

NONLINEAR LEFT BI-SKEW LIE TYPE DERIVATIONS ON *-ALGEBRAS

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ABSTRACT. Let \mathcal{A} be a unital $*$ -algebra over \mathbb{C} (the field of complex number). For any $\alpha, \beta \in \mathcal{A}$, define $\alpha \circ \beta = \alpha^* \beta - \beta^* \alpha$. In this article, it is shown that a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (need not be linear) satisfies $\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n)$ for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$ -derivation. As applications, we apply our main result to various special classes of unital $*$ -algebras, such as prime $*$ -algebras, factor von Neumann algebras and von Neumann algebra with no central summands of type I_1 .

1. INTRODUCTION

Let \mathcal{A} be a $*$ -algebra on the complex field \mathbb{C} . A map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ is called an additive derivation if $\vartheta(\alpha + \beta) = \vartheta(\alpha) + \vartheta(\beta)$ and $\vartheta(\alpha\beta) = \vartheta(\alpha)\beta + \alpha\vartheta(\beta)$ hold for all $\alpha, \beta \in \mathcal{A}$. An algebra with an involution ‘ $*$ ’ is called $*$ -algebra, then an additive derivation $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ is called an additive $*$ -derivation, if $\vartheta(\alpha^*) = \vartheta(\alpha)^*$ hold for all $\alpha \in \mathcal{A}$. For $\alpha, \beta \in \mathcal{A}$, describe the skew Lie product and bi-skew Lie product of α and β by $[\alpha, \beta]_* = \alpha\beta - \beta\alpha^*$ and $[\alpha, \beta]_{\nabla} = \alpha\beta^* - \beta\alpha^*$, respectively. A map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (need not be linear) is said to be nonlinear skew Lie derivation (resp. nonlinear skew Lie triple derivation) if $\vartheta([\alpha, \beta]_*) = [\vartheta(\alpha), \beta]_* + [\alpha, \vartheta(\beta)]_*$ (resp. $\vartheta([\alpha, \beta]_*, \gamma)_* = [[\vartheta(\alpha), \beta]_*, \gamma]_* + [\alpha, \vartheta(\beta)]_*, \gamma]_* + [[\alpha, \beta]_*, \vartheta(\gamma)]_*$), holds for all $\alpha, \beta, \gamma \in \mathcal{A}$. Analogously, a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is called nonlinear bi-skew Lie derivation (resp. nonlinear bi-skew Lie triple derivation) if $\vartheta([\alpha, \beta]_{\nabla}) = [\vartheta(\alpha), \beta]_{\nabla} + [\alpha, \vartheta(\beta)]_{\nabla}$ (resp. $\vartheta([\alpha, \beta]_{\nabla}, \gamma)_{\nabla} = [[\vartheta(\alpha), \beta]_{\nabla}, \gamma]_{\nabla} + [[\alpha, \vartheta(\beta)]_{\nabla}, \gamma]_{\nabla} + [[\alpha, \beta]_{\nabla}, \vartheta(\gamma)]_{\nabla}$) holds for

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all $\alpha, \beta, \gamma \in \mathcal{A}$. Many mathematicians have spent the last decade to the studying mappings involving different products on various types of rings and algebras. Yu et al. in [15], proved that every $*$ -Lie derivation on factor von Neumann algebras is an additive $*$ -derivation. Li et al. [9] discussed the skew Lie triple derivation between factor von Neumann algebras and showed that it is an additive $*$ -derivation. Kong et al. [7], extended the results of [15] to prime $*$ -rings. Similarly, skew Jordan derivations, skew Jordan triple derivations and skew Jordan-type derivations have been studied under certain conditions on various types of algebras such as factor von Neumann algebras, von Neumann algebras and $*$ -algebras (see [8, 10, 14, 16, 17]).

For any $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ and integer $n \geq 2$, define a sequence of polynomials as follows: $P_1(\lambda_1) = \lambda_1$, $P_2(\lambda_1, \lambda_2) = \lambda_1 \circ \lambda_2 = \lambda_1^* \lambda_2 - \lambda_2^* \lambda_1$ and $P_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) = P_{n-1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) \circ \lambda_n$. The polynomial $P_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n)$ is called the left bi-skew Lie n -product of elements $\lambda_1, \lambda_2, \dots, \lambda_n$. Now, we define a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) is said to be nonlinear left bi-skew Lie n -derivation if

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n)$$

holds for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$. Obviously, a nonlinear left bi-skew Lie 2-derivation is a nonlinear left bi-skew Lie derivation and a nonlinear left bi-skew Lie 3-derivation is a nonlinear left bi-skew Lie triple derivation. Nonlinear left bi-skew Lie 2-derivations, nonlinear left bi-skew Lie 3-derivations and nonlinear left bi-skew Lie n -derivations are collectively known as nonlinear left bi-skew Lie-type derivations. In 2018, Lin [11], studied nonlinear $*$ -Lie type derivations on standard operator algebras and proved that every nonlinear $*$ -Lie n -derivation on a standard operator algebra is an additive $*$ -derivation. In the same year Lin [12], proved that every nonlinear $*$ -Lie n -derivation on von Neumann algebras is an additive $*$ -derivation. Recently Madni et al. [13], proved that if \mathcal{A} is a unital $*$ -algebra and $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ is a skew Lie n -derivation, then ϑ is additive. Moreover, if $\vartheta(\frac{iI}{2})$ is self-adjoint, then ϑ is a $*$ -derivation. Kong et al. in [5], proved that if \mathcal{A} is a factor von Neumann algebra and a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (need not be linear) satisfy $\vartheta([\alpha, \beta]_{\nabla}) = [\vartheta(\alpha), \beta]_{\nabla} + [\alpha, \vartheta(\beta)]_{\nabla}$ if and only if ϑ is an additive $*$ -derivation. Khan [4], proved that every nonlinear bi-skew Lie triple derivation on factor von Neumann algebra is an additive $*$ -derivation. Ashraf et al. [1], generalized the result of [4] and proved that every nonlinear bi-skew Lie n -derivation on factor von Neumann algebra is an additive $*$ -derivation.

Inspired by the above-mentioned works, our focus will be on providing a description of nonlinear left bi-skew Lie n -derivations on $*$ -algebras. More accurately, we show that, under mild assumptions, every nonlinear left bi-skew Lie n -derivation on a unital $*$ -algebra is an additive $*$ -derivation.

2. MAIN RESULTS

The main result of this article is as follows.

Theorem 2.1. *Let \mathcal{A} be a unital *-algebra with the unity I and containing a non-trivial projection P_1 . Write $P_2 = I - P_1$ and suppose that \mathcal{A} satisfies*

$$(\spadesuit) \quad XAP_k = 0 \Rightarrow X = 0 \quad (k = 1, 2),$$

where $X \in \mathcal{A}$. Then a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (not necessarily linear) satisfies

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive *-derivation.

Proof. Assume $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then, by Peirce decomposition of \mathcal{A} , we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly, any $\Pi \in \mathcal{A}$ can be written as $\Pi = \Pi_{11} + \Pi_{12} + \Pi_{21} + \Pi_{22}$, where $\Pi_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. Let $\mathcal{A}^+ = \{A \in \mathcal{A} : A^* = A\}$ and $\mathcal{A}^- = \{A \in \mathcal{A} : A^* = -A\}$, $\mathcal{A}_{12}^- = \{P_1AP_2 + P_2AP_1 : A \in \mathcal{A}^-\}$, $\mathcal{A}_{kk}^- = P_k\mathcal{A}^-P_k$, $k = 1, 2$. Thus, for every $A \in \mathcal{A}^-$, we have $A = A_{11} + A_{12} + A_{22}$, where $A_{11} \in \mathcal{A}_{11}^-$, $A_{12} \in \mathcal{A}_{12}^-$ and $A_{22} \in \mathcal{A}_{22}^-$. Obviously, we only need to prove the necessity part. To complete the proof of the theorem stated above, several lemmas are required. These lemmas are presented as follows. □

Lemma 2.1. $\vartheta(0) = 0$.

Proof. It is obvious that

$$\begin{aligned} \vartheta(0) &= \vartheta(P_n(0, 0, \dots, 0)) \\ &= P_n(\vartheta(0), 0, \dots, 0) + P_n(0, \vartheta(0), \dots, 0) + \dots + P_n(0, 0, \dots, \vartheta(0)) \\ &= 0. \end{aligned} \quad \square$$

Lemma 2.2. $\vartheta(H)^* = -\vartheta(H)$ for any $H \in \mathcal{A}^-$.

Proof. We have the relation $P_n(H, -\frac{I}{2}, \dots, -\frac{I}{2}) = H$ and

$$\begin{aligned} \vartheta(H) &= \vartheta\left(P_n\left(H, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right) \\ &= P_n\left(\vartheta(H), -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(H, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) \\ &\quad + \dots + P_n\left(H, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right) \\ &= \frac{I}{2}(\vartheta(H) - \vartheta(H)^*) + \left(-\vartheta\left(-\frac{I}{2}\right)^* H - H\vartheta\left(-\frac{I}{2}\right)\right) \\ &\quad + \dots + \left(-\vartheta\left(-\frac{I}{2}\right)^* H - H\vartheta\left(-\frac{I}{2}\right)\right) \\ &= \frac{I}{2}(\vartheta(H) - \vartheta(H)^*) + (n-1)\left(-\vartheta\left(-\frac{I}{2}\right)^* H - H\vartheta\left(-\frac{I}{2}\right)\right). \end{aligned}$$

On solving further we get

$$-\vartheta(H)^* = \frac{I}{2}(\vartheta(H) - \vartheta(H)^*) + (n-1)\left(-\vartheta\left(-\frac{I}{2}\right)^* H - H\vartheta\left(-\frac{I}{2}\right)\right) = \vartheta(H). \quad \square$$

Lemma 2.3. *For any $H_{11} \in \mathcal{A}_{11}^-, H_{12} \in \mathcal{A}_{12}^-$ and $H_{22} \in \mathcal{A}_{22}^-$, we have*

$$\vartheta(H_{11} + H_{12}) = \vartheta(H_{11}) + \vartheta(H_{12})$$

and

$$\vartheta(H_{12} + H_{22}) = \vartheta(H_{12}) + \vartheta(H_{22}).$$

Proof. Let $T = \vartheta(H_{11} + H_{12}) - \vartheta(H_{11}) - \vartheta(H_{12})$. We need to demonstrate this $T = 0$. Utilizing the reality $P_n(P_2, H_{11}, -I, -I, \dots, -I) = 0$ and Lemma 2.1, we have

$$\begin{aligned} & \vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \dots, -I)) \\ &= \vartheta(P_n(P_2, H_{11}, -I, -I, \dots, -I)) + \vartheta(P_n(P_2, H_{12}, -I, -I, \dots, -I)) \\ &= P_n(\vartheta(P_2), H_{11}, -I, -I, \dots, -I) + P_n(P_2, \vartheta(H_{11}), -I, -I, \dots, -I) \\ & \quad + P_n(P_2, H_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{11}, -I, -I, \dots, \vartheta(-I)) \\ & \quad + P_n(\vartheta(P_2), H_{12}, -I, \dots, -I) + P_n(P_2, \vartheta(H_{12}), -I, -I, \dots, -I) \\ & \quad + P_n(P_2, H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{12}, -I, -I, \dots, \vartheta(-I)) \\ &= P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \dots, -I) \\ & \quad + P_n(P_2, (\vartheta(H_{11}) + \vartheta(H_{12})), -I, -I, \dots, -I) \\ & \quad + P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \dots, -I) \\ & \quad + \dots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \dots, \vartheta(-I)). \end{aligned}$$

Further, we have

$$\begin{aligned} & \vartheta(P_n(P_2, H_{11} + H_{12}, -I, -I, \dots, -I)) \\ &= P_n(\vartheta(P_2), (H_{11} + H_{12}), -I, -I, \dots, -I) + P_n(P_2, (\vartheta(H_{11} + H_{12})), -I, -I, \dots, -I) \\ & \quad + P_n(P_2, (H_{11} + H_{12}), \vartheta(-I), -I, \dots, -I) \\ & \quad + \dots + P_n(P_2, (H_{11} + H_{12}), -I, -I, \dots, \vartheta(-I)). \end{aligned}$$

On equating the above two expressions, we obtain that $P_n(P_2, T, -I, \dots, -I) = 0$. This gives us $T_{12} = T_{21} = T_{22} = 0$.

Invoking the fact $P_n(P_2 - P_1, H_{12}, -I, \dots, -I) = 0$ and using Lemma

$$\begin{aligned} & \vartheta(P_n((P_2 - P_1), H_{11} + H_{12}, -I, \dots, -I)) \\ &= \vartheta((P_n(P_2 - P_1), H_{11}, -I, \dots, -I)) + \vartheta(P_n((P_2 - P_1), H_{12}, -I, \dots, -I)) \\ &= P_n(\vartheta(P_2 - P_1), H_{11}, -I, \dots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}), -I, \dots, -I) \\ & \quad + P_n((P_2 - P_1), H_{11}, \vartheta(-I), \dots, -I) + \dots + P_n((P_2 - P_1), H_{11}, -I, \dots, \vartheta(-I)) \\ & \quad + P_n(\vartheta(P_2 - P_1), H_{12}, -I, \dots, -I) + P_n((P_2 - P_1), \vartheta(H_{12}), -I, \dots, -I) \\ & \quad + P_n((P_2 - P_1), H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n((P_2 - P_1), H_{12}, -I, \dots, \vartheta(-I)) \\ &= P_n(\vartheta(P_2 - P_1), H_{11} + H_{12}, -I, \dots, -I) + P_n((P_2 - P_1), \vartheta(H_{11}) + \vartheta(H_{12}), \\ & \quad -I, \dots, -I) + P_n((P_2 - P_1), H_{11} + H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n((P_2 - P_1), \\ & \quad H_{11} + H_{12}, -I, \dots, \vartheta(-I)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \vartheta(P_n((P_2 - P_1), H_{11} + H_{12}, -I, \dots, -I)) \\ &= P_n(\vartheta(P_2 - P_1), H_{11} + H_{12}, -I, \dots, -I) + P_n((P_2 - P_1), \vartheta(H_{11} + H_{12}), -I, \dots, -I) \\ & \quad + P_n((P_2 - P_1), H_{11} + H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n((P_2 - P_1), H_{11} + H_{12}, \\ & \quad -I, \dots, \vartheta(-I)). \end{aligned}$$

When comparing the two expressions above, we obtain that

$$P_n((P_2 - P_1), T, -I, \dots, -I) = 0.$$

This further implies that $T_{11} = 0$. Hence $T = 0$, that is,

$$\vartheta(H_{11} + H_{12}) = \vartheta(H_{11}) + \vartheta(H_{12}).$$

Similarly, we can show that $\vartheta(H_{12} + H_{22}) = \vartheta(H_{12}) + \vartheta(H_{22})$. □

Lemma 2.4. *For any $H_{11} \in \mathcal{A}_{11}^-$, $H_{12} \in \mathcal{A}_{12}^-$ and $H_{22} \in \mathcal{A}_{22}^-$, we have*

$$\vartheta(H_{11} + H_{12} + H_{22}) = \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{22}).$$

Proof. Let $T = \vartheta(H_{11} + H_{12} + H_{22}) - \vartheta(H_{11}) - \vartheta(H_{12}) - \vartheta(H_{22})$. We show that $T = 0$. Using the fact that $P_n(P_1, H_{22}, -I, \dots, -I) = 0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned} & \vartheta(P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\ &= \vartheta(P_n(P_1, H_{11} + H_{12}, -I, \dots, -I)) + \vartheta(P_n(P_1, H_{22}, -I, \dots, -I)) \\ &= P_n((\vartheta(P_1), H_{11} + H_{12}, -I, \dots, -I)) + P_n(P_1, \vartheta(H_{11} + H_{12}), -I, \dots, -I) \\ & \quad + P_n(P_1, H_{11} + H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, H_{11} + H_{12}, -I, \dots, \vartheta(-I)) \\ & \quad + P_n(\vartheta(P_1), H_{22}, -I, \dots, -I) + P_n(P_1, \vartheta(H_{22}), -I, \dots, -I) \\ & \quad + P_n(P_1, H_{22}, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, H_{22}, -I, \dots, \vartheta(-I)) \\ &= P_n((\vartheta(P_1), H_{11} + H_{12}, -I, \dots, -I)) + P_n(P_1, \vartheta(H_{11}) + \vartheta(H_{12}), -I, \dots, -I) \\ & \quad + P_n(P_1, H_{11} + H_{12}, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, H_{11} + H_{12}, -I, \dots, \vartheta(-I)) \\ & \quad + P_n(\vartheta(P_1), H_{22}, -I, \dots, -I) + P_n(P_1, \vartheta(H_{22}), -I, \dots, -I) \\ & \quad + P_n(P_1, H_{22}, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, H_{22}, -I, \dots, \vartheta(-I)) \\ &= P_n((\vartheta(P_1), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\ & \quad + P_n(P_1, \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{22}), -I, \dots, -I) \\ & \quad + P_n(P_1, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\ & \quad + \dots + P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \vartheta(P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\ &= P_n((\vartheta(P_1), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\ & \quad + P_n(P_1, \vartheta(H_{11} + H_{12} + H_{22}), -I, \dots, -I) \end{aligned}$$

$$\begin{aligned}
 &+ P_n(P_1, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\
 &+ \dots + P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).
 \end{aligned}$$

On solving the above two expressions for $\vartheta(P_n(P_1, H_{11} + H_{12} + H_{22}, -I, \dots, -I))$, we obtain that $P_n(P_1, T, -I, \dots, -I) = 0$ which further implies that $T_{11} = T_{12} = T_{21} = 0$. Using the fact that $P_n(P_2, H_{11}, -I, \dots, -I) = 0$ and Lemmas 2.1 and 2.3, we find that

$$\begin{aligned}
 &\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
 &= \vartheta(P_n(P_2, H_{11}, -I, \dots, -I)) + \vartheta(P_n(P_2, H_{12} + H_{22}, -I, \dots, -I)) \\
 &= P_n((\vartheta(P_2), H_{11}, -I, \dots, -I)) + P_n(P_2, \vartheta(H_{11}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{11}, -I, \dots, \vartheta(-I)) \\
 &\quad + P_n(\vartheta(P_2), H_{12} + H_{22}, -I, \dots, -I) + P_n(P_2, \vartheta(H_{12}) + \vartheta(H_{22}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{12} + H_{21}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{12} + H_{21}, -I, \dots, \vartheta(-I)) \\
 &= P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
 &\quad + P_n(P_2, \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{22}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\
 &\quad + \dots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 &\vartheta(P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
 &= P_n((\vartheta(P_2), H_{11} + H_{12} + H_{22}, -I, \dots, -I)) \\
 &\quad + P_n((P_2, \vartheta(H_{11}) + H_{12} + H_{22}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11} + H_{12} + H_{22}, \vartheta(-I), \dots, -I) \\
 &\quad + \dots + P_n(P_2, H_{11} + H_{12} + H_{22}, -I, \dots, \vartheta(-I)).
 \end{aligned}$$

On comparing the above two expressions, we obtain that $P_n(P_2, T, -I, \dots, -I) = 0$ which further implies that $T_{22} = 0$. Thus, $T = 0$, that is,

$$\vartheta(H_{11} + H_{12} + H_{21} + H_{22}) = \vartheta(H_{11}) + \vartheta(H_{12}) + \vartheta(H_{21}) + \vartheta(H_{22}). \quad \square$$

Lemma 2.5. *For any $H_{12}, H'_{12} \in \mathcal{A}_{12}^-$ we have*

$$\vartheta(H_{12} + H'_{12}) = \vartheta(H_{12}) + \vartheta(H'_{12}).$$

Proof. Take $H_{12} = A_{12} - A_{12}^*$ and $H'_{12} = B_{12} - B_{12}^*$ for $A_{12}, B_{12} \in \mathcal{A}_{12}$. Using the facts $P_n(iP_1 + iA_{12} + iA_{12}^*, iP_2 + iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, -\frac{I}{2}) = H_{12} + H'_{12} + H_{12}(H'_{12})^* - H'_{12}H_{12}^*$, where $H_{12}(H'_{12})^* - H'_{12}H_{12}^* = (A_{12}^*B_{12} - B_{12}^*A_{12}) + (A_{12}B_{12}^* - B_{12}A_{12}^*)$ and making use of Lemma 2.4, we have

$$\begin{aligned}
 &\vartheta(H_{12} + H'_{12}) + \vartheta(H_{12}(H'_{12})^* - H'_{12}H_{12}^*) \\
 &= \vartheta(H_{12} + H'_{12} + H_{12}(H'_{12})^* - H'_{12}H_{12}^*)
 \end{aligned}$$

$$\begin{aligned}
 &= \vartheta \left(P_n \left(iP_1 + iA_{12} + iA_{12}^*, iP_2 + iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \right) \\
 &= P_n \left(\vartheta(iP_1) + \vartheta(iA_{12} + iA_{12}^*), iP_2 + iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \\
 &\quad + P_n \left(iP_1 + iA_{12} + iA_{12}^*, \vartheta(iP_2) + \vartheta(iB_{12} + iB_{12}^*), -\frac{I}{2}, \dots, -\frac{I}{2} \right) \\
 &\quad + P_n \left(iP_1 + iA_{12} + iA_{12}^*, iP_2 + iB_{12} + iB_{12}^*, \vartheta \left(-\frac{I}{2} \right), \dots, -\frac{I}{2} \right) \\
 &\quad + \dots + P_n \left(iP_1 + iA_{12} + iA_{12}^*, iP_2 + iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, \vartheta \left(-\frac{I}{2} \right) \right) \\
 &= \vartheta \left(P_n \left(iP_1, iP_2, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \right) + \vartheta \left(P_n \left(iP_1, iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \right) \\
 &\quad + \vartheta \left(P_n \left(iA_{12} + iA_{12}^*, iP_2, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \right) \\
 &\quad + \vartheta \left(P_n \left(iA_{12} + iA_{12}^*, iB_{12} + iB_{12}^*, -\frac{I}{2}, \dots, -\frac{I}{2} \right) \right) \\
 &= \vartheta(H_{12}) + \vartheta(H'_{12}) + \vartheta(H_{12}(H'_{12})^* - H'_{12}H_{12}^*).
 \end{aligned}$$

Hence, $\vartheta(H_{12} + H'_{12}) = \vartheta(H_{12}) + \vartheta(H'_{12})$ for any $H_{12}, H'_{12} \in \mathcal{A}_{12}^-$. □

Lemma 2.6. For any $H_{ii}, H'_{ii} \in \mathcal{A}_{ii}$ for $i = 1, 2$, we have

$$\vartheta(H_{11} + H'_{11}) = \vartheta(H_{11}) + \vartheta(H'_{11}) \quad \text{and} \quad \vartheta(H_{22} + H'_{22}) = \vartheta(H_{22}) + \vartheta(H'_{22}).$$

Proof. Let $T = \vartheta(H_{11} + H'_{11}) - \vartheta(H_{11}) - \vartheta(H'_{11})$, we show that $T = 0$. Using the fact that $P_n(P_2, H_{11}, -I \dots, -I) = P_n(P_2, H'_{11}, -I, \dots, -I) = 0$ and making use of Lemma 2.1, we obtain

$$\begin{aligned}
 &\vartheta(P_n(P_2, H_{11} + H'_{11}, -I \dots, -I)) \\
 &= \vartheta(P_n(P_2, H_{11}, -I \dots, -I)) + \vartheta(P_n(P_2, H'_{11}, -I \dots, -I)) \\
 &= P_n(\vartheta(P_2), H_{11}, -I, \dots, -I) + P_n(P_2, \vartheta(H_{11}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{11}, -I, \dots, \vartheta(-I)) \\
 &\quad + P_n(\vartheta(P_2), H'_{11}, -I, \dots, -I) + P_n(P_2, \vartheta(H'_{11}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H'_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H'_{11}, -I, \dots, \vartheta(-I)) \\
 &= P_n(\vartheta(P_2), H_{11} + H'_{11}, -I, \dots, I) + P_n(P_2, \vartheta(H_{11}) + \vartheta(H'_{11}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11} + H'_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{11} + H'_{11}, -I, \dots, \vartheta(-I)).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 &\vartheta(P_n(P_2, H_{11} + H'_{11}, -I \dots, -I)) \\
 &= P_n(\vartheta(P_2), H_{11} + H'_{11}, -I, \dots, I) + P_n(P_2, \vartheta(H_{11} + H'_{11}), -I, \dots, -I) \\
 &\quad + P_n(P_2, H_{11} + H'_{11}, \vartheta(-I), \dots, -I) + \dots + P_n(P_2, H_{11} + H'_{11}, -I, \dots, \vartheta(-I)).
 \end{aligned}$$

Comparing the above two expressions, we find that $P_n(P_2, T, -I, \dots, -I) = 0$, which gives us $T_{12} = T_{21} = T_{22} = 0$.

Next, we show that $T_{11} = 0$. Let $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$ and it is easy to observe that $P_n(X, H_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}), P_n(X, H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}) \in \mathcal{A}_{12}$. Thus, using Lemma 2.5, we find that

$$\begin{aligned} & \vartheta\left(P_n\left(X, H_{11} + H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right) \\ = & \vartheta\left(P_n\left(X, H_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right) + \vartheta\left(P_n\left(X, H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right) \\ = & P_n\left(\vartheta(X), H_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(X, \vartheta(H_{11}), -\frac{I}{2}, \dots, -\frac{I}{2}\right) \\ & + P_n\left(X, H_{11}, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) + \dots + P_n\left(X, H_{11}, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right) \\ & + P_n\left(\vartheta(X), H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(X, \vartheta(H'_{11}), -\frac{I}{2}, \dots, -\frac{I}{2}\right) \\ & + P_n\left(X, H'_{11}, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) + \dots + P_n\left(X, H'_{11}, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right) \\ = & P_n\left(\vartheta(X), H_{11} + H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(X, \vartheta(H_{11}) + \vartheta(H'_{11}), -\frac{I}{2}, \dots, -\frac{I}{2}\right) \\ & + P_n\left(X, H_{11} + H'_{11}, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) \\ & + \dots + P_n\left(X, H_{11} + H'_{11}, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \vartheta\left(P_n\left(X, H_{11} + H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right) \\ = & P_n\left(\vartheta(X), H_{11} + H'_{11}, -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(X, \vartheta(H_{11} + H'_{11}), -\frac{I}{2}, \dots, -\frac{I}{2}\right) \\ & + P_n\left(X, H_{11} + H'_{11}, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) \\ & + \dots + P_n\left(X, H_{11} + H'_{11}, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right). \end{aligned}$$

From the last two expressions, we get $P_n(X, T, -\frac{I}{2}, \dots, -\frac{I}{2}) = 0$, on solving further, we obtain $-XT + TX = 0$. Multiplying by P_1 and P_2 from left and right, we obtain $P_1TP_1AP_2 = 0$ for all $A \in \mathcal{A}$. Application of the condition (\spadesuit) yields $T_{11} = 0$. Hence, $T = 0$, that is, $\vartheta(H_{11} + H'_{11}) = \vartheta(H_{11}) + \vartheta(H'_{11})$. Symmetrically, we can prove that $\vartheta(H_{22} + H'_{22}) = \vartheta(H_{22}) + \vartheta(H'_{22})$. □

Lemma 2.7. ϑ is additive on \mathcal{A}^- .

Proof. For any $H, K \in \mathcal{A}^-$, we have $H = H_{11} + H_{12} + H_{22}$ and $K = K_{11} + K_{12} + K_{22}$. With the help of Lemmas 2.4, 2.5 and 2.6, we obtain

$$\begin{aligned} \vartheta(H + K) &= \vartheta(H_{11} + H_{12} + H_{22} + K_{11} + K_{12} + K_{22}) \\ &= \vartheta(H_{11} + K_{11}) + \vartheta(H_{12} + K_{12}) + \vartheta(H_{22} + K_{22}) \\ &= \vartheta(H_{11}) + \vartheta(K_{11}) + \vartheta(H_{12}) + \vartheta(K_{12}) + \vartheta(H_{22}) + \vartheta(K_{22}) \\ &= \vartheta(H_{11} + H_{12} + H_{22}) + \vartheta(K_{11} + K_{12} + K_{22}) \\ &= \vartheta(H) + \vartheta(K). \end{aligned} \quad \square$$

Lemma 2.8. $\vartheta(-I) = 0$.

Proof. Using Lemmas 2.2 and 2.7, it follows that $2^{n-1}V = P_n(V, -I, -I, \dots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned} 2^{n-1}\vartheta(V) &= \vartheta(P_n(V, -I, -I, \dots, -I)). \\ &= P_n(\vartheta(V), -I, -I, \dots, -I) + P_n(V, \vartheta(-I), -I, \dots, I) \\ &\quad + P_n(V, -I, \vartheta(-I), \dots, -I) + \dots + P_n(V, -I, -I, \dots, \vartheta(-I)) \\ &= 2^{n-1}\vartheta(V) + 2^{n-2}(n-1)\{-A\vartheta(-I) - \vartheta(-I)^*V\}. \end{aligned}$$

On simplifying further, we obtain

$$0 = V\vartheta(-I) + \vartheta(-I)^*V.$$

Taking $V = iI$ in the above expression, we obtain $\vartheta(-I) = -\vartheta(-I)^*$, which implies $\vartheta(-I) \in \mathcal{A}^-$. We have now $V\vartheta(-I) = \vartheta(-I)V$ for all $V \in \mathcal{A}^-$. For every $F \in \mathcal{A}$, we have $F = F_1 + iF_2$, where $F_1 = \frac{F-F^*}{2} \in \mathcal{A}^-$ and $F_2 = \frac{F+F^*}{2i} \in \mathcal{A}^-$. Thus, we get $F\vartheta(-I) = \vartheta(-I)F$ for all $F \in \mathcal{A}$, this further implies that

$$(2.1) \quad \vartheta(-I) \in Z(\mathcal{A}) \cap \mathcal{A}^-.$$

Using Lemmas 2.1 and 2.2, it follows that $0 = P_n(V, -iI, -I, \dots, -I)$ for any $V \in \mathcal{A}^-$, we have

$$\begin{aligned} 0 &= \vartheta(P_n(V, -iI, -I, \dots, -I)) \\ &= P_n(\vartheta(V), -iI, -I, \dots, -I) + P_n(V, \vartheta(-I), -I, \dots, I) \\ &\quad + P_n(V, -I, \vartheta(-iI), \dots, -I) + \dots + P_n(V, -iI, -I, \dots, \vartheta(-I)) \\ &= 2^{n-2}(-V\vartheta(-iI) - \vartheta(-iI)^*V). \end{aligned}$$

By following similar steps as done earlier, we get

$$(2.2) \quad \vartheta(-iI) \in Z(\mathcal{A}) \cap \mathcal{A}^-.$$

Assume $\vartheta(P_1) = L_1 + iL_2$, where $L_1, L_2 \in \mathcal{A}^+$, with the help of equation (2.1) and fact $P_n(-I, P_1, -\frac{I}{2}, \dots, -\frac{I}{2}) = 0$, we get

$$0 = \vartheta\left(P_n\left(-I, P_1, -\frac{I}{2}, \dots, -\frac{I}{2}\right)\right)$$

$$\begin{aligned}
&= P_n\left(\vartheta(-I), P_1, -\frac{I}{2}, \dots, -\frac{I}{2}\right) + P_n\left(-I, \vartheta(P_1), -\frac{I}{2}, \dots, -\frac{I}{2}\right) \\
&\quad + P_n\left(-I, P_1, \vartheta\left(-\frac{I}{2}\right), \dots, -\frac{I}{2}\right) + \dots + P_n\left(-I, P_1, -\frac{I}{2}, \dots, \vartheta\left(-\frac{I}{2}\right)\right) \\
&= -2\vartheta(-I)P_1 - 2iL_2.
\end{aligned}$$

Thus, we obtain

$$(2.3) \quad \vartheta(P_1) = L_1 - \vartheta(-I)P_1.$$

With the help of equations (2.2) and (2.3) and fact $P_n(-iI, P_1, -I, \dots, -I) = 2^{n-1}iP_1$, we obtain

$$\begin{aligned}
2^{n-1}\vartheta(iP_1) &= \vartheta(P_n(-iI, P_1, -I, \dots, -I)) \\
&= P_n(\vartheta(-iI), P_1, -I, \dots, -I) + P_n(-iI, \vartheta(P_1), -I, \dots, -I) \\
&\quad + P_n(-iI, P_1, \vartheta(-I), \dots, -I) + \dots + P_n(-iI, P_1, -I, \dots, \vartheta(-I)) \\
&= -2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}iL_1.
\end{aligned}$$

On solving further, we obtain

$$(2.4) \quad \vartheta(iP_1) = -\vartheta(-iI)P_1 + iL_1.$$

Applying (2.3) and (2.4), we get

$$\begin{aligned}
2^{n-1}\vartheta(iP_1) &= \vartheta(P_n(P_1, iP_1, -I, \dots, -I)) \\
&= P_n(\vartheta(P_1), iP_1, -I, \dots, -I) + P_n(P_1, \vartheta(iP_1), -I, \dots, -I) \\
&\quad + P_n(P_1, iP_1, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, iP_1, -I, \dots, \vartheta(-I)) \\
&= -2^{n-1}\vartheta(-iI)P_1 + 2^{n-1}i(P_1L_1 + L_1P_1).
\end{aligned}$$

Therefore, we get

$$(2.5) \quad \vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1 + L_1P_1).$$

Comparing (2.4) and (2.5), we obtain $L_1 = P_1L_1 + L_1P_1$, implies $P_1L_1P_1 = P_2L_1P_2 = 0$. Invoking the last relation in (2.3), we obtain

$$(2.6) \quad \vartheta(P_1) = P_1L_1P_2 + P_2L_1P_1 - \vartheta(-I)P_1.$$

Taking $X = A_{12} - A_{12}^*$ for any $A_{12} \in \mathcal{A}_{12}$, then $X \in \mathcal{A}^-$. Invoking Lemmas 2.2 and 2.7, we possess

$$\begin{aligned}
2^{n-2}\vartheta(X) &= \vartheta(P_n(P_1, X, -I, \dots, -I)) \\
&= P_n(\vartheta(P_1), X, -I, \dots, -I) + P_n(P_1, \vartheta(X), -I, \dots, -I) \\
&\quad + P_n(P_1, X, \vartheta(-I), \dots, -I) + \dots + P_n(P_1, X, -I, \dots, \vartheta(-I)) \\
&= 2^{n-2}(\vartheta(P_1)^*X + X\vartheta(P_1)) + 2^{n-2}(P_1\vartheta(X) + \vartheta(X)P_1).
\end{aligned}$$

Accordingly, we get

$$(2.7) \quad \vartheta(X) = \vartheta(P_1)^*X + X\vartheta(P_1) + P_1\vartheta(X) + \vartheta(X)P_1.$$

Multiplying (2.7) by P_1 and P_2 from left and right, respectively, and applying (2.6), we obtain $\vartheta(-I)A_{12}$. Implementation of the condition (\spadesuit) gives us $\vartheta(-I)P_1 = 0$, employing (2.1) and condition (\spadesuit) in the last relation, we get $\vartheta(-I)P_2 = 0$, finally we obtain $\vartheta(-I) = 0$. □

Lemma 2.9. *For any $S, S_1, S_2 \in \mathcal{A}^+$*

- (1) $\vartheta(S)^* = \vartheta(S)$;
- (2) $\vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S$;
- (3) ϑ is additive on \mathcal{A}^+ ;
- (4) $\vartheta(S_1 + iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$.

Proof. (1) Using the fact that $P_n(-I, S, -I, \dots, -I) = 0$ and Lemma 2.8, where $S \in \mathcal{A}^+$, we obtain

$$0 = \vartheta(P_n(-I, S, -I, \dots, -I)) = P_n(-I, \vartheta(S), -I, \dots, -I) = 2^{n-2}(\vartheta(S)^* - \vartheta(S)),$$

which allows us $\vartheta(S)^* = \vartheta(S)$.

(2) Using the fact that $P_n(-iI, S, -I, \dots, -I) = 2^{n-1}iS$, together with Lemmas 2.7, 2.8 and equation (2.2), where $S \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}\vartheta(iS) &= \vartheta(P_n(-iI, S, -I, \dots, -I)) \\ &= P_n(\vartheta(-iI), S, -I, \dots, -I) + P_n(-iI, \vartheta(S), -I, \dots, -I) \\ &= 2^{n-1}i(\vartheta(S) - \vartheta(-iI)S), \end{aligned}$$

which gives us

$$(2.8) \quad \vartheta(iS) = i\vartheta(S) - \vartheta(-iI)S.$$

(3) For any $S_1, S_2 \in \mathcal{A}^+$, invoking Lemma 2.7 and (2.8), we have

$$\vartheta(iS_1 + iS_2) = \vartheta(iS_1) + \vartheta(iS_2) = i\vartheta(S_1) - \vartheta(-iI)S_1 + i\vartheta(S_2) - \vartheta(-iI)S_2.$$

On the other hand, we have $\vartheta(i(S_1 + S_2)) = i\vartheta(S_1 + S_2) - \vartheta(-iI)(S_1 + S_2)$. Comparing the above two expressions for $\vartheta(i(S_1 + S_2))$, we obtain $\vartheta(S_1 + S_2) = \vartheta(S_1) + \vartheta(S_2)$.

(4) Using the fact that $P_n(S_1 + iS_2, -I, -I, \dots, -I) = 2^{n-1}iS_2$, together with Lemmas 2.7, 2.8 and (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) &= 2^{n-1}\vartheta(iS_2) = \vartheta(P_n(S_1 + iS_2, -I, -I, \dots, -I)) \\ &= P_n(\vartheta(S_1 + iS_2), -I, -I, \dots, -I) \\ &= 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*). \end{aligned}$$

Thus, we have

$$(2.9) \quad 2^{n-1}(i\vartheta(S_2) - \vartheta(-iI)S_2) = 2^{n-2}(\vartheta(S_1 + iS_2) - \vartheta(S_1 + iS_2)^*).$$

Invoking the fact that $P_n(-iI, S_1 + iS_2, -I, \dots, -I) = 2^{n-1}iS_1$ and using Lemmas 2.7, 2.8 and (2.2), (2.8), where $S_1, S_2 \in \mathcal{A}^+$, we obtain

$$\begin{aligned} 2^{n-1}(i\vartheta(S_1) - \vartheta(-iI)S_1) &= 2^{n-1}\vartheta(iS_1) \\ &= \vartheta(P_n(-iI, S_1 + iS_2, -I, \dots, -I)) \end{aligned}$$

$$\begin{aligned}
&= P_n(\vartheta(-iI), S_1 + iS_2, -I, \dots, -I) \\
&\quad + P_n(-iI, \vartheta(S_1 + iS_2), -I, \dots, -I) \\
&= -2^{n-1}\vartheta(-iI)S_1 + 2^{n-2}i(\vartheta(S_1 + iS_2) + \vartheta(S_1 + iS_2)^*).
\end{aligned}$$

On solving further, we obtain

$$(2.10) \quad 2^{n-1}\vartheta(S_1) = 2^{n-2}(\vartheta(S_1 + iS_2) + \vartheta(S_1 + iS_2)^*).$$

By adding (2.9) and (2.10), we get $\vartheta(S_1 + iS_2) = \vartheta(S_1) + i\vartheta(S_2) - \vartheta(-iI)S_2$. \square

Lemma 2.10. For all $A \in \mathcal{A}$

- (1) $\vartheta(A^*) = \vartheta(A)^*$;
- (2) ϑ is additive on \mathcal{A} .

Proof. (1) For $A \in \mathcal{A}$, we have $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$ and using Lemma 2.9 and (2.2), we obtain

$$\begin{aligned}
\vartheta(A)^* &= \vartheta(A_1 + iA_2)^* = (\vartheta(A_1) + i\vartheta(A_2) - \vartheta(-iI)A_2)^* \\
&= \vartheta(A_1) - i\vartheta(A_2) + \vartheta(-iI)A_2 = \vartheta(A_1 + i(-A_2)) = \vartheta(A_1 - iA_2) = \vartheta(A^*).
\end{aligned}$$

(2) For $A, B \in \mathcal{A}$, then $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_1, A_2, B_1, B_2 \in \mathcal{A}^+$. Using Lemma 2.9, we obtain

$$\begin{aligned}
\vartheta(A + B) &= \vartheta(A_1 + iA_2 + B_1 + iB_2) \\
&= \vartheta((A_1 + B_1) + i(A_2 + B_2)) \\
&= \vartheta(A_1 + B_1) + i\vartheta(A_2 + B_2) - \vartheta(-iI)(A_2 + B_2) \\
&= \vartheta(A_1) + \vartheta(B_1) + i\vartheta(A_2) + i\vartheta(B_2) - \vartheta(-iI)A_2 - \vartheta(-iI)B_2 \\
&= (\vartheta(A_1) + i\vartheta(A_2) - \vartheta(-iI)A_2) + (\vartheta(B_1) + i\vartheta(B_2) - \vartheta(-iI)B_2) \\
&= \vartheta(A) + \vartheta(B).
\end{aligned}$$

\square

Lemma 2.11. For all $A \in \mathcal{A}$

- (1) $\vartheta(-iI) = 0$;
- (2) $\vartheta(iA) = i\vartheta(A)$.

Proof. (1) Equation (2.4) implies

$$(2.11) \quad \vartheta(iP_1) = -\vartheta(-iI)P_1 + i(P_1L_1P_2 + P_2L_1P_1).$$

Observe that $P_n(iP_1, (A_{12} - A_{12}^*), -I, \dots, -I) = -2^{n-2}i(A_{12} + A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned}
-2^{n-2}\vartheta(i(A_{12} + A_{12}^*)) &= \vartheta(P_n(iP_1, (A_{12} - A_{12}^*), -I, \dots, -I)) \\
&= P_n(\vartheta(iP_1), (A_{12} - A_{12}^*), -I, \dots, -I) \\
&\quad + P_n(iP_1, \vartheta(A_{12} - A_{12}^*), -I, \dots, -I).
\end{aligned}$$

Using (2.11), in the above expression, we obtain

$$-2^{n-2}\vartheta(i(A_{12} + A_{12}^*)) = 2^{n-2}(\vartheta(-iI)A_{12} - iP_2L_1A_{12} + iP_1L_1A_{12}^* + iA_{12}L_1P_1)$$

$$+ A_{12}^* \vartheta(-iI) - iA_{12}^* L_1 P_2 - iP_1 \vartheta(A_{12}) + iP_1 \vartheta(A_{12})^* - i\vartheta(A_{12})^* P_1 + i\vartheta(A_{12}) P_1).$$

On the other hand, we have

$$-2^{n-2} \vartheta(i(A_{12} + A_{12}^*)) = -2^{n-2} i\vartheta(A_{12} + A_{12}^*) + \vartheta(-iI)(A_{12} + A_{12}^*).$$

Comparing the above two expressions, we obtain

$$\begin{aligned} & -i\vartheta(A_{12}) - i\vartheta(A_{12})^* + \vartheta(-iI)A_{12} + \vartheta(-iI)A_{12}^* \\ & = \vartheta(-iI)A_{12} - iP_2 L_1 A_{12} + iP_1 L_1 A_{12}^* + iA_{12} L_1 P_1 + A_{12}^* \vartheta(-iI) \\ & \quad - iA_{12}^* L_1 P_2 - iP_1 \vartheta(A_{12}) + iP_1 \vartheta(A_{12})^* - i\vartheta(A_{12})^* P_1 + i\vartheta(A_{12}) P_1. \end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, we obtain

$$(2.12) \quad P_1 \vartheta(A_{12})^* P_2 = 0.$$

Alternatively observed that $P_n(iP_1, i(A_{12} + A_{12}^*), -I, \dots, -I) = 2^{n-2}(A_{12} - A_{12}^*)$, for any $A_{12} \in \mathcal{A}_{12}$. Using Lemmas 2.8, 2.9 and 2.10, we find that

$$\begin{aligned} 2^{n-2} \vartheta(A_{12} - A_{12}^*) & = \vartheta(P_n(iP_1, i(A_{12} + A_{12}^*), -I, \dots, -I)) \\ & = P_n(\vartheta(iP_1), i(A_{12} + A_{12}^*), -I, \dots, -I) \\ & \quad + P_n(iP_1, \vartheta(i(A_{12} + A_{12}^*)), -I, \dots, -I). \end{aligned}$$

Using (2.11) in the above expression, we obtain

$$\begin{aligned} \vartheta(A_{12} - A_{12}^*) & = i\vartheta(-iI)A_{12} + P_2 L_1 A_{12} + P_1 L_1 A_{12}^* - iA_{12} L_1 P_1 \\ & \quad - iA_{12}^* \vartheta(-iI) - iA_{12}^* L_1 P_2 + P_1 \vartheta(A_{12}) + P_1 \vartheta(A_{12})^* \\ & \quad + iP_1 \vartheta(-iI)A_{12} - \vartheta(A_{12})P_1 - \vartheta(A_{12})^* P_1 - i\vartheta(-iI)A_{12}^*. \end{aligned}$$

Multiplying the above expression by P_1 and P_2 from left and right, respectively, and applying (2.12), we obtain $\vartheta(-iI)A_{12} = 0$. Implementation of the condition (\spadesuit) gives us $\vartheta(-iI)P_1 = 0$, employing (2.2) and condition (\spadesuit) in $\vartheta(-iI)A_{12} = 0$, we get $\vartheta(-I)P_2 = 0$, hence we obtain $\vartheta(-iI) = 0$.

(2) Applying the fact $\vartheta(-iI) = 0$ in the Lemma 2.9 (2), we obtain $\vartheta(iS) = i\vartheta(S)$ for all $S \in \mathcal{A}^+$. Let any $A \in \mathcal{A}$ then $A = A_1 + iA_2$, where $A_1, A_2 \in \mathcal{A}^+$

$$\begin{aligned} \vartheta(iA) & = \vartheta(iA_1 - A_2) = \vartheta(iA_1) + \vartheta(-A_2) = i\vartheta(A_1) - \vartheta(A_2) \\ & = i(\vartheta(A_1) + i\vartheta(A_2)) = i\vartheta(A_1 + iA_2) = i\vartheta(A). \end{aligned} \quad \square$$

Lemma 2.12. $\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$, for all $A, B \in \mathcal{A}$.

Proof. For any $A, B \in \mathcal{A}$, applying the fact

$$P_n(A, iB, -I, \dots, -I) = 2^{n-2}i(A^*B + B^*A),$$

and using Lemmas 2.8, 2.10 (2) and 2.11 (2), we obtain

$$2^{n-2}i\vartheta(A^*B + B^*A) = \vartheta(P_n(A, iB, -I, \dots, -I))$$

$$\begin{aligned} &= P_n(\vartheta(A), iB, -I, \dots, -I) + P_n(A, \vartheta(iB), -I, \dots, -I) \\ &= 2^{n-2}i(\vartheta(A)^*B + B^*\vartheta(A)) + 2^{n-2}i(A^*\vartheta(B) + \vartheta(B)^*A). \end{aligned}$$

Hence, we obtain

$$(2.13) \quad \vartheta(A^*B + B^*A) = \vartheta(A)^*B + B^*\vartheta(A) + A^*\vartheta(B) + \vartheta(B)^*A.$$

Implementing the fact $P_n(A, B, -I, \dots, -I) = 2^{n-2}(A^*B - B^*A)$ and using Lemmas 2.8 and 2.10 (2), we obtain

$$\begin{aligned} 2^{n-2}\vartheta(A^*B - B^*A) &= \vartheta(P_n(A, B, -I, \dots, -I)) \\ &= P_n(\vartheta(A), B, -I, \dots, -I) + P_n(A, \vartheta(B), -I, \dots, -I) \\ &= 2^{n-2}(\vartheta(A)^*B - B^*\vartheta(A)) + 2^{n-2}(A^*\vartheta(B) - \vartheta(B)^*A). \end{aligned}$$

Therefore, we obtain

$$(2.14) \quad \vartheta(A^*B - B^*A) = \vartheta(A)^*B - B^*\vartheta(A) + A^*\vartheta(B) - \vartheta(B)^*A.$$

Adding equations (2.13) and (2.14) and adopting Lemma 2.10 (1), we obtain

$$\vartheta(A^*B) = \vartheta(A^*)B + A^*\vartheta(B). \quad \square$$

Accordingly, we obtain $\vartheta(AB) = \vartheta(A)B + A\vartheta(B)$.

Invoking Lemmas 2.10 and 2.12, we obtain that ϑ is an additive $*$ -derivation on \mathcal{A} . Hence, the proof of Theorem 2.1, is completed.

3. COROLLARIES

Recall that an algebra \mathcal{A} is prime if for any $\alpha, \beta \in \mathcal{A}$, $\alpha\mathcal{A}\beta = \{0\}$ implies that either $\alpha = 0$ or $\beta = 0$. It is simple to check that every prime $*$ -algebra satisfies (\spadesuit). Hence, as a direct significance of Theorem 2.1, we have the following result.

Corollary 3.1. *Let \mathcal{A} be a unital prime $*$ -algebra containing a nontrivial projection. Then, a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ (need not be linear) satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$ -derivation.

It is commonly known that a factor von Neumann algebra is prime, thus, it always satisfies (\spadesuit). Thus, due to the quick aftereffects of Corollary 3.1, we obtain the subsequent result.

Corollary 3.2. *Let \mathcal{A} be a factor von Neumann algebra with $\dim(\mathcal{A}) \geq 2$. Then, a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive $*$ -derivation.

Further, it is well-known that every von Neumann algebra with no central summands of type I_1 satisfies (\spadesuit) (see [3, 8] for details). Therefore, applying Theorem 2.1, we have the following result.

Corollary 3.3. *Let \mathcal{A} be a von Neumann algebra with no central summands of type I_1 . Then, a map $\vartheta : \mathcal{A} \rightarrow \mathcal{A}$ satisfies*

$$\vartheta(P_n(\lambda_1, \lambda_2, \dots, \lambda_n)) = \sum_{i=1}^n P_n(\lambda_1, \dots, \lambda_{i-1}, \vartheta(\lambda_i), \lambda_{i+1}, \dots, \lambda_n),$$

for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathcal{A}$ if and only if ϑ is an additive *-derivation.

4. CONCLUSION

In this paper, we have studied nonlinear left bi-skew Lie n -derivations on arbitrary *-algebras. In fact, it is shown that, under certain assumptions, every nonlinear left bi-skew Lie n -derivation on a unital *-algebra is an additive *-derivation.

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