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NEW GENERALIZED BIPARAMETRIC FUBINI-TYPE POLYNOMIALS OF LEVEL-m

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Abstract. This paper introduces two new biparametric families of generalized Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x,y,a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x,y,a)$. We give some algebraic and differential properties, as well as relationships between these classes of polynomials with other polynomials and numbers. In addition, we introduce the new generalized biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$. As a result, we derive the product formula for $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$ and provide some factorizations of biparametric Fubini-type polynomials matrix $\mathscr{D}_c^{[m-1,\alpha]}(x,y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x,y;a)$, which involve the generalized Pascal matrix.

1. Introduction

This section presents some generating functions that are well-known Fubini numbers and Fubini polynomials, which have many applications in several fields of mathematics, physics, and engineering. The Fubini numbers $w_g(n)$, were studied, and are given by the following generating function (see, [3, p. 11, Eq. (1)]):

$$\frac{1}{2-e^z}=\sum_{n=0}^{+\infty} w_g(n)\frac{z^n}{n!}, \quad |z|<\ln 2,$$

where $w_g(0)=1$.

Key words and phrases. Generalized parametric Fubini-type polynomials, Fubini polynomials, Apostol-Bernoulli polynomials of higher order, Apostol-Genocchi polynomials of higher order, Stirling numbers of the second kind, generalized Pascal matrix.

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On the other hand, the Fubini-type numbers were defined and are given by the following generating function (see, [10, p. 397, Eq. (10.21)]):

$$\frac{e^z - 1}{2 - e^z} = \sum_{n=0}^{+\infty} w_M(n) \frac{z^n}{n!}, \quad |z| < \ln 2.$$

Also, a new family of numbers a_n was defined, which are obtained by modifying the numbers $w_g(n)$, give by (see, [4, p. 1609, Eq. (14)]):

$$\frac{2}{(2 - e^z)^2} = \sum_{n=0}^{+\infty} a_n \frac{z^n}{n!}, \quad |z| < \ln 2.$$

Recently, the generalized Fubini-type polynomial $a_n^{(\alpha)}(x)$ of order α, was defined by means of the following generating function (see, [4, p. 1611, Eq. (18)]):

$$\left(\frac{2}{(2 - e^z)^2}\right)^{\alpha} e^{xz} = \sum_{n=0}^{+\infty} a_n^{(\alpha)}(x) \frac{z^n}{n!}, \quad |z| < \ln 2,$$

where $\alpha \in \mathbb{N}_0$. Observe that $a_n^{(\alpha)}(0) = a_n^{(\alpha)}$ denotes the Fubini-type numbers of order α.

Two new families of biparametric Fubini-type polynomials were defined, which are given by the following generating function (see, [14, Eq. (2.2) and Eq. (2.3)]):

$$\left(\frac{2}{(2 - e^z)^2}\right)^{\alpha} e^{xz} \cos(yz) = \sum_{n=0}^{+\infty} a_n^{(c,\alpha)}(x, y) \frac{z^n}{n!}, \tag{1.1}$$

$$\left(\frac{2}{(2 - e^z)^2}\right)^{\alpha} e^{xz} \sin(yz) = \sum_{n=0}^{+\infty} a_n^{(s,\alpha)}(x, y) \frac{z^n}{n!}. \tag{1.2}$$

Another important family in this area of study is the Genocchi polynomials $G_n(x)$, a sequence of polynomials related to a generalization of the Genocchi numbers. These polynomials arise in number theory and are related to the Riemann zeta function and Fourier series. Their generating function is given (see, [12, Eq. (2)]):

$$\sum_{n=0}^{+\infty} \frac{G_n(x)}{n!} t^n = \frac{2te^t}{e^t + 1} e^{xt}, \quad |t| < \pi.$$

In the past, some authors have introduced and studied new families of biparametric polynomials in which they demonstrate interesting properties and presented useful applications in this area of mathematics (see, [1,5,7,8,13,14]). One of the applications introduced by a large number of authors in various contexts is the matrix approach of several families of polynomials [6,16,17]. This paper will introduce two new classes of biparametric Fubini-type polynomials of level-m (see, [4,11,14,17]). Finally, we will show the relationships between these polynomials and other polynomials and numbers, as well as the generalized biparametric Fubini-type polynomials matrices.

This paper will be organized as follow. Section 2 contains the definitions and some auxiliary results. Section 3 will define the generalized biparametric Fubini-type of level-m polynomials and proved some algebraic and differential properties of them, as well as their relationships with other families of polynomials and the Stirling numbers of the second kind. Part 4 will introduce the generalized biparametric Fubini-type polynomial matrix derived a product formula for it. In Section 5 we will give some factorizations for such a matrix, which involves summation matrices and the generalized Pascal matrix of the first kind, respectively. Finally, the conclusion of the paper will be presented.

2. Previous Definitions and Notations

Throughout this paper, we use the following standard notions by: $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural, nonnegative integer, real and complex numbers, respectively. Furthermore, we denote the symbol of Pochhammer by

$$(\lambda)_k = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1),$$

where $(\lambda)_0 = 1$ and $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. For the complex logarithm, we consider the principal branch. All matrices are in $M_{n+1}(\mathbb{K})$, the set of all $(n+1)\times(n+1)$ matrices over the field $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$. Also, for i, j any nonnegative integers we adopt the following convention $\binom{i}{j} = 0$, whenever $j > i$. Aditionally, we take that $0^0 = 1$ and $0^n = 0$ if $n \in \mathbb{N}$.

For real parameters x and y the Taylor series representation in $z = 0$ of the following functions $e^{xz}\cos(yz)$ and $e^{xz}\sin(yz)$ are given by (see, [14]):

$$\begin{aligned} F_c(z;x;y) =& e^{xz}\cos(yz) = \sum_{k=0}^{+\infty} C_k(x,y)\frac{z^k}{k!}, \\ F_s(z;x;y) =& e^{xz}\sin(yz) = \sum_{k=0}^{+\infty} S_k(x,y)\frac{z^k}{k!}, \end{aligned}$$

where the expressions $C_k(x,y)$ and $S_k(x,y)$ are given by:

$$\begin{aligned} C_k(x;y) =& \sum_{j=0}^{\left[\frac{k}{2}\right]}(-1)^j\binom{k}{2j}x^{k-2j}y^{2j}, \\ S_k(x;y) =& \sum_{j=0}^{\left[\frac{k-1}{2}\right]}(-1)^j\binom{k}{2j+1}x^{k-2j-1}y^{2j+1}. \end{aligned}$$

Proposition 2.1. *For $n \in \mathbb{N}$, let $\{a_n^{(c,\alpha)}(x,y)\}_{n\geq 0}$ and $\{a_n^{(s,\alpha)}(x,y)\}_{n\geq 0}$ be the sequences of Fubini-type biparametric polynomials of order α, respectively. Then, the following statements hold* (*see,* [4, 14]).

(a) *Special values. For $\alpha, \beta \in \mathbb{N}$,*

$$a_n^{(\alpha+\beta)} = \sum_{k=0}^{n}\binom{n}{k}a_k^{(\alpha)}a_{n-k}^{(\beta)}.$$

(b) *Summation formulas:*

$$a_n^{(c,\alpha)}(x+y,y) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(c,\alpha)}(x,y)y^{n-k}, \quad \alpha \in \mathbb{N}_0,$$

$$a_n^{(s,\alpha)}(x+y,y) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(s,\alpha)}(x,y)y^{n-k}, \quad \alpha \in \mathbb{N}_0,$$

$$a_n^{(\alpha)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(\alpha)}(x),$$

$$\frac{\partial^m}{\partial x^m} a_n^{(\alpha)}(x) = m!\binom{n}{m} a_{n-m}^{(\alpha)}(x), \quad n,m \in \mathbb{N}, n \geqslant m,$$

$$\int_0^1 a_{n-m}^{(\alpha)}(x)dx = \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{a_k^{(\alpha)}}{n+1-k-m}.$$

For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, the Stirling numbers of second kind $S(n,k)$ are defined by means to of the following expansion (see, [2, p. 207]):

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k).$$

The Jacobi polynomial of the degree n and order (α,β), with $\alpha, \beta > -1$, $P_n^{(\alpha,\beta)}(x)$ may be defined through Rodrigues' formula (see, [11, p. 395]):

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\alpha}\right\}.$$

The relationship between the n-th monomial x^n and the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be written as

$$x^n = n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}} P_k^{(\alpha,\beta)}(1-2x). \tag{2.1}$$

Proposition 2.2. *For $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$, $\{G_n(x)\}_{n\geq 0}$ and $\{\mathscr{E}_n(x;\lambda)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level-m, Genocchi polynomials and Apostol-Euler polynomials, respectively. Then, we have the relationships:*

(a) [15, Eq. (4)]

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x), \tag{2.2}$$

(b) [12, Eq. (8)]

$$x^n = \frac{1}{2(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} G_k(x) + G_{n+1}(x)\right], \tag{2.3}$$

(c) [9, Eq. (32)]

$$x^n = \frac{1}{2}\left[\lambda \sum_{k=0}^{n} \binom{n}{k} \mathscr{E}_k(x;\lambda) + \mathscr{E}_n(x;\lambda)\right]. \tag{2.4}$$

Let x be any nonzero real number. The generalized Pascal matrix of first kind $P[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by (see, [17, Definition 1]):

$$p_{i,j}(x) := \begin{cases} \binom{i}{j}(x)^{i-j}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Let the $(n+1)\times(n+1)$ Fibonacci matrix be $\mathscr{F} = [f_{i,j}]$, $i,j = 0,1,2,\ldots,n$, whose entries are given by (see, [18, Eq. (16)]):

$$f_{i,j} := \begin{cases} F_{i-j+1}, & \text{if } i-j+1 \geq 0, \\ 0, & \text{if } i-j+1 < 0, \end{cases}$$

where F_n is the n-th Fibonacci number. Also, they gave the inverse of $\mathscr{F}$ as follows. If $\mathscr{F}^{-1} = [f'_{i,j}]$, where $i,j = 0,1,2,\ldots,n$, then

$$f'_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i = j+1, i = j+2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Let $\{L_n\}_{n\geq 1}$ be the Lucas numbers sequence, i.e., $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ with initial conditions $L_1 = 1$ and $L_2 = 3$. The Lucas matrix $\mathscr{L} \in M_{n+1}(\mathbb{R})$ is the matrix whose entries are given by (see, [19, Eq. (2)]):

$$l_{i,j} := \begin{cases} L_{i-j+1}, & \text{if } i-j \geq 0, \\ 0, & \text{if } i-j < 0, \end{cases}$$

where L_n is the n-th Lucas number.

3. Generalized Biparametric Fubini Type Polynomials of Level-m

In this section, we introduce two new biparametric family of Fubini type polynomials of level-m and prove some algebraic and differential properties of them.

Definition 3.1. For $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$, $a \in \mathbb{R}^+$ and $z \in \mathbb{C}\setminus\{0\}$, the generalized biparametric Fubini-type polynomials in the variable x and y, parameter a order α and level-m, are defined through the following generating functions

$$\left[\frac{\displaystyle\sum_{h=0}^{m-1} \frac{(z\ln a)^h}{h!} + 1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz) = \sum_{n=0}^{+\infty} \mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!}, \qquad |z| < \ln 2, \tag{3.1}$$

$$\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^\alpha e^{xz}\sin(yz)=\sum_{n=0}^{+\infty}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!},\quad |z|<\ln 2. \tag{3.2}$$

For $x=0$, $y=0$ in (3.1) and $x=0$, $y=\frac{\pi}{2z}$ in (3.2) we obtain the generalized biparametric Fubini-type numbers of parameter a order α and level-m

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(a):=\mathscr{A}_{n,c}^{[m-1,\alpha]}(0;0;a),$$
$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(a):=\mathscr{A}_{n,s}^{[m-1,\alpha]}(0;\frac{\pi}{2z};a).$$

By comparing Definition 3.1 with (1.1) and (1.2) we have

$$\mathscr{A}_{n,c}^{[0,\alpha]}(x;y;e)=a_n^{(c,\alpha)}(x,y),$$
$$\mathscr{A}_{n,s}^{[0,\alpha]}(x;y;e)=a_n^{(s,\alpha)}(x,y).$$

Example 3.1. Using (3.3) for any $m=2$ and $a=2$, the first generalized biparametric Fubini-type numbers, order α and level-m are:

$$\mathscr{A}_{0,c}^{[1,\alpha]}(0;0;2)=2^\alpha,$$
$$\mathscr{A}_{1,c}^{[1,\alpha]}(0;0;2)=2^\alpha\left(\frac{42272111787129327}{18014398509481984}\alpha\right),$$
$$\mathscr{A}_{2,c}^{[1,\alpha]}(0;0;2)=2^\alpha\left(\frac{1786931434943558184171171423472929}{324518553658426726783156020576256}\right)\alpha^2 + 2^\alpha\left(\frac{1259095235339315236735153314706143}{324518553658426726783156020576256}\right)\alpha.$$

Example 3.2. Using (3.3) for any $m=2$ and $a=2$, the first generalized biparametric Fubini-type polynomials in the variable x and y, order α and level-m are:

$$\mathscr{A}_{0,c}^{[1,\alpha]}(x;y;2)=2^\alpha,$$
$$\mathscr{A}_{1,c}^{[1,\alpha]}(x;y;2)=2^\alpha\left(\frac{42272111787129327}{18014398509481984}\alpha+x\right),$$
$$\mathscr{A}_{2,c}^{[1,\alpha]}(x;y;2)=2^\alpha\left(\frac{1786931434943558184171171423472929}{324518553658426726783156020576256}\right)\alpha^2 + 2^\alpha\left(\frac{1523013335141436710453162369089536}{324518553658426726783156020576256}\right)\alpha x + 2^\alpha\left(\frac{1259095235339315236735153314706143}{324518553658426726783156020576256}\right)\alpha + 2^\alpha\left(x^2-y^2\right).$$

Example 3.3. Using (3.4) for any $m=4$, and $a=3$, the first generalized biparametric Fubini-type polynomials in the variable x and y, order α and level-m are:

$$\mathscr{A}_{0,s}^{[3,\alpha]}(x;y;3)=2^\alpha y,$$

$$
\begin{aligned}
\mathscr{A}_{1,s}^{[3,\alpha]}(x;y;3) &= 2^{\alpha}\left(\frac{11481054201676165}{4503599627370496}\alpha y + xy\right),\\
\mathscr{A}_{2,s}^{[3,\alpha]}(x;y;3) &= 2^{\alpha}\left(\frac{3954438167454777672872865873216 75}{6084722881095501127184175385 8048}\right)\alpha^2 y\\
&\quad + 2^{\alpha}\left(\frac{31023642854693546475776800456704 0}{60847228810955011271841753858048}\right)\alpha x y\\
&\quad + 2^{\alpha}\left(\frac{26174879013924692794648460905197 9}{60847228810955011271841753858048}\right)\alpha y\\
&\quad + 2^{\alpha}\left(x^2 y - \frac{20282409603651670423947251286016}{60847228810955011271841753858048}y^3\right).
\end{aligned}
$$

Theorem 3.1. *For $m \in \mathbb{N}$, let $\{\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\}_{n\geq 0}$ and $\{\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\}_{n\geq 0}$ be the sequence of generalized biparametric Fubini-type polynomials, with parameter $a \in \mathbb{R}^{+}$, order α and level-m. Then, the following statements hold.*

(a) *For every $\alpha = 0$ and $n \in \mathbb{N}_0$*

$$
\mathscr{A}_{n,c}^{[m-1,0]}(x;y;a) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \tag{3.3}
$$

$$
\mathscr{A}_{n,s}^{[m-1,0]}(x;y;a) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \tag{3.4}
$$

(b) *For $\alpha \in \mathbb{N}$ and $n, k \in \mathbb{N}_0$, we have the relationship*

$$
\begin{aligned}
\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a) &= \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(a) C_k(x,y)\\
&= \sum_{k=0}^{n} \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{n}{k}\binom{k}{2j} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(a) x^{k-2j} y^{2j},\\
\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a) &= \sum_{k=0}^{n} \binom{n}{k} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(a) S_k(x,y)\\
&= \sum_{k=0}^{n} \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{n}{k}\binom{k}{2j+1} (-1)^j \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(a) x^{k-2j-1} y^{2j+1}.
\end{aligned}
$$

(c) *Differential relationships. For $m \in \mathbb{N}$ and $n, k \in \mathbb{N}_0$, with $0 \leq k \leq n$, we have*

$$
\begin{aligned}
\frac{\partial^k}{\partial x^k}\left[\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\right] &= \frac{n!}{(n-k)!}\, \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a),\\
\frac{\partial^k}{\partial x^k}\left[\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a))\right] &= \frac{n!}{(n-k)!}\, \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a).
\end{aligned}
$$

(d) *Integral formulas. For $m \in \mathbb{N}$, is fulfilled*

$$\int_{x_0}^{x_1} \mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\,dx = \frac{\left[\mathscr{A}_{n+1,c}^{[m-1,\alpha]}(x_1;y;a) - \mathscr{A}_{n+1,c}^{[m-1,\alpha]}(x_0;y;a)\right]}{n+1},$$

$$\int_{x_0}^{x_1} \mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\,dx = \frac{\left[\mathscr{A}_{n+1,s}^{[m-1,\alpha]}(x_1;y;a) - \mathscr{A}_{n+1,s}^{[m-1,\alpha]}(x_0;y;a)\right]}{n+1}.$$

(e) *Addition theorem of the argument*

$$\mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,c}^{[m-1,\beta]}(y;0,a), \tag{3.5}$$

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\lfloor \frac{k}{2}\rfloor}\binom{n}{k}\binom{k}{2j}\mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a)y^k, \tag{3.6}$$

$$C_n(x+y,y) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k,c}^{[m-1,-\alpha]}(y;0;a), \tag{3.7}$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha+\beta]}(x+y;y;a) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{k,s}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,s}^{[m-1,\beta]}\left(y;\frac{\pi}{2z},a\right),$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\binom{n}{k}\binom{k}{2j+1}\mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a)y^k,$$

$$S_n(x+y,y) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k,s}^{[m-1,-\alpha]}\left(y;\frac{\pi}{2z};a\right).$$

Proof. For (3.5), from Definition 3.1, we have

$$\begin{aligned}
\sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a)\frac{z^n}{n!} &= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{(\alpha+\beta)} e^{(x+y)z}\cos(yz) \\
&= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz) \\
&\quad\times \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\beta} e^{yz}\cos(0) \\
&= \sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!}\sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\beta]}(y;0;a)\frac{z^n}{n!} \\
&= \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{n-k,c}^{[m-1,\beta]}(y;0;a)\frac{z^n}{n!}.
\end{aligned}$$

□

Proof. We give proof of (3.7)

$$\begin{aligned}\sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\alpha+\beta]}(x+y;y;a)\frac{z^n}{n!} &= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{(\alpha+\beta)} e^{(x+y)z}\cos(yz)\\
&= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\alpha} e^{xz}\cos(yz)\\
&\quad\times\left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}+1}{(2-e^z)^2}\right]^{\beta} e^{yz}\cos(0)\\
&= \sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\frac{z^n}{n!}\sum_{n=0}^{+\infty}\mathscr{A}_{n,c}^{[m-1,-\alpha]}(y;0;a)\frac{z^n}{n!}\\
&= e^{(x+y)z}\cos(yz)\\
&= \sum_{n=0}^{+\infty}C_n(x+y,y)\frac{z^n}{n!}.\end{aligned}$$

□

From (2.1) and Proposition 2.2, we have deduced some algebraic relationship connecting the polynomials $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, with other families of polynomials.

Theorem 3.2. *For $m\in\mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, are related with the Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$, by means of the following identity:*

$$\begin{aligned}(3.8)\quad \mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{n-k}(-1)^v\binom{n}{k}\binom{k}{2j}\binom{n-k+\alpha}{n-k-v}\\
&\quad\times\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k-1}}\mathscr{A}_{k,c}^{[m-1,\alpha]}(x;y;a)P_v^{(\alpha,\beta)}(1-2y),\end{aligned}$$

$$\begin{aligned}(3.9)\quad \mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{n-k}(-1)^v\binom{n}{k}\binom{k}{2j+1}\binom{n-k+\alpha}{n-k-v}\\
&\quad\times\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k-1}}\mathscr{A}_{k,s}^{[m-1,\alpha]}(x;y;a)P_v^{(\alpha,\beta)}(1-2y).\end{aligned}$$

Proof. By replacing (2.1) into the right-hand side of (3.6), we obtain

$$\begin{aligned}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) &= \sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\binom{n}{k}\binom{k}{2j}\mathscr{A}_{k}^{[m-1,\alpha]}(x;y;a)(n-k)!\\
&\quad\times\sum_{v=0}^{n-k}(-1)^v\binom{n-k+\alpha}{n-k-v}\frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k+1}}P_v^{(\alpha,\beta)}(1-2y)\end{aligned}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{n-k} \binom{n}{k} \binom{k}{2j} \binom{n-k+\alpha}{n-k-v} (-1)^v (n-k)!$$
$$\times \frac{(1+\alpha+\beta+2v)}{(1+\alpha+\beta+v)_{n-k+1}} \mathscr{A}_{k,c}^{[m-1,\alpha]}(x,y;a) P_v^{(\alpha,\beta)}(1-2y).$$

Therefore, (3.8) holds. The proof of (3.9) is similar. □

Theorem 3.3. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$, and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$, are related with the generalized Bernoulli polynomials of level-m $B_n^{[m-1]}(x)$, by means of the following identity:*

(3.10)
$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k} \frac{v!}{(v+m)!} \binom{n}{k} \binom{k}{2j} \binom{k}{v} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) B_{k-v}^{[m-1]}(y),$$

(3.11)
$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k} \frac{v!}{(v+m)!} \binom{n}{k} \binom{k}{2j} \binom{k}{v} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) B_{k-v}^{[m-1]}(y).$$

Proof. By substituting (2.2) into the right-hand side of (3.6) and making the corresponding modifications, we obtain (3.10). The proof of (3.11) is similar. □

Theorem 3.4. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y,a)$and $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y,a)$, are related with the Genocchi polynomials $G_n(x)$, by means of*

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j} \binom{k+1}{v} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) G_v(y)$$
$$+ \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j} \mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x;y;a) G_{v+1}(y), \tag{3.12}$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j+1} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) G_{v+1}(y)$$
$$+ \frac{1}{2} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{v=0}^{k+1} \binom{n}{k} \binom{k}{2j+1} \binom{k+1}{v} \mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x;y;a) G_v(y). \tag{3.13}$$

Proof. By substituting (2.3) into the right-hand side of (3.6), we obtain the proof (3.12). The proof of (3.13) is similar. □

Theorem 3.5. *For $m \in \mathbb{N}$, the generalized biparametric Fubini-type polynomials of level-m and $\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)$, and $\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)$are related with the Apostol-Euler* *polynomials $\mathscr{E}_n(x;\lambda)$, by means of the following identity:*

(3.14)

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j}\binom{k}{v}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_v(y;\lambda)$$
$$+\frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j}\mathscr{A}_{n,c}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_k(y;\lambda),$$

(3.15)

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j+1}\binom{k}{v}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_v(y;\lambda)$$
$$+\frac{\lambda}{2}\sum_{k=0}^{n}\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{v=0}^{k}\binom{n}{k}\binom{k}{2j+1}\mathscr{A}_{n,s}^{[m-1,\alpha]}(x;y;a)\mathscr{E}_k(y;\lambda).$$

Proof. By substituting (2.4) into the right-hand side of (3.6), it suffices to follow the proof given in Theorem 3.2, making the corresponding modifications. The proof of (3.14) and (3.15) sre similar. □

Proposition 3.1. *For $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$, $a \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$, we have*

$$\mathscr{A}_{n,c}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\left[\frac{k}{2}\right]}\sum_{v=0}^{k} v!\binom{n}{k}\binom{k}{2j}\binom{x}{v}\mathscr{A}_{n-k,c}^{[m-1,\alpha]}(x,y;a)S(k,v),$$

$$\mathscr{A}_{n,s}^{[m-1,\alpha]}(x+y;y;a) = \sum_{k=0}^{n}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\sum_{v=0}^{k} v!\binom{n}{k}\binom{k}{2j+1}\binom{x}{v}\mathscr{A}_{n-k,s}^{[m-1,\alpha]}(x,y;a)S(k,v).$$

4. The Generalized Biparametric Fubini-type Polynomials Matrix

Inspired by [17], in which the authors introduce the generalized Euler polynomial matrix, in this section we addressed the generalized biparametric Fubini-type polynomials matrix and we will show you some of their properties.

Definition 4.1. The generalized $(n+1)\times(n+1)$ biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ with $m \in \mathbb{N}$, $\alpha \in \mathbb{N}$ and a positive real numbers are defined by

$$\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y;a) = \begin{cases} \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathscr{D}_{i,j,s}^{[m-1,\alpha]}(x;y;a) = \begin{cases} \binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Since, $\mathscr{A}_{n,c}^{[m-1,0]}(x;0;a)=x^n$ and $\mathscr{A}_{n,s}^{[m-1,0]}\left(x;\frac{\pi}{2z};a\right)=x^n$, we have

$$\begin{aligned}\mathscr{D}_c^{[m-1,0]}(x;0;a)=&P[x],\\ \mathscr{D}_s^{[m-1,0]}\left(x;\frac{\pi}{2z};a\right)=&P[x].\end{aligned}$$

Theorem 4.1. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ *satisfy the following product formula*

$$\begin{aligned}\mathscr{D}_c^{[m-1,\alpha+\beta]}(x+y;y;a)=&\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_c^{[m-1,\beta]}(y;0;a)\\ =&\mathscr{D}_c^{[m-1,\beta]}(x;y;a)\,\mathscr{D}_c^{[m-1,\alpha]}(y;0;a)\\ =&\mathscr{D}_c^{[m-1,\alpha]}(y;0;a)\,\mathscr{D}_c^{[m-1,\beta]}(x;y;a),\end{aligned}\tag{4.1}$$

$$\begin{aligned}\mathscr{D}_s^{[m-1,\alpha+\beta]}(x+y;y;a)=&\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_s^{[m-1,\beta]}\left(y;\frac{\pi}{2z};a\right)\\ =&\mathscr{D}_s^{[m-1,\beta]}(x;y;a)\,\mathscr{D}_s^{[m-1,\alpha]}\left(y;\frac{\pi}{2z};a\right)\\ =&\mathscr{D}_s^{[m-1,\alpha]}\left(y;\frac{\pi}{2z};a\right)\,\mathscr{D}_s^{[m-1,\beta]}(x;y;a).\end{aligned}\tag{4.2}$$

Proof. Let $V_{i,j,c}^{[m-1,\alpha,\beta]}(a)(x,y)$ be the (i,j)-th entry of the matrix product $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)\,\mathscr{D}_c^{[m-1,\beta]}(y;0;a)$. Then, by the addition formula (3.5) we have

$$\begin{aligned}V_{i,j,c}^{[m-1,\alpha,\beta]}(a)(x,y)=&\sum_{k=0}^{n}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =&\sum_{k=j}^{i}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =&\sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a)\mathscr{A}_{k-j,c}^{[m-1,\beta]}(y;0;a)\\ =&\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathscr{A}_{i-j-k,c}^{[m-1,\alpha]}(x;y,a)\mathscr{A}_{k,c}^{[m-1,\beta]}(y;0;a)\\ =&\binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha+\beta]}(x+y;y;a),\end{aligned}$$

which implies (4.1). The second and third equalities of the theorem and (4.2), can be derived in a similar way. □

Corollary 4.1. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ *satisfy the following relation*

$$\begin{aligned}\mathscr{D}_c^{[m-1,\alpha]}(x+y;y;a)=&\ \mathscr{D}_c^{[m-1,\alpha]}(x;y;a)P[y]=P[x]\mathscr{D}_c^{[m-1,\alpha]}(y;y;a)\\ =&\ \mathscr{D}_c^{[m-1,\alpha]}(y;y;a)P[x],\\ \mathscr{D}_s^{[m-1,\alpha]}(x+y;y;a)=&\ \mathscr{D}_s^{[m-1,\alpha]}(x;y;a)P[y]=P[x]\mathscr{D}_s^{[m-1,\alpha]}(y;y;a)\end{aligned}$$

$$= \mathscr{D}_s^{[m-1,\alpha]}(y;y;a)P[x],$$

$$\mathscr{D}_c^{[m-1]}(x+y;y) = P[x]\mathscr{D}_c^{[m-1]}(y;y;a) = P[y]\mathscr{D}_c^{[m-1]}(x;y,a).$$

Example 4.1. Using (3.3), for $m=1$, $\alpha=1$ the first four polynomials $\mathscr{A}_{k,c}^{[1-1,\alpha]}(x;y;a)$, $k=0,1,2,3$ are

$$\begin{aligned}
\mathscr{A}_{0,c}^{[1-1,1]}(x;y,a) &= 2,\\
\mathscr{A}_{1,c}^{[1-1,1]}(x;y;a) &= 2x+4,\\
\mathscr{A}_{2,c}^{[1-1,1]}(x;y;a) &= 2x^2+8x+16-2y^2,\\
\mathscr{A}_{3,c}^{[1-1,1]}(x;y;a) &= 2x^3+12x^2+48x-12y^2-6xy^2+88.
\end{aligned}$$

Hence, for $n=3$ we have

$$\mathscr{D}_c^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 & 0\\ 2x+4 & 2 & 0 & 0\\ 2x^2+8x+16-2y^2 & 4x+8 & 2 & 0\\ 2x^3+12x^2+48x-12y^2-6xy+88 & 6x^2+24x+48-6y^2 & 6x+12 & 2 \end{bmatrix}.$$

Example 4.2. Using (3.4), for $m=1$, $\alpha=1$ the first four polynomials $\mathscr{A}_{k,s}^{[1-1,\alpha]}(x;y;a)$, $k=0,1,2,3$ are:

$$\begin{aligned}
\mathscr{A}_{0,s}^{[1-1,1]}(x;y,a) &= 2y,\\
\mathscr{A}_{1,s}^{[1-1,1]}(x;y;a) &= 2xy+4y,\\
\mathscr{A}_{2,s}^{[1-1,1]}(x;y;a) &= 2x^2y+8x+16y-\frac{2}{3}y^2,\\
\mathscr{A}_{3,s}^{[1-1,1]}(x;y;a) &= 2yx^3+12x^2y+48xy-4y^3-2xy^3+88y.
\end{aligned}$$

Hence, for $n=3$, we have

$$\mathscr{D}_s^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2y & 0 & 0 & 0\\ 2xy+4y & 2y & 0 & 0\\ 2x^2y+8x+16y-\frac{2}{3}y^2 & 4xy+8y & 2y & 0\\ 2yx^3+12x^2y+48xy-4y^3-2xy^3+88y & 6x^2y+24x+48y-2y^2 & 6xy+12y & 2y \end{bmatrix}.$$

5. Matrix Factorization of the New Family of Biparametric Fubini-type Polynomials of Level-m Trough the Fibonacci and Lucas Matrices

For $m\in\mathbb{N}$, $a\in\mathbb{R}^+$, $\alpha\in\mathbb{N}_0$ and $0\le i,j\le n$, let $\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a)$ be the matrices whose entries are defined as follows. Inspired by the ideas of [17], the entries of our auxiliary matrices are (see, [17, Eq. (4.39)]):

$$\begin{aligned}
\tilde{r}_{i,j,c}^{[m-1,\alpha]}(x;y;a) = &\binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - \binom{i-1}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\
&- \binom{i-2}{j}\mathscr{A}_{i-j-2,c}^{[m-1,\alpha]}(x;y;a),
\end{aligned}$$

$$\tilde{r}^{[m-1,\alpha]}_{i,j,s}(x;y;a) = \binom{i}{j}\mathscr{A}^{[m-1,\alpha]}_{i-j,s}(x;y;a) - \binom{i-1}{j}\mathscr{A}^{[m-1,\alpha]}_{i-j-1,s}(x;y;a) - \binom{i-2}{j}\mathscr{A}^{[m-1,\alpha]}_{i-j-2,s}(x;y;a).$$

On the other hand, $\mathscr{J}^{[m-1,\alpha]}_{c}(x;y;a)$ and $\mathscr{J}^{[m-1,\alpha]}_{s}(x;y;a)$ are the matrices whose entries are given by (see, [17, Eq. (4.40)])

$$\tilde{s}^{[m-1,\alpha]}_{i,j,c}(x;y;a) = \binom{i}{j}\mathscr{A}^{[m-1,\alpha]}_{i-j,c}(x;y;a) - \binom{i}{j+1}\mathscr{A}^{[m-1,\alpha]}_{i-j-1,c}(x;y;a) - \binom{i}{j+2}\mathscr{A}^{[m-1,\alpha]}_{i-j-2,c}(x;y;a),$$

$$\tilde{s}^{[m-1,\alpha]}_{i,j,s}(x;y;a) = \binom{i}{j}\mathscr{A}^{[m-1,\alpha]}_{i-j,s}(x;y;a) - \binom{i}{j+1}\mathscr{A}^{[m-1,\alpha]}_{i-j-1,s}(x;y;a) - \binom{i}{j+2}\mathscr{A}^{[m-1,\alpha]}_{i-j-2,s}(x;y;a).$$

Using the definitions of $\mathbb{K}^{[m-1,\alpha]}_{Fub,c}(x;y;a)$, $\mathbb{K}^{[m-1,\alpha]}_{Fub,s}(x;y;a)$, $\mathscr{J}^{[m-1,\alpha]}_{c}(x;y;a)$ and $\mathscr{J}^{[m-1,\alpha]}_{s}(x;y;a)$, it is observed that

$$\begin{aligned}
\tilde{r}^{[m-1,\alpha]}_{0,0,c}(x;y;a) &= \tilde{r}^{[m-1,\alpha]}_{1,1,c}(x;y;a) = \tilde{s}^{[m-1,\alpha]}_{0,0,c}(x;y;a)\\
&= \tilde{s}^{[m-1,\alpha]}_{1,1,c}(x;y;a) = \mathscr{A}^{[m-1,\alpha]}_{0,c}(x,y,a),\\
\tilde{r}^{[m-1,\alpha]}_{0,j,c}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{0,j,c}(x;y;a) = 0, \quad j \geq 1,\\
\tilde{r}^{[m-1,\alpha]}_{1,0,c}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{1,0,c}(x;y;a) = \mathscr{A}^{[m-1,\alpha]}_{1,c}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{0,c}(x;y;a),\\
\tilde{r}^{[m-1,\alpha]}_{1,j,c}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{1,j,c}(x;y,a) = 0, \quad j \geq 2,\\
\tilde{r}^{[m-1,\alpha]}_{i,0,c}(x;y;a) &= \mathscr{A}^{[m-1,\alpha]}_{i,c}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-1,c}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-2,c}(x;y;a), \quad i \geq 2,\\
\tilde{s}^{[m-1,\alpha]}_{i,0,c}(x;y;a) &= \mathscr{A}^{[m-1,\alpha]}_{i,c}(x;y;a) - 2\mathscr{A}^{[m-1,\alpha]}_{i-1,c}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-2,c}(x;y;a), \quad i \geq 2,\\
\tilde{r}^{[m-1,\alpha]}_{0,0,s}(x;y;a) &= \tilde{r}^{[m-1,\alpha]}_{1,1,s}(x;y;a) = \tilde{s}^{[m-1,\alpha]}_{0,0,s}(x;y;a)\\
&= \tilde{s}^{[m-1,\alpha]}_{1,1,s}(x;y;a) = \mathscr{A}^{[m-1,\alpha]}_{0,s}(x,y,a),\\
\tilde{r}^{[m-1,\alpha]}_{0,j,s}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{0,j,s}(x;y;a) = 0, \quad j \geq 1,\\
\tilde{r}^{[m-1,\alpha]}_{1,0,s}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{1,0,s}(x;y;a) = \mathscr{A}^{[m-1,\alpha]}_{1,s}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{0,s}(x;y;a),\\
\tilde{r}^{[m-1,\alpha]}_{1,j,s}(x;y;a) &= \tilde{s}^{[m-1,\alpha]}_{1,j,s}(x;y,a) = 0, \quad j \geq 2,\\
\tilde{r}^{[m-1,\alpha]}_{i,0,s}(x;y;a) &= \mathscr{A}^{[m-1,\alpha]}_{i,s}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-1,s}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-2,s}(x;y;a), \quad i \geq 2,\\
\tilde{s}^{[m-1,\alpha]}_{i,0,s}(x;y;a) &= \mathscr{A}^{[m-1,\alpha]}_{i,s}(x;y;a) - 2\mathscr{A}^{[m-1,\alpha]}_{i-1,s}(x;y;a) - \mathscr{A}^{[m-1,\alpha]}_{i-2,s}(x;y;a), \quad i \geq 2.
\end{aligned}$$

For $m \in \mathbb{N}$, $a \in \mathbb{R}^+$, $\alpha \in \mathbb{N}_0$, $0 \leq i, j \leq n$, let $\mathscr{L}_{1,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{L}_{1,s}^{[m-1,\alpha]}(x;y;a)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,1,c}^{[m-1,\alpha]}(x;y;a) = &\binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - 3\binom{i-j}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\ &+ 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}\mathscr{A}_{k-j,c}^{[m-1,\alpha]}(x;y;a),\\ \hat{l}_{i,j,1,s}^{[m-1,\alpha]}(x;y;a) = &\binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - 3\binom{i-j}{j}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a)\\ &+ 5\sum_{k=j}^{i-2}(-1)^{i-k}2^{i-k-2}\binom{k}{j}\mathscr{A}_{k-j,s}^{[m-1,\alpha]}(x;y;a).\end{aligned}$$

Similarly, let $\mathscr{L}_{2,c}^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{L}_{2,s}^{[m-1,\alpha]}(x;y;a)$, $(n+1)\times(n+1)$ be the matrices whose entries are given by

$$\begin{aligned}\hat{l}_{i,j,2,c}^{[m-1,\alpha]}(x;y;a) = &\binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - 3\binom{i}{j+1}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\ &+ 5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k}\mathscr{A}_{i-k,c}^{[m-1,\alpha]}(x;y;a),\\ \hat{l}_{i,j,2,s}^{[m-1,\alpha]}(x;y;a) = &\binom{i}{j}\mathscr{A}_{i-j,s}^{[m-1,\alpha]}(x;y;a) - 3\binom{i}{j+1}\mathscr{A}_{i-j-1,s}^{[m-1,\alpha]}(x;y;a)\\ &+ 5\sum_{k=j+1}^{i}(-1)^{k-j}2^{k-j-2}\binom{i}{k}\mathscr{A}_{i-k,s}^{[m-1,\alpha]}(x;y;a).\end{aligned}$$

Next we will show, factorizations of the matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$, involving the Fibonacci and Lucas matrices, respectively.

Theorem 5.1. *The generalized biparametric Fubini-type polynomials matrices $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ and $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ can be factored in terms of the Fibonacci matrix $\mathscr{F}$ as follows:*

$$\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) = \mathscr{F}\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a), \tag{5.1}$$

$$\mathscr{D}_s^{[m-1,\alpha]}(x;y;a) = \mathscr{F}\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a), \tag{5.2}$$

$$\mathscr{D}_c^{[m-1,\alpha]}(x;y,a) = \mathscr{J}_c^{[m-1,\alpha]}(x;y;a)\mathscr{F}, \tag{5.3}$$

$$\mathscr{D}_s^{[m-1,\alpha]}(x;y,a) = \mathscr{J}_s^{[m-1,\alpha]}(x;y;a)\mathscr{F}. \tag{5.4}$$

Proof. It suffices to prove that $\mathscr{F}^{-1}\mathscr{D}_c^{[m-1,\alpha]}(x;y;a) = \mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a)$. Recall that $\mathscr{F}^{-1} = f'_{i,j}$, where $f'_{i,j}$ is given in equation (2.5). Since $f'_{0,j} = 0$ for $j \geq 1$, we have:

$$f'_{0,0}\mathscr{D}_{0,0,c}^{[m-1,\alpha]}(x;y,a) = \mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a)$$

and

$$\tilde{r}_{0,0,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a) = \sum_{k=0}^{n} f'_{0,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y;a).$$

Since $\mathscr{D}_{0,j,c}^{[m-1,\alpha]}(x;y,a) = 0$ and $f'_{0,j} = 0$ for $j \geq 1$, we obtain:

$$\sum_{k=0}^{n} f'_{0,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) = f'_{0,0}\mathscr{D}_{0,j,c}^{[m-1,\alpha]}(x;y,a) = \tilde{r}_{0,j,c}^{[m-1,\alpha]}(x;y;a), \quad \text{for } j \geq 1.$$

Next, for $f'_{1,0} = -1$, $f'_{1,1} = 1$, and $f'_{1,j} = 0$ for $j \geq 2$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{1,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& f'_{1,0}\mathscr{D}_{0,0,c}^{[m-1,\alpha]}(x;y,a) + f'_{1,1}\mathscr{D}_{1,0,c}^{[m-1,\alpha]}(x;y,a),\\
\sum_{k=0}^{n} f'_{1,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& -\mathscr{A}_{0,c}^{[m-1,\alpha]}(x;y;a) + \mathscr{A}_{1,c}^{[m-1,\alpha]}(x;y;a) = \tilde{r}_{1,0,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

From equation (2.5), for $i = 2, 3, \ldots, n$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,0,c}^{[m-1,\alpha]}(x;y,a) =& \mathscr{A}_{i,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-1,c}^{[m-1,\alpha]}(x;y;a) - \mathscr{A}_{i-2,c}^{[m-1,\alpha]}(x;y;a)\\
=& \tilde{r}_{i,0,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

Now, for $i \geq 2$ and $j \geq 1$, using equation (2.5) and the definition of $\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y,a)$, we have:

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& f'_{i,i}\mathscr{D}_{i,j,c}^{[m-1,\alpha]}(x;y,a) + f'_{i,i-1}\mathscr{D}_{i-1,j,c}^{[m-1,\alpha]}(x;y,a)\\
&+ f'_{i,i-2}\mathscr{D}_{i-2,j,c}^{[m-1,\alpha]}(x;y,a),
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& \binom{i}{j}\mathscr{A}_{i-j,c}^{[m-1,\alpha]}(x;y;a) - \binom{i-1}{j}\mathscr{A}_{i-j-1,c}^{[m-1,\alpha]}(x;y;a)\\
&- \binom{i-2}{j}\mathscr{A}_{i-j-2,c}^{[m-1,\alpha]}(x;y;a),\\
\sum_{k=0}^{n} f'_{i,k}\mathscr{D}_{k,j,c}^{[m-1,\alpha]}(x;y,a) =& \tilde{r}_{i,j,c}^{[m-1,\alpha]}(x;y;a).
\end{aligned}$$

Therefore, we conclude that

$$\mathscr{F}^{-1}\mathscr{D}_{c}^{[m-1,\alpha]}(x;y;a) = \mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a).$$

The proof is complete. □

The proofs of (5.2), (5.3) and (5.4) follow an analogous path.

The expressions (5.1) (5.2), (5.3) and (5.4) allow us to deduce the following identity:

$$\mathbb{K}_{Fub,c}^{[m-1,\alpha]}(x;y;a) = \mathscr{F}^{-1}\,\mathscr{J}_{c}^{[m-1,\alpha]}(x;y,a)\,\mathscr{F},$$

$$\mathbb{K}_{Fub,s}^{[m-1,\alpha]}(x;y;a) = \mathscr{F}^{-1}\mathscr{J}_s^{[m-1,\alpha]}(x;y,a)\mathscr{F}.$$

An analogous reasoning used in the proof of Theorem 5.1, allows us to prove the results below.

Example 5.1. For $m=1$ and $\alpha=1$, the matrices, for $n=2$, $K_{Fub,c}^{[m-1,1]}(x;y,a)$ and $\mathscr{F}$ are

$$\mathbb{K}_{Fub,c}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+2 & 2 & 0 \\ 2x^2+6x+10-2y^2 & 4x+6 & 2 \end{bmatrix},$$

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathscr{F}\mathbb{K}_{Fub,c}^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}$$

and

$$\mathscr{D}_c^{[m-1,1]}(x,y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.$$

This is a particular case of the statement (5.1) of Theorem 5.1.

Example 5.2. For $m=1$ and $\alpha=1$, the matrices, for $n=2$, $\mathscr{J}^{[m-1,1]}(x;y;a)$ and $\mathscr{F}$ are

$$\mathscr{J}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+2 & 2 & 0 \\ 2x^2+4x+6-2y^2 & 4x+6 & 2 \end{bmatrix},$$

$$\mathscr{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathscr{J}^{[m-1,1]}(x;y;a)\mathscr{F} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}$$

and

$$\mathscr{D}_c^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.$$

This is a particular case of the statement (5.3) of Theorem 5.1.

Theorem 5.2. *The generalized biparametric Fubini-type polynomials matrices* $\mathscr{D}_c^{[m-1,\alpha]}(x;y;a)$ *and* $\mathscr{D}_s^{[m-1,\alpha]}(x;y;a)$ *can be factored in terms of the Lucas matrix*

$\mathscr{L}$ *of the following form:*

$$\begin{aligned} \mathscr{D}_c^{[m-1,\alpha]}(x;y;a) &= \mathscr{L}\mathscr{L}_{1,c}^{[m-1,\alpha]}(x;y;a), \\ \mathscr{D}_c^{[m-1,\alpha]}(x;y;a) &= \mathscr{L}_{2,c}^{[m-1,\alpha]}(x;y;a)\mathscr{L}, \\ \mathscr{D}_s^{[m-1,\alpha]}(x;y;a) &= \mathscr{L}\mathscr{L}_{1,s}^{[m-1,\alpha]}(x;y;a), \\ \mathscr{D}_s^{[m-1,\alpha]}(x;y;a) &= \mathscr{L}_{2,s}^{[m-1,\alpha]}(x;y;a)\mathscr{L}. \end{aligned} \tag{5.5}$$

Example 5.3. For $m = 1$ and $\alpha = 1$, the matrices, for $n = 2$ $\mathscr{L}_{1,c}^{[m-1,1]}(x;y;a)$ and $\mathscr{L}$ are

$$\mathscr{L}_{1,c}^{[m-1,1]}(x;y,a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x-2 & 2 & 0 \\ 2x^2+2x+14-2y^2 & 4x+2 & 2 \end{bmatrix},$$

$$\mathscr{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix},$$

$$\mathscr{L}\mathscr{L}_{1,c}^{[m-1,1]}(x;y;a) = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}$$

and

$$\mathscr{D}_c^{[m-1,1]} = \begin{bmatrix} 2 & 0 & 0 \\ 2x+4 & 2 & 0 \\ 2x^2+8x+16-2y^2 & 4x+8 & 2 \end{bmatrix}.$$

This is a particular case of the statement (5.5) of Theorem 5.2.

6. Conclusion

In this paper, we have introduced two new biparametric families of generalized Fubini-type polynomials of level-m, and explored their key algebraic and differential properties. By leveraging these properties, we have derived several formulas and identities that contribute to the understanding of these polynomials. Additionally, we have presented a new matrix representation for the generalized biparametric Fubini-type polynomials, offering a novel approach to their application. Given the broad range of applications of special polynomials in mathematics, physics, and engineering, the results of this study have significant potential for further development in these fields. However, future research may focus on exploring the specific advantages and limitations of these new polynomials in practical applications.

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