.

3. Main Results and Proofs

In order to obtain our first result. We start by introducing the following diagonal operator matrix denoted by $\mathcal{M}_{\mathbf{d}}$ expressed as well: $\mathcal{M}_{\mathbf{d}} := \mathrm{diag}\ (A_1, A_2, A_3)$.

Our first lemma reads as follows.

Lemma 3.1. *Let* $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$. *Then, for every* $B_1 \in \mathcal{L}(F, E)$, $B_2 \in \mathcal{L}(G, F)$ *and* $C_1 \in \mathcal{L}(G, E)$ *we have:*

$$\sigma_{Br}(\mathcal{M}) \subseteq \sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \sigma_{Br}(A_i).$$

Proof. We start with the following factorization of the operator matrices $\mathcal{M}_{\frac{1}{k}}$ written as follows:

$$\mathcal{M}_{\frac{1}{k}} := \mathfrak{D}_1 \, \mathcal{M} \, \mathfrak{D}_2, \text{ for every } k \in \mathbb{N}^* \tag{3.1}$$

where: $\mathfrak{D}_1 := \begin{pmatrix} I & 0 & 0 \\ 0 & kI & 0 \\ 0 & 0 & kI \end{pmatrix}$ and $\mathfrak{D}_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{1}{k}I & 0 \\ 0 & 0 & \frac{1}{k}I \end{pmatrix}$.

Since $\|\mathcal{M}_{1/k} - \mathcal{M}_{\mathbf{d}}\| \to 0$ as $k \to +\infty$, and given that the Browder spectrum σ_{Br} is upper semi-continuous (see Theorem 2 in [16]), it follows that $\sigma_{Br}(\mathcal{M}) \subseteq \sigma_{Br}(\mathcal{M}_{\mathbf{d}})$.

The second equality is evident. □

Currently, we express the following proposition which will be widely used in the sequel.

Proposition 3.1. *Presume that:*

(**i**) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$;

(**ii**) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F, E), \mathcal{L}(G, F), \mathcal{L}(G, E)\Big)$.

Then, we have:

$$\operatorname{Acc} \sigma_{\mathrm{Br}}(\mathcal{M}) \subseteq \operatorname{Acc} \sigma_{\mathrm{Br}}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{\mathrm{Br}}(A_i).$$

Proof. It is obvious that $\operatorname{Acc} \sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$, since we have $\sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \sigma_{Br}(A_i)$. Without loss of generality, let consider $0 \notin \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$, then, there exists $\alpha > 0$ such that $\tau I - A_1$, $\tau I - A_2$ and $\tau I - A_3$ are Browder operators for every τ, $0 < |\tau| < \alpha$. Furthermore, according to Lemma 3.1, we get $\tau I - \mathcal{M}$ is Browder for every τ, $0 < |\tau| < \alpha$. This concludes that $0 \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M})$. □

The presented example illustrates that the inclusion $\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$ can indeed be strict, meaning that the accumulation points of the Browder spectrum of $\mathcal{M}$ may not coincide with the union of those of its diagonal entries.

Example 3.1. Let $A_1, A_2, A_3, C_1 \in \mathcal{L}(\ell^2)$ be defined by:

$$A_1 e_k = A_2 e_k = e_{k+1}, \quad B_1 = B_2 = 0, \quad C_1 = e_0 \otimes e_0, \quad A_3 = A_1^*,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is the canonical orthonormal basis of the Hilbert space ℓ^2.

It is well known that $\sigma_{Br}(A_1) = \sigma_{Br}(A_2) = \{\tau \in \mathbb{C} : |\tau| \leq 1\}$, hence

$$\operatorname{Acc} \sigma_{Br}(A_1) = \operatorname{Acc} \sigma_{Br}(A_2) = \{\tau \in \mathbb{C} : |\tau| \leq 1\}.$$

Since the operator matrix $\mathcal{M}$ constructed from these entries is unitary, it follows that $\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \subseteq \{\tau \in \mathbb{C} : |\tau| = 1\}$. This implies that $0 \notin \operatorname{Acc}\sigma_{Br}(\mathcal{M})$, although $0 \in \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$.

We also remark that the operator $A_3 = A_1^*$ does not possess the Single-Valued Extension Property (SVEP).

In what follows, we introduce the following definition and we state our next lemma which plays a crucial role in proving our next result.

Definition 3.1. Let $T \in \mathcal{L}(E)$. We say that the operator T has the property (Br^a) at $\tau \in \mathbb{C}$ if $\tau \notin \operatorname{Acc}\sigma_{Br}(T)$.

Lemma 3.2. *If any three of operators A_1, A_2, A_3, and $\mathcal{M}$ have the property (Br^a) at 0, then the fourth does as well.*

Proof. (**i**) If the operators A_1, A_2 and A_3 have the property (Br^a) at 0, then, according to Proposition 3.1, $\mathcal{M}$ has the property (Br^a) at 0.

(**ii**) If the operators A_1, A_2 and $\mathcal{M}$ have the property (Br^a) at 0, this means $0 \notin \operatorname{Acc}\sigma_{Br}(A_1)$, $0 \notin \operatorname{Acc}\sigma_{Br}(A_2)$ and $0 \notin \operatorname{Acc}\sigma_{Br}(\mathcal{M})$. Hence, there exists $\alpha > 0$ such that $\tau I - A_1$, $\tau I - A_2$ and $\tau I - \mathcal{M}$ are Browder operators for every τ, $0 < |\tau| < \alpha$. Thus, by combining Proposition 3.5 in [8] and Corollary 5 in [11], we obtain that $\tau I - A_3$ is Browder operator for every τ, $0 < |\tau| < \alpha$, that is $0 \notin \operatorname{Acc}\sigma_{Br}(A_3)$.

(**iii**) If the operators A_2, A_3 and $\mathcal{M}$ have the property (Br^a) at 0, then A_1 has also the property (Br^a) at 0. We adopt the same reasoning treated in the previous case.

(**iv**) If A_1, A_3 and $\mathcal{M}$ have the property (Br^a) at 0, then A_2 has the property (Br^a) at 0. The proof is similar to (**ii**). □

Our next aim in this section is to prove our first main result by means of localized SVEP. This theorem will guide us to a sufficient condition that guaranties the desired equality shown in Corollary 3.1.

Theorem 3.1. *Assume that:*

(**i**) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$;

(**ii**) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F,E), \mathcal{L}(G,F), \mathcal{L}(G,E)\Big)$.

Then, we have:

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M}) \cup \Big[\Big[\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)\Big] \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big] = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

Proof. Theorem 2.1 shows that

$$\sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)\Big] \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big] = \bigcup_{i=1}^{3} \sigma_{Br}(A_i),$$

for every $B_1 \in \mathcal{L}(F,E)$, $B_2 \in \mathcal{L}(G,F)$ and $C_1 \in \mathcal{L}(G,E)$. Hence,

$$\bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i) = \Big\{\sigma_{Br}(\mathcal{M}) \cup \Big[\,(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big]\Big\}$$

$$\cap \left\{ \bigcup_{i=1}^{3} \operatorname{Iso} \sigma_{Br}(A_i) \right\}^c . \tag{3.2}$$

In addition, it is obvious that

$$\left[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \right] \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma(A_i) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i) \tag{3.3}$$

and

$$\begin{aligned} \left[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \right] &= \left[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \right] \\ &\quad \cap \left\{ \bigcup_{i=1}^{3} \operatorname{Iso} \sigma_{Br}(A_i) \right\}^c . \end{aligned} \tag{3.4}$$

First, we need to demonstrate that

$$\begin{aligned} &\left\{ \sigma_{Br}(\mathcal{M}) \cup \left[\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \right] \cap \mathrm{Svep}(A_3) \right] \right\} \cap \left\{ \bigcup_{i=1}^{3} \mathrm{Iso} \sigma_{Br}(A_i) \right\}^c \\ &\subseteq \operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \left[\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \right] \cap \mathrm{Svep}(A_3) \right]. \end{aligned}$$

As a matter of fact, let $\tau \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \left[\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \right] \cap \mathrm{Svep}(A_3) \right]$. Without loss of generality, we can suppose that $\tau = 0$. Hence, $0 \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M})$ and $0 \notin \left[\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \right] \cap \mathrm{Svep}(A_3) \right]$. Thus, there exists $\alpha > 0$ such that $\tau I - \mathcal{M}$ is Browder for every $0 < |\tau| < \alpha$. Therefore, for every $0 < |\tau| < \alpha$, $\tau I - A_3$ is lower semi-Browder operator and $\tau I - A_1$ is upper semi-Browder operator. Consequently, $0 \notin \left[\operatorname{Acc} \sigma_{Br^+}(A_1) \cap \operatorname{Acc} \sigma_{Br^+}(A_2) \right] \cup \operatorname{Acc} \sigma_{Br^-}(A_3)$. On the other side

$$0 \notin \left[\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \right] \cap \mathrm{Svep}(A_3) \right].$$

There are two possible cases.

• Case 1: $0 \notin \mathrm{S}_{\mathrm{vep}}(A_1^*) \cap \mathrm{S}_{\mathrm{vep}}(A_3)$.

▪ $0 \in \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$.

We keep into account that $\sigma_{Br}(A_1^*) = \mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \sigma_{Br^-}(A_1^*)$. So, if $0 \in \sigma_{Br}(A_1^*)$ then $0 \in \sigma_{Br^-}(A_1^*)$. Since $0 \notin \operatorname{Acc} \sigma_{Br^+}(A_1) = \operatorname{Acc} \sigma_{Br^-}(A_1^*)$, we obtain that 0 is an isolated point of $\sigma_{Br^-}(A_1^*)$. Furthermore, $\overline{\mathrm{S}_{\mathrm{vep}}(A_1^*)} \subseteq \sigma_{Br}(A_1^*) = \sigma_{Br^-}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_1^*)$, thus $\partial \mathrm{S}_{\mathrm{vep}}(A_1^*) \subseteq \sigma_{Br^-}(A_1^*)$. Since $\sigma_{Br}(A_1^*) = \mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \sigma_{Br^-}(A_1^*)$ and $0 \in \mathrm{Iso}\sigma_{Br^-}(A_1^*)$. Therefore, 0 is an isolated point of $\sigma_{Br}(A_1) = \sigma_{Br}(A_1^*)$. On the other side, if $0 \in \sigma_{Br}(A_3)$ then $0 \in \sigma_{Br^-}(A_3)$. As $0 \notin \operatorname{Acc} \sigma_{Br^-}(A_3)$, it follows that $0 \in \mathrm{Iso}\sigma_{Br^-}(A_3)$. This shows that $0 \in \mathrm{Iso}\sigma_{Br}(A_3)$.

▪ $0 \notin \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$.

If $0 \notin \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$. Then, $0 \notin \sigma_{Br}(A_1^*) = \sigma_{Br}(A_1)$ and $0 \notin \sigma_{Br}(A_3)$. Hence, $0 \notin \operatorname{Acc} \sigma_{Br}(A_1)$ and $0 \notin \operatorname{Acc} \sigma_{Br}(A_3)$. This proves that $0 \notin \operatorname{Acc} \sigma_{Br}(A_2)$. So either $0 \notin \sigma_{Br}(A_2)$ or $0 \in \mathrm{Iso}\sigma_{Br}(A_2)$, i.e., $0 \notin \sigma_{Br}(\mathcal{M})$ or $0 \in \mathrm{Iso}\sigma_{Br}(A_2)$.

• Case 2: $0 \notin \mathrm{S}_{\mathrm{vep}}(A_2^*) \cap \mathrm{S}_{\mathrm{vep}}(A_3)$.

We adopt the same reasoning treated in the first case. It suffices to replace A_1 by A_2.

On the basis of Proposition 3.1, (3.2), (3.3) and (3.4), we get:

$$\begin{aligned}\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i) &= \Big\{\sigma_{Br}(\mathcal{M})\cup\Big[\left(\mathrm{S}_{\mathrm{vep}}(A_1^*)\cup\mathrm{S}_{\mathrm{vep}}(A_2^*)\right)\cap\mathrm{S}_{\mathrm{vep}}(A_3)\Big]\Big\}\cap\left\{\bigcup_{i=1}^{3}\operatorname{Iso}\sigma_{Br}(A_i)\right\}^{c}\\ &\subseteq \operatorname{Acc}\sigma_{Br}(\mathcal{M})\cup\Big[\left(\mathrm{S}_{\mathrm{vep}}(A_1^*)\cup\mathrm{S}_{\mathrm{vep}}(A_2^*)\right)\cap\mathrm{S}_{\mathrm{vep}}(A_3)\Big]\\ &\subseteq \bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i).\end{aligned}$$

Thus, we complete the proof. □

Taking into account the previous theorem, the following corollary provides a sufficient condition which ensures that $\operatorname{Acc}\sigma_{Br}(\mathcal{M})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$ for every bounded operators $(B_1,B_2,C_1)\in\Big(\mathcal{L}(F,E),\mathcal{L}(G,F),\mathcal{L}(G,E)\Big)$.

Corollary 3.1. *Suppose that:*

(i) $(A_1,A_2,A_3)\in\Big(\mathcal{L}(E),\mathcal{L}(F),\mathcal{L}(G)\Big)$*;*

(ii) $\Big[S_{\mathrm{vep}}(A_1^*)\cup S_{\mathrm{vep}}(A_2^*)\Big]\cap S_{\mathrm{vep}}(A_3)=\emptyset$.

Then, for every $B_1\in\mathcal{L}(F,E), B_2\in\mathcal{L}(G,F)$ *and* $C_1\in\mathcal{L}(G,E)$ *we have:*

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

Remark 3.1. The above result makes it easy to confirm that the limit point set of the Browder spectrum of the operator matrix $\mathcal{M}$ is the union of the limit point set of the Browder spectrum of its diagonal entries if A_3 has SVEP or A_1^* and A_2^* have SVEP.

The efficacy of the previously presented result is illustrated by the following example, which serves as a direct application of Corollary 3.1.

Example 3.2. Let consider $\mathcal{S}_u$ the simple unilateral shift operator defined on $\ell^2(\mathbb{N})$. We define also the following operators as follows:

$$A_1:=(\mathcal{S}_u\oplus{\mathcal{S}_u}^*)+2I,\quad A_2:=\mathcal{S}_u\oplus{\mathcal{S}_u}^*\quad\text{and}\quad A_3:=(\mathcal{S}_u\oplus{\mathcal{S}_u}^*)-2I.$$

Then,

$$\begin{aligned}\sigma_{Br}(A_1)&:=\{\tau\in\mathbb{C}:0\leq|\tau-2|\leq1\},\\ \sigma_{Br}(A_2)&:=\{\tau\in\mathbb{C}:0\leq|\tau|\leq1\},\\ \sigma_{Br}(A_3)&:=\{\tau\in\mathbb{C}:0\leq|\tau+2|\leq1\}.\end{aligned}$$

It follows that:

$$\begin{aligned}\mathrm{S}_{\mathrm{vep}}(A_1)&:=\{\tau\in\mathbb{C}:0\leq|\tau-2|<1\},\\ \mathrm{S}_{\mathrm{vep}}(A_2)&:=\{\tau\in\mathbb{C}:0\leq|\tau|<1\},\\ \mathrm{S}_{\mathrm{vep}}(A_3)&:=\{\tau\in\mathbb{C}:0\leq|\tau+2|<1\}.\end{aligned}$$

Consequently, $\Big[\mathrm{S_{vep}}(A_1^*) \cup \mathrm{S_{vep}}(A_2^*)\Big] \cap \mathrm{S_{vep}}(A_3) = \emptyset$. This proves that

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i).$$

The next two lemmas are the key to our second main outcome and are founded in [17]. We consider two compact subsets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ such that $\mathcal{U} \subseteq \mathcal{V}$.

Lemma 3.3. *Assume further that $\partial\mathcal{V} \subseteq \mathcal{U}$. Then, $\partial\mathrm{Acc}\,\mathcal{V} \subseteq \mathrm{Acc}\,\mathcal{U}$.*

Lemma 3.4. *Assume further that $\eta(\mathcal{U}) = \eta(\mathcal{V})$. Then, $\eta(\mathrm{Acc}\,\mathcal{U}) = \eta(\mathrm{Acc}\,\mathcal{V})$.*

Finally, we are in a position to prove the following theorem which says that the passage from $\bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$ to $\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})$ is the punching of some set in $\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$.

Theorem 3.2. *Imply that:*
(i) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$*;*
(ii) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F,E), \mathcal{L}(G,F), \mathcal{L}(G,E)\Big)$.
Then, we obtain:

$$\mathrm{Acc}\,\sigma_{\mathrm{Br}}(\mathcal{M}) \cup \mathcal{W}_{\mathrm{Acc}\,\sigma_{\mathrm{Br}}} = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{\mathrm{Br}}(A_i),$$

where $\mathcal{W}_{\mathrm{Acc}\,\sigma_{\mathrm{Br}}}$ is the union of certain holes in $\mathrm{Acc}\,\sigma_{\mathrm{Br}}(\mathcal{M})$, that occur to be subsets of $\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{\mathrm{Br}}(A_i)$.

Proof. First notice that the inclusion

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\} \subseteq \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$$

holds for every $B_1 \in \mathcal{L}(F,E)$, $B_2 \in \mathcal{L}(G,F)$ and $C_1 \in \mathcal{L}(G,E)$.

The reverse inclusion follows from the following equivalence:

$$\begin{aligned}
&\tau \notin \mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\} \\
&\Leftrightarrow \tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_1)^c\} \quad \text{or} \quad \tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_2)^c\} \\
&\qquad \text{or} \quad \tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_3)^c\} \\
&\Leftrightarrow \tau \in \mathrm{Acc}\,\sigma_{Br}(A_1)^c \text{ and } \tau \in \mathrm{Acc}\,\sigma_{Br}(A_2)^c \text{ and } \tau \in \mathrm{Acc}\,\sigma_{Br}(A_3)^c \quad \text{(Lemma 3.2)} \\
&\Leftrightarrow \tau \notin \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i).
\end{aligned}$$

We deduce that

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\} = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i). \tag{3.5}$$

Moreover, getting inspired by the proof of Theorem 2.6 in [18] and by applying Lemmas 3.3 and 3.4, we get

$$\partial\left(\bigcap_{i=1}^{3}\sigma_{Br}(A_i)\right)\subseteq\sigma_{Br}(\mathcal{M}),$$

$$\eta\left(\operatorname{Acc}\sigma_{Br}(\mathcal{M})\right)=\eta\left(\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)\right). \tag{3.6}$$

Consequently, (3.6) proves that the passage from $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ to $\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$ is the filling in certain of the holes in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$. But, $\left(\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)\right)\backslash\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ is contained in $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$, according to (3.5). This shows that the filling in certain of the holes in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ should take place in $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$. □

Nevertheless, we have the following corollary.

Corollary 3.2. *Suppose that:*
(**i**) $(A_1,A_2,A_3)\in\Big(\mathcal{L}(E),\mathcal{L}(F),\mathcal{L}(G)\Big)$*;*
(**ii**) $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i)$ *has no interior points.*
Then, for every $B_1\in\mathcal{L}(F,E),B_2\in\mathcal{L}(G,F)$ *and* $C_1\in\mathcal{L}(G,E)$ *we have:*

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

We would like to finish this work with the following question.

Question 3.3. Let consider the following bounded full 3×3 block operator matrix

$$\widehat{\mathcal{M}}:=\begin{pmatrix}A_1&B_1&B_2\\C_1&A_2&B_3\\C_2&C_3&A_3\end{pmatrix}.$$

The following question is being asked: what are the conditions that will be placed on the input parts of the operator matrices $\widehat{\mathcal{M}}$ to achieve that

$$\operatorname{Acc}\sigma_{Br}(\widehat{\mathcal{M}})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)?$$

4. Conclusion

In this paper, we investigated the accumulation points of the Browder spectrum for 3×3 upper triangular operator matrices acting on Banach spaces. By applying the tools of local spectral theory, we established a precise relationship between the accumulation set of the matrix and those of its diagonal entries. In particular, we described the structure of a supplementary set of "holes" that appear within the intersection of the accumulation points of the diagonal Browder spectra, allowing us to characterize the accumulation behavior of the full matrix spectrum. Our results provide a refined description that extends the work of A. Tajmouati et al. [17] by

considering a more general setting and offering a deeper analysis of the spectral accumulation structure. This contribution enhances the understanding of spectral properties of operator matrices and may serve as a foundation for further developments in the theory.

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ERROR ANALYSIS OF THE SEMI-DISCRETIZED DOUBLY NONLINEAR NON-LOCAL THERMISTOR PROBLEM

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Abstract. In this paper, we study a doubly nonlinear parabolic equation obtained from the reduction of the wellknown nonlocal thermistor problem. Therefore, we focus our study on proving the existence of the solution to the semi-discrete problem. We also establish the stability and error estimates for a family of time discretization schemes. We investigate a time discretization of the continuous problem by the backward Euler scheme.

1. Intorduction

Thermistors are a type of resistor that can be found in a variety of goods and applications in our modern society. They have been around since the 1830s, when the industrial revolution came to an end. Back then, they were used to measure temperature sensing, as self-resetting over current protection and to limit inrush current, among other things. Thermistors were first discovered by Michael Faraday, a British scientist and chemist, who is credited with inventing the first NTC thermistor. Faraday is also known for his contributions to electrochemistry and electromagnetic induction. In 1833, Faraday wrote a treatise on the behavior (semiconducting) of Ag2S (silver sulfide), which is said to be the first thermistor ever.

There are two types of thermistors PTC and NTC, which have Positive and Negative Temperature Coefficient, respectively. In the former, the electrical conductivity decreases with increasing temperature, while in the latter it increases with increasing temperature. PTC thermistors can be found in a variety of applications, including

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switches and electric surge protection devices. The PTC electric surge device operates in the following manner: upon a sudden increase in current in the circuit, there is a rise in temperature that leads to a significant reduction in the electrical conductivity of the device. As a result, the circuit is effectively switched off. Once the current surge subsides, the device cools down, its electrical conductivity increases and the circuit resumes normal operation, as stated by Bartosz in [3]. However, it has been found that the rapid rise in temperature can generate large thermal stresses that can compromise the integrity of the device [19, 23] which can lead to breakage and failure of the device. Since the nonlinearities of the system were in the electrical conductivity, Joule heating and viscous heating terms, it was assumed that the constitutive behavior of the material was linear. The "Thermistor Problem" can be roughly defined as a nonlinear parabolic equation describing the temperature coupled with an elliptic equation describing the quasi-static evolution of the electric potential. The non-linearity of the problem is primarily due to the coupling between these equations, which is partly due to the significant influence of temperature on electrical conductivity. Mathematically, the problem can be expressed as follows:

$$\begin{aligned} v_t &= \nabla \cdot (\kappa(v)\nabla v) + \rho(v)|\nabla\psi|^2, \\ \nabla \cdot (\rho(v)\nabla\psi) &= 0. \end{aligned} \tag{1.1}$$

In the above model, κ denotes the thermal conductivity and $\rho(v)$ the electrical conductivity, which is normally a positive function of real values. Furthermore, the temperature of the conductor is represented by v, while the electric potential is represented by ψ. The first equation describes heat diffusion, while the second governs charge conservation. Here, $\kappa(v)$ and σ represent the thermal and electrical conductivity respectively, with their specific functional forms determined by the physical properties of the system. Under suitable conditions, this coupled system can be reformulated as a parabolic problem if it is supplemented by suitable boundary and initial conditions. For detailed treatments of such systems we refer to[7, 20, 26]. In this paper, we will deal with the following non-local model

$$\begin{cases} \dfrac{\partial b(v)}{\partial t} - \Delta v = \dfrac{\lambda g(v)}{\left(\int_\Omega g(v)dx\right)^2}, & \text{in } Q, \\ v(x,0) = v_0, & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma\times]0, M[, \end{cases} \tag{1.2}$$

which is considered as a generalization of the problem from the work of A. A. Lacey [23], where $g(v)$ is the electrical resistance of the conductor and $\frac{g(v)}{\left(\int_\Omega g(v)dx\right)^2}$ represents the non-local term of (1.2). Whereas$Q$ is defined as follows $Q := \Omega \times [0, M]$ where Ω is an open bounded subset of $\mathbb{R}^m$, $m \geq 2$, and T is a positive constant. The literature on the problems (1.1) and (1.2) is extensive (see [1, 5, 9–12, 15–18, 21, 24, 25, 27, 30, 32]). Our motivation is stimulated by various applications. A thermistor has been widely used in electronic circuits to protect, control and compensate temperature. On the other hand, b is a nonlinear function that can grow faster than any function at infinity

$b(v) = e^{(v+1)^2 e^v}$, for example). Furthermore, it does not have to be strictly growing on any part of $\mathbb{R}$. Thus, the evolution equation can become stationary in a subdomain of $\Omega\times]0, M[$, where $]0, M[$ denotes the time horizon. Our aim is to prove the existence of a solution to the problem (1.2). We will also prove some stability results and provide error estimates.

The rest of the paper is organised as follows. Section 2 starts with a brief reminder of some notations and hypotheses that we need in this paper. In Section 3 we show the existence and uniqueness of a solution to a semi-discrete problem. In Section 4 we prove some stability results. Section 5 is dedicated to error analysis. Finally, in Section 6 we present conclusions and perspectives for future research.

2. Notations and Hypotheses

Throughout this paper, we assume the following.

(H1) $g : \mathbb{R} \to \mathbb{R}$ is a L_1-Lipschitzian function.

(H2) There exists a positive constant σ such that for all $s \in \mathbb{R}$, we have

$$\sigma \leq g(s). \tag{2.1}$$

(H3) b is an increasing Lipchitz function with $b(0) = 0$.

(H4) We assume that exists $C_\sigma \in \left]0, \frac{(\sigma\cdot \mathrm{meas}(\Omega))^2}{\tau\lambda}\right[$ such that

$$|g(x) - g(y)| \leq C_\sigma |b(x) - b(y)|. \tag{2.2}$$

Throughout this paper, we will choose λ large enough to get that

$$\frac{(\sigma \cdot \mathrm{meas}(\Omega))^2}{\tau\lambda} < 1.$$

(H5) $v_0 \in L^\infty(\Omega)$. We define for $r \in \mathbb{R}$

$$\Psi(r) = \int_0^r b(t)dt.$$

Then, the Legendre transform Ψ^* of Ψ is defined by

$$\Psi^*(r) = \sup_{s\in\mathbb{R}}\{rs - \Psi(s)\}.$$

Note that $\Psi^*(b(r)) + \Psi(r) = rb(r)$. Ω will denote an open bounded set of $\mathbb{R}^m$, $m \geq 2$ with smooth boundary and $\langle\cdot,\cdot\rangle$ denotes the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Throughout this paper, C_i and C will denote various positive constants.

Remark 2.1. Under hypotheses (H1), we can see that $g(s) \leq C_1|s| + C_2$. Indeed,

$$g(s) \leq |g(s) - g(0)| + g(0) \leq L_1|s| + g(0) \leq C_1|s| + C_2, \tag{2.3}$$

where $L_1 = C_1$ and $g(0) = C_2$.

3. Existence of Solution for a Semi-discrete Problem

For the time discretization of the problem (3.1) we will use the following notations. We denote the time step $\tau = T/N, t^n = n\tau$, and $J_n := (t^{n-1}, t^n)$ for $n = 1, \dots, N$. We consider the corresponding discrete scheme (backward Euler scheme) related to (1.2), which is represented by

$$\begin{cases} b(v_n) - \tau\Delta v_n = b(v_{n-1}) + \dfrac{\tau\lambda g(v_n)}{(\int_\Omega g(v_n)dx)^2}, & \text{in } Q, \\ v_n(x,0) = v_{0,n}, & \text{in } \Omega, \\ v_n = 0, & \text{on } \Gamma\times]0, M[. \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that hypotheses* (H1)-(H4) *hold. Then, for each integer n, there exists a solution v_n to* (3.1).

Proof. To show the existence of a solution, Schauder fixed point theorem will be applied. For simplicity, putting $v = v_n$ substituting in the first equation of the system (3.1), we get that for a fixed w in $H_0^1(\Omega)$

$$b(v) - \tau\Delta v = b(v_{n-1}) + \frac{\tau\lambda g(w)}{(\int_\Omega g(w)dx)^2}.$$

Taking the inner product with the function v to get

$$\langle b(v), v\rangle - \tau\langle \Delta v, v\rangle = \mu\langle h(w), v\rangle, \tag{3.2}$$

where

$$h(w) = b(v_{n-1}) + \frac{\tau\lambda g(w)}{(\int_\Omega g(w)dx)^2} \quad \text{and} \quad \mu \in [0,1].$$

Let us consider the operator $A(\mu, w) = v,$ for all $\mu \in [0,1]$, being the solution to the following problem

$$\begin{cases} b(v) - \tau\Delta v = \mu h(w), & \text{in } Q, \\ v(x,0) = v_0, & \text{in } \Omega, \\ v = 0, & \text{on } \Gamma\times]0, M[. \end{cases} \tag{3.3}$$

If $v \geq b(v)$, by using (3.2), we obtain that

$$\int_\Omega (b(v))^2 \, \mathrm{d}x + \tau \int_\Omega |\nabla v|^2 \, \mathrm{d}x \leq \mu\langle h(w), v\rangle. \tag{3.4}$$

Using hypothesis (H2) and Young's inequality, we get the following estimates

$$\begin{aligned} \|b(v)\|_2^2 + \tau \|\nabla v\|_2^2 &\leq \int_\Omega (C_3 + C_4|w|)\,|v|dx \\ &\leq C_4 \|w\|_2 \|v\|_2 + C_5 \|v\|_2 \\ &\leq C_6 \|w\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} + C_5 \|v\|_2 \\ &\leq \frac{\tau}{2} \|v\|_{H_0^1(\Omega)}^2 + \frac{1}{2\tau} C_6^2 \|w\|_{H_0^1(\Omega)}^2 + C_5 \|v\|_2 . \end{aligned}$$

Then,

$$\|b(v)\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \leq \frac{1}{2\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + C_5\|v\|_2.$$

It follows that

$$2\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5\|v\|_2.$$

Hence,

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5\|v\|_2.$$

Using again Poincaré's inequality, we obtain

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5C_7\|\nabla v\|_2.$$

By using Young's inequality, we get that

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{(2C_5C_7)^2}{\tau} + \frac{\tau}{2}\|\nabla v\|_2^2.$$

Then,

$$\|b(v)\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{(2C_5C_7)^2}{\tau},$$

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq 2\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{2}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{2(2C_5C_7)^2}{\tau}.$$

Then, there exist positive constants C_8 and C_9, such that

$$\|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.5}$$

Then,

$$\|b(v)\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.6}$$

Since w is bounded in $H_0^1(\Omega)$, then there exists a positive constant C_{10} such that

$$\|v\|_2^2 \leq C_{10}. \tag{3.7}$$

Recall from (3.5), that

$$\tau\|\nabla v\|_2^2 \leq \|b(v)\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_9. \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_8\|w\|_{H_0^1(\Omega)}^2 + C_{11}. \tag{3.9}$$

Else we have $v \leq b(v)$, then from (3.2), we get

$$\begin{aligned}\|v\|_2^2 + \tau\|\nabla v\|_2^2 &\leq \int_\Omega (C_3 + C_4|w|)\,|v|dx \\ &\leq C_4\|w\|_2\|v\|_2 + C_5\|v\|_2 \\ &\leq C_6\|w\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)} + C_5\|v\|_2 \\ &\leq \frac{\tau}{2}\|v\|_{H_0^1(\Omega)}^2 + \frac{1}{2\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + C_5\|v\|_2.\end{aligned} \tag{3.10}$$

Then,

$$\|v\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \leq \frac{1}{2\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + C_5\|v\|_2.$$

Hence,

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq 2\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 2C_5\|v\|_2.$$

Using together Poincaré's inequality with Young's inequality, we obtain

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{2(C_5C_7)^2}{\tau} + \frac{\tau}{2}\|\nabla v\|_2^2.$$

It follows that

$$\|v\|_2^2 + \frac{\tau}{2}\|\nabla v\|_2^2 \leq \frac{1}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + \frac{2(C_5C_7)^2}{\tau}.$$

Then,

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq 2\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq \frac{2}{\tau}C_6^2\|w\|_{H_0^1(\Omega)}^2 + 4\frac{(C_5C_7)^2}{\tau}.$$

Then, there exist positive constants C_{12} and C_{13}, such that

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_{12}\|w\|_{H_0^1(\Omega)}^2 + C_{13}. \tag{3.11}$$

From (3.9) and (3.11), there exist positive constants C_{14} and C_{15} such that

$$\|v\|_2^2 + \tau\|\nabla v\|_2^2 \leq C_{14}\|w\|_{H_0^1(\Omega)}^2 + C_{15}. \tag{3.12}$$

We can see that $\mu \mapsto A(\mu, v)$ is a continuous function and we have that $A(0, v) = V$ for any v if and only if $V = 0$. By using the theory of degree topology, we get that $v \in H_0^1(\Omega)$ and satisfies Problem (3.1). □

The following lemmas play a key role in the proof of Theorem 4.1.

Lemma 3.1. *If $v_0 \in L^\infty(\Omega)$, then $v_n \in L^\infty(\Omega)$ for all $n = 1, 2, 3, \ldots, N$.*

The proof of Lemma 3.1, is similar to the one used by de Thelin in [6] in a different problem; we shall give here only a sketch. Suppose $m \geq 2$ and define

$$\gamma = \begin{cases} \frac{2m}{m-2}, & \text{if } m > 2, \\ 4, & \text{if } m = 2. \end{cases}$$

Let $p_1 = \gamma$ and let

$$p_k = \left\{\left(\frac{\gamma}{2}\right)^{k-1}(\gamma - 2)\right\}\frac{\gamma}{\gamma - 2}, \quad k \geq 2.$$

Then, we have

$$p_{k+1} = p_k\frac{\gamma}{2}, \quad \text{for all } k \in \mathbb{N}^*.$$

Lemma 3.2. *For k in $\mathbb{N}^*$, $v_n \in L^{p_k}(\Omega)$ and*

$$\|v_n\|_\infty = \lim_{k\to+\infty}\sup\|v_n\|_{p_k} < +\infty. \tag{3.13}$$

Proof. For simplicity, we write $V = v_n$, $K(x) = b(v_{n-1})$. From (3.1), we have

$$b(V) - \tau\Delta V = K(x) + \frac{\tau\lambda g(V)}{(\int_\Omega g(V)dx)^2} =: H(V).$$

Multiplying the above identity by $|V|^{p_k-2}V$, to obtain

$$\int_\Omega b(V)|V|^{p_k-2}V\ dx - \tau\int_\Omega \Delta V|V|^{p_k-2}Vdx = \int_\Omega H(V)\cdot|V|^{p_k-2}Vdx. \tag{3.14}$$

In view of the fact that

$$\begin{aligned}
&-\tau\int_\Omega \Delta V|V|^{p_k-2}V\ dx = \tau\int_\Omega \nabla V\cdot\nabla(|V|^{p_k-2}V)\ dx\\
=&\tau\int_\Omega |\nabla V|^2|V|^{p_k-2}\ dx + \tau\int_\Omega (p_k-2)|\nabla V|^2|V|^{p_k-2}\ dx\\
=&\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx,
\end{aligned}$$

and from (3.14), we get that

$$\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx \le -\int_\Omega b(V)|V|^{p_k-2}V + \int_\Omega H(V)|V|^{p_k-2}V\ dx. \tag{3.15}$$

Keeping (3.15) in mind and using the fact that $H(V) \le C_0|V| + C_1$, we obtain that

$$\begin{aligned}
&\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx\\
&\le -\int_\Omega b(V)|V|^{p_k-2}V\ dx + \int_\Omega (C_0|V|+C_1)|V|^{p_k-1}\ dx.
\end{aligned} \tag{3.16}$$

Let us deal with the first term on the right side of the inequality (3.16). If $b(V) \le V$ then we get

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le -\int_\Omega (b(V))^2|V|^{p_k-2}\ dx \le 0,$$

else we have $b(V) \ge V$,

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le -\int_\Omega V^2|V|^{p_k-2}\ dx \le 0.$$

In both cases, we have

$$-\int_\Omega b(V)|V|^{p_k-2}V\ dx \le 0.$$

This implies that

$$\begin{aligned}
\tau\int_\Omega (p_k-1)|\nabla V|^2|V|^{p_k-2}\ dx &\le \int_\Omega (C_0|V|+C_1)|V|^{p_k-1}\ dx\\
&\le C_0\int_\Omega |V|^{p_k}\ dx + C_1\int_\Omega |V|^{p_k-1}\ dx\\
&\le C_0\int_\Omega |V|^{p_k}\ dx + C_1\int_\Omega |V|^{p_k-1}\ dx.
\end{aligned} \tag{3.17}$$

Using Young's inequality, we get that

$$C_1\int_\Omega |V|^{p_k-1}\ dx \le \int_\Omega C_1^{p_k}\ dx + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k} \le C_{16} + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k}. \tag{3.18}$$

Combining (3.17) with (3.18), we get

$$\tau \int_\Omega (p_k-1)|\nabla V|^2 \cdot |V|^{p_k-2}\, dx \le C_{16} + \frac{p_k-1}{p_k}||V||_{p_k}^{p_k} + C_0 \int_\Omega |V|^{p_k}\, dx \le C_{16} + C_{17}||V||_{p_k}^{p_k}. \tag{3.19}$$

On the other hand, by using Poincaré's inequality, we have for all $V \in H_0^1(\Omega)$ the following inequality

$$||V||_\gamma \le C_{18}||\nabla V||_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned} \|V\|_{p_{k+1}}^{p_k} &= \left(\int_\Omega \|V\|^{p_{k+1}}\, dx\right)^{\frac{p_k}{p_{k+1}}} = \left(\int_\Omega \|V\|^{\frac{p_k\gamma}{2}}\, dx\right)^{\frac{2}{\gamma}} \\ &= \left(\int \left(|V|^{1+\frac{p_k-2}{2}}\right)^\gamma dx\right)^{\frac{2}{\gamma}} = \left\||V|^{\frac{p_k}{2}}\right\|_\gamma^2 \\ &\le C_{18}^2 \int_\Omega \left|\nabla\left(|V|^{\frac{p_k}{2}}\right)\right|^2 \\ &\le \left(\frac{p_k}{2}\right)^2 C_{18}^2 \int_\Omega |\nabla V|^2 |V|^{p_k-2} dx. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$\|V\|_{p_{k+1}}^{p_k} \le p_k^2 \left(C_{19} + C_{20}||V||_{p_k}^{p_k}\right).$$

By induction on k, we get that $V \in L^{p_k}(\Omega)$, $1 \le k \le m+1$. Then, we have the following

$$\left(\|V\|_{p_{m+1}}^{p_{m+1}}\right)^{\frac{2}{\gamma}} \le \left(C_{19} + C_{20}\,\|V\|_{p_m}^{p_m}\right) p_m^2. \tag{3.21}$$

The rest of the proof follows the same lines as in [29]. As follows: for $\delta =: p_1$, we remark that

$$p_m \le \delta \left(\frac{\delta}{2}\right)^{m-1}.$$

Setting $a =: \frac{\delta}{2}$, $b =: \frac{\delta}{2}\log\max\{1, C_{19}+C_{20}\}$, $E_m = p_m \log\max\{1, ||V||_{p_m}\}$ and $r_m = b + \delta(m-1)\log a$. We obtain that

$$\begin{aligned} E_{m+1} &\le r_m + aE_m \le r_m + ar_{m-1} + \cdots + a^{m-1}r_1 + a^m E_1, \\ E_{m+1} &\le a^m \left\{E_1 + \frac{b}{a-1} + \frac{\delta\log a}{(a-1)^2}\right\} =: da^m. \end{aligned}$$

From which we conclude that

$$\|V\|_\infty \le \limsup_{m\to+\infty} \exp\left(\frac{E_m}{p_m}\right) \le \limsup_{m\to+\infty} \exp\left(\frac{da^{m-1}}{p_m}\right) \le \exp\left(\frac{d}{\delta}\right).$$

This concludes the proof of (3.13). □

4. Stability Results

In this section, we focus on the stability analysis for the semi-discretized problem. For this purpose, we consider V_0 in $L^\infty(\Omega)$.

Theorem 4.1. *Under hypotheses* (H1)-(H3) *there exists positive constants* $C(M, V_0)$ *and* $C_i(M, V_0)$, $i = 1, 2$, *such that*

$$\|V_n\|_\infty \leq C(M, V_0), \tag{4.1}$$

$$\int_\Omega \psi^*(b(V_n))\ dx + \tau \sum_{k=1}^{n} \|V_k\|_{1,p}^p \leq C_1(M, V_0), \tag{4.2}$$

$$\sum_{k=1}^{n} \|b(V_k) - b(V_{k-1})\|_2^2 \leq C_2(M, V_0). \tag{4.3}$$

Proof. From Lemma 3.2, we have that $V_n \in L^\infty(\Omega)$. Multiplying the first equation of (3.1) by $|b(V_n)|^k b(V_n)$, we get

$$\begin{aligned} &\int_\Omega |b(V_n)|^k |b(V_n)|^2\ dx - \tau \int_\Omega \Delta V_n |b(V_n)|^k b(V_n)\ dx \\ &= \int_\Omega b(V_{n-1}) |b(V_n)|^k b(V_n)\ dx + \int_\Omega h(V_n) |b(V_n)|^k b(V_n)\ dx. \end{aligned} \tag{4.4}$$

It can be shown that

$$-\int_\Omega \Delta V_n |b(V_n)|^k b(V_n) \leq 0. \tag{4.5}$$

Combining (4.4) with (4.5), we get

$$\|b(V_n)\|_{k+2}^{k+2} \leq \int_\Omega b(V_{n-1}) |b(V_n)|^k b(V_n)\ dx + \frac{\tau\lambda}{(\sigma\ \text{meas}\ (\Omega))^2} \int_\Omega g(V_n) |b(V_n)|^{k+1}\ dx.$$

It follows that

$$\|b(V_n)\|_{k+2}^{k+2} \leq \|b(V_n)\|_{k+2}^{k+1} \|b(V_{n-1})\|_{k+2} + \frac{\tau\lambda}{(\sigma\, \text{meas}(\Omega))^2} \int_\Omega g(V_n) |b(V_n)|^{k+1}\ dx. \tag{4.6}$$

To deal with the second term on the above inequality, using hypotheses (H3)-(H4), we obtain

$$\begin{aligned} \int_\Omega g(V_n) |b(V_n)|^{k+1}\ dx &\leq \int_\Omega (|g(V_n)) - g(0)| + g(0))|b(V_n)|^{k+1}\ dx \\ &\leq \int_\Omega |g(V_n) - g(0)| \cdot |b(V_n)|^{k+1}\ dx + g(0) \int_\Omega |b(V_n)|^{k+1}\ dx \\ &\leq C_\sigma \int_\Omega |b(V_n) - b(0)| \cdot |b(V_n)|^{k+1}\ dx + g(0) \int_\Omega |b(V_n)\Big|^{k+1} dx \\ &\leq C_\sigma \int_\Omega |b(V_n)|^{k+2}\ dx + g(0) \int_\Omega |b(V_n)|^{k+1} dx \\ &\leq C_\sigma \|b(V_n)\|_{k+2}^{k+2} + g(0) \int_\Omega |b(V_n)|^{k+1} dx. \end{aligned}$$

Then, inequality (4.6) becomes

$$\|b(V_n)\|_{k+2}^{k+2} \leq \|b(V_n)\|_{k+2}^{k+1} \|b(V_{n-1})\|_{k+2} + C_\sigma \|b(V_n)\|_{k+2}^{k+2} + g(0) \int_\Omega |b(V_n)|^{k+1} \, dx.$$

Under the hypothesis (H4), we get that

$$(1 - C_\sigma) \|b(V_n)\|_{k+2}^{k+2} \leq \|b(V_n)\|_{k+2}^{k+1} \|b(V_{n-1})\|_{k+2} + C_{21} \|b(V_n)\|_{k+1}^{k+1}.$$

Then, there exists a positive constant C such that

$$C \|b(V_n)\|_{k+2} \leq \|b(V_{n-1})\|_{k+2} + C_{22}.$$

By induction, we get that

$$\|b(V_n)\|_{k+2} \leq \left(\frac{1}{C}\right)^N \|b(V_0)\|_{k+2} + C_{23}N.$$

We tend k to infinity, we get

$$\|b(V_n)\|_\infty \leq \left(\frac{1}{C}\right)^N \|b(V_0)\|_\infty + C_{23}N.$$

This implies that

$$||V_n||_\infty \leq C(M, V_0).$$

Now, we prove the second inequality (4.2) in Theorem 4.1. Toward this end, multiplying (3.1) by V_k and integrating over Ω, we get

$$\int_\Omega (b(V_k) - b(V_{k-1})) \cdot V_k \, dx - \tau \int_\Omega \Delta V_k V_k \, dx = \lambda \int_\Omega \frac{g(V_k) V_k}{\left(\int_\Omega g(V_k) dx\right)^2} \, dx. \tag{4.7}$$

Since

$$\int_\Omega \Psi^*(b(V_k)) \, dx - \int_\Omega \Psi^*(b(V_{k-1})) \, dx \leq (b(V_k) - b(V_{k-1}), V_k)_{L^2(\Omega)}. \tag{4.8}$$

Keeping (4.7)–(4.8) and hypotheses on g in mind, we obtain

$$\int_\Omega \Psi^*(b(V_k)) \, dx - \int_\Omega \psi^*(b(V_{k-1})) \, dx + \tau \|\nabla V_k\|_{L^2(\Omega)}^2 \leq \frac{\tau\lambda}{(\sigma \operatorname{meas}(\Omega))^2} \int_\Omega |g(V_k)| \cdot |V_k| \, dx. \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega g(V_k) |V_k| \, dx &\leq \int_\Omega \frac{\tau\lambda}{(\sigma \operatorname{meas}(\Omega))^2} (|g(V_k) - g(0)| \cdot |V_k| + |g(0)| \cdot |V_k|) \, dx \\
&\leq \frac{\tau\lambda L_1}{(\sigma \operatorname{meas}(\Omega))^2} \left[\int_\Omega |V_k|^2 \, dx + g(0) \int_\Omega |V_k| \, dx\right] \\
&\leq C_{24} \|V_k\|_{L^2(\Omega)} + C_{25} \|V_k\|_{L^1(\Omega)}.
\end{aligned}$$

From the above inequality and Höder's inequality combining with (4.9), it yields that

$$\int_\Omega \Psi^*(b(V_k)) \, dx - \int_\Omega \Psi^*(b(V_{k-1})) \, dx + \tau \|\nabla V_k\|_{L^2(\Omega)}^2 \leq C_{26} \|V_k\|_{L^2(\Omega)}. \tag{4.10}$$

Summing (4.10) from $k=1$ to n, we obtain

$$\sum_{k=0}^{n}\left[\int_{\Omega}\Psi^{*}\left(b\left(V_{k}\right)\right)dx-\int_{\Omega}\Psi^{*}(b\left(V_{k-1}\right))dx\right]+\tau\sum_{k=0}^{n}\left\|\nabla V_{k}\right\|_{2}^{2}\leq C_{26}\sum_{k=0}^{n}\left\|V_{k}\right\|_{2}.$$

Then,

$$\int_{\Omega}\Psi^{*}\left(b\left(V_{n}\right)\right)dx-\int_{\Omega}\Psi^{*}(b\left(V_{0}\right))dx+\tau\sum_{k=0}^{n}\left\|\nabla V_{k}\right\|_{2}^{2}\leq C_{26}\sum_{k=0}^{n}\left\|V_{k}\right\|_{2}.$$

Hence,

$$\int_{\Omega}\Psi^{*}\left(b\left(V_{n}\right)\right)dx+\tau\sum_{k=0}^{n}\left\|\nabla V_{k}\right\|_{2}^{2}\leq C_{26}\sum_{k=0}^{n}\left\|V_{k}\right\|_{2}+\int_{\Omega}\Psi^{*}(b\left(V_{0}\right))dx.$$

Keeping this in mind and using (4.1), there exists a positive constant $C_1(M,V_0)$ such that

$$\int_{\Omega}\Psi^{*}\left(b\left(V_{n}\right)\right)dx+\tau\sum_{k=0}^{n}\left\|\nabla V_{k}\right\|_{2}^{2}\leq C_{1}(M,V_{0}). \tag{4.11}$$

Now, let us prove that

$$\sum_{k=1}^{n}\left\|b(V^{k})-b\left(V^{k-1}\right)\right\|_{2}^{2}\leq C_{2}\left(M,V_{0}\right).$$

To this end, multiplying the first equation of (3.1) by $b(V_k)$ and integrating over Ω, we get that

$$\int_{\Omega}\left(b\left(V_{k}\right)-b\left(V_{k-1}\right)\right)b\left(V_{k}\right)\ dx+\tau\langle-\Delta V_{k},b\left(V_{k}\right)\rangle=\tau\lambda\int_{\Omega}\frac{g\left(V_{k}\right)b\left(V_{k}\right)}{(\int_{\Omega}g(V_{k})dx)^{2}}\ dx.$$

Applying the following identity $2a(a-b)=a^{2}-b^{2}+(a-b)^{2}$ in the first term on the above identity, we get that

$$\begin{aligned}&\left\|b\left(V_{k}\right)\right\|_{2}^{2}-\left\|b\left(V_{k-1}\right)\right\|_{2}^{2}+\left\|b\left(V_{k}\right)-b\left(V_{k-1}\right)\right\|_{L^{2}(\Omega)}^{2}-\tau\int_{\Omega}\Delta V_{k}b\left(V_{k}\right)\\ =&\tau\lambda\int_{\Omega}\frac{g\left(V_{k}\right)b\left(V_{k}\right)}{(\int_{\Omega}g(V_{k})dx)^{2}}\ dx.\end{aligned}$$

Knowing that

$$-\tau\int_{\Omega}\Delta V_{k}b\left(V_{k}\right)\ dx=\tau\int_{\Omega}\nabla V_{k}\nabla\left(b\left(V_{k}\right)\right)\ dx=\tau\int_{\Omega}\left|\nabla V_{k}\right|^{2}b'(V_{k})\ dx\geq 0.$$

It yields that

$$\left\|b\left(V_{k}\right)\right\|_{2}^{2}-\left\|b\left(V_{k-1}\right)\right\|_{2}^{2}+\left\|b\left(V_{k}\right)-b\left(V_{k-1}\right)\right\|_{L^{2}(\Omega)}^{2}\leq\lambda M\int_{\Omega}\frac{g\left(V_{k}\right)b\left(V_{k}\right)}{(\int_{\Omega}g(V_{k})dx)^{2}}\ dx. \tag{4.12}$$

On the other hand, by using hypotheses on g, we get that

$$\begin{aligned}\int_\Omega g\left(V_k\right)\left|b\left(V_k\right)\right| dx &\leq \int_\Omega \left[\left|g\left(V_k\right)-g(0)\right|+|g(0)|\right]\cdot\left|b\left(V_k\right)\right| \ dx\\ &\leq C_\sigma \int_\Omega \left|b\left(V_k\right)\right|^2 dx+|g(0)|\int_\Omega \left|b\left(V_k\right)\right| dx\\ &\leq C_\sigma \left\|b\left(V_k\right)\right\|_2^2+|g(0)|\cdot\left\|b\left(V_k\right)\right\|_{L^1(\Omega)}.\end{aligned}$$

Then,

$$\int_\Omega g\left(V_k\right)\left|b\left(V_k\right)\right| dx \leq C_\sigma \left\|b\left(V_k\right)\right\|_2^2+|g(0)|\cdot\left\|b\left(V_k\right)\right\|_{L^1(\Omega)}. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\begin{aligned}&\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2+\left\|b\left(V_k\right)-b\left(V_{k-1}\right)\right\|_{L^2(\Omega)}^2\\ \leq&\frac{MC_\sigma\lambda}{\left(\sigma \operatorname{meas}(\Omega)\right)^2}\left\|b\left(V_k\right)\right\|_2^2+\frac{M|g(0)|\lambda}{\left(\sigma \operatorname{meas}(\Omega)\right)^2}\left\|b\left(V_k\right)\right\|_{L^1(\Omega)}.\end{aligned}$$

We know that $(V_k)_k$ is a bounded sequence in $L^\infty(\Omega)$. Furthermore, we have that b is a Liptshitz function, then there exists a positive constant C_{27} such that

$$\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2+\left\|b\left(V_k\right)-b\left(V_{k-1}\right)\right\|_{L^2(\Omega)}^2 \leq C_{27}.$$

This implies that

$$\sum_{k=1}^n\left(\left\|b\left(V_k\right)\right\|_2^2-\left\|b\left(V_{k-1}\right)\right\|_2^2\right)+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq nC_{27}.$$

It follows that

$$\left\|b\left(V_n\right)\right\|_2^2-\left\|b\left(V_0\right)\right\|_2^2+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq nC_{27}.$$

Hence,

$$\left\|b\left(V_n\right)\right\|_2^2+\sum_{k=1}^n\left\|b(V_k)-b\left(V_{k-1}\right)\right\|_2^2 \leq nC_{27}+\left\|b\left(V_0\right)\right\|_2^2.$$

Then, there exists a positive constant $C_2\left(M, V_0\right)$ such that

$$\sum_{k=1}^n\left\|b(V^k)-b\left(V^{k-1}\right)\right\|_2^2 \leq C_2\left(M, V_0\right).$$

This concludes the proof. □

5. Error Estimates for Solutions

In this section, we will study the error estimate. If y is a continuous function (resp., summable), defined in $]0, M[$, with values in $H^{-1}(\Omega)$ or $L^2(\Omega)$ or $H_0^1(\Omega)$, we define $y^n=y\left(t^n, \cdot\right)$, $\bar{y}^n=(1/\tau)\int_{J_n} y(t, \cdot)dt$, $\bar{y}^0=y^0=y(0, \cdot)$ and the error $e_v=v(t)-V_n$, $e_b=b(v(t))-b\left(V_n\right)$ for all $t \in I^n, n=1, \ldots, N$. For notational simplicity, we denote $u_n=b\left(v_n\right)$ and $U_n=b\left(V_n\right)$ and introduce the local errors e_v^n and e_b^n, defined by

$$e_v^n=\bar{v}_n-V_n, \quad e_b^n=b\left(v_n\right)-b\left(V_n\right)=u_n-U_n.$$

Let's $(-\Delta)^{-1}$ be the Green operator satisfying

$$\left\langle \nabla(-\Delta)^{-1}v, \nabla w \right\rangle = \langle v, w\rangle, \quad \text{for all } v \in H_0^1(\Omega), w \in H^{-1}(\Omega). \tag{5.1}$$

By arguments inspired from [4, 13, 28], we get the following result hold.

Theorem 5.1. *Assume that the hypotheses* (H1)-(H5) *holds. Then,*

$$\|e_b^m\|_{H^{-1}(\Omega)} \leq C_{28}\tau^{1/2}, \tag{5.2}$$

$$\|e_b^n\|^2_{L^\infty(0,M,H^{-1}(\Omega))} + \int_0^M \|e_b^n\|_2^2\, dt \leq C_{29}\tau, \tag{5.3}$$

$$\left\| \nabla \int_0^{t^m} e_v dt \right\|^2_{L^2(\Omega)} \leq C_{30}\tau^{1/4}.$$

Proof. The discrete problem (3.1) has the following variational formulation

$$\begin{aligned} &\langle b(v_n) - b(v_{n-1}), \varphi\rangle_{L^2(\Omega)} + \tau \langle \nabla v_n, \nabla\varphi\rangle_{L^2(\Omega)} \\ =&\frac{\tau\lambda}{(\int_\Omega (g(v_n)dx)^2} \langle g(v_n), \varphi\rangle, \quad \text{for all } \varphi \in H_0^1(\Omega). \end{aligned} \tag{5.4}$$

Integrating the continuous problem (1.2) over J_n, we obtain that its solution verifies the following variational identity with respect to the notation introduced in the introduction

$$b(V_n) - b(V_{n-1}) - \Delta\left(\int_{J_n} V_n dt\right) = \lambda \int_{J_n} \frac{g(V_n)}{\left(\int_\Omega g(V_n)dx\right)^2} dt.$$

Then,

$$b(V_n) - b(V_{n-1}) - \tau\Delta\bar{V}_n = \lambda \int_{J_n} \frac{g(V_n)}{\left(\int_\Omega g(V_n)\, dx\right)^2} dt. \tag{5.5}$$

Multiplying (5.5) by φ and integrating over Ω, we get

$$\langle b(V_n) - b(V_{n-1}), \varphi\rangle - \tau\left\langle \Delta\bar{V}^n, \varphi\right\rangle = \left\langle \lambda \int_{J_n} \frac{g(V_n)}{\left(\int_\Omega g(V_n)\, dx\right)^2} dt, \varphi \right\rangle,$$

for all $\varphi \in H_0^1(\Omega)$. From Green's formula, we obtain

$$\langle b(V_n) - b(V_{n-1}), \varphi\rangle + \tau\left\langle \nabla\bar{V}_n, \nabla\varphi\right\rangle = \lambda\left\langle \int_{J_n} \frac{g(V_n)}{(\int_\Omega (g(V_n)dx)^2} dt, \varphi \right\rangle,$$

for all $\varphi \in H_0^1(\Omega)$. Then,

$$\langle b(V_n) - b(V_{n-1}), \varphi\rangle + \tau\left\langle \nabla\bar{V}_n, \nabla\varphi\right\rangle = \frac{\tau\lambda}{(\int_\Omega g(V_n)dx)^2} \langle \overline{g(V_n)}, \varphi\rangle, \tag{5.6}$$

for all $\varphi \in H_0^1(\Omega)$. Substracting (5.6) from (5.4), we get

$$\begin{aligned} &\langle b(v_n) - b(v_{n-1}) - (b(V_n) - b(V_{n-1})), \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \tau\left\langle \nabla\bar{V}_n - \nabla v_n, \nabla\varphi\right\rangle_{L^2(\Omega)} \\ =&\tau\lambda\left\langle \frac{g(v_n)}{(\int_\Omega (g(v_n)dx)^2} - \frac{\overline{g(V_n)}}{(\int_\Omega g(V_n)dx)^2}, \varphi \right\rangle. \end{aligned}$$

Then,

$$\begin{aligned}&\sum_{n=1}^{m}\left\langle e_b^n-e_b^{n-1},\varphi\right\rangle+\tau\sum_{n=1}^{m}\left(\nabla e_v^n,\nabla\varphi\right)\\ \leq& C_{31}\tau\left|\sum_{n=1}^{m}\left(\overline{g(v)}^n-g\left(V_n\right),\varphi\right)\right|+C_{32}\tau\left|\sum_{n=1}^{m}\left(g\left(V_n\right),\varphi\right)\right|.\end{aligned}\tag{5.7}$$

We set

$$\begin{aligned}I_1^n&=\left\langle b\left(v_n\right)-b\left(V_n\right)-\left(b\left(v_{n-1}\right)-b(V_{n-1})\right),\varphi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\&=\left\langle e_b^n-e_b^{n-1},\varphi\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},\\I_2^n&=\tau\langle\nabla e_v^n,\nabla\varphi\rangle_{L^2(\Omega)},\\I_3^m&=C_{31}\tau\left|\sum_{n=1}^{m}\left(\overline{g(v)}^n-g\left(V_n\right),\varphi\right)\right|+C_{32}\tau\left|\sum_{n=1}^{m}\left(g\left(V_n\right),\varphi\right)\right|=I_{3,1}^m+I_{3,2}^m.\end{aligned}$$

Choosing $\varphi=\nabla(-\Delta)^{-1}e_b^n$ and adding from $n=1$ to m with $m\leq N$, we have

$$\begin{aligned}\sum_{n=1}^{m}I_1^n=&\sum_{n=1}^{m}\left\langle\nabla(-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right],\nabla(-\Delta)^{-1}e_b^n\right\rangle_{L^2(\Omega)}\\=&\frac{1}{2}\left\langle\nabla(-\Delta)^{-1}e_b^m,\nabla(-\Delta)^{-1}e_b^m\right\rangle_{L^2(\Omega)}\\&+\frac{1}{2}\sum_{n=1}^{m}\left\langle\nabla\left((-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right]\right),\nabla\left((-\Delta)^{-1}\left[e_b^n-e_b^{n-1}\right]\right)\right\rangle\\=&\frac{1}{2}\left\|e_b^m\right\|_{H^{-1}(\Omega)}^2+\frac{1}{2}\sum_{n=1}^{m}\left\|e_b^n-e_b^{n-1}\right\|_{H^{-1}(\Omega)}^2.\end{aligned}$$

From (5.1), we get

$$I_2^n=\tau\langle\nabla e_v^n,\nabla\varphi\rangle_{L^2(\Omega)}=\tau\left\langle\nabla e_v^n,\nabla(-\Delta)^{-1}e_b^n\right\rangle_{L^2(\Omega)}=\left\langle e_v^n,e_b^n\right\rangle_{L^2(\Omega)}.$$

Again adding from $n=1$ to m with $m\leq N$, we obtain

$$\begin{aligned}\sum_{n=1}^{m}I_2^n=&\tau\sum_{n=1}^{m}\left\langle e_v^n,e_b^n\right\rangle_{L^2(\Omega)}\\=&\sum_{n=1}^{m}\left\langle\int_{J_n}\left(v(t)-V_n\right)dt,u(t)-U_n\right\rangle_{L^2(\Omega)}\\&+\sum_{n=1}^{m}\int_{J_n}\left\langle v(t)-V_n,u_n-u(t)\right\rangle_{L^2(\Omega)}dt\\=&I_4^m+I_5^m.\end{aligned}$$

Since b is an increasing function, it follows that

$$I_4^m=\sum_{n=1}^{m}\int_{J_n}\left\langle v(t)-v_n,b(v(t))-b(v_n)\right\rangle_{L^2(\Omega)}\geq 0.$$

Now, we split I_5^m into two parts and we estimate each one separately

$$\begin{aligned} I_5^m =& \sum_{n=1}^{m} \int_{J_n} \langle v(t) - V_n, u_n - u(t) \rangle_{L^2(\Omega)} \, dt \\ =& \sum_{n=1}^{m} \int_{J_n} \langle v(t), u(t) - U_n \rangle_{L^2(\Omega)} \, dt \\ & - \sum_{n=1}^{m} \int_{J_n} \langle V_n, u_n - u(t) \rangle_{L^2(\Omega)} \, dt \\ =& I_6^m + I_7^m. \end{aligned}$$

We start by estimating I_6^m

$$\begin{aligned} |I_6^m| &= \left| \sum_{n=1}^{m} \int_{J_n} \left\langle v(t), \int_t^{t^n} \frac{\partial u}{\partial s} ds \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \right| \\ &\leq \sum_{n=1}^{m} \int_{J_n} \left(\int_t^{t^n} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)} ds \right) \cdot \|v(t)\|_{H_0^1(\Omega)} \, dt \\ &\leq \sum_{n=1}^{m} \int_{J_n} \left(\int_t^{t^n} \left\| \frac{\partial u}{\partial s} \right\|_{H^{-1}(\Omega)} ds \right) \cdot \|v(t)\|_{H_0^1(\Omega)} \, dt \\ &\leq \tau \left\| \frac{\partial b(u)}{\partial s} \right\|_{L^2(0,t^m H^{-1}(\Omega))} \leq C_{33} \tau. \end{aligned} \tag{5.8}$$

In similar manner, to derive an estimate of I_7^m, we use inequality (4.2) to get

$$\begin{aligned} |I_7^m| &= \left| \sum_{n=1}^{m} \int_{J_n} \left\langle V_n, \int_t^{t^n} \frac{\partial u}{\partial s} ds \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \right| \\ &\leq \tau \left\| \frac{\partial b(u)}{\partial s} \right\|_{L^2(0,t^m;H^{-1}(\Omega))} \left(\tau \sum_{n=1}^{m} \|V_n\|_{H_0^1(\Omega)}^2 \right)^{1/2} \\ &\leq C_{34} \tau. \end{aligned} \tag{5.9}$$

From (5.8)–(5.9), we get that

$$|I_5^m| \leq C_{35} \tau. \tag{5.10}$$

Next, we estimate $I_{3,1}^m$ by using Hölder's, Young's inequalities and hypotheses on g

$$\begin{aligned} \left| I_{3,1}^m \right| &\leq \left| \sum_{n=1}^{m} \left\langle \int_{J_n} \left[g(v) - g\left(V_n\right) \right] dt, (-\Delta)^{-1} \left(e_b^n \right) \right\rangle \right| \\ &\leq C_{36} \tau^{1/2} \sum_{n=1}^{m} \left(\int_{J_n} \|g(v) - g\left(V_n\right)\|_2^2 \, dt \right)^{1/2} \|e_b^n\|_{H^{-1}(\Omega)} \\ &\leq \eta \sum_{n=1}^{m} \left(\int_{J_n} \|g(v) - g\left(V_n\right)\|_2^2 \, dt \right) + \frac{C_{37}}{\eta} \tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 \end{aligned}$$

$$\leq C_{38}\eta \sum_{n=1}^{m} \left(\int_{J_n} \|b(v) - b(V_n)\|_2^2 \, dt \right) + \frac{C_{37}}{\eta} \tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

Knowing that b is an increasing Lipschitz function, we get

$$|I_{3,1}^m| \leq C_{39}\eta \sum_{n=1}^{m} \int_{J_n} \int_{\Omega} (b(v) - b(V_n))(v - V_n) dx dt + c(\eta)\tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

Moreover, we have

$$|I_{3,2}^m| \leq C\tau + \tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 .$$

From (5.7), we get

$$\sum_{n=1}^{m} I_1^n + \sum_{n=1}^{m} I_2^n \leq I_{3,1}^m + I_{3,2}^m. \tag{5.11}$$

It follows that

$$\begin{aligned} &\sum_{n=1}^{m} I_1^n - C\tau + \sum_{n=1}^{m} \int_{J_n} \int_{\Omega} (b(v) - b(V_n))(v - V_n) dx dt \\ \leq & C_{39}\eta \sum_{n=1}^{m} \int_{J_n} \int_{\Omega} (b(v) - b(V_n))(v - V_n) dx dt + c(\eta)\tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 . \end{aligned}$$

Then choosing η small enough to absorb the first term of the right hand side of the above estimate, we get that

$$\begin{aligned} &\frac{1}{2} \|e_b^m\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \sum_{n=1}^{m} \left\| e_b^n - e_b^{n-1} \right\|_{H^{-1}(\Omega)}^2 + \sum_{n=1}^{m} \int_{J_n} \int_{\Omega} (b(v) - b(V_n))(v - V_n) dx dt \\ \leq & C\tau + C_{40}\tau \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 . \end{aligned} \tag{5.12}$$

Putting

$$y^m = \sum_{n=1}^{m} \|e_b^n\|_{H^{-1}(\Omega)}^2 , \quad y^m - y^{m-1} \leq C \cdot r + C_{40} \cdot \tau \cdot y^m$$

and applying discrete Gronwall's inequality, we get that $y^m \leq C(T)$

$$\|e_b^m\|_{H^{-1}(\Omega)} \leq C_{41}\tau^{1/2},$$

which proves the inequality (5.2). On the other hand, we have

$$\sup_{t \in (0,t_m)} \|e_b^n(t)\|_{H^{-1}(\Omega)} - c_{41}\tau^{1/2} \leq \max_{1 \leq n \leq m} \|e_b^n(t^n)\|_{H^{-1}(\Omega)} = \max_{1 \leq n \leq m} \|e_b^n\|_{H^{-1}(\Omega)} .$$

Thus,

$$\|e_b^n\|_{L^\infty(0,T;H^{-1}(\Omega))} - C_{41}\tau^{1/2} \leq \max_{1 \leq n \leq m} \|e_b^n\|_{H^{-1}(\Omega)} .$$

From the above inequality, we obtain

$$\|e_b\|_{L^\infty(0,T;H^{-1}(\Omega))}^2 + \sum_{n=1}^{N} \int_{J_n} \left(b(v(t)) - b\left(V_n\right), v(t) - V_n \right)_{L^2(\Omega)} dt \leq C_{42}\tau.$$

That achieve the proof of the second point in Theorem 5.3.

Now, return to the inequality (5.7) and taking $\varphi = \sum_{n=1}^{m} e_v^n = \sum_{n=1}^{m}(\bar{v}_n - V_n)$ as test function, to get

$$\begin{aligned}
&\tau \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dxdt + \tau^2 \left\| \sum_{n=1}^{m} \nabla (\bar{v}_n - V_n) \right\|^2_{L^2(\Omega)} \\
&\leq C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} (\overline{g(v)})^n - g(V_n) \right) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \Bigg| \\
&\quad + C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\tau \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dxdt + \left\| \nabla \int_0^{t^m} e_v dt \right\|^2_{L^2(\Omega)} \\
&\leq C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} (\overline{g(v)})^n - g(V_n) \right) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \Bigg| \\
&\quad + C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right|.
\end{aligned}$$

Then,

$$\begin{aligned}
\tau^2 \left\| \sum_{n=1}^{m} \nabla (\bar{v}_n - V_n) \right\|^2_{L^2(\Omega)} &= \left\| \nabla \int_0^{t^m} e_v dt \right\|^2_{L^2(\Omega)} \\
&\leq \tau \left| \int_\Omega (b(v) - b(V_n)) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right| \\
&\quad + C_{43}\tau^2 \left| \int_\Omega \sum_{n=1}^{m} \left(\overline{g(v)})^n - g(V_n) \right) \left(\sum_{n=1}^{m} (\bar{v}_n - V_n) \right) dx \right| \\
&\quad + C_{44}\tau^2 \left| \sum_{n=1}^{m} \left(g(V_n), \sum_{n=1}^{m} (\bar{v}_n - V_n) \right) \right| \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

We begin by estimating J_2

$$\begin{aligned}
J_2 &\leq \left(\int_\Omega \left(\sum_{n=1}^{m} \int_{J_n} (g(v) - g(V_n))\, dt \right)^2 dx \right)^{1/2} \\
&\quad \times \left(\int_\Omega \left(\sum_{n=1}^{m} \int_{J_n} (v(t) - V_n)\, dt \right)^2 dx \right)^{1/2} \\
&\leq T^2 \left(\sum_{n=1}^{m} \int_{J_n} (||g(v) - g(V_n)||_2)^2 dt \right)^{1/2} \left(\sum_{n=1}^{m} \int_{J_n} ||v(t) - V_n||_2^2 dt \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq T^2 \left(\sum_{n=1}^{m} \int_{J_n} ||g(v) - g(V_n)||_2^2 dt \right)^{1/2} \\
&\quad \times \left(2 \|v\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} + 2\tau \sum_{n=1}^{m} ||V_n||_2^2 \right)^{1/2} \\
&\leq C_{45} \tau^{1/2}.
\end{aligned}$$

The above inequality follows by using respectively the L^∞-estimate of $v(t), V^n$ and the error bound given in Theorem 4.1. Arguing as in the previous estimate, we get

$$J_3 \leq T^2 \left(\sum_{n=1}^{m} \int_{I_n} ||g(V_n)||_2^2 dt \right)^{1/2} \left(2 \|v\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} + 2\tau \sum_{n=1}^{m} \|V_n\|_2^2 \right)^{1/2}.$$

Combining (4.2) with the estimates above and using hypothesis (H1), to obtain $J_3 \leq C_{46}\tau^{1/2}$ and

$$J_1 \leq \|e_b^m\|_{H^{-1}(\Omega)} \left(\sum_{n=1}^{m} \int_{I^n} \|v(t)\|_{H_0^1(\Omega)} \, dt + \tau \sum_{n=1}^{m} \|V_n\|_{H_0^1(\Omega)} \right). \tag{5.13}$$

Finally, collecting these results, it follows that

$$\left\| \nabla \int_0^T e_v^n dt \right\|_2^2 \leq C_{30} \tau^{1/2}.$$

By accumulating all of the previous results, the proof of Theorem 5.1 is completed. □

Remark 5.1. As an example of the application, we can take $b(v) = v$. If we then choose a field that satisfies the hypotheses (H1)-(H5), we will apply Theorem 3.1 to obtain the existence of a solution to the semi-discrete problem associated with the continuous problem (1.2). This result is confirmed by the result obtained in [14] with the appropriate value of "b". We can also consider the following example:

$$g_1(x) = \sqrt{C_\sigma^2 x^2 + 5},$$

and the function $b(x) = x$. It can be shown that g_1 and b satisfy assumptions (H1) to (H5).

We now exhibit an example showing that many functions satisfy (H1)-(H4).

Example 5.1. Consider b and g two functions such that:

- $b(x) = kx$ (increasing, Lipschitz, $b(0) = 0$);
- $g(x) = \sigma + \frac{L_1}{2} \tanh(x)$ (bounded below by σ, Lipschitz).

Verification.

(a) (H1) g is Lipschitz because $\tanh(x)$ has derivative $\operatorname{sech}^2(x) \leq 1$, so $L_1 = \frac{L_1}{2} \cdot 1$.

(b) (H2) $\tanh(x) \in]-1, 1[$, so $g(x) \geq \sigma - \frac{L_1}{2}$. To ensure $g(x) \geq \sigma$, we must pick L_1 small enough (e.g., $L_1 \leq 2\sigma$).

(c) (H3) b is increasing, Lipschitz, and $b(0) = 0$.

(d) (H4)

i) Since $b(x) - b(y) = k(x - y)$, we have:

$$|g(x) - g(y)| = \frac{L_1}{2}|\tanh(x) - \tanh(y)| \leq \frac{L_1}{2}|x - y|.$$

ii) Meanwhile, $|b(x) - b(y)| = k|x - y|$.

iii) Thus, $|g(x) - g(y)| \leq \left(\frac{L_1}{2k}\right)|b(x) - b(y)|$.

Choose k small enough to ensure $\frac{L_1}{2k} < \frac{(\sigma \cdot \text{meas}(\Omega))^2}{\tau\lambda}$.

Observe that all hypotheses of the preceding theorem are satisfied, from which the error estimates immediately follow.

6. Conclusion and Perspective

In this work, we have shown the existence and uniqueness of a solution to the steadystate problem, which is the time discretization for the continuous problem (1.2). We have also proved some stability results and error estimates for a family of time discretization schemes. We would like to point out that this study is accompanied by effective numerical computations.

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NUMERICAL MODELING OF LAMINAR CASSON FLUID FLOW IN ECCENTRIC ANNULAR ENTRANCE REGIONS

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Abstract. The laminar flow of an incompressible Casson fluid in the entrance region of an eccentric annulus was analyzed numerically. Such asymmetric annular structures are critical in applications like drilling, heat exchangers and biomedical simulations. The Casson fluid model, which is relevant for biological (e.g., blood), industrial (e.g., drilling fluids) and environmental (e.g., mudflows) systems, was investigated using the Finite Difference Method (FDM) with variable mesh size. The Navier-Stokes equations in bipolar coordinates were solved to calculate pressure gradients and axial velocities for varying radius ratios S, relative eccentricities e and yield stress values τ_D. The results for the entrance region, which were validated against the literature on fully developed flow, showed an accuracy of 97.89% to 99.68%. The entrance region pressure gradient exceeded the fully developed zone by 6.16 to 50.52 times, while the axial velocity ranged from 43.75% to 53.65% of its fully developed value. These findings underscore the importance of entrance region dynamics in engineering design.

1. Introduction

In heat exchangers, well drilling, and plastic extrusion, flow in the entrance region of concentric and/or eccentric annuli is important from an industrial standpoint. In engineering, however, the amount of power required to pump fluids through such systems is critical. The two factors that determine how much power is required to pump an incompressible fluid are the volumetric flow rate and the pressure drop. However, the pressure drop in the entrance region is greater than in the fully developed zone due to the acceleration of the fluid motion and the higher velocity gradients at the

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wall within this region. Since the fully developed pressure drop for the entire flow zone is used in the design calculations, the pumping power is underestimated. As a result, estimating the excess pressure gradient at the entrance region relative to the fully developed one is critical. In addition, the Casson fluid model is often used to describe the flow behavior of non-Newtonian fluids with yield stress, such as blood, chocolate, and certain industrial slurries [1]. It finds applications in biomedical engineering to model blood flow in vessels [2], in food processing to optimize the handling of viscous products such as chocolate [3] and in drilling operations to predict the behavior of muds and cement pastes [4]. This model helps engineers design efficient systems by accurately capturing the fluid's resistance to flow until a critical stress is reached [5].

However, numerous studies on laminar incompressible flow have focused on the entrance region of the concentric annular duct. For instance, an early work conducted theoretical studies on the velocity distribution, pressure drop and hydrodynamic entry length of pipes with annular space [6]. In addition, using the Casson stress-strain relation as a guide, the Runge-Kutta method was used to ascertain how blood flows in the entrance region of annuli [1]. The Finite Difference Method (FDM) was used to solve the axial, incompressible, isothermal, laminar and steady flow of a power-law fluid in a concentric annulus [3]. The entrance region for Herschel-Bulkley and Bingham non-Newtonian fluids in concentric annuli with a rotating inner wall and a stationary outer one was studied using the FDM [7, 8]. Recent studies have extended this focus to include advanced non-Newtonian models, such as the exploration of temperature-dependent thermophysical properties under bioconvection effects [9] and heat transport in Casson fluid flows within Darcy-Forchheimer media [10]. In addition, the entrance effects in concentric rings with inner ring rotation for the blood model Casson non-Newtonian fluid were also investigated [2]. They also investigated how the Casson fluid affects heat and mass transmission as the inner walls of the concentric annuli moved [11]. The non-Newtonian Bingham fluid was also investigated in the entry region of the annular space between two rotating coaxial cylinders [12].

On the other hand, several researchers have studied the problem of eccentric annuli in detail. Examples include the study of non-Newtonian fluid in eccentric annuli for highpolymer aqueous solutions of CMC, HEC, and MC [13]. The yield-power law fluids'velocity and viscosity profiles were derived using FDM by solving the governing laminar flow equation [4]. In their innovative method [14], the laminar eccentric annular flow of non-Newtonian fluids was investigated, where the eccentric annulus is represented by an infinite number of concentric annuli with varied outer radii. A fully developed laminar flow of a Newtonian fluid across an eccentric annulus was investigated computationally and experimentally [15]. In addition, when Newtonian and non-Newtonian fluids were examined in an eccentric annulus, the case of inner cylinder rotation was also taken into consideration [16].

Current research has further explored complex flow phenomena in eccentric geometries, including bioconvection flows with triple stratification [17], irreversibility analysis in porous media [18], and radiative mixed convection in hybrid nanofluids

[19]. Furthermore, using an approximation of the velocity distribution, the eccentric annulus was treated as a slit with variable height in order to assess the loss coefficient for power-law and Bingham fluids [20]. Furthermore, the eccentric or concentric annular duct provided a computational solution to the fully developed Casson model problem [5]. The FDM was also used to solve the hydrodynamic entrance region problem for the eccentric annulus [21]. Other references have investigated related aspects, such as non-Newtonian flow properties [22], slot-flow approximations for eccentric annuli [23] and viscoplastic fluid dynamics in annular geometries [24]. More recent advancements in non-Newtonian fluid dynamics, such as non-Fourier diffusion models [25], entropy generation in 3D flows [26], magnetohydrodynamic Sutterby nanofluid flows [27], stratified Carreau nanofluid studies [28], the dynamics of chemically reactive magneto-hybrid nanomaterials with heat radiation in porous media [29] and radiative convective flows with magnetic fields [30], underscore the ongoing relevance of such studies.

Therefore, this study attempts to fill a gap in the literature by proposing a numerical solution to the Casson model's entrance problem in eccentric or concentric annuli ducts. The fluid velocity and pressure gradients were evaluated at the entrance and fully developed regions and the ratios between them were calculated. In addition, comparisons of the calculated values with those found in the literature revealed a high degree of consistency. In contrast to previous studies that focused primarily on the fully developed flow or simpler concentric geometries, this work introduces a novel approach by tackling the complex entrance region dynamics of Casson fluid flow within eccentric annuli using a finite difference method with a variable mesh size. This method, combined with the adoption of bipolar coordinates, enables a more precise resolution of the non-uniform flow fields inherent to eccentric configurations. This challenge was often overlooked in previous studies that focused on Newtonian fluids or uniform annuli. Furthermore, the study's emphasis on the Casson fluid, a non-Newtonian model with yield stress relevant for industrial and biological applications, distinguishes it from traditional analyses, offering new insights into the pressure and velocity behaviors under varying radius ratios (S), relative eccentricities (e) and yield stress values (τ_D). By quantifying these parameters in the entrance region, which has been insufficiently explored in previous literature, this research provides a significant advance. When validated against established work, accuracy rates of 97.89% to 99.68% are achieved for fully developed regions, increasing the reliability of design predictions for engineering systems.

Furthermore, this paper is divided into five sections. Section 2 provides the problem formulation and solution, while Section 3 introduces the numerical model and application. Sections 4 and 5 include the results, along with discussion and conclusion, respectively.

2. Problem Formulation

2.1. **Governing Equations.** Figure 1 shows the geometry of the problem. The eccentric annulus comprises two cylindrical surfaces with inner radius R_1 and outer radius R_2, which are offset by the eccentricity d. It does not involve rotating disks or mechanical transmission; the study focuses on the transition of steady laminar flow from the entrance region to the fully developed region. In the entrance region of an eccentric or concentric annulus, the fluid is assumed to be laminar, steady, incompressible and non-Newtonian, in particular a Casson fluid with constant physical properties. The continuity (2.1) and momentum equations (2.2)–(2.4) in Cartesian

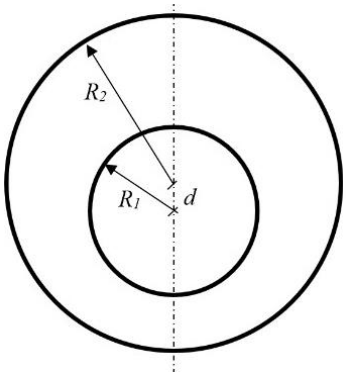


Figure 1. Geometry of studied eccentric annulus

coordinates are transformed into bipolar coordinates for the eccentric geometry, with the focus on the axial flow (z-direction) as the primary, with transverse components (u, v) secondary per boundary layer assumptions [21, 31]. Neglecting the body forces, the continuity equation is represented as follows

$$\nabla \cdot \mathbf{w} = 0, \tag{2.1}$$

while the momentum equations are given by

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right], \tag{2.2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right], \tag{2.3}$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}\right], \tag{2.4}$$

where the apparent viscosity, μ, is given by:

$$\mu = \left(K_c + \left(\frac{\tau_0}{\sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}} \right)^{\frac{1}{2}} \right)^2, \tag{2.5}$$

where K_c is the model viscosity and τ_0 is the yield stress.

Considering the no-slip condition at the walls, and due to symmetry, the solution can be evaluated for only one-half of the eccentric annulus. This yields the following

$$\begin{aligned} &\text{Inner conditions:} \quad w = w_e, \quad u = v = 0, \\ &\text{Boundary conditions:} \quad w = u = v = 0, \\ &\text{Symmetry lines:} \quad \frac{\partial w}{\partial n} = \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0. \end{aligned}$$

2.2. **Bipolar Coordinates System Formulation.** It is convenient to represent the governing equations in the bipolar coordinate system, (ξ, η, z) when analyzing flow in eccentric annuli. The eccentric annulus, which consists of sets of orthogonal circles, is represented by the two coordinates (ξ, η), while the z-axis is perpendicular to the paper's plane. The walls of the eccentric annulus are represented by $\xi = \xi_1$ and $\xi = \xi_2$. Moreover, the relation between the rectangular coordinates x, y, z and the bipolar coordinates ξ, η, z is given by [22] as:

$$\begin{aligned} &\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a} - \coth\xi\right)^2 = \operatorname{csch}^2\xi, \quad \text{for } -\infty < \xi < +\infty, \\ &\left(\frac{x}{a} - \cot\eta\right)^2 + \left(\frac{y}{a}\right)^2 = \csc^2\eta, \quad \text{for } 0 < \eta < 2\pi, \\ &x = \frac{a\sinh\xi}{\psi}, \quad y = -\frac{a\sin\eta}{\psi}, \end{aligned}$$

where

$$\begin{aligned} &z = za, \quad R_1\sinh\xi_1 = R_2\sinh\xi_2, \quad \psi = \cosh\xi - \cos\eta, \\ &\xi_1 = \cosh^{-1}\left(\frac{1 - S^2 - M^2}{2SM}\right), \\ &\xi_2 = \cosh^{-1}\left(\frac{1 - S^2 + M^2}{2M}\right). \end{aligned} \tag{2.6}$$

The relative displacement $M = \frac{d}{R_2}$ or the relative eccentricity $e = \frac{d}{R_2 - R_1}$, with the radius ratio $S = \frac{R_1}{R_2}$, are the two factors that characterize the annulus. Using the values of S, e, and M, which are between 0 and 1.0, the preceding 2.6 can be used to calculate ξ_1 and ξ_2. Due to the problem's symmetry, only one half, which lies between $\eta = 0$ and $\eta = \pi$, is taken into consideration. The bipolar coordinate system is explained in further detail in [31, 32], where the continuity equation and the three

momentum equations can be expressed as:

$$\frac{\partial}{\partial\xi}(hu)+\frac{\partial}{\partial\eta}(hv)+\frac{\partial}{\partial z}(h^2w)=0, \tag{2.7}$$

$$\begin{aligned}
\rho\left(\frac{u}{h}\cdot\frac{\partial u}{\partial\xi}+\frac{v}{h}\cdot\frac{\partial u}{\partial\eta}+w\frac{\partial u}{\partial z}\right)=&-\frac{\partial p}{\partial\xi}+\frac{1}{h}\cdot\frac{\partial}{\partial\xi}\left[2\frac{\mu}{h^2}\cdot\frac{\partial(hu)}{\partial\xi}\right]+\frac{1}{h}\cdot\frac{\partial}{\partial\eta}\left[\frac{\mu}{h^2}\left(\frac{\partial(hv)}{\partial\xi}+\frac{\partial(hu)}{\partial\eta}\right)\right]\\
&+\frac{1}{h}\cdot\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial\xi}+\frac{\partial(hu)}{\partial z}\right)\right],\\
\rho\left(\frac{u}{h}\cdot\frac{\partial v}{\partial\xi}+\frac{v}{h}\cdot\frac{\partial v}{\partial\eta}+w\frac{\partial v}{\partial z}\right)=&-\frac{\partial p}{\partial\eta}+\frac{1}{h}\cdot\frac{\partial}{\partial\xi}\left[\frac{\mu}{h^2}\left(\frac{\partial(hv)}{\partial\xi}+\frac{\partial(hu)}{\partial\eta}\right)\right]+\frac{1}{h}\cdot\frac{\partial}{\partial\eta}\left[2\frac{\mu}{h^2}\cdot\frac{\partial(hv)}{\partial\eta}\right]\\
&+\frac{1}{h}\cdot\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial\eta}+\frac{\partial(hv)}{\partial z}\right)\right],\\
\rho\left(\frac{u}{h}\cdot\frac{\partial w}{\partial\xi}+\frac{v}{h}\cdot\frac{\partial w}{\partial\eta}+w\frac{\partial w}{\partial z}\right)=&-\frac{\partial p}{\partial z}+\frac{1}{h^2}\cdot\frac{\partial}{\partial\xi}\left[\mu\left(\frac{\partial w}{\partial\xi}+\frac{\partial(hu)}{\partial z}\right)\right]\\
&+\frac{1}{h^2}\cdot\frac{\partial}{\partial\eta}\left[\mu\left(\frac{\partial(hv)}{\partial z}+\frac{\partial w}{\partial\eta}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}\right],
\end{aligned}$$

where

$$\mu=\left(K_c+\left(\frac{\tau_0 h}{\sqrt{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2}}\right)^{1/2}\right)^2,\quad h=\frac{a}{\cosh\xi-\cos\eta}. \tag{2.8}$$

Hughes and Gaylord [33] provided the general orthogonal curvilinear coordinate formulas along with the required bipolar coordinate system for (2.7) to (2.8) and (2.5). Feldman et al. [21] proposed an order of magnitude analysis that follows the boundary layer assumptions assuming that the length of the entrance region to the duct hydraulic diameter ratio is significantly greater than 1. As a result, the model assumes a uniform entrance velocity profile and disregards the transverse pressure gradient. Additionally, the continuity equation is unaffected by the order of magnitude analysis, and all three scalar components are replaced by a simplified form of the axial momentum equation. In bipolar coordinates, the continuity equation and the three reduced momentum equations are

$$\frac{\partial}{\partial\xi}(hu)+\frac{\partial}{\partial\eta}(hv)+\frac{\partial}{\partial z}(h^2w)=0, \tag{2.9}$$

$$\rho\left(\frac{u}{h}\cdot\frac{\partial w}{\partial\xi}+\frac{v}{h}\cdot\frac{\partial w}{\partial\eta}+w\frac{\partial w}{\partial z}\right)=-\frac{dp}{dz}+\frac{1}{h^2}\left[\frac{\partial}{\partial\xi}\mu\frac{\partial w}{\partial\xi}+\frac{\partial}{\partial\eta}\mu\frac{\partial w}{\partial\eta}\right]. \tag{2.10}$$

The pressure gradient, $\frac{dp}{dz}$, is a total derivative and a function of only z, whereas u, v, and w in (2.9) and (2.10) are functions of ξ, η, and z, respectively. Given that the pressure gradient is only one of three unknown of velocitycomponents, a comprehensive mathematical model cannot be formed by (2.9) and (2.10) only. Similar challenges existed when Carlson and Hornbeck [34] solved the hydrodynamic entrance region of

a square-sectioned duct. By assuming that all transverse flow was conducted along straight lines that crossed the center of the duct, they were able to create a relationship between the two transverse velocity components in the square cross-section. As stated in Feldman et al. [21], a brief examination of the process that causes transverse flow in an annular duct provides important insights into its behavior. When a flow with a uniform axial profile and no transverse components enters a circular tube, the flow near the wall is retarded, and the transverse flow is directed radially and symmetrically toward the center. Similarly, external axial flow along a cylinder's lateral surface causes radially symmetrical transverse flow outward from the cylinder's center. In a concentric annulus formed by inserting a cylinder inside the tube, the two transverse flows collide and vanish along a surface of revolution. If we consider two boundary layers growing concurrently along the two walls, then this surface lies between the two boundary layers, and the cylindrical surface with constant radius serves as a good approximation. The ridge of maximum axial velocities for fully developed flow is cut by this cylinder. This representation is accurate at axial points far from the duct inlet and reasonable near the duct inlet.

Similar mechanisms operate in the eccentric case as well. The expected paths of the transverse flow were described in Feldman et al. [21]. It would follow the radial lines originating from its surface if the transverse flow of either surface could act independently of the other one. The transverse flows of each surface, however, are forced to curve away from their respective radial lines by the presence of the other surface. As they approach each other, the two flows follow paths that tend to transfer flow from the narrower side of the annulus to the wider side of the annulus. The cylindrical surface that intersects the ridge of the maximum axial velocity for the fully developed flow can be used to approximate the surface along which the two transverse flows merge under conditions similar to those proposed for the concentric case. This surface can be represented mathematically by one with a constant radius, R^*. However, as absolute eccentricity increases, this curve significantly deviates from a constant radius, adopting an egg-shaped form. Fortunately, as will be shown later in this paper, the solution's sensitivity to the transverse flow prescription is negligible. For a concentric annulus, the classical closed-form representation of R^* is

$$R^* = R_2 \frac{\sqrt{1-S^2}}{\ln(1/S^2)}, \quad \text{for } e \neq 0.9,\ S \neq 0.1,$$

$$R^* = \frac{R_2}{2}, \quad \text{for } e = 0.9,\ S = 0.1.$$

For all but the most eccentric geometry ($e = 0.9$, $S = 0.1$), this representation is also used to approximate R^*. The circle formed by the R^* curve in this extreme case does not contain the center of the curve defining the outer wall due to the use of the chosen R^* representation. Consequently, the model fails because the radial lines originating from the inner and outer walls do not intersect as shown. R^* is therefore assumed in this case to be equal to 50% of the outer wall's radius [21].

Based on the above description and assumptions, a mathematical relationship for the transverse velocity ratio, v/u, was developed as

$$\frac{u}{v} = F\frac{(t_2 - t_1)}{1 + t_1 t_2}, \quad \xi \neq \xi^*, \tag{2.11}$$

where

$$F = \frac{\operatorname{csc}(\xi_2) - \operatorname{csch}(\xi^*)}{\operatorname{csch}(\xi) - \operatorname{csch}(\xi^*)}, \quad \text{for } \xi < \xi^* \quad \text{or} \quad F = \frac{\operatorname{csc}(\xi_1) - \operatorname{csch}(\xi^*)}{\operatorname{csch}(\xi) - \operatorname{csch}(\xi^*)}, \quad \text{for } \xi > \xi^*, \tag{2.12}$$

$$t_2 = \frac{H(1-S)\sinh(\xi)}{0.5\sinh(\xi_2)\cot(\eta) - H(1-S)\sin(\eta)}, \quad \xi^* = \sinh^{-1}\left(\frac{a}{R^*}\right), \tag{2.13}$$

$$t_1 = \frac{H(1-S)\sin(\eta)}{H(1-S)\sinh(\xi) - \cosh(\xi_2)} \quad \text{or} \tag{2.14}$$

$$t_1 = \frac{H(1-S)\sin(\eta)}{H(1-S)\sinh(\xi) - \cosh(\xi_1)}, \quad \text{for } \xi > \xi^*,$$

$$H = \frac{0.5\sinh(\xi_2)}{(1-S)(\cosh(\xi) - \cos(\eta))}, \tag{2.15}$$

$$\text{for} \quad \xi = \xi^*, \quad u = 0. \tag{2.16}$$

A complete model of the hydrodynamic entrance region is provided by (2.9), (2.10), and (2.11) to (2.16) along with zero-velocity boundary conditions and a uniform inlet velocity profile. However, several hydrodynamic entrance region solutions, such as those for parallel plates by Bodoia [35], a circular tube by Hornbeck [36], and a square duct by Carlson and Hornbeck [34], have also used the integral form of the continuity equation as noted by Hornbeck [36] and Feldman and Hornbeck [21]. These solutions all contain the differential version of the continuity equation, a reduced axial momentum equation in which the pressure gradient is a total derivative, and, in the case of the square duct, a transverse flow model, making them identical to consistent with the present model. Nevertheless, the integral form of the continuity equation is mathematically redundant because it can simply be produced by applying the divergence theorem to the differential form. This form is not represented in all nodal regions close to the duct walls in the finite difference solution, avoidingredundancy. The integral continuity equation has the following bipolar coordinate form

$$\frac{\pi}{2}a^2(1-S)^2 w_e \operatorname{csch}^2(\xi_2) = \int_0^{\pi}\int_{\xi_2}^{\xi_1} w h^2 d\xi d\eta, \tag{2.17}$$

where w_e represents the volume rate of flow per unit of cross-sectional area, was calculated using the average axial velocity. Equation (2.17), which imposes a physical constraint and excludes the transverse velocity components, is particularly useful [21].

2.3. **Method of Solution.** Here, it is more convenient to represent the variables and parameters in dimensionless form for the model as

$$\begin{aligned} &U = \frac{uR_e}{w_e}, \quad V = \frac{vR_e}{w_e}, \quad W = \frac{w}{w_e}, \quad P = \frac{p - p_e}{\rho w_e^2}, \quad Z = \frac{z}{D_h R_e}, \quad H = \frac{h}{D_h}, \\ &\mu_r = \frac{\mu R_e}{\rho w_e D_h}, \quad \tau_D = \frac{\tau_o R_e}{\rho w_e^2} \quad \text{and} \quad R_e = \frac{\rho}{k_c^2} w_e D_h, \end{aligned} \tag{2.18}$$

where p_e is the pressure at the entrance, R_e is the Reynolds number, τ_D is the dimensionless yield stress, and D_h is the hydraulic diameter given by

$$D_h = 2(R_2 - R_1) = 2a(1 - S)\operatorname{csch}\xi_2. \tag{2.19}$$

The following equations are obtained by substituting (2.18) and (2.19) into (2.5), (2.9), (2.10) and (2.11) to (2.17).

Assumed by the dimensionless form of viscosity

$$\mu_r = \left(K_c + \left(\frac{H\tau_D}{\sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}} \right)^{\frac{1}{2}} \right)^{1/2}.$$

The dimensionless forms continuity and momentum equations now take the form

$$\frac{\partial}{\partial \xi}(HU) + \frac{\partial}{\partial \eta}(HV) + \frac{\partial}{\partial z}(H^2 W) = 0,$$

$$\frac{U}{H} \cdot \frac{\partial W}{\partial \xi} + \frac{V}{H} \cdot \frac{\partial W}{\partial \eta} + W\frac{\partial W}{\partial z} = -\frac{dP}{dZ} + \frac{1}{H^2}\left[\frac{\partial}{\partial \xi}\left(\mu_r \frac{\partial W}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(\mu_r \frac{\partial W}{\partial \eta} \right) \right]. \tag{2.20}$$

All other variables in (2.12) to (2.16) are already dimensionless due to the dimensionless nature of u and v in (2.18), which makes u/v equal to U/V. Thus, (2.11) through (2.16) are not altered, except that (2.11) and (2.16) are updated to:

$$\frac{U}{V} = F\frac{t_2 - t_1}{1 + t_1 t_2}, \quad \text{for } \xi \neq \xi^*, \quad \text{and} \quad U = 0, \quad \text{for } \xi = \xi^*. \tag{2.21}$$

Then, the dimensionless integral form of the continuity equation (2.17) is as follows

$$\frac{8(1 - S)}{\pi(1 + S)} \int_0^{\pi} \int_{\xi_2}^{\xi_1} W H^2 \, d\xi \, d\eta = 1.0, \tag{2.22}$$

which employs the trapezoidal rule for the double integral. Moreover, the dimensionless boundary conditions are:

$$\begin{aligned} U = V = W = 0, &\quad \text{at } \xi = \xi_1 \text{ and } \xi = \xi_2, \\ W = 1, &\quad \text{at } Z = 0, \\ \frac{\partial W}{\partial \eta} = 0, &\quad \text{at } \eta = 0 \text{ and } \eta = \pi. \end{aligned}$$

3. Numerical Model and Implementation

3.1. **Numerical Model.** A numerical marching technique that divides the hydrodynamic entrance region into a number of parallel planes that are perpendicular to the z-axis provides a finite difference solution to the model. To create a finite difference grid that covers half of the symmetric annular cross section, sets of constant ξ curves (with index i) and constant η curves (with index j) are used to segment off the region defined by $\xi_1 \le \xi \le \xi_2$. This grid was selected to be fine enough to produce solutions with acceptably low truncation errors without requiring excessive computational cost or storage. Starting from either wall, the ξ interval from ξ_1 to ξ^* and the one from ξ^* to ξ_2 is divided into m variable segments. The steep velocity gradients near the walls are accommodated very fine grid spacing. The η interval from 0 to π is divided into l equal segments. Therefore, there are $(2m-1)l$ interior grid points, with indices (i,j), for which values of the three velocity components, $U_{i,j}$, $V_{i,j}$, and $W_{i,j}$, are required, while $(l+1)$ points on each ξ boundary for which have all values of $U_{i,j}$, $V_{i,j}$, and $W_{i,j}$ set to zero. Non-uniform step lengths in the ξ direction (variable mesh size), uniform step lengths in the η direction, and non-uniform step lengths in the z direction are each represented by $\Delta\xi_i$ (where $\Delta\xi_i = \xi_{i+1} - \xi_i$), $\Delta\eta$, and Δz, respectively. Consequently, the continuity equation, using central difference, takes the following form

$$
\begin{aligned}
&\frac{H_{i,j-1}U_{i,j-1,k+1} - H_{i-1,j}U_{i-1,j,k+1} - H_{i-1,j-1}U_{i-1,j-1,k+1} + H_{i,j}U_{i,j,k+1}}{2\Delta\xi_{i-1}} \\
&+ \frac{-H_{i,j-1}V_{i,j-1,k+1} + H_{i-1,j}V_{i-1,j,k+1} - H_{i-1,j-1}V_{i-1,j-1,k+1} + H_{i,j}V_{i,j,k+1}}{2\Delta\eta} \\
&+ \frac{H_{i,j}^2W_{i,j,k+1} + H_{i-1,j}^2W_{i-1,j,k+1} + H_{i,j-1}^2W_{i,j-1,k+1} + H_{i-1,j-1}^2W_{i-1,j-1,k+1}}{4\Delta z} \\
&- \frac{H_{i,j}^2W_{i,j,k} + H_{i-1,j}^2W_{i-1,j,k} + H_{i,j-1}^2W_{i,j-1,k} + H_{i-1,j-1}^2W_{i-1,j-1,k}}{4\Delta z} = 0,
\end{aligned} \tag{3.1}
$$

while the momentum equation in the finite difference form is:

(3.2)

$$
\begin{aligned}
&\frac{U_{i,j,k}}{H_{i,j}}\left(\frac{W_{i+1,j,k+1}}{d_2\left(\Delta\xi_i^2\right)} - \frac{W_{i-1,j,k+1}}{d_2\left(\Delta\xi_{i-1}^2\right)} + \frac{d_3W_{i,j,k+1}}{d_2}\right) + \frac{V_{i,j,k}}{H_{i,j}}\left(\frac{W_{i,j+1,k+1} - W_{i,j-1,k+1}}{2\Delta\eta}\right) \\
&+ \frac{W_{i,j,k}}{\Delta z}\left(W_{i,j,k+1} - W_{i,j,k}\right) \\
&= -\frac{dP}{dZ} + \frac{1}{H_{i,j}^2}\left[\frac{\mu_{r_{i+1/2},j,k+1}(W_{i+1,j,k+1} - W_{i,j,k+1})}{d_1\left(\Delta\xi_i\right)} - \frac{\mu_{r_{i-1/2},j,k+1}(W_{i,j,k+1} - W_{i-1,j,k+1})}{d_1\left(\Delta\xi_{i-1}\right)}\right] \\
&+ \frac{1}{H_{i,j}^2}\left[\frac{\mu_{r_{i,j+1/2},k+1}(W_{i,j+1,k+1} - W_{i,j,k+1}) + \mu_{r_{i,j-1/2},k+1}(W_{i,j,k+1} - W_{i,j-1,k+1})}{\Delta\eta^2}\right],
\end{aligned}
$$

where

$$
d_1 = \frac{\Delta\xi_i + \Delta\xi_{i-1}}{2}, \quad d_2 = \frac{1}{\Delta\xi_i} + \frac{1}{\Delta\xi_{i-1}}, \quad d_3 = \frac{1}{\Delta\xi_{i-1}^2} - \frac{1}{\Delta\xi_i^2}. \tag{3.3}
$$

3.2. **Problem Implementation.** The previously obtained results were coded in MATLAB [37]. However, the established difference scheme generates a system of linear algebraic equations in V, W, and $\frac{dP}{dZ}$. Subsequently, a set of $(2m+1)(l+1)+1$ simultaneous linear algebraic equations in W and $\frac{dP}{dZ}$ are obtained by combining (2.22) with (3.1) and (3.2) for $i = 2$ to $2m-1$ and $j = 2$ to l. At the first plane beyond the inlet, the hydrodynamic model is solved independently of all subsequent planes. The solution of the second plane depends only on the solution of the first one, and the solution of the $(k+1)$-th plane depends only on the solution of the k-th plane, which is its direct predecessor. As a result, the entire hydrodynamic entrance region is resolved by successively solving each plane in turn. The matrix form of the linear algebraic equations is as follows

$$AX = B, \tag{3.4}$$

where X denotes the unknowns of the algebraic equations, A represents their coefficients, and B contains their residuals. Then the matrix form of (3.4) may be expressed in the following way

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \cdot \begin{bmatrix} W \\ P \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{3.5}$$

where A_1 is a $(2m-1)\times(l+1)$ matrix that includes the coefficients of $W_{i,j}$ in (3.2), while A_2 contains the coefficients of P in (3.2) and A_3 has the coefficients of the axial velocity $W_{i,j}$ derived from the integral (2.22). Also,

$$A_4 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Additionally, B_1 is a $(2m-1)\times(l+1)$ vector representing the residuals of (3.2) whereas B_2 is the vector that represents the residuals of (2.22). In order to solve nonsingular linear systems with coefficient matrices, Keller [38] investigated the behavior of the bordering algorithm (3.5). This algorithm has been used to solve a broad class of differential equations boundary value problems that are discretized. The matrix form (3.5) can be split into the following two equations using this technique

$$A_1 W + A_2 P = B_1, \tag{3.6}$$

$$A_3 W + A_4 P = B_2. \tag{3.7}$$

In which, the unknown W may be put in the form

$$W = \bar{W} - \hat{W} P. \tag{3.8}$$

Then (3.6) may be written as

$$A_1(\bar{W} - \hat{W}P) + A_2 P = B_1, \tag{3.9}$$

i.e.,

$$A_1 \bar{W} + P(A_2 - A_1 \hat{W}) = B_1. \tag{3.10}$$

In order to eliminate P from (3.10), $\hat{W}$ is solved using

$$A_1\hat{W} = A_2, \tag{3.11}$$

and then $\bar{W}$ is obtained through solving

$$A_1\bar{W} = B_1. \tag{3.12}$$

However, the five-band system represented by (3.11) and (3.12) can be solved using the relaxation method to obtain both $\bar{W}$ and $\hat{W}$. The sparsity of matrix A_1 with many zero elements that are ignored during the solution steps makes this method appropriate for solving such systems. Therefore, the relaxation equation is expressed as

$$\alpha_{2i,j} = \frac{w_e}{a_{i,j}}\left(a_{i,j-1}\alpha_{2i,j-1} + a_{i,j+1}\alpha_{2i,j+1} + a_{i-1,j}\alpha_{2i-1,j} + a_{i+1,j}\alpha_{2i+1,j}\right) + (1 - w_e)\alpha_{1i,j}, \tag{3.13}$$

where α represents an element in the unknowns vectors $\bar{W}$ and $\hat{W}$, the subscript 1 means the known values of the variables 1, and the subscript 2 means the unknown values of the variables one. w_e is the relaxation factor, a is an element of matrix A_1, and b is an element of matrix A_2 or matrix B_1. Substitute (3.8) into (3.7) yields $A_3(\bar{W} - \hat{W}P) = B_2$. Then,

$$A_3\hat{W}P = A_3\bar{W} - B_2.$$

Equation (3.8) is used to determine W once P is known. Following the acquisition of all $W_{i,j}$ for a plane, the unknown $U_{i,j}$ and $V_{i,j}$ in the plane are all directly evaluated without any iterations using the numerical approximation to (3.1) and (2.21).

Starting with uniform axial velocity as a starting point, the flow in the annulus develops during the computations. The process is repeated at the next plane after resolving all unknowns at the $(k+1)$ plane. Repeating this process until the flow has fully developed, the relaxation parameter and the mesh size (axial and transverse) affect the stability and convergence speed. Furthermore, the model's finite difference equations are stable and consistent based on the available numerical findings. At a certain axial position on z, the truncation error of the difference scheme in (3.2) is $O(\Delta\xi + \Delta\eta^2)$. As a result, since $\Delta\xi$ and $\Delta\eta$ converge to zero, the difference equation (3.2) tends to the differential equation (2.20). The very tiny axial step size Δz, no larger than 0.25×10^5, was used near the inlet to account for the high axial gradients typically occurring at the duct entrance. Z was increased to a maximum of 0.001 increased.

4. Results and Discussion

The exact solution of Snyder and Goldstein [39] for the magnitude of discharge, Q, has been compared to the results of the current study while considering various values of e. As shown in Table 1, the comparison has examined a fully developed Newtonian fluid ($\tau_D = 0.0$), with $S = 0.5$. Different dimensionless parameters were considered

when calculating the discharge. Additionally, the accuracy of the current study was evaluated, revealing that agreement with the exact solution increases as e increases.

TABLE 1. Comparison of numerical results of discharge, Q with the exact solution of Synder and Goldstein

e	**Present Study**	**Snyder and Goldstein**	**Accuracy (%)**
$S = 0.5$			
0.2	0.051157	0.05226	97.88940
0.4	0.060041	0.06054	99.17575
0.6	0.073739	0.074052	99.57732
0.8	0.092061	0.09236	99.67627

Moreover, Table 2 compares the discharge values to those reported by Ateia et al. [5] at $S = 0.5$ considering various values of τ_D and e. As e is increased or decreased, the accuracy of the discharge values from the current study was evaluated and they were found to be consistent.

TABLE 2. Comparison of numerical results of discharge with the numerical solution of Ateia et al. for different values of τ_D

Reference	$\tau_D = 0$	$\tau_D = 0.025$	$\tau_D = 0.05$
Present study $S = 0.5$, $e = 0.2$	0.01197	0.020446	0.05116
[5]	0.01222	0.020845	0.05224
Accuracy (%)	97.9326	97.9542	98.0859
Present study $S = 0.5$, $e = 0.4$	0.01555	0.025390	0.06004
[5]	0.01537	0.025310	0.06049
Accuracy (%)	99.6839	99.2561	98.8289
Present study $S = 0.5$, $e = 0.6$	0.02068	0.032686	0.07398
[5]	0.02110	0.033011	0.07415
Accuracy (%)	99.0155	99.7707	98.0095
Present study $S = 0.5$, $e = 0.8$	0.02801	0.042758	0.09232
[5]	0.02859	0.043263	0.09237
Accuracy (%)	98.8327	99.9459	97.9713

This study's dimensionless entrance solution for pressure is contrasted with that of Feldman and Hornbeck [21]. Figure 2 depicts $e = 0.9$ and $S = 0.1$ for Newtonian fluid flow ($\tau_D = 0.0$), and the current findings are in good agreement with the earlier conclusion.

However, Figures 3, 4 and 5 examine the effect of S, e, and τ_D on the pressure gradient, respectively. Figure 3 illustrates the impact of the radius ratio ($S = 0.1, 0.5$ and 0.9) on the pressure gradient $-\frac{dP}{dZ}$ for constant values of relative eccentricity $e = 0.5$ and yield stress $\tau_D = 0$. It showed that as S decreases, the pressure gradient grows and quickly reaches its maximum. For $S = 0.1, 0.5$ and 0.9, the ratio

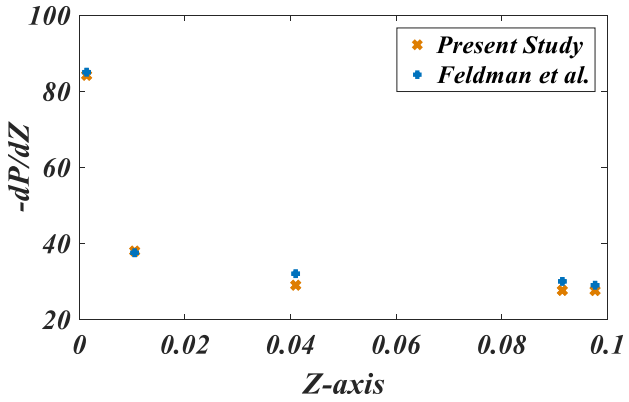


FIGURE 2. Comparison of the pressure gradient with the results of Feldman and Hornbeck, $e = 0.9$, $s = 0.1$ and $\tau_D = 0.0$

of pressure gradient values at $Z = 0$ to those at $Z = 1$ was 6.16, 7.90, and 17.65 times, respectively. This indicates how high the pressure gradient at the entrance region is. These significant increases (up to 17.65 times the fully developed values) stem from steep velocity gradients near the inlet. In Figure 3, the pressure gradient range expands as S increases, due to narrower annular gaps intensifying wall shear effects. Moreover, Figure 4 shows how the relative eccentricity ($e = 0.3, 0.5$ and 0.7) affects the pressure gradient $-\frac{dP}{dZ}$ for constant values of the radius ratio $S = 0.5$ and $\tau_D = 0$. As e decreases, the pressure gradient increases and quickly reaches its maximum. The pressure gradient values at $Z = 0$ were 50.52, 35.44, and 21.99 times greater than those at $Z = 1$ for $e = 0.3, 0.5$ and 0.7, respectively. This reflects the significant increase in the pressure gradient within the entrance region. For a Newtonian fluid $\tau_D = 0$, decreasing e narrows the annular gap, intensifying shear and resistance near the inner wall, which sharply elevates the pressure gradient at the inlet. Additionally, the pressure gradient $-\frac{dP}{dZ}$ for fixed values of the radius ratio $S = 0.5$ and the relative eccentricity $e = 0.5$ is affected by the yield stress $\tau_D = 0.0, 0.05, 0.5$ and 1, as shown in Figure 5. It increases as τ_D increases and gradually develops to its full value, which is expected given increased viscosity and resulting elevation in pressure gradient. For $\tau_D = 0.0, 0.05, 0.5$ and 1, the pressure gradient ratios at the entrance region relative to the steady fully developed zone were 10.64, 11.96, 14.83, and 16.86 times, respectively. This draws the designers' attention to the extremely high pressure in the entrance areas. For a Casson fluid, higher τ_D enhances the yield stress, requiring greater pressure to initiate flow, particularly at the entrance, where undeveloped flow amplifies viscous resistance.

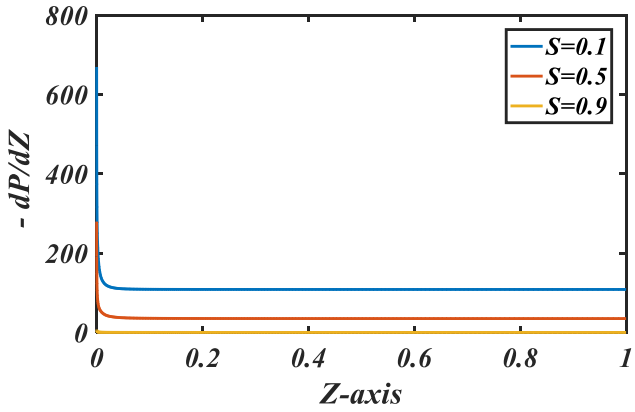


FIGURE 3. Distribution of the pressure gradient $(\frac{-dP}{dZ})$ versus Z for different values of S for $e = 0.5$ and $\tau_D = 0.0$

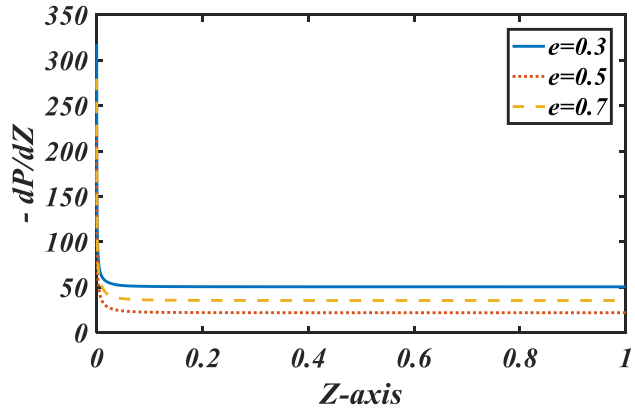


FIGURE 4. Distribution of the pressure gradient $(\frac{-dP}{dZ})$ versus Z for different values of e for $S = 0.5$ and $\tau_D = 0.0$

On the other hand, Figures 6, 7, and 8 demonstrate the effects of S, e, and τ_D on the axial velocity. Figure 6 depicts the impact of the radius ratio ($S = 0.1$, 0.5, and 0.9) on the axial velocity for constant values of the relative eccentricity $e = 0.5$ and the yield stress $\tau_D = 0.0$. As S decreases, the axial velocity increases to accommodate the increased area. The ratio of the entrance zone velocity to the fully developed one was 0.443626, 0.47152, and 0.437509 for $S = 0.1$, 0.5, and 0.9, respectively. For a Newtonian fluid $\tau_D = 0$, a lower S (smaller inner radius relative to the outer) widens

the annular gap, reducing flow resistance and boosting axial velocity near the entrance. However, as the flow develops, the velocity profile adjusts, leading to the observed ratios reflecting the transition from entrance to fully developed conditions.

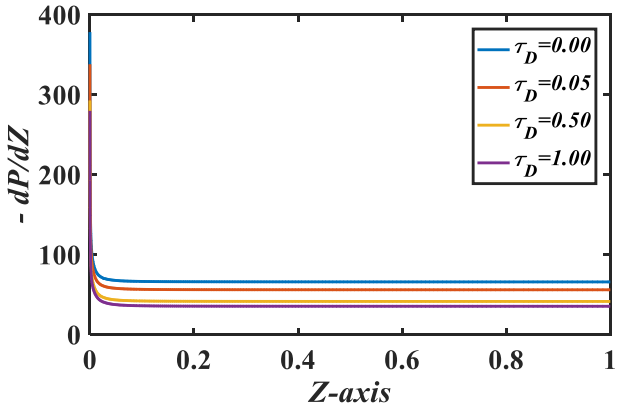


FIGURE 5. Distribution of the pressure gradient $(\frac{-dP}{dZ})$ versus Z for different values of $e = 0.5$ for $S = 0.5$ for different values of yield stress

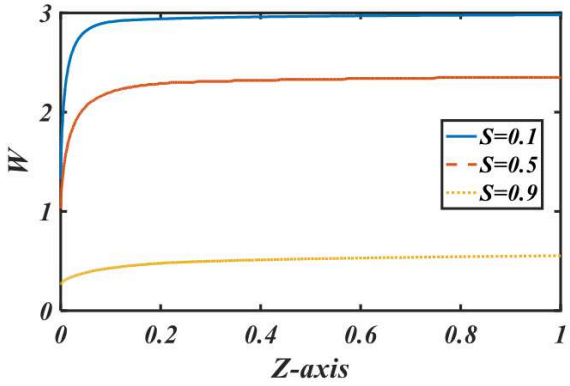


FIGURE 6. Distribution of the velocity, w versus Z for different values of S for $e = 0.5$ and $\tau_D = 0.0$

Additionally, Figure 7 shows the influence of relative eccentricity, $e = 0.3$, 0.5 and 0.7 on axial velocity for fixed values of radius ratio, $S = 0.5$, and yield stress, $\tau_D = 0$. It was found that as e increases (due to an increase in area), the axial velocity decreases. According to the results, the entrance zone velocity was 0.469391, 0.438141 and 0.437509 times its value in the fully developed region, respectively. For a Newtonian fluid $\tau_D = 0$, increasing e shifts the inner cylinder off-center, narrowing the gap on one side and widening it on the other. This asymmetry increases resistance

in the narrow region, reducing the axial velocity near the entrance despite the larger overall area, while the fully developed flow adjusts to a more uniform profile.

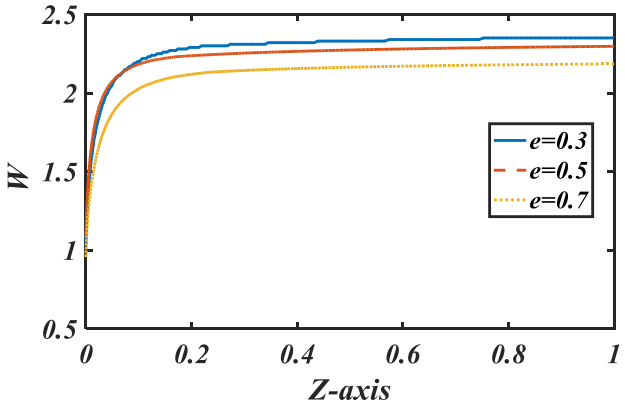


FIGURE 7. Distribution of the velocity, w versus Z for different values of e at $S = 0.5$ and $\tau_D = 0.0$

Furthermore, Figure 8 illustrates how the yield stress, $\tau_D = 0, 0.5$ and 1.0 affects the axial velocity for constant values of the radius ratio, $S = 0.1$ and relative eccentricity, $e = 0.9$. As τ_D increases, the viscosity rises and the axial velocity decreases correspondingly . The ratios of the entrance region velocity to the fully developed velocity were 0.496551714, 0.505679036 and 0.53654 for $\tau_D = 0, 0.5$ and 1.0, respectively. For a Casson fluid, higher τ_D increases the yield stress, enhancing resistance to flow initiation and reducing axial velocity, particularly near the entrance where flow is undeveloped. This effect is pronounced at low S and high e, as the narrow annular gap and eccentricity amplify viscous opposition, though the fully developed flow partially mitigates this reduction.

5. Concluding Remarks

The numerical findings were reported for the three-dimensional, steady, laminar fluid flow under hydrodynamically developed conditions. The fluid was assumed to be either Newtonian or non-Newtonian flowing through concentric or eccentric annulus duct. Computer software was developed that utilized two-dimensional storage and a marching strategy for solving differential equations line-by-line using the relaxation method. Subsequently, the effects of the axial position Z, the relative e, and the radius ratio S on the axial velocity profiles were studied.

Through this work, it was found that the pressure gradient is directly proportional to the yield stress; τ_D, and inversely proportional to both the radius ratio; S, and the relative eccentricity; e, whereas the axial velocity is inversely proportional to S, e, and

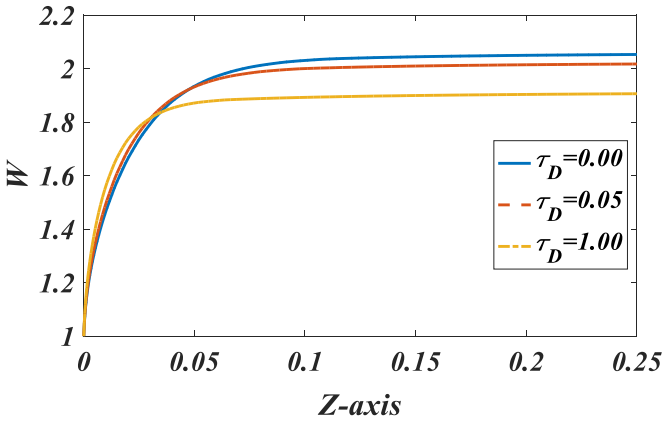


FIGURE 8. Distribution of the velocity, w versus Z for different values of τ_D at $e = 0.9$ and $S = 0.1$

τ_D. It was observed that the pressure gradient in the entrance region is significantly greater than the fully developed one. This ratio ranged from 6.16 for $S = 0.1$, $e = 0.5$ and $\tau_D = 0$ to 50.52 for $S = 0.5$, $e = 0.3$ and $\tau_D = 0$. In contrast to the pressure gradient, the axial velocity in the entrance region is consistently lower than in the fully developed zone,with its ratio varying between 0.437509 for $S = 0.5$, $e = 0.7$ and $\tau_D = 0$ and 0.53654 for $S = 0.1$, $e = 0.9$ and $\tau_D = 1$. This emphasizes the importance of investigating the entrance region for system designers.

The accuracy of the current results was compared to values reported in the literature showing good agreement.

Notations. We have used the following list of notations.

Dimensionless velocity components	U, V, W
Coefficient matrix	A
Velocity components	u, v, w
Coefficients sub-matrices	A_1, A_2, A_3
Velocity vector	W
Residuals vector	B
Intermediate matrices	$\bar{W}, \hat{W}$
Residuals sub-vectors	B_1, B_2
Volume rate of flow per unit	w_e
Hydraulic diameter	D_h
Eccentricity	d
Dimensionless z component	Z
Pressure gradient	$\frac{dp}{dz}$

Greek Symbols	
Model viscosity	K_c^2
Apparent viscosity	μ
Plane number	k
Dimensionless viscosity	μ_r
Relative eccentricity	M
Bipolar system coordinates	ξ, η, z
Dimensionless pressure	P
Fluid density	ρ
Entrance's pressure gradient	p_e
Yield stress	τ_D
Inner and outer radii	R_1, R_2
Distance between two points	Δ
Reynolds's number	R_e
Gradient derivative	∇
Radius ratio	S

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ON UNIQUE MINIMAL SOBOLEV NORM ELEMENT OF BANACH SPACES OF FUNCTIONS WHICH TAKES A GIVEN VALUE IN A FIXED POINT

TOMASZ ŁUKASZ ŻYNDA

Abstract. First, it will be shown that some Banach spaces V of functions, which are subspaces subspaces of Sobolev spaces satisfy the c-minimal norm property, i.e., in any set

$$V_{z,c} := \{f \in V \mid f(z) = c\},$$

if non-empty, there is exactly one element with t minimal Sobolev norm. Later, it will be proved that this element depends continuously on the deformation of the norm and on an increasing sequence of domains in a precisely defined sense. We conclude with applications to the theory of linear partial differential equations.

1. Introduction

Classical theory states that the solution of a partial differential equation is unique in many cases if the boundary or initial conditions are given. Here we introduce a different idea - if we want a solution of a homogeneous linear partial differential equation to be unique, we can consider the solution with the minimal Sobolev norm among the solutions which take given value at a fixed point.

First, we prove a theorem showing sufficient conditions for the Banach space of functions to have the minimal norm property, i.e., the property of the existence of a unique element with minimal norm in the subset of elements taking a certain value at a fixed point. Then we show that some Banach spaces of functions with Sobolev norm satisfy the assumptions of this theorem. Next, we prove our main results - theorems

Key words and phrases. Sobolev spaces, minimal norm element, continuous dependence, linear partial differential equation

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stating that the unique minimal norm element of a Banach space of functions with Sobolev norm continuously depends on an integration weight, i.e., on the deformation of the norm and on an increasing sequence of domains.

2. Preliminaries

It is known that a Hilbert space V of functions defined on a domain D is a reproducing kernel Hilbert space if and only if it satisfies the minimal norm property, e.g., in any set

$$V_z := \{f \in V \mid f(z) = 1\},$$

if it is not empty, there is exactly one element with the minimal norm. In fact, the property of the minimal norm is equivalent to a formally stronger property: c-minimal norm property, i.e., in every set

$$V_{z,c} := \{f \in V \mid f(z) = c\},$$

if non-empty, there is exactly one element with minimal norm. Indeed, if V satisfies the property of c-minimal norm, then it also satisfies the property of minimal norm. On the other hand, if φ_z is unique element of V_z with the minimal norm, then $c\varphi_z$ is the unique element of $V_{z,c}$ with the minimal norm.

For more details and proofs see e.g. [6]. For examples of Hilbert spaces that do not satisfy the minimal norm property, i.e., are not reproducing kernel Hilbert spaces, see e.g. [6] or [3].

The aim of this paper is to show that some subspaces of functions of Sobolev spaces, which are not necessarily Hilbert spaces, also satisfy the minimal norm property. Moreover, we will show that the unique minimal norm element in the set V_z depends continuously on a weight function and on the integration domain. These results can be somehow treated as a continuation of the research introduced in [6] and [4].

Throughout the paper, the unique minimal norm element of V_z will be denoted by φ_z without further recalling.

3. Proof that some Subspaces of Functions of Sobolev Spaces Satisfy the Minimal Norm Property

Let V be a Banach space of functions which is a subspace of a Sobolev space $W^{k,p}(U)$, $U \subset \mathbb{R}^n$, U an open bounded domain with boundary of class C^1, $1 < p < +\infty$, equipped with the same norm, i.e.,

$$||f|| := ||f||_{W^{k,p}(U)} = \left(\sum_{|i| \leq k} ||D^i f||^p_{L^p(U)} \right)^{\frac{1}{p}},$$

where $i = (i_1, i_2, \ldots, i_m)$, $|i| = i_1 + i_2 + \cdots + i_m$ and

$$D^i f = \frac{\partial^{|i|} f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}.$$

It is well-known that each space L^p, $1 < p < +\infty$, is strictly convex, therefore also V is strictly convex as a closed subspace of L^p. Moreover, the following theorem holds.

Theorem 3.1 ([1], Theorem 6 in Section 5.6.3.)**.** *If $f \in W^{k,p}(U)$, with additional assumptions above and $k > \frac{n}{p}$, then f is also an element of Hölder's space $C^{k-\left[\frac{n}{p}\right]-1,\gamma}(U)$, where γ is equal to $\left[\frac{n}{p}\right] + 1 - \frac{n}{p}$, if $\frac{n}{p} \notin \mathbb{Z}$ and is any positive number lower than 1, otherwise.*

Moreover, the following holds.

$$||f||_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})} \leq C||f||_{W^{k,p}(U)}. \tag{3.1}$$

A direct consequence of this theorem is the fact that the functionals of the point evaluation in V are continuous if only its assumptions are fulfilled, i.e. for every $z \in U$ there exists constant C_z, such that

$$|f(z)| \leq C_z||f||_{W^{k,p}(U)},$$

for any $f \in V$.

Now we are ready to prove the main theorems of this section.

Theorem 3.2. *Let V be a Banach space of functions defined on U satisfying the following conditions:*

(i) functionals of point evaluation are continuous;

(ii) V is strictly convex.

Then, for any $z \in U$, if there exist $f_z \in V$ such that $f_z(z) \neq 0$, we have

$$\inf_{\{f \in V | f(z)=1\}} ||f|| > 0. \tag{3.2}$$

Moreover, if the infimum is reached, then it is only reached for one function, i.e, then V satisfies the minimal norm property.

Proof. Let $z \in U$ be a point for which there exist functions in V that take a nonzero value at z, and let $f \in V$ be such a function. Then, by continuity of functionals of the point evaluation we have

$$0 < \frac{1}{C_z} \leq \left|\left|\frac{f}{f(z)}\right|\right|.$$

Therefore, (3.2) holds.

Now, let us assume that the infimum in (3.2) is reached for some functions $f, g \in V_z$. We will show that $f = g$. First, let us define

$$h = \frac{1}{2}(f + g) \in V_z,$$

i.e., $h(z) = 1$. Next, by the triangle inequality and the fact that $||f|| = ||g||$, we obtain

$$||h|| = \left|\left|\frac{1}{2}(f + g)\right|\right| \leq \frac{1}{2}(||f|| + ||g||) = ||f||.$$

On the other hand, by our assumptions,

$$||h|| \geq ||f||,$$

therefore

$$||h|| = ||f||. \tag{3.3}$$

Since V is strictly convex and the triangle inequality in the above case is in fact equality, we conclude that there exists $\alpha \in \mathbb{C}$, such that $f = \alpha g$. Then,

$$||h|| = \left|\left|\frac{1}{2}(f+g)\right|\right| = \frac{1}{2}(\alpha+1)||f||.$$

By (3.3) we get that $\alpha = 1$ and in conclusion $f = g$. □

Corollary 3.1. *Let V be a Banach space of functions equipped with the Sobolev norm, which satisfies assumptions of Theorem* 3.1*. Then, V satisfies the minimal norm property.*

Proof. As it was stated before, V is strictly convex and functionals of point evaluation for that space are continuous. Therefore,

$$\inf_{f \in V_z} ||f||_{W^{k,p}(U)} > 0,$$

and if it is reached, then it is reached only for one function. It is sufficient to show that the infimum above is in fact minimum. Let $\{f_n\} \subset V_z$ be a sequence which approximates this infimum, i.e., such that

$$\lim_{n \to +\infty} ||f_n||_{W^{k,p}(U)} = \inf_{f \in V_z} ||f||_{W^{k,p}(U)}.$$

It is obvious that $\{f_n\}$ is bounded in the $||\cdot||_{W^{k,p}(U)}$ norm. By Theorem 3.1, it is also bounded in a Hölder space. Therefore, the sequence $\{f_n\}$ satisfies the assumptions of the Arzelà-Ascoli theorem and there is a convergent subsequence of it. Let us denote its limit by f_0. By Fatou's lemma $||f_0||_{W^{k,p}(U)} \leq \inf_{f \in V_z} ||f||_{W^{k,p}(U)}$, so f_0 is the unique element of $W^{k,p}(U)$, such that

$$||f_0||_{W^{k,p}(U)} = \min_{f \in V_z} ||f||_{W^{k,p}(U)}.$$

□

Lemma 3.1. *Let V be a Banach space of functions defined on D which satisfies the minimal norm property. Then, the functionals of the point evaluation for that space are continuous. Moreover, for any $z \in D$ and any functional $E_z : V \ni f \mapsto f(z) \in \mathbb{C}$,*

$$||E_z||^* = \frac{1}{||\varphi_z||}.$$

Proof. By the minimal norm property for any $z \in D$ there exists $C_z > 0$ such that for any $f \in V$ for which $f(z) \neq 0$ we have

$$C_z \leq \left|\left|\frac{f}{f(z)}\right|\right|.$$

Therefore,

$$|f(z)| \leq C_z||f||.$$

If $f(z) = 0$, then the above inequality is trivially satisfied. Moreover, for any g such that $g(z) = 1$, we have

$$|g(z)| \leq \frac{1}{||\varphi_z||}||g||.$$

For any function $f \in V$, such that $f(z) = c$, $c \neq 0, 1$ there exists function $g \in V$, such that $g(z) = 1$ and $f = cg$. So, we have

$$|f(z)| = |cg(z)| \leq \frac{1}{||\varphi_z||}||cg|| = \frac{1}{||\varphi_z||}||f||.$$

Therefore, $||E_z||^* \leq \frac{1}{||\varphi_z||}$. Since the above inequality for $f = \varphi_z$ is in fact equality, we conclude that $||E_z||^* = \frac{1}{||\varphi_z||}$. □

4. Dependence of Minimal Norm Element on a Weight of Integration

Let $\mu_1, \mu_2, \ldots, \mu_r$ be measurable positive functions defined on a domain U such that there exist positive constants c_1, c_2 for which

$$c_1 < \mu_i(w) < c_2, \tag{4.1}$$

for any $i = 1, 2, \ldots, r$. $\mu := (\mu_1, \mu_2, \ldots, \mu_r)$ will be called a weight. We will say that a sequence of weights $\{\mu^N\}$ converges almost everywhere to the weight μ if the coefficients of μ^N converge almost everywhere to the corresponding coefficients of μ.

Let

$$||f||_{L^p_{\mu_i}(U)} = \left(\int_U |f(w)|^p \mu_i(w) \mathrm{d}w\right)^{\frac{1}{p}}.$$

Now let us define weighted Sobolev norm $W^{k,p}_\mu$

$$||f|| := ||f||_{W^{k,p}_\mu(U)} = \left(\sum_{|i|\leq k} ||D^i f||^p_{L^p_{\mu_i}(U)}\right)^{\frac{1}{p}}.$$

It is clear that in $W^{k,p}_\mu$ functionals of point evaluation are continuous. Indeed, using Theorem 3.1, we get that

$$||u||_{C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{U})} \leq C||u||_{W^{k,p}(U)} \leq C\frac{1}{c_1^{\frac{1}{p}}}||u||_{W^{k,p}_\mu(U)}.$$

A direct consequence of this inequality is the fact that any sequence that is bounded in the weighted Sobolev norm has a convergent subsequence, by Arzelá-Ascoli Theorem. Therefore, by Theorem 3.2, Banach space of functions with weighted Sobolev norm with the additional assumptions of Theorem 3.1 satisfies the minimal norm property.

Theorem 4.1. *Let μ^N be a sequence of weights that converges to the weight μ almost everywhere. Let V_0 be a Banach space of functions with weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu}(U)}$ satisfying the assumptions of Theorem* 3.1. *For any $N \in \mathbb{N}$, let V^N be the same as a vector space as V_0, but equipped with the weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu^N}(U)}$ and analogously let V be the same as a vector space as V_0 but equipped with the weighted Sobolev norm $||\cdot||_{W^{k,p}_{\mu}(U)}$. Let $\varphi^{\mu^N}_{z,c}$ denote the minimal norm element in the set*

$$\{f \in V^N \mid f(z) = c\},$$

and similarly $\varphi^{\mu}_{z,c}$ denote the minimal norm element in the set $\{f \in V \mid f(z) = c\}$. Then,

$$\lim_{N\to+\infty} \varphi^{\mu^N}_{z,c} = \varphi^{\mu}_{z,c}$$

and the limit above is uniform on U.

By (4.1) all spaces V^N, V and V^0 are the same as topological vector spaces. In particular they are all complete, i.e., they are Banach spaces.

Note that by the minimal norm property it is sufficient to show this theorem for $c = 1$.

Lemma 4.1. *If $\mu^1 := (\mu^1_1, \mu^1_2, \dots, \mu^1_r)$ and $\mu^2 := (\mu^2_1, \mu^2_2, \dots, \mu^2_r)$ are weights such that $\mu^1_i \le \mu^2_i$ for any $1 \le i \le r$, then*

$$||\varphi^{\mu^1}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^2}(U)}.$$

Proof. By its very own definition

$$||\varphi^{\mu^1}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^1}(U)} \le ||\varphi^{\mu^2}_{z,c}||_{W^{k,p}_{\mu^2}(U)}.$$

□

Now we are going to give the proof of the main theorem.

Proof. First, let us show that the limit of $\varphi^{\mu^N}_z$ exists. Let us take

$$\mu^0 := (2c_2, 2c_2, \dots, 2c_2)$$

(see (4.1)). By (4.1) and Lemma 4.1,

$$||\varphi^{\mu^N}_z||_{W^{k,p}(U)} \le \frac{1}{c_1^{\frac{1}{p}}} ||\varphi^{\mu^N}_z||_{W^{k,p}_{\mu^N}(U)} \le ||\varphi^{\mu^0}_z||_{W^{k,p}_{\mu^0}(U)}.$$

So the sequence $||\varphi^{\mu^N}_z||_{W^{k,p}(U)}$ is bounded. Therefore, by (3.1) sequence $\varphi^{\mu^N}_{z,c}$ satisfies the assumptions of the Arzelá-Ascoli theorem and we conclude that there exists a uniformly convergent subsequence of it. Without loss of generality, we can identify it with the entire sequence. Let us denote its limit by g. By the property of the minimal norm, it is sufficient to show that

$$||g||_{W^{k,p}_{\mu}(U)} \le ||\varphi^{\mu}_z||_{W^{k,p}_{\mu}(U)}. \tag{4.2}$$

By Fatou's lemma

$$||g||_{W^{k,p}_{\mu}(U)} \leq \liminf_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}. \tag{4.3}$$

Note that since μ^N for any N and μ are bounded from below and above by non-zero constants, all spaces $W^{k,p}_{\mu^N}(U)$ are equal as topological vector spaces (and in particular as sets). Therefore, for any f that is an element of any of these spaces, by Lemma 3.1, we have

$$|f(z)| \leq \frac{1}{||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}} ||f||_{W^{k,p}_{\mu^N}(U)}.$$

Taking the limit on the right hand side we get that

$$|f(z)| \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}} ||f||_{W^{k,p}_{\mu}(U)}.$$

We can use Lebesgue dominated convergence theorem to show that $||f||_{W^{k,p}_{\mu^N}(U)} \to ||f||_{W^{k,p}_{\mu}(U)}$. Moreover, by Lemma 3.1, again

$$\frac{1}{||\varphi_z^{\mu}||_{W^{k,p}_{\mu}(U)}} \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{\mu^N}||_{W^{k,p}_{\mu^N}(U)}}. \tag{4.4}$$

Combining (4.3) with (4.4), we obtain (4.2). □

5. Dependence of Minimal Norm Element on Increasing Sequence of Domains

Theorem 5.1. *Let $\{U_N\}$ be an increasing sequence of domains with boundaries of class C^1 and $U = \bigcup_{N=1}^{+\infty} U_N$. Let V_1 be the Banach space of functions defined on U_1 equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U_1)}$ satisfying the assumptions of Theorem 3.1. Let V^N for $N > 1$ be the Banach space of functions equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U_N)}$, such that for any $f \in V^N$ we have $f_{|U_{N-k}} \in V_{N_k}$, for $k = 1, 2, \ldots, N-1$. Similarly, let V be the Banach space of functions equipped with the Sobolev norm $||\cdot||_{W^{k,p}(U)}$, such that for any $f \in V$ we have $f_{|U_N} \in V_N$ for any N. Let $\varphi_{z,c}^{U_N}$ denote the unique minimal norm element of the set*

$$\{f \in V^N \mid f(z) = c\}$$

and similarly $\varphi_{z,c}^{U}$ denote the unique minimal norm element of the set

$$\{f \in V \mid f(z) = c\}.$$

Then,

$$\lim_{N\to+\infty} \varphi_{z,c}^{U^N} = \varphi_{z,c}^{U}$$

and the limit above is locally uniform on U.

As in the previous section, by the minimal norm property, it is sufficient to show the above theorem only for $c = 1$. We can also define $\varphi_{z,c}^{U^N}(w) \equiv 0$ for $w \in U \setminus U^N$.

Lemma 5.1. *If U_1 and U_2 are domains such that $U_1 \subseteq U_2$, then $||\varphi_{z,c}^{U^1}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_2)}$.*

Proof. By its very own definition $||\varphi_{z,c}^{U^1}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_1)} \leq ||\varphi_{z,c}^{U_2}||_{W^{k,p}(U_2)}$, it is clear that $\varphi_{z,c}^{U_2} \in W^{k,p}(U_1)$. □

We will now prove the main theorem.

Proof of Theorem 5.1. First, let us show that the locally uniform limit of $\varphi_z^{U^N}$ exists on U. For any compact set $X \subset U$ there exists $j \in \mathbb{N}$ such that $X \subset U_N$ for $N > j$. By properties of the integral and Lemma 4.1 for $N > j$ we have

$$||\varphi_z^{U_N}||_{W^{k,p}(X)} \leq ||\varphi_z^{U_N}||_{W^{k,p}(U_N)} \leq ||\varphi_z^{U}||_{W^{k,p}(U)}.$$

So, the sequence $||\varphi_z^{U_N}||_{W^{k,p}(X)}$ is bounded. Therefore, by (3.1) sequence $\varphi_z^{\mu^N}$ satisfies the assumptions of the Arzelá-Ascoli theorem on X and we conclude that there exists a locally uniformly convergent subsequence of it. Without loss of generality, we can identify it with the entire sequence. Let us denote its limit by g. By the property of the minimal norm, it is sufficient to show that

$$||g||_{W^{k,p}(U)} \leq ||\varphi_z^{U}||_{W^{k,p}(U)}. \tag{5.1}$$

By Fatou's lemma

$$||g||_{W^{k,p}(U)} \leq \liminf_{N\to+\infty} ||\varphi_z^{U^N}||_{W^{k,p}(U_N)}. \tag{5.2}$$

Note that we have $W^{k,p}(U) \subseteq W^{k,p}(U_{j+1}) \subseteq W^{k,p}(U_j)$. Therefore, for any $f \in W^{k,p}(U)$, by Lemma 3.1, we have

$$|f(z)| \leq \frac{1}{||\varphi_z^{U_N}||_{W^{k,p}(U_N)}} ||f||_{W^{k,p}(U_N)}.$$

Taking the limit on the right hand side we get that

$$|f(z)| \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{U_N}||_{W^{k,p}(U_N)}} ||f||_{W^{k,p}(U)}.$$

(We can use Lebesgue dominated convergence theorem to show that

$$||f||_{W^{k,p}(U_N)} \to ||f||_{W^{k,p}(U)}.$$

We just need to consider integrals $\int_U |f(w)|^p \mathbb{I}_{U_n} \mathrm{d}w$, where $\mathbb{I}_X$ is indicator function of set X, and notice that the integrated functions can be dominated by $|f|^p$.) Moreover, by Lemma 3.1, again

$$\frac{1}{||\varphi_z^{U}||_{W^{k,p}(U)}} \leq \frac{1}{\lim_{N\to+\infty} ||\varphi_z^{U_N}||_{W^{k,p}(U_N)}}. \tag{5.3}$$

Combining (5.2) with (5.3), we obtain (5.1). □

6. Examples of Applications

Usually, we want the solutions of differential equations to be elements of some Sobolev spaces. Classical theory states that in many cases, if boundary or initial conditions are given, then the solution is unique. If the only set of solutions of the differential equation under consideration is a vector space that is closed in the corresponding Sobolev norm, and we want to have a unique solution, we can look for the solution with minimal possible Sobolev norm in the set of functions that take given value at a fixed point.

In the rest of the section, we assume without further recalling that our Sobolev norm satisfies the assumptions of Theorem 3.1. By solution we mean a classical solution, i.e., a sufficiently smooth function.

Note also that the term 'minimal solution' of a differential equation is usually used in a different sense (see e.g. [2, 5]).

Example 6.1. Let us consider the set of solutions of Laplace's equation that have a finite Sobolev norm. It is clear that this set forms a vector space. We will show that this vector space is closed in the Sobolev norm topology.

Let (f_n) be a sequence of harmonic functions converging to f in the Sobolev norm topology on a bounded domain U. Then, f_n in particular converges to f in the $L^p(U)$ topology. It is known that the L^p-limit of harmonic functions implies a locally uniform limit of the same sequence to the same limit. Indeed, by Mean Value Theorem for subharmonic functions,

$$|f(x)|^p \leq C_x \int_{B(x,r)} |f(x)|^p \mathrm{d}x,$$

where $B(x,r)$ is the ball with centre in x and sufficiently small radius $r > 0$ to lie with its boundary in U. Moreover, $C_x = \frac{1}{L(r)}$, where $L(r)$ is Lebesgue measure of a ball of radius r, i.e. C_x depends only on distance of x to the boundary. It means that for any compact set $X \subset U$ there is constant C_X, such that for any $x \in X$

$$|f(x)|^p \leq C_X \int_{B(x,r)} |f(x)|^p \mathrm{d}x.$$

Therefore, if f_n converges to f in the topology $L^p(U)$, it also converges to the same limit in the topology of locally uniform convergence. It is known that the locally uniform limit of harmonic functions is also a harmonic function, therefore the vector space of solutions of the Laplace equation with finite Sobolev norm is closed in the topology $W^{k,p}(U)$. This means that there is exactly one harmonic function which takes a given value in a fixed point with minimal Sobolev norm. Moreover, that extremal function depends continuously on a weight of integration and an increasing sequence of domains in precisely defined sense. Note that $V_{z,c}$ is never empty in this example, since the constant function is always its element.

Example 6.2. Now let $p = 2$, i.e., let our Sobolev norm be a norm generated by the inner product. Moreover, let us consider any homogeneous elliptic partial differential

equation with C^∞ coefficients. It is obvious that the set of solutions of such equation, which have a finite $W^{k,2}$ norm, is a vector space. Let (f_n) be a sequence of elements of that space that converges to f in the topology $W^{k,2}(U)$, where U is a bounded domain. Then, in particular, f_n converges to f in $L^2(U)$. It can be shown that the L^2-limit of the classical solutions of an elliptic equation with smooth coefficients is also a classical solution of the same equation (see [6] for the proof). Therefore, the vector space of solutions with finite $W^{k,2}$ norm of the homogeneous elliptic equation with C^∞coefficients is closed in $W^{k,2}(U)$ topology. This means that if we want our solution of the elliptic homogeneous equation with C^∞ coefficients to be unique, we may look for the solution with the minimal $W^{k,2}(U)$ norm among the solutions which take given value at a fixed point. Moreover, if the equation also has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y = 0,$$

then $V_{z,c}$ is never empty, because constant function is always its element.

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FURTHER REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS AND OPERATOR NORM OF HILBERT SPACE OPERATORS

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Abstract. The main purpose of this paper is to give some reverse inequalities for the numerical radius of bounded linear operators on a Hilbert space, in a way that complements many celebrated inequalities in the literature.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on Hilbert space $\mathbb{H}$, with identity I. For $A \in \mathbb{B}(\mathbb{H})$, let $r(A)$, $\omega(A)$, and $\|A\|$ denote the spectral radius, the numerical radius and the operator norm of A, respectively. It is well known that for every $A \in \mathbb{B}(\mathbb{H})$,

$$r(A) \leq \omega(A) \leq \|A\|.$$

Recall that $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A, $\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$ and $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$. The Crawford number of $A \in \mathbb{B}(\mathbb{H})$ is defined by $m(A) = \inf_{\|x\|=1} |\langle Ax, x\rangle|$.

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the operator norm. In fact, the following more precise result holds for every $A \in \mathbb{B}(\mathbb{H})$, as one can find in [11, Theorem 1.3-1],

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

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The inequalities in (1.1) are sharp. Indeed, if $A^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if A is normal.

For recent reverses and refinements of the inequalities in (1.1), see [13–16, 18, 19, 25, 27]. In addition, for a comprehensive account of the numerical radius, the reader is referred to [11]. Finding reverse inequalities for already established results enhances our understanding of these notions, and provides new tools and auxiliary results that we might benefit from. The above references, and many others, have discussed several forms of reverse inequalities.

While the operator norm is sub-multiplicavive in the sense that $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{B}(\mathbb{H})$, the numerical radius is not. However, the numerical radius enjoys the power inequality stating that, for $A \in \mathbb{B}(\mathbb{H})$,

$$\omega\left(A^p\right) \leq \omega^p(A), \tag{1.2}$$

for $p = 1, 2, \ldots$ This was first considered by Berger in [1]. However, the problem of finding the smallest constant c such that

$$\omega^p(A) \leq c\, \omega\left(A^p\right), \quad c \geq 1,$$

is still unsolved.

A weaker version of sub-multiplicativity for the numerical radius asserts that [11, Theorem 2.5-2]:

$$\omega(AB) \leq 4\omega(A)\omega(B). \tag{1.3}$$

This paper presents a general discussion that leads to several relations in the above setting. More precisely, we find some additive reverses of celebrated inequalities related to the numerical radius and the operator norm. For example, we find a constant γ_1, that depends on $A, B \in \mathbb{B}(\mathbb{H})$, such that $\omega(AB) \leq \omega(A)\omega(B) + \gamma_1$. This provides a weak sub-multiplicative behavior of the numerical radius.

Further, we show that

$$\frac{1}{2}\left\||A|^2 + |A^*|^2\right\| - \omega^2\left(A\right) \leq \gamma_2,$$

for some constant γ_2, providing an additive reverse of the celebrated inequality [17]

$$\omega^2\left(A\right) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|. \tag{1.4}$$

Many other relations are proved similarly, covering a wide selection of celebrated results in this direction.

We will need the following lemmas to achieve some of our results. The first lemma presents a refinement and a reverse for the simple inequality $a^p \leq b^p$, whenever $a \leq b$ and $p \geq 1$.

Lemma 1.1. *Let $a, b, m, M \in \mathbb{R}$ and let $p \geq 1$. Then,*

$$0 < m \leq a \leq b \leq M \;\Rightarrow\; \begin{cases} a^p + pm^{p-1}\left(b-a\right) \leq b^p, \\ b^p \leq a^p + pM^{p-1}\left(b-a\right). \end{cases}$$

Proof. Applying Lagrange theorem (Mean Value Theorem) on the function $f(t) = t^p$ implies

$$f(b) - f(a) = p\theta^{p-1}(b-a),$$

for some $\theta \in (a,b)$. This immediately gives the desired conclusion. □

Lemma 1.2 (Buzano's inequality [2]). *Let $a, b, e \in \mathbb{H}$ with $\|e\| = 1$. Then,*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{2}\left(\|a\| \cdot \|b\| + |\langle a, b\rangle|\right).$$

Lemma 1.3 (Hölder-McCarthy inequality [22]). *Let $A \in \mathbb{B}(\mathbb{H})$ be a positive operator. Then, for any unit vector $x \in \mathbb{H}$,*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle, \quad r \geq 1.$$

Lemma 1.4 (Mixed Schwarz inequality [12, pp. 75–76]). *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x, y \in \mathbb{H}$ be any vectors. Then,*

$$|\langle Ax, y\rangle| \leq \sqrt{\langle |A| x, x\rangle \langle |A^*| y, y\rangle}.$$

For $\alpha, \beta \in \mathbb{C}$ and $A \in \mathbb{B}(\mathbb{H})$, we define the following transform

$$C_{\alpha,\beta}(A) := (A^* - \overline{\alpha} I)(\beta I - A).$$

This transformation received some attention in the literature, as seen in [5,10]. In fact, it was introduced in [5] as a tool for discussing some operator bounds. In [10], it was discussed in further details as an indicator of some relations among different operator forms. Also, in [23], it was applied to obtain certain Cassels type inequalities.

We recall that a bounded linear operator A on the complex Hilbert space $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ is called accretive if $\operatorname{Re}\langle Ax, x\rangle \geq 0$ for any $x \in \mathbb{H}$, where $\operatorname{Re}(\cdot)$ denotes the real part operation.

It has been shown in [7, Lemma 86] that if $A \in \mathbb{B}(\mathbb{H})$ is such that $C_{\alpha,\beta}(A)$ is accretive, then

$$\left\| A - \frac{\alpha+\beta}{2} I \right\| \leq \frac{|\alpha - \beta|}{2}. \tag{1.5}$$

The Euclidean operator radius shows some benefits when dealing with the numerical radius. We recall that if $A_1, \ldots, A_n \in \mathbb{B}(\mathbb{H})$, then the Euclidean operator radius of these operators is defined by [24]

$$\omega_e(A_1, A_2, \ldots, A_n) = \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle A_i x, x\rangle|^2 \right)^{\frac{1}{2}}.$$

The following upper bound for the Euclidean operator radius is well known [3, Theorem 1]

$$\omega_e^2(A, B) \leq \left\| |A|^2 + |B|^2 \right\|. \tag{1.6}$$

Related inequalities that include a reversed version of this will also be discussed in the sequel.

2. Upper Bounds for $\left\||A|^2+|B|^2\right\|$

First, we prove the following general form that provides a reversed version of the simple inequality $\omega^2(A)+\omega^2(B)\leq\|A\|^2+\|B\|^2$.

Theorem 2.1. *Let $A,B\in\mathbb{B}(\mathbb{H})$. Then, for any $\alpha,\beta,\lambda,\mu\in\mathbb{C}$,*

$$\begin{aligned}\left\||A|^2+|B|^2\right\|\leq\omega^2(A)+\omega^2(B)+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\+\left\|A-\frac{\alpha+\beta}{2}I\right\|^2+\left\|B-\frac{\lambda+\mu}{2}I\right\|^2.\end{aligned}$$

Proof. On account of [8, Theorem 3], we have

$$\|x\|^2-|\langle x,e\rangle|^2\leq\frac{|\alpha-\beta|^2}{4}+\left\|x-\frac{\alpha+\beta}{2}e\right\|^2, \tag{2.1}$$

where $x,e\in\mathbb{H}$, $\|e\|=1$ and $\alpha,\beta\in\mathbb{C}$. Replacing x by Ax and e by x, with $\|x\|=1$ in (2.1), we obtain

$$\|Ax\|^2\leq|\langle Ax,x\rangle|^2+\frac{|\alpha-\beta|^2}{4}+\left\|\left(A-\frac{\alpha+\beta}{2}I\right)x\right\|^2. \tag{2.2}$$

Similarly, we can show that

$$\|Bx\|^2\leq|\langle Bx,x\rangle|^2+\frac{|\lambda-\mu|^2}{4}+\left\|\left(B-\frac{\lambda+\mu}{2}I\right)x\right\|^2, \tag{2.3}$$

for any $\mu,\lambda\in\mathbb{C}$. Combining (2.2) and (2.3) implies that

$$\begin{aligned}\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle&=\left\langle|A|^2x,x\right\rangle+\left\langle|B|^2x,x\right\rangle\\&=\|Ax\|^2+\|Bx\|^2\\&\leq|\langle Ax,x\rangle|^2+|\langle Bx,x\rangle|^2+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&\quad+\left\|\left(A-\frac{\alpha+\beta}{2}I\right)x\right\|^2+\left\|\left(B-\frac{\lambda+\mu}{2}I\right)x\right\|^2\\&\leq\omega^2(A)+\omega^2(B)+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&\quad+\left\|A-\frac{\alpha+\beta}{2}I\right\|^2+\left\|B-\frac{\lambda+\mu}{2}I\right\|^2,\end{aligned} \tag{2.4}$$

where we have used the definitions of the numerical radius and the operator norm to obtain the second inequality above. We deduce the desired inequality by taking the supremum over $x\in\mathbb{H}$ with $\|x\|=1$. □

The approach we adopt in this work can be also applied to obtain bounds for $\omega^2\left(|A|+\mathrm{i}\,|B|\right)$, as follows.

Corollary 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\omega^2(|A| + \mathrm{i}\,|B|) \leq \omega^2(A) + \omega^2(B) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

Proof. This follows from Theorem 2.1 noting that $\omega^2(|A| + \mathrm{i}|B|) \leq \left\||A|^2 + |B|^2\right\|$. □

Remark 2.1. Set $\alpha = \beta$ and $\lambda = \mu$, in Corollary 2.1, and then take the infimum over α and λ, we get

$$\omega^2(|A| + \mathrm{i}\,|B|) \leq \omega^2(A) + \omega^2(B) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2.$$

In particular,

$$\frac{1}{2}\omega^2(|A| + \mathrm{i}\,|A^*|) \leq \omega^2(A) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2.$$

Notice that the above inequality provides a reverse for the first inequality in [20, Corollary 2.2].

Another upper bound for $\left\||A|^2 + |B|^2\right\|$ in terms of $\omega^2(A) + \omega^2(B)$ can be stated in the following form.

Proposition 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\||A|^2 + |B|^2\right\| \leq \frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. If we put $a = Ax$ (resp. Bx), $b = A^*x$ (resp. B^*x), and $e = x$, in Lemma 1.2, we obtain

$$\begin{aligned}
&|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 \\
\leq&\frac{1}{2}\left(\|Ax\| \cdot \|A^*x\| + \left|\left\langle A^2x, x\right\rangle\right| + \|Bx\| \cdot \|B^*x\| + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
=&\frac{1}{2}\left(\sqrt{\left\langle |A|^2x, x\right\rangle\left\langle |A^*|^2x, x\right\rangle} + \left|\left\langle A^2x, x\right\rangle\right| + \sqrt{\left\langle |B|^2x, x\right\rangle\left\langle |B^*|^2x, x\right\rangle} + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
\leq&\frac{1}{4}\left\langle\left(|A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right)x, x\right\rangle + \frac{1}{2}\left(\left|\left\langle A^2x, x\right\rangle\right| + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
\leq&\frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right),
\end{aligned}$$

where the second inequality is obtained from the arithmetic-geometric mean inequality. So, we obtain the desired result after adding (2.2) and (2.3) together and then taking the supremum over all unit vectors $x \in \mathbb{H}$. □

Remark 2.2. (i) If we set $B = A$ and $\alpha = \beta = \lambda = \mu$, in Proposition 2.1, we get

$$\|A\|^2 \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\omega\left(A^2\right) + \inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2.$$

(ii) If we set $B = A^*$ and $\alpha = \beta, \lambda = \mu = \overline{\alpha}$, in Proposition 2.1, we get

$$\frac{1}{2}\left\||A|^2 + |A^*|^2\right\| \leq \omega\left(A^2\right) + 2\inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2 \leq \omega\left(A\right)^2 + 2\inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2.$$

This provides a reverse and a refinement of the inverse of (1.4).

Remark 2.3. Both Theorem 2.1 and Proposition 2.1 provided upper bounds for $\left\||A|^2 + |B|^2\right\|$. In this remark, we show that the two bounds are incomparable in the sense that neither upper bound is always better than the other. For this, we consider two choices of A, B.

First, if we let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we find that $\omega^2(A)+\omega^2(B) = 2$ and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) = 2.5,$$

showing that Theorem 2.1 is sharper than Proposition 2.1 for these A, B.

On the other hand, letting $A = \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, we find that $\omega^2(A) + \omega^2(B) \approx 15.3218$ and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) \approx 14.4357.$$

We point out that for $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the two bounds are equal.

Proposition 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\||A|^2 + |B|^2\right\| \leq \omega\left(|A| + \mathrm{i}\,|B|\right)\omega\left(|A^*| + \mathrm{i}\,|B^*|\right) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Then,

$$\begin{aligned}
&|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 \\
&\leq \langle |A|\,x, x\rangle\,\langle |A^*|\,x, x\rangle + \langle |B|\,x, x\rangle\,\langle |B^*|\,x, x\rangle \quad \text{(by Lemma 1.4)} \\
&\leq \sqrt{\left(\langle |A|\,x, x\rangle^2 + \langle |B|\,x, x\rangle^2\right)\left(\langle |A^*|\,x, x\rangle^2 + \langle |B^*|\,x, x\rangle^2\right)} \\
&\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= |\langle |A|\,x, x\rangle + \mathrm{i}\,\langle |B|\,x, x\rangle| \cdot |\langle |A^*|\,x, x\rangle + \mathrm{i}\,\langle |B^*|\,x, x\rangle| \\
&\qquad \text{(since } |a + \mathrm{i}b| = \sqrt{a^2+b^2},\ a, b \in \mathbb{R}\text{)}
\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= |\langle (|A| + \mathrm{i}\,|B|)\, x, x\rangle| \cdot |\langle (|A^*| + \mathrm{i}\,|B^*|)\, x, x\rangle| \\
&\leq \omega\left(|A| + \mathrm{i}\,|B|\right) \omega\left(|A^*| + \mathrm{i}\,|B^*|\right).
\end{aligned}
$$

From (2.2) and (2.3) above we have

$$
\|Ax\|^2 \leq |\langle Ax, x\rangle|^2 + \frac{|\alpha - \beta|^2}{4} + \left\| \left(A - \frac{\alpha + \beta}{2} I \right) x \right\|^2. \tag{2.6}
$$

Similarly, we can show that

$$
\|Bx\|^2 \leq |\langle Bx, x\rangle|^2 + \frac{|\lambda - \mu|^2}{4} + \left\| \left(B - \frac{\lambda + \mu}{2} I \right) x \right\|^2. \tag{2.7}
$$

Noting that $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle |A|^2 x, x\rangle$ and that $\|Bx\|^2 = \langle Bx, Bx\rangle = \langle |B|^2 x, x\rangle$, then (2.6) and (2.7), and using (2.5) imply

$$
\begin{aligned}
\left\langle (|A|^2 + |B|^2)x, x \right\rangle \leq & \omega\left(|A| + \mathrm{i}\,|B|\right) \omega\left(|A^*| + \mathrm{i}\,|B^*|\right) \\
& + \frac{|\alpha - \beta|^2}{4} + \left\| \left(A - \frac{\alpha + \beta}{2} I \right) x \right\|^2 + \frac{|\lambda - \mu|^2}{4} \\
& + \left\| \left(B - \frac{\lambda + \mu}{2} I \right) x \right\|^2.
\end{aligned}
$$

Taking the supremum over all unit vectors $x \in \mathbb{H}$ implies the desired conclusion. □

Remark 2.4. If we set $B = A^*$ and $\alpha = \beta, \lambda = \mu = \overline{\alpha}$, in Proposition 2.2, we obtain

$$
\begin{aligned}
\left\| |A|^2 + |A^*|^2 \right\| &\leq \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(|A^*| + \mathrm{i}\,|A|\right) + 2\|A - \alpha I\|^2 \\
&= \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(\mathrm{i}\left(|A^*| + \mathrm{i}\,|A|\right)\right) + 2\|A - \alpha I\|^2 \\
&= \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(\mathrm{i}\,|A^*| - |A|\right) + 2\|A - \alpha I\|^2 \\
&= \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(\left(\mathrm{i}\,|A^*| - |A|\right)^*\right) + 2\|A - \alpha I\|^2 \\
&= \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(-\mathrm{i}\,|A^*| - |A|\right) + 2\|A - \alpha I\|^2 \\
&= \omega\left(|A| + \mathrm{i}\,|A^*|\right) \omega\left(-\left(|A| + \mathrm{i}\,|A^*|\right)\right) + 2\|A - \alpha I\|^2 \\
&= \omega^2\left(|A| + \mathrm{i}\,|A^*|\right) + 2\|A - \alpha I\|^2.
\end{aligned}
$$

Namely,

$$
\left\| |A|^2 + |A^*|^2 \right\| \leq \omega^2\left(|A| + \mathrm{i}\,|A^*|\right) + 2 \inf_{\alpha \in \mathbb{C}} \|A - \alpha I\|^2.
$$

The above inequality provides a reverse for the second inequality in [20, Corollary 2.2].

Another upper bound for $\left\| |A|^2 + |B|^2 \right\|$ is found next. This bound relates the two quantities $\left\| |A|^2 + |B|^2 \right\|$ and $\left\| |A^*|^2 + |B^*|^2 \right\|$. A reversed version is obtained by changing A to A^* and B to B^* in this result.

Corollary 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\||A|^2 + |B|^2\right\| \leq \left\||A^*|^2 + |B^*|^2\right\| + \frac{|\alpha-\beta|^2}{2} + \frac{|\lambda-\mu|^2}{2} + 2\left(\left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2\right).$$

Proof. Direct calculations show that $\omega^2(|A| + \mathrm{i}|B|) \leq \left\||A|^2 + |B|^2\right\|$. Employing this inequality in Proposition 2.2, we obtain

$$\left\||A|^2 + |B|^2\right\| \leq \sqrt{\left\||A|^2 + |B|^2\right\| \cdot \left\||A^*|^2 + |B^*|^2\right\|} + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

We obtain the desired result if we apply the arithmetic-geometric mean inequality on the square root on the right side. □

Remark 2.5. If we set $\alpha = \beta$, $\lambda = \mu$, and then taking infimum over α and λ, in Corollary 2.2, we obtain

$$\left\||A|^2 + |B|^2\right\| \leq \left\||A^*|^2 + |B^*|^2\right\| + 2\left(\inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2\right). \tag{2.8}$$

Of course, if we replace A and B by A^* and B^*, in (2.8), we get

$$\left\||A^*|^2 + |B^*|^2\right\| \leq \left\||A|^2 + |B|^2\right\| + 2\left(\inf_{\alpha\in\mathbb{C}} \|A^* - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|B^* - \lambda I\|^2\right).$$

Letting $B = A$, $\alpha = \lambda$ and $\beta = \mu$, in Theorem 2.1, we infer that

$$\|A\|^2 - \omega^2(A) \leq \frac{|\alpha-\beta|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2. \tag{2.9}$$

In the following, we present an upper bound for the non-negative difference $\|A\|^2 - \omega^2(A)$. We remark that the fact of Corollary 2.3 was pointed out in [4, Corollary 1].

Corollary 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$. Then,*

$$\|A\|^2 - \omega^2(A) \leq \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2. \tag{2.10}$$

Proof. Since α, β are arbitrary, we can choose $\alpha = \beta$ in (2.9) to get

$$\|A\|^2 - \omega^2(A) \leq \|A - \alpha I\|^2.$$

By taking infimum over $\alpha \in \mathbb{C}$, we infer (2.10). □

Remark 2.6. We have seen that Proposition 2.2 has added a new upper bound for $\left\||A|^2 + |B|^2\right\|$. This remark shows that this bound is independent of both bounds in Theorem 2.1 and Proposition 2.1. For this purpose, let us use the notations

$$R_1(A, B) = \omega^2(A) + \omega^2(B),$$

$$R_2(A,B) = \frac{1}{4}\left\||A|^2+|A^*|^2+|B|^2+|B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right)+\omega\left(B^2\right)\right)$$

and

$$R_3(A,B) = \omega\left(|A|+\mathrm{i}\,|B|\right)\omega\left(|A^*|+\mathrm{i}\,|B^*|\right).$$

If we let $A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}$, then numerical calculations show that, for these matrices,

$$R_1(A,B) \approx 21.1939, \quad R_2(A,B) \approx 19.8777, \quad R_3(A,B) \approx 22.7987,$$

showing that Theorem 2.1 and Proposition 1.1 can be both better than Proposition 2.2. On the other hand, if we let $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$, we find that

$$R_1(A,B) \approx 30.3463, \quad R_2(A,B) \approx 28.3897, \quad R_3(A,B) \approx 25.3546,$$

showing that Proposition 2.2 can be better than both Theorem 2.1 and Proposition 2.1.

Proposition 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ and $p \geq 1$. Then,*

$$\|A\|^{2p} - \omega^{2p}(A) \leq p\|A\|^{p-1} \inf_{\alpha\in\mathbb{C}} \|A-\alpha I\|^2.$$

Proof. We know that $\omega^2(A) \leq \|A\|^2$. We infer by substituting $a = \omega^2(A)$, $b = M = \|A\|^2$, in Lemma 1.1, that

$$\begin{aligned} \|A\|^{2p} &\leq \omega^{2p}(A) + p\|A\|^{p-1}\left(\|A\|^2 - \omega^2(A)\right) \\ &\leq \omega^{2p}(A) + p\|A\|^{p-1} \inf_{\alpha\in\mathbb{C}} \|A-\alpha I\|^2, \end{aligned}$$

where the second inequality is obtained from Corollary 2.3. This completes the proof of the theorem. □

Remark 2.7. Let $A \in \mathbb{B}(\mathbb{H})$. It can be easily seen that there is a scalar $\gamma \in \mathbb{C}$ such that $\inf_{\alpha\in\mathbb{C}} \|A-\alpha I\| = \|A-\gamma I\|$. Put

$$\Gamma_A = \left\{\gamma \in \mathbb{C} : \ \|A-\gamma I\| = \inf_{\alpha\in\mathbb{C}} \|A-\alpha I\|\right\}.$$

Therefore, from Corollary 2.3, we have

$$\|A\|^2 - \omega^2(A) \leq \inf_{\alpha\in\mathbb{C}} \|A-\alpha I\|^2 = \|A-\gamma I\|, \quad \gamma \in \Gamma_A.$$

Letting $B = A^*$, in Theorem 2.1, we get

(2.11)

$$\left\||A|^2+|A^*|^2\right\| \leq 2\omega^2(A) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|A^* - \frac{\lambda+\mu}{2}I\right\|^2.$$

In the following, we aim to find an upper estimate for the non-negative quantity $\frac{1}{2}\left\||A|^2+|A^*|^2\right\| - \omega^2(A)$, motivated by (1.4).

Corollary 2.4 ([26, Corollary 2.2])**.** *Let $A \in \mathbb{B}(\mathbb{H})$. Then,*

$$\frac{1}{2}\left\||A|^2+|A^*|^2\right\|-\omega^2(A)\leq \inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2.$$

Proof. It follows from (2.11) that

$$\frac{1}{2}\left\||A|^2+|A^*|^2\right\|-\frac{1}{2}\left(\|A-\alpha I\|^2+\|A^*-\lambda I\|^2\right)\leq \omega^2(A). \tag{2.12}$$

We obtain the desired result if we put $\lambda=\overline{\alpha}$ in (2.12). □

Proposition 2.4. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{aligned}\left\||A|^2+|B|^2\right\|\leq &\max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&+\left\|A-\frac{\alpha+\beta}{2}I\right\|^2+\left\|B-\frac{\lambda+\mu}{2}I\right\|^2.\end{aligned}$$

Proof. It follows from (3.239) in [7] that

$$|\langle Ax,x\rangle|^2+|\langle Bx,x\rangle|^2\leq \max\left\{\|Ax\|^2,\|Bx\|^2\right\}+|\langle Ax,Bx\rangle|,$$

for any unit vector $x \in \mathbb{H}$. Hence, by adding (2.2) and (2.3) together, we obtain

$$\begin{aligned}\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle\leq &\max\left\{\|Ax\|^2,\|Bx\|^2\right\}+|\langle Ax,Bx\rangle|+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&+\left\|\left(A-\frac{\alpha+\beta}{2}I\right)x\right\|^2+\left\|\left(B-\frac{\lambda+\mu}{2}I\right)x\right\|^2.\end{aligned} \tag{2.13}$$

We deduce the desired result by taking the supremum over all unit vectors $x \in \mathbb{H}$. □

Remark 2.8. By setting $\alpha=\beta$ and $\lambda=\mu$, in Proposition 2.4, we get

$$\left\||A|^2+|B|^2\right\|\leq \max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)+\|A-\alpha I\|^2+\|B-\lambda I\|^2.$$

If we put $B=A^*$, we infer that

$$\left\||A|^2+|A^*|^2\right\|\leq \|A\|^2+\omega\left(A^2\right)+\|A-\alpha I\|^2+\|A^*-\lambda I\|^2,$$

for any $\alpha, \lambda \in \mathbb{C}$. The case $\lambda=\overline{\alpha}$ implies that

$$\left\||A|^2+|A^*|^2\right\|\leq \|A\|^2+\omega\left(A^2\right)+2\inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2.$$

Remark 2.9. If we let $R_4(A,B)=\max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)$, we find that for $A=\begin{bmatrix}2&-1\\3&0\end{bmatrix}$, $B=\begin{bmatrix}-1&-3\\3&-1\end{bmatrix}$, and

$$R_1(A,B)=16,\quad R_2(A,B)\approx 17.9142,\quad R_3(A,B)\approx 23.3246,\quad R_4(A,B)\approx 22.7967,$$

where R_1, R_2 and R_3 are as in Remark 2.6, showing that the bound in Proposition 2.4 can be weaker than those in Theorem 2.1, Proposition 2.1 and Proposition 2.2. On the

other hand, letting $A = \begin{bmatrix} -1 & -3 \\ -2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$ shows that $R_1(A,B) \approx 25.698$,

$$R_2(A,B) \approx 25.8801, \quad R_3(A,B) \approx 24.7839, \quad R_4(A,B) \approx 21.0623,$$

meaning that Proposition 2.4 can also be better than the three aforementioned results.

Proposition 2.5. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\| |A|^2 + |B|^2 \right\| \leq \left\| |A|^2 - |B|^2 \right\| + 2\omega\left(B^*A\right) + \frac{|\alpha-\beta|^2}{2} + \frac{|\lambda-\mu|^2}{2} + 2\left(\left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2 \right).$$

Proof. We can see from (2.13) that

$$\begin{aligned}
&\left\langle \left(|A|^2 + |B|^2 \right) x, x \right\rangle \\
\leq &\max\left\{ \|Ax\|^2, \|Bx\|^2 \right\} + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
&+ \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2 \\
= &\frac{1}{2}\left(\|Ax\|^2 + \|Bx\|^2 + \left| \|Ax\|^2 - \|Bx\|^2 \right| \right) + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
&+ \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2 \\
= &\frac{1}{2}\left(\left\langle \left(|A|^2 + |B|^2 \right) x, x \right\rangle + \left| \left\langle \left(|A|^2 - |B|^2 \right) x, x \right\rangle \right| \right) + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} \\
&+ \frac{|\lambda-\mu|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2.
\end{aligned}$$

We deduce the desired result by taking the supremum over all unit vectors $x \in \mathbb{H}$. □

Remark 2.10. By setting $\alpha = \beta$ and $\lambda = \mu$, in Proposition 2.13, we get

$$\left\| |A|^2 + |B|^2 \right\| \leq \left\| |A|^2 - |B|^2 \right\| + 2\left(\omega\left(B^*A\right) + \left(\|A - \alpha I\|^2 + \|B - \lambda I\|^2 \right) \right).$$

If we put $B = A^*$, we infer that

$$\left\| |A|^2 + |A^*|^2 \right\| \leq \left\| |A|^2 - |A^*|^2 \right\| + 2\left(\omega\left(A^2\right) + \left(\|A - \alpha I\|^2 + \|A^* - \lambda I\|^2 \right) \right),$$

for any $\alpha, \lambda \in \mathbb{C}$. The case $\lambda = \overline{\alpha}$ implies that

$$\left\| |A|^2 + |A^*|^2 \right\| \leq \left\| |A|^2 - |A^*|^2 \right\| + 2\left(\omega\left(A^2\right) + 2\inf_{\alpha \in \mathbb{C}} \|A - \alpha I\|^2 \right).$$

In connection with (1.6), we present the following upper bound for $\left\| |A|^2 + |B|^2 \right\| - \omega_e^2(A,B)$.

Theorem 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

(2.14)
$$\left\| |A|^2 + |B|^2 \right\| - \omega_e^2(A,B) \leq \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2.$$

In particular,

$$\left\| |A|^2 + |B|^2 \right\| \leq \omega_e^2(A,B) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2. \tag{2.15}$$

Proof. It observes from the first inequality in (2.4) that

$$\begin{aligned}
\left\langle \left(|A|^2 + |B|^2 \right) x, x \right\rangle \leq & |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
& + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2 \\
\leq & \omega_e^2(A,B) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
& + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2.
\end{aligned}$$

We get the first inequality by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$.

The second inequality follows from (2.14) by placing $\alpha = \beta$ and $\lambda = \mu$ and then taking the infimum over α and λ. □

If $T = \mathrm{Re}T + \mathrm{i}\mathrm{Im}T$ is the Cartesian decomposition of T, then from Theorem 2.2, we have the following reverse of (1.4).

Corollary 2.5. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \lambda \in \mathbb{C}$,*

$$\frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\| - \omega^2(A) \leq \inf_{\alpha\in\mathbb{C}} \|\mathrm{Re}A - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|\mathrm{Im}A - \lambda I\|^2.$$

Remark 2.11. If, in Corollary 2.5, ReA and ImA are scalar multiples of the identity, we deduce that $\frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\| = \omega^2(A)$.

3. Upper Bounds for $\|B^*A\|$ and the Power Inequality

Due to sub-multiplicativity of the operator norm, one has $\|B^*A\| \leq \|B\| \cdot \|A\| \leq 4\omega(B)\omega(A)$. In the following result, we present a new form for an upper bound of $\|B^*A\|$ in terms of the numerical radii.

Theorem 3.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{aligned}
\|B^*A\| \leq & \sqrt{\omega^2(A) + \frac{|\alpha-\beta|^2}{4} + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2} \\
& \times \sqrt{\omega^2(B) + \frac{|\lambda-\mu|^2}{4} + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2}.
\end{aligned}$$

Furthermore, the inequality is sharp.

Proof. We can show from (2.2) that

$$\|By\|^2 \leq |\langle By, y\rangle|^2 + \frac{|\lambda-\mu|^2}{4} + \left\|\left(B - \frac{\lambda+\mu}{2}I\right)y\right\|^2, \tag{3.1}$$

for any unit vector $y \in \mathbb{H}$. So, by (2.2) and (3.1), we can write

$$\begin{aligned}
|\langle B^*Ax, y\rangle| &= |\langle Ax, By\rangle| \\
&\leq \|Ax\| \cdot \|By\| \\
&\leq \sqrt{|\langle Ax, x\rangle|^2 + \frac{|\alpha-\beta|^2}{4} + \left\|\left(A - \frac{\alpha+\beta}{2}I\right)x\right\|^2} \\
&\quad \times \sqrt{|\langle By, y\rangle|^2 + \frac{|\lambda-\mu|^2}{4} + \left\|\left(B - \frac{\lambda+\mu}{2}I\right)y\right\|^2} \\
&\leq \sqrt{\omega^2(A) + \frac{|\alpha-\beta|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2} \\
&\quad \times \sqrt{\omega^2(B) + \frac{|\lambda-\mu|^2}{4} + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2}.
\end{aligned}$$

We deduce the desired inequality by taking supremum over $x, y \in \mathbb{H}$ with $\|x\| = \|y\| = 1$. □

Corollary 3.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \lambda \in \mathbb{C}$,*

$$\|B^*A\| \leq \sqrt{\omega^2(A) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2}\sqrt{\omega^2(B) + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2}.$$

Proof. Letting $\alpha = \beta$ and $\lambda = \mu$ and then taking infimum over α and λ, in Theorem 3.1, we obtain the desired inequality. □

Remark 3.1. We notice that if $A = \sigma_1 I$, $B = \sigma_2 I$, for some $\sigma_1, \sigma_2 \in \mathbb{C}$, then by selecting $\alpha = \sigma_1, \gamma = \sigma_2$ in Corollary 3.1, the inequality becomes an equality with both sides equal to $\sigma_1\sigma_2$.

We continue this section with the following result, which yields a reverse for the inequality (1.2) in the case $p = 2$.

Theorem 3.2. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\omega^2(A) - \omega\left(A^2\right) \leq \sqrt{\frac{|\alpha-\beta|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2}\sqrt{\frac{|\lambda-\mu|^2}{4} + \left\|A^* - \frac{\lambda+\mu}{2}I\right\|^2}.$$

Proof. Thanks to [8, Theorem 4], we have

$$|\langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle| \leq \left(\frac{|\alpha-\beta|^2}{4} + \left\|x - \frac{\alpha+\beta}{2}z\right\|^2\right)^{\frac{1}{2}} \tag{3.2}$$

$$\times \left(\frac{|\lambda-\mu|^2}{4} + \left\| y - \frac{\lambda+\mu}{2} z \right\|^2 \right)^{\frac{1}{2}},$$

for all $x, y, z \in \mathbb{H}$, with $\|z\| = 1$, and for every $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. Applying (3.2) for $x = Au$, $y = A^*u$ and $z = u$, we get

$$\begin{aligned}
&|\langle Au, u\rangle|^2 - \left|\left\langle A^2u, u\right\rangle\right| \\
\leq &\left|\left\langle A^2u, u\right\rangle - \langle Au, u\rangle^2\right| \\
\leq &\left(\frac{|\alpha-\beta|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) u \right\|^2 \right)^{\frac{1}{2}} \left(\frac{|\lambda-\mu|^2}{4} + \left\| \left(A^* - \frac{\lambda+\mu}{2} I \right) u \right\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

for any $u \in \mathbb{H}$ with $\|u\| = 1$. Namely,

$$\begin{aligned}
|\langle Au, u\rangle|^2 \leq &\left|\left\langle A^2u, u\right\rangle\right| + \left(\frac{|\alpha-\beta|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) u \right\|^2 \right)^{\frac{1}{2}} \\
&\times \left(\frac{|\lambda-\mu|^2}{4} + \left\| \left(A^* - \frac{\lambda+\mu}{2} I \right) u \right\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

for any $u \in \mathbb{H}$ with $\|u\| = 1$.

We conclude the result by taking the supremum over $u \in \mathbb{H}$ with $\|u\| = 1$ in the above inequality. □

If we put $\alpha = \beta$ and $\lambda = \mu$, in Theorem 3.2, we get

$$\omega^2(A) \leq \omega\left(A^2\right) + \|A - \alpha I\| \cdot \|A^* - \lambda I\|. \tag{3.3}$$

In particular, if $\lambda = \overline{\alpha}$, in (3.3), we infer that [6]

$$\omega^2(A) - \omega\left(A^2\right) \leq \inf_{\alpha \in \mathbb{C}} \|A - \alpha I\|^2.$$

When $x \in \mathbb{H}$ is a unit vector, the Cauchy-Schwarz inequality asserts that $|\langle Ax, x\rangle| \leq \|Ax\|$. In the following, a reversed version of this inequality is presented.

Proposition 3.1. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x \in \mathbb{H}$ be a unit vector. Then, for any $\alpha, \beta \in \mathbb{C}$,*

$$\|Ax\|^2 \leq |\langle Ax, x\rangle|^2 + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 - m^2 \left(A - \frac{\alpha+\beta}{2} I \right),$$

where $m(X) = \inf_{\|x\|=1} |\langle Xx, x\rangle|$, for $X \in \mathbb{B}(\mathbb{H})$.

Proof. Assume that Φ is a unital positive linear map on $\mathbb{B}(\mathbb{H})$. From [21, (4.2)], we know that for any $A \in \mathbb{B}(\mathbb{H})$

$$\Phi\left(|A|^2\right) - |\Phi(A)|^2 = \Phi\left(|A - cI|^2\right) - |\Phi(A - cI)|^2, \quad c \in \mathbb{C}. \tag{3.4}$$

Now, when $x \in \mathbb{H}$ is a unit vector, the mapping $\Phi : \mathbb{B}(\mathbb{H}) \to \mathbb{B}(\mathbb{H})$ defined by $\Phi(A) = \langle Ax, x\rangle$ is a unital positive linear map. Consequently, (3.4) implies, for any unit vector $x \in \mathbb{H}$ and $A \in \mathbb{B}(\mathbb{H})$,

$$\begin{aligned}
\|Ax\|^2 - |\langle Ax, x\rangle|^2 &= \left\langle |A|^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \\
&= \Phi\left(|A|^2\right) - |\Phi(A)|^2 \\
&= \Phi\left(\left|A - \frac{\alpha+\beta}{2}I\right|^2\right) - \left|\Phi\left(A - \frac{\alpha+\beta}{2}I\right)\right|^2 \\
&= \left\langle \left|A - \frac{\alpha+\beta}{2}I\right|^2 x, x\right\rangle - \left|\left\langle \left(A - \frac{\alpha+\beta}{2}I\right)x, x\right\rangle\right|^2 \\
&= \left\|\left(A - \frac{\alpha+\beta}{2}I\right)x\right\|^2 - \left|\left\langle \left(A - \frac{\alpha+\beta}{2}I\right)x, x\right\rangle\right|^2 \\
&\leq \left\|\left(A - \frac{\alpha+\beta}{2}I\right)x\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right),
\end{aligned}$$

as required. □

The next result improves [9, Corollary 2.4].

Corollary 3.2. *Let $A \in \mathbb{B}(\mathbb{H})$. If $C_{\alpha,\beta}(A)$ is accretive, then*

$$\|A\|^2 - \omega^2(A) \leq \frac{|\alpha-\beta|^2}{4} - m^2\left(A - \frac{\alpha+\beta}{2}I\right).$$

Proof. It follows from Proposition 3.1 and the relation (1.5) that

$$\begin{aligned}
\|Ax\|^2 &\leq |\langle Ax, x\rangle|^2 + \left\|\left(A - \frac{\alpha+\beta}{2}I\right)x\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right) \\
&\leq \omega^2(A) + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right) \\
&\leq \omega^2(A) + \frac{|\alpha-\beta|^2}{4} - m^2\left(A - \frac{\alpha+\beta}{2}I\right).
\end{aligned}$$

We conclude the result by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$ in the above inequality. □

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