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# THE PERFECT LOCATING SIGNED ROMAN DOMINATION OF SOME GRAPHS

ABOLAPE DEBORAH AKWU[1], TAYO CHARLES ADEFOKUN[2], AND OPEYEMI OYEWUMI[3]

Abstract. In this paper, we introduce the concept of Perfect locating signed Roman dominating functions in graphs. A perfect locating signed Roman dominating $PLSRD$ function of a graph $G=(V,E)$ is a function $f: V(G)\to\{-1,1,2\}$ satisfying the conditions that for (i) every vertex $v$ with $f(v)=-1$ is adjacent to exactly one vertex $u$ with $f(u)=2$; (ii) any pair of distinct vertices $v,w$ with $f(v)=f(w)=-1$ does not have a common neighbor $u$ with $f(u)=2$ and (iii) $f(v)+\sum_{u\in N(v)} f(u)\geq 1$ for any vertex $v$. The weight of $PLSRD$- function is the sum of its function values over all the vertices. The perfect locating signed Roman domination number of $G$ denoted by $\gamma^P_{LSR}(G)$ is the minimum weight of a $PLSRD$-function in $G$. We present the upper and lower bonds of $PLSRD$- function for trees. In addition, for grid graph $G$, we show that $\gamma^P_{LSR}(G)\leq \frac{3}{4}|G|$.

## 1. Introduction and Preliminaries

In this paper, we continue the study of variant of Roman dominating function. Let $G=(V,E)$ be an undirected graph with vertex set $V$ and edge set $E$. The order and size of graph $G$ is the number of vertices and edges in $G$, respectively. The open neighborhood of vertex $u$ in $G$ is the set of all neighbors of $u$ in $G$; that is $N_G(u)=\{v\in V \mid uv\in E(G)\}$. The closed neighborhood of $u$ in $G$ is $G[u]=\{u\}\cup N_G(u)$. The degree of $u$ is $d_G(u)=|N_G(u)|$. We write $P_n$ for the path of order $n$.

A leaf of a tree is a vertex of degree one and the support vertex is a vertex adjacent to a leaf. Let $S(T)$ and $L(T)$ denotes the set of all support vertices and the set of

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leaves in $T$, respectively. We denote $|L(T)| = l(T)$ and $s(T) = |S(T)|$. Let $L(u)$ denote the set of all leaves adjacent to a support vertex $u$ and $l(u) = |L(u)|$.

Let $G_1$ and $G_2$ be two graphs. The cartesian product of graphs $G_1$ and $G_2$, denoted by $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1), (u_2, v_2) \in G_1 \Box G_2$ are adjacent if either

- $u_1, u_2 \in E(G_1)$ and $v_1 = v_2$, or
- $v_1, v_2 \in E(G_2)$ and $u_1 = u_2$.

The graph $P_n \Box P_m$ has $n$ rows and $m$ columns. If $G = P_n \Box P_m$, then $|G| = |nm|$.

A subset $D \subset V$ is a *dominating set* of $G$ if every vertex in $V \setminus D$ has a neighbor in $D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. Let $\alpha \in \{-1, 1, 2\}$ and for any vertex $u \in G$, we denote the set of vertices with $f(u) = \alpha$ by $V_\alpha$.

The study of locating dominating sets in graphs was first studied by Slater [19, 20] whereby many graph related problems with various types of protection are studied. The objective of the work is to locate the intruder. A locating dominating set $D \subset V(G)$ is a dominating set with the property that for each vertex $u \in V(G) - D$, the set $N(u) \cap D$ is unique. The locating dominating set of $G$ with minimum cardinality is known as locating domination number of $G$. The concept of locating domination has been considered for several domination parameters, for more result, see [6, 8–10].

A function $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function ($RDF$) on $G$ if for every vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to at leaast one vertex $u$ with $f(u) = 2$. The weight of $RDF$ denoted by $w(f)$ is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The $RDF$ on $G$ with minimum weight is known as Roman domination number and denoted by $\gamma_R(G)$. Cockayne et al. [13] introduced Roman domination which was motivated by the work of Re Velle and Rosing [18] and Stewart [21]. More results on Roman domination can be found in [11, 12].

A perfect Roman dominating function ($PRD$-function) is a Roman dominating function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to exactly one vertex $u$ with $f(u) = 2$. The weight of $f$ is the sum $\sum_{v \in V(G)} f(v)$ denoted by $w(f)$. The perfect Roman domination number denoted by $\gamma_R^P(G)$ is the $PRD$-function with minimum weight. Henning et al. [14] first study perfect Roman domination. More work on $PRD$ can be found in [5, 16, 17].

A signed Roman dominating function ($SRD$-function) on a graph $G$ is a function $f : V(G) \to \{-1, 1, 2\}$ with the condition that for every $v \in V(G)$, $f(N[v]) \geq 1$. This concept was introduced by Abdollahzadeh Ahangar in [3]. Further results on $SRD$-function can be found in [1, 2].

A $RD$-function is called a locating Roman dominating function ($LRD$-function) if for any pair of vertices $u, v$ with $f(u) = f(v) = 0, N(u) \cap V_2 \neq N(v) \cap V_2$ where $w \in V(G)$. The minimum weight of $LRD$-function is known as the locating Roman domination number denoted as $\gamma_R^L(G)$. See [15] for more result on $LRD$-function.

In this paper, we consider the case whereby there will be optimal security control, that is, the whole empire will be secured in case of multiple attacks at the same time. This lead to the study of perfect locating signed Roman dominating function.

A perfect locating signed Roman dominating function of a graph $G$, abbrevated $PLSRD$-function is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) every vertex $v$ with $f(v) = -1$ is adjacent to exactly one vertex $u$ with $f(u) = 2$; (ii) for any pair of distinct vertices $v, w$ of $V_{-1}$, $N(v) \cap V_2 \neq N(w) \cap V_2$ and (iii) $f(v) + \sum_{u \in N(v)} f(u) \geq 1$ for any vertex $v \in G$. In Section 2, we present the lower and upper bonds of $PLSRD$-functions for trees and in Section 3, we present the upper bond of $PLSRD$-functions for the grid graph.

## 2. Perfect Locating Signed Roman domination of trees

In this section, we presents the lower and upper bounds of $PLSRD$-functions for trees. We begin with the following observations and existing result.

**Observations**.

- For any star graph $S_n$, $\gamma^P_{LSR}(S_n) = n - 1$.
- If $f$ is a $PLSRD$-function, then $|D| = |V_2|$, where $D$ is a minimum dominating set in $T$.

**Theorem 2.1** ([6]). *For any tree $T$ of order $n \geq 2$, $\gamma_L(T) \geq \lceil \frac{n+1}{3} \rceil$.*

**Lemma 2.1.** *If $T$ is a tree with $l$ leaves, $s$ support vertices and $f : V(T) \to \{-1, 1, 2\}$ is a perfect locating signed dominating function, then $|V_1| \geq l - s$.*

*Proof.* For any support vertex $u$ and an arbitrary vertex $x \in T$ with $f(x) = 1$, we have $|L(u) \cap V_1| \geq l(u) - 1$, then

$$|V_1| \geq \sum_{u \in S} (l(u) - 1) = \sum_{u \in S} l(u) - \sum_{u \in S} 1 = l - s. \qquad \square$$

**Lemma 2.2.** *For any tree $T$ of order $n \geq 2$ with minimum dominating set $D$, $l$ leaves and $s$ support vertices, the $|D| \geq \frac{n-l+2s}{3}$.*

*Proof.* Consider the $PLSRD$-function on the vertices of $T$ by assigning 2 to each support vertex $u$ and $-1$ to only one leaf adjacent to support vertex $u$. Also, assign 1 to the remaining leaves adjacent to support vertex $u$. The assigned values on the support vertices and leaves in $T$ follows from the definition of $PLSRD$-function.

Next, let $T'$ be a tree of order $n'$ obtained from $T$ by deleting all support vertices and leaves, then $n' = n - l - s$. Next, divide the vertices in $T'$ into $q$ connected sets of cardinality 3, i.e. $n' = 3q + r$, where $q \geq 0$ and $0 \leq r \leq 2$. Assign 2 to at least one vertex in each $q$ set. Let the vertices $\{x, v, w\} \in T'$ such that $r$ contains one vertex, say $w$ and $\{v, w\} \in E(T')$. If $f(v) = \{1, 2\}$, then assign 1 to vertex $w$. Also, assign 2 to vertex $w$ if $f(v) = -1$ and there is no vertex $x$ adjacent to $v$ with label 2. If $f(x) = 2$ and $f(v) = -1$, then assign $w$ with label 1.

Let $r$ contain two vertices say $\{w, y\}$ and $\{x, v, w, y\}$ is a path in $T'$. Set $f(w) = -1$ and $f(y) = 2$ if $f(v) = 1$. Also, set $f(w) = -1$ and $f(y) = 1$ if $f(v) = 2$ and there is no adjacent vertex $x$ to $v$ with $f(x) = -1$, otherwise set $f(w) = f(y) = 1$. Set $f(w) = 1 = f(y)$ if $f(v) = -1$.

The assigned values produces $PLSRD$-function $f$. Let $V_2'$ denote the set of vertices in $T'$ with label 2. Also for an arbitrary vertex $u \in T'$ with $f(u) = 2$, $|V_2'| \geq \frac{n'}{3}$, Now, for an arbitrary vertex $u \in T$ with $f(u) = 2$, we have

$$|V_2| \geq \frac{n'}{3} + s = \frac{n-l-s}{3} + s = \frac{n-l+2s}{3}.$$

Applying observation 2 above, we have $|D| = |V_2| \geq \frac{n-l+2s}{3}$. □

**Theorem 2.2.** *For any tree $T$ of order $n \geq 2$ with $l$ leaves and $s$ support vertices, $\gamma^P_{LSR}(T) \geq \frac{n+2l-s}{3}$.*

*Proof.* Let $T$ be the tree of order $n$ and $f : V(T) \to \{-1, 1, 2\}$ be a $PLSRD$-function defined on $T$. The set $V_2$ is a minimum dominating set of $T$. Clearly, $D$ is the locating dominating set for $T$, i.e. $\gamma_L(T) \leq |D|$. By Lemma 2.2, $|D| \geq \frac{n-l+2s}{3}$ which implies that $|V_2| \geq \frac{n-l+2s}{3}$. Also, $|V_2| = |V_{-1}|$ since $f$ is perfect and locating dominating function. Let $x \in V(T)$ such that $f(x) = 1$, then by Lemma 2.1, $|V_1| \geq l - s$. Hence,

$$\begin{aligned}\gamma^P_{LSR}(T) =&|V_{-1}| + |V_1| + |V_2| = 2|V_2| + |V_1| \quad (\text{since } |V_2| = |V_{-1}|)\\ \geq&|V_2| + |V_1| \geq \frac{n-l+2s}{3} + l - s = \frac{n+2l-s}{3}.\end{aligned}$$ □

**Corollary 2.1.** *For any tree $T$ of order $n \geq 2$, $\gamma^P_{LSR}(T) \geq \frac{n}{3}$.*

*Proof.* The proof follows from Theorem 2.2. □

**Theorem 2.3.** *If $T$ is a tree of order $n \geq 4$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$, where $T$ does not contain a star of order greater than 4.*

*Proof.* We proof the result by induction on the order $n$ of the tree. If $n = 3$, then $\gamma^P_{LSR}(T) = 2 \leq \frac{3}{4}n$. Now, let $n \geq 4$, if $T$ is a star graph $S_n$ with maximum degree 3, then $\gamma^P_{LSR}(T) = 3 \leq \frac{3}{4}n$. Observation 1 applies if maximum degree in $S_n$ is greater than 3. Assume that $T^*$ and $T$ are trees of order $n^*$ and $n$ respectively, with $n^* \geq 3$ and $n^* < n$. Then, $\gamma^P_{LSR}(T^*) \leq \frac{3}{4}n^*$.

Let the $\text{diam}(T) \geq 3$. Suppose $\text{diam}(T) = 3$, let $T$ be a double star $S(r, t)$, where $r \geq t \geq 1$ with maximum degree 4. Let $v, w$ be the vertices of $T$ that are not leaves such that $v$ and $w$ has $r$ and $t$ leaf neighbors, respectively. The function $f$ assign 2 to each vertices $v$ and $w$, $-1$ to only one leaf neighbor of each vertex $v$ and $w$, 1 to the remaining leaves in $T$ is a $PLSRD$-function with weight $r + t$. So, $\gamma^P_{LSR}(T) = r + t \leq \frac{3}{4}(r + t + 2) = \frac{3}{4}n$. Hence, assume that $\text{diam}(T) \geq 4$.

Let $v$ and $w$ be two vertices in $T$ with maximum distance apart. This implies that $v$ and $w$ are leaves and $d(v, w) = \text{diam}(T)$. Let root the tree at the vertex $w$ and let $\{v, u, x, y, r, \ldots, w\}$ be a path in $T$. Note that if $\text{diam}(T) = 4$, then $r = w$; otherwise $r \neq w$. The remaining part of the theorem is split into the following claims.

Claim 1. If $d_T(u) \leq 4$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $d_T(u) \leq 4$. Let $T^*$ be the tree obtained from $T$ by deleting vertex $u$ and its children. Let $T^*$ be of order $n^*$, then $n^* = n - d_T(u)$. Note that $n^* \geq 3$ since $\text{diam}(T) \geq 4$. Apply induction on tree $T^*$, $\gamma^P_{LSR}(T^*) \leq \frac{3}{4}n^* \leq \frac{3}{4}(n-3)$. Let $f^*$ be a $\gamma^P_{LSR}(T^*)$-function. If $f^*(x) \in \{1,2\}$, then $f^*$ can be extended to a PLSRD-function $f$ of $T$ by assigning 2 to vertex $u$, weights $-1$ and 1 to vertex $v$ and other leaf neigbhor of $u$ respectively. Furthermore, if $f^*(x) = -1$, this implies that there exist a neighbor vertex of $x$ (say $y$) with weight 2. Then $f$ can be obtained from $f^*$ as follows:

If $d_{T^*}(y) = 1$, re-assigning weights $2, -1$ to vertices $x, y$ respectively. If $d_{T^*}(y) \geq 2$ and $f^*(r) = 1$, re-assign $f^*(x), f^*(y), f^*(r)$ with weights $1, -1, 2$ respectively and leave the weight of the remaining vertices under $f^*$ unchanged. If $d_{T^*}(y) \geq 2$ and $f^*(r) = 2$, re-assign $f^*(x)$ and $f^*(y)$ with 1 and leave the weights of the remaining vertices under $f^*$ unchanged.

Also, if $d_{T^*}(y) \geq 2$ and $y$ has a leaf neighbor, re-assign one of the leaf neighbor of $y$ with $-1$ and re-assign vertex $x$ with weight 1, leave the weight of the remaining vertices under $f^*$ unchanged.

From the illustration above, $f^*(x) = -1$ has been reassign weight 1. Next, extend $f^*$ to a $PLSRD$-function $f$ as given above whenever $f(x) \in \{1,2\}$. Therefore, we have

$$\begin{aligned}
\gamma^P_{LSR}(T) = w(f) &\leq w(f^*) + d_T(u) - 1 \\
&\leq \frac{3}{4}(n - d_T(u)) + d_T(u) - 1 \\
&= \frac{3}{4}n + \frac{d_T(u)}{4} - 1 \\
&\leq \frac{3}{4}n.
\end{aligned}$$

Next, assume that every child of vertex $x$ in $T$ has at most degree 3. For $i = 1,2,3$, let $q_i$ be the number of children of $x$ with degree $i$. The leaf neighbor of vertex $x$ is $q_1$. Note that $q_2 + q_3 \geq 1$ since vertex $u$ has degree 2 or 3.

Claim 2. If $q_3 \geq 1$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose that $q_3 \geq 1$, let $T^*$ be the tree obtained from $T$ by deleting $q_3$ children of vertex $x$ and their leaf neighbors. Let $n^* \geq 3$ be the order of tree $T^*$ and $n^* = n - 3q_3$. Applying induction on $T^*$, $\gamma^P_{LSR}(T^*) \leq \frac{3}{4}n^* = \frac{3}{4}(n - 3q_3)$.

If $f^*(x) \in \{1,2\}$, then we can extend $f^*$ to a PLSRD-function $f$ of $T$ by assigning weight 2 to each child of $x$ with degree 3 and weight 1 and $-1$ to the leaf neighbors of each child of $x$. The resulting function $f$ is a PLSRD-function of $T$ since each vertex with weight $-1$ is adjacent to exactly one neighbor with weight 2, vertices with weight $-1$ do not have a common vertex with weight 2 and the sum of weights of each vertex and its neighbors is greater than 1. The weight $w(f) = w(f^*) + 2q_3 \leq \frac{3}{4}(n - 3q_3) + 2q_3 = \frac{3}{4}n - \frac{q_3}{4} \leq \frac{3}{4}n$.

If $f^*(x) = -1$, only one leaf of $x$ can have weight 2 and each child of $x$ with degree 3 and their leaf neigbhors will have weight 1. Furthermore, if $f^*(x) = -1$, one child of $x$

with degree 3 can be assign weight 2 and the remaining child of $x$ of degree 3 and their leaves neighbors will have weight 1 each. This will produce another $\gamma^P_{LSR}(T^*)$-function that assign larger weight than when $f^*(x) \in \{1, 2\}$. Hence, vertex $x$ cannot have weight $-1$.

Claim 3. If $q_3 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_3 = 0$, then every child of $x$ is a support vertex of degree 2 or a leaf. Let $T^*$ be the tree obtained from $T$ by deleting the vertex $x$ and all its descendants. Let $n^*$ be the order of the tree $T^*$ where $n^* = n - q_1 - 2q_2 - 1$. Note that $n^* \geq 2$ since $\{y, r\} \subseteq V(T^*)$.

Suppose $q_1 = 0$, then the tree $T$ has the order $n = n^* + 2q_2 + 1$, assign $1, 2, -1$ to vertex $x$, support vertex adjacent to $x$ and the leaf adjacent to the child of $x$, respectively. The assigned weight produces a $PLSRD$-function $f$ of $T$ of weight

$$w(f) = w(f^*) + q_2 + 1 = \frac{3}{4}(n - 2q_2 - 1) + q_2 + 1 = \frac{3}{4}n - \frac{q_2}{2} + \frac{1}{4} \leq \frac{3}{4}n.$$

Suppose $q_2 \geq q_1 \geq 1$, then the tree $T$ has the order $n = n^* + 2q_2 + q_1 + 1$. Assign 2 to vertex $x$ and all the support vertices adjacent to $x$. Also, assign $-1$ to only one leaf adjacent to $x$ and the leaf adjacent to child of $x$. Assign 1 to the remaining leaf adjacent to $x$. The assigned weight produces a $PLSRD$-function $f$ of $T$ of weight

$$w(f) = w(f^*) + q_2 + q_1 = \frac{3}{4}(n - 2q_2 - q_1 - 1) + q_2 + q_1 = \frac{3}{4}n - \frac{q_2}{2} + \frac{q_1}{4} - \frac{3}{4} \leq \frac{3}{4}n.$$

Suppose $q_2 = 0$ and $0 < q_1 \leq 3$, then the tree $T$ has the order $n = n^* + q_1 + 1$. Assign 2 to vertex $x$, $-1$ to only one leaf adjacent to $x$ and 1 to the remaining leaf adjacent to $x$. The assigned weight produces a $PLSRD$-function $f$ of $T$ of weight

$$w(f) = w(f^*) + q_1 = \frac{3}{4}(n - q_1 - 1) + q_1 = \frac{3}{4}n + \frac{q_1}{4} - \frac{3}{4} \leq \frac{3}{4}n.$$

Claim 4. If $q_2 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_2 = 0$, then every child of $x$ has a vertex with degree 3 or a leaf. Let $T^*$ be the tree obtained from $T$ by deleting the vertex $x$ and all its descendants. Let $n^*$ be the order of the tree $T^*$ with $n^* = n - 3q_3 - q_1 - 1$. We consider the claim for $q_3 \geq q_1 \geq 1$. Assign 2 to vertex $x$ and each child of $x$ with degree 3. Also assign $-1$ to only one leaf adjacent to vertex $x$ and one leaf adjacent to child of $x$ with degree 3. Assign 1 to the remaining leaves adjacent to vertex $x$ and the child of $x$ with degree 3. The assigned weight produces a $PLSRD$-function $f$ of $T$ of weight

$$\begin{aligned} w(f) &= w(f^*) + 2q_3 + q_1 = \frac{3}{4}(n - 3q_3 - q_1 - 1) + 2q_3 + q_1 = \frac{3}{4}n - \frac{q_3}{4} + \frac{q_1}{4} - \frac{3}{4} \\ &\leq \frac{3}{4}n. \end{aligned}$$

Suppose $q_1 = 0$, then claim 2 holds. If $q_3 = 0$, claim 3 holds.

Claim 5. If $q_1 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_1 = 0$, then every child of $x$ has vertices of degree 2 and 3. Let $T^*$ with order $n^*$ be the tree obtained from $T$ by deleting the vertex $x$ and its descendants.

Then $n^* = n - 3q_3 - 2q_2 - 1$. Assign 1 to vertex $x$ and 2 to the child of $x$ with degree 2 and degree 3. Also assign $-1$ to only one leaf adjacent to child of $x$ and 1 to the remaining leaf adjacent to child of $x$. The assigned weights produced $PLSRD$-function $f$ of $T$ of weight

$$\begin{aligned} w(f) &= w(f^*) + q_2 + 2q_3 + 1 \\ &= \frac{3}{4}(n - 3q_3 - 2q_2 - 1) + 2q_3 + q_2 + 1 = \frac{3}{4}n - \frac{q_3}{4} - \frac{q_2}{2} - \frac{1}{4} \\ &\leq \frac{3}{4}n. \end{aligned}$$

Suppose $q_2 = 0$, then claim 2 holds. If $q_3 = 0$, then claim 3 holds.

In all these cases, $w(f) \leq \frac{3}{4}n$. Hence, for $n \geq 3$, $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$. This complete the proof. □

## 3. Perfect Locating Signed Roman Domination of Cartesian Product Graph

In this section, we present an upper bond for the perfect locating signed Roman domination number of the Grid graph $G = P_n \Box P_m$. Let $i, 1 \leq i \leq n$ and $j, 1 \leq j \leq m$ denotes the rows and columns in the graph $P_n \Box P_m$ We denote the vertex in the row $i$ and column $j$ by $u_{ij}$.

**Theorem 3.1.** *Let $n > 5$ and $m \geq 2$. If $G = P_n \Box P_m$, then $\gamma^P_{LSR}(G) \leq \frac{3}{4}|G|$.*

*Proof.* Define the function $f : V(G) \to \{-1, 1, 2\}$ as shown in figure 1 as follows: For vertex $u_{ij} \in V(G)$, we have

$$f(u_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 6 \text{ and } j \text{ even}, \\ 1, & \text{if } i \equiv 3 \text{ or } 4 \pmod 6 \text{ and } j \text{ odd}, \\ -1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 6 \text{ and } j \text{ odd}, \\ -1, & \text{if } i \equiv 3 \text{ or } 4 \pmod 6 \text{ and } j \text{ even}, \\ 2, & \text{if } i \equiv 2 \pmod 3 \text{ for all } j. \end{cases}$$

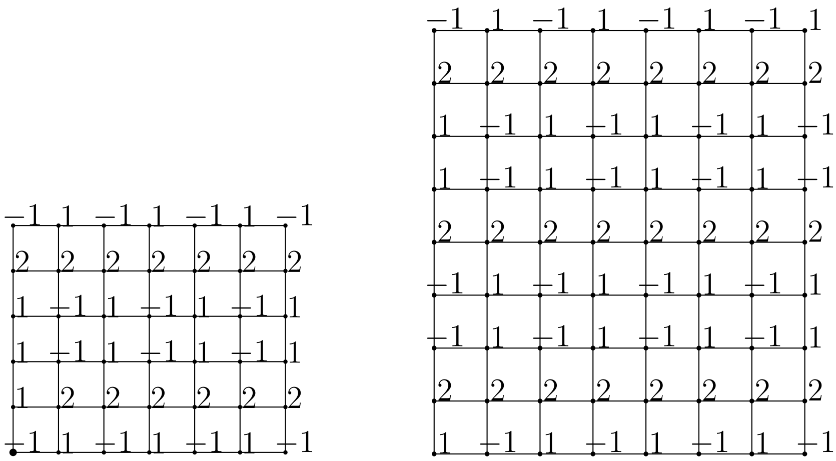
The function $f$ has a pattern that reoccur at every six rows and every two columns. The above function $f$ define on the vertices of $G$ gives the $PLSRD$-function on $G$, since each vertex with label $-1$ is adjacent to only one vertex with label 2, any pair of vertices $u_{ij}, v_{ij}$ with label $-1$ does not have a common neighbor vertex with label 2 and $f(u_{ij}) + \sum_{v_{ij} \in N(u_{ij})} f(v_{ij}) \geq 1$. The result will be prove in the following cases.

Case 1: when $n \equiv 0 \pmod 6$ and $m$ a positive integer.

From the above function $f$, the total sum of labels on each column $j$ is $\frac{4n}{6}$. Therefore, we have

$$w(f) = \frac{4nm}{6} \leq \frac{2}{3}nm + \frac{1}{12}nm = \frac{3}{4}nm = \frac{3}{4}|G|.$$

Case 2: when $n \equiv 1 \pmod 6$ and $m$ a positive integer.



(a) $n \equiv 0 \pmod 6$ (b) $n \equiv 3 \pmod 6$

FIGURE 1. The function $f$, $m$ odd in (a) and $m$ even in (b)

Define a function $f^* : V(G) \to \{-1, 1, 2\}$ as follows:

$$f^*(u_{ij}) = \begin{cases} 1, & \text{if } i = n \text{ and } j \text{ odd}, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The above function $f^*$ gives $PLSRD$-function which follows from the definition of $PLSRD$-function. From the function $f^*$, the total sum of the labels on each column $j$ is $\frac{4(n-1)}{6} + 1$.

Hence, we have

$$w(f^*) = m\left(\frac{4(n-1)}{6} + 1\right) \leq \frac{2}{3}nm + \frac{1}{12}nm = \frac{3}{4}nm = \frac{3}{4}|G|.$$

Case 3: when $n \equiv 2 \pmod 6$ and $m$ a positive integer.

Define a function $f^* : V(G) \to \{-1, 1, 2\}$ as follows:

$$f^*(u_{ij}) = \begin{cases} 1, & \text{if } i = n \text{ and j even}, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The above function $f^*$ gives $PLSRD$-function. From the function $f^*$, the total sum of the labels on each odd column $j$ is $\frac{4(n-2)}{6} + 1$ and $\frac{4(n-2)}{6} + 2$ on even column $j$.

If $m$ is odd, we have

$$\begin{aligned} w(f^*) &= \frac{m+1}{2}\left(\frac{4(n-2)}{6}+1\right)+\frac{m-1}{2}\left(\frac{4(n-2)}{6}+2\right)=\frac{2}{3}nm+\frac{1}{6}m-\frac{1}{2} \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

If $m$ is even, we have

$$\begin{aligned} w(f^*) &= \frac{m}{2}\left(\frac{4(n-2)}{6}+1\right)+\frac{m}{2}\left(\frac{4(n-2)}{6}+2\right)=\frac{2}{3}nm+\frac{1}{6}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Case 4: when $n \equiv 3 \pmod 6$ and $m$ a positive integer.

From the above function $f$, the total sum of the labels on each column $j$ is $\frac{4(n-3)}{6}+2$. Therefore, we have

$$w(f)=m\left(\frac{4(n-3)}{6}+2\right)\leq \frac{2}{3}nm+\frac{1}{12}nm=\frac{3}{4}nm=\frac{3}{4}|G|.$$

Case 5: when $n \equiv 4 \pmod 6$ and $m$ a positive integer.

Define a function $f^*: V(G) \to \{-1,1,2\}$ as follows:

$$f^*(u_{ij})=\begin{cases} 1, & \text{if } i=n \text{ and } j \text{ even}, \\ f(u_{ij}), & \text{otherwise}. \end{cases}$$

The above function $f^*$ gives $PLSRD$-function. From the function $f^*$, the sum of the labels on each column $j$ is $\frac{4(n-4)}{6}+3$.

Hence, we have

$$\begin{aligned} w(f^*) &= m\left(\frac{4(n-4)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{3}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Case 6: when $n \equiv 5 \pmod 6$ and $m$ a positive integer.

Define a function $f^*: V(G) \to \{-1,1,2\}$ as follows:

$$f^*(u_{ij})=\begin{cases} 1, & \text{if } i=n \text{ and } j \text{ odd}, \\ f(u_{ij}), & \text{otherwise}. \end{cases}$$

The above function $f^*$ gives $PLSRD$-function which follows from the definition. From the function $f^*$, if $j$ is odd, the sum of the labels on each column $j$ is

$$\frac{4(n-5)}{6}+4 \quad \text{and} \quad \frac{4(n-5)}{6}+3.$$

for each even column $j$.

Therefore, if $m$ is odd, we have

$$\begin{aligned} w(f^*) &= \frac{m+1}{2}\left(\frac{4(n-5)}{6}+4\right)+\frac{m-1}{2}\left(\frac{4(n-5)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{6}m+\frac{1}{2} \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Also, if $m$ is even, we have

$$\begin{aligned} w(f^*) &= \frac{m}{2}\left(\frac{4(n-5)}{6}+4\right)+\frac{m}{2}\left(\frac{4(n-5)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{6}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Hence, the result follows. $\square$

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[1]Department of Mathematics,
University Federal University of Agriculture,
Makurdi, Nigeria
*Email address*: abolape.akwu@uam.edu.ng

[2]Department of Computer and Mathematical Sciences,
Crawford University,
Nigeria
*Email address*: tayoadefokun@crawforduniversity.edu.ng

[3]Department Mathematics,
Stellenbosch University,
Stellenbosch, South Africa
*Email address*: opeyemioluwaoyewumi@gmail.com

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# STUDY OF A STOCHASTIC DIFFERENTIAL SYSTEM OF ARBITRARY ORDER UNDER G-BROWNIAN MOTION

EL-HACÈNE CHALABI[1], AMAR OUAOUA[2], AND SALIM MESBAHI[3]

Abstract. In this paper, we study the existence and uniqueness of the solution for a class of stochastic differential systems of arbitrary order driven by G-Brownian motion. We prove under certain suitable conditions that our system has a unique solution. We also prove a stability theorem for our system.

## 1. Introduction

The theory of nonlinear expectations are a generalization of the classical mathematical concept of expectation. Unlike the classical expectation, which is linear, the nonlinear expectation allows for nonlinearity, which makes it a useful tool in modeling situations involving uncertainty and risk. Nonlinear expectations have found many applications in finance, where they are used to model and measure risk. In particular, they are used in the context of super-hedging, which is a risk management strategy used to minimize the potential losses of an investment portfolio. By using them, investors can account for the possibility of extreme market events, which may not be captured by traditional linear models. They also have applications in other fields, such as decision theory, statistics, and machine learning. They are used to model situations where the outcome depends on a combination of factors, rather than just a single factor, and where there is uncertainty about the relationship between these factors and the outcome. We find models and applications in different fields in Denis et al. [2], Y. Lin [9], Peng [13], Ren et al. [15,16], Soumana-Hima [17], Yang [18] and

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corresponding references therein; where we also find techniques and methods used to discuss such problems.

Peng [14] (for more details see Peng [11]-[13]) introduced the theory of nonlinear expectation, G-Brownian motion and defined the related stochastic calculus, especially stochastic integrals of Itô's type with respect to G-Brownian motion and derived the related Itô's formula. Also, the notion of G-normal distribution plays the same important role in the theory of nonlinear expectation as that of normal distribution with the classical probability. Gao [5] studied pathwise properties and homeomorphic property with respect to the initial values for stochastic differential equations driven by G-Brownian motion. Later Faizullah et al. extended this theory, see for example [3] and [4].

The existence and uniqueness theorem for some stochastic differential equations under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients was developed by Peng and Gao. This theorem is established by using the stated method under the Lipschitz and the linear growth conditions.

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \sum_{i,j=1}^{d} \int_0^t g_{i,j}(s, X(s))\, d\left\langle B^i, B^j \right\rangle (s) + \sum_{j=1}^{d} \int_0^t h_j(s, X(s))\, dB^j(s), \quad t \in [0, T], \tag{1.1}$$

where $T$ is a positive constant.

The existence and uniqueness of the solution $X(t)$ for G-SDEs (1.1) under different conditions was proved by Bai and Y. Lin [1], Faizullah [3,4], Gao [5], Q. Lin [7], Y. Lin [8], Peng and Falei [10], Ren et al. [16], Zhang and Chen [19]. In this paper, we study the existence, uniqueness and stability of the solution for the following stochastic differential system driven by G-Brownian motion (SG-SDEs)

$$\begin{cases} X_1(t) = & X_1(0) + \int_0^t f_1(s, X_1(s), \ldots, X_n(s))\, ds \\ & + \sum_{i,j=1}^{d} \int_0^t f_{1,i,j}(s, X_1(s), \ldots, X_n(s))\, d\langle B^i, B^j \rangle (s) \\ & + \sum_{j=1}^{d} \int_0^t f_{1,j}(s, X_1(s), \ldots, X_n(s))\, dB^j(s), \\ \vdots & \\ X_n(t) = & X_n(0) + \int_0^t f_n(s, X_1(s), \ldots, X_n(s))\, ds \\ & + \sum_{i,j=1}^{d} \int_0^t f_{n,i,j}(s, X_1(s), \ldots, X_n(s))\, d\langle B^i, B^j \rangle (s) \\ & + \sum_{j=1}^{d} \int_0^t f_{n,j}(s, X_1(s), \ldots, X_n(s))\, dB^j(s), \end{cases} \tag{1.2}$$

where $(X_1(0), \ldots, X_n(0))$ is a given initial condition, $(\langle B^i, B^j \rangle (t))_{t \geq 0}$ is the quadratic variation process of the G-Brownian motion $(B(t))_{t \geq 0}$ and all $f_k(t, x_1, \ldots, x_n)$,

$f_{k,i,j}(t, x_1, \ldots, x_n)$, $f_{k,j}(t, x_1, \ldots, x_n)$ for $t \in [0, T]$, $k = 1, 2, \ldots, n$ and $i, j = 1, 2, \ldots, d$ are the integral-Lipschitz coefficients with respect to $(x_1, \ldots, x_n)$.

The paper is organized as follows. In the following section, we provide some definitions, remarks and lemmas necessary to fully understand the content of this work. The third section is devoted to our first contribution, where we prove the existence and uniqueness of the solution of System (1.2). The last section is devoted to our second contribution, where we prove another important result on the stability of solutions.

## 2. Preliminaries

In this section, we recall some of the basic concepts, definitions, and lemmas that we will use in this work. More details can be found in Gao [5], Hu and Li [6].

Let $\Omega$ be a given non-empty set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ such that any arbitrary constant is an element of $\mathcal{H}$, and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

**Definition 2.1** ([14])**.** A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all $X$, $Y$ in $\mathcal{H}$, $c$ in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:
(i) (Monotonicity): if $X \leq Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$;
(ii) (Constant preserving): $\mathbb{E}[c] = c$;
(iii) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
(iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
The triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is called sublinear expectation space.

We assume that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^n)$, where $C_{l,\text{Lip}}(\mathbb{R}^n)$ is the space of linear functions $\varphi$ defined as follows, for all $x, y \in \mathbb{R}^n$

$$C_{l,\text{Lip}}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R} : |\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|\},$$

where $C$ is a positive constant and $m \in \mathbb{N}^\star$ dependent only on $\varphi$.

**Definition 2.2** ([13])**.** Let $X$, $Y$ be two $n$-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if $\mathbb{E}_2[\varphi(Y)] = \mathbb{E}_1[\varphi(X)]$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$.

**Definition 2.3** ([6])**.** In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ if

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}], \quad \text{for all } \varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

*$\tilde{X}$ is called an independent copy of $X$, if $\tilde{X} \overset{d}{=} X$ and $\tilde{X}$ is independent from $X$.*

*Remark* 2.1. Under a sublinear expectation space, $Y$ is independent from $X$ means that the distributional uncertainty of $Y$ does not change after the realization of $X = x$. Or, in other words, the conditional sublinear expectation of $Y$ knowing $X$ is

$\mathbb{E}\left[\varphi\left(x,Y\right)\right]_{x=X}$. In the case of linear expectation, this notion of independence is just the classical one.

*Remark* 2.2. It is important to note that $Y$ is independent from $X$ does not imply that $X$ is independent from $Y$.

Let $S^d$ be the space of $d\times d$ symmetric matrices. $\Gamma$ is a given non-empty, bounded and closed subset of $S^d$. For $A=\left(A_{i,j}\right)_{i,j=1}^{d}\in S^d$ given, we define $G:S^d\to\mathbb{R}$ by

$$G\left(A\right)=\frac{1}{2}\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\gamma\gamma^{Tr}A\right),$$

where $\gamma^{Tr}$ is the transpose matrix of $\gamma$, and $\operatorname{tr}\left(\gamma\gamma^{Tr}A\right)$ is the trace of a matrix $\left(\gamma\gamma^{Tr}A\right)$.

**Definition 2.4** ([8]). In a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$, a $d$-dimensional vector of random variables $X\in\mathcal{H}^d$ is G-normal distributed, if for each $\varphi\in C_{l,Lip}(\mathbb{R}^d)$, the function $u\left(t,x\right)=\mathbb{E}\left(\varphi\left(x+\sqrt{t}X\right)\right)$ is the unique viscosity solution of the following parabolic equation called the G-heat equation

$$\frac{\partial u}{\partial t}=G\left(D^2u\right),\qquad \text{with } u\left(0,x\right)=\varphi\left(x\right),\left(t,x\right)\in\mathbb{R}_+\times\mathbb{R}^d,$$

where $D^2u=\left(\partial^2_{x_ix_j}u\right)_{i,j}^{d}$ is the Hessian matrix of $u$.

*Remark* 2.3. In fact, if $d=1$ we have $G\left(\alpha\right)=\frac{1}{2}\left(\overline{\sigma}^2\alpha^+-\underline{\sigma}^2\alpha^-\right)$, where $\overline{\sigma}^2=\mathbb{E}[X^2]$, $\underline{\sigma}^2=-\mathbb{E}[-X^2]$, $\alpha^+=\max\left\{\alpha,0\right\}$ and $\alpha^-=\max\left\{-\alpha,0\right\}$. We write $X\sim\mathcal{N}\left(0;[\underline{\sigma}^2,\overline{\sigma}^2]\right)$.

**Definition 2.5** ([14]). A process $\left(B\left(t\right)\right)_{t\geq0}$ in a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied:

(i) $B\left(0\right)=0$;

(ii) for each $t,s\geq0$, the increment $B\left(t+s\right)-B\left(t\right)$ is $\mathcal{N}(0;[\underline{\sigma}^2s,\overline{\sigma}^2s])$-distributed and is independent from $\left(B\left(t_1\right),\ldots,B\left(t_n\right)\right)$ for each $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n\leq t$.

*Remark* 2.4. For any $a\in\mathbb{R}^d$, $B^a\left(t\right):=\sum_{k=1}^{d}a_kB^k\left(t\right)$ is a one-dimensional $G_a$-Brownian motion, where

$$G_a\left(\beta\right)=\frac{1}{2}\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\beta\gamma\gamma^{Tr}aa^{Tr}\right)=\frac{1}{2}\left(\sigma_{aa^{Tr}}\beta^++\sigma_{-aa^{Tr}}\beta^-\right),\quad \beta\in\mathbb{R},$$

and

$$\sigma_{aa^{Tr}}=\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\gamma\gamma^{Tr}aa^{Tr}\right),\quad \sigma_{-aa^{Tr}}=-\sup_{\gamma\in\Gamma}\operatorname{tr}\left(-\gamma\gamma^{Tr}aa^{Tr}\right).$$

We denote by $\Omega=C_0(\mathbb{R})$ the space of all $\mathbb{R}$-valued continuous functions $\omega$ defined on $\mathbb{R}_+$ such that $\omega\left(0\right)=0$, equipped with the distance

$$\rho(\omega_1,\omega_2)=\sum_{k=1}^{+\infty}\frac{1}{2^k}\max_{t\in[0,k]}\left[\left|\left(\omega_1\left(t\right)-\omega_2\left(t\right)\right)\right|\wedge1\right].$$

For each fixed $T > 0$, let $\Omega_T = \{\omega(\cdot_{\wedge T}) : \omega \in \Omega\}$,

$$\mathrm{Lip}(\Omega_T) = \{\varphi(B(t_1), \dots, B(t_m)) : m \geq 1,\ t_1, \dots, t_m \in [0,T],\ \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^m)\},$$

and let

$$\mathrm{Lip}(\Omega) = \bigcup_{n=1}^{+\infty} \mathrm{Lip}(\Omega_n).$$

Peng [13] constructs a sublinear expectation $\mathbb{E}$ on $(\Omega, \mathrm{Lip}(\Omega))$ under which the canonical process $(B(t))_{t\geq 0}$ (i.e., $B(t,\omega) = \omega(t)$) is a G-Brownian motion. In what follows, we consider this G-Brownian motion.

We denote by $L_G^p(\Omega_T)$, $p \geq 1$, the completion of $\mathrm{Lip}(\Omega_T)$ under the norm $||X||_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote $L_G^p(\Omega)$ the completion of $\mathrm{Lip}(\Omega)$. We can represent this sublinear expectation by the following theorem.

**Theorem 2.1** ([13]). *In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a sublinear expectation $\mathbb{E}[\cdot]$ has the following representation: there exist a family of probability measures $\mathcal{P}$ on $\Omega$ such that*

$$\mathbb{E}(X) = \sup_{P\in\mathcal{P}} E^P[X], \quad \textit{for } X \in L_G^1(\Omega),$$

*where $E^P$ stands for the linear expectation under the probability $P$.*

For a finite partition of $[0,T]$, $\pi_T = \{t_0, t_1, \dots, t_N\}$, we set

$$\mu(\pi_T) = \max\{|t_{k+1} - t_k| : k = 0, 1, \dots, N-1\}.$$

Consider the collection $M_G^{p,0}(0,T)$ of simple processes defined by

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k})$, $k = 0, 1, \dots, N-1$, and $p \geq 1$.

The completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\| = \left(\frac{1}{T}\int_0^T \mathbb{E}(|\eta_t|^p)\,dt\right)^{\frac{1}{p}}$$

is denoted by $M_G^p(0,T)$. Note that $M_G^q(0,T) \subset M_G^p(0,T)$ for $1 \leq p \leq q$.

**Definition 2.6.** For each $\eta \in M_G^{2,0}(0,T)$, the G-Itô's integral is defined by

$$I(\eta) = \int_0^T \eta(s)\,dB^a(s) := \sum_{k=0}^{N-1} \xi_k(B^a(t_{k+1}) - B^a(t_k)).$$

The mapping $\eta \mapsto I(\eta)$ can be extended continuously to $M_G^2(0,T)$.

**Definition 2.7.** The increasing continuous process $(\langle B^a\rangle(t))_{t\geq 0}$, with $\langle B^a\rangle(0) = 0$ defined by

$$\langle B^a\rangle(t) := \lim_{\mu(\pi_t^N)\to 0} \sum_{k=0}^{N-1} \left(B^a\left(t_{k+1}^N\right) - B^a\left(t_k^N\right)\right)^2 = (B^a(t))^2 - 2\int_0^t B^a(s)\,dB^a(s)$$

is called the quadratic variation process of $(B^a(t))_{t\geq 0}$.

**Definition 2.8.** Define a mapping $Q_{0,T} : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ as follows

$$Q_{0,T}(\eta) = \int_0^T \eta(s)\, d\langle B^a\rangle(s) := \sum_{k=0}^{N-1} \xi_k(\langle B^a\rangle(t_{k+1}) - \langle B^a\rangle(t_k)).$$

Then $Q_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$Q_{0,T}(\eta) = \int_0^T \eta(s)\, d\langle B^a\rangle(s), \quad \eta \in M_G^1(0,T).$$

Burkholder-Davis-Gundy (BDG) inequalities play an important role in the study of G-stochastic differential equations. There has been an increased interest in the following lemmas, see Gao [5].

**Lemma 2.1** ([5])**.** *Let $p \geq 1$, $\eta \in M_G^p(0,T)$, $a$, $\overline{a} \in \mathbb{R}^d$ and $0 \leq s \leq t \leq T$. Then*

$$\begin{aligned} &\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, d\left\langle B^a, B^{\overline{a}}\right\rangle(r)\right|^p\right] \\ &\leq \left(\frac{\sigma_{(a+\overline{a})(a+\overline{a})^{Tr}} + \sigma_{(a-\overline{a})(a-\overline{a})^{Tr}}}{4}\right)^p (t-s)^{p-1}\int_s^t \mathbb{E}\left[|\eta(u)|^p\right] du. \end{aligned}$$

**Lemma 2.2** ([5])**.** *Let $p \geq 2$, $\eta \in M_G^p(0,T)$, $a \in \mathbb{R}^d$ and $0 \leq s \leq t \leq T$. Then*

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, dB^a(r)\right|^p\right] \leq C_p \sigma_{aa^{Tr}}^{\frac{p}{2}} |t-s|^{\frac{p}{2}-1}\int_s^t \mathbb{E}\left[|\eta(u)|^p\right] du,$$

*where $C_p > 0$ is a constant independent of $\eta$ and $a$.*

In the following, we also need the following two important lemmas.

**Lemma 2.3** (Bihari's inequality, [1])**.** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and increasing function that $\varphi(0^+) = 0$ and $\int_0^1 \frac{ds}{\varphi(s)} = +\infty$. Let $u$ be a measurable and nonnegative function defined on $\mathbb{R}_+$ that satisfies*

$$u(t) \leq a + \int_0^t \alpha(s)\varphi(u(s))\, ds,$$

*where $a \in \mathbb{R}^+$ and $\alpha$ is a positive function and Lebesgue integrable. We have the following.*

*i) If $a = 0$, then $u(t) = 0$, $t \in \mathbb{R}^+$.*

*ii) If $a > 0$, then*

$$u(t) \leq v^{-1}\left(v(a) + \int_0^t \alpha(s)\, ds\right),$$

*where*

$$v(t) := \int_0^t \frac{ds}{\varphi(s)}, \quad t \in \mathbb{R}^+.$$

**Lemma 2.4** ([11])**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing concave function, then for each $X \in L_G^1(\Omega)$ and $t \geq 0$, we have the following Jensen inequality holds*

$$\varphi(\mathbb{E}[X \mid \Omega_t]) \geq \mathbb{E}[\varphi(X) \mid \Omega_t].$$

## 3. Existence and Uniqueness Result

In this section, we will present our first contribution to this paper, which results from the study of the existence and uniqueness of the solution of SG-SDEs (1.2), where $(X_1(0),\ldots,X_n(0)) \in \left(\mathbb{R}^d\right)^n$, and for $k=1,\ldots,n$ and $i,j=1,\ldots,d$, $f_k, f_{k,i,j}, f_{k,j} \in M_G^2\left(0,T;\mathbb{R}^d\right)$ the completion of the collection $M_G^2\left(0,T;\mathbb{R}^d\right)$ of simple processes defined by

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k,t_{k+1}[}(t), \quad \omega \in \Omega^d,$$

under the norm

$$\|\eta\| = \left(\frac{1}{T}\int_0^T \mathbb{E}\left(|\eta_t|^2\right) dt\right)^{\frac{1}{2}},$$

where $\xi_k \in L_G^2(\Omega_{t_k})$, $k=0,1,\ldots,N-1$.

We assume the following assumptions (A1) and (A2) about $J=f_k, f_{k,j}$ or $f_{k,i,j}$, $k=1,\ldots,n$ and $i,j=1,\ldots,d$.

(A1)

$$|J(t,x_1,x_2,\ldots,x_n)|^2 \le |\alpha_1(t)|^2 + \alpha_2^2\left(\sum_{k=1}^n |x_k|^2\right),$$

for each $x_1,\ldots,x_n \in \mathbb{R}^d$, $t\in[0,T]$, $\alpha_1 \in M_G^2(0,T)$ and $\alpha_2 \in \mathbb{R}_+$.

(A2)

$$|J(t,x_1,\ldots,x_n) - J(t,y_1,\ldots,y_n)|^2 \le |\alpha(t)|^2 \varphi\left(\sum_{k=1}^n |x_k-y_k|^2\right),$$

for each $x_1, y_1, \ldots, x_n, y_n \in \mathbb{R}^d$ and $t\in[0,T]$, $\alpha$ is a positive function square integrable on $[0,T]$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous, increasing and concave function satisfying

$$\varphi\left(0^+\right)=0, \quad \int_0^1 \frac{ds}{\varphi(s)} = +\infty.$$

The space of processes in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ equipped with the norm

$$||(X_1,\ldots,X_n)|| = \mathbb{E}^{\frac{1}{2}}\left[\sup_{0\le t\le T}\left(\sum_{k=1}^n |x_k(t)|^2\right)\right]$$

is a Banach space.

Now we can state our first contribution of this work, it is the following theorem.

**Theorem 3.1.** *Under assumptions* (A1) *and* (A2), *System* (1.2) *has a unique solution*

$$(X_1(t),\ldots,X_n(t)) \in \left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n.$$

*Proof.* We will prove the theorem in four steps.

**Step 1.** Suppose that $(X_1(t),\dots,X_n(t))$ and $(Y_1(t),\dots,Y_n(t))$ are two solutions of System (1.2) with initial conditions $(X_1(0),\dots,X_n(0))$ and $(Y_1(0),\dots,Y_n(0))$, respectively. Then, we have for $1\le k\le n$

$$\begin{aligned}
&X_k(t)-Y_k(t)\\
=&X_k(0)-Y_k(0)+\int_0^t\left[f_k(s,X_1(s),\dots,X_n(s))-f_k(s,Y_1(s),\dots,Y_n(s))\right]ds\\
&+\sum_{i,j=1}^d\int_0^t\left[f_{k,i,j}(s,X_1(s),\dots,X_n(s))-f_{k,i,j}(s,Y_1(s),\dots,Y_n(s))\right]d\left\langle B^i,B^j\right\rangle(s)\\
&+\sum_{j=1}^d\int_0^t\left[f_{k,j}(s,X_1(s),\dots,X_n(s))-f_{k,j}(s,Y_1(s),\dots,Y_n(s))\right]dB^j(s).
\end{aligned}$$

By using the inequality, $(a+b+c+d)^2\le 4(a^2+b^2+c^2+d^2)$, we obtain

$$\begin{aligned}
&|X_k(t)-Y_k(t)|^2\\
\le&4|X_k(0)-Y_k(0)|^2+4\left|\int_0^t\left[f_k(s,X_1(s),\dots,X_n(s))-f_k(s,Y_1(s),\dots,Y_n(s))\right]ds\right|^2\\
&+4\left|\sum_{i,j=1}^d\int_0^t\left[f_{k,i,j}(s,X_1(s),\dots,X_n(s))-f_{k,i,j}(s,Y_1(s),\dots,Y_n(s))\right]d\left\langle B^i,B^j\right\rangle_s\right|^2\\
&+4\left|\sum_{j=1}^d\int_0^t\left[f_{k,j}(s,X_1(s),\dots,X_n(s))-f_{k,j}(s,Y_1(s),\dots,Y_n(s))\right]dB^j_s\right|^2.
\end{aligned}$$

We use the fact that $\left(\sum\limits_{i=1}^d a_i\right)^2\le d\sum\limits_{i=1}^d a_i^2$, for each positive constants $a_i,\ i=1,\dots,d$, we have

$$\begin{aligned}
&|X_k(t)-Y_k(t)|^2\\
\le&4|X_k(0)-Y_k(0)|^2+4\left|\int_0^t\left[f_k(s,X_1(s),\dots,X_n(s))-f_k(s,Y_1(s),\dots,Y_n(s))\right]ds\right|^2\\
&+4d^2\sum_{i,j=1}^d\left|\int_0^t\left[f_{k,i,j}(s,X_1(s),\dots,X_n(s))-f_{i,j,j}(s,Y_1(s),\dots,Y_n(s))\right]d\left\langle B^i,B^j\right\rangle(s)\right|^2\\
&+4d\sum_{j=1}^d\left|\int_0^t\left[f_{k,j}(s,X_1(s),\dots,X_n(s))-f_{k,j}(s,Y_1(s),\dots,Y_n(s))\right]dB^j(s)\right|^2.
\end{aligned}$$

Taking the supremum and the G-expectation, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}|X_k(s)-Y_k(s)|^2\right]\\
\le&4|X_k(0)-Y_k(0)|^2
\end{aligned}$$

$$+ 4\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s \left[f_k\left(r, X_1\left(r\right), \ldots, X_n\left(r\right)\right) - f_k\left(r, Y_1\left(r\right), \ldots, Y_n\left(r\right)\right)\right] dr\right|^2\right]$$

$$+ 4d^2\sum_{i,j=1}^{d}\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s \left[f_{k,i,j}\left(r, X_1\left(r\right), \ldots, X_n\left(r\right)\right) - f_{k,i,j}\left(r, Y_1\left(r\right), \ldots, Y_n\left(r\right)\right)\right]$$

$$d\left\langle B^i, B^j\right\rangle\left(r\right)\Big|^2\Big]$$

$$+ 4d\sum_{j=1}^{d}\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s \left[f_{k,j}\left(r, X_1\left(r\right), \ldots, X_n\left(r\right)\right) - f_{k,j}\left(r, Y_1\left(r\right), \ldots, Y_n\left(r\right)\right)\right] dB^j\left(r\right)\right|^2\right]$$

By the Hölder inequality and Lemmas 2.1 and 2.2, we have

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right]$$
$$\le 4\left|X_k\left(0\right) - Y_k\left(0\right)\right|^2$$
$$+ 4T\int_0^t \mathbb{E}\left[\left|f_k\left(s, X_1\left(s\right), \ldots, X_n\left(s\right)\right) - f_k\left(s, Y_1\left(s\right), \ldots, Y_n\left(s\right)\right)\right|^2 ds\right]$$
$$+ 4C_1Td^2\sum_{i,j=1}^{d}\int_0^t \mathbb{E}\left[\left|f_{k,i,j}\left(s, X_1\left(s\right), \ldots, X_n\left(s\right)\right) - f_{k,i,j}\left(s, Y_1\left(s\right), \ldots, Y_n\left(s\right)\right)\right|^2 ds\right]$$
$$+ 4C_2d\sum_{j=1}^{d}\int_0^t \mathbb{E}\left[\left|f_{k,j}\left(s, X_1\left(s\right), \ldots, X_n\left(s\right)\right) - f_{k,j}\left(s, Y_1\left(s\right), \ldots, Y_n\left(s\right)\right)\right|^2 ds\right],$$

and by assumption (A2), we obtain

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right]$$
$$\le 4\left|X_k\left(0\right) - Y_k\left(0\right)\right|^2 + 4T\int_0^t \left|\alpha\left(s\right)\right|^2 \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right)\right] ds$$
$$+ 4C_1Td^2\sum_{i,j=1}^{d}\int_0^t \left|\alpha\left(s\right)\right|^2 \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right)\right] ds$$
$$+ 4C_2d\sum_{j=1}^{d}\int_0^t \left|\alpha\left(s\right)\right|^2 \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right)\right] ds.$$

Then

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right]$$
$$\le 4\left|X_k\left(0\right) - Y_k\left(0\right)\right|^2 + 4\left(T + C_1Td^4 + C_2d^2\right)\int_0^t \left|\alpha\left(s\right)\right|^2 \mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right) - Y_k\left(s\right)\right|^2\right)\right] ds$$

and

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right] \leq& \sum_{k=1}^{n}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_k(s)-Y_k(s)\right|^2\right] \\
\leq& 4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2+4\left(T+C_1Td^4+C_2d^2\right)n \\
&\times\int_0^t\left|\alpha(s)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k(s)-Y_k(s)\right|^2\right)\right]ds.
\end{aligned}
$$

Since, $\varphi$ is a concave function, by Lemma 2.4, we have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right] \leq& 4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2+4\left(T+C_1Td^4+C_2d^2\right)n \\
&\times\int_0^t\left|\alpha(s)\right|^2\varphi\left(\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k(s)-Y_k(s)\right|^2\right)\right)ds.
\end{aligned}
$$

Taking $(X_1(0),\ldots,X_n(0))=(Y_1(0),\ldots,Y_n(0))$, we get

$$
4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2=0.
$$

By Lemma 2.3, we see that for $t\in[0,T]$

$$
\mathbb{E}\left[\left(\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right)\right]=0,
$$

which implies

$$
(X_1(t),\ldots,X_n(t))=(Y_1(t),\ldots,Y_n(t)),\quad \text{for each } t\in[0,T].
$$

**Step 2.** We define a Picard sequence $(X_1^m(\cdot),\ldots,X_n^m(\cdot))_{m\in\mathbb{N}}$ in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ by the following

$$
\left(X_1^0(t),\ldots,X_n^0(t)\right)=(x_1,\ldots,x_n),
$$

and for each integer $k=1,\ldots,n$ and $t\in[0,T]$

$$
\begin{aligned}
X_k^{m+1}(t)=&x_k+\int_0^t f_k\left(s,X_1^m(s),\ldots,X_n^m(s)\right)ds \\
&+\sum_{i,j=1}^{d}\int_0^t f_{k,i,j}\left(s,X_1^m(s),\ldots,X_n^m(s)\right)d\left\langle B^i,B^j\right\rangle(s) \\
&+\sum_{j=1}^{d}\int_0^t f_{k,j}\left(s,X_1^m(s),\ldots,X_n^m(s)\right)dB^j(s).
\end{aligned}
$$

We will prove that is a Cauchy sequence for each $t\in[0,T]$. First, we prove an a priory estimate for $\left(\mathbb{E}\left[\sum\limits_{k=1}^{n}\left|X_k^m(t)\right|^2\right]\right)_{m\in\mathbb{N}}$. By the same arguments, we have for each

$m \in \mathbb{N}$,

$$\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] \leq 4|x_k|^2 + 4\left(T + C_1Td^2 + C_2d\right)\left(\int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right]ds + \alpha_2^2\int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right]ds\right),$$

then

$$\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] \leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right]ds + \alpha_2^2\int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right]ds\right),$$

where $C = \max\{4, 4(T + C_1Td^2 + C_2d)\}$.

Let

$$q(t) = Ce^{C\alpha_2^2 t}\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right]ds\right),$$

then $q(\cdot)$ is a solution of the following ordinary differential equation

$$q(t) = C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right]ds + \alpha_2^2\int_0^t p(s)\,ds\right).$$

By induction, it is easy that for each $m \in \mathbb{N}$

$$\mathbb{E}\left[\sum_{k=1}^n |X_k^m(t)|^2\right] \leq nq(t).$$

Suppose for each $m$, $\mathbb{E}\left[|X_k^m(t)|^2\right] \leq q(t)$, then

$$\begin{aligned}\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] &\leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[\alpha_1^2(s)\right]ds + \alpha_2^2\int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right]ds\right)\\ &\leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[\alpha_1^2(s)\right]ds + \alpha_2^2\int_0^t p(s)\,ds\right) = q(t)\end{aligned}$$

and

$$\mathbb{E}\left[\sum_{k=1}^n |X_k^m(t)|^2\right] \leq \sum_{k=1}^n \mathbb{E}\left[|X_k^m(t)|^2\right] \leq nq(t).$$

**Step 3.** For each $l, m \in \mathbb{N}$, by the definition of $(X_1^m(.), \ldots, X_n^m(\cdot))$, we have

$$\begin{aligned}&X_k^{l+1+m}(t) - X_k^{l+1}(t)\\ =&\int_0^t \left[f_k\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_k\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right]ds\\ &+\sum_{i,j=1}^d \int_0^t \left[f_{k,i,j}\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,i,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right]d\left\langle B^i, B^j\right\rangle(s)\\ &+\sum_{j=1}^d \int_0^t \left[f_{k,j}\left(s, X^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right]dB^j(s).\end{aligned}$$

Using the inequality, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain

$$\left|X_k^{l+1+m}(t) - X_k^{l+1}(t)\right|^2$$

$$\leq 3\left|\int_0^t\left[f_k\left(s,X_1^{l+m}(s),\ldots,X_n^{l+m}(s)\right)-f_k\left(s,X_1^l(s),\ldots,X_n^l(s)\right)\right]ds\right|^2$$
$$+3\left|\sum_{i,j=1}^d\int_0^t\left[f_{k,i,j}\left(s,X_1^{l+m}(s),\ldots,X_n^{l+m}(s)\right)-f_{k,i,j}\left(s,X_1^l(s),\ldots,X_n^l(s)\right)\right]d\left\langle B^i,B^j\right\rangle(s)\right|^2$$
$$+3\left|\sum_{j=1}^d\int_0^t\left[f_{k,j}\left(s,X^{l+m}(s),\ldots,X_n^{l+m}(s)\right)-f_{k,j}\left(s,X_1^l(s),\ldots,X_n^l(s)\right)\right]dB^j(s)\right|^2$$

Taking the supremum and the G-expectation, by using the Hölder inequality and lemmas 2.1, 2.2, 2.4 and assumption (A2), we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq t}\left[\left|X_k^{l+1+m}(s)-X_k^{l+1}(s)\right|^2\right]\right)$$
$$\leq 3\left(T+C_1Td^4+C_2d^2\right)\int_0^t|\alpha(s)|^2\varphi\left(\sum_{k=1}^n\left|X_k^{l+m}(s)-X_k^l(s)\right|^2\right)ds.$$

Let

$$h_{n,l}(t)=\sup_{m\in\mathbb{N}}\left[\mathbb{E}\left(\sup_{0\leq s\leq t}\sum_{k=1}^n\left|X_k^{l+m}(s)-X_k^l(s)\right|^2\right)\right],\quad 0\leq t\leq T,$$
$$0\leq h_{n,l+1}(t)\leq C_3\int_0^t|\alpha(s)|^2\varphi\left(h_{n,l}(s)\right)ds.$$

We define

$$g(t):=\lim_{l\to+\infty}\sup_{0\leq t\leq T}h_{n,l}(t), \tag{3.1}$$

which is uniformly bounded by $4nq(t)$. By using the Fatou-Lebesgue theorem to (3.1), we deduce

$$0\leq g(t)\leq C_3\int_0^t|\alpha(s)|^2\varphi(g(s))\,ds.$$

By Lemma 2.3, we obtain

$$g(t)=0,\quad 0\leq t\leq T,$$

which implies that $\left(X_1^m(\cdot),\ldots,X_n^m(\cdot)\right)_{m\in\mathbb{N}}$ is a Cauchy sequence under the norm

$$\sup_{0\leq t\leq T}\left[\mathbb{E}\left(\sum_{k=1}^n|X_k(\cdot)|^2\right)\right]^{\frac{1}{2}}.$$

**Step 4.** We will prove that the limite $(X_1(t),\ldots,X_n(t))$ in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ of $\left(X_1^m(t),\ldots,X_n^m(t)\right)$ is the solution of system (1.2). By the same arguments as those used in Step 1, we have for each $m\in\mathbb{N}$

$$\sup_{0\leq t\leq T}\left[\mathbb{E}\left(\sum_{k=1}^n|X_k^m(t)-X_k(t)|^2\right)\right]\leq C_3\int_0^T|\alpha(t)|^2\varphi\left(\mathbb{E}\left(\sum_{k=1}^n|X_k^m(t)-X_k(t)|^2\right)\right)dt$$
$$\leq C_3\int_0^T|\alpha(t)|^2\varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^n|X_k^m(t)-X_k(t)|^2\right)\right)dt$$

$$\leq C\varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right).$$

By the continuity of $\varphi$ and $\varphi(0^+)=0$, we know that

$$\varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right)\to 0,$$

and $\sup\limits_{0\leq t\leq T}\left[\mathbb{E}\left(\sum\limits_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right]$ converge to 0. Thus, $\left(X_1^m(\cdot),\ldots,X_n^m(\cdot)\right)_{m\in\mathbb{N}}$ is a successive approximation to $\left(X_1(t),\ldots,X_n(t)\right)$, which is a solution to SG-SDEs (1.2) in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$. □

## 4. Stability Theorem

In this section, we prove another important result on the stability of the solutions of (1.2). We consider the following perturbed SG-SDEs (4.1) whith a parameter $\epsilon\geq 0$, for $0\leq t\leq T$

$$\left\{\begin{array}{rl}X_1^\epsilon(t)= & X_1^\epsilon(0)+\int_0^t f_1^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)ds\\ & +\sum\limits_{i,j=1}^{d}\int_0^t f_{1,i,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)d\left\langle B^i,B^j\right\rangle(s)\\ & +\sum\limits_{j=1}^{d}\int_0^t f_{1,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)dB^j(s),\\ & \vdots\\ X_n^\epsilon(t)= & X_n^\epsilon(0)+\int_0^t f_n^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)ds\\ & +\sum\limits_{i,j=1}^{d}\int_0^t f_{n,i,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)d\left\langle B^i,B^j\right\rangle(s)\\ & +\sum\limits_{j=1}^{d}\int_0^t f_{n,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)dB^j(s),\end{array}\right. \tag{4.1}$$

where $\left(X_1^\epsilon(0),\ldots,X_n^\epsilon(0)\right)\in\left(\mathbb{R}^d\right)^n$ and $f_k,f_{k,i,j},f_{k,j}\in M_G^2\left(0,T;\mathbb{R}^d\right)$.

Now, we make the following assumptions.

For any $\epsilon\geq 0$, $x_k\in\mathbb{R}^d$, $J^\epsilon=f_k^\epsilon,f_{k,i,j}^\epsilon$ or $f_{k,j}^\epsilon\in M_G^2\left(0,T;\mathbb{R}^d\right)$, $X_k(0)\in\mathbb{R}^d$, $1\leq k\leq n$ and $1\leq i,j\leq d$

(B1)

$$\left|J^\epsilon\left(t,x_1,\ldots,x_n\right)\right|^2\leq\left|\alpha_1(t)\right|^2+\alpha_2^2\left(\sum_{k=1}^{n}\left|x_k\right|^2\right),$$

for each $x_1,\ldots,x_n\in\mathbb{R}^d$, where $\alpha_1\in M_G^2(0,T)$ and $\alpha_2\in\mathbb{R}_+$.

(B2)

$$\left|J^\epsilon\left(t,x_1,\ldots,x_n\right)-J^\epsilon\left(t,y_1,\ldots,y_n\right)\right|^2\leq\left|\alpha(t)\right|^2\varphi\left(\sum_{k=1}^{n}\left|x_k-y_k\right|^2\right),$$

for each $x_1, y_1, \ldots, x_n, y_n \in \mathbb{R}^d$, where $\alpha$ is a positive and square integrable function on $[0,T]$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, increasing and concave function satisfying $\varphi(0^+) = 0$, $\int_0^1 \frac{ds}{\varphi(s)} = +\infty$.

(B3)

(i) For all $t \in [0,\ T]$,

$$\lim_{\epsilon\to 0}\int_0^t \mathbb{E}\left[\left|J^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right) - J^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right|^2\right] ds = 0.$$

(ii) $\lim_{\epsilon\to 0}\left(X_1^\epsilon(0), \ldots, X_n^\epsilon(0)\right) = \left(X_1^0(0), \ldots, X_n^0(0)\right).$

*Remark* 4.1. Assumptions (B1) and (B2) guarantee, for any $\epsilon \geq 0$, the existence of unique solution $\left(X_1^\epsilon(t), \ldots, X_n^\epsilon(t)\right) \in \left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ of our system while assumption (B3) will allows us to deduce the following stability theorem for the system.

**Theorem 4.1.** *Under assumptions* (B1), (B2) *and* (B3) *we have*

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\sum_{k=1}^n\left|X_k^\epsilon(t) - X_k^0(t)\right|^2\right] = 0, \quad \textit{for all } t \in [0,T].$$

*Proof.* For all $1 \leq k \leq n$, we have

$$\begin{cases} X_k^\epsilon(t) = & X_k^\epsilon(0) + \int_0^t f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) ds \\ & + \sum\limits_{i,j=1}^d \int_0^t f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) d\left\langle B^i, B^j\right\rangle(s) \\ & + \sum\limits_{j=1}^d \int_0^t f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) dB_s^j(s), \\ & \vdots \\ X_k^0(t) = & X_k^0(0) + \int_0^t f_k^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) ds \\ & + \sum\limits_{i,j=1}^d \int_0^t f_{k,i,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) d\left\langle B^i, B^j\right\rangle(s) \\ & + \sum\limits_{j=1}^d \int_0^t f_{k,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) dB^j(s), \end{cases}$$

and

$$\begin{aligned} & X_k^\epsilon(t) - X_k^0(t) \\ =& X_k^\epsilon(0) - X_k^0(0) + \int_0^t \left[f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_k^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] ds \\ & + \sum_{i,j=1}^d \int_0^t \left[f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_{k,i,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] d\left\langle B^i, B^j\right\rangle_s \\ & + \sum_{j=1}^d \int_0^t \left[f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_{k,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] dB^j s. \end{aligned}$$

We have

$$\left|X_k^\epsilon(t) - X_k^0(t)\right|^2$$

$$\begin{aligned}
\leq & 4\left|X_k^\epsilon(0)-X_k^0(0)\right|^2+4\left|\int_0^t\left[f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_k^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] ds\right|^2 \\
& +4\left|\sum_{i,j=1}^d \int_0^t\left[f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_{k,i,j}^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] d\left\langle B^i, B^j\right\rangle(s)\right|^2 \\
& +4 \sum_{j=1}^d\left|\int_0^t\left[f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_{k,j}^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] dB^j(s)\right|^2
\end{aligned}$$

and

$$\begin{aligned}
& \left|X_k^\epsilon(t)-X_k^0(t)\right|^2 \\
\leq & 4\left|X_k^\epsilon(0)-X_k^0(0)\right|^2+4\left|\int_0^t\left[f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_k^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] ds\right|^2 \\
& +4d^2 \sum_{i,j=1}^d\left|\int_0^t\left[f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_{k,i,j}^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] d\left\langle B^i, B^j\right\rangle(s)\right|^2 \\
& +4d \sum_{j=1}^d\left|\int_0^t\left[f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_{k,j}^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] dB^j(s)\right|^2 .
\end{aligned}$$

Taking the supremum and the G-expectation, we have

$$\begin{aligned}
& \mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_k^\epsilon(s)-X_k^0(s)\right|^2\right] \\
\leq & 4\mathbb{E}\left[\left|X_k^\epsilon(0)-X_k^0(0)\right|^2\right] \\
& +4\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left[f_k^\epsilon\left(r, X_1^\epsilon(r), \ldots, X_n^\epsilon(r)\right)-f_k^\epsilon\left(r, X_1^0(r), \ldots, X_n^0(r)\right)\right] dr\right|^2 \\
& +4d^2 \sum_{i,j=1}^d \mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left[f_{k,i,j}^\epsilon\left(r, X_1^\epsilon(r), \ldots, X_n^\epsilon(r)\right)\right.\right. \\
& \left.\left.-f_{k,i,j}^\epsilon\left(r, X_1^0(r), \ldots, X_n^0(r)\right)\right] d\left\langle B^i, B^j\right\rangle(r)\right|^2 \\
& +4d \sum_{j=1}^d \mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s\left[f_{k,j}^\epsilon\left(r, X_1^\epsilon(r), \ldots, X_n^\epsilon(r)\right)\left|-f_{k,j}^\epsilon\left(r, X_1^0(r), \ldots, X_n^0(r)\right)\right] dB^j(r)\right|\right|^2 .
\end{aligned}$$

By lemmas 2.1, 2.2 and Hölder's inequality, we have

$$\begin{aligned}
& \mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_k^\epsilon(s)-X_k^0(s)\right|^2\right] \\
\leq & \; 4\mathbb{E}\left[\left|X_k^\epsilon(0)-X_k^0(0)\right|^2\right] \\
& +8T\int_0^t \mathbb{E}\left|\left[f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right)-f_k^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right]\right|^2 ds \\
& +8T\int_0^t \mathbb{E}\left|\left[f_k^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right)-f_k^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right]\right|^2 ds
\end{aligned}$$

$$+ 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^{\epsilon}(s), \ldots, X_n^{\epsilon}(s)\right) - f_{k,i,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds$$

$$+ 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^{\epsilon}(s), \ldots, X_n^0(s)\right) - f_{k,i,j}^{0} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds$$

$$+ 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^{\epsilon}(s), \ldots, X_n^{\epsilon}(s)\right) - f_{k,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds$$

$$+ 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^{\epsilon}(s), \ldots, X_n^0(s)\right) - f_{k,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds,$$

then by assumption (B2) we obtained

$$\mathbb{E} \left[ \sup_{0 \le s \le t} \left| X_k^{\epsilon}(s) - X_k^0(s) \right|^2 \right] \le C_{k,\epsilon}(T) + C_d(T) \int_0^t |\alpha(s)|^2 \times \mathbb{E} \left[ \varphi \left( \sum_{k=1}^{n} \left| X_k^{\epsilon}(s) - X_k^0(s) \right|^2 \right) \right] ds,$$

where

$$\begin{aligned} &C_{k,\epsilon}(t) \\ =& 4\mathbb{E} \left[ \left| X_k^{\epsilon}(0) - X_k^0(0) \right|^2 \right] \\ &+ 8T \int_0^t \mathbb{E} \left| f_k^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) - f_k^0 \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds \\ &+ 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) - f_{k,i,j}^{0} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds \\ &+ 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) - f_{k,j}^{\epsilon} \left(s, X_1^0(s), \ldots, X_n^0(s)\right) \right|^2 ds, \end{aligned}$$

and $C_d(t) = 8\left(t + C_1td^2 + C_2d\right)$, then

$$\begin{aligned} &\mathbb{E} \left[ \sum_{k=1}^{n} \left| X_k^{\epsilon}(t) - X_k^0(t) \right|^2 \right] \\ \le& \sum_{k=1}^{n} \mathbb{E} \left| X_k^{\epsilon}(t) - X_k^0(t) \right|^2 \le \sum_{k=1}^{n} \mathbb{E} \left[ \sup_{0 \le s \le t} \left| X_k^{\epsilon}(s) - X_k^0(s) \right|^2 \right] \\ \le& C_{n,\epsilon}(T) + nC_d(T) \int_0^t |\alpha(s)|^2 \mathbb{E} \left[ \varphi \left( \sum_{k=1}^{n} \left| X_k^{\epsilon}(s) - X_k^0(s) \right|^2 \right) \right] ds, \end{aligned}$$

where $C_{n,\epsilon}(T) = \sum\limits_{k=1}^{n} C_{k,\epsilon}(T)$.

Since $\varphi$ is a continuous concave function, by Lemma 2.4, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k^{\epsilon}(t)-X_k^{0}(t)\right|^2\right]\\ \leq & C_{n,\epsilon}(T)+nC_d(T)\int_0^t|\alpha(s)|^2\varphi\left(\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k^{\epsilon}(s)-X_k^{0}(s)\right|^2\right]\right)ds.\end{aligned}$$

Since $C_{n,\epsilon}(T)\to 0$ when $\epsilon\to 0$, we get by Lemma 2.3

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\sum_{j=1}^{n}\left|X_j^{\epsilon}(t)-X_j^{0}(t)\right|^2\right]=0,\quad \text{for all } t\in[0,T],$$

that is what we want to prove. □

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[1]Department of Pharmacy,
Faculty of Medicine,
Salah Boubnider University,
Constantine, Algeria
*Email address*: `elhacene25@gmail.com`

[2]Department of Sciences and Technology,
Faculty of Technology,
20 August 1955 University,
Skikda, Algeria
*Email address*: `ouaouaama21@gmail.com`

[3]Department of Mathematics,
Faculty of Sciences,
Ferhat Abbas University,
Setif, Algeria
*Email address*: `salimbra@gmail.com`

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# CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY LINEAR MULTIPLIER FRACTIONAL $q$-DIFFERENTIAL OPERATOR

C. R. KRISHNA[1], N. RAVIKUMAR[2], AND B. A. FRASIN[3]

Abstract. This paper introduces a novel subclass of analytic and bi-univalent functions that are linked to a linear multiplier fractional $q$-differential operator, defined in the open unit disk $\mathbb{D}$. The authors establish the upper bounds for the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to this new subclass and its subclasses.

## 1. Introduction and preliminaries

Let the class of functions $\mathcal{A}$ be of the form:

$$\eta(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1.1}$$

which are analytic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $S$ indicates the functions of all subclasses in $\mathcal{A}$, which are univalent in $\mathbb{D}$. Since univalent functions are one-to-one, they are invertible. Although the inverse functions of single-valued functions are inverse functions, they do not need to be defined for the entire unit disk $\mathbb{D}$. Certainly, according to Koebe's quarter theorem [1], the disk with radius $\frac{1}{4}$ is in the image $\mathbb{D}$. Thus, every univalent function $\eta$ has an inverse $\eta^{-1}$ that satisfies $\eta^{-1}(\eta(z)) = z$, $z \in \mathbb{D}$, and $\zeta(w) = \eta^{-1}(\eta(w)) = w$, $|w| < r_0(\eta)$, $r_0(\eta) \geq \frac{1}{4}$, where

$$\eta^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

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A function $\eta \in \mathcal{A}$ is bi-univalent in $\mathbb{D}$ if both $\eta(z)$ and $\eta^{-1}(z)$ are univalent $\mathbb{D}$. Let $\Sigma$ be the class of bi-univalent functions on $\mathbb{D}$ given by (1.1). Example of functions in the class $\Sigma$ are

$$\frac{z}{1-z}, \quad \log \frac{1}{1-z}, \quad \log \sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of $\Sigma$. Other common examples of functions in $\mathbb{D}$ such as

$$\frac{2z-z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of $\Sigma$.

The widely-cited by Srivastava et al. [2] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (see, for example, [3–23]).

If $|q| < 1$, the $q$-shifted factorial, also known as the $q$-Pochhammer symbol, is defined for all $n \in \mathbb{N}$ by

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

where $a$ and $q$ are complex numbers. When $n = +\infty$, the product becomes

$$(a;q)_{+\infty} = \prod_{k=0}^{+\infty} (1 - aq^k).$$

If $|q| < 1$, then the product converges absolutely, and we can define the $q$-shifted factorial for $n = +\infty$ as the limit of the sequence of partial products

$$(a;q)_{+\infty} = \lim_{n\to+\infty} (a;q)_n = \lim_{n\to+\infty} \prod_{k=0}^{n-1} (1 - aq^k).$$

Therefore, when $|q| < 1$, the $q$-shifted factorial remains meaningful for $n = +\infty$ as a convergent infinite product.

The $q$-gamma function is a $q$-analogue of the gamma function, defined by the recurrence relation $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$, where $[y]_q = \frac{(1-q^y)}{(1-q)}$ is the $q$-analogue of $y$.

Jackson's [24] $q$-derivative and $q$-integral of a function $\eta$ defined on a subset of $\mathbb{C}$ are given by

$$D_q^a \eta(x) = \frac{\eta(q^a x) - \eta(x)}{(1-q^a)x - x}, \quad I_q^a \eta(x) = (1-q^a)x \sum_{n=0}^{+\infty} q^{an} \eta(q^n x),$$

where $a \in \mathbb{C}$ is a fixed parameter. These operators are also known as the $q$-difference and $q$-integral operators, respectively. The theory of $q$-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, $q$-difference and $q$-integral equations, as well as geometric function theory of complex analysis. The application of $q$-calculus was initiated by Jackson [24]. Recently, many researchers studied $q$-calculus such as Srivastava et al.

[25], Muhammad and Darus [26], Kanas and Răducanu [27], (see also, [28–33]) and also the reference cited therein.

**Definition 1.1** ([34])**.** The fractional integral operator $I^{\delta}_{q,z}$ of order $\delta > 0$, for the function $\eta(z)$ is defined by

$$I^{\delta}_{q,z} = D^{-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-rq)_{1-\delta}\eta(r)d_qr,$$

where $\eta(z)$ is the analytic of the simply connected regions of the $z$ plane containing the origin. Here, the term $(z-rq)_{\delta-1}$ is a $q$-binomial function defined by

$$(z-rq)_{\delta-1} = z^{\delta-1}\prod_{k=0}^{+\infty}\left[\frac{1-(\frac{rq}{z})q^k}{1-(\frac{rq}{z})q^{\delta}+k-1}\right] = z^{\delta}\ {}_1\phi_0\left[q^{-\delta+1};-;q,\frac{rq^{\delta}}{z}\right].$$

**Definition 1.2.** The fractional $q$-derivative operator $D^{\delta}_{q,z}$ of a $\eta(z)$ of order $0 \leq \delta < 1$, is defined by

$$D^{\delta}_{q,z}\eta(z) = D_{q,z}I^{1-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z-rq)_{-\delta}\eta(r)d_qr,$$

where $\eta(z)$ is suitably constrained and the multiplicity of $(z-rq)_{-\delta}$ is removed as in Definition 1.1 above.

**Definition 1.3.** Under the hypotheses of Definition 1.2, the fractional $q$-derivative for the function $\eta(z)$ of order $\delta$ is defined by

$$D^{\delta}_{q,z}\eta(z) = D^{n}_{q,z}I^{n-\delta}_{q,z}\eta(z),$$

where $n-1 \leq \delta < n$, $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

**Definition 1.4** ([35])**.** The definition of the fractional $q$-differintegral operator $\Omega^{\delta}_{q,z}$ is as follows. For a function $\eta(z)$ of the form (1.1), we define

$$\Omega^{\delta}_{q}\eta(z) = \Gamma_q(2-\delta)z^{\delta}D^{\delta}_{q,z}\eta(z),$$

where $D^{\delta}_{q,z}$ denotes the fractional $\delta$ order of the $q$-integral $\eta(z)$ when $-\infty < \delta < 0$ and the fractional $\delta$ order $q$-derivative of $\eta(z)$ if $0 < \delta < 2$.

The expression for $\Omega^{\delta}_{q}\eta(z)$ in terms of the coefficients $a_k$ of the power series expansion of $\eta(z)$ is given by

$$\Omega^{\delta}_{q}\eta(z) = z + \sum_{k=2}^{+\infty}\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}a_kz^k.$$

**Definition 1.5** ([34])**.** A linear multiplier fractional $q$-differintegral operator is defined as

$$\begin{aligned} \mathcal{L}_{q,\lambda}^{\delta,0}\eta(z) =& \eta(z), \\ \mathcal{L}_{q,\lambda}^{\delta,1}\eta(z) =& (1-\lambda)\Omega_q^{\delta}\eta(z) + \lambda z \mathcal{L}_q \left(\Omega_q^{\delta}\eta(z)\right), \\ \mathcal{L}_{q,\lambda}^{\delta,2}\eta(z) =& \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,1}\eta(z)\right), \\ & \vdots \\ \mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) =& \mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,n-1}\eta(z)\right). \end{aligned} \tag{1.3}$$

We note that if $f \in \mathcal{A}$ is given by (1.1), then by (1.3), we have

$$\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) = z + \sum_{k=2}^{+\infty} C\left(k,\delta,\lambda,n,q\right) a_k z^k,$$

where

$$C\left(k,\delta,\lambda,n,q\right) = \left(\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}\Big[\left([k]_q-1\right)\lambda+1\Big]\right)^n.$$

We define two new subclasses of the function class $\Sigma$ by utilizing the linear multiplier fractional $q$-differential operator of a function $\eta \in \mathcal{A}$. Then, we provide coefficient estimates for $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses of the function class $\Sigma$.

First, we have to follow the lemma to get the main results.

**Lemma 1.1** ([36])**.** *Let $\mathcal{H}$ be the family of all functions $\mathfrak{h}$ that are analytic in the open unit disk $\mathbb{D}$ and satisfy $\mathfrak{h}(0)=1$ and $\Re(\mathfrak{h}(z))>0$ for all $z \in \mathbb{D}$. If a function $\mathfrak{h} \in \mathcal{H}$ is given by $\mathfrak{h}(z) = 1 + d_1 z + d_2 z^2 + \cdots$ for $z \in \mathbb{D}$, then $|d_k| \leq 2$ for all $k \in \mathbb{N}$.*

## 2. Coefficient Bounds for the Function Class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$

**Definition 2.1.** A function $\eta(z)$ given by (1.1) is said to be in the class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$ if the following conditions are satisfied: $\eta \in \Sigma$ and

$$\left|\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau z D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right| < \frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $z\in\mathbb{D}$, and

$$\left|\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau w D_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right| < \frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $w\in\mathbb{D}$ and function $\zeta$ is given by

$$\zeta(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2.1}$$

We note that the following hold.

(a) When we set $\delta = 0$, $\lambda = 1$, and $q \to 1^-$, the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$ reduces to the class $S_\Sigma^{n,\tau}(\alpha)$, where $0 < \alpha \leq 1$, $0 \leq \tau < 1$, and $n \in \mathbb{N}_0$. This class was previously introduced and studied by Jothibasu [37].

(b) If we set $\delta = 0$, $\lambda = 1$, $q \to 1^-$, $n = 0$, and $\tau = 0$ in the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$, it reduces to the class of strongly bi-starlike functions $S_\Sigma^\star(\alpha)$ of order $\alpha$ introduced and studied by Brannan and Taha [38], where $0 < \alpha \leq 1$ .

**Theorem 2.1.** *Let $\eta(z)$ given by (1.1) be in the class $M_\Sigma(q, \alpha, \tau, \delta, \lambda, n)$, $0 < \alpha \leq 1$, $0 \leq \tau < 1$, $\delta \leq 2$, $\lambda > 0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Yq(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)\left[\tau q + 1\right] + X^2(1-\alpha)^2(1-\tau)^2}} \tag{2.2}$$

*and*

$$|a_3| \leq \frac{4\alpha^2}{X^2 q^2 (1-\tau)^2} + \frac{2\alpha}{Yq(q+1)(1-\tau)}, \tag{2.3}$$

*where $X = C(2, \delta, \lambda, n, q)$ and $Y = C(3, \delta, \lambda, n, q)$.*

*Proof.* It follows from the Definition 2.1

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} = \left[s(z)\right]^\alpha \tag{2.4}$$

and

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} = \left[t(w)\right]^\alpha, \tag{2.5}$$

respectively, where $s(z)$ and $t(w)$ satisfy the following inequalities: $\Re(s(z)) > 0$, $z \in \mathbb{D}$, and $\Re(t(w)) > 0$, $w \in \mathbb{D}$.

Furthermore, the functions $s(z)$ and $t(w)$ have the forms

$$s(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \tag{2.6}$$

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots. \tag{2.7}$$

Now, equating the coefficients in (2.4) and (2.5), we get

$$a_2 Xq(1-\tau) = \alpha s_1, \tag{2.8}$$

$$a_3 Yq(q+1)(1-\tau) - a_2^2 X^2 q(1-\tau)[\tau q + 1] = \alpha s_2 + \frac{\alpha(\alpha-1)}{2}s_1^2, \tag{2.9}$$

$$-a_2 Xq(1-\tau) = \alpha t_1 \tag{2.10}$$

and

$$\begin{aligned} &- a_3 Yq(q+1)(1-\tau) + 2a_2^2 Yq(q+1)(1-\tau) - a_2^2 X^2 q(1-\tau)[\tau q + 1] \\ &= \alpha t_2 + \frac{\alpha(\alpha-1)}{2}t_1^2. \end{aligned} \tag{2.11}$$

From (2.8) and (2.10), we get

$$s_1 = -t_1 \tag{2.12}$$

and

$$2a_2^2X^2q^2(1-\tau)^2 = \alpha^2\left(s_1^2 + t_1^2\right). \tag{2.13}$$

From (2.9), (2.11) and (2.13), we obtain

$$a_2^2 = \frac{\alpha^2(s_2+t_2)}{2\alpha Yq(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)\left[\tau q+1\right] + X^2(1-\alpha)q^2(1-\tau)^2}.$$

Applying Lemma 1.1 to the coefficients $s_2$ and $t_2$, we immediately get

$$|a_2| \le \frac{2\alpha}{\sqrt{2\alpha Yq(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)\left[\tau q+1\right] + X^2(1-\alpha)q^2(1-\tau)^2}}.$$

This gives the value of $|a_2|$ as shown in (2.2)

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.9), we get

$$\begin{aligned} &2a_3Yq(q+1)(1-\tau) - 2a_2^2Yq(q+1)(1-\tau) \\ =&\alpha(s_2-t_2) + \frac{\alpha(\alpha-1)}{2}\left(s_1^2 - t_1^2\right). \end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$|a_3| = \frac{\alpha^2(s_1^2+t_1^2)}{2X^2q^2(1-\tau)^2} + \frac{\alpha(s_2-t_2)}{2Yq(q+1)(1-\tau)}.$$

Applying Lemma 1.1 again to the coefficients $s_1$, $s_2$, $t_1$ and $t_2$, we easily get

$$|a_3| \le \frac{4\alpha^2}{X^2q^2(1-\tau)^2} + \frac{2\alpha}{Yq(q+1)(1-\tau)}.$$

This end the proof of Theorem 2.1. □

Utilizing the parameters setting of Definition 2.1 in the Theorem 2.1, we get the following corollaries.

**Corollary 2.1.** *If $\eta(z)$ given by (1.1) be in the class $S_\Sigma^{n,\tau}(\alpha)$, $0 < \alpha \le 1$, $0 \le \tau < 1$ and $n \in \mathbb{N}_0$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\tau)3^n + \left[2\alpha(\tau^2-1) - (\alpha-1)(1-\tau)^2\right]2^{2n}}}$$

*and*

$$|a_3| \le \frac{\alpha}{3^n(1-\tau)} + \frac{4\alpha^2}{2^{2n}(1-\tau)^2}.$$

**Corollary 2.2.** *If $\eta(z)$ given by (1.1) and in the class $S_\Sigma^\star(\alpha)$, $0 < \alpha \le 1$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \quad \textit{and} \quad |a_3| \le 4\alpha^2 + \alpha.$$

## 3. Coefficient Bounds for the Function Class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$

**Definition 3.1.** A function $\eta(z)$ given by (1.1) is said to be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ if the following conditions are satisfied: $\eta \in \Sigma$ and

$$\Re\left(\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right)>\gamma,$$

where $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $z\in\mathbb{D}$, and

$$\Re\left(\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right)>\gamma,$$

where $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $w\in\mathbb{D}$.

The function $\zeta$ is defined as given in equation (2.1).

(a) If we set $\delta=0$, $\lambda=1$, and $q\to 1^-$ in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, it reduces to the class $S_\Sigma^{n,\tau}(\gamma)$ introduced and studied by Jothibasu [37], where $0\le\gamma<1$, $0\le\tau<1$ and $n\in\mathbb{N}_0$.
(b) When $\delta=0$, $\lambda=1$, $q\to 1^-$, $n=0$ and $\tau=0$, the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ simplifies to the class of strongly bi-starlike functions $S_\Sigma^{\star}(\gamma)$ of order $\gamma$ introduced and studied by Brannan and Taha [38].

**Theorem 3.1.** *Let $\eta(z)$ given by (1.1) be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$. Then*

$$|a_2|\le\sqrt{\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)-X^2q(1-\tau)[\tau q+1]}}\tag{3.1}$$

*and*

$$|a_3|\le\frac{4(1-\gamma)^2}{X^2q^2(1-\tau)^2}+\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)},\tag{3.2}$$

*where $X=C(2,\delta,\lambda,n,q)$ and $Y=C(3,\delta,\lambda,n,q)$.*

*Proof.* It follows from the Definition 3.1 that there exist $s(z)$ and $t(w)\in\mathcal{H}$ such that

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}=\gamma+(1-\gamma)\,s(z),\tag{3.3}$$

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}=\gamma+(1-\gamma)\,t(w),\tag{3.4}$$

where $s(z)$ and $t(w)$ in $\mathcal{H}$ and have the forms (2.6) and (2.7), respectively.

Equating the coefficients in (3.3) and (3.4) yields

$$a_2 Xq\,(1-\tau) = (1-\gamma)\,s_1, \tag{3.5}$$

$$a_3 Yq(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\,s_2, \tag{3.6}$$

$$-a_2 Xq\,(1-\tau) = (1-\gamma)\,t_1 \tag{3.7}$$

and

$$\begin{aligned} & -a_3 Yq(q+1)\,(1-\tau) + 2a_2^2 Yq(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] \\ & = (1-\gamma)\,t_2. \end{aligned} \tag{3.8}$$

From (3.5) and (3.7), we get $s_1 = -t_1$ and

$$2a_2^2 X^2 q^2\,(1-\tau)^2 = (1-\gamma)^2\left(s_1^2 + t_1^2\right). \tag{3.9}$$

Also, from (3.6) and (3.8), we find that

$$2a_2^2 Yq(q+1)\,(1-\tau) - 2a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\,(s_2+t_2)\,.$$

Applying Lemma 1.1 to the coefficients $s_2$ and $t_2$, we immediately get

$$|a_2| \leq \sqrt{\frac{2\,(1-\gamma)}{Yq(q+1)\,(1-\tau) - X^2 q\,(1-\tau)\,[\tau q+1]}},$$

which is the bound on $|a_2|$ as given in (3.1). Then, to get the limit of $|a_3|$ by subtracting (3.8) from (3.6),

$$2a_3 Yq(q+1)\,(1-\tau) - 2a_2^2 Yq(q+1)\,(1-\tau) = (1-\gamma)\,(s_2-t_2)\,,$$

or, equivalently

$$a_3 = a_2^2 + \frac{(1-\gamma)\,(s_2-t_2)}{2Yq(q+1)\,(1-\tau)}.$$

Substituting the values of $a_2^2$ into (3.9), we get

$$a_3 = \frac{(1-\gamma)^2\,(s_1^2+t_1^2)}{2X^2 q^2\,(1-\tau)^2} + \frac{(1-\gamma)\,(s_2-t_2)}{2Yq(q+1)\,(1-\tau)}.$$

After applying Lemma 1.1 to the coefficients $s_1$, $s_2$, $t_1$ and $t_2$, we get

$$|a_3| \leq \frac{4\,(1-\gamma)^2}{X^2 q^2\,(1-\tau)^2} + \frac{2\,(1-\gamma)}{Yq(q+1)\,(1-\tau)}.$$

This completes the proof of Theorem 3.1. □

Utilizing the parameters setting of Definition 3.1 in the Theorem 3.1, we get the following corollaries.

**Corollary 3.1.** *If $\eta(z)$ given by (1.1) is in the class $S_\Sigma^{n,\tau}(\gamma)$, $0 \leq \gamma < 1$, $0 \leq \tau < 1$ and $n \in \mathbb{N}_0$, then*

$$|a_2| \leq \sqrt{\frac{2\,(1-\gamma)}{2^{2n}\,(\tau^2-1) + 2\,(1-\tau)\,3^n}}$$

*and*

$$|a_3| \leq \frac{4\left(1-\gamma\right)^2}{2^{2n}\left(1-\tau\right)^2} + \frac{\left(1-\gamma\right)}{3^n\left(1-\tau\right)}.$$

**Corollary 3.2.** *If* $\eta(z)$ *given by* (1.1) *and in the class* $S^{\star}_{\Sigma}(\gamma)$, $0 \leq \gamma < 1$, *then*

$$|a_2| \leq \sqrt{2\left(1-\gamma\right)} \quad \textit{and} \quad |a_3| \leq 4\left(1-\gamma\right)^2 + \left(1-\gamma\right).$$

## 4. Conclusions

The main contribution of this paper is the introduction of new subclasses of bi-univalent functions defined by the linear multiplier fractional $q$-differential operator. Additionally, we provide upper bounds for the coefficients $|a_2|$ and $|a_3|$ for functions belonging to this new subclass and its subclasses.

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*Email address*: `krishnargowda91@gmail.com`
*Email address*: `ravisn.kumar@gmail.com`
*Email address*: `bafrasin@yahoo.com`

[1]PG Department of Mathematics,
Bharathi College PG and RC, Maddur Tq., Mandya - 571 422, India

[2]PG Department of Mathematics,
JSS College of Arts, Science and Commerce, Mysore 570 006, India

[3]Faculty of Science, Department of Mathematics,
Al al-Bayt University, Mafraq, Jordan

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# LIPSCHITZ $p$-APPROXIMATE SCHAUDER FRAMES

K. MAHESH KRISHNA[1] AND P. SAM JOHNSON[2]

Abstract. With the aim of representing subsets of Banach spaces as an infinite series using Lipschitz functions, we study a variant of metric frames which we call Lipschitz $p$-approximate Schauder frames (Lipschitz $p$-ASFs). We characterize Lipschitz $p$-ASFs and their duals completely using the canonical Schauder basis for classical sequence spaces. Similarity of Lipschitz $p$-ASF is introduced and characterized.

## 1. Introduction

Grochenig in 1991 introduced the notion of Banach frames [17] as a generalization of notion of frames for Hilbert spaces introduced by Duffin and Schaeffer in 1952 [11]. This notion originated from the study of atomic decompositions and coorbit spaces arising from square integrable representations of locally compact groups developed by Feichtinger and Grochenig in 1980's [13–15]. Casazza, Han and Larson in 2000 explored the connection between Banach frames and atomic decompositions and introduced the notion of (unconditional) Schauder frames [8]. In 2001, Aldroubi, Sun and Tang introduced the notion of $p$-frames and $p$-Riesz bases for Banach spaces, $1 \leq p < +\infty$ [1]. These notions have been generalized by Casazza, Christensen and Stoeva by introducing the notion of $\mathcal{X}_d$-frames [5, 9]. A slight variant notion of $\mathcal{X}_d$-frames for Banach spaces was given by Terekhin [24–26]. In 2014, Thomas, Freeman, Odell, Schlumprecht and Zsak [16, 27, 28] introduced the notion of approximate Schauder frames as a generalization of notion of Schauder frames by Casazza, Dilworth, Odell, Schlumprecht and Zsak [4] (also see [10]). In 2021, Krishna and Johnson characterized some classes of

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approximate Schauder frames [21]. In 2022, Krishna and Johnson introduced metric frames which have surprising connections with subsets of Banach spaces using Lipschitz-free Banach spaces [22]. We now ask the following question which is the main motivation for writing the paper.

(1.1) Can we represent a subset (which need not be a subspace) of a Banach space as an infinite series using Lipschitz maps and elements of the set?

Note that we can not demand linear functionals in the above problem as we are not considering subspaces. Motivated from 1.1 we study representation of subsets (need not be subspaces) of Banach spaces using Lipschitz functions.

The paper is organized as follows. We introduce the notion of Lipschitz $p$-approximate Schauder frame (Lipschitz $p$-ASF) for subsets of Banach spaces in Definition 2.1. Followed by interesting Examples 2.1, 2.2 and 2.3, factorization property of Lipschitz frame map is derived in Theorem 2.1. Lipschitz $p$-ASFs are characterized in Theorem 2.2. Next we introduce the notion of dual frames in Definition 2.2 and classify them in Theorem 2.4. Definition 2.3 introduces the notion of similarity and Theorem 2.5 gives an operator-theoretic characterization for similarity. Orthogonality of frames is introduced in Definition 2.4 and interpolation result is derived in Theorem 2.6. We end by formulating an open Problem in Section 3.

## 2. Lipschitz $p$-Approximate Schauder Frames

Let $\mathcal{X}$ be a real or complex Banach space and $\mathcal{M}$ be a non-empty subset of $\mathcal{X}$. The identity operator on $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$. The set of all Lipschitz functions from $\mathcal{M}$ to $\mathcal{X}$ is denoted by $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$. For $1 \leq p < +\infty$, the canonical Schauder basis for $\ell^p(\mathbb{N})$ is denoted by $\{e_n\}_n$ and its coordinate functionals are denoted by $\{\zeta_n\}_n$. We introduce the following important notion as a first step in answering Motivation 1.1.

**Definition 2.1.** For $1 \leq p < +\infty$, let $\mathcal{X}$ be a Banach space and $\mathcal{M}$ be a subset (need not be a subspace) of $\mathcal{X}$. Let $\{\tau_n\}_n$ be a sequence in $\mathcal{M}$ and $\{f_n\}_n$ be a sequence in $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a **Lipschitz $p$-approximate Schauder frame** (we write Lipschitz $p$-ASF) if the following conditions hold.

(i) The map (**analysis map**)
$$\theta_f : \mathcal{M} \ni x \mapsto \theta_f x \coloneqq \{f_n(x)\}_n \in \ell^p(\mathbb{N})$$
is a well-defined Lipschitz map.

(ii) The map (**synthesis operator**)
$$\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau \{a_n\}_n \coloneqq \sum_{n=1}^{+\infty} a_n \tau_n \in \mathcal{X}$$
is a well-defined bounded linear operator.

(iii) The map (**Lipschitz frame map**)

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^{+\infty} f_n(x)\tau_n \in \mathcal{M}$$

is a well-defined invertible bi-Lipschitz map and

$$x = \sum_{n=1}^{+\infty} f_n(x)S_{f,\tau}^{-1}\tau_n, \quad \text{for all } x \in \mathcal{M}. \tag{2.1}$$

If $S_{f,\tau} = I_{\mathcal{M}}$, then we say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a **Lipschitz $p$-Schauder frame** (we write Lipschitz $p$-SF). If we do not impose the condition 'invertible bi-Lipschitz' and Equation (2.1) in (iii), then we say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a **Lipschitz $p$-Bessel sequence** (we write Lipschitz $p$-BS) for $\mathcal{M}$.

Whenever $\mathcal{M} = \mathcal{X}$, and $f_n$'s are all linear, Definition 2.1 reduces to definition of $p$-ASF given in [21]. It is important to note that the partial sums of series in (iii) of Definition 2.1 need not be inside $\mathcal{M}$ (which may not be as it is only a subset) but only demanding limit has to be inside $\mathcal{M}$. Definition 2.1 says that there are $a, b, c, d > 0$ satisfying following:

$$a\|x-y\| \leq \left\|\sum_{n=1}^{+\infty}(f_n(x) - f_n(y))\tau_n\right\| \leq b\|x-y\|, \quad \text{for all } x, y \in \mathcal{M},$$

$$\left(\sum_{n=1}^{+\infty} |f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \leq c\|x-y\|, \quad \text{for all } x, y \in \mathcal{M},$$

$$\left\|\sum_{n=1}^{+\infty} a_n\tau_n\right\| \leq d\left(\sum_{n=1}^{+\infty} |a_n|^p\right)^{\frac{1}{p}}, \quad \text{for all } \{a_n\}_n \in \ell^p(\mathbb{N}).$$

We call $a$ as lower Lipschitz frame bound, $b$ as upper Lipschitz frame bound, $c$ as Lipschitz analysis bound and $d$ as Lipschitz synthesis bound. We give various interesting examples of Lipschitz $p$-ASFs.

*Example* 2.1. Let $\mathcal{X} := \mathbb{C}$, $p = 1$ and

$$\mathcal{M} := \left\{z \in \mathbb{C} : |z| \leq \frac{1}{2}|z+1|\right\} = \left\{x + iy : x, y \in \mathbb{R}, \left(x - \frac{1}{3}\right)^2 + y^2 \leq \left(\frac{2}{3}\right)^2\right\}.$$

For $n \in \mathbb{N}$, define

$$f_n : \mathcal{M} \ni z \mapsto f_n(z) := \left(\frac{z}{1+z}\right)^n \in \mathbb{C}, \quad \tau_n := 1 \in \mathcal{M}.$$

We first show that $f_n$ is Lipschitz for all $n$. For $z \in \mathcal{M}$,

$$1 - |z+1| \leq \Big|1 - |z+1|\Big| \leq \Big|1 - (z+1)\Big| = |z| \leq \frac{1}{2}|z+1|.$$

Hence,

$$|z+1| \geq \frac{2}{3}, \quad \text{for all } z \in \mathcal{M}.$$

Let $z, w \in \mathcal{M}$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned}
|f_n(z) - f_n(w)| &= \left|\left(\frac{z}{1+z}\right)^n - \left(\frac{w}{1+w}\right)^n\right| \\
&= \left|\frac{z}{1+z} - \frac{w}{1+w}\right| \cdot \left|\left(\frac{z}{1+z}\right)^{n-1} + \cdots + \left(\frac{w}{1+w}\right)^{n-1}\right| \\
&\leq \frac{|z-w|}{|1+z| \cdot |1+w|} \cdot \frac{n}{2^{n-1}} \\
&\leq \frac{9}{4} \cdot \frac{n}{2^{n-1}} |z-w|.
\end{aligned}$$

Therefore, each $f_n$ is Lipschitz. Set

$$r := \sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} < +\infty.$$

We then see that for $z, w \in \mathcal{M}$,

$$\begin{aligned}
\|\theta_f z - \theta_f w\| &= \sum_{n=1}^{+\infty} |f_n(z) - f_n(w)| = \sum_{n=1}^{+\infty} \left|\left(\frac{z}{1+z}\right)^n - \left(\frac{w}{1+w}\right)^n\right| \\
&\leq \sum_{n=1}^{+\infty} \frac{9}{4} \cdot \frac{n}{2^{n-1}} |z-w| = \frac{9}{4} r |z-w|.
\end{aligned}$$

Therefore, $\theta_f$ is Lipschitz. Clearly,

$$\theta_\tau : \ell^1(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{+\infty} a_n \cdot 1 \in \mathbb{C}$$

is a well-defined bounded linear operator. Finally, we observe that for $z \in \mathcal{M}$, we have $\frac{|z|}{|z+1|} < 1$ and hence

$$S_{f,\tau} z = \sum_{n=1}^{+\infty} f_n(z) \tau_n = \sum_{n=1}^{+\infty} \left(\frac{z}{1+z}\right)^n \cdot 1 = \frac{1}{1 - \frac{z}{1+z}} - 1 = z, \quad \text{for all } z \in \mathcal{M}.$$

Thus, we proved that $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz 1-SF for $\mathcal{M}$.

*Example* 2.2. Let $\mathcal{X} := \mathbb{R}$, $p = 1$ and $\mathcal{M} := [1, +\infty)$ For $n \in \mathbb{N} \cup \{0\}$, define $f_n : \mathcal{M} \to \mathbb{R}$ by

$$\begin{aligned}
f_0(x) &:= 1, \quad \text{for all } x \in \mathcal{M}, \\
f_n(x) &:= \frac{(\log x)^n}{n!}, \quad \text{for all } x \in \mathcal{M}, \text{ for all } n \geq 1
\end{aligned}$$

and $\tau_n \coloneqq 1 \in \mathcal{M}$. Then $f_n'(x) = \frac{(\log x)^{(n-1)}}{(n-1)!x}$, for all $x \in \mathcal{M}$, for all $n \geq 1$. Since $f_n'$ is bounded on $\mathcal{M}$ for all $n \geq 1$, $f_n$ is Lipschitz on $\mathcal{M}$ for all $n \geq 1$. For $x, y \in \mathcal{M}$, with $x < y$, we see that

$$\begin{aligned}\|\theta_f x - \theta_f y\| &= \sum_{n=0}^{+\infty} |f_n(x) - f_n(y)| = \sum_{n=0}^{+\infty} \frac{(\log y)^n}{n!} - \sum_{n=0}^{+\infty} \frac{(\log x)^n}{n!} \\ &= e^{\log y} - e^{\log x} = y - x = |x - y|.\end{aligned}$$

Therefore, $\theta_f$ is Lipschitz. It is clear that $\theta_\tau$ is a well-defined bounded linear operator. For $x \in \mathcal{M}$,

$$S_{f,\tau} x = \sum_{n=1}^{+\infty} f_n(x)\tau_n = \sum_{n=0}^{+\infty} \frac{(\log x)^n}{n!} \cdot 1 = x.$$

Hence, $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz 1-SF for $\mathcal{M}$.

*Example* 2.3. For $1 \leq p < +\infty$, let $\mathcal{X}$ be a Banach space and $\mathcal{M}$ be a subset of $\mathcal{X}$. Assume that there is a Lipschitz map $U : \mathcal{M} \to \ell^p(\mathbb{N})$, a bounded linear operator $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, $VU : \mathcal{M} \to \mathcal{M}$ is an invertible bi-Lipschitz map and

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1} Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define

$$f_n \coloneqq \zeta_n U, \quad \tau_n \coloneqq Ve_n, \quad \text{for all } n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is Lipschitz $p$-ASF for $\mathcal{M}$. If $VU = I_{\mathcal{M}}$, then $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz $p$-SF for $\mathcal{M}$.

We show in the sequel that (in Theorem 2.2) every Lipschitz $p$-ASF can be written in the form of Example 2.3. Following theorem gives various fundamental factorization properties of Lipschitz $p$-ASFs whose proof is a direct calculation.

**Theorem 2.1.** *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz $p$-ASF for $\mathcal{M} \subseteq \mathcal{X}$. Then the following hold.*

(i) *We have*

$$x = \sum_{n=1}^{+\infty} (f_n S_{f,\tau}^{-1})(x)\tau_n, \quad \text{for all } x \in \mathcal{M}. \tag{2.2}$$

(ii) $(\{f_n S_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ *is a Lipschitz $p$-ASF for $\mathcal{M}$.*

(iii) *The analysis map $\theta_f$ is injective.*

(iv) *The synthesis operator $\theta_\tau$ is surjective.*

(v) *Lipschitz frame map $S_{f,\tau}$ factors as $S_{f,\tau} = \theta_\tau \theta_f$.*

(vi) $P_{f,\tau} \coloneqq \theta_f S_{f,\tau}^{-1} \theta_\tau : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ *is a Lipschitz projection onto $\theta_f(\mathcal{M})$.*

Holub characterized frames for Hilbert spaces using standard orthonormal basis for the standard Hilbert space [20]. This result has been derived for Banach spaces in [21]. We show that such a result can be derived for Lipschitz $p$-ASFs.

**Theorem 2.2.** *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz $p$-ASF for $\mathcal{M} \subseteq \mathcal{X}$ if and only if*

$$f_n = \zeta_n U, \quad \tau_n = Ve_n, \quad \text{for all } n \in \mathbb{N},$$

*where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, $VU : \mathcal{M} \to \mathcal{M}$ is an invertible bi-Lipschitz map and*

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1}Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

*Proof.* ($\Leftarrow$) Clearly $\theta_f$ is Lipschitz and $\theta_\tau$ is a bounded linear operator. Now let $x \in \mathcal{M}$. Then

$$S_{f,\tau}x = \sum_{n=1}^{+\infty} f_n(x)\tau_n = \sum_{n=1}^{+\infty} \zeta_n(Ux)Ve_n = V\left(\sum_{n=1}^{+\infty} \zeta_n(Ux)e_n\right) = VUx. \tag{2.3}$$

Hence, $S_{f,\tau}$ is an invertible bi-Lipschitz map.

($\Rightarrow$) Define $U := \theta_f$, $V := \theta_\tau$. Then $(\zeta_n U)(x) = (\zeta_n \theta_f)(x) = \zeta_n(\{f_k(x)\}_k) = f_n(x)$, for all $x \in \mathcal{M}$, $Ve_n = \theta_\tau e_n = \tau_n$, for all $n \in \mathbb{N}$ and $VU = \theta_\tau\theta_f = S_{f,\tau}$ which is an invertible bi-Lipschitz map. ☐

**Corollary 2.1.** (i) *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz $p$-SF for $\mathcal{M} \subseteq \mathcal{X}$ if and only if $f_n = \zeta_n U, \tau_n = Ve_n$, for all $n \in \mathbb{N}$, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $VU = I_{\mathcal{M}}$.*

(ii) *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz $p$-BS for $\mathcal{M} \subseteq \mathcal{X}$ if and only if $f_n = \zeta_n U, \tau_n = Ve_n$, for all $n \in \mathbb{N}$, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$ and $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$.*

Equations (2.1) and (2.2) lead us to define the notion of dual frame as follows.

**Definition 2.2.** Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz $p$-ASF for $\mathcal{M} \subseteq \mathcal{X}$. A Lipschitz $p$-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$ is said to be a **dual** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$x = \sum_{n=1}^{+\infty} g_n(x)\tau_n = \sum_{n=1}^{+\infty} f_n(x)\omega_n, \quad \text{for all } x \in \mathcal{M}.$$

We can give a characterization of dual frames by using analysis map and synthesis operator.

**Proposition 2.1.** *Given two Lipschitz $p$-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$, the following are equivalent:*

(a) $(\{g_n\}_n, \{\omega_n\}_n)$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$*;*
(b) $\theta_\tau\theta_g = \theta_\omega\theta_f = I_{\mathcal{M}}$.

Equations (2.1) and (2.2) show that the Lipschitz $p$-ASF $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$. We call $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ as the **canonical dual** for $(\{f_n\}_n, \{\tau_n\}_n)$. With this notion, the following theorem is evident.

**Theorem 2.3.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$ *with frame bounds a and b. Then the following statements hold good.*

(a) *The canonical dual for the canonical dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *is itself.*
(b) $\frac{1}{b}, \frac{1}{a}$ *are frame bounds for the canonical dual for* $(\{f_n\}_n, \{\tau_n\}_n)$.
(c) *If* $a, b$ *are optimal frame bounds for* $(\{f_n\}_n, \{\tau_n\}_n)$*, then* $\frac{1}{b}, \frac{1}{a}$ *are optimal frame bounds for its canonical dual.*

In 1995, Li derived a characterization of dual frames using standard orthonormal basis for $\ell^2(\mathbb{N})$ [23]. For Banach spaces, such a characterization using canonical Schauder basis for $\ell^p(\mathbb{N})$ is derived in [21]. Now we derive such characterization for Lipschitz $p$-ASF.

**Lemma 2.1.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then a Lipschitz p-ASF* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M}$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *if and only if*

$$g_n = \zeta_n U, \quad \omega_n = Ve_n, \quad \text{for all } n \in \mathbb{N},$$

*where* $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz right-inverse of* $\theta_\tau$ *and* $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a linear bounded left-inverse of* $\theta_f$ *such that* $VU(\mathcal{M}) \subseteq \mathcal{M}$*,* $Ve_n \in \mathcal{M}$ *for all* $n \in \mathbb{N}$*,* $VU$ *is an invertible bi-Lipschitz map and*

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1}Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

*Proof.* ($\Leftarrow$) Using the 'if' part of proof of Theorem 2.2, we get that $(\{g_n\}_n, \{\omega_n\}_n)$ is a Lipschitz $p$-ASF for $\mathcal{M}$. We check for duality of $(\{g_n\}_n, \{\omega_n\}_n)$: $\theta_\tau\theta_g = \theta_\tau U = I_{\mathcal{M}}$, $\theta_\omega\theta_f = V\theta_f = I_{\mathcal{M}}$.

($\Rightarrow$) Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a dual Lipschitz $p$-ASF for $(\{f_n\}_n, \{\tau_n\}_n)$. Then $\theta_\tau\theta_g = I_{\mathcal{M}}$, $\theta_\omega\theta_f = I_{\mathcal{M}}$. Define $U \coloneqq \theta_g, V \coloneqq \theta_\omega$. Then $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz right-inverse of $\theta_\tau$ and $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a linear bounded left-inverse of $\theta_f$ such that the operator $VU = \theta_\omega\theta_g = S_{g,\omega}$ is invertible. Further,

$$(\zeta_n U)x = \zeta_n\left(\sum_{k=1}^{+\infty} g_k(x)e_k\right) = \sum_{k=1}^{+\infty} g_k(x)\zeta_n(e_k) = g_n(x), \quad \text{for all } x \in \mathcal{M},$$

and $Ve_n = \theta_\omega e_n = \omega_n$, for all $n \in \mathbb{N}$. □

**Lemma 2.2.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then,*

(i) $R : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz right-inverse of* $\theta_\tau$ *if and only if*

$$R = \theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U$$

*where* $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz map;*

(ii) $L : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded left-inverse of* $\theta_f$ *if and only if*

$$L = S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right),$$

*where* $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded linear operator.*

*Proof.* (i) ($\Leftarrow$) $\theta_\tau\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U\right) = I_{\mathcal{M}} + \theta_\tau U - I_{\mathcal{M}}\theta_\tau U = I_{\mathcal{M}}$. Therefore, $\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U$ is a Lipschitz right-inverse of $\theta_\tau$.

($\Rightarrow$) Define $U \coloneqq R$. Then,

$$\begin{aligned}\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U =& \theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) R\\ =& \theta_f S_{f,\tau}^{-1} + R - \theta_f S_{f,\tau}^{-1} = R.\end{aligned}$$

(ii) ($\Leftarrow$) $\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\theta_f = I_{\mathcal{M}} + V\theta_f - V\theta_f I_{\mathcal{M}} = I_{\mathcal{M}}$. Therefore, $S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)$ is a bounded left-inverse of $\theta_f$.

($\Rightarrow$) Define $V \coloneqq L$. Then,

$$\begin{aligned}S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) =& S_{f,\tau}^{-1}\theta_\tau + L\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\\ =& S_{f,\tau}^{-1}\theta_\tau + L - S_{f,\tau}^{-1}\theta_\tau = L.\end{aligned}$$ ☐

**Theorem 2.4.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then a Lipschitz p-ASF* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M}$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *if and only if*

$$\begin{aligned}g_n &= f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,\\ \omega_n &= S_{f,\tau}^{-1}\tau_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

*such that*

$$S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U$$

*is an invertible bi-Lipschitz map, where* $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz map,* $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded linear operator,* $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ *for all* $n \in \mathbb{N}$ *and*

$$\begin{aligned}&\sum_{n=1}^{+\infty}\zeta_n\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)x\left[S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U\right]^{-1}\\ &\times\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)e_n = x, \quad \text{for all } x \in \mathcal{M}.\end{aligned}$$

*Proof.* Lemmas 2.1 and 2.2 give the characterization of dual frame as

$$\begin{aligned}g_n &= \zeta_n\theta_f S_{f,\tau}^{-1} + \zeta_n U - \zeta_n\theta_f S_{f,\tau}^{-1}\theta_\tau U = f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,\\ \omega_n &= S_{f,\tau}^{-1}\theta_\tau e_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\theta_\tau e_n = S_{f,\tau}^{-1}\tau_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

such that

$$\left(S_{f,\tau}^{-1}\theta_\tau+V\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1}+\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)$$

is an invertible bi-Lipschitz map, where $U:\mathcal{M}\to\ell^p(\mathbb{N})$ is a Lipschitz map, $V:\ell^p(\mathbb{N})\to\mathcal{X}$ is a bounded linear operator, $VU(\mathcal{M})\subseteq\mathcal{M}$, $Ve_n\in\mathcal{M}$ for all $n\in\mathbb{N}$ and for all $x\in\mathcal{M}$

$$\begin{aligned}&\sum_{n=1}^{+\infty}\zeta_n\left(\theta_f S_{f,\tau}^{-1}+\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)x\\ &\times W^{-1}\left(S_{f,\tau}^{-1}\theta_\tau+V\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)e_n=x,\end{aligned}$$

where

$$W\coloneqq\left(S_{f,\tau}^{-1}\theta_\tau+V\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1}+\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right).$$

Through expansion and simplification we get

$$\begin{aligned}&\left(S_{f,\tau}^{-1}\theta_\tau+V\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1}+\left(I_{\ell^p(\mathbb{N})}-\theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)\\ =&S_{f,\tau}^{-1}+VU-V\theta_f S_{f,\tau}^{-1}\theta_\tau U.\end{aligned}$$ □

Balan introduced the notion of similarity for frames for Hilbert space which gives an equivalence relation on frames [3]. It has been done for Banach spaces by Krishna and Johnson in [21]. We define the same for Lipschitz $p$-ASF as follows.

**Definition 2.3.** Two Lipschitz $p$-ASFs $(\{f_n\}_n,\{\tau_n\}_n)$ and $(\{g_n\}_n,\{\omega_n\}_n)$ for $\mathcal{M}\subseteq\mathcal{X}$ are said to be **similar** or **equivalent** if there exist invertible bi-Lipschitz map $T_{f,g}:\mathcal{M}\to\mathcal{M}$ and an invertible bounded linear operator $T_{\tau,\omega}:\mathcal{X}\to\mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M})\subseteq\mathcal{M}$ and

$$g_n=f_nT_{f,g},\quad \omega_n=T_{\tau,\omega}\tau_n,\qquad \text{for all } n\in\mathbb{N}.$$

Since maps giving similarity are invertible, similarity is an equivalence relation on the set $\{(\{f_n\}_n,\{\tau_n\}_n):(\{f_n\}_n,\{\tau_n\}_n)$ is a Lipschitz $p$-ASF for $\mathcal{M}\}$. Observe that for every Lipschitz $p$-ASF $(\{f_n\}_n,\{\tau_n\}_n)$, both $(\{f_nS_{f,\tau}^{-1}\}_n,\{\tau_n\}_n)$ and $(\{f_n\}_n,\{S_{f,\tau}^{-1}\tau_n\}_n)$ are Lipschitz $p$-ASFs and are similar to $(\{f_n\}_n,\{\tau_n\}_n)$. Balan gave an operator algebraic characterization of similarity in Hilbert spaces [3] and it is extended to Banach spaces by Krishna and Johnson in [21]. We derive Lipschitz version in the following theorem.

**Theorem 2.5.** *For two Lipschitz $p$-ASFs $(\{f_n\}_n,\{\tau_n\}_n)$ and $(\{g_n\}_n,\{\omega_n\}_n)$ for $\mathcal{M}\subseteq\mathcal{X}$, the following are equivalent:*

(a) *$g_n=f_nT_{f,g}$, $\omega_n=T_{\tau,\omega}\tau_n$, for all $n\in\mathbb{N}$, for some invertible bi-Lipschitz map $T_{f,g}:\mathcal{M}\to\mathcal{M}$, for some invertible linear map $T_{\tau,\omega}:\mathcal{X}\to\mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M})\subseteq\mathcal{M}$;*

(b) *$\theta_g=\theta_fT_{f,g}$, $\theta_\omega=T_{\tau,\omega}\theta_\tau$, for some invertible bi-Lipschitz map $T_{f,g}:\mathcal{M}\to\mathcal{M}$, for some invertible linear map $T_{\tau,\omega}:\mathcal{X}\to\mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M})\subseteq\mathcal{M}$;*

(c) $P_{g,\omega} = P_{f,\tau}$.

*If one of the above conditions is satisfied, then invertible maps in* (i) *and* (ii) *are unique and given by* $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g$, $T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. *In the case that* $(\{f_n\}_n, \{\tau_n\}_n)$ *is a Lipschitz p-SF, then* $(\{g_n\}_n, \{\omega_n\}_n)$ *is a Lipschitz p-SF if and only if* $T_{\tau,\omega}T_{f,g} = I_{\mathcal{M}}$ *if and only if* $T_{f,g}T_{\tau,\omega} = I_{\mathcal{M}}$.

*Proof.* (i) $\Rightarrow$ (ii) Let $x \in \mathcal{M}$ and $\{a_n\}_n \in \ell^p(\mathbb{N})$. Then

$$\begin{aligned}\theta_g x =&\{g_n(x)\}_n = \{f_n(T_{f,g}x)\}_n = \theta_f(T_{f,g}x),\\ \theta_\omega(\{a_n\}_n) =& \sum_{n=1}^{+\infty} a_n\omega_n = \sum_{n=1}^{+\infty} a_n T_{\tau,\omega}\tau_n = T_{\tau,\omega}\theta_\tau\{a_n\}_n.\end{aligned}$$

(ii) $\Rightarrow$ (iii) $S_{g,\omega} = \theta_\omega\theta_g = T_{\tau,\omega}\theta_\tau\theta_f T_{f,g} = T_{\tau,\omega}S_{f,\tau}T_{f,g}$ and

$$P_{g,\omega} = \theta_g S_{g,\omega}^{-1}\theta_\omega = (\theta_f T_{f,g})(T_{\tau,\omega}S_{f,\tau}T_{f,g})^{-1}(T_{\tau,\omega}\theta_\tau) = P_{f,\tau}.$$

(ii) $\Rightarrow$ (i) $\sum_{n=1}^{+\infty} g_n(x)e_n = \theta_g(x) = \theta_f(T_{f,g}x) = \sum_{n=1}^{+\infty} f_n(T_{f,g}x)e_n$, for all $x \in \mathcal{M}$. This gives (i).

(iii) $\Rightarrow$ (ii) $\theta_g = P_{g,\omega}\theta_g = P_{f,\tau}\theta_g = \theta_f(S_{f,\tau}^{-1}\theta_\tau\theta_g)$ and $\theta_\omega = \theta_\omega P_{g,\omega} = \theta_\omega P_{f,\tau} = (\theta_\omega\theta_f S_{f,\tau}^{-1})\theta_\tau$. We show that $S_{f,\tau}^{-1}\theta_\tau\theta_g$ and $\theta_\omega\theta_f S_{f,\tau}^{-1}$ are invertible. For,

$$\begin{aligned}(S_{f,\tau}^{-1}\theta_\tau\theta_g)(S_{g,\omega}^{-1}\theta_\omega\theta_f) &= S_{f,\tau}^{-1}\theta_\tau P_{g,\omega}\theta_f = S_{f,\tau}^{-1}\theta_\tau P_{f,\tau}\theta_f = I_{\mathcal{M}},\\ (S_{g,\omega}^{-1}\theta_\omega\theta_f)(S_{f,\tau}^{-1}\theta_\tau\theta_g) &= S_{g,\omega}^{-1}\theta_\omega P_{f,\tau}\theta_g = S_{g,\omega}^{-1}\theta_\omega P_{g,\omega}\theta_g = I_{\mathcal{M}}\end{aligned}$$

and

$$\begin{aligned}(\theta_\omega\theta_f S_{f,\tau}^{-1})(\theta_\tau\theta_g S_{g,\omega}^{-1}) &= \theta_\omega P_{f,\tau}\theta_g S_{g,\omega}^{-1} = \theta_\omega P_{g,\omega}\theta_g S_{g,\omega}^{-1} = I_{\mathcal{M}},\\ (\theta_\tau\theta_g S_{g,\omega}^{-1})(\theta_\omega\theta_f S_{f,\tau}^{-1}) &= \theta_\tau P_{g,\omega}\theta_f S_{f,\tau}^{-1} = \theta_\tau P_{f,\tau}\theta_f S_{f,\tau}^{-1} = I_{\mathcal{M}}.\end{aligned}$$

Let $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ be invertible bi-Lipschitz maps and $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n$, for all $n \in \mathbb{N}$. Then $\theta_g = \theta_f T_{f,g}$ says that $\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g}$ which implies $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g$. Similarly, $\theta_\omega = T_{\tau,\omega}\theta_\tau$ says that $\theta_\omega\theta_f = T_{\tau,\omega}\theta_\tau\theta_f = T_{\tau,\omega}S_{f,\tau}$. Hence, $T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. ☐

In Definition 2.2 we defined the notion of dual frames [2,18,19] and for Banach spaces in [21]. We can define the orthogonality for Lipschitz $p$-ASFs as follows.

**Definition 2.4.** Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz $p$-ASF for $\mathcal{M} \subseteq \mathcal{X}$. A Lipschitz $p$-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M}$ is said to be **orthogonal** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$0 = \sum_{n=1}^{+\infty} g_n(x)\tau_n = \sum_{n=1}^{+\infty} f_n(x)\omega_n, \quad \text{for all } x \in \mathcal{M}.$$

Similar to Proposition 2.1 we have the following proposition.

**Proposition 2.2.** *Given two Lipschitz p-ASFs* $(\{f_n\}_n, \{\tau_n\}_n)$ *and* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M} \subseteq \mathcal{X}$, *the following are equivalent:*

(a) $(\{g_n\}_n, \{\omega_n\}_n)$ *is orthogonal for* $(\{f_n\}_n, \{\tau_n\}_n)$;

(b) $\theta_\tau\theta_g = \theta_\omega\theta_f = 0$.

Using orthogonality we derive following interpolation result. For the Hilbert space frames this is derived by Han and Larson in [19] and for Banach spaces in [21].

**Theorem 2.6.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *and* $(\{g_n\}_n, \{\omega_n\}_n)$ *be two Lipschitz $p$-SF for* $\mathcal{M} \subseteq \mathcal{X}$ *which are orthogonal. If* $A, B : \mathcal{M} \to \mathcal{M}$ *are bi-Lipschitz maps,* $C, D : \mathcal{X} \to \mathcal{X}$ *are bounded linear operators,* $C(\mathcal{M}) \subseteq \mathcal{M}$, $D(\mathcal{M}) \subseteq \mathcal{M}$ *and* $CA + DB = I_{\mathcal{M}}$, *then* $(\{f_nA + g_nB\}_n, \{C\tau_n + D\omega_n\}_n)$ *is a Lipschitz $p$-SF for* $\mathcal{M}$. *In particular, if scalars* $a, b, c, d$ *satisfy* $ca+db = 1$, *then* $(\{af_n+bg_n\}_n, \{c\tau_n+d\omega_n\}_n)$ *is a Lipschitz $p$-SF for* $\mathcal{M}$.

*Proof.* We find

$$\begin{aligned}\theta_{fA+gB}x &= \{(f_nA + g_nB)(x)\}_n = \{f_n(Ax)\}_n + \{g_n(Bx)\}_n \\ &= \theta_f(Ax) + \theta_g(Bx), \quad \text{for all } x \in \mathcal{M}\end{aligned}$$

and

$$\begin{aligned}\theta_{C\tau+D\omega}\{a_n\}_n &= \sum_{n=1}^{+\infty} a_n(C\tau_n + D\omega_n) \\ &= C\theta_\tau\{a_n\}_n + D\theta_\omega\{a_n\}_n, \quad \text{for all } \{a_n\}_n \in \ell^p(\mathbb{N}).\end{aligned}$$

So

$$\begin{aligned}S_{fA+gB,C\tau+D\omega} &= \theta_{C\tau+D\omega}\theta_{fA+gB} = (C\theta_\tau + D\theta_\omega)(\theta_fA + \theta_gB) \\ &= C\theta_\tau\theta_fA + C\theta_\tau\theta_gB + D\theta_\omega\theta_fA + D\theta_\omega\theta_gB \\ &= CS_{f,\tau}A + 0 + 0 + DS_{g,\omega}B = CI_{\mathcal{M}}A + DI_{\mathcal{M}}B = I_{\mathcal{M}}. \qquad \square\end{aligned}$$

We use Theorem 2.5 to relate three notions duality, similarity and orthogonality.

**Proposition 2.3.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz $p$-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then the canonical dual* $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ *is the only dual Lipschitz $p$-ASF that is similar to* $(\{f_n\}_n, \{\tau_n\}_n)$.

*Proof.* Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a Lipschitz $p$-ASF for $\mathcal{M} \subseteq \mathcal{X}$ which is both similar and dual for $(\{f_n\}_n, \{\tau_n\}_n)$. Then there exist invertible bi-Lipschitz maps $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ such that $g_n = f_nT_{f,g}, \omega_n = T_{\tau,\omega}\tau_n,$ for all $n \in \mathbb{N}$. Theorem 2.5 then gives

$$T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g = S_{f,\tau}^{-1}I_{\mathcal{M}} = S_{f,\tau}^{-1} \text{ and } T_{\tau,\omega} = \theta_\omega\theta_fS_{f,\tau}^{-1} = I_{\mathcal{M}}S_{f,\tau}^{-1} = S_{f,\tau}^{-1}.$$

Hence, $(\{g_n\}_n, \{\omega_n\}_n)$ is the canonical dual for $(\{f_n\}_n, \{\tau_n\}_n)$. $\square$

**Proposition 2.4.** *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *and* $(\{g_n\}_n, \{\omega_n\}_n)$ *be two similar Lipschitz $p$-ASFs for* $\mathcal{M} \subseteq \mathcal{X}$. *Then* $(\{f_n\}_n, \{\tau_n\}_n)$ *is not orthogonal for* $(\{g_n\}_n, \{\omega_n\}_n)$.

*Proof.* Since $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ similar, there exist invertible bi-Lipschitz maps $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ such that $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n,$ for all $n \in \mathbb{N}$. Theorem 2.5 then says $\theta_g = \theta_f T_{f,g}$, $\theta_\omega = T_{\tau,\omega}\theta_\tau$. Therefore,

$$\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g} \neq 0.$$

Orthogonality condition demands $\theta_\tau\theta_g = 0$ whereas above equation says it is not true. □

Another use of orthogonal frames is to take direct sum. Given Lipschitz maps $f, g : \mathcal{M} \to \mathbb{K}$, we define $f \oplus g : \mathcal{M} \oplus \mathcal{M} \ni x \oplus y \mapsto f(x) + g(y) \in \mathbb{K}$.

**Theorem 2.7.** *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two Lipschitz $p$-ASFs for $\mathcal{M} \subseteq \mathcal{X}$ which are orthogonal. Then $(\{f_n \oplus g_n\}_n, \{\tau_n \oplus \omega_n\}_n)$ is a Lipschitz $p$-ASF for $\mathcal{M} \oplus \mathcal{M} \subseteq \mathcal{X} \oplus \mathcal{X}$.*

*Proof.* Let $x \oplus y \in \mathcal{M} \oplus \mathcal{M}$. Then,

$$\begin{aligned}
S_{f\oplus g,\tau\oplus\omega}(x \oplus y) &= \sum_{n=0}^{+\infty}(f_n \oplus g_n)(x \oplus y)(\tau_n \oplus \omega_n)\\
&= \left(\sum_{n=0}^{+\infty} f_n(x)\tau_n + \sum_{n=0}^{+\infty} g_n(x)\tau_n\right) \oplus \left(\sum_{n=0}^{+\infty} f_n(x)\omega_n + \sum_{n=0}^{+\infty} g_n(x)\omega_n\right)\\
&= (S_{f,\tau}x + 0) \oplus (0 + S_{g,\omega}y) = (S_{f,\tau} \oplus S_{g,\omega})(x \oplus y).
\end{aligned}$$ □

## 3. An Open Problem

Motivated from the approximation properties of Banach spaces (Schauder basis problem) [6,12] and from the failure of atomic decompositions for (even separable) Banach spaces (see [7]), we formulate the following interesting (high-end) problem: Can anyone classify subsets of a Banach space having a Lipschitz $p$-ASF, for some $1 \leq p < +\infty$? In particular, does every subset of a Banach space have Lipschitz $p$-ASF, for some $1 \leq p < +\infty$?

## 4. Conclusions

In the literature, only frames coming from inner products and linear functionals are studied. The paper [22] is the first one to introduce and make a systematic study of frames for metric spaces by using Lipschitz functions. In this paper, we define a class of non-linear frames for subsets (need not be subspaces) of Banach spaces which can be characterized using standard Schauder basis and Lipschitz functions on sequence spaces. We derived Holub's theorem [20] in non-linear form. Duals and similar frames in non-linear form are also characterized.

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[1]Statistics and Mathematics Unit
Indian Statistical Institute, Bangalore Centre
Karnataka 560 059 India
*Email address*: kmaheshak@gmail.com

[2]Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
Mangaluru 575 025, India
*Email address*: sam@nitk.edu.in

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# TOPOLOGICAL DEGREE METHOD FOR A CLASS OF Ψ-CAPUTO FRACTIONAL DIFFERENTIAL LANGEVIN EQUATION

HAMID LMOU[1*], KHALID HILAL[1], AND AHMED KAJOUNI[1]

Abstract. This paper deals with the existence and uniqueness of solution for a new class of Ψ-Caputo fractional differential Langevin equation. The suggested study is based on some basic definitions of topological degree theory and fractional calculus. We established the existence result by using the topological degree method for condensing maps, and by means of Banach's fixed point theorem we obtained the uniqueness result. As application, we give an illustrative example to demonstrate our theoretical result.

## 1. Introduction

Newly, fractional differential equations have attracted the interest of many mathematicians, because it can represent and verified to be effective modeling of many phenomena in several fields of science as physics, mechanics, biology, chemistry, and control theory, and other domains for exemple, see [8, 13, 16, 19, 27, 33].

In 1908 Paul Langevin, introduced the Langevin equation of the form $m\frac{d^2w}{d\tau^2} = -\lambda\frac{dw}{d\tau} + \eta(\tau)$ where, $\frac{dw}{d\tau}$ is the velocity of the particle, and $m$ is its mass and a noise term $\eta(\tau)$ representing the effect of the collisions with the molecules of the fluid. For the removal of the noise term, mathematicians used fractional order differential equations, for this reason it is very important to study Langevin equations via fractional derivatives, for more details see [3, 4, 22–25, 28, 31].

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There are several definitions of fractional integrals and derivatives, the popular definitions are the Riemann-Liouville and the Caputo fractional derivatives, in [15], Almeida introduce the generalization of these derivatives under the name of $\Psi$-Caputo fractional derivative, for more details for $\Psi$-Caputo fractional derivative, we direct readers to the papers [2,17,20,21,29,30]. Furthermore, distinct version of fixed point theorems are commonly utilized to prove the existence and uniqueness of solutions for various classes of fractional differential equations, Isaia [12] proved a new fixed theorem that was obtained via coincidence degree theory for condensing operators. This fixed point theorem due to Isaia was utilized by researchers to establish the existence of solutions for several classes of nonlinear differential equations [1,5,11,14,18].

Recently, Baitiche et al. [6], discussed the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the $\Psi$-Caputo fractional derivative with multi-point boundary conditions based on the technique of topological degree theory for condensing maps and the Banach contraction principle. Faree and Panchal [9], investigated the existence and uniqueness of solutions to boundary value problems involving the Caputo fractional derivative in Banach space by topological structures with some appropriate conditions. Hilal et al. [10], discussed the existence and uniqueness of solution for a boundary value problem for the Langevin equation and inclusion, based on Krasnoselskii's fixed point theorem, Banach's contraction principle and Leray-Schauder's alternative. Rizwan [26], considered a non local boundary value problem of nonlinear fractional Langevin equation with non-instantaneous impulses. Baitiche et al. [32], proved the Ulam-Hyers stability of solutions for a new form of nonlinear fractional Langevin differential equations involving two fractional orders in the $\Psi$-Caputo sense.

Motivated by the mentioned works, and by using topological degree methods we investigate the existence and uniqueness result for the following problem

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{p;\Psi}\left[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi}+\lambda\right]w(\tau)=\varphi(\tau,w(\tau)), \quad \tau\in\Upsilon:=[a,b], \\ w(a)=0, \quad w'(a)=0, \quad w(b)=\sum_{i=1}^{n}\iota_i\mathfrak{I}_{a^+}^{\beta_i;\Psi}w(\kappa_i). \end{cases} \tag{1.1}$$

The originality of this work is studing a new and a challenging case of fractional derivative named the $\Psi$-Caputo fractional derivative [15], this kind of fractional derivative generalize the well-known fractional derivatives, for different values of function $\Psi$ such as the following.

- ⋆ If $\Psi(\tau)=\tau$, then Problem (1.1) reduces to Caputo-type fractional derivative.
- ⋆ If $\Psi(\tau)=\log(\tau)$, then Problem (1.1) reduces to Caputo-Hadamard-type fractional derivative.
- ⋆ If $\Psi(\tau)=\tau^\rho$, then Problem (1.1) reduces to Caputo-Katugampola-type fractional derivative.

The rest of the paper is organized as follows. In Section 2, we recall some theorems, notations, lemmas, and definitions from fractional calculus and important results of topological degree method that will be used throughout this study. In Section 3, based on the application of topological degree method, we discuss the existence result for the problem (1.1), and by making use of Banach's contraction principle we prove the uniqueness of solution. In Section 4, we give an example to support the main result.

## 2. Preliminaries

In this section, we introduce some definitions, lemmas and useful notations that we can used throughout this paper.

Denote by $X$ a Banach space and $\Gamma_X$ the class of non-empty and bounded subsets of $X$. $\mathcal{C}(\Upsilon,\mathbb{R})$ denote the Banach space of all continuous functions from $\Upsilon$ into $\mathbb{R}$ with the norm defined by $\|\varphi\| = \sup_{\tau\in\Upsilon}\{|\varphi(\tau)|\}$. We denote by $\mathcal{C}^n(\Upsilon,\mathbb{R})$ the $n$-times absolutely continuous functions given by $\mathcal{C}^n(\Upsilon,\mathbb{R}) = \Big\{\varphi : \Upsilon \to \mathbb{R} : \varphi^{(n-1)} \in \mathcal{C}(\Upsilon,\mathbb{R})\Big\}$. $\mathcal{B}_\rho(0)$ denote the closed ball centered at 0 with radius $\rho$.

**Definition 2.1** ([15]). For $p>0$, $\varphi \in \mathbb{L}^1(\Upsilon,\mathbb{R})$ and $\Psi \in \mathcal{C}^n(\Upsilon,\mathbb{R})$, with $\Psi'(\tau)>0$, for all $\tau\in\Upsilon$, the $\Psi$-Riemann-Liouville fractional integral of order $p$ of a function $\varphi$ is defined by

$$\mathfrak{I}^{p;\Psi}_{a^+}\varphi(\tau) = \frac{1}{\Gamma(p)}\int_a^\tau \Psi'(\tau)(\Psi(\tau)-\Psi(s))^{p-1}\varphi(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ represents the gamma function.

**Definition 2.2** ([15]). For $p>0$, $\varphi \in \mathcal{C}^{n-1}(\Upsilon,\mathbb{R})$ and $\Psi \in \mathcal{C}^n(\Upsilon,\mathbb{R})$, with $\Psi'(\tau)>0$, for all $\tau\in\Upsilon$, the $\Psi$-Caputo fractional derivative of order $p$ of a function $\varphi$ is defined by

$$^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}\varphi(\tau) = \mathfrak{I}^{n-p;\Psi}_{a^+}\varphi^{[k]}_\Psi(\tau) = \frac{1}{\Gamma(n-p)}\int_a^\tau \Psi'(\tau)(\Psi(\tau)-\Psi(s))^{n-p-1}\varphi^{[k]}_\Psi(s)ds,$$

where $\varphi^{[k]}_\Psi(\tau) = \left(\frac{1}{\Psi'(\tau)}\cdot\frac{d}{d\tau}\right)^n$, $n-1<p<n$, $n=[p]+1$ and $[p]$ denotes the integer part of the real number $p$.

**Lemma 2.1** ([15]). *Let $p,q>0$. Then we have the following semigroup property given by*

$$\mathfrak{I}^{p;\Psi}_{a^+}\mathfrak{I}^{q;\Psi}_{a^+}\varphi(\tau) = \mathfrak{I}^{p+q;\Psi}_{a^+}\varphi(\tau), \quad \tau>a. \tag{2.2}$$

**Proposition 2.1** ([15]). *Let $p>0$, $\upsilon>0$ and $\tau\in\Upsilon$. Then*

(i) $\mathfrak{I}^{p;\Psi}_{a^+}(\Psi\tau)-\Psi(a))^{\upsilon-1} = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+p)}(\Psi(\tau)-\Psi(a))^{\upsilon+p-1}$;

(ii) $^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}(\Psi(\tau)-\Psi(a))^{\upsilon-1} = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-p)}(\Psi(\tau)-\Psi(a))^{\upsilon-p-1}$;

(iii) $^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}(\Psi(\tau)-\Psi(a))^{k} = 0$, *for all $k<n\in\mathbb{N}$.*

**Lemma 2.2** ([15]). *If $\varphi\in\mathcal{C}^n(\Upsilon,\mathbb{R})$, $n-1<p<n$, then*

$$\mathfrak{I}^{p;\Psi}_{a^+}(^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}\varphi)(\tau) = \varphi(\tau) - \sum_{k=0}^{n-1}\frac{\varphi^{[k]}_\Psi}{k!}(\Psi(\tau)-\Psi(a))^k, \tag{2.3}$$

*for all* $\tau \in \Upsilon$, *where* $\varphi_{\Psi}^{[k]}(\tau) := \left(\frac{1}{\Psi'(\tau)} \cdot \frac{d}{d\tau}\right)^k \varphi(\tau)$.

**Definition 2.3** ([7]). The Kuratowski measure of non-compactness is the mapping $\vartheta : \Gamma_X \to \mathbb{R}_+$ defined by $\vartheta(B) = \inf\{\xi > 0 :$ B can be covered by finitely many sets with diameter less or equal to $\xi\}$.

**Proposition 2.2** ([7]). *The Kuratowski measure of noncompactness* $\vartheta$ *satisfies the following properties*

1. $\vartheta(A) = 0$ *if and only if* $A$ *is relatively compact;*
2. $A \subset B \Rightarrow \vartheta(A) \leq \vartheta(B)$*;*
3. $\vartheta(A) = \vartheta(\overline{A}) = \vartheta(\mathrm{conv}(A))$, *where* $\overline{A}$ *and* $\mathrm{conv}(A)$ *denote the closure and the convex hull of* $A$*, respectively;*
4. $\vartheta(A + B) \leq \vartheta(A) + \vartheta(B)$*;*
5. $\vartheta(kA) = |k|\vartheta(A)$, $k \in \mathbb{R}$.

**Definition 2.4.** Let $\mathcal{F} : A \to X$ be a continuous bounded map. The operator $\mathcal{F}$ is said to be $\vartheta$-Lipschitz if there exists $l \geq 0$ such that

$$\vartheta(\mathcal{F}(B)) < l\vartheta(B), \quad \text{for every } B \subset A. \tag{2.4}$$

Furthermore, if $l < 1$, then $\mathcal{F}$ is a strict $\vartheta$-contraction.

**Definition 2.5.** $\mathcal{F} : A \to X$ is called $\vartheta$-condensing if

$$\vartheta(\mathcal{F}(B)) < \vartheta(B), \tag{2.5}$$

for every bounded and nonprecompact subset $B$ of $A$, with $\vartheta(B) > 0$.

**Definition 2.6.** We say that the function $\mathcal{F} : A \to X$ is Lipschitz if there exists $l > 0$ such that

$$\|\mathcal{F}(w) - \mathcal{F}(v)\| \leq l\|u - v\|, \quad \text{for all } w, v \in A, \tag{2.6}$$

Furthermore, if $l < 1$, then $\mathcal{F}$ is a strict contraction.

**Proposition 2.3** ([7, 12]). *If* $\mathcal{F}, \mathcal{Y} : A \to X$ *are* $\vartheta$*-Lipschitz mapping with constants* $l_1$ *and* $l_2$ *respectively, then* $\mathcal{F} + \mathcal{Y} : A \to X$ *is* $\vartheta$*-Lipschitz mapping with constant* $l_1 + l_2$.

**Proposition 2.4** ([7, 12]). *If* $\mathcal{F} : A \to X$*, is compact, then* $\mathcal{F}$ *is* $\vartheta$*-Lipschitz mapping with constant* $l = 0$.

**Proposition 2.5** ([7, 12]). *If* $\mathcal{F} : A \to X$ *is Lipschitz mapping with constant* $l$*, then* $\mathcal{F}$ *is* $\vartheta$*-Lipschitz mapping with the same constant* $l$.

**Theorem 2.1** ( [12]). *Let* $\mathcal{W} : A \to X$ *be* $\vartheta$*-condensing and*

$$\Pi_\epsilon = \{w \in X : w = \epsilon\mathcal{W}w, \text{ for some } 0 \leq \epsilon \leq 1\}. \tag{2.7}$$

*If* $\Pi_\epsilon$ *is a bounded set in* $X$*, so there exists* $r > 0$*, such that* $\Pi_\epsilon \in \mathcal{B}_r(0)$*, then the degree*

$$\deg\left(I - \epsilon\mathcal{W}, \mathcal{B}_r(0), 0\right) = 1, \quad \text{for all } \epsilon \in [0, 1]. \tag{2.8}$$

*Consequently, $\mathcal{W}$ has at least one fixed point and the set of the fixed points of $\mathcal{W}$ lies in $\mathcal{B}_r(0)$.*

## 3. Main Result

**Definition 3.1.** A function $w \in \mathcal{C}(\Upsilon, \mathbb{R})$ is said to be a solution of Problem (1.1), if $w$ satisfies the equation ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{p;\Psi}\left[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} + \lambda\right] w(\tau) = \varphi(t, w(\tau))$, a.e. on $\Upsilon$ with the conditions $w(a) = 0$, $w'(a) = 0$, $w(b) = \sum\limits_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{\beta_i;\Psi} w(\kappa_i)$.

**Lemma 3.1.** *Let $a \geq 0$, $0 < p \leq 1$, $0 < q \leq 2$ and $h \in \mathcal{C}(\Upsilon, \mathbb{R})$. Then the function $w$ is a solution of the following boundary value problem*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{p;\Psi}\left[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} + \lambda\right] w(\tau) = h(\tau), \quad \tau \in \Upsilon := [a, b], \\ w(a) = 0, \quad w'(a) = 0, \quad w(b) = \sum\limits_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{\beta_i;\Psi} w(\kappa_i), \quad a < \kappa_i < b, \end{cases} \tag{3.1}$$

*if and only if*

$$\begin{aligned} w(\tau) =& \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + \frac{(\Psi(\tau) - \Psi(a))^q}{\Delta\Gamma(q+1)} \Big( \mathfrak{I}_{a^+}^{p+q;\Psi} h(b) \\ & - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(b) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} h(\kappa_i) + \lambda \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{q+\beta_i;\Psi} w(\kappa_i) \Big), \end{aligned} \tag{3.2}$$

*where*

$$\Delta = \sum_{i=1}^{n} \iota_i \frac{(\Psi(\kappa_i) - \Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)} - \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} \neq 0. \tag{3.3}$$

*Proof.* Applying the Ψ-Riemann-Liouville fractional integral of order $p$ to both sides of (3.1) and by using Lemma 2.2 we get

$${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} w(\tau) + \lambda w(\tau) = \mathfrak{I}_{a^+}^{p;\Psi} h(\tau) + d_0, \quad \tau \in \Upsilon, \tag{3.4}$$

where $d_0$ is a constant, applying the Ψ-Riemann-Liouville fractional integral of order $q$ to both sides of (3.4) we obtain by using Lemma 2.2

$$w(\tau) = \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + d_0 \frac{(\Psi(\tau) - \Psi(a))^q}{\Gamma(q+1)} + d_1 + d_2(\Psi(\tau) - \Psi(a)), \tag{3.5}$$

where $d_1$ and $d_2$ are constants, next by using the boundary condition $w(a) = 0$ in (3.5) we obtain that $d_1 = 0$. Then, we get

$$w(\tau) = \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + d_0 \frac{(\Psi(\tau) - \Psi(a))^q}{\Gamma(q+1)} + d_2(\Psi(\tau) - \Psi(a)). \tag{3.6}$$

It follows that

$$w'(\tau) = \left(\mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau)\right)' - \lambda \left(\mathfrak{I}_{a^+}^{q;\Psi} w(\tau)\right)' + d_0 \frac{q\Psi'(\tau)(\Psi(\tau) - \Psi(a))^{q-1}}{\Gamma(q+1)} + d_2 \Psi'(\tau), \tag{3.7}$$

by using $w'(a)=0$, in (3.7) we find $d_2=0$ (Definition 2.2: $\Psi'(\tau)>0$, for all $\tau\in\Upsilon$). Then,

$$w(\tau)=\mathfrak{I}_{a^+}^{p+q;\Psi}h(\tau)-\lambda\mathfrak{I}_{a^+}^{q;\Psi}w(\tau)+d_0\frac{(\Psi(\tau)-\Psi(a))^q}{\Gamma(q+1)}. \tag{3.8}$$

By making use of the boundary condition $w(b)=\sum_{i=1}^{n}\iota_i w(\kappa_i)$, in (3.8) we find

$$d_0=\frac{1}{\Delta}\left(\mathfrak{I}_{a^+}^{p+q;\Psi}h(b)-\lambda\mathfrak{I}_{a^+}^{q;\Psi}w(b)-\sum_{i=1}^{n}\iota_i\mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}h(\kappa_i)+\lambda\sum_{i=1}^{n}\iota_i\mathfrak{I}_{a^+}^{q+\beta_i;\Psi}w(\kappa_i)\right). \tag{3.9}$$

Substituting the value of $d_0$ in (3.8) we obtain the integral equation in (3.2). The converse follows by direct computation. □

In this part, we deal with the existence and uniqueness of solution for the problem (1.1), for that to simplify the computations, we use the following notation

$$\begin{aligned}\Lambda_1=&\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}\\&+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\left(\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)}\right),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Lambda_2=&|\lambda|\Bigg\{\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}\\&+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\left[\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right]\Bigg\}.\end{aligned} \tag{3.11}$$

Assume that the following hold.

$(H_1)$ There exists a constant $\mathcal{L}_\varphi>0$ such that

$$|\varphi(\tau,w)-\varphi(\tau,v)|\le\mathcal{L}_\varphi|w-v|,\quad \text{for each } \tau\in\Upsilon \text{ and } w,v\in\mathcal{C}(\Upsilon,\mathbb{R}). \tag{3.12}$$

$(H_2)$ There exist two constants $\mathcal{K}_\varphi,\mathcal{N}_\varphi>0$ and $\alpha\in(0,1)$ such that

$$|\varphi(\tau,w)|\le\mathcal{K}_\varphi|w|^\alpha+\mathcal{N}_\varphi,\quad \text{for each } \tau\in\Upsilon \text{ and } w\in\mathcal{C}(\Upsilon,\mathbb{R}). \tag{3.13}$$

From Lemma 3.1 we define the operators $\mathcal{F},\mathcal{Y}:\mathcal{C}(\Upsilon,\mathbb{R})\to\mathcal{C}(\Upsilon,\mathbb{R})$ by

$$\begin{aligned}\mathcal{F}w(\tau)=&\mathfrak{I}_{a^+}^{p+q;\Psi}\varphi(\tau,w(\tau))+\frac{(\Psi(\tau)-\Psi(a))^q}{\Delta\Gamma(q+1)}\\&\times\left(\mathfrak{I}_{a^+}^{p+q;\Psi}\varphi(b,w(b))-\sum_{i=1}^{n}\iota_i\mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}\varphi(\kappa_i,w(\kappa_i))\right),\quad \tau\in\Upsilon,\end{aligned} \tag{3.14}$$

and

$$\mathcal{Y}w(\tau)=-\lambda\mathfrak{I}_{a^+}^{q;\Psi}w(\tau)+\frac{(\Psi(\tau)-\Psi(a))^q}{\Delta\Gamma(q+1)}\left(-\lambda\mathfrak{I}_{a^+}^{q;\Psi}w(b)+\lambda\sum_{i=1}^{n}\iota_i\mathfrak{I}_{a^+}^{q+\beta_i;\Psi}w(\kappa_i)\right), \tag{3.15}$$

then, the fractional integral equation (3.2) can be written as follows

$$\mathcal{W}w(\tau)=\mathcal{F}w(\tau)+\mathcal{Y}w(\tau),\quad \tau\in\Upsilon. \tag{3.16}$$

**Theorem 3.1.** *Suppose that* $(H_1)$ *and* $(H_2)$ *are satisfied, then Problem* (1.1) *has at least one solution* $w\in\mathcal{C}(\Upsilon,\mathbb{R})$ *as long as* $\Lambda_2<1$. *Moreover, the set of the solution of Problem* (1.1) *is bounded in* $\mathcal{C}(\Upsilon,\mathbb{R})$.

As a way to prove Theorem 3.1, we will demonstrate it in several lemmas.

**Lemma 3.2.** $\mathcal{Y}$ *is* $\vartheta$*-Lipschitz with the constant* $\Lambda_2$, *where* $\Lambda_2$ *is given by* (3.11).

*Proof.* Let $w,v\in\mathcal{C}(\Upsilon,\mathbb{R})$, then we get

$$\begin{aligned}
|\mathcal{Y}w(\tau)-\mathcal{Y}v(\tau)|\leq&|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(\tau)-v(\tau)|+\frac{(\Psi(\tau)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\Big(|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(b)-v(b)|\\
&+|\lambda|\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{a^+}^{q+\beta_i;\Psi}|w(\kappa_i)-v(\kappa_i)|\Big)\\
\leq&|\lambda|\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}|w(\tau)-v(\tau)|+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\Bigg(|\lambda|\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}|w(b)-v(b)|\\
&+|\lambda|\sum_{i=1}^{n}|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}|w(\kappa_i)-v(\kappa_i)|\Bigg)\\
\leq&|\lambda|\Bigg\{\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\left(\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\sum_{i=1}^{n}|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right)\Bigg\}\|w-v\|,\\
\leq&\Lambda_2\|w-v\|,
\end{aligned}$$

where $\Lambda_2$ is given by (3.11). Taking the supremum over $\tau$, we obtain

$$\|\mathcal{Y}w-\mathcal{Y}v\|\leq\Lambda_2\|w-v\|.$$

Then, $\mathcal{Y}$ is Lipschitz with the constant $\Lambda_2$ and by Proposition 2.5, $\mathcal{Y}$ is $\vartheta$-Lipschitz with the same constant $\Lambda_2$. □

**Lemma 3.3.** $\mathcal{F}$ *is continuous and satisfies the inequality given below*

$$\|\mathcal{F}w\|\leq\Lambda_1\Big(\mathcal{K}_\varphi\|w\|^\alpha+\mathcal{N}_\varphi\Big), \tag{3.17}$$

*where* $\Lambda_1$ *is given by* (3.10).

*Proof.* Let $w_n,w\in\mathcal{C}(\Upsilon,\mathbb{R})$ such that $w_n$ converging to $w$ in $\mathcal{C}(\Upsilon,\mathbb{R})$, implies that there exists $\mu>0$ such that $\|w_n\|\leq\mu$ for all $n\geq1$, in addition by taking limits, we get $\|w\| \leq \mu$. By using the fact that $\varphi$ is continuous and $(H_2)$, for every $\tau \in \Upsilon$ we have

$$|\varphi(\tau, w_n(\tau)) - \varphi(\tau, w(\tau))| \leq |\varphi(\tau, w_n(\tau))| + |\varphi(\tau, w(\tau)| \leq 2(\mathcal{K}_\varphi \mu^\alpha + \mathcal{N}_\varphi).$$

The function $s \mapsto 2(\mathcal{K}_\varphi \mu^\alpha + \mathcal{N}_\varphi)$ is integrable for $s \in [0, \tau]$, $\tau \in \Upsilon$ by making use of Lebesgue dominated convergence theorem we get

$$\begin{aligned}|\mathcal{F}w_n(\tau) - \mathcal{F}w(\tau)| \leq& \mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(\tau, w_n(\tau)) - \varphi(\tau, w(\tau))| + \frac{(\Psi(\tau) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ &\times \Bigg(\mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(b, w_n(b)) - \varphi(b, w(b))| \\ &+ \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} |\varphi(\kappa_i, w_n(\kappa_i)) - \varphi(\kappa_i, w(\kappa_i))|\Bigg) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Then, $\|\mathcal{F}w_n - \mathcal{F}w\| \to 0$ as $n \to \infty$, implies that $\mathcal{F}$ is continuous.

In addition, for every $\tau \in \Upsilon$ we get

$$\begin{aligned}|\mathcal{F}w(\tau)| \leq& \mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(\tau, w(\tau))| + \frac{(\Psi(\tau) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ &\times \left(\mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(b, w(b))| + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} |\varphi(\kappa_i, w(\kappa_i)|\right), \\ \leq& \Bigg\{ \frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ &\times \left( \frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)} \right) \Bigg\} \Big(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi\Big),\end{aligned}$$

implies that $\|\mathcal{F}w\| \leq \Lambda_1 \Big(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi\Big)$. □

**Lemma 3.4.** *$\mathcal{F}$ is compact, as a consequence $\mathcal{F}$ is $\vartheta$-Lipschitz with zero constant.*

*Proof.* To prove that $\mathcal{F}$ is compact, let $\mathcal{M}$ be a bounded set, such that $\mathcal{M} \subset \mathcal{B}_\rho$. It remain to prove that $\mathcal{F}(\mathcal{M})$ is relatively compact in $\mathcal{C}(\Upsilon, \mathbb{R})$. For this reason let $w \in \mathcal{M} \subset \mathcal{B}_\rho$ and by making use of (3.17), we obtain

$$\|\mathcal{F}w\| \leq \Lambda_1 \Big(\mathcal{K}_\varphi \rho^\alpha + \mathcal{N}_\varphi\Big) := \upsilon. \tag{3.18}$$

Then, $\mathcal{F}(\mathcal{M}) \subset \mathcal{B}_\upsilon$, as a consequence, $\mathcal{F}(\mathcal{M})$ is bounded.

For the equicontinuity of $\mathcal{F}$, let $\tau_1, \tau_2 \in \Upsilon$ with $\tau_1 < \tau_2$ and for $w \in \mathcal{M}$ we have

$$\begin{aligned}&|\mathcal{F}w(\tau_2) - \mathcal{F}w(\tau_1)| \\ \leq& \mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(\tau_2, w(\tau_2)) - \varphi(\tau_1, w(\tau 1))| + \frac{(\Psi(\tau_2) - \Psi(a))^q - (\Psi(\tau_1) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ &\times \left(\mathfrak{I}_{a^+}^{p+q;\Psi} |\varphi(b, w(b))| + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} |\varphi(\kappa_i, w(\kappa_i)|\right)\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{\left(\mathcal{K}_\varphi \rho^\alpha+\mathcal{N}_\varphi\right)}{\Gamma(p+q)}\Big| \int_a^{\tau_1} \Psi'(s)\Big((\Psi(\tau_2)-\Psi(s))^{p+q-1} \\
& -(\Psi(\tau_1)-\Psi(s))^{p+q-1}\Big) ds+\int_{\tau_1}^{\tau_2} \Psi'(s)(\Psi(\tau_2)-\Psi(s))^{p+q-1} ds\Big| \\
& +\frac{(\Psi(\tau_2)-\Psi(a))^q-\Psi(\tau_1)-\Psi(a))^q}{|\Delta| \Gamma(q+1)} \\
& \times\left(\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\sum_{i=1}^n|\iota_i| \frac{(\Psi(\kappa_i)-\Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)}\right)\Big(\mathcal{K}_\varphi \rho^\alpha+\mathcal{N}_\varphi\Big),
\end{aligned}
$$

By using the continuity of the function $\Psi$, the right hand side of the above inequality tends to 0 as $\tau_2$ tends to $\tau_1$, this implies that $\mathcal{F}(\mathcal{M})$ is equicontinuous. It follows by using Arzelá-Ascoli's theorem that $\mathcal{F}$ is compact as a consequence of Proposition 2.4, $\mathcal{F}$ is $\vartheta$-Lipschitz with zero constant. □

Since all the conditions are satisfied we demonstrate the validity of our main result as Theorem 3.1.

*Proof of Theorem* 3.1. Let $\mathcal{F}$, $\mathcal{Y}$ and $\mathcal{W}$, be the operators given by (3.14), (3.15), (3.16), respectively. These operators are continuous and bounded. Furthermore, by making use of Lemma 3.2 , $\mathcal{Y}$ is is $\vartheta$-Lipschitz with constant $\Lambda_2$, and by using Lemma 3.4, $\mathcal{F}$ is $\vartheta$-Lipschitz with constant zero, hence $\mathcal{W}$ is a strict $\vartheta$-contraction with constant $\Lambda_2$, finally $\mathcal{W}$ is $\vartheta$-condensing because $\Lambda_2 < 1$.

Next, let us consider the following set

$$\Pi_\epsilon=\{w \in X: w=\epsilon \mathcal{W} w, \text{ for some } 0 \leq \epsilon \leq 1\}. \tag{3.19}$$

It remains to show that the set $\Pi_\epsilon$ is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$, for that let $w \in \Pi_\epsilon$ then we have $w=\epsilon \mathcal{W} w=\epsilon(\mathcal{F} w+\mathcal{Y} w)$. It follows, by using Lemma 3.3 and 3.2,

$$
\begin{aligned}
\|w\| &= \epsilon\|\mathcal{F} w+\mathcal{Y} w\|, \\
&\leq\|\mathcal{F} w\|+\|\mathcal{Y} w\| \leq \Lambda_1\Big(\mathcal{K}_\varphi\|w\|^\alpha+\mathcal{N}_\varphi\Big)+\Lambda_2\|w\| \leq \frac{\Lambda_1\Big(\mathcal{K}_\varphi\|w\|^\alpha+\mathcal{N}_\varphi\Big)}{1-\Lambda_2},
\end{aligned}
$$

where $\Lambda_1$ and $\Lambda_2$ are given by (3.10) and (3.11), respectively. Then, the set $\Pi_\epsilon$ is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$. If the set $\Pi_\epsilon$ is not bounded, then we suppose that $\chi:=\|w\| \rightarrow +\infty$ and by using the above inequality we get

$$1 \leq \lim_{\chi \rightarrow+\infty} \frac{\Lambda_1\Big(\mathcal{K}_\varphi \chi^\alpha+\mathcal{N}_\varphi\Big)}{\chi(1-\Lambda_2)}=0, \tag{3.20}$$

which is a contradiction. Thus by using Theorem 2.1, $\mathcal{W}$ has at least one fixed point which is the solution of Problem (1.1). Moreover, the set of solution of Problem (1.1) is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$. □

To deal with the uniqueness of solution for Problem (1.1), we use Banach's contraction principle.

**Theorem 3.2.** *Assume that* $(H_1)$ *hold. If* $\mathcal{L}_\varphi\Lambda_1+\Lambda_2<1$*, then Problem* (1.1) *has a unique solution on* $\mathcal{C}(\Upsilon,\mathbb{R})$.

*Proof.* For every $w,v\in\mathcal{C}(\Upsilon,\mathbb{R})$ and $\tau\in\Upsilon$ we have

$$\begin{aligned}
&|\mathcal{W}w(\tau)-\mathcal{W}v(\tau)|\\
\leq&|\mathcal{F}w(\tau)-\mathcal{F}v(\tau)+\mathcal{Y}w(\tau)-\mathcal{Y}v(\tau)|\\
\leq&|\mathcal{F}w(\tau)-\mathcal{F}v(\tau)|+|\mathcal{Y}w(\tau)-\mathcal{Y}v(\tau)|\\
\leq&\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(\tau,w(\tau))-\varphi(\tau,v(\tau))|+\frac{(\Psi(\tau)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\Bigg(\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(b,w(b))-\varphi(b,v(b))|\\
&+\sum_{i=1}^n|\iota_i|\mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}|\varphi(\kappa_i,w(\kappa_i))-\varphi(\kappa_i,v(\kappa_i))|\Bigg)\\
&+|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(\tau)-v(\tau)|+\frac{(\Psi(\tau)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\Bigg[|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(b)-v(b)|\\
&+|\lambda|\sum_{i=1}^n|\iota_i|\mathfrak{I}_{a^+}^{q+\beta_i;\Psi}|w(\kappa_i)-v(\kappa_i)|\Bigg]\\
\leq&\Bigg\{\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\left(\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\sum_{i=1}^n|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)}\right)\Bigg\}\mathcal{L}_\varphi|w(\tau)-v(\tau)|\\
&+|\lambda|\Bigg\{\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\left(\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\sum_{i=1}^n|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right)\Bigg\}|w(\tau)-v(\tau)|\\
\leq&(\mathcal{L}_\varphi\Lambda_1+\Lambda_2)\|w-v\|,
\end{aligned}$$

where $\Lambda_1$ and $\Lambda_2$ are given by (3.10) and (3.11), respectively. Then, by taking the supremum over $\tau$, we get $\|\mathcal{W}w-\mathcal{W}v\|\leq(\mathcal{L}_\varphi\Lambda_1+\Lambda_2)\|w-v\|$. Using the fact that $\mathcal{L}_\varphi\Lambda_1+\Lambda_2<1$, it follows that $\mathcal{W}$ is a contraction. Finally, by the Banach fixed point theorem, $\mathcal{W}$ has a unique fixed point which is a unique solution of Problem (1.1). □

## 4. Example

Consider the following problem

$$\begin{cases}{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{1}{2};e^\tau}\left({}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{3}{2};e^\tau}+\frac{1}{5}\right)w(\tau)=\frac{e^{-\tau}}{e^\tau+10}\left(1+\frac{|w(\tau)|}{1+|w(\tau)|}\right), & 0\leq\tau\leq1,\\ w(0)=0,\quad w'(0)=0,\quad w(1)=\frac{3}{5}\mathfrak{I}_{0^+}^{\frac{2}{7};e^\tau}w\left(\frac{1}{4}\right)+\frac{4}{5}\mathfrak{I}_{0^+}^{\frac{2}{5};e^\tau}w\left(\frac{1}{2}\right),\end{cases}\tag{4.1}$$

where $p = \frac{1}{2}$, $q = \frac{3}{2}$, $\lambda = \frac{1}{5}$, $a = 0$, $b = 1$, $\Upsilon = [0, 1]$, $n = 2$, $\iota_1 = \frac{3}{5}$, $\iota_2 = \frac{4}{5}$, $\beta_1 = \frac{2}{7}$, $\beta_2 = \frac{2}{5}$, $\kappa_1 = \frac{1}{4}$, $\kappa_2 = \frac{1}{2}$ and $\Psi(\tau) = e^{\tau}$.

For $(\tau, w) \in \Upsilon \times \mathbb{R}_+$, we define $\varphi(\tau, w) = \frac{e^{-\tau}}{e^{\tau}+10}\Big(1 + \frac{w(\tau)}{1+w(\tau)}\Big)$. Function $\varphi$ is a continuous function, in addition for every $\tau \in \Upsilon$ and for every $w, v \in \mathbb{R}_+$ we have

$$|\varphi(\tau, w) - \varphi(\tau, v)| \leq \left|\frac{e^{-\tau}}{e^{\tau} + 10}\right| \cdot \left|\frac{w - v}{(1 + w)(1 + v)}\right| \leq \frac{1}{11}|w - v|.$$

Then, Hypotheses $(H_1)$ holds with $\mathcal{L}_{\varphi} = \frac{1}{11} > 0$. In addition, for every $\tau \in \Upsilon$ and $w, v \in \mathbb{R}_+$ we have

$$|\varphi(\tau, w)| \leq \left|\frac{e^{-\tau}}{e^{\tau} + 10}\right|\Big(1 + |w|\Big) \leq \frac{1}{11}\Big(1 + |w|\Big).$$

Then, Hypotheses $(H_2)$ holds with $\mathcal{K}_{\varphi} = \mathcal{N}_{\varphi} = \frac{1}{11} > 0$ and $\alpha = 1$ moreover $\Lambda_2 = 0.70263036 < 1$. Finally, all the conditions of Theorem 3.1 are satisfied, consequently Problem (4.1) has at least one solution defined on $[0, 1]$.

To deal with the uniqueness we use the data given above, we get, $|\Delta| \simeq 1.74138859$, $\Lambda_1 = 2.9745821$, $\Lambda_2 = 0.70263036 < 1$, and $\mathcal{L}_{\varphi} = \frac{1}{11} = 0.09$. Then, $\mathcal{L}_{\varphi}\Lambda_1 + \Lambda_2 = 0.09 \times 2.9745821 + 0.70263036 = 0.97034275 < 1$.

Accordingly, by Theorem 3.2, Problem (4.1) has a unique solution on $[0, 1]$.

## 5. Conclusion

In this article, we have studied and discussed the existence and uniqueness of solution for a class of Ψ-Caputo fractional differential Langevin equation. The suggested study is based on some basic definitions of fractional calculus and topological degree theory. The novelty of this work is that it is more general than the works based on the well-known fractional derivatives such as (Caputo fractional derivative, Caputo-Hadamard fractional derivative and Caputo-Katugampola fractional derivative) for different values of function Ψ. Additionally, as a scope of future direction, by studying this specific case of fractional derivative, it can be used as an overview to studying the general case known by the Ψ-Hilfer fractional derivative. In this paper we proved the existence and uniqueness results for Problem (1.1), by using the topological degree method and Banach's fixed point theorem. Finally, a numerical example is presented to clarify the theoretical result.

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[1]Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
Beni Mellal, Morocco

*Corresponding Author
*Email address*: hamid.lmou@usms.ma
*Email address*: hilalkhalid2005@yahoo.fr
*Email address*: kajjouni@gmail.com

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# A FINITE DIFFERENCE TECHNIQUE FOR NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM IN ODES OF ORDER THREE

PRAMOD KUMAR PANDEY

Abstract. In the article, we study the approximate numerical solution to the boundary value problem in ordinary differential equations. In the present article, a third-order two-point boundary value problem is considered for discussion. We developed a second order accurate finite difference method for the approximate numerical solution of the considered problem. We took a special boundary condition; we did not find this boundary condition in the literature. We have discussed the standard convergence analysis of the proposed method. Numerical experiments on linear, nonlinear, and obstacle problems approve the order of accuracy and efficiency of the method.

## 1. Introduction

The present article is aimed at finite difference method for the numerical solution of the third-order boundary value problems of the following form

$$u'''(x) = f(x, u), \quad a < x < b, \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u'(b) = \beta \quad \text{and} \quad u''(b) = \gamma,$$

where $\alpha$, $\beta$ and $\gamma$ are real constants.

The importance of third order boundary value problems is well-established in the physical and natural sciences. The analytical solution to such problems is subject

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to a variety of boundary conditions, and restricted forcing function $f(x,u)$ has been studied by many researchers. But for an arbitrary forcing function $f(x,u)$, it is difficult to find closed-form analytical solution.

The theory on the existence and uniqueness of the solution of higher order boundary value problems can be found in [1]. The existence and uniqueness of the solution, especially for the third order boundary value problem (1.1) in detail are discussed in [2–5] and references therein. So, we have assumed the existence and uniqueness of the solution for problem (1.1) with the considered boundary conditions.

There are a variety of approximation techniques available in the literature for numerical solutions to third-order two-point boundary value problems. But not many researchers considered the problem with the boundary conditions as described in this article. For instance, among a substantial number of works, we refer to work reported by [6, 7] and references therein for numerical approximation of the solution using the finite difference method with different boundary conditions.

Based on the idea in [8], the purpose of the present article is to develop an algorithm using the finite difference method to deal with the numerical solution of the third-order boundary value problems that is accurate, inexpensive, and simple in its computational efforts. We hope the present technique will supplement the existing literature on the solution of third-order boundary value problems.

In this article, we have organised our work as follows. In Section 2, we have derived our finite difference method. In Section 3, we have discussed and analysed the standard convergence of the proposed method. The computational work presented in Section 4 and a discussion on the computational performance of the proposed method is presented in Section 5.

## 2. The Difference Method

We define $a \leq x_0 < x_1 < x_2 < \cdots < x_N \leq b$, $N-1$ number of nodal points in the domain $[a,b]$ of the problem (1.1). In this domain we wish to determine an approximate numerical solution of the problem (1.1), using uniform step length $h$ such that $x_i = a+ih$, $i = 0,1,2,\ldots,N$. We wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1.1) at these discrete nodal points $x_i$, $i = 1,2,\ldots,N$. We denote the numerical approximation of $u(x)$ by $u_i$ and source function $f(x,u(x))$ by $f_i$ at nodes $x = x_i$, $i = 1,2,\ldots,N$. Thus, the boundary value problem (1.1) at node $x = x_i$ may be written as

$$u_i''' = f_i, \quad a \leq x_i \leq b, \tag{2.1}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_N' = \beta \quad \text{and} \quad u_N'' = \gamma.$$

Let we define nodes $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}$, $i = 1,2,\ldots,N-1$, and denote the solution of the problem (1.1) at these nodes as $u_{i\pm\frac{1}{2}}$. Following the idea in [8] and using method of undetermined coefficients and Taylor series expansion, we discretize problem (2.1) at these nodes in $[a, b]$ as follows

$$
\begin{aligned}
(2.2)\quad & 15u_{i-\frac{1}{2}} - 10u_{i+\frac{1}{2}} + 3u_{i+\frac{3}{2}} = 8u_{i-1} + \frac{h^3}{16}(15f_{i-\frac{1}{2}} + 25f_{i+\frac{1}{2}}) + T_i, \quad i = 1,\\
& -u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} = \frac{h^3}{2}(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + T_i, \quad 2 \le i \le N-2,\\
& -u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} = -h^2u''_{i+1} - \frac{h^3}{24}(25f_{i-\frac{3}{2}} - 61f_{i-\frac{1}{2}}) + T_i, \quad i = N-1,\\
& -u_{i-\frac{3}{2}} + u_{i-\frac{1}{2}} = hu'_i - h^2u''_i - \frac{h^3}{24}(36f_{i-\frac{3}{2}} - 49f_{i-\frac{1}{2}}) + T_i, \quad i = N,
\end{aligned}
$$

where $T_i$, $i = 1, 2, \ldots, N$ are truncating terms. Also, in discretization we have used boundary conditions in a natural way.

After neglecting the terms $T_i$ in (2.2), at nodal points $x_{i-\frac{1}{2}}$, $i = 1, 2, \ldots, N$ we will obtain the $N$ linear or nonlinear system of equations in $N$ unknown namely $u_{i-\frac{1}{2}}$ which depends on the source function $f(x, u)$. We have applied Gauss Seidel and Newton-Raphson iterative method to solve system of linear and system of nonlinear equations, respectively.

We compute numerical value of $u_i$, $i = 1, 2, \ldots, N$ by using following second order approximation

$$
u_i = \begin{cases} \frac{1}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}), & 1 \le i \le N-1,\\ \frac{1}{2}(3u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}}), & i = N. \end{cases} \tag{2.3}
$$

## 3. Convergence Analysis

We will consider following linear test equation for convergence analysis of the proposed method (2.2)

$$
u'''(x) = f(x, u), \quad a < x < b, \tag{3.1}
$$

subject to the boundary conditions

$$
u_0 = \alpha, \quad u'_N = \beta \quad \text{and} \quad u''_N = \gamma.
$$

Let $u_{i-\frac{1}{2}}$ and $U_{i-\frac{1}{2}}$ for $i = 1, 2, \ldots, N$ are, respectively approximate and exact solution of (2.2). Let us define

$$
F_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}), \quad i = 1, 2, \ldots, N,
$$

and error that occur in approximate solution

$$
\epsilon_{i-\frac{1}{2}} = U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}, \quad i = 1, 2, \ldots, N.
$$

Let we linearize source function $f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}})$ by application of Taylor series expansion, i.e.,

$$f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}) - f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}) = (U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}})\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, \quad i = 1, 2, \ldots, N.$$

Thus, using these above definitions and boundary condition, we can write an equation in (2.2) as follows

$$15\epsilon_{i-\frac{1}{2}} - 10\epsilon_{i+\frac{1}{2}} + 3\epsilon_{i+\frac{3}{2}} = \frac{h^3}{16}\left(128\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}} + 25\left(\frac{\partial f}{\partial u}\right)_{i+\frac{1}{2}}\right) + T_i, \quad i = 1.$$

Similarly we can write remaining equations in (2.2). Thus, we can write proposed method (2.2) in the matrix form as

$$\mathbf{JE} = \mathbf{T}, \tag{3.2}$$

where $\mathbf{J}$, $\mathbf{E}$ and $\mathbf{T}$ are matrices. These matrices are defined as

$$\mathbf{J} = \mathbf{A} + \mathbf{L},$$

$$\mathbf{A} = \begin{pmatrix} 15 & -10 & 3 & & & 0 \\ -1 & 3 & -3 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -3 & 1 \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 1 \end{pmatrix}_{N\times N},$$

$$\mathbf{L} = -\frac{h^3}{48}\begin{pmatrix} 384\delta_{\frac{1}{2}} & 75\delta_{\frac{3}{2}} & & & & 0 \\ & 24\delta_{\frac{3}{2}} & 24\delta_{\frac{5}{2}} & & & \\ & \ddots & \ddots & & & \\ & & 24\delta_{N-\frac{5}{2}} & 24\delta_{N-\frac{3}{2}} & & \\ & & -50\delta_{N-\frac{5}{2}} & 122\delta_{N-\frac{3}{2}} & & \\ 0 & & & -72\delta_{N-\frac{3}{2}} & 98\delta_{N-\frac{1}{2}} & \end{pmatrix}_{N\times N},$$

where $\delta = \frac{\partial f}{\partial u}$ ,

$$\mathbf{E} = \left(\epsilon_{\frac{1}{2}}, \epsilon_{\frac{3}{2}}, \ldots, \epsilon_{N-\frac{3}{2}}, \epsilon_{N-\frac{1}{2}}\right)^T \quad \text{and} \quad \mathbf{T} = (T_1, T_2, \ldots, T_{N-1}, T_N)^T,$$

where

$$T_i = \begin{cases} -\frac{19}{40}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = 1, \\ \frac{1}{240}h^7 u^{(7)}_{i-\frac{1}{2}}, & 2 \le i \le N-2, \\ -\frac{13}{12}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = N-1, \\ -\frac{27}{40}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = N. \end{cases}$$

Thus, we note from (3.2) that the convergence of the difference method (2.2) depends on matrix $\mathbf{J}$. So, we have determined $\mathbf{A}^{-1} = (a_{i,j})$ explicitly where

$$a_{i,j} = \begin{cases} \frac{(4j-1)+4(i-1)(2j-i)}{8}, & i < j < N, \\ \frac{2i-1}{2}, & i \leq j = N, \\ \frac{1}{8}, & 1 = j \leq i \leq N, \\ \frac{4j^2-1}{8}, & 1 < j < i \leq N, \\ \frac{4j^2-1}{8}, & 1 < j = i < N. \end{cases} \tag{3.3}$$

We observed that $a_{i,j} > 0$ for all $i$ and $j$. Also, we have calculated the row sum of $\mathbf{A}^{-1}$ which are given as

$$R_i = \frac{4i^3 - 12i^2 + 32i - 18}{24} + \frac{(N-1)(2i-1)(2N-2i+1)}{8}.$$

Thus, we have obtained

$$R_N = \max_{1 \leq i \leq N} |R_i| = \frac{4N^3 - N^2 + 23N - 15}{24}. \tag{3.4}$$

Hence, it is easy from (3.4) to prove that

$$||\mathbf{A}^{-1}|| < \frac{(b-a)^3}{6h^3}. \tag{3.5}$$

Let square matrix $\mathbf{M}$ and identity matrix $\mathbf{I}$ have the same order and $||\mathbf{M}|| < 1$. Then square matrix $(\mathbf{I} + \mathbf{M})$ is invertible [9–11] and

$$||(\mathbf{I} + \mathbf{M})^{-1}|| < \frac{1}{1 - ||\mathbf{M}||}.$$

Let us assume $||\mathbf{A}^{-1}\mathbf{L}|| < 1$. Thus, from (3.2), we have

$$||\mathbf{E}|| < \frac{1}{1 - ||\mathbf{A}^{-1}\mathbf{L}||} ||\mathbf{A}^{-1}|| \cdot ||\mathbf{T}||. \tag{3.6}$$

Let $V = \max_{x \in [a,b]} |u^{(5)}(x)|$, $v = \max_{x \in [a,b]} \delta_{i-\frac{1}{2}}$ and $v > 0$. Thus, from (3.5) and (3.6) we obtained

$$||\mathbf{E}|| < \frac{52(b-a)^3 V h^2}{9(32 - 51v(b-a)^3)}. \tag{3.7}$$

From equation (3.7), we conclude $||\mathbf{E}||$ is bounded above and as $h \to 0$ implies $||\mathbf{E}|| \to 0$. Thus, we have established the convergence of our proposed method (2.2). The order of convergence of the proposed method is at least quadratic.

## 4. Numerical Results

To test the computational efficiency and validity of the theoretical development of the proposed method (2.2), we have considered two linear, a nonlinear, and an obstacle model problem. In each model problem, we took a uniform step size $h$. In Table 1–4, we have shown the maximum absolute error $MAE$ in the computed solution $u(x)$ of the problem for different values of $N$. We have used the following formula in the computation of $MAE$

$$MAE = \max_{1 \leq i \leq N} |U(x_i) - u_i|,$$

where $U(x)$ is the exact solution of the problem. All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compilers (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{-10}$ or the number of iterations reached $10^6$.

*Problem* 4.1. The linear model problem in [12] with different boundary conditions is given by

$$u'''(x) = xu(x) + (x^3 - 2x^2 - 5x - 3)\exp(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 1 \quad \text{and} \quad u''(1) = -4\exp(1).$$

The analytical solution of the problem is $u(x) = x(1-x)\exp(x)$. The $MAE$ computed by method (2.2) for different values of $N$ are presented in Table 1.

Table 1. Maximum absolute error in solution of Problem 1.1.

| | N | | | |
|---|---|---|---|---|
| | 128 | 256 | 512 | 1024 |
| $MAE$ | .21027867e-3 | .61334344e-4 | .15507685e-4 | .38828002e-5 |

*Problem* 4.2. The linear model problem in [13] with different boundary conditions is given by

$$u'''(x) = -u(x) + (7 - x^2)\cos(x) + (x^2 - x - 1)\sin(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 2\sin(1) \quad \text{and} \quad u''(1) = 2\sin(1) + 4\cos(1).$$

The analytical solution of the problem is $u(x) = (x^2 - 1)\sin(x)$. The $MAE$ computed by method (2.2) for different values of $N$ are presented in Table 2.

TABLE 2. Maximum absolute error in solution of Problem 2.1.

| | N | | | |
|---|---|---|---|---|
| | 128 | 256 | 512 | 1024 |
| MAE | .98107150e-4 | .22081891e-4 | .55033015e-5 | .34570694e-5 |

*Problem* 4.3. The nonlinear model problem in [14] with different boundary conditions is given by

$$u'''(x) = -2\exp(-3u(x)) + 4(1+x)^{-3}, \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u'(1) = \frac{1}{2} \quad \text{and} \quad u''(1) = -\frac{1}{4}.$$

The analytical solution of the problem is $u(x) = \ln(1+x)$. The $MAE$ computed by method (2.2) for different values of $N$ are presented in Table 3.

TABLE 3. Maximum absolute error in solution of Problem 2.2.

| | N | | | |
|---|---|---|---|---|
| | 32 | 64 | 128 | 256 |
| MAE | .70750713e-4 | .24229288e-4 | .68414956e-5 | .19595027e-5 |

*Problem* 4.4. Let consider the following third-order obstacle problems [15]

$$u'''(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ u(x) - 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} \le x \le 1, \end{cases}$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 0 \quad \text{and} \quad u''(1) = a_5.$$

The analytical solution of the problem is

$$u(x) = \begin{cases} \frac{1}{2}a_1x^2, & 0 \le x \le \frac{1}{4}, \\ 1 + a_2\exp(x) + \exp(\frac{-x}{2})\left(a_3\cos\left(\frac{\sqrt{3}}{2}x\right) + a_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right), & \frac{1}{4} \le x \le \frac{3}{4}, \\ \frac{1}{2}a_5x(x-2) + a_6, & \frac{3}{4} \le x \le 1, \end{cases}$$

where $a_i$, $i = 1, 2, \ldots, 6$ are constants. To determine these constants, we apply a continuity condition to the solution, first and second derivatives of the solution of the problem. Hence, we shall get a system of linear equations and solve the system of equations in variable $a_i$, $i = 1, 2, \ldots, 6$. The maximum absolute error in domain $D_1 = [0, \frac{1}{4}]$, $D_2 = [\frac{1}{4}, \frac{3}{4}]$ and $D_3 = [\frac{3}{4}, 1]$ in computed solution are presented in Table 4, by the proposed method (2.2) for the different values of $N$. Hence, we presented the maximum absolute error in the computed solution in the considered domain $D = [0, 1]$ of the problem in Table 4.

TABLE 4. The maximum absolute error in solution of Problem 4.4.

| N | Maximum Absolute Error in | | | |
|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | D |
| 16 | .10749675e -3 | .97104762e -4 | .35469564e -3 | .35469564e -3 |
| 32 | .28237705e -4 | .27046958e -4 | .88673911e -4 | .88673911e -4 |
| 64 | .72459166e -5 | .71096102e -5 | .22168478e -4 | .22168478e -4 |
| 128 | .18359854e -5 | .18398562e -5 | .55421195e -5 | .55421195e -5 |
| 256 | .46230928e -6 | .48803843e -6 | .13855249e -5 | .13855249e -5 |
| 512 | .14240736e -6 | .14589006e -6 | .34638247e -6 | .34638247e -6 |

We have tested our finite difference method for the approximate numerical solution of linear and nonlinear model problems. Observing the numerical result in the tables, we found error in the computed solution decreases with a decrease in step size $h$ in each considered model problem. The order of accuracy in computed solution of Problem 4.1, 4.2 is quadratic, and the order of accuracy computed solution of Problem 4.3 is non quadratic. We observed from the tabulated result for Problem 4.4, that the order of accuracy in the computed solution in the domain is quadratic. Hence, the maximum absolute error in the computed solution is in the domain $D_3$. We have noted in numerical experiments that our method is efficient, convergent, and consistent with theoretical development.

## 5. Conclusion

We considered a third-order two-point boundary value problem in ordinary differential equations for the approximate numerical solution. There are numerous techniques for the approximate solution in the literature of numerical analysis. Hence, we have developed an algorithm of quadratic order exact using the finite difference method for the approximate numerical solution of third order boundary value problems. The main concern in the present article is the boundary conditions. Some work with these boundary conditions has been reported in the literature for a closed analytical solution of the problem, but no algorithm or technique has been developed for an approximate solution of the problem. We converted a differential equation, a continuous problem, into a difference equation, a discrete problem, i.e., we discretized the problem at discrete nodal points in the domain of the considered problem. Hence, we have obtained a system of algebraic equations (2.2) and the solution of system of equations (2.2) is an approximate numerical solution of the considered problem (1.1). We considered four model problems, including an obstacle problem, to test the efficiency and accuracy of the proposed method (2.2). The numerical experiments produced a good approximate numerical solution for model problems. The numerical experiments approved the theoretical discussion on the order of accuracy and efficiency of the proposed method (2.2). Thus, we arrived at the conclusion that our method is computationally efficient and the order of accuracy is quadratic. The idea presented in this article is simple and leads to the possibility of developing higher order finite difference methods. Work in these directions and areas is in progress.

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Department of Mathematics,
Dyal Singh College (Univ. of Delhi),
Lodhi Road, New Delhi-110003, India
*Email address*: pramod_10p@hotmail.com

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# FROM MONOTONICITY OF A CLASS OF BESSEL DISTRIBUTION FUNCTIONS TO NEW BOUNDS FOR RELATED FUNCTIONALS

DRAGANA JANKOV MAŠIREVIĆ[1] AND TIBOR K. POGÁNY[2]

*Dedicated to Academician Gradimir V. Milovanović on the occasion of his 75th birthday*

Abstract. In this note we prove a monotonicity result with respect to the parameter $\nu$ of the cumulative distribution function for the McKay $I_\nu$ Bessel distribution and uniform upper bounds for a bilinear expression containing modified Bessel function of the first kind $I_\nu$. Certain implications, among others with the Horn function $\Phi_2$ and for the Gaussian hypergeometric function close the exposition.

## 1. Introduction

The first results about probability distributions involving Bessel functions can be traced back to the early work of McKay [4] in 1932 who considered two classes of continuous distributions called Bessel distributions.

For reader's convenience, let us recall the definition of the modified Bessel function of the first kind $I_\nu$ of the order $\nu$ [6, p. 249, Eq. **10.25.2**]

$$I_\nu(z) = \sum_{k\geq 0} \frac{1}{\Gamma(\nu+k+1)\,k!}\left(\frac{z}{2}\right)^{2k+\nu}, \quad \mathrm{Re}\,(\nu) > -1,\, z \in \mathbb{C}\,.$$

On a standard probability space $(\Omega, \mathcal{F}, \mathsf{P})$ we consider a random variable (r.v.) $\xi$ which follows a distribution which is a McNolty's variant of the McKay $I_\nu$ Bessel law.

*Key words and phrases.* Modified Bessel function of the first kind, McKay $I_\nu$ Bessel distribution, Horn function $\Phi_2$, Moments, Gaussian hypergeometric function ${}_2F_1$, Turán inequality.

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This means that $\xi$ is a nonnegative r.v. with the following probability density function (density in short) [5, p. 496, Eq. (13)]

$$f_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}\, \mathrm{e}^{-bx} x^\nu I_\nu(ax), \quad x \geq 0.$$

The density $f_I$ depends on three real parameters $a, b, \nu$, where $\nu > -1/2$ and $b > a > 0$.

The corresponding distribution function of $\xi$ is as follows:

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^x \mathrm{e}^{-bt} t^\nu I_\nu(at)\,\mathrm{d}t, \quad x \geq 0. \tag{1.1}$$

In the sequel we use any of the notations $\xi \sim \mathrm{McKayI}(a,b,\nu)$, $\xi \sim f_I(x;a,b;\nu)$, $\xi \sim F_I(x;a,b;\nu)$.

Recently, Jankov Maširević and Pogány [2] reported on the expression of the distribution function $F_I$, see (1.1), in terms of the Horn confluent hypergeometric function [8, p. 25, Eq. (17)]

$$\Phi_2(b,b';c;x,y) = \sum_{m,n\geq 0} \frac{(b)_m (b')_n}{(c)_{m+n}} \cdot \frac{x^m\, y^n}{m!\, n!}, \quad \max\{|x|,|y|\} < +\infty\,.$$

So, for all $\nu > -1/2$, $b > a > 0$ and for all $x \geq 0$ this result is [2, p. 149, Theorem 3]

$$F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2} x^{2\nu+1}}{\Gamma(2\nu+2)}\, \Phi_2\Big(\nu+\tfrac{1}{2}, \nu+\tfrac{1}{2}; 2\nu+2; (a-b)x, -(a+b)x\Big). \tag{1.2}$$

It is natural to ask about important characteristics of the Bessel distribution (1.1). While, as we know, the positive integer order moments play a great role in Probability and Statistics, here we can find an explicit expression for the moment $m_s$ of order $s$, for $s \in \mathbb{C}$. Thus,

$$m_s = \mathsf{E}\,[\xi^s] = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{+\infty} \mathrm{e}^{-bx} x^{\nu+s}\, I_\nu(ax)\mathrm{d}x\,.$$

We see that up to a constant factor, $m_s$ is the Laplace transform of the input function $x^{\nu+s}\, I_\nu(ax)$. Applying a result [7, p. 313, Eq. 3.15.1.2.] for complex valued $\mu, \nu, p, \alpha$, we obtain

$$\int_0^{+\infty} \mathrm{e}^{-px} x^\mu\, I_\nu(\alpha x)\,\mathrm{d}x = \frac{\alpha^\nu\, \Gamma(\mu+\nu+1)}{2^\nu p^{\mu+\nu+1} \Gamma(\nu+1)}\, {}_2F_1\left[\begin{array}{c} \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu)+1 \\ \nu+1 \end{array} \Bigg| \frac{\alpha^2}{p^2}\right].$$

This formula is valid for all $\mu$, $\nu$, $p$, $\alpha$, provided $\mathrm{Re}\,(\mu+\nu) > -1$, $\mathrm{Re}\,(p) > |\mathrm{Re}\,(\alpha)|$. Now together with the Legendre duplication formula for the gamma function, we conclude that for all $\mathrm{Re}\,(s) > -2\nu-1$ there holds

$$m_s = \frac{(b^2-a^2)^{\nu+1/2} \Gamma(2\nu+s+1)}{\Gamma(2\nu+1)\, b^{2\nu+s+1}}\, {}_2F_1\left[\begin{array}{c} \nu+\frac{1}{2}(s+1), \nu+\frac{s}{2}+1 \\ \nu+1 \end{array} \Bigg| \frac{a^2}{b^2}\right]. \tag{1.3}$$

One of our goals is to prove the monotonicity of the distribution function $F_I$ with respect to $\nu$. This result implies an attractive uniform bound upon a bilinear function built with modified Bessel functions of the first kind which orders are contiguous with the input parameter $\nu$ occuring in McKayI$(a, b, \nu)$. We end the presentation with Turán type inequalities for Gauss hypergeometric function derived by certain moment inequalities.

## 2. Main Results

Sun et al. in [9] proved the next integral inequality. Let $X$ and $Y$ be positive independent random variables (r.v.), where $X$ is absolutely continuous with density function $f_X$, while $Y$ is arbitrary, either continuous or discrete; no density at the latter case. Let further, $g : (0, +\infty) \to (0, +\infty)$ be a nondecreasing positive function. Then, provided $F_Y(0) < 1$ and the integrals exist, compare [9, p. 1169, Lemma 1] (actually, this inequality is a consequence of the fact that if $X$ and $Y$ are positive r.v.s, $X + Y$ is *stochastically larger* than $X$), the following inequality holds true for each $x > 0$:

$$\int_x^{+\infty} g(t)\, f_{X+Y}(t)\,\mathrm{d}t > \int_x^{+\infty} g(t)\, f_X(t)\,\mathrm{d}t. \tag{2.1}$$

With the help of this inequality we prove a strict monotonicity of the generalized distribution function (1.2) and two consequences of this monotone behaviour of $F_I$.

**Theorem 2.1.** *For all $\nu_1 > -\frac12$, $\nu_2 > -\frac12$ and $b > a > 0$ there holds*

$$F_I\left(x; a, b; \nu_1 + \nu_2 + \frac12\right) < F_I(x; a, b; \nu_1), \quad x \ge 0\,. \tag{2.2}$$

*Moreover, for the same parameter range, the following inequality holds true*

$$\frac{I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)}{I_{\nu_1}(ax) \mp I_{\nu_1+1}(ax)} < \frac{\Gamma(\nu_1+\nu_2+2)}{\Gamma(\nu_1+\frac32)}\left(\frac{2a}{(b^2-a^2)x}\right)^{\nu_2+1/2}.$$

*Finally, for all $x > 0$ we have*

$$x^{2\nu_2+1}\,\frac{\Phi_2^{[\nu_1+\nu_2+1]}(x)}{\Phi_2^{[\nu_1+\frac12]}(x)} < \frac{\Gamma(2\nu_1+2\nu_2+3)}{(b^2-a^2)^{\nu_2+\frac12}\,\Gamma(2\nu_1+2)}\,, \tag{2.3}$$

*where we have used the quantity*

$$\Phi_2^{[\eta]}(x) = \Phi_2\Big(\eta, \eta; 2\eta+1; (a-b)x, -(a+b)x\Big)\,.$$

*Proof.* The moment generating function of the r.v. $\xi \sim$ McKayI$(a, b; \nu)$ equals

$$\begin{aligned} M_\xi(s) = \mathsf{E}\,[\mathrm{e}^{s\xi}] &= \int_0^{+\infty} \mathrm{e}^{sx} f_I(x; a, b; \nu)\,\mathrm{d}x \\ &= \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac12\right)} \int_0^{+\infty} \mathrm{e}^{-(b-s)x} x^\nu I_\nu(ax)\,\mathrm{d}x \\ &= \left(1 - \frac{s(2b-s)}{b^2-a^2}\right)^{-\nu-\frac12}, \quad s \in \mathbb{R}, |b-s| > a, \end{aligned}$$

see again the Laplace transform [7, p. 313, Eq. **3.15.1.3**]. Clearly, the moment generating function $M_\xi$ exists if we find a proper interval of zero, say $(-s_l, s_r)$, where $s_l > 0$, $s_r > 0$, such that for all $s \in (-s_l, s_r)$ it is $M_\xi(s) < +\infty$.

Now, letting $X \sim f_I(x; a, b; \nu_1)$ and $Y \sim f_I(x; a, b; \nu_2)$ be two independent r.v.s. Hence, the moment generating function of the r.v. $X + Y$ becomes

$$M_{X+Y}(s) = M_X(s) M_Y(s) = \left(1 - \frac{s(2b-s)}{b^2 - a^2}\right)^{-\nu_1-\nu_2-1}, \quad |b - s| > a,$$

which implies that r.v. $X + Y \sim f_I(x; a, b; \nu_1 + \nu_2 + 1/2)$. Rewriting the inequality (2.1) in the form

$$\int_0^x g(t) f_{X+Y}(t)\,\mathrm{d}t < \int_0^x g(t) f_X(t)\,\mathrm{d}t, \tag{2.4}$$

and taking $g(x) = 1$ for all $x > 0$ we conclude

$$\int_0^x f_I(t; a, b; \nu_1 + \nu_2 + 1/2)\,\mathrm{d}t < \int_0^x f_I(t; a, b; \nu_1)\,\mathrm{d}t,$$

which is equivalent to the first stated result.

As to the second inequality, observe that from (2.4) there follows

$$\begin{aligned}&\frac{(b^2-a^2)^{\nu_2+1/2}\Gamma(\nu_1+1/2)}{(2a)^{\nu_2+1/2}\Gamma(\nu_1+\nu_2+1)} \int_0^x g(t)\mathrm{e}^{-bt}\, t^{\nu_1+\nu_2+1/2} I_{\nu_1+\nu_2+1/2}(at)\,\mathrm{d}t \\ &< \int_0^x g(t)\mathrm{e}^{-bt}\, t^{\nu_1} I_{\nu_1}(at)\,\mathrm{d}t,\end{aligned}$$

and choosing the positive non-decreasing function $g(x) = \mathrm{e}^{(b\pm a)x}$ we conclude

$$\begin{aligned}&\frac{(b^2-a^2)^{\nu_2+1/2}\Gamma(\nu_1+1/2)}{(2a)^{\nu_2+1/2}\Gamma(\nu_1+\nu_2+1)} \int_0^x \mathrm{e}^{\pm at} t^{\nu_1+\nu_2+1/2} I_{\nu_1+\nu_2+1/2}(at)\,\mathrm{d}t \\ &< \int_0^x \mathrm{e}^{\pm at} t^{\nu_1} I_{\nu_1}(at)\,\mathrm{d}t.\end{aligned}$$

By virtue of [6, p. 259, Eq. **10.43.7**]

$$\int_0^x \mathrm{e}^{\pm t} t^{\nu} I_\nu(t)\,\mathrm{d}t = \frac{\mathrm{e}^{\pm x} x^{\nu+1}}{2\nu+1}(I_\nu(x) \mp I_{\nu+1}(x)), \quad \mathrm{Re}\,(\nu) > -1/2\,,$$

and applying the substitution $at \mapsto u$ it follows that

$$\begin{aligned}&\frac{\Gamma(\nu_1+3/2)}{\Gamma(\nu_1+\nu_2+2)} \left(\frac{(b^2-a^2)x}{2a}\right)^{\nu_2+1/2} \Big(I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)\Big) \\ &< I_{\nu_1}(ax) \mp I_{\nu_1+1}(ax).\end{aligned}$$

The rest is obvious.

Finally, inserting the Horn function representation (1.2) of the distribution function $F_I$ into (2.2), we arrive at (2.3). $\square$

To close the exposition we apply the well-known Turán inequality for the raw moments $m_s = \mathsf{E}[\xi^s]$, $s > 0$, of non-negative random variables [3, p. 28, Eqs. (1.4.6)] $m_{s+r}^2 \le m_s\, m_{s+2r}$, $s, r > 0$, which is an immediate consequence of the CBS inequality. Firstly, we define the Turánian ratio for the moment $m_s$ with respect to the increment $r > 0$ as

$$\mathfrak{T}_r(m_s) := \frac{m_{s+r}^2}{m_s \cdot m_{s+2r}}\,,$$

which one transforms the previous inequality into

$$\mathfrak{T}_r(m_s) \le 1. \tag{2.5}$$

To establish the bounding inequality for the Gaussian hypergeometric function ${}_2F_1$, we insert into (2.5) the expression (1.3).

**Proposition 2.1.** *For all $b > a > 0$, $\nu > -1/2$ and $s, r > 0$ we have*

$$\frac{\big\{{}_2F_1[s+r]\big\}^2}{{}_2F_1[s] \cdot {}_2F_1[s+2r]} \le \frac{\Gamma(2\nu+s+1)\Gamma(2\nu+s+2r+1)}{\Gamma^2(2\nu+s+r+1)}\,,$$

*where the abbreviation*

$${}_2F_1[s] := {}_2F_1\left[\begin{matrix} \nu+\frac12(s+1),\, \nu+\frac{s}{2}+1 \\ \nu+1 \end{matrix}\,\middle|\,\frac{a^2}{b^2}\right].$$

However, to derive another bound for ${}_2F_1[s]$ we take into account the integral moment inequality [1, p. 143, Theorem 192]

$$\mathfrak{M}_r(h,p) < \mathfrak{M}_s(h,p), \quad 0 < r < s, \tag{2.6}$$

where

$$\mathfrak{M}_r(h,p) = \int_\alpha^\beta h^r(t)\, p(t)\, \mathrm{d}t\,,$$

for a suitable, integrable non-negative input function $h$, the integration interval $(\alpha, \beta)$ is either finite or infinite, and the non-negative weight function $p$ has integral $\int_\alpha^\beta p(t)\,\mathrm{d}t = 1$. In our case the shorthand $\mathfrak{M}_s(x^s, f_I) = (m_s)^{1/s}$ is adopted to the McKayI$(a, b, \nu)$ distribution, $(\alpha, \beta) = \mathbb{R}_+$. Inserting $m_s$ from (1.3) into moment inequality (2.6) we obtain the following result.

**Proposition 2.2.** *For all $b > a > 0$, $\nu > -1/2$ and $s > r > 0$ there holds true*

$$\frac{\big\{{}_2F_1[r]\big\}^{1/r}}{\big\{{}_2F_1[s]\big\}^{1/s}} \le \Big(1-\frac{a^2}{b^2}\Big)^{(\nu+1/2)(1/s-1/r)}\, \frac{(2\nu+1)_s^{1/s}}{(2\nu+1)_r^{1/r}}\,,$$

*where the hypergeometric terms remain the same as in the previous proposition.*

*Remark* 2.1. According to Lukacs [3, p. 393, a)] for all $0 < r \le s$ there holds the moment inequality $m_{s+r}^2 \le m_{2s} \cdot m_{2r}$. We notice that this inequality is implied by virtue of the CBS inequality, using re-scaling of the integrand in $m_{s+r}$. However, to imply another bound for ${}_2F_1[s]$ via this inequality and/or the Lyapunov inequality we leave to the interested reader.

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[1]School of Applied Mathematics and Informatics
University of Osijek
Trg Ljudevita Gaja 6, 31000 Osijek, Croatia
*Email address*: djankov@mathos.hr

[2]Óbuda University,
John von Neumann Faculty of Informatics, Institute of Applied Mathematics,
Bécsi út 96/b, 1034 Budapest, Hungary
*Email address*: pogany.tibor@nik.uni-obuda.hu

[2]University of Rijeka,
Faculty of Maritime Studies,
Studentska 2, 51000 Rijeka, Croatia
*Email address*: tibor.poganj@uniri.hr

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# STABILITY OF A SOLUTION FOR A HYBRID FRACTIONAL DIFFERENTIAL EQUATION

MOHAMED HANNABOU[1], MOHAMED BOUAOUID[2], AND KHALID HILAL[2]

Abstract. This study focuses on examining the existence, uniqueness, and U-lam stability of a solution for a hybrid fractional equation by utilizing the derivative of Caputo-Hadamard (C-H). The primary tools used in our research are the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem. Additionally, we provide an example to demonstrate our results.

## 1. Introduction

The definitions like Riemann-Liouville (1832), Grunwald-Letnikov (1867), Hadamard (1891, [14]) and Caputo (1997) are used to model problems in engineering and applied sciences and the formulations are used to model the physical systems and has given more accurate results. In 1891, Hadamard introduced the new derivative. For more details one can refer [6, 23, 25, 27] and the references cited therein. A new approach called Caputo-Hadamard derivative [22], obtained from the Hadamard derivative and is applied to solve for physically interpretable initial condition problems. For the recent results in Caputo-Hadamard derivative, one can cite [2, 7, 12, 16, 34–36] and the references therein.

The theory of fractional calculus is an interesting field to be explored in recent years. Also, this theory has many applications to describe many events in the real world and deal with a group of phenomena in several fields such as blood flow phenomena, mechanics, biophysics, automatic, aerodynamics, some branches of medicine, and electronics. For instance, the authors [10] discussed the applicability of fractional differential equations in electric circuits, and in 2019 M. Saqib et al. applied the

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fractional differential equation to heat transfer in hybrid nanofluid see [26]. For more details, one can refer to [24, 25, 37]. In addition to the great importance of studying the existence of solutions to fractional differential equations using the many theories of the fixed point, several studies have been conducted over the years to investigate how stability concepts such as the Mittag-Leffler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified previously unknown types of stability known as Ulam-stability [1]. This example is not exclusive, many similar works can be found in [3, 5, 11, 17, 31]. In 2008, Benchohra et al. [13], discussed the following boundary value problem

$$
\begin{gathered}
{}^cD^p\varphi(\vartheta) = f_1(\vartheta, \vartheta(t)), \quad \text{for } \vartheta \in [0, T], 0 < p \leq 1,\\
a_1\vartheta(0) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^cD^p$ denotes the Caputo fractional derivative of order $p$, $f_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a_1, b_1, c_1 \in \mathbb{R}$ such that $a_1 + b_1 \neq 0$.

In 2017, Arioua et al. [9] proved the existence of solution for the boundary value problem of nonlinear differential equation of fractional order

$$
{}^cD^p_{1^+}\vartheta(t) + f_1(t, \vartheta(t)) = 0, \quad \text{for } 1 < t < e, 2 < p \leq 3,
$$

with the fractional boundary conditions:

$$
\vartheta(1) = \vartheta^{'}(1) = 0, \quad ({}^cD^{p-1}_{1^+}\vartheta)(e) = ({}^cD^{p-2}_{1^+}\vartheta)(e) = 0,
$$

where ${}^cD^p$ denotes the Caputo-Hadamard (C-H) fractional derivatives of order $p$, a continuous function $f_1 : [1, e] \times \mathbb{R} \to \mathbb{R}$.

In 2018, Benhamida et al. [14] investigated the following Caputo-Hadamard fractional differential equations with the boundary conditions:

$$
\begin{gathered}
{}^c_HD^p\vartheta(t) = f_1(t, \vartheta(t)), \quad \text{for } t \in [1, T], 0 < p \leq 1,\\
a_1\vartheta(1) + b_1\vartheta(T) = c_1,
\end{gathered}
$$

where ${}^c_HD^p$ denotes the Caputo-Hadamard (C-H) fractional derivative of order $p$, a given continuous function $f_1 : [1, T] \times \mathbb{R} \to \mathbb{R}$ and the real constants $a_1, b_1$ and $c_1$ such that $a_1 + b_1 \neq 0$.

The present paper is a continuation of the work see [19], we consider the system of hybrid nonlinear Caputo-Hadamard (C-H) fractional differential equations:

$$
{}^c_HD^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \xi(\hat{\varkappa}))}\right] = \Lambda(\hat{\varkappa}, \xi(\hat{\varkappa})), \quad \hat{\varkappa} \in [1, T], 0 < \gamma_1 \leq 1, \tag{1.1}
$$

supplemented with the boundary condition;

$$
\lambda\frac{\xi(1)}{\varpi(1, \xi(1))} + \mu\frac{\xi(T)}{\varpi(T, \xi(T))} = \nu, \tag{1.2}
$$

where ${}^c_HD^{\gamma_1}$, denote the Caputo-Hadamard (C-H) fractional derivatives of orders $\gamma_1$. The given continuous functions $\Lambda : [1, T] \times \mathbb{R} \to \mathbb{R}$, with $\lambda, \mu$ and $\nu \in \mathbb{R}$, $\varpi : [1, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

The paper is organized as follows. Section 2 provides an overview of preliminary concepts and an auxiliary lemma related to the problem. Section 3 presents the main proof for the existence of solutions to Problem (1.1)–(1.2). Section 4 examines the Ulam-Hyers stability of the fractional differential equations (1.1)–(1.2). In Section 5, an example is presented to further illustrate the findings of the study. Lastly, in Section 6, we conclude and discuss future work that can be done in this area.

## 2. Preliminaries

**Definition 2.1** ([23])**.** Let $h_1$: $[1, +\infty) \to \mathbb{R}$ be an integrable function. The Hadamard fractional integral of $h_1$ of order $q_1$ is defind by

$$_H I^{q_1} h_1(\hat{\varkappa}) = \frac{1}{\Gamma(q_1)} \int_1^{\hat{\varkappa}} \left(\ln \frac{\hat{\varkappa}}{s}\right)^{q_1 - 1} \frac{h_1(s)}{s} ds, \quad q_1 > 0, \hat{\varkappa} > 1.$$

**Definition 2.2** ([22])**.** The C-H fractional derivative of order $q_1$ where $q_1 \geq 0$, $n-1 < q_1 < n$, with $n = [q_1] + 1$ and $h_1 \in AC_\delta^n[1, +\infty)$

$$({}_H^c D^{q_1} h_1)(\hat{\varkappa}) = \frac{1}{\Gamma(n - q_1)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s}\right)^{n - q_1 - 1} \delta^n h_1(s) \frac{ds}{s} =_H I^{n - q_1}(\delta^n h_1)(\hat{\varkappa}).$$

**Lemma 2.1** ([22])**.** *Let $h_1 \in AC_\delta^n[1, +\infty)$ and $q_1 > 0$. Then*

$$_H I^{q_1} ({}_H^c D^{q_1} h_1)(\hat{\varkappa}) = h_1(\hat{\varkappa}) - \sum_{i=0}^{n-1} \frac{\delta^i x(1)}{i!} (\log t)^i.$$

**Lemma 2.2.** *Let the function $h_1 : [1, +\infty) \to \mathbb{R}$. The function $\xi$ is a solution of the following equation*

$$\xi(\hat{\varkappa}) = \varpi(\hat{\varkappa}, \xi(\hat{\varkappa})) \Bigg( \frac{1}{\Gamma(\gamma)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s}\right)^{\gamma - 1} h_1(s) \frac{ds}{s} \tag{2.1}$$

$$- \frac{\mu}{\Gamma(\gamma)(\lambda + \mu)} \int_1^T \left(\log \frac{T}{s}\right)^{\gamma - 1} h_1(s) \frac{ds}{s} + \frac{\nu}{\lambda + \mu} \Bigg) \tag{2.2}$$

*if and only if*

$${}_H^c D^{\gamma_1} \left[ \frac{\xi(t)}{\varpi(\hat{\varkappa}, \xi(t))} \right] = h_1(\hat{\varkappa}), \quad 0 < \gamma_1 < 1, \tag{2.3}$$

*and*

$$\lambda \frac{\xi(1)}{\varpi(1, \xi(1))} + \mu \frac{\xi(T)}{\varpi(T, \xi(T))} = \nu. \tag{2.4}$$

*Proof.* Suppose that $\xi$ satisfies (2.3). Then

$$\left[ \frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \xi(\hat{\varkappa}))} \right] =_H I^{\gamma_1} h_1(\hat{\varkappa}) + \alpha_1, \tag{2.5}$$

when we apply the boundary condition (2.4), we get

$$\begin{aligned}
\frac{\xi(1)}{\varpi(1,\xi(1))} =&\alpha_1,\\
\frac{\xi(T)}{\varpi(T,\xi(T))} =&{}_HI^{\gamma_1}h_1(T)+\alpha_1,\\
\lambda\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu\frac{\xi(T)}{\varpi(T,\xi(T))} =&\nu,\\
\lambda\alpha_1+\mu\left[{}_HI^{\gamma_1}h_1(T)+\frac{\xi(1)}{\varpi(1,\xi(1))}\right] =&\nu,\\
\lambda\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu_HI^{\gamma_1}h_1(T)+\mu\frac{\xi(1)}{\varpi(1,\xi(1))} =&\nu,\\
(\lambda+\mu)\frac{\xi(1)}{\varpi(1,\xi(1))}+\mu_HI^{\gamma_1}h_1(T) =&\nu,\\
\frac{\xi(1)}{\varpi(1,\xi(1))} =&\frac{\nu-\mu_HI^{\gamma_1}h(T)}{(\lambda+\mu)},
\end{aligned}$$

which leads to the solution (2.1) that

$$\xi(t)=\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Big({}_HI^{\gamma_1}h_1(\hat{\varkappa})-\frac{\mu}{(\lambda+\mu)}{}_HI^{\gamma_1}h_1(T)+\frac{\nu}{\lambda+\mu}\Big).$$

Conversely, $\xi$ has to satisfy equation (2.1) and then equation (2.3)–(2.4) hold. ☐

## 3. Main Results

Let us now consider the Banach space $\mathfrak{S}$ of all continuous functions $f:[1,T]\to\mathbb{R}$ endowed with the norm $\|\tilde{\xi}\|_\infty=\sup\{|\tilde{\xi}(\hat{\varkappa})|:1\leq\hat{\varkappa}\leq T\}$.

Let consider thr following assumtions:

- $(F_1)$ The function $\xi\mapsto\frac{\xi}{\varpi(\hat{\varkappa},\xi)}$ is increasing for every $\hat{\varkappa}\in[1,T]$.
- $(F_2)$ There is numbers $L>0$ such that $|\varpi(\hat{\varkappa},\xi)|\leq L$ for all $(\hat{\varkappa},\xi)\in[1,T]\times\mathbb{R}$.
- $(F_3)$ Let $\Lambda:[1,T]\times\mathbb{R}\to\mathbb{R}$ be continuous and bounded functions and there is constants $\pi$, such that, for all $\hat{\varkappa}\in[1,T]$ and $\rho,\varrho\in\mathbb{R}$
$$|\Lambda(\hat{\varkappa},\rho)-\Lambda(\hat{\varkappa},\varrho)|\leq\pi|\rho-\varrho|.$$
- $(F_4)$ $\sup_{\hat{\varkappa}\in[1,T]}\Lambda(\hat{\varkappa},0)=\mathcal{M}<+\infty$.
- $(F_5)$ There is $N>0$ such that $|\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))|\leq N$. For ease of computation, we set
$$\tau=\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)},\quad Q=\frac{|\nu|}{|\lambda+\mu|}<1.$$

Let consider the operator $\Theta : \mathfrak{S} \to \mathfrak{S}$ associated with Problem (1.1)–(1.2) as follows:

$$\begin{aligned}\Theta(\xi,\vartheta)(\hat{\varkappa}) =&\varpi(\hat{\varkappa},\xi)\Big(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Lambda(s,\xi(s),\vartheta(s))\frac{ds}{s}\\ &-\frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}\Lambda(s)\frac{ds}{s}+\frac{\nu}{\lambda+\mu}\Big).\end{aligned}\tag{3.1}$$

**Theorem 3.1.** *Suppose that conditions* $(F_1)$ *to* $(F_5)$ *hold. Then* $\Theta\bar{\mathcal{B}}_r \subset \bar{\mathcal{B}}_r$, *where* $\bar{\mathcal{B}}_r = \{\xi \in \mathfrak{S} : \|(\xi)\|_\infty \le r\}$ *is a closed ball with* $r = L\tau\pi < 1$. *Problem* (1.1)–(1.2) *has a unique solution on* $[1,T]$.

*Proof.* For $\xi \in \bar{\mathcal{B}}_r$ and $\hat{\varkappa} \in [1,T]$, it follows by $(F_3)$ that

$$|\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))| \le |\Lambda(\hat{\varkappa},\xi(\hat{\varkappa}))-\Lambda(\hat{\varkappa},0)| \le \pi||\xi||_\infty.$$

Then we have

$$\begin{aligned}|\Theta(\xi)(\hat{\varkappa})| \le&L\max_{\hat{\varkappa}\in[1,T]}\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}\Big|\Lambda(s,\xi(s))-\Lambda(s,0))+\Lambda(s,0)\Big|\frac{ds}{s}\\ &-\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}|\Lambda(s,\xi(s))-\Lambda(s,0)+\Lambda(s,0)|\frac{ds}{s}\\ &+\frac{|\nu|}{|\lambda+\mu|}\Bigg]\\ \le&L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\Big(\log\frac{\hat{\varkappa}}{s}\Big)^{\gamma_1-1}(\pi|\xi|+\mathcal{M})\frac{ds}{s}\\ &+\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^{T}\Big(\log\frac{T}{s}\Big)^{\gamma_1-1}(\pi|\xi|+\mathcal{M})\frac{ds}{s}+Q\Bigg]\\ \le&L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\Big(\pi|\xi|+\mathcal{M}\Big)+LQ.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Theta(\xi,\vartheta)\|_\infty \le&L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\Big(1+\frac{|\mu|}{|\lambda+\mu|}\Big)\Big(\pi|\xi|+\mathcal{M}\Big)+LQ\\ \le&L\Big(\tau(\pi\|\xi\|_\infty+\mathcal{M})+Q\Big) \le L\Big(\tau\pi r+\tau\mathcal{M}+Q\Big) \le L\Big(\tau\pi r+\tau\mathcal{M}+Q\Big).\end{aligned}$$

From the foregoing estimates for $\Theta$ it follows that $||\Theta(\xi)(\hat{\varkappa})||_\infty \leq r$. Next, for $\xi_1, \xi_2 \in \mathfrak{S}$ and $\hat{\varkappa} \in [1, T]$, we get

$$\begin{aligned}
|\Theta(\xi_2)(\hat{\varkappa}) - \Theta(\xi_1)(\hat{\varkappa})| \leq & L_1 \Bigg( \frac{1}{\Gamma(\gamma_1)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s}\right)^{\gamma_1 - 1} |\Lambda(s, \xi_2(s)) - \Lambda(s, \xi_1(s))| \frac{ds}{s} \\
& + \frac{|\mu|}{\Gamma(\gamma_1)|\lambda + \mu|} \int_1^T \left(\log \frac{T}{s}\right)^{\gamma_1 - 1} |\Lambda(s, \xi_2(s)) - \Lambda(s, \xi_1(s))| \frac{ds}{s} \Bigg) \\
\leq & L \left(1 + \frac{\mu}{\lambda + \mu}\right) \frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1 + 1)} \left(\pi \|\xi_2 - \xi_1\|_\infty\right) \\
= & L\tau\pi \|\xi_2 - \xi_1\|_\infty,
\end{aligned}$$

which implies that

$$\|\Theta(\xi_2)(\hat{\varkappa}) - \Theta(\xi_1)\|_\infty \leq L\tau\pi \|\xi_2 - \xi_1\|_\infty. \tag{3.2}$$

In view of condition $L\tau\pi < 1$, it follows that the operator $\Theta$ possesses a unique fixed point. This leads to the conclusion that Problem (1.1)–(1.2) has a unique solution on $[1, T]$. □

The fixed point theorem of Schaefer can now be used to demonstrate the existence of solutions to Problem (1.1)–(1.2).

**Theorem 3.2.** *Suppose the hypothesis* $(F_1)$-$(F_4)$. *Then the boundary value Problem* (1.1)–(1.2) *has at least one solution on* $[1, T]$.

*Proof.* Several steps will be involved in the proof.

**Step I.** $\Theta : \mathfrak{S} \to \mathfrak{S}$ is continuous. Notice that continuity of $\Lambda$ and $\varpi$ show that $\Theta$ is bounded. Let $\xi_n$ be a sequence of points in $\mathfrak{S}$ converging to a point $\xi \in \mathfrak{S}$. We get

$$\begin{aligned}
& |\Theta(\xi_n)(\hat{\varkappa}) - \Theta(\xi)(\hat{\varkappa})| \\
\leq & L \Bigg( \frac{1}{\Gamma(\gamma_1)} \int_1^{\hat{\varkappa}} \left(\log \frac{\hat{\varkappa}}{s}\right)^{\gamma_1 - 1} |\Lambda(s, \xi_n(s)) - \Lambda(s, \xi(s))| \frac{ds}{s} \\
& - \frac{|\mu_1|}{\Gamma(\gamma_1)|\lambda + \mu|} \int_1^T \left(\log \frac{T}{s}\right)^{\gamma_1 - 1} |\Lambda(s, \xi_n(s)) - \Lambda(s, \xi(s))| \frac{ds}{s} \Bigg) \\
\leq & L_1 \left(1 + \frac{|\mu|}{|\lambda + \mu|}\right) \frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1 + 1)} \|\Lambda(\cdot, \xi_n(\cdot)) - \Lambda(\cdot, \xi(\cdot))\|_\infty,
\end{aligned}$$

since $\Lambda$ is continuous, we have $\|\Theta(\xi_n) - \Theta(\xi)\|_\infty \to 0$ as $n \to +\infty$ for all $\hat{\varkappa} \in [1, T]$. Hence, it follows from the foregoing inequalities satisfied by above that $\Theta$ is continuous.

**Step II.** $\Theta : \mathfrak{S} \to \mathfrak{S}$ maps bounded sets into bounded sets. Let $\xi \in \mathcal{B}_{\nu_1^*} := \{\xi \in \mathfrak{S} : \|\xi\|_\infty \le \nu_1^*\}$, certainly for any $\nu_1^* > 0$, we have

$$\begin{aligned}|\Theta(\xi)(\hat{\varkappa})| \le& L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s}\\ &+\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s}+\frac{\nu}{\lambda+\mu}\Bigg],\\ \|\Theta(\xi)\|_\infty \le& L\left[\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}N_1+\frac{|\nu|}{|\lambda+\mu|}\right] := \mathcal{L}.\end{aligned}$$

Thus, we deduce that $\|\Theta(\xi)\|_\infty \le \mathcal{L}$. Hence, it follows from the foregoing inequalities above hence the operator $\Theta$ is uniformly bounded.

**Step III.** Next we prove that $\Theta$ is equicontinuous sets. Let $r_1, r_2 \in [1,T]$ with $r_1 < r_2$,

$$\begin{aligned}|\Theta(\xi(r_2)) - \Theta(\xi(r_1))| \le& L\Big[\frac{1}{\Gamma(\gamma_1)}\int_1^{r_1}\left((\log\frac{r_2}{s})^{\gamma_1-1}-(\log\frac{r_1}{s})^{\gamma_1-1}\right)|\Lambda(s,\xi(s))|\,\frac{ds}{s}\\ &+\frac{1}{\Gamma(\gamma_1)}\int_{r_1}^{r_2}(\log\frac{r_2}{s})^{\gamma_1-1}\,|\Lambda(s,\xi(s))|\,\frac{ds}{s}\Big]\\ \le&\frac{LN}{\Gamma(\gamma_1+1)}\Big((\log r_2)^{\gamma_1}-(\log r_1)^{\gamma_1}\Big)\\ \to& 0 \quad \text{as} \quad r_1 \to r_2.\end{aligned}$$

Therefore, the operator $\Theta$ is equicontinuous and hence the operator $\Theta(\xi)$ is completely continuous.

**Step IV .** To show that the set $\mathcal{P} = \{\xi \in \mathfrak{S} : \xi = \gamma\Theta(\xi), 0 < \gamma < 1\}$ is bounded (Apriori bounds). Let $\xi \in \mathcal{P}$ and $\hat{\varkappa} \in [1,T]$. Then it follows from $\xi(\hat{\varkappa}) = \gamma\Theta(\xi)(\hat{\varkappa})$ that

$$\begin{aligned}|\xi(\hat{\varkappa})| \le& L\Bigg[\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s}\\ &-\frac{|\mu|}{\Gamma(\gamma_1)|\lambda+\mu|}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}|\Lambda(s,\xi(s))|\frac{ds}{s}+\frac{\nu}{\lambda+\mu}\Bigg]\\ \le& L\left[\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}N+\frac{|\nu|}{|\lambda+\mu|}\right] := R\end{aligned}$$

and

$$\|\xi(t)\|_\infty \le R. \tag{3.3}$$

Hence, $\mathcal{P}$ is bounded and therefore by Theorem 3.2, $\Theta$ has a fixed point, then Problem (1.1)–(1.2) has at least one solution on $[1,T]$. The proof is completed. □

## 4. Stability Results for the Problem

We analyse the Ulam-Hyers stability for Problem (1.1)–(1.2) in this section. Consider the following definitions of nonlinear operators $\mathcal{Z}_1 \in \mathcal{C}([1,T],\mathbb{R}) \to \mathcal{C}([1,T],\mathbb{R})$, where $\xi$ is defined by (2.1)

$$ {}_H^cD^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\right] - \Lambda(\hat{\varkappa},\xi(\hat{\varkappa})) = \mathcal{Z}_1(\xi)(\hat{\varkappa}), \quad \hat{\varkappa} \in [1,T],\ 0<\gamma_1\leq 1. $$

For some $\varsigma_1 > 0$, we consider the following inequality:

$$ ||\mathcal{Z}_1(\xi)||_\infty \leq \varsigma_1. \tag{4.1} $$

**Definition 4.1.** The system (1.1)–(1.2) is U-H stable if $\mathcal{M}_1 > 0$, for every solution $\xi^* \in \mathcal{C}([1,T],\mathbb{R})$ of inequality (4.1) there exists a unique solution on $\xi \in \mathcal{C}([1,T],\mathbb{R})$, of Problem (1.1)–(1.2) with $||\xi - \xi^*||_\infty \leq \mathcal{M}_1\varsigma_1$.

**Theorem 4.1.** *Suppose that* $(F_4)$ *is satisfied, then the BVP* (1.1)–(1.2) *is U-H stable if* $L\tau\pi > 1$.

*Proof.* Let $\xi \in \mathcal{C}([1,T],\mathbb{R})$, be the solution of the BVP (1.1)–(1.2) satisfying (3.1).

Let $\xi$ be any solution satisfying (4.1):

$$ {}_H^cD^{\gamma_1}\left[\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa},\xi(\hat{\varkappa}))}\right] = \Lambda(\hat{\varkappa},\xi(\hat{\varkappa})) + \mathcal{Z}_1(\xi)(\hat{\varkappa}), \quad \hat{\varkappa} \in [1,T],\ 0<\gamma_1\leq 1. $$

Therefore,

$$ \begin{aligned} \xi^*(\hat{\varkappa}) =& \Theta(\xi^*)(\hat{\varkappa}) + \varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Bigg(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} \\ &- \frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg), \end{aligned} $$

it follows that

$$ \begin{aligned} |\xi^*(\hat{\varkappa}) - \Theta(\xi^*)| \leq& \varpi(\hat{\varkappa},\xi(\hat{\varkappa}))\Bigg(\frac{1}{\Gamma(\gamma_1)}\int_1^{\hat{\varkappa}}\left(\log\frac{\hat{\varkappa}}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} \\ &- \frac{\mu}{\Gamma(\gamma_1)(\lambda+\mu)}\int_1^T\left(\log\frac{T}{s}\right)^{\gamma_1-1}\mathcal{Z}_1(\xi)(\hat{\varkappa})\frac{ds}{s} + \frac{\nu}{\lambda+\mu}\Bigg) \\ \leq& L\frac{(\log T)^{\gamma_1}}{\Gamma(\gamma_1+1)}\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\varsigma_1. \end{aligned} $$

Consequently, based on the fixed point property of the operator $\Theta$, provided in (3.1), we derive that

$$\begin{aligned}|\xi(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| &= |\xi(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa}) + \Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq |\Theta(\xi)(\hat{\varkappa}) - \Theta(\xi^*)(\hat{\varkappa})| + |\Theta(\xi^*)(\hat{\varkappa}) - \xi^*(\hat{\varkappa})| \\ &\leq L\tau\pi\|(\xi) - (\xi^*)\| + L\tau\varsigma_1. \end{aligned} \tag{4.2}$$

From (4.2) it follows that

$$\|\xi - \xi^*\| \leq L\pi\tau\|\xi - \xi^*\| + L\tau\varsigma_1 \leq \frac{L\tau\varsigma_1}{1 - L\tau\pi} \leq \mathcal{M}_1\varsigma_1,$$

with

$$\mathcal{M}_1 = \frac{L\tau}{1 - L\tau\pi}.$$

Hence, Problem (1.1)–(1.2) is U-H stable. □

## 5. Example

*Example* 5.1. Consider the following system of coupled fractional differential equations:

$$\begin{aligned} &{}^cD^{1/2}\Big(\frac{\xi(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \xi(\hat{\varkappa}))}\Big) = \frac{1}{100}\Big(\xi(\hat{\varkappa}) + \frac{1}{2}\Big) + \frac{5}{200} \cdot \frac{\xi(\hat{\varkappa})}{1 + \xi(\hat{\varkappa})} + e^{-2}, \\ &\frac{\xi(1)}{\varpi(1, \xi(1))} + \frac{\xi(e)}{\varpi(e, \xi(e))} = 0. \end{aligned} \tag{5.1}$$

Here $\gamma_1 = \frac{1}{2}$, $T = e$, $\lambda = \mu = 1$, $\nu = 0$ and $\varpi(\hat{\varkappa}, \xi) = \frac{\xi+1)}{100}\Big(\sin\xi + \frac{|\xi|}{1+|\xi|} + 3\Big) + e^{-1}$, $\Lambda(\hat{\varkappa}, \xi(\hat{\varkappa})) = \frac{1}{100}\Big(\xi(\hat{\varkappa}) + \frac{1}{2}\Big) + \frac{5}{200} \cdot \frac{\xi(\hat{\varkappa})}{1+\xi(\hat{\varkappa})} + e^{-2}$, $\pi = \frac{2}{53}$.

From the given data, we find that $\tau = 1.6930$ and $L\tau\pi = 0.123456987 < 1$. By the Theorem (3.1), Problem (1.1)–(1.2), with the given $\Theta(\hat{\varkappa}, \xi)$ has at least one solution on $[1, T]$.

## Conclusion

Fractional differential equations are commonly used to model various natural phenomena, and their diverse types enable us to study the integration of many phenomena in different fields. In this study, we focused on examining the existence, uniqueness, and stability of solutions for Caputo-Hadamard hybrid fractional differential equations. Additionally, we extended our results for new classes of fractional boundary conditions by utilizing the Hadamard-Caputo sequential derivative and author fixed point theorem. For future research, we recommend exploring other types of fractional derivative operators such as the generalized Hilfer fractional derivative. Furthermore, those interested in this subject can also investigate the existence and uniqueness of solutions for coupled systems.

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[1]Department of Mathematics and Computer Sciences,
Sultan Moulay Slimane University, Multidisciplinary faculty,
Beni Mellal, Morocco

[2]Laboratoire de Mathématiques Appliquées & Calcul Scientifique,
Université Sultan Moulay Slimane, BP 523, 23000 Beni Mellal, Morocco
*Email address*: hnnabou@gmail.com
*Email address*: hilalkhalid2005@yahoo.fr
*Email address*: Bouaouidfst@gmail.com

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# GENERALISATION OF COMPANION OF OSTROWSKI'S TYPE INEQUALITY VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL FOR MAPPINGS WHOSE $1^{st}$ DERIVATIVES ARE BOUNDED WITH APPLICATIONS

FARAZ MEHMOOD*[1,2], ASIF R. KHAN[3], MUHAMMAD AWAIS SHAIKH[3],
AND MOHAMMAD W. ALOMARI[4]

Abstract. We apply Riemann-Liouville fractional integral to get generalisation of companion of Ostrowski's type integral inequality for differentiable mappings whose 1st derivatives are bounded. The present article recapture all results of M. W. Alomari's article and also for one more article of different authors. Applications are also deduced for numerical integration, probability theory and special means.

## 1. Introduction

In the development of mathematics, inequalities are one of the most powerful tools. From two decades back, scholars researched on fractional calculus because of its importance in inequalities.

We quote from [4]: "The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

Due to worth of fractional integral inequalities, many scholars have mentioned certain generalisations of fractional integral inequalities (see [3, 17–19]).

*Key words and phrases.* Fractional Calculus, Riemann-Liouville fractional integral operator, Ostrowski's inequality, Grüss inequality, Differentiable mapping, Bounded mapping, Numerical integration, Probability density function, Special means.

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In 1938, A. M. Ostrowski gave an inequality (see [16]). Now-a-days this inequality is called Ostrowski inequality and this result had obtained by applying the Montgomery identity. For more discussion about Ostrowski inequality (see [9–12]).

Here, we present an inequality from article [6] that is given below. Throughout the article $I \subset \mathbb{R}$ and $I^\circ$ is the interior of the interval $I$.

**Proposition 1.1.** *Suppose $g : I \to \mathbb{R}$ is a differentiable mapping in the interval $I^o$ such that $g' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|g'(\theta)| \leq M$ for all $\theta \in (a,b)$, where $M > 0$ is constant. Then*

$$\left| g(\theta) - \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(\theta - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].$$

*The value $\frac{1}{4}$ is the best possible constant that this can not be replaced by the smallest one.*

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss inequality [9, 14].

**Proposition 1.2.** *Let $f, g : [a,b] \to \mathbb{R}$ be both integrable functions such that $m_1 \leq f(\tau) \leq M_1$ and $m_2 \leq g(\tau) \leq M_2$ for all $\tau \in [a,b]$, where $m_1, M_1, m_2, M_2$ are real constants, then*

$$\left| \frac{1}{b-a}\int_a^b f(\tau)g(\tau)d\tau - \frac{1}{b-a}\int_a^b f(\tau)d\tau \cdot \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

In [7], S. S. Dragomir has derived the following companion of the Ostrowski inequality.

**Proposition 1.3.** *Let $g : I \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, where $a, b \in I$. Then we have the inequalities*

$$\begin{aligned}
&\left| \frac{g(\theta) + g(a+b-\theta)}{2} - \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \\
&\leq \begin{cases}
\left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|g'\|_\infty, \quad g' \in L_\infty[a,b], \\[2ex]
\frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\theta-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2} - \theta}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|g'\|_{[a,b],p}\ , \\
p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \text{and } g' \in L_p[a,b], \\[2ex]
\left[\frac{1}{4} + \left|\frac{\theta - \frac{3a+b}{4}}{b-a}\right|\right]\|g'\|_{[a,b],1},
\end{cases}
\end{aligned} \tag{1.1}$$

*for all $\theta \in [a, \frac{a+b}{2}]$.*

In 2011, M. W. Alomari has proved the following result about a companion inequality for differentiable functions whose derivatives are bounded (see [1]).

**Proposition 1.4.** *Let $g : I \to \mathbb{R}$ be a differentiable function in the interval $I^\circ$ and let $a, b \in I$ with $a < b$. If $g' \in L^1[a,b]$ and $m_2 \leq g'(\theta) \leq M_2$, for all $\theta \in [a,b]$, then the following inequality holds*

$$\left| \frac{g(\theta) + g(a+b-\theta)}{2} - \frac{1}{b-a} \int_a^b g(\tau) d\tau \right| \leq (b-a) \left[ \frac{1}{16} + \left( \frac{\theta - \frac{3a+b}{4}}{b-a} \right)^2 \right] (M_2 - m_2),$$

*for all $\theta \in [a, \frac{a+b}{2}]$.*

We need here to define Riemann-Liouville fractional integral (RLFI) (see [8]) for proving our next main result in the second section.

**Definition 1.1.** The Riemann-Liouville fractional integral operator of order $\gamma > 0$ is stated as

$$J_a^\gamma g(\theta) = \frac{1}{\Gamma(\gamma)} \int_a^\theta (\theta - \tau)^{\gamma-1} g(\tau) d\tau, \quad J_a^0 g(\theta) = g(\theta),$$

where gamma function $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma) = \int_0^\infty \theta^{\gamma-1} e^{-\theta} d\theta.$$

In 2009, Z. Liu [13] introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. In 2009, Barnett et. al [5] have derived some companions for Ostrowski inequality and the generalised trapezoid inequality. In 2012, M. W. Alomari [2] obtained a companion inequality of Ostrowski's type using Grüss result with applications. Recently, authors [15] gave a companion of weighted Ostrowski's type inequality using Grüss result with application.
In the present article we would prove a companion of weighted Fractional Ostrowski's type inequality by applying Grüss result and then we would give its applications.

## 2. Generalisation of Companion of Ostrowski's Type Inequality Via Riemann-Liouville Fractional Integral

Under present section we would give our results about companion of Ostrowski's type inequality which are as follow.

**Theorem 2.1.** *Suppose $g : [a,b] \to \mathbb{R}$ is a differentiable mapping in the interval $(a,b)$ and $a < b$ and $w : [a,b] \to \mathbb{R}$ is a probability density function. If $g' \in L^1[a,b]$ and*

*$m_2 \leq g'(\tau) \leq M_2$, for all $\tau \in [a,b]$, then*

$$\begin{aligned}
&\Bigg| g(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&\quad - (b-\theta)^{1-\gamma}\Gamma(\gamma)J_a^\gamma(w(b)g(b)) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)g(b))\Bigg| \\
&\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2) \qquad (2.1)
\end{aligned}$$

*holds for all $\theta \in [a, \frac{a+b}{2}]$.*

*Proof.* For the sake of proof we state the weighted kernel as

$$P(\theta,\tau) = (b-\theta)^{1-\gamma}\Gamma(\gamma)\begin{cases} \displaystyle\int_a^\tau w(u)du, & \text{if } \tau \in [a,\theta], \\ \displaystyle\int_{\frac{a+b}{2}}^\tau w(u)du, & \text{if } \tau \in (\theta, a+b-\theta], \\ \displaystyle\int_b^\tau w(u)du, & \text{if } \tau \in (a+b-\theta, b], \end{cases} \qquad (2.2)$$

for all $\theta \in [a, \frac{a+b}{2}]$.

Applying RLFI operator and by parts formula of integration, obtain

$$\begin{aligned}
J_a^\gamma(P(\theta,b)g(b)) =& \frac{1}{\Gamma(\gamma)}\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau \\
=& g(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&- (b-\theta)^{1-\gamma}\Gamma(\gamma)J_a^\gamma(w(b)g(b)) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)g(b)).
\end{aligned}$$

It is clear that for all $\tau \in [a,b]$ and $\theta \in [a, \frac{a+b}{2}]$, we have

$$\theta - \frac{a+b}{2} \leq P(\theta,\tau) \leq \theta - a.$$

Applying Proposition (1.2) to the mappings $P(\theta,\cdot)$ and $(b-\cdot)^{\gamma-1}g'(\cdot)$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\gamma)}\left(\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau - \int_a^b P(\theta,\tau)d\tau\cdot\frac{1}{b-a}\int_a^b (b-\tau)^{\gamma-1}g'(\tau)d\tau\right)\right| \\
\leq& \frac{1}{4\Gamma(\gamma)}\left(\theta - a - \left(\theta - \frac{a+b}{2}\right)\right) = \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2),
\end{aligned} \qquad (2.3)$$

for all $\theta \in [a, \frac{a+b}{2}]$. Since $\int_a^b P(\theta,\tau)d\tau = 0$, then (2.3) implies

$$\left|\frac{1}{\Gamma(\gamma)}\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau\right| \leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2). \qquad (2.4)$$

Finally, we obtain desired result (2.1) from (2.4). $\square$

*Remark* 2.1. If we put $\gamma = 1$ and $w = \frac{1}{b-a}$ in Theorem 2.1, then we recapture the Theorem 2.1 of [2].

*Remark* 2.2. If we put $\gamma = 1$ in Theorem 2.1, then we recapture the result of Theorem 2.1 of [15].

**Corollary 2.1.** *In the inequality* (2.1)*, select*
(i) $\theta = \frac{a+b}{2}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{a+b}{2}\right)\int_a^b w(\tau)d\tau - \left(\frac{b-a}{2}\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))\right.\\&\left.+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{a+b}{2},b\right)g(b)\right)\right|\\ \leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2);\end{aligned}$$

(ii) $\theta = \frac{3a+b}{4}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{3a+b}{4}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + 3^{1-\gamma}g(\frac{a+3b}{4})\int_{\frac{a+b}{2}}^b w(\tau)d\tau\right.\\&\left.-\left(\frac{3}{4}(b-a)\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{3a+b}{4},b\right)g(b)\right)\right|\\ \leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2);\end{aligned}\tag{2.5}$$

(iii) $\theta = \frac{2a+b}{3}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{2a+b}{3}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + 2^{1-\gamma}g(\frac{a+2b}{3})\int_{\frac{a+b}{2}}^b w(\tau)d\tau\right.\\&\left.-\left(\frac{2}{3}(b-a)\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{2a+b}{3},b\right)g(b)\right)\right|\\ \leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2).\end{aligned}$$

In the following we present special case of (iii) of Corollary 2.1.
**Special Case.** If put $w = \frac{1}{b-a}$ and $\gamma = 1$ in (iii) of Corollary 2.1, then we get

$$\left|\frac{g(\frac{2a+b}{3})+g(\frac{a+2b}{3})}{2} - \frac{1}{b-a}\int_a^b g(\tau)d\tau\right| \leq \frac{1}{8}(b-a)(M_2-m_2).$$

*Remark* 2.3. (i) First by putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Theorem 2.1 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.1 (a) of [2].

(ii) By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in (i) of Corollary 2.1, we recapture Corollary 2.1 (c) of [2].

(iii) By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in (ii) of Corollary 2.1, we recapture Corollary 2.1 (b) of [2].

*Remark* 2.4. (i) First by putting $\gamma = 1$ in Theorem 2.1 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.3 (i) of [15].

(ii) By putting $\gamma = 1$ in (i) of Corollary 2.1, we recapture Corollary 2.3 (ii) of [15].

(iii) By putting $\gamma = 1$ in (ii) of Corollary 2.1, we recapture Corollary 2.3 (iii) of [15].

(iv) By putting $\gamma = 1$ in (iii) of Corollary 2.1, we recapture Corollary 2.3 (iv) of [15].

Ostrowski's type inequality can be defined in the form of following corollary.

**Corollary 2.2.** *Let the suppositions of Theorem* 2.1 *be valid. Further, if $g$ is symmetric about the $\theta$-axis, i.e., $g(a + b - \theta) = g(\theta)$, then*

$$\begin{aligned}
&\bigg| g(\theta) \int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(\theta)(b-\theta)^{1-\gamma}(\theta - a)^{\gamma-1} \int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&\quad - (b-\theta)^{1-\gamma}\Gamma(\gamma) J_a^{\gamma}(w(b)g(b)) + (\gamma - 1) J_a^{\gamma-1}(P(\theta, b)g(b)) \bigg| \\
&\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2 - m_2) \qquad (2.6)
\end{aligned}$$

*holds.*

*Remark* 2.5. First by putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 2.2 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.2 of [2].

*Remark* 2.6. First by putting $\gamma = 1$ in Corollary 2.2 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.5 of [15].

## 3. Application to Numerical Integration

Let $I_n : a = \theta_0 < \theta_1 < \cdots < \theta_n = b$ be division of interval $[a, b]$ and $h_i = \theta_{i+1} - \theta_i$, $i = 0, 1, 2, \dots, n-1$.

Consider the quadrature formula

$$\begin{aligned}
Q_n(I_n, g) := \sum_{i=0}^{n-1} \Bigg[ & g\Big(\frac{3\theta_i + \theta_{i+1}}{4}\Big) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \\
& \times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma - 1) J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i + \theta_{i+1}}{4}, \theta_{i+1}\right) g(\theta_{i+1})\right)\Bigg].
\end{aligned}$$

We give following result.

**Theorem 3.1.** *Suppose $g : I \to \mathbb{R}$ is a differentiable mapping in interval $I^\circ$ and $w : [a,b] \to \mathbb{R}$ is a probability density function, where $a, b \in I$ with $a < b$. If $g' \in L^1[a,b]$ and $m_2 \leq g'(\theta) \leq M_2$, for all $\theta \in [a,b]$, then the following holds*

$$\Gamma(\gamma) \sum_{i=0}^{n-1} \left(\frac{3}{4} h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1}) g(\theta_{i+1})) = Q_n(I_n, g) + R_n(I_n, g), \tag{3.1}$$

*where $Q_n(I_n, g)$ is stated as above and the following remainder $R_n(I_n, g)$ satisfies the estimates*

$$|R_n(I_n, g)| \leq \frac{1}{8\Gamma(\gamma)} (M_2 - m_2) h_i. \tag{3.2}$$

*Proof.* Applying inequality (2.5) on the intervals $[\theta_i, \theta_{i+1}]$, we get

(3.3)
$$\begin{aligned} R_i(I_i, g) =& \Gamma(\gamma) \left(\frac{3}{4} h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1}) g(\theta_{i+1})) \\ & - \left[ g\left(\frac{3\theta_i + \theta_{i+1}}{4}\right) \int_{\theta_i}^{\frac{\theta_i + \theta_{i+1}}{2}} w(\tau) d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \int_{\frac{\theta_i + \theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau) d\tau \right. \\ & \left. + (\gamma - 1) J_{\theta_i}^{\gamma - 1} \left( P\left(\frac{3\theta_i + \theta_{i+1}}{4}, \theta_{i+1}\right) g(\theta_{i+1}) \right) \right]. \end{aligned}$$

Summing (3.3) over $i$ from 0 to $n-1$, then

$$\begin{aligned} R_n(I_n, g) =& \Gamma(\gamma) \sum_{i=0}^{n-1} \left(\frac{3}{4} h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1}) g(\theta_{i+1})) \\ & - \sum_{i=0}^{n-1} \left[ g\left(\frac{3\theta_i + \theta_{i+1}}{4}\right) \int_{\theta_i}^{\frac{\theta_i + \theta_{i+1}}{2}} w(\tau) d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \right. \\ & \left. \times \int_{\frac{\theta_i + \theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau) d\tau + (\gamma - 1) J_{\theta_i}^{\gamma - 1} \left( P\left(\frac{3\theta_i + \theta_{i+1}}{4}, \theta_{i+1}\right) g(\theta_{i+1}) \right) \right], \end{aligned}$$

which follows the form of (2.5), i.e.,

$$\begin{aligned}|R_n(I_n,g)| =&\Bigg|\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma}J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\&-\sum_{i=0}^{n-1}\left[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}}w(\tau)d\tau+3^{1-\gamma}g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\right.\\&\left.\times\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}}w(\tau)d\tau+(\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\right]\Bigg|\\\leq&\frac{1}{8\Gamma(\gamma)}(M_2-m_2)\sum_{i=0}^{n-1}h_i.\end{aligned}$$

This completes the required proof. □

*Remark* 3.1. By putting $\gamma=1$ and $w=\frac{1}{b-a}$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [2].

*Remark* 3.2. By putting $\gamma=1$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [15].

## 4. Applications to Probability Theory

Throughout this section we consider $w:[a,b]\to[0,1]$. Suppose $Y$ is a random variable taking values in the finite interval $[a,b]$ with probability density function $g:[a,b]\to[0,1]$ and with cumulative distribution function $G:[a,b]\to[0,1]$ is introduced and defined by us, i.e.,

$$G(\theta)=P(Y\leq\theta)=\Gamma(\gamma)J_a^{\gamma}(w(\theta)g(\theta))=\int_a^{\theta}(\theta-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau,\quad a\leq\theta\leq\frac{a+b}{2},$$

and

$$\begin{aligned}E(Y)=\int_a^b\tau g(\tau)d\tau,\quad E_w(Y)&=\int_a^b\tau w(\tau)g(\tau)d\tau,\\E_{wf}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau,\\E_{wf1}(Y)=&\Gamma(\gamma)J_a^{\gamma-1}(bw(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-2}w(\tau)g(\tau)d\tau,\\E_{wf2}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw'(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w'(\tau)g(\tau)d\tau,\\E_{wf3}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw(b)g'(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w(\tau)g'(\tau)d\tau,\end{aligned}$$

are the expectation, weighted expectation and weighted fractional expectation of random variable ‘$Y$’ in interval $[a,b]$, respectively. Then we can write the following theorem.

**Theorem 4.1.** *Suppose $g : [a,b] \to \mathbb{R}$ is a differentiable mapping in the interval $(a,b)$ and $a < b$. If $g' \in L^1[a,b]$ and $m_2 \leq g'(\tau) \leq M_2$, for all $\tau \in [a,b]$. Further, suppose that function $w$ is differentiable, then*

(4.1)

$$\begin{aligned}&\Bigg| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\ &\quad -(b-\theta)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)G(b))\Bigg| \\ &\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2)\end{aligned}$$

*holds for all $\theta \in [a, \frac{a+b}{2}]$.*

*Proof.* Select $g = G$, we obtain (4.1), by applying the identity

$$\begin{aligned}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b)) &= \int_a^b (b-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau \\ &= (\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y).\end{aligned}$$

Since $G(a) = 0$ and $G(b) = 1$.

We left the details to research scholars. □

**Corollary 4.1.** *Select $\gamma = 1$ in Theorem 4.1. Then get the following*

$$\begin{aligned}&\left| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(a+b-\theta)\int_{\frac{a+b}{2}}^b w(\tau)d\tau + E_w(Y) + \int_a^b \tau w'(\tau)G(\tau)d\tau - bw(b)\right| \\ &\leq \frac{1}{8}(b-a)(M_2-m_2)\end{aligned}$$

*holds for all $\theta \in [a, \frac{a+b}{2}]$, where $E_w(Y)$ is the weighted expectation of $Y$.*

*Remark* 4.1. If we put $w = \frac{1}{b-a}$ in Corollary 4.1 and taking the expectation $E(Y) = \int_a^b \tau G(\tau)d\tau = b - \int_a^b G(\tau)d\tau$, we recapture Theorem 4.1 of [2].

**Corollary 4.2.** *Select $\theta = \frac{3a+b}{4}$ in Theorem 4.1, we get*

$$\begin{aligned}&\Bigg| G\left(\frac{3a+b}{4}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G\left(\frac{a+3b}{4}\right)\left(\frac{3}{4}(b-a)\right)^{1-\gamma}\left(\frac{b-a}{4}\right)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\ &\quad -\left(\frac{3}{4}(b-a)\right)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big) \\ &\quad +(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{3a+b}{4}, b\right)G(b)\right)\Bigg| \\ &\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2).\end{aligned}$$

*Remark* 4.2. By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 4.2, we recapture Corollary 4.1 of [2].

**Corollary 4.3.** *In Theorem* 4.1*, if* $G$ *is symmetric about the* $\theta-$*axis, i.e.,* $G(a+b-\theta) = G(\theta)$*, then*

$$\begin{aligned}
&\Bigg| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&- (b-\theta)^{1-\gamma}\Bigg((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Bigg) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)G(b))\Bigg| \\
\leq &\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2)
\end{aligned}$$

*holds for all* $\theta \in [a, \frac{a+b}{2}]$.

*Remark* 4.3. By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 4.3, we recapture Corollary 4.2 of [2].

Before application to special means, we would present some special means and these means will apply in the 5th section.

**Special Means.** These means can be found in [20].

(a) The Arithmetic Mean

$$A(a,b) = \frac{a+b}{2}, \quad a, b \geq 0.$$

(b) The Geometric Mean

$$G = G(a,b) = \sqrt{ab}, \quad a, b \geq 0.$$

(c) The Harmonic Mean

$$H = H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a, b > 0.$$

(d) The Logarithmic Mean

$$L = L(a,b) = \begin{cases} a, & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b, \end{cases} \quad a, b > 0.$$

(e) Identric Mean

$$I = I(a,b) = \begin{cases} a, & \text{if } a = b, \\ \ln\left(\frac{\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}}{e}\right), & \text{if } a \neq b, \end{cases} \quad a, b > 0.$$

(f) $p$-Logarithmic Mean

$$L_p = L_p(a,b) = \begin{cases} a, & \text{if } a = b, \\ \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1, 0\}$, $a, b > 0$. It is known that $L_p$ monotonically increasing over $p \in \mathbb{R}$, $L_0 = I$ and $L_{-1} = L$.

## 5. Application to Special Means

*Example* 5.1. Consider $\gamma = 1$, $g(\theta) = \theta^p$, $p \in \mathbb{R}\backslash\{-1, 0\}$. Then for $a < b$, we have

$$\frac{1}{(b-a)} \int_a^b g(\tau) d\tau = L_p^p(a, b), \quad \frac{g(a) + g(b)}{2} = A(a^p, b^p),$$

and $\frac{a+b}{2} = A(a, b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left| \frac{\theta^p + (2A - \theta)^p}{2} - L_p^p(a, b) \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left| A(a^p, b^p) - L_p^p(a, b) \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

*Example* 5.2. Consider $\gamma = 1$, $g(\theta) = \frac{1}{\theta}$, $\theta \neq 0$. Then

$$\frac{1}{b-a} \int_a^b g(\tau) d\tau = L^{-1}(a, b), \quad \frac{g(a) + g(b)}{2} = \frac{A}{G^2},$$

and $\frac{a+b}{2} = A(a, b)$, where $\theta \in [a, \frac{a+b}{2}] \subset (0, \infty)$.

Therefore, (2.1) becomes

$$\left| \frac{A}{\theta(a + b - \theta)} - L^{-1}(a, b) \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left| \frac{A}{G^2} - L^{-1}(a, b) \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

*Example* 5.3. Consider $\gamma = 1$, $g(\theta) = \ln \theta$, $\theta \in (0, \infty)$. Then

$$\frac{1}{b-a} \int_a^b g(\tau) d\tau = \ln(I(a, b)), \quad \frac{g(a) + g(b)}{2} = \ln G,$$

and $\frac{a+b}{2} = A(a, b)$, where $\theta \in [a, \frac{a+b}{2}] \subset (0, \infty)$. Therefore, (2.1) becomes

$$\left| \ln \left[ \frac{[\theta(2A - \theta)]^{\frac{1}{2}}}{I(a, b)} \right] \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left| \ln \left[ \frac{G}{I(a, b)} \right] \right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

*Example* 5.4. Consider $\gamma = 1$, $g(\theta) = e^\theta$, $\theta \in (-\infty, \infty)$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = \frac{e^b - e^a}{b-a}, \quad \frac{g(a)+g(b)}{2} = A(e^a, e^b),$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left|\frac{e^\theta + e^{(2A-\theta)}}{2} - \frac{e^b - e^a}{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|A(e^a, e^b) - \frac{e^b - e^a}{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

*Example* 5.5. Consider $\gamma = 1$, $g(\theta) = \tan\theta$, $\theta \neq \frac{\pi}{2} \pm n\pi$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}, \quad \frac{g(a)+g(b)}{2} = A(\tan a, \tan b),$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left|\frac{\tan\theta + \tan(2A-\theta)}{2} - \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|A(\tan a, \tan b) - \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

## 6. Conclusion

In this article our target was to generalise the results of [2] and [15]. We have obtained generalisation of companion of Ostrowski's type integral inequality for differentiable mappings whose 1st derivatives are bounded by using the Riemann-Liouville fractional integral. By applying suitable substitutions we have recaptured all results of M. W. Alomari's article [2] and also recaptured all results of one more article [15] of different authors. Moreover, we have given applications to numerical integration, probability theory and special means.

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[1]Department of Mathematics,
Samarkand State University,
University boulevard 15, Samarkand 140104, Uzbekistan

[2]Department of Mathematics,
Dawood University of Engineering and Technology,
New M. A. Jinnah Road, Karachi-74800, Pakistan
*Email address*: faraz.mehmood@duet.edu.pk

[3]Department of Mathematics,
University of Karachi,
University Road, Karachi-75270 Pakistan
*Email address*: asifrk@uok.edu.pk
*Email address*: m.awaisshaikh2014@gmail.com

[4]Department of Mathematics,
Faculty of Science and Information Technology, Jadara University,
P.O. Box 733, Irbid, P.C. 21110, Jordan
*Email address*: mwomath@gmail.com

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# EVALUATION SUBGROUPS OF A MAP BETWEEN RATIONAL FINITE H-SPACES

ABDELHADI ZAIM

Abstract. We use the theory of Sullivan minimal models and derivation to compute the evaluation subgroups and moreover the relative evaluation subgroups of a map $f : X \to Y$ between rational finite H-spaces. As a consequence, we show that the $G$-sequence is exact if $f$ induces a zero map on rational homotopy groups.

## 1. Introduction

In this paper all spaces are assumed to be simply connected CW-complex and are of finite type over $\mathbb{Q}$, that is, have finite dimensional rational cohomology in each degree.

An important problem in homotopy theory is the computation of the Gottlieb groups. It is often difficult to describe these groups fully and the best that can be hoped for is some partial information about them. As is well known, the homotopy theory of rational spaces, i.e., spaces whose homotopy groups are vector spaces over $\mathbb{Q}$, is equivalent to the homotopy theory of minimal commutative differential graded algebras over $\mathbb{Q}$. More precisely, there is an equivalence between the homotopy category of rational spaces and the homotopy category of minimal cdgas. However, it is known that the category of continuous map between rational spaces is equivalent to the category of morphism between corresponding models.

Now, let us recall the definition of Gottlieb groups. Given a based space $X$, the $n$-th Gottlieb group or the $n$-th evaluation subgroups of $X$ is the subgroup of $\pi_n(X)$ consisting of homotopy classes of map $h : \mathbb{S}^n \to X$ such that the wedge

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$(h \vee Id_X) : \mathbb{S}^n \vee X \to X$ extends to a map $H : \mathbb{S}^n \times X \to X$. Alternately, it is known that

$$G_n(X) = \mathrm{Im}\left(\mathrm{ev}_\sharp : \pi_n(\mathrm{aut}_1(X)) \to \pi_n(X)\right),$$

where $\mathrm{aut}_1(X)$ is the set of self-homotopy equivalences of $X$ which are homotopic to the identity map. These groups were discovered and studied by Gottlieb in the early 1960's (the interested reader may consult [4]). They have led to many interesting results in algebraic topology, especially, in theory of fibrations.

The generalization of Gottlieb groups was initiated by Woo and Kim [10]. Let $f : X \to Y$ be a based map of spaces, then the $n$-th evaluation subgroups of $f$, also called the $n$-th generalized Gottlieb group is given by:

$$G_n(Y, X; f) = \mathrm{Im}\left(\mathrm{ev}_\sharp : \pi_n(map(X, Y; f)) \to \pi_n(Y)\right).$$

Here $\mathrm{map}(X, Y; f)$ means that the space of all maps from $X$ to $Y$ that are homotopic to $f$. Note that

$$G_n(X, X; Id_X) = G_n(X),$$

and in general we have $G_n(Y) \subset G_n(Y, X; f)$.

The Gottlieb groups and the generalized Gottlieb groups play a profound role in the homotopy theory of fibrations. But, until now, there are not many explicit computations of $G_*(X)$ and $G_*(Y, X; f)$. Since a map of spaces does not necessarily induce a corresponding homomorphism of Gottlieb groups, Woo and Lee was introduced the $n$-th relative evaluation subgroups $G_n^{rel}(Y, X; f)$, also called the $n$-th relative Gottlieb group [11]. The authors showed also that these groups fit in the following G-sequence

$$\cdots \to G_{n+1}^{rel}(Y, X; f) \to G_n(X) \to G_n(Y, X; f) \to G_n^{rel}(Y, X; f) \to \cdots \tag{1.1}$$

The computation of rational relative evaluation subgroups have been receiving a growing attention and have become a popular subject of study with a lot of progresses (see [3, 6, 12, 13] for instance). Our goal in this paper is to compute these subgroups in some new cases. So, by using Sullivan minimal models and other invariants in rational homotopy theory we compute the relative Gottlieb groups of a map between rational finite H-spaces. These spaces form a very well-studied and interesting class of spaces which appear abundantly in geometry and topology. They include products of rational spheres.

The paper is organized as follows. In Section 2, we introduce our notation and recall some background of rational homotopy theory, namely Sullivan minimal models, derivations and mapping cone. Next, we use them to recall the algebraic version of all terms involved in the G-sequence (1.1). Section 3 is devoted to our results and their proofs.

## 2. Preliminaries

We will work with $\mathbb{Q}$ as ground field and our principal tools are Sullivan minimal models. A detailed description of these and the standard tools of rational homotopy theory can be found in [1]. For our purposes, we recall the following.

A commutative differential graded algebra (abbreviated cdga) $(A, d)$ consists of graded vector spaces $A = \oplus_{i\geq 0} A^i$ with a multiplication $A^i \otimes A^j \to A^{i+j}$ satisfying $ab = (-1)^{ij}\, ba$ and a map $d : A^i \to A^{i+1}$, such that $d^2 = 0$ and $d\,(ab) = d\,(a)\, b + (-1)^i\, ad\,(b)$ for all $a \in A^i$ and $b \in A^j$. A cdga $(A, d)$ is called simply connected if $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. Denote by

$$V^{even} = \oplus_{i\geq 1} V^{2i} \quad \text{and} \quad V^{odd} = \oplus_{i\geq 1} V^{2i+1}.$$

We say that a simply connected cdga $(A, d)$ is a Sullivan minimal algebra if it is of the form

$$(\Lambda V, d) = (S(V^{even}) \otimes E(V^{odd}), d),$$

such that $dV \subset \Lambda^{\geq 2} V$.

For a simply connected CW-complex $X$ of finite type, there is a cdga $A_{PL}(X)$ verifying

$$H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q}).$$

Then the Sullivan minimal model of $X$ is defined to be the Sullivan minimal model of $A_{PL}(X)$ which is unique up to isomorphism [1]. Moreover, the rational homotopy type of $X$ is completely determined by its Sullivan minimal model $(\Lambda V, d)$, that is, $Hom(V, \mathbb{Q}) \cong \pi_*(X) \otimes \mathbb{Q}$ as graded vector spaces.

Now, we go to generalize this situation to any map $f : X \to Y$ of simply connected spaces. A Sullivan model of $f$ is a morphism of cdga's

$$\phi : (\Lambda W, d_W) \to (\Lambda V, d_V),$$

where $(\Lambda V, d_V)$ and $(\Lambda W, d_W)$ are the Sullivan minimal models of $X$ and $Y$, respectively.

Next, consider $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ a morphism of cdga's. A $\phi$-derivation $\theta$ of degree $n$ is a linear map

$$\theta : (\Lambda W)^m \to (\Lambda V)^{m-n}$$

verifying $\theta\,(xy) = \theta\,(x)\, \phi\,(y) + (-1)^{n|x|}\, \phi\,(x)\, \theta\,(y)$ for $x$ and $y$ are in $W$. In the following, we denote by $Der_n\,(\Lambda W, \Lambda V; \phi)$ the vector space of $\phi$-derivations of degree $n$ and when $n = 1$, we require additionally that all derivations are cycles, that is,

$$d_V \circ \theta = -\theta \circ d_W.$$

There is a differential

$$\delta : Der_n\,(\Lambda W, \Lambda V; \phi) \to Der_{n-1}\,(\Lambda W, \Lambda V; \phi),$$

given by

$$\delta\,(\theta) = d_V \circ \theta - (-1)^n\, \theta \circ d_W.$$

Moreover, let

$$Der_*\,(\Lambda W, \Lambda V; \phi) = \oplus_n Der_n\,(\Lambda W, \Lambda V; \phi),$$

and in particular

$$Der_*\,(\Lambda V, \Lambda V; Id_{\Lambda V}) = Der_*\,(\Lambda V).$$

Further, it is easy to see that there is an isomorphism of graded vector spaces

$$Der_*\left(\Lambda W,\Lambda V;\phi\right)\cong Hom_*\left(W,\Lambda V\right).$$

Thus, we denote by $(w,v)\in Der_{|w|-|v|}\left(\Lambda W,\Lambda V;\phi\right)$ the unique $\phi$-derivation sending an element $w\in W$ to $v\in\Lambda V$ and the other generators to zero, and in particular $(w,1)=w^*$ for an element $w\in W$.

In the remainder of this section, we give a description in rational homotopy theory of all terms involved in the G-sequence (1.1) (see [5]). For this, let us consider the augmentation map $\varepsilon:\Lambda V\to\mathbb{Q}$ which induces $\varepsilon_*: Der_*\left(\Lambda W,\Lambda V;\phi\right)\to Der_*\left(\Lambda W,\mathbb{Q};\varepsilon\right)$. Thus, the $n$-th evaluation subgroups of $\phi$ is given by

$$G_n\left(\Lambda W,\Lambda V;\phi\right)=\operatorname{Im}\left\{H_n\left(\varepsilon_*\right):H_n\left(Der\left(\Lambda W,\Lambda V;\phi\right)\right)\to Hom_n\left(W,\mathbb{Q}\right)\right\},\quad \text{for } n\geq 2.$$

So, an element $w^*$ in $Hom_n\left(W,\mathbb{Q}\right)$ is in $G_n\left(\Lambda W,\Lambda V;\phi\right)$ if and only if $w^*$ extends to a cycle-derivation $\theta$ of $Der_n\left(\Lambda W,\Lambda V;\phi\right)$.

A special case of the preceding that is of interest to us is the one in which $\Lambda W\cong\Lambda V$ and $\phi=Id_{\Lambda V}$. In this case, we get:

$$G_n\left(\Lambda V\right)=\operatorname{Im}\left\{H_n\left(\varepsilon_*\right):H_n\left(Der\left(\Lambda V\right)\right)\to\ Hom_n\left(V,\mathbb{Q}\right)\right\},\quad \text{for } n\geq 2,$$

which is called the $n$-th Gottlieb group of $(\Lambda V,d_V)$.

The following result is due to S.B. Smith which shows that the rational evaluation subgroups of a map are completely determined only by the graded vector space of derivations (see [9]).

**Theorem 2.1.** *Suppose $\phi:(\Lambda W,d_W)\to(\Lambda V,d_V)$ is a Sullivan model of a map $f:X\to Y$ between simply connected CW-complexes such that $X$ is finite. Then*

$$G_n\left(Y,X;f\right)\otimes\mathbb{Q}\ \cong\ G_n\left(\Lambda W,\Lambda V;\phi\right),\quad n\geq 2.$$

Next, let us remind the notion of mapping cone which is very useful to characterize the rational relative evaluation subgroups in terms of derivations.

Suppose $\phi:(A,d_A)\to(B,d_B)$ is a map of differential graded vector spaces. We define differential graded vector spaces $\left(Rel_*\left(\phi\right),D\right)$, called the mapping cone of $\phi$, as follows

$$Rel_n\left(\phi\right)=A_{n-1}\oplus B_n.$$

The differential

$$D:Rel_n\left(\phi\right)\to Rel_{n-1}\left(\phi\right)$$

is given by

$$D\left(a,b\right)=\left(-d_A\left(a\right),\phi\left(a\right)+d_B\left(b\right)\right).$$

Furthermore, we define the following maps

$$J:B_n\to Rel_n\left(\phi\right)\qquad\text{and}\qquad P:Rel_n\left(\phi\right)\to A_{n-1}$$

by

$$J\left(b\right)=\left(0,b\right)\qquad\text{and}\qquad P\left(a,b\right)=a,\qquad\text{for } (a,b)\in A\times B.$$

Next, suppose given a differential graded algebra map $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ of a map $f : X \to Y$. Then, we can constructed a map of chain complexes

$$\phi^* : Der_* (\Lambda V) \to Der_* (\Lambda W, \Lambda V; \phi), \quad \text{given by } \phi^*(\theta) = \phi \circ \theta.$$

The consideration above induces the following commutative diagram

$$\begin{array}{ccccc} Der_* (\Lambda V) & \xrightarrow{\phi^*} & Der_* (\Lambda W, \Lambda V; \phi) & \xrightarrow{J} & Rel_*(\phi^*) \\ \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle (\varepsilon_*, \varepsilon_*)} \\ Der_* (\Lambda V, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & Der_* (\Lambda W, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{J}} & Rel_*(\widehat{\phi^*}) \end{array}.$$

In this diagram, $\varepsilon$ denotes the augmentation of either $\Lambda V$ or $\Lambda W$. So, the $n$-th relative evaluation subgroups of $\phi$ is defined as follows

$$G_n^{rel} (\Lambda W, \Lambda V; \phi) = \mathrm{Im} \left\{ H_n (\varepsilon_*, \varepsilon_*) : H_n(Rel(\phi^*)) \to H_n(Rel(\widehat{\phi^*})) \right\}, \quad \text{for } n \geq 2.$$

We finish this section by some notations and conventions. The cohomology of a cdga $(A, d)$ is denoted $H^* (A, d)$ or just $H^* (A)$ and let $[x] \in H^* (A, d)$ stand for the cohomolgy class of the cocycle $x \in A$. In the sequel, all spaces appearing in this paper are assumed to be *rational simply connected CW-complexes*, i.e., all spaces satisfy $X = X_{\mathbb{Q}}$.

## 3. Relative Evaluation Subgroups of a Map Between Rational Finite H-Spaces

Our aim in this section is devoted to compute in terms of Sullivan minimal models and derivations the rational evaluation subgroups and moreover the rational relative evaluation subgroups of a map between finite H-spaces.

3.1. **Evaluation subgroups of a map between rational finite H-spaces.** In the first place, we are interested in determining the evaluation subgroups of a rational H-space.

**Proposition 3.1.** *Suppose $X$ is a rational H-space. Then $G_* (X) \cong \pi_* (X)$.*

*Proof.* It is easy, but for the sake of completeness we write a proof. It is well known that a rational H-space $X$ has the rational homotopy type of a product of Eilenberg-MacLane spaces, i.e., $X \simeq_{\mathbb{Q}} \prod_i K(\mathbb{Q}, n_i)$ (see [8, Corollary 1]). It follows that

$$G_* (X) \cong G_* \left( \prod_i K(\mathbb{Q}, n_i) \right) \cong \oplus_i G_* (K(\mathbb{Q}, n_i)) \cong \oplus_i \pi_* (K(\mathbb{Q}, n_i)) \cong \pi_*(X),$$

as required. $\square$

We note that Proposition 3.1 can also be showed by using the Sullivan minimal model of a rational H-space.

Now let's move on to evaluation subgroups of a map between rational finite H-spaces.

**Proposition 3.2.** *Let $f : X \to Y$ be a map between rational finite H-spaces, then*

$$G_* (Y, X; f) \cong \pi_* (Y).$$

*Proof.* From [1, Example 3, p. 143] we know that the Sullivan minimal model of $Y$ is given by

$$(\Lambda W, 0) = (\Lambda (y_1, y_2, \ldots, y_p), 0)$$

and the Sullivan minimal model of $X$ is of the form

$$(\Lambda V, 0) = (\Lambda (x_1, x_2, \ldots, x_q), 0),$$

where $p = \dim \pi_*(Y)$ and $q = \dim \pi_*(X)$. Now, denote by $(y_j, 1)$ the derivation $\theta_j$ in $Der_{|y_j|} (\Lambda W, \Lambda V; \phi)$ for $1 \leq j \leq p$. Since the differential on $\Lambda V$ and $\Lambda W$ are trivial, it follows that the differential $\delta$ on $Der_* (\Lambda W, \Lambda V; \phi)$ is trivial. It is therefore automatic that $\theta_j$ is closed and not a boundary. Further, we consider

$$\varepsilon_* : Der_* (\Lambda W, \Lambda V; \phi) \to Hom(W, \mathbb{Q}),$$

which is given by $\varepsilon_*(\theta_j) = y_j^*$ for $1 \leq j \leq p$. Hence, combining the preceding we obtain

$$G_* (\Lambda W, \Lambda V; \phi) = \langle y_1^*, y_2^*, \ldots, y_p^* \rangle \cong Hom(W, \mathbb{Q}),$$

which completes the proof. ☐

3.2. **Relative evaluation subgroups of a map between rational finite H-spaces.** In this subsection, we will present our main result concerning the relative evaluation subgroups of a map between rational finite H-spaces. First, we offer one example to illustrate the general idea and then give a summary result.

*Example* 3.1. Suppose $f : X \to Y$ is a map of rational H-spaces which its Sullivan model

$$\phi : (\Lambda W, 0) = (\Lambda (x_3, y_4, z_8), 0) \to (\Lambda V, 0) = (\Lambda (u_3, v_5, w_9), 0)$$

is given on generators by $\phi(x) = u$, $\phi(y) = 0$ and $\phi(z) = uv$. In both Sullivan minimal models, subscripts denote degrees. We compute $\mathrm{G}_*^{rel} (\Lambda W, \Lambda V; \phi)$ as follows.

Let us consider

$$\phi^* : Der_* (\Lambda V) \to Der_* (\Lambda W, \Lambda V; \phi),$$

which is given by $\phi^* (u^*) = x^* + (z, v)$, $\phi^* (v^*) = (z, u)$ and $\phi^* (w^*) = 0$. Thus, we have immediately

$$D (u^*, 0) = (0, x^* + (z, v)), \quad D (v^*, 0) = (0, (z, u)) \quad \text{and} \quad D (w^*, 0) = 0.$$

Further, it is easy to see that

$$D (0, x^*) = D (0, y^*) = D (0, z^*) = 0.$$

An easy argument shows that the elements $[(w^*, 0)]$, $[(0, y^*)]$ and $[(0, z^*)]$ are nonzero in $H_* (Rel (\phi^*))$. Next, denote by $\varepsilon_* (a^*) = \widehat{a^*}$ for an element $a$ in $W$ or $V$. Since, we have $H_*(\varepsilon_*,\varepsilon_*)([(w^*,0)]) = \left[\left(\widehat{w^*},0\right)\right]$, $H_*(\varepsilon_*,\varepsilon_*)([(0,y^*)]) = \left[\left(0,\widehat{y^*}\right)\right]$ and also $H_*(\varepsilon_*,\varepsilon_*)([(0,z^*)]) = \left[\left(0,\widehat{z^*}\right)\right]$ are nonzero in $H_*\left(Rel\left(\widehat{\phi^*}\right)\right)$. It follows that

$$G_*^{rel}(\Lambda W,\Lambda V;\phi) = \left\langle \left[\left(\widehat{w^*},0\right)\right], \left[\left(0,\widehat{y^*}\right)\right], \left[\left(0,\widehat{z^*}\right)\right] \right\rangle.$$

The following discussion will fix our notation. Suppose $f : X \to Y$ is a map between rational finite H-spaces and denote by $\phi : (\Lambda W, 0) \to (\Lambda V, 0)$ the Sullivan model of $f$. Let $\{y_1,\ldots,y_r,y_{r+1},\ldots,y_s,y_{s+1},\ldots,y_p\}$ be a basis for $W$ and let us consider $\{x_1,\ldots,x_r,x_{r+1},\ldots,x_t,x_{t+1},\ldots,x_q\}$ be a basis for $V$. By using part (a) of ([2], Proposition 2.2) and a change of KS-basis, we can write $\phi(y_{i_0}) = x_{i_0}$ for $1 \leq i_0 \leq r$, $\phi(y_{i_1}) \in \Lambda^{\geq 2}(x_1,\ldots,x_t)$ -contains only decomposable elements- for $r+1 \leq i_1 \leq s$ and $\phi(y_{i_2}) = 0$ for $s+1 \leq i_2 \leq p$. Hence, it is easy to see that $\phi$ is well defined.

Now, we may extend the argument explained in Example 3.1 to give our main result in this section.

**Theorem 3.1.** *With the same notation as above, let $f : X \to Y$ be a map between rational finite H-spaces and $\phi : (\Lambda W, 0) \to (\Lambda V, 0)$ its Sullivan model. Then*

$$G_*^{rel}(\Lambda W,\Lambda V;\phi) = \left\langle \left[\left(\widehat{x_{j_2}^*},0\right)\right], \left[\left(0,\widehat{y_{i_1}^*}\right)\right], \left[\left(0,\widehat{y_{i_2}^*}\right)\right] \right\rangle,$$

*for $t+1 \leq j_2 \leq q$, $r+1 \leq i_1 \leq s$ and $s+1 \leq i_2 \leq p$.*

*Proof.* First, as recalled above denote by

$$\phi : (\Lambda(y_1,\ldots,y_p),0) \to (\Lambda(x_1,\ldots,x_q),0)$$

the Sullivan model of $f$ in which is given by $\phi(y_{i_0}) = x_{i_0}$ for $1 \leq i_0 \leq r$, $\phi(y_{i_1}) \in \Lambda^{\geq 2}(x_1,\ldots,x_t)$ for $r+1 \leq i_1 \leq s$ and $\phi(y_{i_2}) = 0$ for $s+1 \leq i_2 \leq p$. Thus, it induces the following map

$$\phi^* : Der_*(\Lambda V) \to Der_*(\Lambda W,\Lambda V;\phi),$$

where $\phi^*\left(x_{i_0}^*\right) = y_{i_0}^* + (y_{i_1}, v_{i_1})$ such that $v_{i_1} = x_{i_0}^*(\phi(y_{i_1}))$, $\phi^*\left(x_{j_1}^*\right) = (y_{i_1}, w_{i_1})$ such that $w_{i_1} = x_{j_1}^*(\phi(y_{i_1}))$ for $r+1 \leq j_1 \leq t$ and finally, $\phi^*\left(x_{j_2}^*\right) = 0$ for $t+1 \leq j_2 \leq q$. Here, since $\phi(y_{i_1})$ is decomposable, we note that the elements $v_{i_1}$ and $w_{i_1}$ are in $\Lambda^+(x_1,\ldots,x_t)$. Further, an easy computation gives that

$$D\left(x_{i_0}^*,0\right) = \left(0, y_{i_0}^* + (y_{i_1},v_{i_1})\right), \quad D\left(x_{j_1}^*,0\right) = (0,(y_{i_1},w_{i_1})) \quad \text{and} \quad D\left(x_{j_2}^*,0\right) = 0.$$

Hence, for $r+1 \leq i_1 \leq s$ and $s+1 \leq i_2 \leq p$, we have

$$D\left(0,y_{i_1}^*\right) = D\left(0,y_{i_2}^*\right) = 0.$$

Next, an easy argument by contradiction shows that $\left(x_{j_2}^*,0\right)$, $\left(0,y_{i_1}^*\right)$ and $\left(0,y_{i_2}^*\right)$ are not D-boundaries. This means that $\left[\left(x_{j_2}^*,0\right)\right]$, $\left[\left(0,y_{i_1}^*\right)\right]$ and $\left[\left(0,y_{i_2}^*\right)\right]$ are non null in $H_*(Rel(\phi^*))$. Otherwise, we see that

$$Rel_*\left(\widehat{\phi^*}\right) = Der_{*-1}(\Lambda V,\mathbb{Q};\varepsilon) \oplus Der_*(\Lambda W,\mathbb{Q};\varepsilon).$$

Moreover, we recall that for $a \in V$

$$\varepsilon_* : Der_*(\Lambda V) \to Der_*(\Lambda V, \mathbb{Q}; \varepsilon), \quad \varepsilon_*(a^*) = \widehat{a^*},$$

and let also for $b \in W$

$$\varepsilon_* : Der_*(\Lambda W, \Lambda V; \phi) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon), \quad \varepsilon_*(b^*) = \widehat{b^*}$$

Next to determine $H_*\left(Rel\left(\widehat{\phi^*}\right)\right)$, we need to compute the differential $\widehat{D}$ in $\mathrm{Rel}_*\left(\widehat{\phi^*}\right)$. For this, let

$$\widehat{\phi^*} : Der_*(\Lambda V, \mathbb{Q}; \varepsilon) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon),$$

which is defined as follows: $\widehat{\phi^*}\left(\widehat{x^*_{i_0}}\right) = \widehat{y^*_{i_0}}$ and $\widehat{\phi^*}\left(\widehat{x^*_{j_1}}\right) = \widehat{\phi^*}\left(\widehat{x^*_{j_2}}\right) = 0$. Hence, in a similar fashion as above, we prove that the elements $\left(\widehat{x^*_{j_2}}, 0\right)$, $\left(0, \widehat{y^*_{i_1}}\right)$ and $\left(0, \widehat{y^*_{i_2}}\right)$ are cycles which are not $\widehat{D}$-boundaries. Then, by summarizing all the above we get

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(x^*_{j_2}, 0\right)\right]\right) = \left[\left(\widehat{x^*_{j_2}}, 0\right)\right], \quad \text{for } t+1 \leq j_2 \leq q,$$

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(0, y^*_{i_1}\right)\right]\right) = \left[\left(0, \widehat{y^*_{i_1}}\right)\right], \quad \text{for } r+1 \leq i_1 \leq s,$$

and also

$$H_*(\varepsilon_*, \varepsilon_*)\left(\left[\left(0, y^*_{i_2}\right)\right]\right) = \left[\left(0, \widehat{y^*_{i_2}}\right)\right], \quad \text{for } s+1 \leq i_2 \leq p.$$

In summary, we have proved that

$$G^{rel}_*(\Lambda W, \Lambda V; \phi) = \left\langle \left[\left(\widehat{x^*_{j_2}}, 0\right)\right], \left[\left(0, \widehat{y^*_{i_1}}\right)\right], \left[\left(0, \widehat{y^*_{i_2}}\right)\right] \right\rangle,$$

for $t+1 \leq j_2 \leq q$, $r+1 \leq i_1 \leq s$ and $s+1 \leq i_2 \leq p$. □

**Proposition 3.3.** *Given a map $f : X \to Y$ of rational finite H-spaces. Suppose that $f$ induces an injective morphism on rational homotopy groups, then*

$$G^{rel}_*(Y, X; f) \cong \pi_*(Y) \big/ \pi_*(X).$$

*Proof.* Denote by $(\Lambda W, 0)$ and $(\Lambda V, 0)$ the Sullivan minimal models respectively of $Y$ and $X$. Let $\{x_1, \ldots, x_q\}$ be a homogeneous basis for $V$. As $f$ induces an injective morphism on rational homotopy groups, we may choose $y_{q+1}, \ldots, y_p$ such that $\{x_1, \ldots, x_q, y_{q+1}, \ldots, y_p\}$ is a homogeneous basis for $W$. Further, let us denote by

$$\phi : (\Lambda(x_1, \ldots, x_q, y_{q+1}, \ldots, y_p), 0) \to (\Lambda(x_1, \ldots, x_q), 0)$$

the Sullivan model of $f$ which is defined as follows: $\phi(x_i) = x_i$ for $1 \leq i \leq q$ and $\phi(y_j) = 0$ for $q+1 \leq j \leq p$. Of course, we have $\phi^*(x_i^*) = x_i^*$ for $1 \leq i \leq q$. Then by using a similar argument given in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} G^{rel}_*(\Lambda W, \Lambda V; \phi) = & \left\langle \left[(0, \widehat{y_j{}^*})\right] \text{ for } q+1 \leq j \leq p \right\rangle \\ \cong & \pi_*(Y) \big/ \pi_*(X), \quad \text{as graded vector spaces.} \end{aligned}$$ □

Various conditions are known under which the $G$-sequence of a map is exact [5, 7]. However, in general there is not information about the exactness of the G-sequence.

**Corollary 3.1.** *Let $f : X \to Y$ be a map between rational finite H-spaces in which $f$ induces an injective morphism on rational homotopy groups. Then the G-sequence of $f$ splits into short exact sequence*

$$0 \to G_*(X) \to G_*(Y, X; f) \to G_*^{rel}(Y, X; f) \to 0.$$

*Proof.* It follows directly from Proposition 3.1, Proposition 3.2 and Proposition 3.3 together with the G-sequence (1.1). □

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Faculty of Sciences Ain Chock,
University Hassan II,
Km 8 Route d'El Jadida, Maarif, B.P.5366, Casablanca, Morocco
*Email address*: abdelhadi.zaim@gmail.com

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# A STUDY OF FUNCTIONS ON THE TORUS AND MULTI-PERIODIC FUNCTIONS

DHAOU LASSOUED[1]

Abstract. In this paper, we are concerned with functions defined on the cube $Q^m = [-\pi, \pi]^m$ and functions defined on the torus $\mathbb{T}^m$. Especially, the harmonic analysis of Sobolev-type spaces is carefully studied. We analyze in particular periodic distributions and distributions on the torus. We introduce a space similar to $H^1_0$, for which we prove a Poincaré-Wirtinger inequality. We prove that the usual Rellich-Kondrachov result does not hold for these last space because of the lack of compactness. A result of absolute continuity and density of regular functions is then established and a theorem of traces is obtained.

## 1. Introduction

Functions which repeat themselves after a fixed length of their arguments, so-called period, are called the periodic functions. Common examples of the periodic functions are the trigonometric sine and cosine functions with period each. Geometrically, a periodic function is the one whose graph displays a translational symmetry. In particular, a function is periodic with period if the graph remains invariant under translation in the direction by a distance.

Periodic functions appear in many practical problems. In most of the cases, they are more complicated than the ordinary sine and cosine functions. Indeed, periodic functions are a vital part of all scientific, engineering, technological and mathematical processes. In all branches of mathematics, they have well-defined analogues. These functions are used in modeling many dynamical, physical, and biological processes. They have wide-range applications in different fields of science, mathematics and

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engineering to study and characterize phenomena like conduction of heat, mechanical vibrations, electric circuits and electromagnetic waves etc.

Based on the periodic functions, there exist their periodic extensions. In fact, periodic extensions of the periodic functions are another class of the functions which are used in modeling more complex physical, biological and more advanced systems. Most of the studies on the periodic functions and their extensions have been limited to a single period. Their transformation to the functions called the periodic extension are the half-wave rectification, full-wave rectifications in electrical engineering. The graph of such a function is obtained by periodic repetition of its graph in any interval of the length of its period. We call these periodic extensions as the multiperiodic functions. Multiperiodic functions are commonly used to model complex dynamic systems in applied and pure sciences such as population dynamics and weather forecasting.

Historically, the Fourier transform was first introduced by Joseph Fourier in his study of the heat equation. In this context, Fourier showed how a periodic function could be decomposed into a sum of sines and cosines which represent the frequencies of the function. From a modern point of view, the Fourier transform is a transformation which accepts a function and returns a new function, defined via the frequency data of the original function. As a bridge between the physical domain and the frequency domain, the Fourier transform seems to be the main tool of use in harmonic analysis.

Moreover, Sobolev spaces are the main tools in the modern theory of partial differential equations. They give a very natural functional analytical framework for the study of existence, regularity and qualitative properties of the boundary value problems. Indeed, Sobolev spaces are ubiquitous in harmonic analysis and Partial differential equations, where they appear naturally in problems about regularity of solutions or well-posedness. Tightly connected to these problems are certain embedding theorems that relate the norms of Lebesgue and Sobolev spaces for appropriate indices. The appeal of Sobolev spaces is due to the simplicity of their definition which captures both the regularity and size of a distribution. On the other hand, an efficient tool when dealing with Sobolev spaces and partial differential equations is the Poincaré-Wirtinger inequality that provides norm equivalences under appropriate assumptions. These inequalities usually provide Sobolev embeddings and compactness results (see Adams [1]).

Besides, the Rellich-Kondrachov compactness theorem, which gives compact embeddings of Sobolev spaces [2, Theorem 6.2], is fundamental for the study of elliptic boundary value problems.

On the other hand, due to its geometric and topological structure, the torus has shown an interesting means to study periodicity/quasi-periodicity of solutions in many applications. The interest of considering spaces of functions on the torus is because these can be identified with periodic functions, so it is natural to look for solutions of partial differential equations with periodic border conditions in these spaces. Now, boundary conditions are understood in the sense of the trace, which is related to the trace method and the trace spaces in the theory of interpolation.

In this paper, we aim to study the functions defined on the cube $[-\pi,\pi]^m$ and those defined on the torus $\mathbb{T}^m$. That means the behaviours of multiperiodic functions and functions on the torus are analyzed. We give a complete harmonic analysis of Sobolev-type spaces. The distributions on the torus and the periodic distributions are studied. Namely, we introduce a space similar to the classical space $H_0^1$ and we prove that on this introduced space, a Poincaré-Wirtinger inequality holds true while the known Rellich-Kondrachov is no longer valid due to the lack of compactness. We finally get a result on absolute continuity and density of regular functions and a theorem of traces.

## 2. Notations and Preliminary results

Throughout this manuscript, we denote by $\mathbb{N}^*$ or by $\mathbb{Z}_+$ the set of all positive integers and by $\mathbb{Z}$ the set of all integers. The set of real numbers is denoted by $\mathbb{R}$ and that of all complex numbers is denoted by $\mathbb{C}$.

We shall fix an integer $m \geq 2$ and a vector $\omega \in \mathbb{R}^m$ whose components are linearly independent on $\mathbb{Z}$ and are strictly positive. We denote also by $Q^m$ the cube of dimension $m$, that is, $Q^m := [-\pi,\pi]^m$. The notation $\mathbb{T}^m$ shall be deserved for the torus of dimension $m$, that is $\mathbb{T}^m := \mathbb{R}^m/(2\pi\mathbb{Z})^m$.

For a vector $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, we denote by $x_{-j}$ the vector of $\mathbb{R}^{m-1}$ defined by $x_{-j} := (x_1,\dots,x_{j-1},x_{j+1},\dots,x_m)$. The couples $(x_{-j},x_j)$ and $(x_j,x_{-j})$ denote both $x$.

For $x,y \in \mathbb{K}^m$, we denote by $x\cdot y$ and by $|x|$ respectively the usual inner product of $x$ and $y$ and its associated euclidean norm:

$$x\cdot y = \sum_{j=1}^m x_j y_j, \quad |x| = \sqrt{x\cdot x}.$$

The notation $e_\nu : \mathbb{R}^m \to \mathbb{C}$, for $\nu \in \mathbb{R}$, is reserved for the function defined for all $x \in \mathbb{R}^m$ by $e_\nu(x) := \exp(i\nu\cdot x)$.

We shall denote by $\mathbb{Z}\langle u_1,\dots,u_p\rangle$ ($u_i$ belongs to a linear space $F$ over $\mathbb{R}$) [9, p. 81], $\mathbb{Z}$-modulus generated by $u_1,\dots u_p$, that is,

$$\mathbb{Z}\langle u_1,\dots,u_p\rangle := \left\{\sum_{j=1}^p k_j u_j : (k_j)_j \in \mathbb{Z}^p\right\}.$$

We denote by $\tau_p(u)$, for a given function $u : \mathbb{R}^m \to \mathbb{E}$ and a given $p \in \mathbb{R}^m$, the translated function of $u$ defined on $\mathbb{R}^m$ by $\tau_p(u)(x) := u(x+p)$.

If the space $\mathbb{E}$ is a topological space, we use the standard notation, for a given subset $A \subset \mathbb{E}$, $\mathrm{int}(A)$ the interior of the set $A$.

Let us consider a linear normed space $X$ over $\mathbb{R}$ of finite dimension $m$.

### 2.1. First notions.

We give here the following definition that can be found for example in [4, p. 55] or in [15, p. 64].

**Definition 2.1.** A function $F : X \to \mathbb{E}$ is said to be periodic if there exists a non zero $p \in X$ such that for all $x \in X$, we have $F(x + p) = F(x)$. Such a vector $p$, as well as 0, is called a period of $F$.

The set of all periods of a given function is an abelian subgroup of $(X, +)$, which becomes also a closed set if the function is continuous. We denote by $Per(F)$ the set of periods of a function $F$.

**Proposition 2.1.** *Let $F : X \to \mathbb{E}$ be a periodic function, $Y$ be a normed linear space of dimension $m$ and $L$ be a linear isomorphism from $Y$ into $X$. Let us define the function $G := F \circ L : Y \to \mathbb{E}$. Then, $G$ is a periodic function, and $Per(G) = L^{-1}(Per(F))$.*

*Proof.* Take $p$ one period of $F$ and $y \in Y$. We have $G(z+L^{-1}(p)) = F\circ L(z+L^{-1}(p)) = F(L(z) + p) = F(L(z)) = G(z)$, which shows that $G$ is a periodic function and that $L^{-1}(Per(F)) \subset Per(G)$. The converse inclusion can be obtained by interchanging $F$ and $G$. □

2.2. **Lecture in a basis.** We consider the canonical basis of $X = \mathbb{R}^m$, denoted by $(e_j)_{1\le j\le m}$. We denote $(e_j^*)_{1\le j\le m}$ its dual basis and $\chi$ the isomoprhism from $X$ into $(\mathbb{R}^m)^*$ defined by:

$$\chi(x) = (e_j^*(x))_{1\le j\le m}.$$

We denote also:

$$\chi^{-1}(x_1, \dots, x_m) = \sum_{i=1}^{m} x_i e_i.$$

We shall now analyze the link between the function $F$ and the function $f := F \circ \chi^{-1}$ defined on $\mathbb{R}^m$.

**Definition 2.2.** The function $f$ is said to be $2\pi$-periodic in each variable if for all $j = 1, \dots, m$, $x_{-j} \in \mathbb{R}^{m-1}$, and for all $x_j \in \mathbb{R}$, we have $f(x_{-j}, x_j + 2\pi) = f(x_{-j}, x_j)$, which is also equivalent to say that for all $k \in \mathbb{Z}^m$, and for any $x \in \mathbb{R}^m$, we have $f(x + 2\pi k) = f(x)$.

Therefore, we have the following.

**Proposition 2.2.** *The following assertions are equivalent.*

(a) *$F$ is periodic and $Per(F) \supset 2\pi\mathbb{Z}\langle e_1, \dots, e_m\rangle$.*
(b) *$f$ is $2\pi$-periodic in each variable.*

*Proof.* We refer to Proposition 2.3 which is more general. □

2.3. **Change of basis effects.** Let actually take another basis of $\mathbb{R}^m$, $(b_j)_{1\le j\le m}$. We denote by $(b_j^*)_{1\le j\le m}$ its dual basis and by $\chi_1$ the isomorphism from $X$ into $(\mathbb{R}^m)^*$ defined for all $x \in \mathbb{R}^m$ as:

$$\chi_1(x) = (b_j^*(x))_{1\le j\le m}.$$

Keeping the same previous notations, we set $f_1 := F \circ \chi_1^{-1}$.

**Proposition 2.3.** *The following assertions are equivalent.*

(a) *$F$ is periodic and $Per(F) \supset 2\pi b_j$.*
(b) *For all $x_{-j} \in \mathbb{R}^{m-1}$, $f_1(x_{-j}, \cdot)$ is $2\pi$-periodic from $\mathbb{R}$ into $\mathbb{E}$.*

*Proof.* (a) $\Rightarrow$ (b) Let us fix arbitrarily $x_{-j} \in \mathbb{R}^{m-1}$.
Then, for all $x_j \in \mathbb{R}$, we have:

$$\begin{aligned} f_1\left(x_{-j}, x_j + 2\pi\right) &= F \circ \chi_1^{-1}\left(x_{-j}, x_j + 2\pi\right) = F\left(2\pi b_j + \sum_{i=1}^{m} x_i b_i\right) \\ &= F\left(\sum_{i=1}^{m} x_i b_i\right) = F \circ \chi_1^{-1}\left(x_{-j}, x_j\right) = f_1\left(x_{-j}, x_j\right). \end{aligned}$$

Therefore, (a) implies (b).
(b) $\Rightarrow$ (a) Now, we fix an arbitrary $x \in \mathbb{R}^m$. We have

$$\begin{aligned} F(x + 2\pi b_j) &= F \circ \chi_1^{-1}\left(b^*_{-j}(x), b^*_j(x) + 2\pi\right) \\ &= f_1\left(b^*_{-j}(x), b^*_j(x) + 2\pi\right) = f_1\left(b^*_{-j}(x), b^*_j(x)\right) \\ &= F \circ \chi_1^{-1}\left(b^*_{-j}(x), b^*_j(x)\right) = F\left(x\right). \end{aligned}$$

This completes the proof. □

*Notation* 2.1. We introduce $Q^m \subset X$ as:

$$Q^m := \left\{\sum_{i=1}^{m} x_i e_i : \text{for all } i = 1, \dots, m,\ x_i \in [-\pi, \pi]\right\}.$$

We have $\chi(Q^m) = [-\pi, \pi]^m$. We denote by $K^m := \chi_1(Q^m)$.

2.4. **Change of basis and integrals.** We keep the previous notations, by assuming in addition that both basis are orthonormal.

We set, for $y_{-1} \in \mathbb{R}^{m-1}$,

$$K(y_{-1}) := \{y_1 \in \mathbb{R} : (y_1, y_{-1}) \in K^m\}$$

and

$$D := \left\{y_{-1} \in \mathbb{R}^{m-1} : K(y_{-1}) \neq \emptyset\right\}.$$

The theorem of change of variables consecutively with change of basis under the integrals allow us to get the following.

**Proposition 2.4.** *For any continuous function $f : \mathbb{R}^m \to \mathbb{E}$, we have:*

$$\int_{[-\pi,\pi]^m} f(x)dx = \int_D \left(\int_{K(y_{-1})} f_1(y)dy_1\right) dy_2 \cdots dy_m.$$

*Proof.* The functions $f$, $f_1$ and $F$ are continuous. By Fubini's theorem, we have:

$$\int_{\chi_1(Q^m)} f_1(y)dy = \int_D \left(\int_{K(y_{-1})} f_1(y)dy_1\right) dy_2 \cdots dy_m.$$

Besides, since the isomorphisms $\chi$ and $\chi_1$ are orthogonal, their determinant is in absolute value equal to 1, which gives the following identities by applying the change of variables formula:

$$\int_{\chi_1(Q^m)} f_1(y)dy = \int_{Q^m} F(x)dx = \int_{\chi(Q^m)} f(x)dx.$$

We deduce the result then by comparison of these inequalities and by using $\chi(Q^m) = [-\pi, \pi]^m$. □

**Lemma 2.1.** *For all $y_{-1} \in D$, $K(y_{-1})$ is a closed interval of $\mathbb{R}$ with diameter equal to $2\pi\sqrt{m}$.*

*Proof.* Let us fix $y_{-1} \in D$. $K(y_{-1})$ is closed convex set of $\mathbb{R}$, because it is the reciprocal image of the convex set $K^m$ via the affine application $\phi(y_1) = (y_1, y_{-1})$. We deduce that $K(y_{-1})$ is a closed interval. Since $K(y_{-1}) \subset \chi_1(Q^m)$, we have:

$$\operatorname{diam}[K(y_{-1})] \leq \operatorname{diam}[\chi_1(Q^m)].$$

Since $\chi_1$ is orthogonal, we know, due to the formula of change of variable for the integrals, that:

$$\operatorname{diam}[\chi_1(Q^m)] = \operatorname{diam}[Q^m].$$

Now, if $x, y \in Q^m$, we have:

$$|x-y|^2 = \sum_{i=1}^{m}(x_i - y_i)^2 \leq m(2\pi)^2,$$

the upper bound is reached (for example) for $x = (-\pi, \ldots, -\pi)$ and $y = (\pi, \ldots, \pi)$. We conclude that:

$$\operatorname{diam}[Q^m] = 2\pi\sqrt{m},$$

which completes the proof of the lemma. □

2.5. **Change of basis and derivation.** $(e_i)_i$ denotes the canonical basis of $X$, and we denote by $(b_j)_j$ another orthonormal basis such that $b_1 = \omega/|\omega|$.

Let $U : \mathbb{R}^m \to \mathbb{E}$ be a function which is $2\pi$-periodic in each variable, $F := U \circ \chi$ and $V := F \circ \chi_1^{-1}$.

**Definition 2.3** ([8, p. 251])**.** Let $x \in \mathbb{R}^m$ and $\phi$ be a function defined on an open set $U$ of $\mathbb{R}^m$ containing $x$ and with values in $\mathbb{E}$. The function $\phi$ admits a directional derivative (called also *Gâteaux-variation*) in the direction $v$ if

$$\frac{\phi(x+\theta v) - \phi(x)}{\theta}$$

has a limit when $\theta$ tends to 0.

This limit, denoted by $\vec{D}\phi(x, v)$, is called *directional derivative* (or also *Gâteaux-variation*) of $\phi$ in the direction $v$.

**Definition 2.4.** We define the Percival derivation operator (cf. [11,12]) for $U$ differentiable in the direction of $\omega$ by:

$$d_\omega U(x) := \vec{D}U(x,\omega).$$

*Remark* 2.1. When $U$ is in addition Fréchet-differentiable in $x$, we have

$$d_\omega U(x) = \sum_{i=1}^{m} \omega_i \frac{\partial U}{\partial x_i}(x).$$

The link between the notions of derivation is analyzed in the following proposition.

**Proposition 2.5.** *Let $U$ be differentiable in the direction of $\omega$. Then, $V$ is differentiable with the respect to the first variable, and we have the relation*

$$d_\omega U(x_1,\ldots,x_m) = |\omega| \frac{\partial V}{\partial y_1}(y_1,\ldots,y_m).$$

*Proof.* Since $\omega = |\omega| b_1$, we have:

$$\vec{D}F(x,\omega) = |\omega|\vec{D}F(x,b_1).$$

Besides, we have $\vec{D}F(x,\omega) = d_\omega U(x)$ and $\vec{D}F(x,b_1) = \frac{\partial V}{\partial y_1}(y_1,\ldots,y_m)$. This achieves the proof of the proposition. ☐

## 3. Functions on the Torus and Functions on $Q^m$

3.1. **Functions defined on the torus.** We give here a (non-geometric) definition of the functions defined on the torus $\mathbb{T}^m$ and we study how to extend a function defined on $Q^m$ into a function defined on the torus. Here, the torus is not seen as a geometric object, but as a notation to specify the periodicity with respect to each variable of the functions involved.

For "regular" functions, we therefore set, when $k \in \mathbb{N} \cup \{+\infty\}$:

- $C^k(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k(\mathbb{R}^m,\mathbb{E})$ which are $2\pi$-periodic in each variable;
- $C^k_\omega(\mathbb{T}^m,\mathbb{E})$ is the space of continuous functions, $k$ times continuously differentiable in the direction of $\omega$ and $2\pi$-periodic in each variable;
- $C^k_c(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k(\mathbb{T}^m,\mathbb{E})$ which vanish on an open neighborhood of $\partial Q^m$;
- $C^k_{c,\omega}(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k_\omega(\mathbb{T}^m,\mathbb{E})$ which vanish on an open neighborhood of $\partial Q^m$.

For the functions of Lebesgue spaces, we shall first define the notion of periodicity.

**Definition 3.1.** For a function $u$ (strongly) measurable from $\mathbb{R}^m$ into $\mathbb{E}$, we call that $u$ admits the vector $p$ as a period if : $\tau_p u = u$. We call $u$ is periodic if it has a non zero period and we denote

$$L^0(\mathbb{T}^m,\mathbb{E}) := \left\{ u \in L^0(\mathbb{R}^m,\mathbb{E}) : 2\pi\mathbb{Z}\langle (e_i)_i\rangle \in Per(u) \right\}.$$

The space $L^0(\mathbb{R}^m,\mathbb{E})$ denotes here the space of measurable functions.

For the other Lebesgue spaces $L^\alpha$, we set, when $\alpha \in [1, +\infty]$:

$$L^\alpha(\mathbb{T}^m, \mathbb{E}) := \left\{ u \in L^\alpha_{loc}(\mathbb{R}^m, \mathbb{E}) : 2\pi\mathbb{Z}\langle (e_i)_i \rangle \rangle \in Per(u) \right\}.$$

We endow this space with the norm:

$$\|u\|_{L^\alpha} := \left( \int_{Q^m} |u(x)|^\alpha_{\mathbb{E}} dx \right)^{1/\alpha},$$

if $\alpha$ is finite, and when $\alpha = \infty$ with the norm:

$$\|u\|_\infty = \operatorname{ess\,sup} |u|_{\mathbb{E}} := \inf \{ \alpha : |u|_{\mathbb{E}} \leq \alpha \text{ a.e.} \}.$$

These spaces are Banach spaces. We remind that we note sup instead of ess sup.

*Notation* 3.1. In the sequel, we use the notation $\| \cdot \|$ instead of $\| \cdot \|_{L^2}$.

*Remark* 3.1. For a given function $u \in L^1(\mathbb{T}^m, \mathbb{E})$, the integral $\int_{\mathbb{T}^m} u(x)dx$ denotes $\int_{Q^m} u(x)dx$. The relation between this integral and the integral with respect to the Haar measure on the torus is:

$$\int_{\mathbb{T}^m} u(x) d\mu_m(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} u(x)dx.$$

Moreover, we equip the space $L^2(\mathbb{T}^m, \mathbb{H})$ with the following inner product

$$\langle u; v \rangle := \int_{\mathbb{T}^m} u(x) \cdot_{\mathbb{H}} v(x)dx.$$

$L^2(\mathbb{T}^m, \mathbb{H})$ is then a Hilbert space.

We immediately have the following result.

**Proposition 3.1.** *For all $p \in \mathbb{R}^m$, the following assertions hold.*

1. *For all $k \in \mathbb{N} \cup \{+\infty\}$, $\tau_p(C^k(\mathbb{T}^m, \mathbb{E})) \subset C^k(\mathbb{T}^m, \mathbb{E})$.*
2. *For all $\alpha \in [1, +\infty]$, $\tau_p(L^\alpha(\mathbb{T}^m, \mathbb{E})) \subset L^\alpha(\mathbb{T}^m, \mathbb{E})$.*

3.2. **Extension theorems.** The extension of functions defined on $Q^m$ into functions defined on the torus can be done using the following lemma.

**Lemma 3.1.** *The following statements hold true.*

1. *For all $x \in \mathbb{R}^m$, there exists $k \in \mathbb{Z}^m$, such that $x - 2\pi k \in Q^m$.*
2. *The boundary of $Q^m$ is given by*

$$\partial Q^m = \left\{ p \in Q^m : \textit{there exists } j \in \{1, \cdots, m\},\ p_j \in \{-\pi, \pi\} \right\}$$

   *and is Lebesgue-negligible in $Q^m$ and in $\mathbb{R}^m$.*
3. *If $f$ is $2\pi$-periodic in each variable, $f$ satisfies the following condition at the boundary:*

   **(CF)** *For all $i \in \{1, \ldots, m\}$ and for all $x_{-i} \in \mathbb{R}^{m-1}$, $f(x_{-i}, -\pi) = f(x_{-i}, \pi)$, which can also be written as: for all $\xi, \zeta \in Q^m$, [for all $i \in \{1, \ldots, m\}, \xi_i = \zeta_i$ or ($\xi_i \in \{-\pi, \pi\}$ and $\zeta_i \in \{-\pi, \pi\}$)], this implies that $f(\xi) = f(\zeta)$.*

*Proof.* **3.** Results from the definition of periodicity.

For the assertion **1**, given $x \in \mathbb{R}^m$, we set for each $j = 1, \ldots, m$,

$$k_j = E\left(\frac{x_j + \pi}{2\pi}\right),$$

where $E$ denotes the integer function. We verify that $k \in \mathbb{Z}^m$ and for all $j = 1, \ldots, m$, we have $-\pi \leq x_j - 2\pi k_j < \pi$, and so $x - 2\pi k \in Q^m$.

Let us now prove **2.** We shall show that $\operatorname{Int} Q^m = (-\pi, \pi)^m$.

Firstly, we have $\operatorname{Int} Q^m \supset (-\pi, \pi)^m$ because $(-\pi, \pi)^m$ is an open set of $\mathbb{R}^m$ contained in $Q^m$. If the inclusion is strict, there is a $p \in \operatorname{Int} Q^m$ and a $j_0$ such that $p_{j_0} \in \{-\pi, \pi\}$. Without loss of generality, we assume that $p_{j_0} = \pi$.

The sequence $\left((p_j + \frac{1}{n})_{1 \leq j \leq m}\right)_n$ converges to $p$, but none of its elements are in $Q^m$. Therefore, $p \notin \operatorname{Int} Q^m$.

This ends the proof of this lemma. $\square$

Let us start with the study of the extension in the case of Lebesgue spaces.

**Proposition 3.2.** *The map $\mathcal{J} : L^\alpha(\mathbb{T}^m, \mathbb{E}) \to L^\alpha(Q^m, \mathbb{E})$ defined by: $\mathcal{J}(u) := u_{|Q^m}$ is an isometric isomorphism of Banach spaces (and even Hilbert if $\alpha = 2$ and $\mathbb{E} = \mathbb{H}$).*

*Proof.* $\mathcal{J}$ is obviously an isometric linear map.

We shall now prove that it is bijective.

**Surjectivity.** Let us take a function $f \in L^\alpha(Q^m, \mathbb{E})$. Even if it means modifying $f$ on the boundary of $Q^m$ (which is Lebesgue-negligible), we can assume that $f$ is zero on $\partial Q^m$.

Let $x \in \mathbb{R}^m$. If there exists $k, l$ in $\mathbb{Z}^m$, distinct, for which we simultaneously have $x - 2\pi k \in Q^m$ and $x - 2\pi l \in Q^m$ ans so for all $i$ such that $k_i \neq l_i$, we have ($k_i = 0$ and $l_i = 2\pi$) or ($k_i = 2\pi$ and $l_i = 0$).

Therefore, by **(CF)**, $f(x - 2\pi k) = f(x - 2\pi l)$, and it is then possible to define $\tilde{f}(x) = f(x - 2\pi k)$ where $k$ is arbitrarily chosen in $\mathbb{Z}^m$ so that $x - 2\pi k \in Q^m$.

Let us show now that the function $\tilde{f}$ previously defined is periodic. If $p \in \mathbb{Z}^m$ and $x \in \mathbb{R}^m$ are given, let us take $k \in \mathbb{Z}^m$ such that: $x - 2\pi k \in Q^m$. We have $(x + 2\pi p) - 2\pi(k + p) \in Q^m$. Hence, $\tilde{f}(x + 2\pi p) = f((x + 2\pi p) - 2\pi(k + p)) = f(x - 2\pi k) = \tilde{f}(x)$.

Finally, it remains to verify that it belongs to $L^\alpha$. The restriction of $\tilde{f}$ to each $Q^m + 2\pi k$, where $k \in \mathbb{Z}^m$ has the form of $x \mapsto f(x + 2\pi k)$, and hence $\tilde{f} \in L^\alpha_{loc}(\mathbb{R}^m, \mathbb{E})$, which shows that $\tilde{f} \in L^\alpha(\mathbb{T}^m, \mathbb{E})$ and verifies $\mathcal{J}(\tilde{f}) = f$.

**Injectivity.** Let $f \in L^\alpha(Q^m, \mathbb{E})$ and $f_1$ and $f_2$ be two functions such that $\mathcal{J}(f_1) = \mathcal{J}(f_2) = f$. We may suppose that the two functions $f_i$ are equal on $Q^m$.

Let $x \in \mathbb{R}^m$. There exists $k \in \mathbb{Z}^m$ such that $x - 2\pi k \in Q^m$. We have then $f_1(x) = f_1(x - 2\pi k) = f_2(x - 2\pi k) = f_2(x)$.

The proposition is finally proved. $\square$

We analyze now the case of continuous functions. We have precisely to study what is happening on the border.

**Lemma 3.2.** *Let $k$ and $l$ be two different elements of $\mathbb{Z}^m$ and let $p \in (Q^m + 2\pi k) \cap (Q^m + 2\pi l)$. Then,*

$$p \in \partial(Q^m + 2\pi k) \cap \partial(Q^m + 2\pi l) = (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l).$$

*Proof.* There exist $\xi, \zeta \in Q^m$ such that $p = \xi + 2\pi k = \zeta + 2\pi l$. Since $k \neq l$, there exists $j$ such that $\xi_j \neq \zeta_j$. Besides, as $\xi_i + 2\pi k_i = \zeta_i + 2\pi l_i$ and $|\xi_i - \zeta_i| \leq 2\pi$, we have $|k_i - l_i| \leq 1$.

**First case.** $k_i = l_i$. Then, $\xi_i = \zeta_i$.

**Second case.** $k_i = l_i \pm 1$. Then, $\xi_i = \zeta_i \pm 2\pi$, that is, since $\xi, \zeta \in Q^m$, one of the two is equal to $-\pi$ and the other one is equal to $\pi$.

Therefore, we have shown that $i \in \{1, \dots, m\}$, $\xi_i = \zeta_i$ or $\xi_i, \zeta_i \in \{-\pi; \pi\}$ and there exists $j$ such that $\xi_j \neq \zeta_j$. Finally, $p \in (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l)$ and the lemma is proven. $\square$

**Proposition 3.3.** *Let $f \in C^0(Q^m, \mathbb{E})$. The following statements are equivalent.*

1. *There exists a unique $\tilde{f} \in C^0(\mathbb{T}^m, \mathbb{E})$ such that $\tilde{f}_{|Q^m} = f$.*
2. *$f$ satisfies* **(CF)**.

*Proof.* The implication [**1. implies 2.**] is obvious.

We have to show the implication [**2. implies 1.**].

**Existence.** For $x \in Q^m + 2\pi k$, we set $f_k(x) = f(x - 2\pi k)$. When $(Q^m + 2\pi k) \cap (Q^m + 2\pi l) \neq \emptyset$, with $k \neq l$, we have $(Q^m + 2\pi k) \cap (Q^m + 2\pi l) = (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l)$ in virtue of the lemma.

Hence, thanks to **(CF)**, we have $f_k(x) = f_l(x)$. Let us introduce $A_k := Q^m + 2\pi k$. The family $(A_k)_k$ forms a recovery of $\mathbb{R}^m$ such that if $A_k \cap A_l \neq \emptyset$, $f_k(x) = f_l(x)$. We can define the function $\tilde{f}$ as $\tilde{f}(x) = f_k(x)$ if $x \in A_k$. Since each $f_k$ is continuous and as the recovery $(A_k)_k$ is closed and locally finite, we know that $\tilde{f}$ is continuous (cf. [13, p. 20]). Its periodicity is obvious. Therefore, the existence is shown.

**Uniqueness.** Two solutions take the same values on $Q^m$, and so that, they are equal on $\mathbb{R}^m$, by periodicity. $\square$

**Proposition 3.4.** *For all function $f \in C^k_c(Q^m, \mathbb{E})$, there exists a unique $\tilde{f} \in C^k(\mathbb{T}^m, \mathbb{E})$ such that $\tilde{f}_{|Q^m} = f$.*

*Proof.* Uniqueness is acquired by Proposition 3.3. Moreover, Proposition 3.3 gives us, for all $j \leq k$, a unique $\tilde{f}_j \in C^0\left(\mathbb{T}^m, \mathcal{L}^j_{sym}\left((\mathbb{R}^m)^j; \mathbb{E}\right)\right)$ such that $\tilde{f}_{j_{|Q^m}} = f^{(j)}$. Let $\tilde{f} = \tilde{f}_0$. We aim to prove that this function belongs to $C^k(\mathbb{T}^m, \mathbb{E})$.

Let $x \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$ such that $x \in Q^m + 2\pi l$. We can distinguish two cases.

**First case.** $x \in \operatorname{Int}(Q^m + 2\pi l)$. In this case, near to $x$, $\tilde{f} = f \circ \tau_{-2\pi l}$ is of class $C^k$ as composition of a map from $C^k$ and an application from $C^\infty$.

**Second case.** $x \in \partial (Q^m + 2\pi l)$. Let $\Lambda := \{\lambda \in \mathbb{Z}^m : x \in \partial Q^m + 2\pi\lambda\}$. $\Lambda$ is a non empty finite set and since $\lambda \in \Lambda$, we have $\operatorname{supp}\left(\tilde{f}_{|Q^m+2\pi\lambda}\right) = \operatorname{supp}(f) + 2\pi\lambda$ and since

$x \notin \operatorname{supp}\left(\tilde{f}_{|Q^m+2\pi\lambda}\right)$, we can consider

$$r := \min_{\lambda \in \Lambda} d(x; \operatorname{supp}(f) + 2\pi\lambda),$$

which is a strictly positive real and obtain then that $B(x;r)$, $\tilde{f} = 0$. Therefore, it belongs to $C^k$ at the neighborhood of $x$.

This completes the proof. □

### 3.3. **Some other properties of the spaces of functions defined on the torus.**

**Proposition 3.5.** *Each function of $C^0(\mathbb{T}^m, \mathbb{E})$ is uniformly continuous on $\mathbb{R}^m$.*

*Proof.* Let $r$ be a fixed positive real. The set $K := \{x \in \mathbb{R}^m : d(x, Q^m) \leq r\}$ is compact, so that due to Lemma of Heine: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, z \in K$, if $|x - z| \leq \delta$, then $|f(x) - f(z)|_{\mathbb{E}} \leq \varepsilon$.

We fix an arbitrary $\varepsilon > 0$ and a $\delta$ given by the previous inequality. We put $\delta' := \min\{r; \delta\}$. Let $x, z$ be such that $|x - z| \leq \delta'$. There exists $k \in \mathbb{Z}^m$ such that $x - 2\pi k \in Q^m$ and so $z - 2\pi k \in K$.

So, we have: $|f(x) - f(z)|_{\mathbb{E}} = |f(x - 2\pi k) - f(z - 2\pi k)|_{\mathbb{E}} \leq \varepsilon$, which is exactly the uniform continuity. □

We shall prove now some density theorems. For this aim, we introduce the convolution product.

**Proposition 3.6.** *Let $j \in \mathbb{N} \cup \{+\infty\}$, $u \in C_c^j(\mathbb{T}^m, \mathcal{A})$ and $v \in L^\alpha(\mathbb{T}^m, \mathcal{B})$ with $\alpha \in [1, +\infty]$. So, $u * v \in C^j(\mathbb{T}^m, \mathcal{C})$.*

*Proof.* Since $v \in L^\alpha(\mathbb{T}^m, \mathcal{B})$, we get $v \in L^1_{loc}(\mathbb{R}^m, \mathcal{B})$ and the convolution product $u * v$ is given by:

$$u * v(z) = \int_{\mathbb{R}^m} u(x) \diamond v(z - x)dx.$$

Moreover, it is well defined on $\mathbb{R}^m$ and $u * v \in C^j(\mathbb{R}^m, \mathcal{C})$.

Let us verify that $2\pi\mathbb{Z}^m \subset Per(u * v)$, which will complete the demonstration.

Let $p \in 2\pi\mathbb{Z}^m$. $p$ is a period of $v$, and

$$u * v(z + p) = \int_{\mathbb{R}^m} u(x)v(z + p - x)dx = \int_{\mathbb{R}^m} u(x)v(z - x)dx = u * v(z).$$

This is what had to be demonstrated. □

**Proposition 3.7.** *Let $j \in \mathbb{N} \cup \{+\infty\}$. $C^j(\mathbb{T}^m, \mathbb{E})$ and $C_c^j(\mathbb{T}^m, \mathbb{E})$ are dense in $L^2(\mathbb{T}^m, \mathbb{E})$.*

*Proof.* It suffices to prove this result on $C_c^j(\mathbb{T}^m, \mathbb{E})$.

Indeed, we recall that $C_c^\infty(\operatorname{Int}(Q^m), \mathbb{E})$ is dense in $L^2(Q^m, \mathbb{E})$ (the proof in Brezis's book [3, p. 71] can be adapted to Banach spaces). Fixing $u \in L^2(\mathbb{T}^m, \mathbb{E})$, we denote by $\underline{u}$ its restriction to $Q^m$. Hence, for a given $\varepsilon > 0$, there exists $w \in C_c^\infty(\operatorname{Int}(Q^m), \mathbb{E})$ such that

$$\int_{Q^m} |w(x) - \underline{u}(x)|^2_{\mathbb{E}} dx \leq \varepsilon^2.$$

Moreover, using Proposition 3.4, we can extend in a unique way $w$ to an element $z \in C_c^\infty(\mathbb{T}^m, \mathbb{E})$. We get

$$\int_{\mathbb{T}^m} |z(x) - u(x)|_{\mathbb{E}}^2 dx = \int_{Q^m} |w(x) - \underline{u}(x)|_{\mathbb{E}}^2 dx \le \varepsilon^2,$$

which completes the proof of our proposition. □

**Lemma 3.3.** *Let $f \in L^1(\mathbb{T}^m, \mathbb{E})$ and $\beta \in \mathbb{R}^m$. So, we have*

$$\int_{Q^m} f(x+\beta)dx = \int_{Q^m} f(x)dx.$$

*Proof.* We will prove the result by induction on $m$.

For $m = 1$, we have successively

$$\begin{aligned}\int_{-\pi}^{\pi} f(t+\beta)dt &= \int_{\beta-\pi}^{\beta+\pi} f = \int_{\beta-\pi}^{-\pi} f + \int_{-\pi}^{\pi} f + \int_{\pi}^{\pi+\beta} f \\ &= -\int_{-\pi}^{\beta-\pi} f + \int_{-\pi}^{\pi} f + \int_{-\pi}^{\beta-\pi} f(t+2\pi)dt = \int_{-\pi}^{\pi} f.\end{aligned}$$

This is the desired result.

Now, suppose the result is true for 1 and $m-1$. We have

$$\int_{Q^m} f(x+\beta)dx = \int_{-\pi}^{\pi} \left[ \int_{[-\pi,\pi]^{m-1}} f(x_1+\beta_1, x_{-1}+\beta_{-1})dx_{-1} \right] dx_1.$$

From the result at rank $m-1$, the right side is

$$\int_{-\pi}^{\pi} \left[ \int_{[-\pi,\pi]^{m-1}} f(x_1+\beta_1, x_{-1})dx_{-1} \right] dx_1.$$

Due to Fubini's theorem then from the result to row 1, this integral is equal to $\int_{Q^m} f(x)dx$, which means that the proposition is proven. □

**Proposition 3.8.** *Let consider $u \in L^2(\mathbb{T}^m, \mathbb{E})$. The function from $\mathbb{R}^m$ into $L^2(\mathbb{T}^m, \mathbb{E})$ defined by $\beta \mapsto \tau_\beta u$ is uniformly continuous.*

*Proof.* Let us fix an $\varepsilon > 0$, and a function $v \in C_c^0(\mathbb{T}^m, \mathbb{E})$ such that $\|u - v\| \le \varepsilon/3$.

Since $v$ is uniformly continuous on $\mathbb{R}^m$, $\beta \mapsto \tau_\beta$ is uniformly continuous from $\mathbb{R}^m$ into $C^0(\mathbb{T}^m, \mathbb{E})$. Hence, we can find $\eta > 0$ such that if $\gamma$ and $\beta$ are in $\mathbb{R}^m$ such that $|\gamma - \beta| \le \eta$, then

$$\sup_{x\in\mathbb{R}^m} |v(x+\gamma) - v(x+\beta)|_{\mathbb{E}} \le \frac{\varepsilon}{3(2\pi)^m}.$$

Let choose $\gamma$ and $\beta$. We then get

$$\begin{aligned}\|u(\cdot+\beta) - u(\cdot+\gamma)\| \le& \|u(\cdot+\beta) - v(\cdot+\beta)\| + \|v(\cdot+\beta) - v(\cdot+\gamma)\| \\ &+ \|v(\cdot+\gamma) - u(\cdot+\gamma)\| \\ \le& 2\|u - v\| + \|v(\cdot+\beta) - v(\cdot+\gamma)\|,\end{aligned}$$

using Lemma 3.3. This last term being less than $\varepsilon$, the proof is achieved. □

## 4. Construction of Sobolev-type Spaces

We aim to present the construction of Sobolev type spaces adapted to our problems. We will start by introducing a notion of weak derivative of Percival for the elements of $L^2(\mathbb{T}^m, \mathbb{E})$ as infinite generator of a (semi-)group of contractions. The domain of this unbounded operator is the Sobolev type space that we build. We shall explain the relation between distributions on the torus and $2\pi$-periodic distributions in each variable, and we shall show that the different ways of introducing the weak derivative of Percival coincide.

4.1. **Construction and first properties of the space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.** Due to Proposition 3.1, for each $u \in L^2(\mathbb{T}^m, \mathbb{E})$ and all $\beta \in \mathbb{R}^m$, we have $\tau_\beta u \in L^2(\mathbb{T}^m, \mathbb{E})$.

Thus, we can define, for all $t \in \mathbb{R}^+$, $T(t)$ from $L^2(\mathbb{T}^m, \mathbb{E})$ into $L^2(\mathbb{T}^m, \mathbb{E})$ by setting

$$T(t)u := \tau_{t\omega} u, \quad \text{for all } t \in \mathbb{R}^+, \text{ for all } u \in L^2(\mathbb{T}^m, \mathbb{R}^N).$$

It can be easily verified that $T(t)$ is a linear isometry of $L^2(\mathbb{T}^m, \mathbb{E})$.

**Proposition 4.1.** *The following statements hold.*

**(1)** *For all $s, t \in \mathbb{R}^+$, $T(s+t) = T(t) \circ T(s)$.*
**(2)** $T(0) = id$.
**(3)** *For all $u \in L^2(\mathbb{T}^m, \mathbb{E})$, $[t \mapsto T(t)u] \in C^0(\mathbb{R}^+, L^2(\mathbb{T}^m, \mathbb{E}))$.*

*Proof.* **(1)** We have for all $s, t \geq 0$,

$$\begin{aligned} T(t+s)u = \tau_{(s+t)\omega} u = u(\cdot + (s+t)\omega) = \tau_{s\omega} u(\cdot + t\omega) = T(s)[T(t)u] \\ = [T(s) \circ T(t)](u). \end{aligned}$$

**(2)** is obvious.

**(3)** $[t \mapsto t\omega]$ is continuous, which implies that this assertion is a consequence of Proposition 3.8. □

Hence, following [7, p. 614], the family $(T(t))_{t \in \mathbb{R}^+}$ is a strongly continuous semi-group of $\mathcal{L}(L^2(\mathbb{T}^m, \mathbb{E}); L^2(\mathbb{T}^m, \mathbb{E}))$.

We denote by $\nabla_\omega$ the infinitesimal generator of this semi-group, and by $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ its domain. So, we have

$$H^1_\omega(\mathbb{T}^m, \mathbb{E}) := \left\{ u \in L^2(\mathbb{T}^m, \mathbb{E}) : \lim_{t \to 0+} \frac{T(t)u - u}{t} \text{ exists in } L^2(\mathbb{T}^m, \mathbb{E}) \right\}$$

and for $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, this limit is denoted by $\nabla_\omega u$.

We obtain from the theory of strongly continuous semi-groups, cf. [7].

**Proposition 4.2.** *The following assertions are true.*

1. *$H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is a linear subspace of $L^2(\mathbb{T}^m, \mathbb{E})$ and $\nabla_\omega$ is a linear operator from $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ into $L^2(\mathbb{T}^m, \mathbb{E})$.*
2. *If $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and if $t \in \mathbb{R}^+$, then $\tau_{t\omega} u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and*

$$\frac{d}{dt}(\tau_{t\omega} u) = \nabla_\omega(\tau_{t\omega} u) = \tau_{t\omega}(\nabla_\omega u).$$

**3.** *If* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ *and if* $0 \leq s < t < +\infty$, *then*

$$\tau_{t\omega}u - \tau_{s\omega}u = \int_s^t \tau_{r\omega}(\nabla_\omega u)dr.$$

**4.** *If* $t \in \mathbb{R}^+$ *and if* $g \in L^1(\mathbb{R}, \mathbb{R})$ *is continuous in* $t$, *then*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} g(s)\tau_{s\omega}uds = g(t)\tau_{t\omega}u.$$

**5.** $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ *is dense in* $L^2(\mathbb{T}^m, \mathbb{E})$ *and* $\nabla_\omega$ *is of closed graph in* $L^2(\mathbb{T}^m, \mathbb{E}) \times L^2(\mathbb{T}^m, \mathbb{E})$.

*Remark* 4.1. The integrals considered in the previous proposition are integrals of continuous functions with values in the Banach space $L^2(\mathbb{T}^m, \mathbb{E})$.

The space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is endowed with the norm

$$\|u\|_{H^1_\omega(\mathbb{T}^m,\mathbb{E})} = \sqrt{\|u\|^2 + \|\nabla_\omega u\|^2},$$

which we denote also by $\|u\|_{1,\omega}$ if there is no ambiguity on $\mathbb{E}$.

If in addition $\mathbb{E} = \mathbb{H}$ is a Hilbert space, the space $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ is equipped with the following bi-linear form

$$\langle u; v\rangle_{H^1_\omega(\mathbb{T}^m,\mathbb{H})} := \langle u; v\rangle + \langle \nabla_\omega u; \nabla_\omega v\rangle.$$

Again, if there is no ambiguity on $\mathbb{H}$, we shall denote $\langle u; v\rangle_{1,\omega}$ instead of $\langle u; v\rangle_{H^1_\omega(\mathbb{T}^m,\mathbb{H})}$.

**Proposition 4.3.** *Equipped with the bi-linear form* $\langle \cdot, \cdot\rangle_{1,\omega}$, $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ *is a Hilbert space.*

*Proof.* We have only to prove the completeness of this space. Now, if $(u_n)_n$ is a Cauchy sequence in $H^1_\omega(\mathbb{T}^m, \mathbb{H})$, each of the sequences $(u_n)_n$ and $(\nabla_\omega u_n)_n$ is also of Cauchy in the complete space $L^2(\mathbb{T}^m, \mathbb{H})$, hence, they converge. We denote by $u$ and $v$ their respective limits. Since $\nabla_\omega$ is of closed graph, we deduce that $v = \nabla_\omega u$, and then the sequence $(u_n)_n$ converges in $H^1_\omega(\mathbb{T}^m, \mathbb{H})$. □

We now verify that we correctly recover the usual notion for regular functions.

**Proposition 4.4.** *If* $u \in C^1(\mathbb{T}^m, \mathbb{E})$, *then* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, *and* $\nabla_\omega u(x) = u'(x) \cdot \omega$ *for Lebesgue-almost all* $x$.

*Proof.* Since $u'$ is continuous on $\mathbb{T}^m$, it is uniformly continuous, and so if we give and $\varepsilon > 0$, there exists $\eta > 0$ such that: for all $\xi, \zeta \in \mathbb{R}^m$, if $|\xi - \zeta| \leq \eta$, we have $|u'(\zeta) - u'(\xi)|_{\mathbb{E}} \leq \frac{\varepsilon}{|\omega|}$.

Let us fix a such $\eta$, $x \in \mathbb{R}^m$ and let $t \in (0, \eta/|\omega|)$. By the mean inequality applied to the function $y \mapsto u(y) - u'(x) \cdot y$ between $x$ and $x + t\omega$, we get

$$|u(x + t\omega) - u(x) - u'(x) \cdot (t\omega)|_{\mathbb{E}} \leq \frac{\varepsilon}{|\omega|}t|\omega|.$$

We divide by $t$, and we integrate the square of the inequality on $Q^m$. It comes, for all $t \in (0, \eta/|\omega|)$

$$\left\| \frac{\tau_{t\omega} u - u}{t} - u'(\cdot) \cdot \omega \right\| \leq \varepsilon (2\pi)^{m/2},$$

which completes the proof of our proposition. □

4.2. **Convolution and density theorems.** We call a regularizing sequence a sequence $(\rho_j)_{j \geq 0}$ of functions of $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ verifying

**1.** for all $x \in \mathbb{R}^m$, for all $j \in \mathbb{N}$, $\rho_j(x) \geq 0$;
**2.** for all $j \in \mathbb{N}$, $\int_{\mathbb{R}^m} \rho_j = 1$;
**3.** for all $j \in \mathbb{N}$, $\operatorname{supp}(\rho_j) \subset \operatorname{Int}(Q^m)$ and $\lim_{j \to +\infty} \operatorname{diam}[\operatorname{supp}(\rho_j)] = 0$.

**Proposition 4.5.** *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and $\rho \in C^1_c(\mathbb{R}^m, \mathbb{K})$ such that* $\operatorname{supp}(\rho) \subset \operatorname{Int}(Q^m)$. *We have $(d_\omega \rho) * u = \rho * (\nabla_\omega u)$.*

*Proof.* **First step.** Let $t > 0$. We firstly remark that

$$\int_{\mathbb{R}^m} \rho(x) \left[ \frac{\tau_{t\omega} u - u}{t} \right] (z - x) dx = \int_{\mathbb{R}^m} u(z - x) \left[ \frac{\tau_{t\omega} \rho - \rho}{t} \right] (x) dx.$$

**Second step.** Let us prove that, as $t$ tends to 0, $\int_{\mathbb{R}^m} \rho(x) \left[ \frac{\tau_{t\omega} u - u}{t} \right] (z - x) dx$ tends to $\rho * (\nabla_\omega u)(z)$. Indeed, since $\int_{\mathbb{R}^m} \rho = 1$, the difference between this integral and $\rho * (\nabla_\omega u)(z)$ is equal to

$$\int_{\mathbb{R}^m} \rho(x) \left[ \frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u \right] (z - x) dx = \int_{Q^m} \rho(x) \left[ \frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u \right] (z - x) dx,$$

which is dominated by (if we denote by $I = \left[ \int_{Q^m} \rho^2 \right]^{1/2}$)

$$I \left[ \int_{Q^m} \left| \frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u \right|^2_{\mathbb{E}} (z - x) dx \right]^{1/2} \leq I \left\| \frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u \right\|$$

and this last term tends to 0 as $t$ tends to 0.

**Third step.** We aim to prove that, if $t$ tends to 0, $\int_{\mathbb{R}^m} u(z - x) [\frac{\tau_{t\omega} \rho - \rho}{t}](x) dx$ tends to $(d_\omega \rho) * u$. Since $\operatorname{supp}(\rho) \subset \operatorname{Int} Q^m$, there exists a real $r > 0$ such that if $t \leq r$, then $\operatorname{supp}(\tau_{t\omega} \rho) \subset \operatorname{Int} Q^m$.

Since $\operatorname{supp}(\rho')$ is compact, $\rho'$ is uniformly continuous so by using the mean inequality, for all $\varepsilon > 0$, there exists $t' < r$ such that if $t \in (0; t')$, we have

$$\left| \frac{\tau_{t\omega} \rho - \rho}{t}(x) - d_\omega \rho(x) \right| \leq \varepsilon |\omega|.$$

We integrate $\mathbb{R}^m$ this inequality multiplied before by $u(z - x)$, and we get, taking into account $\operatorname{supp} \left[ \frac{\tau_{t\omega} \rho - \rho}{t} \right] \subset Q^m$

$$\left| \int_{\mathbb{R}^m} \left( \frac{\tau_{t\omega} \rho - \rho}{t}(x) - d_\omega \rho(x) \right) u(z - x) dx \right|_{\mathbb{E}} \leq \int_{Q^m} \varepsilon |\omega| \cdot |u(z - x)|_{\mathbb{E}}.$$

Hence, the result of this step is valid.

**Fourth step.** Conclusion, we take the limit in the inequality of the first step to end the proof. ☐

**Proposition 4.6.** *Let $(\rho_j)_j$ regularizing sequence and $u \in L^2(\mathbb{T}^m, \mathbb{E})$. Then,*

$$\lim_{j \to +\infty} \|\rho_j * u - u\| = 0.$$

*Proof.* Since $\rho_j$ is positive and with support contained in $Q^m$, we have for all $z \in \mathbb{R}^m$

$$|\rho_j * u(z)|_{\mathbb{E}} \leq \int_{Q^m} \rho_j(x)|u(z-x)|_{\mathbb{E}} dx \leq \sqrt{\int_{Q^m} \rho_j(x)|u(z-x)|_{\mathbb{E}}^2 dx},$$

where the last inequality is obtained using the inequality of Cauchy-Schwarz. Hence, increasing to the square, integrating over $\mathbb{T}^m$ then using Fubini's theorem, we get

$$\|\rho_j * u - u\|^2 \leq \int_{Q^m} \rho_j(x) \left( \int_{\mathbb{T}^m} |u(z-x)|_{\mathbb{E}}^2 dz \right) dx.$$

But, due to Lemma 3.3, the inside integral is equal to $\|u\|^2$, and so that we get

$$\|\rho_j * u - u\|^2 \leq \|u\|^2.$$

As $C_c^0(\operatorname{Int} Q^m, \mathbb{E})$ is dense in $L^2(Q^m, \mathbb{E})$, we know that $C_c^0(\mathbb{T}^m, \mathbb{E})$ is dense in $L^2(\mathbb{T}^m, \mathbb{E})$ and so that for a given $\varepsilon > 0$, there is $\varphi \in C_c^0(\mathbb{T}^m, \mathbb{E})$ such that $\|u - \varphi\| < \varepsilon$. Also, we have $\|\rho_j * u - \rho_j * \varphi\| \leq \varepsilon$. We deduce then

$$\|\rho_j * u - u\| \leq 2\varepsilon + \|\rho_j * \varphi - \varphi\|.$$

It remains to show that $\|\rho_j * \varphi - \varphi\| \leq \varepsilon$ for $j$ large enough. The uniform continuity of $\varphi$ allows us to find an $\eta > 0$ such that if $|\xi - \zeta| \leq \eta$, then $|\varphi(\xi) - \varphi(\zeta)|_{\mathbb{E}} \leq \varepsilon$. Let $j$ be large enough in order to get $z \in \operatorname{supp}(\rho_j)$ implies $|z| \leq \eta$. Consider then an arbitrary $z \in \mathbb{R}^m$. So, from

$$\rho_j * \varphi(z) - \varphi(z) = \int_{\mathbb{R}^m} \rho_j(x) \left(\varphi(z-x) - \varphi(z)\right) dx,$$

as the integral only relates to $\operatorname{supp}(\rho_j)$, we deduce

$$|\rho_j * \varphi(z) - \varphi(z)|_{\mathbb{E}} \leq \varepsilon \int_{\operatorname{supp}(\rho_j)} \rho_j(x) dx \leq \varepsilon.$$

For $j$ large enough, we obtain finally $\|\rho_j * u - u\| \leq 3\varepsilon$, which achieves the proof. ☐

**Proposition 4.7.** *Let $u \in L^2(\mathbb{T}^m, \mathbb{E})$ and $\rho \in C_c^1(\mathbb{R}^m, \mathbb{K})$. Then*

$$d_\omega(\rho * u) = (d_\omega \rho) * u.$$

*Proof.* Thanks to (cf. [17, p. 122]), we know that

$$\frac{\partial}{\partial x_i}(\rho * u) = \left( \frac{\partial}{\partial x_i} \rho \right) * u.$$

We multiply by $\omega_i$ and add up over $i$ to get the result. ☐

**Proposition 4.8.** *$C^1(\mathbb{T}^m,\mathbb{E})$ is dense in $H^1_\omega(\mathbb{T}^m,\mathbb{E})$.*

*More precisely, if $u\in H^1_\omega(\mathbb{T}^m,\mathbb{E})$, then the sequence $(\rho_j * u)_j$ tends to $u$ in $H^1_\omega(\mathbb{T}^m,\mathbb{E})$ for every regularizing sequence $(\rho_j)_j$.*

*Proof.* By Proposition 3.6, we have $\rho_j * u \in C^1(\mathbb{T}^m,\mathbb{E})$ and $\rho_j*(\nabla_\omega u)\in C^1(\mathbb{T}^m,\mathbb{E})$. Now, by Proposition 4.6, we have

$$\lim_{j\to+\infty}\|\rho_j*u-u\|=0 \quad\text{and}\quad \lim_{j\to+\infty}\|\rho_j*(\nabla_\omega u)-(\nabla_\omega u)\|=0.$$

But, due to Propositions 4.5 and 4.7, we get $\rho_j*(\nabla_\omega u)=\nabla_\omega(\rho_j*u)$ and finally $\lim_{j\to+\infty}\|\rho_j*u-u\|_{1,\omega}=0$. ☐

**Proposition 4.9.** *The following assertions are true.*

1. *For all $f\in C^1(\mathbb{T}^m,\mathbb{E})$, $\int_{\mathbb{T}^m}\partial_\omega f(x)dx=0$.*
2. *For all $u\in H^1_\omega(\mathbb{T}^m,\mathbb{E})$, $\int_{\mathbb{T}^m}\nabla_\omega u(x)dx=0$.*

*Proof.* **1.** By periodicity, we have for all $i$

$$\int_{\mathbb{T}^m}\frac{\partial f}{\partial x_i}(x)dx=0,$$

from which we deduce the assertion 1. by linearity.

**2.** By density, we can find, for all $\varepsilon>0$, a function $f\in C^1(\mathbb{T}^m,\mathbb{E})$ such that $\|f-u\|_{1,\omega}<\varepsilon$.

Therefore, $\|d_\omega f-\nabla_\omega u\|<\varepsilon$, and so that $\|d_\omega f-\nabla_\omega u\|_{L^1(\mathbb{T}^m,\mathbb{E})}<\varepsilon(2\pi)^m$. Hence, using **1**, it can be seen that $|\int_{\mathbb{T}^m}\nabla_\omega u(x)dx|_{\mathbb{E}}<\varepsilon(2\pi)^m$. ☐

**Proposition 4.10.** *Let $\varphi\in C^1(\mathbb{T}^m,\mathcal{A})$ and $u\in H^1_\omega(\mathbb{T}^m,\mathcal{B})$. Then, $\varphi\cdot u\in H^1_\omega(\mathbb{T}^m,\mathcal{C})$ and we have $\nabla_\omega(\varphi\diamond u)=(d_\omega\varphi)\diamond u+\varphi\diamond(\nabla_\omega u)$.*

*Proof.* **First case.** $(\mathcal{A},\mathcal{B})=(\mathbb{E}',\mathbb{E})$ (or $(\mathbb{E},\mathbb{E}')$ which can be treated in the same manner).

**First step in this first case.** We show that

$$\lim_{t\to0}\left\|\frac{\tau_{t\omega}\varphi-\varphi}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u\right\|=0.$$

Let us fix an $\varepsilon>0$. Thanks to the uniform continuity $\varphi'$, there exists $t_0>0$ such that if $|t|<t_0$, for all $x\in\mathbb{R}^m$ and for all $\xi\in[x,x+t\omega]$, $|\varphi'(\xi)-\varphi'(x)|_{\mathbb{E}'}\le\varepsilon|\omega|^{-1}$.

Besides, using the mean inequality, we have

$$\begin{aligned}&\left|\frac{\varphi(x+t\omega)-\varphi(x)}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}u(x+t\omega)-(d_\omega\varphi(x))\cdot_{\mathbb{E}'\times\mathbb{E}}u(x+t\omega)\right|\\ &\le\sup_{\xi\in[x;x+t\omega]}|\varphi'(\xi)-\varphi'(x)|_{\mathbb{E}'}|\omega||u(x+t\omega)|_{\mathbb{E}}.\end{aligned}$$

Therefore, if $|t|<t_0$, this term is less than $\varepsilon|u(x+t\omega)|_E$. Taking the square and integrating, due to Lemma 3.3, we deduce

$$\left\|\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}-d_\omega\varphi\right]\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u\right\|\le\varepsilon\|u\|.$$

Thus, we have shown that

$$\lim_{t\to 0}\left\|\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}-d_\omega\varphi\right]\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u\right\|=0. \tag{4.1}$$

Moreover, we have

$$\|(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}(\tau_{t\omega}u-u)\|^2\le(2\pi)^m\|d_\omega\varphi\|_\infty^2\|\tau_{t\omega}u-u\|^2,$$

and by Proposition 3.8, this term tends to 0 as $t\to 0$.

Due to this result, we deduce that

$$\begin{aligned}&\left\|\frac{\tau_{t\omega}\varphi-\varphi}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u\right\|\\ &\le\left\|\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}-d_\omega\varphi\right]\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u\right\|+\|(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}(\tau_{t\omega}u-u)\|.\end{aligned}$$

The inequality (4.1) allows to deduce the result.

**Second step of first case.** We prove that

$$\lim_{t\to 0}\left\|\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\frac{\tau_{t\omega}u-u}{t}-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\nabla_\omega u\right\|=0.$$

In fact, the term considered is less than

$$(2\pi)^{m/2}\|\varphi\|_\infty\left\|\frac{\tau_{t\omega}u-u}{t}-\nabla_\omega u\right\|,$$

which tends to 0 as $t\to 0$.

**Conclusion.** We get

$$\begin{aligned}&\frac{\tau_{t\omega}(\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}u)-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}u}{t}-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}(\nabla_\omega u)\\ &=\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u\right]+\left[\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\frac{\tau_{t\omega}u-u}{t}-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\nabla_\omega u\right],\end{aligned}$$

and so that it results from the two first steps that the right hand side member tends to 0 when $t$ tends to 0, and the assertion **1.** is consequently proven.

**Second case.** $(\mathcal{A},\mathcal{B})=(\mathbb{K},\mathbb{E})$ (or $(\mathbb{E},\mathbb{K})$ which is the same).

Let $e\in\mathbb{E}'$ and $\varphi_e(x):=\varphi(x)e$. Then, $\varphi_e\in H^1_\omega(\mathbb{T}^m,\mathbb{E}')$ and by assertion 1, we obtain

$$\nabla_\omega[(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u]=d_\omega(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u+(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}(\nabla_\omega u).$$

But, since $e$ is constant, we have $\nabla_\omega[(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u]=(\nabla_\omega\varphi)e\cdot_{\mathbb{E}'\times\mathbb{E}}u$ and $d_\omega(\varphi e)=(d_\omega\varphi)e$. Therefore, we obtain $e\cdot_{\mathbb{E}'\times\mathbb{E}}[\nabla_\omega(\varphi\cdot u)]=e\cdot_{\mathbb{E}'\times\mathbb{E}}[(d_\omega\varphi)u]+e\cdot_{\mathbb{E}'\times\mathbb{E}}[\varphi\cdot(\nabla_\omega u)]$. Since the relation is true for all $e\in\mathbb{E}'$, we conclude that

$$\nabla_\omega(\varphi\cdot u)=(d_\omega\varphi)u+\varphi\cdot(\nabla_\omega u).$$

□

**Proposition 4.11.** *We have the following integration formula by parts. For all* $\varphi\in C^1(\mathbb{T}^m,\mathcal{A})$ *and* $u\in H^1_\omega(\mathbb{T}^m,\mathcal{B})$*, we have*

$$\int_{\mathbb{T}^m}\varphi\diamond(\nabla_\omega u)=-\int_{\mathbb{T}^m}(d_\omega\varphi)\diamond u.$$

*Proof.* Using the previous proposition, we know that $\varphi \diamond u \in H^1_\omega(\mathbb{T}^m, \mathcal{C})$ and that we have

$$\nabla_\omega(\varphi \diamond u) = (d_\omega \varphi) \diamond u + \varphi \diamond (\nabla_\omega u).$$

Integrating this equality, as in Proposition 4.9, the integral of the left hand side is zero, we have

$$\int_{\mathbb{T}^m} \varphi \diamond (\nabla_\omega u) + \int_{\mathbb{T}^m} (d_\omega \varphi) \diamond u = 0,$$

and so that the proof is completed. □

### 4.3. **The space $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$.**

**Definition 4.1.** We define the space $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ as the closure of $C^1_c(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

**Proposition 4.12.** *Endowed with the norm of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$, $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is complete. If in addition the space $\mathbb{E} = \mathbb{H}$ is a Hilbert space, $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a Hilbert space.*

*Proof.* $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a closed linear subspace of the complete space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$. Therefore, it is a complete space. □

**Proposition 4.13.** *$H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is also the closure of $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.*

*Proof.* Let $\tilde{H}$ be the closure of $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

The inclusion $\tilde{H} \subset H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a consequence of $C^1_c(\mathbb{T}^m, \mathbb{E}) \subset C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$.

For the converse sens, we shall prove that the injection of $C^1_c(\mathbb{T}^m, \mathbb{E})$ in $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ is dense for the norm of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$. Let take $\varphi \in C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ and $(\rho_n)_n$ a regularizing sequence. Since $\varphi \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, from the Proposition 3.7 applied on $\varphi$ and $\rho_n$, we get if $\psi_n := \rho_n * \varphi$, we have $\psi_n \in C^1(\mathbb{T}^m, \mathbb{E})$ and

$$\lim_{n \to +\infty} \|\psi_n - \varphi\|_{1,\omega} = 0.$$

Besides, we have

$$\operatorname{supp}(\psi_n) \subset \operatorname{supp}(\varphi) + \operatorname{supp}(\rho_n),$$

and as $\lim_{n \to +\infty} \operatorname{diam}(\operatorname{supp}(\rho_n)) = 0$, for $n$ large enough, we have $\operatorname{supp}(\psi_n) \subset \operatorname{Int} Q^m$. We deduce that $\psi_n \in C^1_c(\mathbb{T}^m, \mathbb{E})$ for $n$ large enough, which ends the proof. □

**Proposition 4.14.** *For all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$, we have the inequality of Poincaré-Wirtinger*

$$\|\nabla_\omega u\| \geq \frac{|\omega|}{\pi\sqrt{m}} \|u\|.$$

*Moreover, the map $u \mapsto \|\nabla_\omega u\|$ is a norm $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$ equivalent to $H^1_\omega(\mathbb{T}^m, \mathbb{H})$.*

We denote $\| \cdot \|_{1,\omega,0}$ the norm given in proposition, which means that for all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$,

$$\|u\|_{1,\omega,0} = \|\nabla_\omega u\|.$$

The proof of this proposition is based essentially on the verification of the inequality indicated for the regular functions, which is the main purpose of the following lemma.

**Lemma 4.1.** *We denote* $\alpha = \frac{|\omega|}{\pi\sqrt{m}}$. *Then, for all* $u \in C_c^1(\mathbb{T}^m, \mathbb{H})$, *we have*

$$\|d_\omega u\| \geq \alpha \|u\|.$$

*Proof of the Proposition* 4.14. Assume for the moment that Lemma 4.1 holds. Let $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$. Consider a sequence $(u_n)_n$ with values in $C_c^1(\mathbb{T}^m, \mathbb{H})$ tending to $u$ and for which we apply the lemma. Hence, for all $n$, we have

$$\|\nabla_\omega u_n\| \geq \alpha \|u_n\|.$$

But, since

$$\lim_{n\to+\infty} \|u - u_n\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\nabla_\omega u - \nabla_\omega u_n\| = 0,$$

we can take the limit to obtain

$$\|\nabla_\omega u\| \geq \alpha \|u\|.$$

From this inequality, we get for all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$

$$\|u\|_{1,\omega,0} \leq \|u\|_{1,\omega} \leq \frac{\sqrt{1+\alpha^2}}{\alpha} \|u\|_{1,\omega,0}$$

which means the equivalence of the norms. This completes the proof of proposition. □

Now, let us prove the lemma.

*Proof of Lemma* 4.1. We use the results and notations of Section 2.1. Let $u \in C_c^1(\mathbb{T}^m, \mathbb{H})$ be a fixed function and $v = u \circ \chi \circ \chi_1^{-1}$. So, we have

$$\|\nabla_\omega u\| = |\omega|^2 \int_{K^m} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy = |\omega|^2 \int_D \left( \int_{K(y_{-1})} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy_1 \right) dy_{-1}.$$

Take $y_{-1} \in D$. We set $[a, b] := K(y_{-1})$ (we remind that $K(y_{-1})$ is a closed interval, cf. Lemma 2.1). We put also $\varphi(y_1) = v(y_1, \dots, y_m)$. We notice that $\varphi \in C_c^1([a, b], \mathbb{H})$ and that

$$\varphi'(y_1) = \frac{\partial v}{\partial y_1}(y_1, \dots, y_m).$$

It comes then

$$\int_{K(y_{-1})} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy_1 = \int_a^b |\varphi'(t)|_{\mathbb{H}}^2 \, dt.$$

But, since $\varphi(a) = 0$, we have

$$|\varphi(t)|_{\mathbb{H}}^2 = \int_a^t \frac{\varphi(s) \cdot_{\mathbb{H}} \varphi'(s)}{2} ds \leq \frac{\|\varphi\|_{L^2([a,b],\mathbb{H})} \cdot \|\varphi'\|_{L^2([a,b],\mathbb{H})}}{2},$$

where we have used the inequality of Cauchy-Schwarz and dominated each integral (of positive functions) by the integral on the integer segment. By integration on $[a, b]$, we deduce that

$$\|\varphi\|_{L^2([a,b],\mathbb{H})} \leq \frac{b-a}{2} \|\varphi'\|_{L^2([a,b],\mathbb{H})},$$

and since $b-a=\operatorname{diam} K(y_{-1}) \leq 2\pi\sqrt{m}$, we get

$$\|\varphi\|_{L^2([a,b],\mathbb{H})} \leq \pi\sqrt{m}\|\varphi'\|_{L^2([a,b],\mathbb{H})}.$$

Going back to $v$, we obtain

$$\int_{K(y_{-1})} \left|\frac{\partial v}{\partial y_1}(y)\right|_{\mathbb{H}}^2 dy \geq \frac{1}{m\pi^2}\int_{K(y_{-1})} |v(y)|_{\mathbb{H}}^2\, dy,$$

or, taking into account $u$

$$\|d_\omega u\| \geq \frac{|\omega|}{\pi\sqrt{m}}\|u\|.$$

This ends the proof of the lemma. □

*Remark* 4.2. The fact that a constant non zero function does not verify the relation of Poincaré-Wirtinger shows that $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ is different of $H^1_\omega(\mathbb{T}^m,\mathbb{H})$.

*Notation* 4.1. We denote by $\alpha_{PW}(m)$ (or $\alpha_{PW}$ if there is no ambiguity on $m$), the best Poincaré-Wirtinger constant, that is to say

$$\begin{aligned}\alpha_{PW}(m) :=& \inf_{u\in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})\setminus\{0\}} \frac{\|\nabla_\omega u\|}{\|u\|}\\ =& \sup\left\{\alpha>0 : \text{for all } u\in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H}),\ \|\nabla_\omega u\|\geq \alpha\|u\|\right\}.\end{aligned}$$

We have then, for all $m$,

$$\alpha_{PW}(m) \geq \frac{|\omega|}{\pi\sqrt{m}}.$$

**Proposition 4.15.** *The canonical injection of $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ in $L^2(\mathbb{T}^m,\mathbb{H})$ is not compact.*

*Remark* 4.3. In other words, the space $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ does not verify a result of the type Rellich-Kondrachov. This lack of compactness makes it more difficult to obtain existence theorems in this space or in usual Sobolev's space $H^1_0(\Omega)$.

*Proof.* Given the characterization of strong compacts, i.e., for the topology of the norm of $L^2(\operatorname{Int} Q^m)$ (cf. [3, p. 74]), to deny Rellich-Kondrachov, it suffices to remark that: there exists $\varepsilon>0$, there exists $\Omega \subset\subset \operatorname{Int} Q^m$, there exists $\delta_0>0$, such that for all $\delta \in (0;\delta_0)$, we can find $h\in\mathbb{R}^m$, and $u\in AB_{H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})}$, such that $|h|\leq\delta$ and $\|\tau_h u - u\| \geq \varepsilon$.

Before moving forward with the proof, let us make three remarks.

- It suffices to construct a counterexample with $\mathbb{H}=\mathbb{R}$.
- Of course, this does not contradict the continuity of translations in $L^2$ because $u$ depends on $\delta$.
- The fundamental idea to remember is that the absence of compactness is due to the absence of control of the derivatives which are not in the direction of $\omega$.

We therefore take the case of $\mathbb{H} = \mathbb{R}$. Let $(b_j)$ be an orthonormal basis such that $b_1 = \frac{\omega}{|\omega|}$. We put $A \subset \operatorname{Int} Q^m$, $L_j$, $j = 1, \ldots, m$, $m$ strictly positive reals such that if

$$K := \left\{ A + \sum_{i=1}^{m} \lambda_i L_i b_i : \lambda = (\lambda_1, \ldots, \lambda_m) \in [0,1]^m \right\}$$

and

$$TK := \bigcup_{\alpha \in [0, \frac{2}{3} L_1]} \tau_\alpha K,$$

we have $TK \subset \operatorname{Int} Q^m$. We can then find an open set $\Omega$ containing $TK$ and so that the closure is contained in $\operatorname{Int} Q^m$ (which means $\Omega \subset\subset \operatorname{Int} Q^m$). We set $\delta_0 := \frac{2}{3} L_1$ and $l = \frac{1}{3} L_1$.

**First step.** In this step (which we only do if $m \geq 3$), we just have to treat the case where $m = 2$.

Let $\phi \in C^0(\mathbb{R}^{m-2}, \mathbb{R}^+)$ be not identically zero, such that $\operatorname{supp}(\phi) \subset \prod_{j=3}^{m} [0, L_j]$. We denote

$$\mathcal{I} := \int_{\mathbb{R}^{m-2}} \phi^2,$$

which is a strictly positive real.

We shall look for $v = u \circ \chi \circ \chi_1^{-1}$ having the form

$$v(y_1, \ldots, y_m) = v_2(y_1, y_2) \phi(y_3, \ldots, y_m).$$

If $\operatorname{supp}(v) \subset K$ and if $h = \delta b_2$ with $\delta \in (0, \delta_0)$, we have $\|(\tau_h v - v)\chi_\Omega\| = \|\tau_h v - v\|$, and using Fubini's theorem we obtain the following identities

$$\|\tau_h u - u\|^2 = \|\tau_h v_2 - v_2\|^2_{L^2(\mathbb{T}^2)} \mathcal{I}$$

and

$$\|\nabla_\omega u\|^2 = |\omega|^2 \|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)} \mathcal{I},$$

and finally,

$$\frac{\|\tau_h u - u\|^2}{\|\nabla_\omega u\|^2} = |\omega|^2 \frac{\|\tau_h v_2 - v_2\|^2_{L^2(\mathbb{T}^2)}}{\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)}}.$$

We see from the previous equality that it is enough to build $v_2$, which amounts to doing the proof in the case $m = 2$.

**Second step.** We shall now construct $v_2$.

Let us fix $\delta \in (0, \delta_0)$. For $i, j \in \{0, 1, 2\}$, we denote

$$A_{i,j} = A + i \frac{L_1}{3} b_1 + j \frac{L_2}{3} \cdot \frac{\delta}{\delta_0} b_2.$$

For all $\lambda \in \mathbb{R}^+$, we define the function $P_\lambda$ on $[0, l]$ by

$$P_\lambda(x) = 2 \frac{\lambda}{l^2} x^2 \chi_{[0, l/2]}(x) + \lambda \left( 1 - 2 \left( 1 - \frac{x}{l} \right)^2 \right) \chi_{[l/2, l]}(x).$$

So, we introduce $f_\lambda$ as $f_\lambda(x) = P_\lambda(x)\chi_{[0,l]}(x) + \lambda\chi_{[l,2l]}(x) + P_\lambda(3l - x)\chi_{[0,l]}(x)$. The function $f_\lambda$ is continuous and it is $C^1$ piecewise. We calculate

$$\|f'_\lambda\|_\infty = 2\frac{\lambda}{l}.$$

We set finally, $v_2(y_1, y_2) = f_{y_2/\delta}(y_1)\chi_{[0,\delta]}(y_2) + f_1(y_1)\chi_{[\delta,2\delta]}(y_2) + f_{3-y_2/\delta}(y_1)\chi_{[2\delta,3\delta]}(y_2)$. So, we have

$$\|\tau_{\delta b_2} v_2 - v_2\|^2_{L^2(\mathbb{T}^2)} \geq \int_{co(A_{1,1};A_{1,2};A_{2,1};A_{2,2})} (1-0)^2 dy = \frac{l\delta}{2}$$

and

$$\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)} \leq 2\left[2\int_{co(A_{0,0};A_{1,0};A_{0,1};A_{1,1})} \frac{4.y_2^2}{l^2\delta^2}dy + \int_{co(A_{0,1};A_{1,1};A_{0,2};A_{2,2})} \frac{4}{l^2}dy\right] = \frac{17\delta}{4l}.$$

Finally, we have

$$\frac{\|\tau_{\delta b_2} v_2 - v_2\|^2_{L^2(\mathbb{T}^2)}}{\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)}} \geq \frac{2l^2}{17},$$

hence we can take in the case arbitrary $m$

$$\varepsilon = \frac{l}{|\omega|}\sqrt{\frac{2}{17}}.$$

This achieves the proof. □

## 5. Fourier Analysis and Comparison of the Different Notions of Derivation

*Remark* 5.1. For convenience, we assume that $\mathbb{K} = \mathbb{C}$. In the real case, this consists in working in the complexification of $\mathbb{E}$, then in obtaining the Fourier coefficients of opposite indices.

We denote, for $u \in L^2(\mathbb{T}^m, \mathbb{E})$ and $\nu \in \mathbb{Z}^m$, $a(u;\nu)$ the element of $\mathbb{E}$

$$a(u;\nu) := \frac{1}{(2\pi)^m}\int_{Q^m} e_{-\nu}(x)u(x)dx.$$

We also denote

$$u \sim \sum_{\nu\in\mathbb{Z}^m} a(u;\nu)e_\nu.$$

We recall the following.

*Remind* 5.1. The map $u \mapsto (a(u;\nu))_{\nu\in\mathbb{Z}^m}$ is an isometric isomorphism from $L^2(\mathbb{T}^m, \mathbb{H})$ into $\ell^2(\mathbb{Z}^m; \mathbb{H})$.

*Remark* 5.2. The function $e_\nu$ is of class $C^1(\mathbb{T}^m, \mathbb{E})$ and

$$d_\omega e_\nu = i(\nu\cdot\omega)e_\nu.$$

**Proposition 5.1.** *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and $\nu \in \mathbb{Z}^m$. We have*

$$a(\nabla_\omega u;\nu) = i(\nu\cdot\omega)a(u;\nu).$$

*Proof.* Fix $\nu \in \mathbb{Z}^m$. The application $a(\cdot;\nu)$ is linear continuous from $L^2(\mathbb{T}^m, \mathbb{E})$ into $\mathbb{E}$, hence

$$a(\nabla_\omega u;\nu) = \lim_{t\to 0} \frac{a(\tau_{t\omega} u;\nu) - a(u;\nu)}{t}.$$

By Lemma 3.3, we have $a(\tau_{t\omega} u;\nu) = e_\nu(t\omega) a(u;\nu)$. □

**Proposition 5.2.** *Let $u \in L^2(\mathbb{T}^m, \mathbb{H})$ such that*

$$\sum_{\nu\in\mathbb{Z}^m} (\nu\cdot\omega)^2 |a(u;\nu)|^2_{\mathbb{H}} < +\infty.$$

*Then, $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$ and $\nabla_\omega u \sim \sum_\nu i(\nu\cdot\omega)a(u;\nu)e_\nu$.*

*Proof.* We know (cf. (5.1)) that there exists $v \in L^2(\mathbb{T}^m, \mathbb{H})$ such that $v \sim \sum_\nu i(\nu\cdot\omega)a(u;\nu)e_\nu$. For $k \in \mathbb{N}^*$, then form the trigonometric polynomial $P_k(x) = \sum_{|\nu|\le k} a(u;\nu)e_\nu(x)$. Thus, $\lim_{k\to+\infty} \|u - P_k\| = 0$ and $\lim_{k\to+\infty} \|v - \nabla_\omega P_k\| = 0$.

Since $(P_k; \nabla_\omega P_k)$ is in the graph $\nabla_\omega$ which is closed, so that we deduce that $v = \nabla_\omega u$ and then $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$. □

**Proposition 5.3.** *Let $u$ and $v$ be two elements of $L^2(\mathbb{T}^m, \mathbb{H})$ such that: for all $\varphi \in C^1(\mathbb{T}^m, \mathbb{C})$,*

$$\int_{\mathbb{T}^m} (d_\omega\varphi) \cdot_{\mathbb{H}} u = -\int_{\mathbb{T}^m} \varphi \cdot_{\mathbb{H}} v,$$

*or equivalently, for all $\varphi \in C^1(\mathbb{T}^m, \mathbb{H})$,*

$$\int_{\mathbb{T}^m} u \cdot_{\mathbb{H}} d_\omega\varphi = -\int_{\mathbb{T}^m} v \cdot_{\mathbb{H}} \varphi,$$

*then $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$ and $\nabla_\omega u = v$.*

*Likewise, let $u$ and $v$ be two elements of $L^2(\mathbb{T}^m, \mathbb{C})$ such that: for all $\varphi \in C^1(\mathbb{T}^m, \mathbb{C})$,*

$$\int_{\mathbb{T}^m} (d_\omega\varphi) u = -\int_{\mathbb{T}^m} \varphi v,$$

*or for all $\varphi \in C^1(\mathbb{T}^m, \mathbb{H})$,*

$$\int_{\mathbb{T}^m} u(d_\omega\varphi) = -\int_{\mathbb{T}^m} v\varphi,$$

*then $u \in H^1_\omega(\mathbb{T}^m, \mathbb{C})$ and $\nabla_\omega u = v$.*

*Remark* 5.3. This way of defining the derivative of an element of $L^2(\mathbb{T}^m, \mathbb{H})$ is analogous to that of Sobolev. Also, we will say that $v$ is the weak derivative of Sobolev. This proposition thus shows that this weak derivative, when it exists, coincides with the notion already introduced. We will show the reciprocal later.

*Proof of Proposition* 5.3. Taking $\varphi = e_\nu$, we obtain that $a(v;\nu) = -i(\nu\cdot\omega)a(u;\nu)$. Since $v \in L^2(\mathbb{T}^m, \mathbb{H})$, we deduce that $((\nu\cdot\omega)a(u;\nu))_\nu \in \ell^2(\mathbb{Z}^m; \mathbb{H})$, and Proposition 5.2 allows us to conclude.

Let $h \in \mathbb{H}$ non zeros. Let $\varphi \in C^1(\mathbb{T}^m, \mathbb{C})$. We apply the hypothesis with the functions $\varphi_h(x) := \varphi(x)h$. We get the result.

It is a special case with $\mathbb{H} := \mathbb{C}$. □

## 6. Link with Periodic Distributions and Distributions on the Torus

6.1. **Preliminary on vector-valued distributions.** We denote by $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ the vector space of class functions $C^\infty$ from $\mathbb{R}^m$ into $\mathbb{K}$ which vanish outside of a compact.

The space of distribution with values into $\mathbb{E}$ is by definition $\mathcal{L}(\mathcal{D}(\mathbb{R}^m, \mathbb{K}), \mathbb{E})$. We will note it $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$.

**Proposition 6.1.** *Each function $f \in L^p_{loc}(\mathbb{R}^m, \mathbb{E})$ defines a vector valued distribution $T_f$, for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$ by*

$$\langle T_f; \varphi \rangle := \int_{\mathbb{R}^m} \varphi(x) f(x) dx.$$

*Proof.* Since $f \in L^p_{loc}(\mathbb{R}^m, \mathbb{E})$, we have $f \in L^1_{loc}(\mathbb{R}^m, \mathbb{E})$ which shows in particular that for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$, $\varphi f \in L^1(\mathbb{R}^m, \mathbb{E})$ and so that $\int_{\mathbb{R}^m} \varphi f \in \mathbb{E}$. Besides, we have for all $e' \in \mathbb{E}'$

$$|e' \cdot_{\mathbb{E}' \times \mathbb{E}} f| \leq |e'|_{\mathbb{E}'} |f|_{\mathbb{E}},$$

and so $e' \cdot_{\mathbb{E}' \times \mathbb{E}} f \in L^1_{loc}(\mathbb{R}^m, \mathbb{R})$ for all $e' \in \mathbb{E}'$. Due to [14, Proposition 19, p. 66], we deduce that $f$ define a vector distribution. $\square$

6.2. **Periodification.** The main reference here is [18].

**Proposition 6.2.** *Let $\varphi : \mathbb{R}^m \to \mathbb{E}$ be a function with compact support. Then, for all $x \in \mathbb{R}^m$,*

$$\varpi(\varphi)(x) = \sum_{\lambda \in 2\pi\mathbb{Z}^m} \tau_\lambda \varphi(x)$$

*is well defined, the function $\varpi(\varphi) : \mathbb{R}^m \to \mathbb{E}$ is periodic, and $2\pi\mathbb{Z}^m \subset Per(\varpi(\varphi))$.*

*Proof.* **Existence.** We will verify that the sum defining $\varpi(\varphi)(x)$ is finite.

Let $x$ be fixed. $\tau_\lambda \varphi(x) \neq 0$ implies $x + \lambda \in \text{supp}(\varphi)$ which gives $\lambda \in (\text{supp}(\varphi) - x) \cap 2\pi\mathbb{Z}^m$ and as this intersection is finite because the support of $\varphi$ is bounded, we deduce that the sum defining $\varpi(\varphi)$ deals only with a finite number of terms, hence the existence of $\varpi(\varphi)$.

**Periodicity.** This property is a direct consequence of the fact that $2\pi\mathbb{Z}^m$ is a group. $\square$

The operator $\varpi$ extends to compactly supported distributions as follows. For $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, we set for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$

$$\langle \varpi T; \varphi \rangle := \langle T; \varpi \varphi \rangle.$$

**Proposition 6.3.** *$\varpi$ applies continuously $\mathcal{D}(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$ and $\mathcal{E}'(\mathbb{R}^m, \mathbb{E})$ in $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$.*

*Proof.* Let $K$ be a fixed compact of $\mathbb{R}^m$. $\varpi$ applies continuously $\mathcal{D}_K(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$, hence $\mathcal{D}(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$. The expression defining $\varpi$ for distributions allows to conclude at the end of the proposition, since $\varpi$ is defined as being its transpose (with an abuse of notations). $\square$

**Proposition 6.4.** *The following statements are true.*

1. *For all $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, $\varpi T$ is periodic and more exactly we have for all $\lambda \in 2\pi\mathbb{Z}^m$ $\varpi(\tau_\lambda T) = \tau_\lambda(\varpi T) = \varpi T$.*
2. *For all $F \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $\psi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$, we have $\varpi(F\psi) = (\varpi\psi).F$.*
3. *For all $f \in C^\infty(\mathbb{T}^m, \mathbb{K})$ and $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, we have $\varpi(fT) = f.(\varpi T)$.*

*Proof.* We refer to [18, p. 62–63], where the proofs can be adapted without problems to the Banach framework as arrival space.

The assertion **1.** is immediate by transposition.

For the assertion **2.**, we have $\tau_\lambda(\psi F) = \tau_\lambda(\psi)\tau_\lambda(F) = \tau_\lambda(\psi)F$ as $F$ is periodic. We can conclude by passing to the sum.

The last assertion can be done as the second one. ☐

**Proposition 6.5.** *The following statements hold true.*

1. *If $\varphi \in C^0_c(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^0(\mathbb{T}^m, \mathbb{E})$.*
2. *If $\varphi \in C^1_c(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^1(\mathbb{T}^m, \mathbb{E})$ and moreover, we have for all $i = 1, \dots, m$*
$$\varpi\left(\frac{\partial\varphi}{\partial x_i}\right) = \frac{\partial\varpi(\varphi)}{\partial x_i} \quad \text{and} \quad \varpi(d_\omega\varphi) = d_\omega\varpi(\varphi).$$
3. *For all $k \in \mathbb{N} \cup \{+\infty\}$, if $\varphi \in C^k_c(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^k(\mathbb{T}^m, \mathbb{E})$.*

*Proof.* We notice that **3.** is a consequence of **2.** by iteration. We also notice that $\operatorname{supp}(\tau_\lambda\varphi) = \operatorname{supp}(\varphi) - \lambda$.

Let fix an $x$. We shall prove that on a ball centered in $x$, we can choose a fix finite set of indexes $\lambda$ for which the terms of the sum are non zero. The assertions of the proposition will follow immediately. Noticing $K$ the compact $\operatorname{supp}(\varphi) - x$, we remark first that $d(x; \operatorname{supp}(\tau_\lambda\varphi)) = d(\lambda; K)$, and for some $r > 0$ being fixed, the set
$$Z := \{\lambda \in 2\pi\mathbb{Z}^m : d(x; \operatorname{supp}(\tau_\lambda\varphi)) < r\}$$
is finite, and since $Z = \{\lambda \in 2\pi\mathbb{Z}^m : \operatorname{Int} B(x,r) \cap \operatorname{supp}(\tau_\lambda\varphi) \neq \emptyset\}$, we have over $\operatorname{Int} B(x,r)$,$\varpi(\varphi) = \sum_{\lambda \in Z} \tau_\lambda\varphi$. Proposition is then proved. ☐

*Remark* 6.1. Previously, we have extended some functions $u : \operatorname{Int} Q^m \to \mathbb{E}$ to functions $\tilde{u} : \mathbb{T}^m \to \mathbb{E}$. Denoting by $u_0$ the extension of $u$ to $\mathbb{R}^m$ by 0, we have $\tilde{u} = \varpi(u_0)$.

6.3. **Periodic distributions and distributions on the torus.** Here also, the main reference is [18] where the proofs can be adapted to the Banach framework. We begin by a lemma of periodic partition of the unit.

**Lemma 6.1.** *There exists a function $\theta \in \mathcal{D}(\mathbb{R}^m, \mathbb{R})$ such that $\varpi\theta = 1$.*

*Proof.* See [18, p. 63]. ☐

**Lemma 6.2** (Surjectivity Lemma)**. 1.** *For all $f \in C^\infty(\mathbb{T}^m, \mathbb{E})$, there exists $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{E})$, such that $f = \varpi(\varphi)$.*

**2.** *For all $F \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, there exists $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, such that $F = \varpi(T)$.*

*Proof.* Taking into account (6.5), we can take $\varphi = \theta f$ and $T = \theta F$. □

**Proposition 6.6.** *The spaces $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$, equipped with the same dual topologies (strong ones or week ones), are algebraically and topologically isomorphic.*

Given the importance of this proposition, we demonstrate it in details.

*Proof.* **1.** $\varpi$ applies continuously $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{E}(\mathbb{R}^m, \mathbb{K})$, and so does apply $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{E}(\mathbb{R}^m, \mathbb{K}) \cap \mathcal{D}'(\mathbb{T}^m, \mathbb{K}) = C^\infty(\mathbb{T}^m, \mathbb{K})$. Its transpose, denoted as $\varpi^T$ and defined for all $L \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, and all $\varphi \in \mathcal{D}(\mathbb{T}^m, \mathbb{E})$ by

$$\langle \varpi^T L; \varphi\rangle = \langle L; \varpi\varphi\rangle_{\mathbb{T}^m}$$

applies then continuously $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$ in $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$. But, $\varpi^T L$ is a periodic distribution, and $\varpi^T$ sends continuously $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$ to $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$.

**2.** Let $\theta$ given by lemma of periodic partition of the unit. The application $f \mapsto \theta f$ applies then continuously $\mathcal{E}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$. Let $\Theta$ be the restriction of this application to $C^\infty(\mathbb{T}^m, \mathbb{K})$. $\Theta$ applies also continuously $C^\infty(\mathbb{T}^m, \mathbb{K})$ in $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$, and so its transpose continuously applies $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$ in $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$. Its restriction to $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$, again noted $\Theta^T$, applies $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ continuously in $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$.

**3.** We check by a simple calculation that $\varpi^T$ and $\Theta^T$ are reciprocal. □

*Remark* 6.2. From now on, we will systematically do this identification.

*Remark* 6.3. We can explain this correspondence.

1. Given $F \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $T$ any distribution with compact verifying $\varpi T = F$, we have for all $f \in C^\infty(\mathbb{T}^m, \mathbb{R})$ $\langle F, f\rangle_{\mathbb{T}^m} = \langle T; f\rangle$.
2. If in addition $F$ is locally integrable, we can take $T = \chi_{Q^m} F$, and

$$\langle F, f\rangle_{\mathbb{T}^m} = \int_{Q^m} F(x) f(x) dx.$$

6.4. **Link with the concepts previously introduced.** We define Percival operators for the distributions.

**Definition 6.1.** We define the following.

1. The operator $\partial_\omega$ on $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ in the following way. If $T \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, we set for all $\varphi \in C^\infty(\mathbb{T}^m, \mathbb{K})$ $\langle \partial_\omega T; \varphi\rangle = -\langle T; d_\omega \varphi\rangle$.
2. For $T \in \mathcal{D}'(\operatorname{Int} Q^m, \mathbb{E})$, $D_\omega T$, for all $\varphi \in C^\infty(\operatorname{Int} Q^m, \mathbb{K})$ by $\langle D_\omega T; \varphi\rangle = -\langle T; d_\omega \varphi\rangle$.

*Remark* 6.4. The following assertions are true.

1. If $\varphi \in C^1(\mathbb{T}^m, \mathbb{E})$, then $\varphi \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $\partial_\omega \varphi = d_\omega \varphi$.
2. If $\varphi \in C^1_c(\operatorname{Int} Q^m, \mathbb{E})$, then $\varphi \in \mathcal{D}'(\operatorname{Int} Q^m, \mathbb{E})$ and $D_\omega \varphi = d_\omega \varphi$.

We now indicate a characterization of $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ in terms of periodic distributions.

**Proposition 6.7.** *The following equality holds true*

$$H^1_\omega(\mathbb{T}^m, \mathbb{H}) = \left\{u \in L^2(\mathbb{T}^m, \mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{H})\right\},$$

*and if* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$*, we have* $\nabla_\omega u = \partial_\omega u$.

*Proof.* We shall suppose that $\mathbb{K} = \mathbb{C}$ for sake of simplicity.

**Inclusion** $H^1_\omega(\mathbb{T}^m, \mathbb{H}) \supset \{u \in L^2(\mathbb{T}^m, \mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{H})\}$.

Let $u \in L^2(\mathbb{T}^m; \mathbb{H})$ such that $\partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{H})$. We remark first that we have for all $\varphi \in C^\infty(\mathbb{T}^m, \mathbb{K})$ the following relation on $\partial_\omega$

$$\int_{\mathbb{T}^m} \varphi \cdot \partial_\omega u = -\int_{\mathbb{T}^m} d_\omega \varphi \cdot u.$$

By Proposition 5.3, we deduce that $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$ and $\partial_\omega u = \nabla_\omega u$.

**Inclusion** $H^1_\omega(\mathbb{T}^m, \mathbb{H}) \subset \{u \in L^2(\mathbb{T}^m, \mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{H})\}$.

From **4.** of Proposition 4.11, if $u \in H^1_\omega(\mathbb{T}^m, \mathbb{H})$, we have for all $\varphi \in C^\infty(\mathbb{T}^m, \mathbb{K})$

$$\int_{\mathbb{T}^m} \varphi \nabla_\omega u = -\int_{\mathbb{T}^m} d_\omega \varphi \cdot u.$$

But, this means that $\nabla_\omega u = \partial_\omega u$, and so $\partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{H})$. ☐

Each function $f \in L^1_{loc}(\mathbb{R}^m, \mathbb{E})$ presents a distribution noted by $T_f$. We denote $D_i$ the distributional partial derivatives on $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$ and $D_\omega$ the operator $D_\omega = \sum_{i=1}^m \omega_i D_i$. We recall that $\partial_\omega u$ has been defined for $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

**Proposition 6.8.** *Let* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$*. Then, we have on* $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$

$$D_\omega T_u = T_{\partial_\omega u},$$

*that is,* $D_\omega u$ *is represented by* $\partial_\omega u$.

To begin with, we notice that $T_{\partial_\omega u}$ is well defined because

$$\partial_\omega u \in L^2(\mathbb{T}^m, \mathbb{E}) \subset L^2_{loc}(\mathbb{R}^m, \mathbb{E}) \subset L^1_{loc}(\mathbb{R}^m, \mathbb{E}).$$

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{K})$ be a fixed function. There exist $\lambda_1, \ldots, \lambda_p$ such that $\operatorname{supp}(\varphi) \subset \cup_{j=1}^p (Q^m + \lambda_j)$. Let $i \in \{1, \ldots, m\}$. We have

$$\langle D_i T_u; \varphi\rangle = -\langle T_u, \frac{\partial \varphi}{\partial x_i}\rangle = -\int_{\mathbb{R}^m} \frac{\partial \varphi}{\partial x_i} u,$$

and since if $i \neq j$, $(Q^m + \lambda_i) \cap (Q^m + \lambda_j)$ is of zero measure, this integral is equal to

$$\begin{aligned} -\sum_{j=1}^p \int_{(Q^m+\lambda_j)} \frac{\partial \varphi}{\partial x_i} u &= -\sum_{j=1}^p \int_{Q^m} \frac{\partial \varphi(x+\lambda_j)}{\partial x_i} u(x+\lambda_j) dx \\ &= -\sum_{j=1}^p \int_{Q^m} \frac{\partial \varphi(x+\lambda_j)}{\partial x_i} u(x) dx \\ &= -\int_{\operatorname{Int} Q^m} \left(\sum_{j=1}^p \tau_{\lambda_j} \frac{\partial \varphi}{\partial x_i}\right) u. \end{aligned}$$

But, if $x \in \operatorname{Int} Q^m$ and $\lambda$ is not a $\lambda_j$, $j = 1, \dots, p$, we have $\varphi(x+\lambda) = 0$ by definition of $\lambda_j$. Thus, the obtained integral is equal to

$$-\int_{\operatorname{Int} Q^m} \varpi\left(\frac{\partial \varphi}{\partial x_i}\right) u.$$

Therefore, we finally obtain

$$\langle D_\omega T_u; \varphi\rangle = -\int_{\operatorname{Int} Q^m} \varpi(d_\omega \varphi) u. \tag{6.1}$$

Moreover, arguing in the same way, we have the following identities

$$\begin{aligned}\int_{\mathbb{R}^m} \varphi \partial_\omega u &= \sum_{j=1}^{p} \int_{Q^m+\lambda_j} \varphi \partial_\omega u = \sum_{j=1}^{p} \int_{\operatorname{Int} Q^m} (\tau_{\lambda_j}\varphi) \partial_\omega u = \int_{\operatorname{Int} Q^m} \left(\sum_{j=1}^{p} \tau_{\lambda_j}\varphi\right) \partial_\omega u \\ &= \int_{\operatorname{Int} Q^m} (\partial_\omega u) \varpi(\varphi) = -\int_{\operatorname{Int} Q^m} u d_\omega \left(\varpi(\varphi)\right) \partial_\omega u = -\int_{\operatorname{Int} Q^m} \varpi\left(d_\omega \varphi\right) u.\end{aligned}$$

Thus, we have shown that

$$\int_{\mathbb{R}^m} \varphi \partial_\omega u = -\int_{\operatorname{Int} Q^m} \varpi\left(d_\omega(\varphi)\right) u.$$

Comparing this equality with the equality (6.1), we finally see that for all $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$

$$\langle D_\omega T_u, \varphi\rangle = \int_{\operatorname{Int} Q^m} \varphi \cdot \partial_\omega u,$$

which ends the proof of proposition. □

## 7. Sobolev Spaces on $\operatorname{Int} Q^m$

**Definition 7.1.** We define

$$H^1_\omega(\operatorname{Int} Q^m, \mathbb{E}) := \left\{u \in L^2(\operatorname{Int} Q^m, \mathbb{E}) : D_\omega u \in L^2(\operatorname{Int} Q^m, \mathbb{E})\right\},$$

which we endow with the norm

$$\|u\|_\omega := \sqrt{\int_{\operatorname{Int} Q^m} |u|^2_{\mathbb{E}} + |D_\omega u|^2_{\mathbb{E}}}.$$

We define an inner product on $H^1_\omega(\operatorname{Int} Q^m, \mathbb{H})$ by setting, for all $u, v \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{H})$

$$(u; v)_\omega := \int_{\operatorname{Int} Q^m} u \cdot_{\mathbb{H}} v + D_\omega u \cdot_{\mathbb{H}} D_\omega v.$$

**Proposition 7.1.** $H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$ *is a Banach space* (*Hilbert space if* $\mathbb{E} = \mathbb{H}$).

*Proof.* Let $(u_n)_n$ be a Cauchy sequence with values in $H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$. Then, the two sequences $(u_n)_n$ and $(D_\omega u_n)_n$ are of Cauchy with values in the complete space $L^2(\operatorname{Int} Q^m, \mathbb{E})$, and so convergent to $u$ and $v$ respectively. Besides, the operator $D_\omega$ is continuous, and so that we can say $v = D_\omega u$, which proves that $u \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$. □

**Definition 7.2.** We define $H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$ as being the closure of $C^1_c(\operatorname{Int} Q^m, \mathbb{E})$ in $H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$.

*Remark* 7.1. We can define also $H^1_\omega(\mathbb{R}^m, \mathbb{E})$.

The following two propositions explain the links between Sobolev spaces on the torus and Sobolev spaces on the cube.

**Proposition 7.2.** *The following assertions hold true.*

**1.** *For all* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, $u_{|\operatorname{Int} Q^m} \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$.

**2.** *For all* $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$, $u_{|\operatorname{Int} Q^m} \in H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$.

*Proof.* **1.** Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$. We set $w = u_{|\operatorname{Int} Q^m}$ and $z = \nabla_\omega u_{|\operatorname{Int} Q^m}$. Let $\varphi \in C^\infty_c(\operatorname{Int} Q^m, \mathbb{K})$. There exists $\phi \in C^\infty_c(\mathbb{T}^m, \mathbb{K})$ such that $\phi_{|\operatorname{Int} Q^m} = \varphi$. We then successively have

$$\int_{\operatorname{Int} Q^m} \varphi z = \int_{\mathbb{T}^m} \phi \cdot (\nabla_\omega u) = -\int_{\mathbb{T}^m} (d_\omega \phi) u = -\int_{\operatorname{Int} Q^m} (d_\omega \varphi) w.$$

This shows that $z = D_\omega w$ and so $w \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$.

**2.** Let $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$. By **1**, $u_{|\operatorname{Int} Q^m} \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$. Let $(f_j)_j$ be a sequence of elements of $C^1_c(\mathbb{T}^m, \mathbb{E})$ converging to $u$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$. We denote by $g_j$ the restriction of $f_j$ to $\operatorname{Int} Q^m$ and $w = u_{|\operatorname{Int} Q^m}$. We have then : $\|w - g_j\|_\omega = \|u - f_j\|_{1,\omega}$ and so the term in the left tends to 0 as $j$ tends to infinity, which means that : $w \in H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$. □

**Proposition 7.3.** *For all* $u \in H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$, *there exists a unique* $\tilde{u} \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ *such that* $\tilde{u}_{|Q^m} = u$. *Moreover,* $\nabla_\omega \tilde{u}_{|Q^m} = D_\omega u$.

*Proof.* Let $u \in H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$ and $(f_j)_j$ be a sequence of $C^1_c(\operatorname{Int} Q^m, \mathbb{E})$ converging to $u$ in $H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$. So, there exists a sequence $(F_j)_j$ of $C^1_c(\mathbb{T}^m, \mathbb{E})$ such that the restriction of $F_j$ to $\operatorname{Int} Q^m$ coincides with $f_j$. The sequence $(F_j)_j$ is of Cauchy in $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ and so converges to a function $U$. We denote $v$ the restriction of $U$ to $\operatorname{Int} Q^m$, which is a function of $H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$ due to the previous proposition.

Besides, we have $\|v - f_j\|_\omega = \|U - F_j\|_{1,\omega}$. Since the right hand side term tends to 0 as $j$ goes to $\infty$, the term of right so is, and then by uniqueness of the limit, we have $v = u$, that is, $u = U_{|\operatorname{Int} Q^m}$ and so we obtain the existence of $\tilde{u}$.

Let us now prove the uniqueness. Let $U_1$ and $U_2$ be two candidates. We have

$$\int_{\mathbb{T}^m} |U_1 - U_2|^2_{\mathbb{E}} = \int_{\operatorname{Int} Q^m} |U_1 - U_2|^2_{\mathbb{E}} = \int_{\operatorname{Int} Q^m} |u - u|^2_{\mathbb{E}} = 0,$$

which ends the proof. □

*Remark* 7.2. The two preceding propositions show in particular that the application of $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ to $H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$ which associates to $u$ the value $u|_{\operatorname{Int} Q^m}$ is an isometric isomorphism. That allows to identify the two Hilbert spaces.

## 8. Higher Order Spaces

Let us quickly point out that we can of course define higher order Sobolev spaces.

**Definition 8.1.** Let $p \in \mathbb{N}^*$. We define the space $H^p_\omega(\mathbb{T}^m, \mathbb{E})$ as the space of $u \in L^2(\mathbb{T}^m, \mathbb{E})$ such that for all $j \leq p$, $\nabla^j_\omega u \in L^2(\mathbb{T}^m, \mathbb{E})$. It is endowed by the norm

$$\|u\|_{p,\omega} := \sqrt{\sum_{j=0}^{p} \|\nabla^j_\omega u\|^2},$$

and is a Hilbert space when $\mathbb{E} = \mathbb{H}$.

Similarly, we define of course $H^p_\omega(\operatorname{Int} Q^m, \mathbb{E})$. We can also define as well other spaces built using $L^p$ where $p \neq 2$. Due to future needs, and since the study of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is already very detailed, we will not dwell more on these spaces.

## 9. On the Absolute Continuity of the Functions of $H^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$

We suppose in this paragraph that $\mathbb{E} = \mathbb{R}^N$.

Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, and let $g := u \circ \chi_1^{-1}$. Then, $g \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$ and $D_1 g \in \mathcal{D}'(\mathbb{R}^m, \mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$. Let now $C^{m-1}$ be a convex set with non empty interior of $\mathbb{R}^{m-1}$, and $\Xi := \chi_1^{-1}(0 \times \operatorname{Int}(C^{m-1}))$.

Fix an $\varepsilon > 0$. We introduce

$$\Omega_n := (n - \varepsilon, n + 1 + \varepsilon) \times \operatorname{Int}(C^{m-1}),$$

this is an open convex subset of $\mathbb{R}^m$, $g_{|\Omega_n}$ and $D_1(g_{|\Omega_n})$ are into $L^2(\Omega_n, \mathbb{R}^N)$.

Set now

$$O_n := \left\{ y_{-1} \in \operatorname{Int}(C^{m-1}) : [y_1 \mapsto g(y_1, y_{-1})] \in AC(\ (n - \varepsilon, n + 1 + \varepsilon)\ , \mathbb{R}^N) \right\}.$$

Due to Necas [10, p. 61], for each integer $n$, $O_n$ is of full measure in $\operatorname{Int}(C^{m-1})$. Since a countable union of negligible set is a negligible set, we deduce that $\bigcap_n O_n$ is also of full measure $\operatorname{Int}(C^{m-1})$. But

$$\bigcap_n O_n = \left\{ y_{-1} \in \operatorname{Int}(C^{m-1}) : [y_1 \mapsto g(y_1, y_{-1})] \in AC_{loc}\left(\mathbb{R}, \mathbb{R}^N\right) \right\}.$$

By remarking that $u(t\omega + \sum_{j=2}^m y_j b_j) = g(t|\omega|, y_{-1})$ and that $\chi_1^{-1}$ is a linear isometry, we have then established the following.

**Lemma 9.1.** *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Then*

$$\Xi' := \left\{ \xi \in \Xi : [t \mapsto u(t\omega + \xi)] \in AC_{loc}(\mathbb{R}, \mathbb{R}^N) \right\}$$

*is of full measure in $\Xi$.*

We shall now establish the following proposition.

**Proposition 9.1.** *Let $u \in H^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$.*

1. *There exists $\Xi_p$ of full measure in $\Xi$ such that if $\xi \in \Xi_p$, for Lebesgue-almost every $t \in \mathbb{R}$, the function $t \mapsto u(t\omega + \xi)$ is differentiable, and*

$$\frac{d^j}{dt^j}[u(t\omega + \xi)] = (\nabla^j_\omega u)(t\omega + \xi), \quad j \in \{0, \dots, p\},\ [t \mapsto u(t\omega + \xi)] \in H^p_{loc}(\mathbb{R}, \mathbb{R}^N).$$

**2.** *If $u \in H^p_\omega(\mathbb{T}^m, \mathbb{R}^N) \cap L^\infty(\mathbb{T}^m, \mathbb{R}^N)$, then there exists $\Xi'_p$ of full measure in $\Xi$ such that if $\xi \in \Xi'_p$*

$$[t \mapsto u(t\omega + \xi)] \in H^p_{loc}(\mathbb{R}, \mathbb{R}^N) \quad \text{and} \quad \sup_{t\in\mathbb{R}} |u(t\omega+\xi)| \le \|u\|_{L^\infty(\mathbb{T}^m,\mathbb{R}^N)}.$$

*Proof.* **First assertion when $p = 1$.**

**1.** For $y_{-1} \in \Xi'$, the function $y_1 \mapsto g(y_1, y_{-1})$ is locally absolutely continuous, and so almost everywhere differentiable, and

$$D_1 g(y_1, y_{-1}) = \frac{\partial g}{\partial y_1}(y_1, y_{-1}).$$

So, for almost every $t \in \mathbb{R}$, we have

$$D_1 g(t|\omega|, y_{-1}) = \frac{\partial g}{\partial y_1}(t|\omega|, y_{-1}).$$

Let $t_0$ arbitrary such that these two members exist. By composition, $t \mapsto g(t|\omega|, y_{-1})$ is differentiable in $t_0$, and we have in this point

$$\frac{d}{dt} g(t|\omega|, y_{-1}) = |\omega| \frac{\partial g}{\partial y_1}(t|\omega|, y_{-1}).$$

Besides, since we have

$$\frac{d}{dt} u\left(t\omega + \sum_{j=2}^m y_j b_j\right) = \frac{d}{dt} g(t|\omega|, y_{-1}),$$

we deduce $\frac{d}{dt} u(t\omega + \sum\limits_{j=2}^m y_j b_j)$ exists $t$-almost everywhere, and then:

$$\frac{d}{dt} u\left(t\omega + \sum_{j=2}^m y_j b_j\right) = \nabla_\omega u\left(t\omega + \sum_{j=2}^m y_j b_j\right),$$

which we looked for.

**2.** We write $\xi = \sum_{j=2}^m y_j b_j$. Since $\chi_1^{-1}$ is a linear isometry and $\nabla_\omega u \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$, $D_1 g = \nabla_\omega u \circ \chi_1^{-1} \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$. Thus, $|D_1 g|^2 \in L^1_{loc}(\mathbb{R}^m, \mathbb{R})$ and so by Fubini' theorem, $|D_1 g(\cdot, y_{-1})|^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ whence $D_1 g(\cdot, y_{-1}) \in L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$. From the previous calculus, we get $[t \mapsto \frac{d}{dt} u(t\omega + \xi)] \in L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, and then $[t \mapsto u(t\omega + \xi)] \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.

**First assertion for any $p$.** We shall proceed by induction. Let $p \ge 2$, and assume the assertion true for 1 and $p-1$. By induction hypothesis for $p-1$, there exists $\Xi_{p-1}$ of full measure in $\Xi$ such that for all $\xi \in \Xi_{p-1}$, $[t \mapsto u(t\omega + \xi)] \in H^{p-1}_{loc}(\mathbb{R}, \mathbb{R}^N)$ and

$$\frac{d^j}{dt^j}[u(t\omega + \xi)] = (\nabla^j_\omega u)(t\omega + \xi), \quad j \in \{0, \cdots, p-1\}.$$

Since $\nabla_\omega^{p-1} u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, by induction hypothesis for the rank 1, there exists $\Xi^*$ of full measure in $\Xi$ such that for all $\xi \in \Xi^*$, $[t \mapsto (\nabla_\omega^{p-1} u)(t\omega + \xi)] \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$ and :

$$\frac{d}{dt}[(\nabla_\omega^{p-1} u)(t\omega + \xi)] = (\nabla_\omega^{p} u)(t\omega + \xi), \quad j \in \{0, \dots, p-1\}$$

which gives the rank $p$ as on the set $\Xi_p := \Xi_{p-1} \cap \Xi^*$ we have

$$\frac{d}{dt}[(\nabla_\omega^{p-1} u)(t\omega + \xi)] = \frac{d}{dt}\left[\frac{d^{p-1}}{dt^{p-1}}[u(t\omega + \xi)]\right] = \frac{d^p}{dt^p}[u(t\omega + \xi)].$$

**Second assertion.** Using the same technique as in the proof of the previous lemma and the positive version of Fubini's theorem, we see that all

$$\{\xi \in \Xi : |u(t\omega + \xi)| \le \|u\|_\infty\}$$

is of full measure in $\Xi$, and so the assertion 2. results from the first one and taking into account the fact that the intersection of two sets of full measures is of full measure, too. □

## 10. Traces Theory

10.1. **Description of the boundary of $Q^m$.** The assertion **2.** of Lemma 3.1 describes the border of the cube. We can decompose it into parts of dimensions $k = 0$ to $m-1$. The part of dimension $k$ is :

$$\{p \in \partial Q^m : \operatorname{card}\{j : |p_j| = \pi\} = m - k\}.$$

We denote $\mathcal{F}^m$ the part (open faces) of dimension $m-1$. It corresponds to the regular border of $Q^m$ (cf. [5, p. 77] and [6, p. 95]). Denoting, for $(i, j) \in \{1, \dots, m\} \times \{1, 2\}$

$$F^i_j := (-\pi, \pi)^{i-1} \times \{(2j-3)\pi\} \times (-\pi, \pi)^{m-i},$$

we then have

$$\mathcal{F}^m = \bigcup_{i,j} F^i_j.$$

We introduce the following notations.

- $R(\partial Q^m) := \partial Q^m + 2\pi\mathbb{Z}^m$ (network generated by $\partial Q^m$).
- If $p \in \mathcal{F}^m$, we denote $\omega(p) := \varepsilon(p)\frac{\omega}{|\omega|}$ where $\varepsilon(p)$ is equal to 1 or -1 so that $\omega(p)$ is returning at $p$ in $Q^m$. $F^i_j$ being a relative open, this has a good sense (If $\omega$ and $-\omega$ were simultaneously leaving (or returning) in $p$, $\omega$ would be tangent, which is contradicted by the freedom of its components). If $p, q \in F^i_j$, $\omega(p) = \omega(q)$, we note $\omega_{i,j}$ the common vector.
- We define an involution $\rho$ on $\partial Q^m$ by setting

  $$\rho(-\pi, x_{-j}) = (\pi, x_{-j}) \quad \text{and} \quad \rho(\pi, x_{-j}) = (-\pi, x_{-j}).$$

  We remark that $\rho(F^i_j) = F^i_{3-j}$ and we call that these faces are opposite.

*Remark* 10.1. $\omega(\rho(p)) = -\omega(p)$ and so $\omega_{i,3-j} = -\omega_{i,j}$.

**Lemma 10.1.** *There exists $\gamma_0 > 0$ such that if $\gamma \in (0, \gamma_0]$, we have the following.*

- *At least one of the intersections is empty*
$$\left(p, p+\gamma\frac{\omega}{|\omega|}\right)\cap R(\partial Q^m), \quad \left(p, p-\gamma\frac{\omega}{|\omega|}\right)\cap R(\partial Q^m).$$
- *If* $(p, p+\gamma\omega(p))\cap R(\partial Q^m)\neq\emptyset$, *then* $(\rho(p), \rho(p)-\gamma\omega(p))\cap R(\partial Q^m)=\emptyset$.
- $co\{F_j^i; F_j^i+\gamma\omega_{i,j}\}\cap co\{F_{3-j}^i; F_{3-j}^i+\gamma\omega_{i,3-j}\}=\emptyset$, *where* $co\{A,B\}:=\{\lambda a+(1-\lambda)b : (\lambda, a, b)\in[0,1]\times A\times B\}$.

*Proof.* Let $\tilde\omega := \frac{\omega}{|\omega|}$.

**First step.** Let
$$\gamma(p) := \sup\Big\{\gamma>0 : (p, p-\gamma\tilde\omega)\cap R(\partial Q^m)=\emptyset \text{ or } (p, p+\gamma\tilde\omega)\cap R(\partial Q^m)=\emptyset\Big\}$$
and $\gamma_1 := \inf_{p\in\mathcal{F}^m}\gamma(p)$. It is clear that $\gamma(p)$ is the upper bound of a nonempty set plus strictly positive reals, it is then into $\mathbb{R}_+\setminus\{0\}$. We will find a strictly positive real lowering all the $\gamma(p)$, which shows that $\gamma_1>0$.

For $\gamma(p)$, let us introduce
$$\gamma_j(p) := \sup\Big\{\gamma>0 : (p_j, p_j-\gamma\tilde\omega_j)\cap(\pi+2\pi\mathbb{Z})=\emptyset$$
$$\text{or } (p_j, p_j+\gamma\tilde\omega_j)\cap(\pi+2\pi\mathbb{Z})=\emptyset\Big\}.$$
Since $(p\in R(\partial Q^m))$ if and only if exists $j$, $p_j\in\pi+2\pi\mathbb{Z}$, we have $\gamma(p)\geq\min_j\gamma_j(p)$.

Let $p\in\mathcal{F}^m$ and assume without loss of generality that $p_1=\pi$. We calculate $\gamma_1(p)=\frac{\pi}{|\tilde\omega_1|}$ and if $j\geq 2$,
$$\gamma_j(p)=\frac{\max\{d(p_j;\pi+2\pi\mathbb{Z}); d(2\pi-p_j;\pi+2\pi\mathbb{Z})\}}{|\tilde\omega_j|}\geq\frac{\pi}{|\tilde\omega_j|}.$$
We conclude that
$$\gamma(p)\geq\min_{1\leq j\leq m}\frac{\pi}{|\tilde\omega_j|}.$$
Therefore, we have proved that $\gamma_1>0$, and all $\gamma_0\leq\gamma_1$ satisfying the first condition.

**Second step.** We shall show that $\gamma_0\leq\gamma_1$ satisfies the second condition.

In fact, the first condition being verified, if $\gamma\leq\gamma_1$ and if $(p, p+\gamma\omega(p))\cap R(\partial Q^m)$ is non empty, then $(\rho(p), \rho(p)-\gamma\omega(p))\cap R(\partial Q^m)$ is empty.

**Third step.** Let us fulfill the last condition.

We set $B_j^i(\gamma) := co\{F_j^i; F_j^i+\gamma\omega_{i,j}\}$. $B_j^i(\gamma)$ and $B_{3-j}^i(\gamma)$ are two parallel bands, of width less than $\gamma|\omega|$; they don't intersect if $2\gamma|\omega|<2\pi$. Setting $\gamma_2 := \frac{\pi}{2|\omega|}$, we ensure that
$$(\gamma\leq\gamma_2)\Rightarrow(B_j^i(\gamma)\cap B_{3-j}^i(\gamma)=\emptyset).$$
Thus, every $\gamma\leq\gamma_2$ permits to fulfill the third condition.

We may then set $\gamma_0 := \min\{\gamma_1;\gamma_2\}$ to conclude. $\square$

*Notations* 10.1. We introduce now the following notations:

- $K_j^i := \{p\in F_j^i : (p, p+\gamma\omega_{i,j})\cap\partial Q^m=\emptyset\}$;

- $L_j^i := F_j^i \cap^c K_j^i$;
- $S_j^i := K_j^i + \omega_{i,j}[0,\gamma]$;
- $K := \cup_{i,j} K_j^i$;
- $L := \cup_{i,j} L_j^i$;
- $S := \cup_{i,j} S_j^i$.

*Remark* 10.2. The last condition of the lemma ensures that $\rho(L_j^i) \subset F_{3-j}^i$.

10.2. **Integration on the cube boundary.** For sake of simplicity, we introduce the following notations.

*Notations* 10.2. For $u \in C^0(\partial Q^m, \mathbb{E})$ and $(i,j) \in \mathbb{N}_m \times \mathbb{N}_2$, we put

- $I_j^i(u) := \displaystyle\int_{[-\pi,\pi]^{m-1}} u((2j-3)\pi, x_{-i}) dx_{-i}$;
- $\displaystyle\int_{\partial Q^m} u d\sigma_i := I_2^i(u) - I_1^i(u)$;
- $\displaystyle\int_{\partial Q^m} u d\sigma_\omega := \sum_{i=1}^m \omega_i \int_{\partial Q^m} u d\sigma_i$.

By density, these continuous linear forms extend to $L^1(\partial Q^m, \mathbb{E})$. We set finally, for $u \in L^2(\partial Q^m, \mathbb{E})$

$$\|u\|_{L^2(\partial Q^m, \mathbb{E})} := \sqrt{\sum_{1 \le i,j \le m} I_j^i(|u|_{\mathbb{E}}^2)}.$$

**Lemma 10.2.** *Let $f \in C^1(Q^m, \mathcal{A})$ and $g \in C^1(Q^m, \mathcal{B})$ with $\mathcal{C} = \mathbb{K}$. We have the following.*

**1.** *For all $i$*

$$\int_{Q^m} \frac{\partial f}{\partial x_i} \diamond g = -\int_{Q^m} f \diamond \frac{\partial g}{\partial x_i} + \int_{\partial Q^m} f \diamond g d\sigma_i.$$

**2.**

$$\int_{Q^m} (d_\omega f) \diamond g = -\int_{Q^m} f \diamond (d_\omega g) + \int_{\partial Q^m} f \diamond g d\sigma_\omega.$$

*Proof.* This is to use the Stokes formula for $Q^m$. This is allowed (cf. [16, p. 343] or [6, Chap. XXIV, n. 14]), and we have then, applying this formula to the differential form $f.g \bigwedge_{j \neq i} dx_j$

$$\int_{Q^m} \frac{\partial}{\partial x_i}(f \diamond g) = \int_{\partial Q^m} f \diamond g d\sigma_i,$$

and so that the first assertion is obtained by developing the derivative of product. From there, the second assertion is immediate by linearity. ☐

10.3. **Traces operators.** We introduce the operator of traces $\tilde{T}_0 : C^1(Q^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E})$, which canonically extends into an operator $T_0 : C^1(\mathbb{T}^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E})$.

10.3.1. *An intermediate estimate.*

**Proposition 10.1.** *There exists a constant $C > 0$ such that, for all $u \in C^1(\mathbb{T}^m, \mathbb{E})$, we have*

$$\|T_0(u)\|_{L^2(\partial Q^m,\mathbb{E})} \leq C\|u\|_{1,\omega}.$$

To prove this proposition, we will start by proving the following lemma.

**Lemma 10.3.** *There exists a constant $C_0 > 0$ such that, for all $u \in C^1(\mathbb{T}^m, \mathbb{E})$, we have :*

$$\int_K |T_0(u)|^2_{\mathbb{E}} d\sigma_\omega \leq C_0^2 \int_S \left[|u|^2_{\mathbb{E}} + |d_\omega u|^2_{\mathbb{E}}\right].$$

*Proof.* Fix $u$, and let $i, j$ be given.

We choose on $F^i_j$ a system of local coordinates $(\xi, \eta)$, where $\xi \in \mathbb{R}^{m-1}$ is tangent to $K^i_j$ and $\eta$ is the coordinate following $\omega_{i,j}$, such that

$$K^i_j \subset \left\{(\xi, 0) : \xi \in \mathbb{R}^{m-1}\right\}.$$

For the sake of simplicity, noting by $u_\eta := \frac{\partial u}{\partial \eta}$, we have for $t \in [0;\gamma]$

$$u(\xi, 0) = \int_t^0 u_\eta(\xi, \eta) d\eta + u(\xi, t)$$

and so,

$$|u(\xi, 0)|^2_{\mathbb{E}} \leq 2\gamma \int_0^t |u_\eta(\xi, \eta)|^2_{\mathbb{E}} d\eta + 2|u(\xi, t)|^2_{\mathbb{E}},$$

and thus,

$$|u(\xi, 0)|^2_{\mathbb{E}} \leq 2\gamma \int_0^\gamma |u_\eta(\xi, \eta)|^2_{\mathbb{E}} d\eta + 2|u(\xi, t)|^2_{\mathbb{E}}.$$

We integrate over $t$ between 0 and $\gamma$, to obtain

$$\gamma|u(\xi, 0)|^2_{\mathbb{E}} \leq 2 \int_0^\gamma \left(\gamma^2 |u_\eta(\xi, \eta)|^2_{\mathbb{E}} + |u(\xi, \eta)|^2_{\mathbb{E}}\right) d\eta.$$

Integrate overt $\xi$ on $K^i_j$. It comes

$$\int_{K^i_j} |u(\xi, 0)|^2_E d\xi \leq \frac{2}{\gamma} \int_{K^i_j} \int_0^\gamma \left(\gamma^2 |u_\eta(\xi, \eta)|^2_{\mathbb{E}} + |u(\xi, \eta)|^2_{\mathbb{E}}\right) d\eta d\xi.$$

Let $\Delta_{i,j}$ be the absolute value of the Jacobian for the transformation $(\xi, \eta) \mapsto (x_1, \ldots, x_m)$. We have

$$\int_{K^i_j} |u(\xi, 0)|^2_{\mathbb{E}} d\xi \leq \frac{2}{\gamma} \Delta_{i,j} \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\} \int_{S^i_j} |u_\eta|^2_{\mathbb{E}} + |u|^2_{\mathbb{E}}.$$

Multiply by $\omega_j$ and sum over $(i, j)$.

Denoting $\Delta := \max\limits_{(i,j)}\{\omega_i \Delta_{i,j}\}$ since each point of $S$ is into at most $2m$ sets $S^i_j$, we have

$$\sum_{i,j} \omega_i \Delta_{i,j} \int_{S^i_j} \leq 2m\Delta \int_S,$$

and finally we get

$$\int_K |u|_{\mathbb{E}}^2 d\sigma_\omega \leq \frac{4}{\gamma} m\Delta \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\} \int_S |u_\eta|_{\mathbb{E}}^2 + |u|_{\mathbb{E}}^2.$$

We can then take

$$C_0 := \sqrt{\frac{4}{\gamma} m\Delta \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\}}. \qquad \square$$

*Proof of Proposition* 10.1. By periodicity of $u$ and by Remark 10.2, we have

$$\int_L |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega \leq \int_K |T_0(u))|_{\mathbb{E}}^2 d\sigma_\omega.$$

Since in addition

$$\int_{\partial Q^m} |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega = \int_L |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega + \int_K |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega,$$

the periodicity of $u$ and the lemma allow to conclude with $C = C_0\sqrt{2}$. $\square$

10.3.2. *Extension to* $H_\omega^1(\mathbb{T}^m, \mathbb{E})$.

**Proposition 10.2.** *The map* $T_0$ *can be extended to a linear continuous map*

$$\gamma_0 : H_\omega^1(\mathbb{T}^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E}).$$

*Proof.* By the previous proposition, the application $T_0$ is linear and continuous of $(C^1(\mathbb{T}^m, \mathbb{E}); \|\cdot\|_{H_\omega^1(\mathbb{T}^m,\mathbb{E})})$ into $L^2(\partial Q^m, \mathbb{E})$ and since $C^1(\mathbb{T}^m, \mathbb{E})$ is dense in $H_\omega^1(\mathbb{T}^m, \mathbb{E})$, $T_0$ can be extended in a unique way to a linear continuous application $\gamma_0$ from $H_\omega^1(\mathbb{T}^m, \mathbb{E})$ into $L^2(\partial Q^m, \mathbb{E})$. $\square$

*Remark* 10.3. $\tilde{T}_0$ extends

$$\tilde{\gamma}_0 : H_\omega^1(Q^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E}),$$

which is linear and continuous.

10.4. **Theorem of traces.** The main purpose here is to prove the theorem of traces, which gives

$$H_{\omega,0}^1(\mathbb{T}^m, \mathbb{E}) = \operatorname{Ker} \gamma_0.$$

Let $u \in L^2(Q^m, \mathbb{E})$. $u$ can be canonically extended to

- $\hat{u} \in L^2(\mathbb{T}^m, \mathbb{E})$;
- $\tilde{u} := \chi_{Q^m}.u$.

**Lemma 10.4.** *We have the following.*

1. *If* $u \in L^2(Q^m, \mathbb{E})$*, then* $\tilde{u} \in L^2(\mathbb{R}^m, \mathbb{E})$.
2. *If* $u \in H_\omega^1(Q^m, \mathbb{E})$ *and* $\tilde{\gamma}_0(u) = 0$*, then, for all* $\varphi \in C^1(Q^m, \mathbb{K})$*, we have*

$$\int_{Q^m} \varphi \cdot \nabla_\omega u = -\int_{Q^m} (d_\omega \varphi) \cdot u.$$

3. *If* $u \in H_\omega^1(Q^m, \mathbb{E})$ *and* $\tilde{\gamma}_0(u) = 0$*, then* $\tilde{u} \in H_\omega^1(\mathbb{R}^m, \mathbb{E})$ *and* $\nabla_\omega \tilde{u} = \widetilde{\nabla_\omega u}$.

*Proof.* **1.** Since $u$ is measurable, $\tilde{u}$ is obviously measurable. We have

$$\int_{\mathbb{R}^m} |\tilde{u}|_{\mathbb{E}}^2 \leq \int_{Q^m} |\tilde{u}|_{\mathbb{E}}^2 \leq \int_{Q^m} |u|_{\mathbb{E}}^2 < +\infty,$$

and so that $\tilde{u} \in L^2(\mathbb{R}^m, \mathbb{E})$.

**2.** There exists $(u_n)_n$ with values into $C^1(Q^m, \mathbb{E})$ converging to $u$.

Since $\gamma_0$ is linear continuous, we see that $(\tilde{\gamma}_0(u_n))_n$ is of Cauchy in $L^2(Q^m, \mathbb{E})$ and so converges, and that the limit is $\tilde{\gamma}_0(u) = 0$.

Stokes formula immediately gives that

$$\int_{Q^m} \varphi d_\omega u_n = -\int_{Q^m} (d_\omega \varphi) u_n + \int_{\partial Q^m} \varphi \cdot \tilde{\gamma}_0(u_n) d\sigma_\omega.$$

Taking the limit, we get the desired result.

**3.** Let $\varphi \in C^1(Q^m, \mathbb{K})$. We have

$$\int_{R^m} \varphi \cdot \nabla_\omega \tilde{u} = -\int_{R^m} (d_\omega \varphi) \tilde{u},$$

due to derivation within the meaning of distributions; but the first term is

$$-\int_{Q^m} (d_\omega \varphi) u = \int_{Q^m} \varphi . (\nabla_\omega u) = \int_{Q^m} \varphi \cdot \widetilde{(\nabla_\omega u)},$$

because of **2.**

Thus, for all $\varphi \in C^1(Q^m, \mathbb{K})$, we have

$$\int_{R^m} \varphi \cdot \nabla_\omega \tilde{u} = \int_{Q^m} \varphi \cdot \widetilde{(\nabla_\omega u)},$$

which gives the assertion **3.** □

**Lemma 10.5.** *Let $u \in L^2(Q^m, \mathbb{E})$. We define for $\alpha > 1$, $\tilde{u}_\alpha : \mathbb{R}^m \to \mathbb{E}$ by*

$$\tilde{u}_\alpha(x) := \tilde{u}(\alpha x).$$

*Then*

**1.** $\tilde{u}_\alpha \in L^2(\mathbb{R}^m, \mathbb{E})$*;*
**2.** $\operatorname{supp}(\tilde{u}_\alpha) \subset \operatorname{Int} Q^m$*;*
**3.** $\lim_{\alpha \to 1^+} \|\tilde{u}_\alpha - \tilde{u}\|_{L^2(\mathbb{R}^m, \mathbb{E})} = 0$*;*
**4.** *If in addition $\tilde{u} \in H^1_\omega(\mathbb{R}^m, \mathbb{E})$, then $\tilde{u}_\alpha \in H^1_\omega(\mathbb{R}^m, \mathbb{E})$ and*

$$\lim_{\alpha \to 1^+} \|\tilde{u}_\alpha - \tilde{u}\|_{H^1_\omega(\mathbb{R}^m, \mathbb{E})} = 0.$$

*Proof.* **1.** It is a consequence of the previous lemma.

**2.** Since $\operatorname{supp}(\tilde{u}) \subset Q^m$, it comes that

$$\operatorname{supp}(\tilde{u}_\alpha) \subset \frac{1}{\alpha} Q^m \subset \operatorname{Int} Q^m.$$

**3.** Suppose, first, that $u$ is in addition continuous.

There exists then in $\mathbb{R}$ the number $M := \sup_{x\in\mathbb{R}^m} |\tilde{u}(x)|_{\mathbb{E}}$. Besides,

$$\|\tilde{u}_\alpha - \tilde{u}\|^2_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \int_{Q^m} |\tilde{u}_\alpha(x) - \tilde{u}(x)|^2_{\mathbb{E}} dx.$$

For a fixed $x$, $|\tilde{u}_\alpha(x) - \tilde{u}(x)|^2_{\mathbb{E}}$ tends to 0 as $\alpha$ tends to 1, and this function is less than the constant $4M^2$, which is integrable on $Q^m$. Lebesgue's dominated convergence theorem allows us to conclude.

Now, let us move on to the general case. Let us fix $\varepsilon > 0$. By density, there exists $\varphi \in C^0(Q^m, \mathbb{E})$ such that $\|\tilde{u} - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/3$. Since $\varphi$ is continuous, there exists $\alpha_0 > 1$ such that if $\alpha \in (1, \alpha_0)$, we have $\|\tilde{\varphi}_\alpha - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/3$. We have then if $\alpha \in (1, \alpha_0)$

$$\|\tilde{u}_\alpha - \tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \|\tilde{\varphi} - \tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|\tilde{\varphi}_\alpha - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|\tilde{\varphi}_\alpha - \tilde{u}_\alpha\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon.$$

**4.** Due to Lemma 10.4, we know that $\tilde{u}_\alpha \in H^1_\omega(\mathbb{R}^m, \mathbb{E})$. By **3**, it suffices to show that

$$\lim_{\alpha\to 1^+} \|\partial_\omega\left(\tilde{u}_\alpha - \tilde{u}\right)\|_{L^2(\mathbb{R}^m,\mathbb{E})} = 0.$$

But, we have

$$\|\partial_\omega\left(\tilde{u}_\alpha - \tilde{u}\right)\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \|(\partial_\omega\tilde{u})_\alpha - \partial_\omega\tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|(\partial_\omega\tilde{u})_\alpha - \partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})}.$$

Besides, by **3**, the first term of the right hand side tends to 0. For the second, we may write

$$\|(\partial_\omega\tilde{u})_\alpha - \partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})} = (\alpha^m - 1)\|\partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})},$$

which is the product of a term tending to 0 by a term bounded at the neighborhood to the right of 1, so the limit is 0, which ends the proof of the lemma. □

**Theorem 10.1** (Theorem of traces)**.** *We have* $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) = \operatorname{Ker}\gamma_0$.

*Proof.* We shall prove firstly that $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) \subset \operatorname{Ker}\gamma_0$.

Let $\hat{u} \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$, $u \in H^1_{\omega,0}(Q^m, \mathbb{E})$ associated to and $(\varphi_n)_n$ be a sequence of $C^1_c(\mathbb{T}^m, \mathbb{E})$ converging to $u$ in $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$. We have for all integers $n$

$$\gamma_0(\varphi_n) = T_0(\varphi_n) = 0,$$

and so by continuity of $\gamma_0$, we have $\gamma_0(u) = 0$, and so that $\gamma_0(\hat{u}) = 0$ and the inclusion is therefore proven.

Conversely, we aim to prove that $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) \supset \operatorname{Ker}\gamma_0$.

Let $\hat{u} \in \operatorname{Ker}\gamma_0$, $u \in H^1_\omega(Q^m, \mathbb{E})$ associated and $\varepsilon > 0$. By Lemma 10.5, there exists $\alpha_0 > 1$ such that $\|\tilde{u} - \tilde{u}_{\alpha_0}\|_{H^1_\omega(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/2$. Let $(\rho_n)_n$ be a regularizing sequence. Then, for all $n$, $\rho_n * \tilde{u}_{\alpha_0} \in C^1_{c,\omega}(\mathbb{R}^m, \mathbb{E})$.

Moreover, since $\operatorname{supp}(\tilde{u}_{\alpha_0}) \subset \operatorname{Int} Q^m$ and $\operatorname{diam}(\operatorname{supp}\rho_n)$ tends to 0 as $n$ goes to $+\infty$, we know that for $n$ large enough, $\operatorname{supp}(\rho_n * \tilde{u}_{\alpha_0}) \subset \operatorname{Int} Q^m$. There exists a $\varphi \in C^1_{c,\omega}(Q^m, \mathbb{E})$ such that $\|\varphi - \tilde{u}_{\alpha_0}\|_{1,\omega} \leq \varepsilon/2$. Finally, we get $\|\varphi - u\|_{1,\omega} \leq \varepsilon$, which proves that $u \in H^1_{\omega,0}(Q^m, \mathbb{E})$, i.e., $\hat{u} \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$. □

## 11. Conclusion

In this work, we have completely studied the relations between functions on the torus and the functions defined on the $m$-dimensional cube $Q = [-\pi, \pi]^m$.

We have in particular presented the spaces derived from Percival's formalism and adapted to them the usual results. We have noticed that whether some results extend, some do not: for example, the Rellich-Kondrachov theorem is no longer valid here.

This study has a number of direct and indirect applications in the search for almost/quasi- periodic solutions of an ordinary differential equation and transforming it to the search of periodic solutions in each variable of a partial differential equation.

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[1]Department of Mathematics, Faculty of Sciences of Gabès,
University of Gabès,
Cité Erriadh 6072 Gabès, Tunisia
Mathematics And Applications Laboratory, LR17ES11.
Faculty of Sciences of Gabès
*Email address*: dhaou06@gmail.com, Dhaou.Lassoued@fsg.rnu.tn, Dhaou.Lassoued@univ-paris1.fr

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