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FURTHER REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS AND OPERATOR NORM OF HILBERT SPACE OPERATORS

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Abstract. The main purpose of this paper is to give some reverse inequalities for the numerical radius of bounded linear operators on a Hilbert space, in a way that complements many celebrated inequalities in the literature.

1. Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on Hilbert space $\mathbb{H}$, with identity I. For $A \in \mathbb{B}(\mathbb{H})$, let $r(A)$, $\omega(A)$, and $\|A\|$ denote the spectral radius, the numerical radius and the operator norm of A, respectively. It is well known that for every $A \in \mathbb{B}(\mathbb{H})$,

$$r(A) \leq \omega(A) \leq \|A\|.$$

Recall that $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A, $\omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|$ and $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$. The Crawford number of $A \in \mathbb{B}(\mathbb{H})$ is defined by $m(A) = \inf_{\|x\|=1} |\langle Ax, x\rangle|$.

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the operator norm. In fact, the following more precise result holds for every $A \in \mathbb{B}(\mathbb{H})$, as one can find in [11, Theorem 1.3-1],

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

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The inequalities in (1.1) are sharp. Indeed, if $A^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if A is normal.

For recent reverses and refinements of the inequalities in (1.1), see [13–16, 18, 19, 25, 27]. In addition, for a comprehensive account of the numerical radius, the reader is referred to [11]. Finding reverse inequalities for already established results enhances our understanding of these notions, and provides new tools and auxiliary results that we might benefit from. The above references, and many others, have discussed several forms of reverse inequalities.

While the operator norm is sub-multiplicavive in the sense that $\|AB\| \le \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{B}(\mathbb{H})$, the numerical radius is not. However, the numerical radius enjoys the power inequality stating that, for $A \in \mathbb{B}(\mathbb{H})$,

$$\omega\left(A^p\right) \le \omega^p(A), \tag{1.2}$$

for $p = 1, 2, \ldots$ This was first considered by Berger in [1]. However, the problem of finding the smallest constant c such that

$$\omega^p(A) \le c\,\omega\left(A^p\right), \quad c \ge 1,$$

is still unsolved.

A weaker version of sub-multiplicativity for the numerical radius asserts that [11, Theorem 2.5-2]:

$$\omega(AB) \le 4\omega(A)\omega(B). \tag{1.3}$$

This paper presents a general discussion that leads to several relations in the above setting. More precisely, we find some additive reverses of celebrated inequalities related to the numerical radius and the operator norm. For example, we find a constant γ_1, that depends on $A, B \in \mathbb{B}(\mathbb{H})$, such that $\omega(AB) \le \omega(A)\omega(B) + \gamma_1$. This provides a weak sub-multiplicative behavior of the numerical radius.

Further, we show that

$$\frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| - \omega^2\left(A\right) \le \gamma_2,$$

for some constant γ_2, providing an additive reverse of the celebrated inequality [17]

$$\omega^2\left(A\right) \le \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\|. \tag{1.4}$$

Many other relations are proved similarly, covering a wide selection of celebrated results in this direction.

We will need the following lemmas to achieve some of our results. The first lemma presents a refinement and a reverse for the simple inequality $a^p \le b^p$, whenever $a \le b$ and $p \ge 1$.

Lemma 1.1. *Let $a, b, m, M \in \mathbb{R}$ and let $p \ge 1$. Then,*

$$0 < m \le a \le b \le M \;\Rightarrow\; \begin{cases} a^p + pm^{p-1}\left(b-a\right) \le b^p, \\ b^p \le a^p + pM^{p-1}\left(b-a\right). \end{cases}$$

Proof. Applying Lagrange theorem (Mean Value Theorem) on the function $f(t) = t^p$ implies

$$f(b) - f(a) = p\theta^{p-1}(b - a),$$

for some $\theta \in (a, b)$. This immediately gives the desired conclusion. □

Lemma 1.2 (Buzano's inequality [2])**.** *Let $a, b, e \in \mathbb{H}$ with $\|e\| = 1$. Then,*

$$|\langle a, e\rangle \langle e, b\rangle| \le \frac{1}{2}\left(\|a\| \cdot \|b\| + |\langle a, b\rangle|\right).$$

Lemma 1.3 (Hölder-McCarthy inequality [22])**.** *Let $A \in \mathbb{B}(\mathbb{H})$ be a positive operator. Then, for any unit vector $x \in \mathbb{H}$,*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle, \quad r \ge 1.$$

Lemma 1.4 (Mixed Schwarz inequality [12, pp. 75–76])**.** *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x, y \in \mathbb{H}$ be any vectors. Then,*

$$|\langle Ax, y\rangle| \le \sqrt{\langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle}.$$

For $\alpha, \beta \in \mathbb{C}$ and $A \in \mathbb{B}(\mathbb{H})$, we define the following transform

$$C_{\alpha,\beta}(A) := (A^* - \overline{\alpha} I)(\beta I - A).$$

This transformation received some attention in the literature, as seen in [5,10]. In fact, it was introduced in [5] as a tool for discussing some operator bounds. In [10], it was discussed in further details as an indicator of some relations among different operator forms. Also, in [23], it was applied to obtain certain Cassels type inequalities.

We recall that a bounded linear operator A on the complex Hilbert space $(\mathbb{H}, \langle \cdot, \cdot\rangle)$ is called accretive if $\operatorname{Re}\langle Ax, x\rangle \ge 0$ for any $x \in \mathbb{H}$, where $\operatorname{Re}(\cdot)$ denotes the real part operation.

It has been shown in [7, Lemma 86] that if $A \in \mathbb{B}(\mathbb{H})$ is such that $C_{\alpha,\beta}(A)$ is accretive, then

$$\left\|A - \frac{\alpha + \beta}{2} I\right\| \le \frac{|\alpha - \beta|}{2}. \tag{1.5}$$

The Euclidean operator radius shows some benefits when dealing with the numerical radius. We recall that if $A_1, \dots, A_n \in \mathbb{B}(\mathbb{H})$, then the Euclidean operator radius of these operators is defined by [24]

$$\omega_e(A_1, A_2, \dots, A_n) = \sup_{\|x\|=1} \left(\sum_{i=1}^n |\langle A_i x, x\rangle|^2\right)^{\frac{1}{2}}.$$

The following upper bound for the Euclidean operator radius is well known [3, Theorem 1]

$$\omega_e^2(A, B) \le \left\||A|^2 + |B|^2\right\|. \tag{1.6}$$

Related inequalities that include a reversed version of this will also be discussed in the sequel.

2. Upper Bounds for $\left\| |A|^2 + |B|^2 \right\|$

First, we prove the following general form that provides a reversed version of the simple inequality $\omega^2(A) + \omega^2(B) \le \|A\|^2 + \|B\|^2$.

Theorem 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\| |A|^2 + |B|^2 \right\| \le \omega^2(A) + \omega^2(B) + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2.$$

Proof. On account of [8, Theorem 3], we have

$$\|x\|^2 - |\langle x, e\rangle|^2 \le \frac{|\alpha - \beta|^2}{4} + \left\| x - \frac{\alpha+\beta}{2} e \right\|^2, \tag{2.1}$$

where $x, e \in \mathbb{H}$, $\|e\| = 1$ and $\alpha, \beta \in \mathbb{C}$. Replacing x by Ax and e by x, with $\|x\| = 1$ in (2.1), we obtain

$$\|Ax\|^2 \le |\langle Ax, x\rangle|^2 + \frac{|\alpha - \beta|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2. \tag{2.2}$$

Similarly, we can show that

$$\|Bx\|^2 \le |\langle Bx, x\rangle|^2 + \frac{|\lambda - \mu|^2}{4} + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2, \tag{2.3}$$

for any $\mu, \lambda \in \mathbb{C}$. Combining (2.2) and (2.3) implies that

$$\begin{aligned} \left\langle \left(|A|^2 + |B|^2 \right) x, x \right\rangle &= \left\langle |A|^2 x, x \right\rangle + \left\langle |B|^2 x, x \right\rangle \\ &= \|Ax\|^2 + \|Bx\|^2 \\ &\le |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} \\ &\quad + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2} I \right) x \right\|^2 \\ &\le \omega^2(A) + \omega^2(B) + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} \\ &\quad + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2, \end{aligned} \tag{2.4}$$

where we have used the definitions of the numerical radius and the operator norm to obtain the second inequality above. We deduce the desired inequality by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$. □

The approach we adopt in this work can be also applied to obtain bounds for $\omega^2(|A| + \mathrm{i}\,|B|)$, as follows.

Corollary 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\omega^2(|A| + \mathrm{i}\,|B|) \le \omega^2(A) + \omega^2(B) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

Proof. This follows from Theorem 2.1 noting that $\omega^2(|A| + \mathrm{i}|B|) \le \left\||A|^2 + |B|^2\right\|$. ☐

Remark 2.1. Set $\alpha = \beta$ and $\lambda = \mu$, in Corollary 2.1, and then take the infimum over α and λ, we get

$$\omega^2(|A| + \mathrm{i}\,|B|) \le \omega^2(A) + \omega^2(B) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2.$$

In particular,

$$\frac{1}{2}\omega^2(|A| + \mathrm{i}\,|A^*|) \le \omega^2(A) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2.$$

Notice that the above inequality provides a reverse for the first inequality in [20, Corollary 2.2].

Another upper bound for $\left\||A|^2 + |B|^2\right\|$ in terms of $\omega^2(A) + \omega^2(B)$ can be stated in the following form.

Proposition 2.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\||A|^2 + |B|^2\right\| \le \frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. If we put $a = Ax$ (resp. Bx), $b = A^*x$ (resp. B^*x), and $e = x$, in Lemma 1.2, we obtain

$$\begin{aligned}
&|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 \\
\le& \frac{1}{2}\left(\|Ax\| \cdot \|A^*x\| + \left|\left\langle A^2x, x\right\rangle\right| + \|Bx\| \cdot \|B^*x\| + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
=& \frac{1}{2}\left(\sqrt{\left\langle |A|^2x, x\right\rangle\left\langle |A^*|^2x, x\right\rangle} + \left|\left\langle A^2x, x\right\rangle\right| + \sqrt{\left\langle |B|^2x, x\right\rangle\left\langle |B^*|^2x, x\right\rangle} + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
\le& \frac{1}{4}\left\langle\left(|A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right)x, x\right\rangle + \frac{1}{2}\left(\left|\left\langle A^2x, x\right\rangle\right| + \left|\left\langle B^2x, x\right\rangle\right|\right) \\
\le& \frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right),
\end{aligned}$$

where the second inequality is obtained from the arithmetic-geometric mean inequality. So, we obtain the desired result after adding (2.2) and (2.3) together and then taking the supremum over all unit vectors $x \in \mathbb{H}$. ☐

Remark 2.2. (i) If we set $B = A$ and $\alpha = \beta = \lambda = \mu$, in Proposition 2.1, we get

$$\|A\|^2 \le \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}\omega\left(A^2\right) + \inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2.$$

(ii) If we set $B = A^*$ and $\alpha = \beta, \lambda = \mu = \overline{\alpha}$, in Proposition 2.1, we get

$$\frac{1}{2}\left\||A|^2 + |A^*|^2\right\| \le \omega\left(A^2\right) + 2\inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2 \le \omega\left(A\right)^2 + 2\inf_{\alpha\in\mathbb{C}}\|A - \alpha I\|^2.$$

This provides a reverse and a refinement of the inverse of (1.4).

Remark 2.3. Both Theorem 2.1 and Proposition 2.1 provided upper bounds for $\left\||A|^2 + |B|^2\right\|$. In this remark, we show that the two bounds are incomparable in the sense that neither upper bound is always better than the other. For this, we consider two choices of A, B.

First, if we let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we find that $\omega^2(A)+\omega^2(B) = 2$ and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) = 2.5,$$

showing that Theorem 2.1 is sharper than Proposition 2.1 for these A, B.

On the other hand, letting $A = \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, we find that $\omega^2(A) + \omega^2(B) \approx 15.3218$ and

$$\frac{1}{4}\left\||A|^2 + |A^*|^2 + |B|^2 + |B^*|^2\right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right) \approx 14.4357.$$

We point out that for $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the two bounds are equal.

Proposition 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{aligned}\left\||A|^2 + |B|^2\right\| &\le \omega\left(|A| + \mathrm{i}\,|B|\right)\omega\left(|A^*| + \mathrm{i}\,|B^*|\right) + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\ &\quad + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2.\end{aligned}$$

Proof. Let $x \in \mathbb{H}$ be a unit vector. Then,

$$\begin{aligned}&|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 \\ &\le \langle |A|\,x, x\rangle\langle |A^*|\,x, x\rangle + \langle |B|\,x, x\rangle\langle |B^*|\,x, x\rangle \quad \text{(by Lemma 1.4)} \\ &\le \sqrt{\left(\langle |A|\,x, x\rangle^2 + \langle |B|\,x, x\rangle^2\right)\left(\langle |A^*|\,x, x\rangle^2 + \langle |B^*|\,x, x\rangle^2\right)} \\ &\qquad \text{(by the Cauchy-Schwarz inequality)} \\ &= |\langle |A|\,x, x\rangle + \mathrm{i}\,\langle |B|\,x, x\rangle| \cdot |\langle |A^*|\,x, x\rangle + \mathrm{i}\,\langle |B^*|\,x, x\rangle| \\ &\qquad \left(\text{since } |a + \mathrm{i}b| = \sqrt{a^2+b^2},\ a, b \in \mathbb{R}\right)\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= |\langle (|A| + \mathrm{i}\,|B|)\,x, x\rangle| \cdot |\langle (|A^*| + \mathrm{i}\,|B^*|)\,x, x\rangle| \\
&\le \omega\,(|A| + \mathrm{i}\,|B|)\,\omega\,(|A^*| + \mathrm{i}\,|B^*|)\,.
\end{aligned}
$$

From (2.2) and (2.3) above we have

$$
\|Ax\|^2 \le |\langle Ax, x\rangle|^2 + \frac{|\alpha-\beta|^2}{4} + \left\|\left(A - \frac{\alpha+\beta}{2} I\right) x\right\|^2. \tag{2.6}
$$

Similarly, we can show that

$$
\|Bx\|^2 \le |\langle Bx, x\rangle|^2 + \frac{|\lambda-\mu|^2}{4} + \left\|\left(B - \frac{\lambda+\mu}{2} I\right) x\right\|^2. \tag{2.7}
$$

Noting that $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle |A|^2 x, x\rangle$ and that $\|Bx\|^2 = \langle Bx, Bx\rangle = \langle |B|^2 x, x\rangle$, then (2.6) and (2.7), and using (2.5) imply

$$
\begin{aligned}
\left\langle (|A|^2 + |B|^2)x, x\right\rangle \le\; & \omega\,(|A| + \mathrm{i}\,|B|)\,\omega\,(|A^*| + \mathrm{i}\,|B^*|) \\
& + \frac{|\alpha-\beta|^2}{4} + \left\|\left(A - \frac{\alpha+\beta}{2} I\right) x\right\|^2 + \frac{|\lambda-\mu|^2}{4} \\
& + \left\|\left(B - \frac{\lambda+\mu}{2} I\right) x\right\|^2.
\end{aligned}
$$

Taking the supremum over all unit vectors $x \in \mathbb{H}$ implies the desired conclusion. □

Remark 2.4. If we set $B = A^*$ and $\alpha = \beta, \lambda = \mu = \overline{\alpha}$, in Proposition 2.2, we obtain

$$
\begin{aligned}
\left\| |A|^2 + |A^*|^2 \right\| &\le \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,(|A^*| + \mathrm{i}\,|A|) + 2\|A - \alpha I\|^2 \\
&= \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,(\mathrm{i}\,(|A^*| + \mathrm{i}\,|A|)) + 2\|A - \alpha I\|^2 \\
&= \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,(\mathrm{i}\,|A^*| - |A|) + 2\|A - \alpha I\|^2 \\
&= \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,((\mathrm{i}\,|A^*| - |A|)^*) + 2\|A - \alpha I\|^2 \\
&= \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,(-\mathrm{i}\,|A^*| - |A|) + 2\|A - \alpha I\|^2 \\
&= \omega\,(|A| + \mathrm{i}\,|A^*|)\,\omega\,(-\,(|A| + \mathrm{i}\,|A^*|)) + 2\|A - \alpha I\|^2 \\
&= \omega^2\,(|A| + \mathrm{i}\,|A^*|) + 2\|A - \alpha I\|^2.
\end{aligned}
$$

Namely,

$$
\left\| |A|^2 + |A^*|^2 \right\| \le \omega^2\,(|A| + \mathrm{i}\,|A^*|) + 2 \inf_{\alpha \in \mathbb{C}} \|A - \alpha I\|^2.
$$

The above inequality provides a reverse for the second inequality in [20, Corollary 2.2].

Another upper bound for $\left\| |A|^2 + |B|^2 \right\|$ is found next. This bound relates the two quantities $\left\| |A|^2 + |B|^2 \right\|$ and $\left\| |A^*|^2 + |B^*|^2 \right\|$. A reversed version is obtained by changing A to A^* and B to B^* in this result.

Corollary 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\||A|^2+|B|^2\right\| \le \left\||A^*|^2+|B^*|^2\right\| + \frac{|\alpha-\beta|^2}{2} + \frac{|\lambda-\mu|^2}{2} + 2\left(\left\|A-\frac{\alpha+\beta}{2}I\right\|^2 + \left\|B-\frac{\lambda+\mu}{2}I\right\|^2\right).$$

Proof. Direct calculations show that $\omega^2(|A|+\mathrm{i}|B|) \le \left\||A|^2+|B|^2\right\|$. Employing this inequality in Proposition 2.2, we obtain

$$\left\||A|^2+|B|^2\right\| \le \sqrt{\left\||A|^2+|B|^2\right\| \cdot \left\||A^*|^2+|B^*|^2\right\|} + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} + \left\|A-\frac{\alpha+\beta}{2}I\right\|^2 + \left\|B-\frac{\lambda+\mu}{2}I\right\|^2.$$

We obtain the desired result if we apply the arithmetic-geometric mean inequality on the square root on the right side. □

Remark 2.5. If we set $\alpha=\beta$, $\lambda=\mu$, and then taking infimum over α and λ, in Corollary 2.2, we obtain

$$\left\||A|^2+|B|^2\right\| \le \left\||A^*|^2+|B^*|^2\right\| + 2\left(\inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2 + \inf_{\lambda\in\mathbb{C}}\|B-\lambda I\|^2\right). \tag{2.8}$$

Of course, if we replace A and B by A^* and B^*, in (2.8), we get

$$\left\||A^*|^2+|B^*|^2\right\| \le \left\||A|^2+|B|^2\right\| + 2\left(\inf_{\alpha\in\mathbb{C}}\|A^*-\alpha I\|^2 + \inf_{\lambda\in\mathbb{C}}\|B^*-\lambda I\|^2\right).$$

Letting $B=A$, $\alpha=\lambda$ and $\beta=\mu$, in Theorem 2.1, we infer that

$$\|A\|^2-\omega^2(A) \le \frac{|\alpha-\beta|^2}{4} + \left\|A-\frac{\alpha+\beta}{2}I\right\|^2. \tag{2.9}$$

In the following, we present an upper bound for the non-negative difference $\|A\|^2 - \omega^2(A)$. We remark that the fact of Corollary 2.3 was pointed out in [4, Corollary 1].

Corollary 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$. Then,*

$$\|A\|^2-\omega^2(A) \le \inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2. \tag{2.10}$$

Proof. Since α, β are arbitrary, we can choose $\alpha=\beta$ in (2.9) to get

$$\|A\|^2-\omega^2(A) \le \|A-\alpha I\|^2.$$

By taking infimum over $\alpha \in \mathbb{C}$, we infer (2.10). □

Remark 2.6. We have seen that Proposition 2.2 has added a new upper bound for $\left\||A|^2+|B|^2\right\|$. This remark shows that this bound is independent of both bounds in Theorem 2.1 and Proposition 2.1. For this purpose, let us use the notations

$$R_1(A,B) = \omega^2(A)+\omega^2(B),$$

$$R_2(A,B) = \frac{1}{4}\left\| |A|^2 + |A^*|^2 + |B|^2 + |B^*|^2 \right\| + \frac{1}{2}\left(\omega\left(A^2\right) + \omega\left(B^2\right)\right)$$

and

$$R_3(A,B) = \omega\left(|A| + \mathrm{i}\,|B|\right)\omega\left(|A^*| + \mathrm{i}\,|B^*|\right).$$

If we let $A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix}$, then numerical calculations show that, for these matrices,

$$R_1(A,B) \approx 21.1939, \quad R_2(A,B) \approx 19.8777, \quad R_3(A,B) \approx 22.7987,$$

showing that Theorem 2.1 and Proposition 1.1 can be both better than Proposition 2.2. On the other hand, if we let $A = \begin{bmatrix} -1 & -1 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$, we find that

$$R_1(A,B) \approx 30.3463, \quad R_2(A,B) \approx 28.3897, \quad R_3(A,B) \approx 25.3546,$$

showing that Proposition 2.2 can be better than both Theorem 2.1 and Proposition 2.1.

Proposition 2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ and $p \geq 1$. Then,*

$$\|A\|^{2p} - \omega^{2p}(A) \leq p\|A\|^{p-1} \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2.$$

Proof. We know that $\omega^2(A) \leq \|A\|^2$. We infer by substituting $a = \omega^2(A)$, $b = M = \|A\|^2$, in Lemma 1.1, that

$$\begin{aligned}\|A\|^{2p} &\leq \omega^{2p}(A) + p\|A\|^{p-1}\left(\|A\|^2 - \omega^2(A)\right) \\ &\leq \omega^{2p}(A) + p\|A\|^{p-1} \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2,\end{aligned}$$

where the second inequality is obtained from Corollary 2.3. This completes the proof of the theorem. □

Remark 2.7. Let $A \in \mathbb{B}(\mathbb{H})$. It can be easily seen that there is a scalar $\gamma \in \mathbb{C}$ such that $\inf_{\alpha\in\mathbb{C}} \|A - \alpha I\| = \|A - \gamma I\|$. Put

$$\Gamma_A = \left\{\gamma \in \mathbb{C} : \ \|A - \gamma I\| = \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|\right\}.$$

Therefore, from Corollary 2.3, we have

$$\|A\|^2 - \omega^2(A) \leq \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 = \|A - \gamma I\|, \quad \gamma \in \Gamma_A.$$

Letting $B = A^*$, in Theorem 2.1, we get

(2.11)

$$\left\| |A|^2 + |A^*|^2 \right\| \leq 2\omega^2(A) + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} + \left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| A^* - \frac{\lambda+\mu}{2} I \right\|^2.$$

In the following, we aim to find an upper estimate for the non-negative quantity $\frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| - \omega^2(A)$, motivated by (1.4).

Corollary 2.4 ([26, Corollary 2.2]). *Let $A \in \mathbb{B}(\mathbb{H})$. Then,*

$$\frac{1}{2}\left\||A|^2+|A^*|^2\right\|-\omega^2(A)\le \inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2.$$

Proof. It follows from (2.11) that

$$\frac{1}{2}\left\||A|^2+|A^*|^2\right\|-\frac{1}{2}\left(\|A-\alpha I\|^2+\|A^*-\lambda I\|^2\right)\le \omega^2(A). \tag{2.12}$$

We obtain the desired result if we put $\lambda=\overline{\alpha}$ in (2.12). □

Proposition 2.4. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha,\beta,\lambda,\mu\in\mathbb{C}$,*

$$\begin{aligned}\left\||A|^2+|B|^2\right\|\le&\max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&+\left\|A-\frac{\alpha+\beta}{2}I\right\|^2+\left\|B-\frac{\lambda+\mu}{2}I\right\|^2.\end{aligned}$$

Proof. It follows from (3.239) in [7] that

$$|\langle Ax,x\rangle|^2+|\langle Bx,x\rangle|^2\le\max\left\{\|Ax\|^2,\|Bx\|^2\right\}+|\langle Ax,Bx\rangle|,$$

for any unit vector $x\in\mathbb{H}$. Hence, by adding (2.2) and (2.3) together, we obtain

$$\begin{aligned}\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle\le&\max\left\{\|Ax\|^2,\|Bx\|^2\right\}+|\langle Ax,Bx\rangle|+\frac{|\alpha-\beta|^2}{4}+\frac{|\lambda-\mu|^2}{4}\\&+\left\|\left(A-\frac{\alpha+\beta}{2}I\right)x\right\|^2+\left\|\left(B-\frac{\lambda+\mu}{2}I\right)x\right\|^2.\end{aligned} \tag{2.13}$$

We deduce the desired result by taking the supremum over all unit vectors $x\in\mathbb{H}$. □

Remark 2.8. By setting $\alpha=\beta$ and $\lambda=\mu$, in Proposition 2.4, we get

$$\left\||A|^2+|B|^2\right\|\le\max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)+\|A-\alpha I\|^2+\|B-\lambda I\|^2.$$

If we put $B=A^*$, we infer that

$$\left\||A|^2+|A^*|^2\right\|\le\|A\|^2+\omega\left(A^2\right)+\|A-\alpha I\|^2+\|A^*-\lambda I\|^2,$$

for any $\alpha,\lambda\in\mathbb{C}$. The case $\lambda=\overline{\alpha}$ implies that

$$\left\||A|^2+|A^*|^2\right\|\le\|A\|^2+\omega\left(A^2\right)+2\inf_{\alpha\in\mathbb{C}}\|A-\alpha I\|^2.$$

Remark 2.9. If we let $R_4(A,B)=\max\left\{\|A\|^2,\|B\|^2\right\}+\omega(B^*A)$, we find that for $A=\begin{bmatrix}2&-1\\3&0\end{bmatrix}$, $B=\begin{bmatrix}-1&-3\\3&-1\end{bmatrix}$, and

$$R_1(A,B)=16,\quad R_2(A,B)\approx 17.9142,\quad R_3(A,B)\approx 23.3246,\quad R_4(A,B)\approx 22.7967,$$

where R_1, R_2 and R_3 are as in Remark 2.6, showing that the bound in Proposition 2.4 can be weaker than those in Theorem 2.1, Proposition 2.1 and Proposition 2.2. On the other hand, letting $A = \begin{bmatrix} -1 & -3 \\ -2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$ shows that $R_1(A,B) \approx 25.698$,

$$R_2(A,B) \approx 25.8801, \quad R_3(A,B) \approx 24.7839, \quad R_4(A,B) \approx 21.0623,$$

meaning that Proposition 2.4 can also be better than the three aforementioned results.

Proposition 2.5. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\| |A|^2 + |B|^2 \right\| \le \left\| |A|^2 - |B|^2 \right\| + 2\omega\left(B^*A\right) + \frac{|\alpha-\beta|^2}{2} + \frac{|\lambda-\mu|^2}{2} + 2\left(\left\| A - \frac{\alpha+\beta}{2} I \right\|^2 + \left\| B - \frac{\lambda+\mu}{2} I \right\|^2 \right).$$

Proof. We can see from (2.13) that

$$\begin{aligned}
&\left\langle \left(|A|^2 + |B|^2\right)x, x \right\rangle \\
&\le \max\left\{ \|Ax\|^2, \|Bx\|^2 \right\} + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
&\quad + \left\| \left(A - \frac{\alpha+\beta}{2}I\right)x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2}I\right)x \right\|^2 \\
&= \frac{1}{2}\left(\|Ax\|^2 + \|Bx\|^2 + \left| \|Ax\|^2 - \|Bx\|^2 \right| \right) + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} + \frac{|\lambda-\mu|^2}{4} \\
&\quad + \left\| \left(A - \frac{\alpha+\beta}{2}I\right)x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2}I\right)x \right\|^2 \\
&= \frac{1}{2}\left(\left\langle \left(|A|^2 + |B|^2\right)x, x \right\rangle + \left| \left\langle \left(|A|^2 - |B|^2\right)x, x \right\rangle \right| \right) + |\langle Ax, Bx\rangle| + \frac{|\alpha-\beta|^2}{4} \\
&\quad + \frac{|\lambda-\mu|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2}I\right)x \right\|^2 + \left\| \left(B - \frac{\lambda+\mu}{2}I\right)x \right\|^2.
\end{aligned}$$

We deduce the desired result by taking the supremum over all unit vectors $x \in \mathbb{H}$. □

Remark 2.10. By setting $\alpha = \beta$ and $\lambda = \mu$, in Proposition 2.13, we get

$$\left\| |A|^2 + |B|^2 \right\| \le \left\| |A|^2 - |B|^2 \right\| + 2\left(\omega\left(B^*A\right) + \left(\|A - \alpha I\|^2 + \|B - \lambda I\|^2 \right) \right).$$

If we put $B = A^*$, we infer that

$$\left\| |A|^2 + |A^*|^2 \right\| \le \left\| |A|^2 - |A^*|^2 \right\| + 2\left(\omega\left(A^2\right) + \left(\|A - \alpha I\|^2 + \|A^* - \lambda I\|^2 \right) \right),$$

for any $\alpha, \lambda \in \mathbb{C}$. The case $\lambda = \overline{\alpha}$ implies that

$$\left\| |A|^2 + |A^*|^2 \right\| \le \left\| |A|^2 - |A^*|^2 \right\| + 2\left(\omega\left(A^2\right) + 2\inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2 \right).$$

In connection with (1.6), we present the following upper bound for $\left\| |A|^2 + |B|^2 \right\| - \omega_e^2(A,B)$.

Theorem 2.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\left\| |A|^2 + |B|^2 \right\| - \omega_e^2(A, B) \le \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} + \left\| A - \frac{\alpha + \beta}{2} I \right\|^2 + \left\| B - \frac{\lambda + \mu}{2} I \right\|^2. \tag{2.14}$$

In particular,

$$\left\| |A|^2 + |B|^2 \right\| \le \omega_e^2(A, B) + \inf_{\alpha \in \mathbb{C}} \|A - \alpha I\|^2 + \inf_{\lambda \in \mathbb{C}} \|B - \lambda I\|^2. \tag{2.15}$$

Proof. It observes from the first inequality in (2.4) that

$$\begin{aligned} \left\langle \left(|A|^2 + |B|^2 \right) x, x \right\rangle \le& |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} \\ &+ \left\| \left(A - \frac{\alpha + \beta}{2} I \right) x \right\|^2 + \left\| \left(B - \frac{\lambda + \mu}{2} I \right) x \right\|^2 \\ \le& \omega_e^2(A, B) + \frac{|\alpha - \beta|^2}{4} + \frac{|\lambda - \mu|^2}{4} \\ &+ \left\| A - \frac{\alpha + \beta}{2} I \right\|^2 + \left\| B - \frac{\lambda + \mu}{2} I \right\|^2. \end{aligned}$$

We get the first inequality by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$.

The second inequality follows from (2.14) by placing $\alpha = \beta$ and $\lambda = \mu$ and then taking the infimum over α and λ. □

If $T = \mathrm{Re}T + \mathrm{i}\mathrm{Im}T$ is the Cartesian decomposition of T, then from Theorem 2.2, we have the following reverse of (1.4).

Corollary 2.5. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \lambda \in \mathbb{C}$,*

$$\frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\| - \omega^2(A) \le \inf_{\alpha \in \mathbb{C}} \|\mathrm{Re}A - \alpha I\|^2 + \inf_{\lambda \in \mathbb{C}} \|\mathrm{Im}A - \lambda I\|^2.$$

Remark 2.11. If, in Corollary 2.5, ReA and ImA are scalar multiples of the identity, we deduce that $\frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\| = \omega^2(A)$.

3. Upper Bounds for $\|B^*A\|$ and the Power Inequality

Due to sub-multiplicativity of the operator norm, one has $\|B^*A\| \le \|B\| \cdot \|A\| \le 4\omega(B)\omega(A)$. In the following result, we present a new form for an upper bound of $\|B^*A\|$ in terms of the numerical radii.

Theorem 3.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{aligned} \|B^*A\| \le& \sqrt{\omega^2(A) + \frac{|\alpha - \beta|^2}{4} + \left\| A - \frac{\alpha + \beta}{2} I \right\|^2} \\ &\times \sqrt{\omega^2(B) + \frac{|\lambda - \mu|^2}{4} + \left\| B - \frac{\lambda + \mu}{2} I \right\|^2}. \end{aligned}$$

Furthermore, the inequality is sharp.

Proof. We can show from (2.2) that

$$\|By\|^2 \le |\langle By, y\rangle|^2 + \frac{|\lambda-\mu|^2}{4} + \left\|\left(B - \frac{\lambda+\mu}{2}I\right)y\right\|^2, \tag{3.1}$$

for any unit vector $y \in \mathbb{H}$. So, by (2.2) and (3.1), we can write

$$\begin{aligned}
|\langle B^*Ax, y\rangle| &= |\langle Ax, By\rangle| \\
&\le \|Ax\| \cdot \|By\| \\
&\le \sqrt{|\langle Ax, x\rangle|^2 + \frac{|\alpha-\beta|^2}{4} + \left\|\left(A - \frac{\alpha+\beta}{2}I\right)x\right\|^2} \\
&\quad \times \sqrt{|\langle By, y\rangle|^2 + \frac{|\lambda-\mu|^2}{4} + \left\|\left(B - \frac{\lambda+\mu}{2}I\right)y\right\|^2} \\
&\le \sqrt{\omega^2(A) + \frac{|\alpha-\beta|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2} \\
&\quad \times \sqrt{\omega^2(B) + \frac{|\lambda-\mu|^2}{4} + \left\|B - \frac{\lambda+\mu}{2}I\right\|^2}.
\end{aligned}$$

We deduce the desired inequality by taking supremum over $x, y \in \mathbb{H}$ with $\|x\| = \|y\| = 1$. □

Corollary 3.1. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \lambda \in \mathbb{C}$,*

$$\|B^*A\| \le \sqrt{\omega^2(A) + \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2}\sqrt{\omega^2(B) + \inf_{\lambda\in\mathbb{C}} \|B - \lambda I\|^2}.$$

Proof. Letting $\alpha = \beta$ and $\lambda = \mu$ and then taking infimum over α and λ, in Theorem 3.1, we obtain the desired inequality. □

Remark 3.1. We notice that if $A = \sigma_1 I$, $B = \sigma_2 I$, for some $\sigma_1, \sigma_2 \in \mathbb{C}$, then by selecting $\alpha = \sigma_1, \gamma = \sigma_2$ in Corollary 3.1, the inequality becomes an equality with both sides equal to $\sigma_1\sigma_2$.

We continue this section with the following result, which yields a reverse for the inequality (1.2) in the case $p = 2$.

Theorem 3.2. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for any $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\omega^2(A) - \omega\left(A^2\right) \le \sqrt{\frac{|\alpha-\beta|^2}{4} + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2}\sqrt{\frac{|\lambda-\mu|^2}{4} + \left\|A^* - \frac{\lambda+\mu}{2}I\right\|^2}.$$

Proof. Thanks to [8, Theorem 4], we have

$$|\langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle| \le \left(\frac{|\alpha-\beta|^2}{4} + \left\|x - \frac{\alpha+\beta}{2}z\right\|^2\right)^{\frac{1}{2}} \tag{3.2}$$

$$\times \left(\frac{|\lambda-\mu|^2}{4} + \left\| y - \frac{\lambda+\mu}{2} z \right\|^2 \right)^{\frac{1}{2}},$$

for all $x, y, z \in \mathbb{H}$, with $\|z\| = 1$, and for every $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. Applying (3.2) for $x = Au$, $y = A^*u$ and $z = u$, we get

$$\begin{aligned} &|\langle Au, u\rangle|^2 - \left|\left\langle A^2u, u \right\rangle\right| \\ &\le \left|\left\langle A^2u, u\right\rangle - \langle Au, u\rangle^2\right| \\ &\le \left(\frac{|\alpha-\beta|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I\right) u \right\|^2 \right)^{\frac{1}{2}} \left(\frac{|\lambda-\mu|^2}{4} + \left\| \left(A^* - \frac{\lambda+\mu}{2} I\right) u \right\|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $u \in \mathbb{H}$ with $\|u\| = 1$. Namely,

$$\begin{aligned} |\langle Au, u\rangle|^2 \le \left|\left\langle A^2u, u\right\rangle\right| &+ \left(\frac{|\alpha-\beta|^2}{4} + \left\| \left(A - \frac{\alpha+\beta}{2} I\right) u \right\|^2 \right)^{\frac{1}{2}} \\ &\times \left(\frac{|\lambda-\mu|^2}{4} + \left\| \left(A^* - \frac{\lambda+\mu}{2} I\right) u \right\|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $u \in \mathbb{H}$ with $\|u\| = 1$.

We conclude the result by taking the supremum over $u \in \mathbb{H}$ with $\|u\| = 1$ in the above inequality. □

If we put $\alpha = \beta$ and $\lambda = \mu$, in Theorem 3.2, we get

$$\omega^2(A) \le \omega\left(A^2\right) + \|A - \alpha I\| \cdot \|A^* - \lambda I\|. \tag{3.3}$$

In particular, if $\lambda = \overline{\alpha}$, in (3.3), we infer that [6]

$$\omega^2(A) - \omega\left(A^2\right) \le \inf_{\alpha\in\mathbb{C}} \|A - \alpha I\|^2.$$

When $x \in \mathbb{H}$ is a unit vector, the Cauchy-Schwarz inequality asserts that $|\langle Ax, x\rangle| \le \|Ax\|$. In the following, a reversed version of this inequality is presented.

Proposition 3.1. *Let $A \in \mathbb{B}(\mathbb{H})$ and let $x \in \mathbb{H}$ be a unit vector. Then, for any $\alpha, \beta \in \mathbb{C}$,*

$$\|Ax\|^2 \le |\langle Ax, x\rangle|^2 + \left\| \left(A - \frac{\alpha+\beta}{2} I \right) x \right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2} I \right),$$

where $m(X) = \inf_{\|x\|=1} |\langle Xx, x\rangle|$, for $X \in \mathbb{B}(\mathbb{H})$.

Proof. Assume that Φ is a unital positive linear map on $\mathbb{B}(\mathbb{H})$. From [21, (4.2)], we know that for any $A \in \mathbb{B}(\mathbb{H})$

$$\Phi\left(|A|^2\right) - |\Phi(A)|^2 = \Phi\left(|A - cI|^2\right) - |\Phi(A - cI)|^2, \quad c \in \mathbb{C}. \tag{3.4}$$

Now, when $x \in \mathbb{H}$ is a unit vector, the mapping $\Phi : \mathbb{B}(\mathbb{H}) \to \mathbb{B}(\mathbb{H})$ defined by $\Phi(A) = \langle Ax, x\rangle$ is a unital positive linear map. Consequently, (3.4) implies, for any unit vector $x \in \mathbb{H}$ and $A \in \mathbb{B}(\mathbb{H})$,

$$\begin{aligned}
\|Ax\|^2 - |\langle Ax, x\rangle|^2 &= \left\langle |A|^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \\
&= \Phi\left(|A|^2\right) - |\Phi(A)|^2 \\
&= \Phi\left(\left|A - \frac{\alpha+\beta}{2}I\right|^2\right) - \left|\Phi\left(A - \frac{\alpha+\beta}{2}I\right)\right|^2 \\
&= \left\langle \left|A - \frac{\alpha+\beta}{2}I\right|^2 x, x\right\rangle - \left|\left\langle \left(A - \frac{\alpha+\beta}{2}I\right) x, x\right\rangle\right|^2 \\
&= \left\|\left(A - \frac{\alpha+\beta}{2}I\right) x\right\|^2 - \left|\left\langle \left(A - \frac{\alpha+\beta}{2}I\right) x, x\right\rangle\right|^2 \\
&\le \left\|\left(A - \frac{\alpha+\beta}{2}I\right) x\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right),
\end{aligned}$$

as required. □

The next result improves [9, Corollary 2.4].

Corollary 3.2. *Let $A \in \mathbb{B}(\mathbb{H})$. If $C_{\alpha,\beta}(A)$ is accretive, then*

$$\|A\|^2 - \omega^2(A) \le \frac{|\alpha-\beta|^2}{4} - m^2\left(A - \frac{\alpha+\beta}{2}I\right).$$

Proof. It follows from Proposition 3.1 and the relation (1.5) that

$$\begin{aligned}
\|Ax\|^2 &\le |\langle Ax, x\rangle|^2 + \left\|\left(A - \frac{\alpha+\beta}{2}I\right) x\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right) \\
&\le \omega^2(A) + \left\|A - \frac{\alpha+\beta}{2}I\right\|^2 - m^2\left(A - \frac{\alpha+\beta}{2}I\right) \\
&\le \omega^2(A) + \frac{|\alpha-\beta|^2}{4} - m^2\left(A - \frac{\alpha+\beta}{2}I\right).
\end{aligned}$$

We conclude the result by taking the supremum over $x \in \mathbb{H}$ with $\|x\| = 1$ in the above inequality. □

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