Kragujevac Journal of Mathematics
Volume 50(2) (2026), Pages 175–185.

THE PERFECT LOCATING SIGNED ROMAN DOMINATION OF SOME GRAPHS

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Abstract. In this paper, we introduce the concept of Perfect locating signed Roman dominating functions in graphs. A perfect locating signed Roman dominating $PLSRD$ function of a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that for (i) every vertex v with $f(v) = -1$ is adjacent to exactly one vertex u with $f(u) = 2$; (ii) any pair of distinct vertices v, w with $f(v) = f(w) = -1$ does not have a common neighbor u with $f(u) = 2$ and (iii) $f(v) + \sum_{u \in N(v)} f(u) \geq 1$ for any vertex v. The weight of $PLSRD$- function is the sum of its function values over all the vertices. The perfect locating signed Roman domination number of G denoted by $\gamma^P_{LSR}(G)$ is the minimum weight of a $PLSRD$-function in G. We present the upper and lower bonds of $PLSRD$- function for trees. In addition, for grid graph G, we show that $\gamma^P_{LSR}(G) \leq \frac{3}{4}|G|$.

1. Introduction and Preliminaries

In this paper, we continue the study of variant of Roman dominating function. Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. The order and size of graph G is the number of vertices and edges in G, respectively. The open neighborhood of vertex u in G is the set of all neighbors of u in G; that is $N_G(u) = \{v \in V \mid uv \in E(G)\}$. The closed neighborhood of u in G is $G[u] = \{u\} \cup N_G(u)$. The degree of u is $d_G(u) = |N_G(u)|$. We write P_n for the path of order n.

A leaf of a tree is a vertex of degree one and the support vertex is a vertex adjacent to a leaf. Let $S(T)$ and $L(T)$ denotes the set of all support vertices and the set of

Key words and phrases. Perfect Roman domination, locating Roman domination, signed Roman domination, Cartesian product graph.

2020 *Mathematics Subject Classification.* Primary: 05C69. Secondary: 05C76, 05C05.

https://doi.org/10.46793/KgJMat2602.175A

Received: July 31, 2023.

Accepted: August 13, 2023.

leaves in T, respectively. We denote $|L(T)| = l(T)$ and $s(T) = |S(T)|$. Let $L(u)$ denote the set of all leaves adjacent to a support vertex u and $l(u) = |L(u)|$.

Let G_1 and G_2 be two graphs. The cartesian product of graphs G_1 and G_2, denoted by $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1), (u_2, v_2) \in G_1 \Box G_2$ are adjacent if either

- $u_1, u_2 \in E(G_1)$ and $v_1 = v_2$, or
- $v_1, v_2 \in E(G_2)$ and $u_1 = u_2$.

The graph $P_n \Box P_m$ has n rows and m columns. If $G = P_n \Box P_m$, then $|G| = |nm|$.

A subset $D \subset V$ is a *dominating set* of G if every vertex in $V \setminus D$ has a neighbor in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Let $\alpha \in \{-1, 1, 2\}$ and for any vertex $u \in G$, we denote the set of vertices with $f(u) = \alpha$ by V_α.

The study of locating dominating sets in graphs was first studied by Slater [19, 20] whereby many graph related problems with various types of protection are studied. The objective of the work is to locate the intruder. A locating dominating set $D \subset V(G)$ is a dominating set with the property that for each vertex $u \in V(G) - D$, the set $N(u) \cap D$ is unique. The locating dominating set of G with minimum cardinality is known as locating domination number of G. The concept of locating domination has been considered for several domination parameters, for more result, see [6, 8–10].

A function $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function (RDF) on G if for every vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to at leaast one vertex u with $f(u) = 2$. The weight of RDF denoted by $w(f)$ is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The RDF on G with minimum weight is known as Roman domination number and denoted by $\gamma_R(G)$. Cockayne et al. [13] introduced Roman domination which was motivated by the work of Re Velle and Rosing [18] and Stewart [21]. More results on Roman domination can be found in [11, 12].

A perfect Roman dominating function (PRD-function) is a Roman dominating function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to exactly one vertex u with $f(u) = 2$. The weight of f is the sum $\sum_{v \in V(G)} f(v)$ denoted by $w(f)$. The perfect Roman domination number denoted by $\gamma_R^P(G)$ is the PRD-function with minimum weight. Henning et al. [14] first study perfect Roman domination. More work on PRD can be found in [5, 16, 17].

A signed Roman dominating function (SRD-function) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ with the condition that for every $v \in V(G)$, $f(N[v]) \geq 1$. This concept was introduced by Abdollahzadeh Ahangar in [3]. Further results on SRD-function can be found in [1, 2].

A RD-function is called a locating Roman dominating function (LRD-function) if for any pair of vertices u, v with $f(u) = f(v) = 0, N(u) \cap V_2 \neq N(v) \cap V_2$ where $w \in V(G)$. The minimum weight of LRD-function is known as the locating Roman domination number denoted as $\gamma_R^L(G)$. See [15] for more result on LRD-function.

In this paper, we consider the case whereby there will be optimal security control, that is, the whole empire will be secured in case of multiple attacks at the same time. This lead to the study of perfect locating signed Roman dominating function.

A perfect locating signed Roman dominating function of a graph G, abbrevated $PLSRD$-function is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) every vertex v with $f(v) = -1$ is adjacent to exactly one vertex u with $f(u) = 2$; (ii) for any pair of distinct vertices v, w of V_{-1}, $N(v) \cap V_2 \neq N(w) \cap V_2$ and (iii) $f(v) + \sum_{u \in N(v)} f(u) \geq 1$ for any vertex $v \in G$. In Section 2, we present the lower and upper bonds of $PLSRD$-functions for trees and in Section 3, we present the upper bond of $PLSRD$-functions for the grid graph.

2. Perfect Locating Signed Roman domination of trees

In this section, we presents the lower and upper bounds of $PLSRD$-functions for trees. We begin with the following observations and existing result.

Observations.

- For any star graph S_n, $\gamma^P_{LSR}(S_n) = n - 1$.
- If f is a $PLSRD$-function, then $|D| = |V_2|$, where D is a minimum dominating set in T.

Theorem 2.1 ([6]). *For any tree T of order $n \geq 2$, $\gamma_L(T) \geq \lceil \frac{n+1}{3} \rceil$.*

Lemma 2.1. *If T is a tree with l leaves, s support vertices and $f : V(T) \to \{-1, 1, 2\}$ is a perfect locating signed dominating function, then $|V_1| \geq l - s$.*

Proof. For any support vertex u and an arbitrary vertex $x \in T$ with $f(x) = 1$, we have $|L(u) \cap V_1| \geq l(u) - 1$, then

$$|V_1| \geq \sum_{u \in S} (l(u) - 1) = \sum_{u \in S} l(u) - \sum_{u \in S} 1 = l - s. \qquad \square$$

Lemma 2.2. *For any tree T of order $n \geq 2$ with minimum dominating set D, l leaves and s support vertices, the $|D| \geq \frac{n-l+2s}{3}$.*

Proof. Consider the $PLSRD$-function on the vertices of T by assigning 2 to each support vertex u and -1 to only one leaf adjacent to support vertex u. Also, assign 1 to the remaining leaves adjacent to support vertex u. The assigned values on the support vertices and leaves in T follows from the definition of $PLSRD$-function.

Next, let T' be a tree of order n' obtained from T by deleting all support vertices and leaves, then $n' = n - l - s$. Next, divide the vertices in T' into q connected sets of cardinality 3, i.e. $n' = 3q + r$, where $q \geq 0$ and $0 \leq r \leq 2$. Assign 2 to at least one vertex in each q set. Let the vertices $\{x, v, w\} \in T'$ such that r contains one vertex, say w and $\{v, w\} \in E(T')$. If $f(v) = \{1, 2\}$, then assign 1 to vertex w. Also, assign 2 to vertex w if $f(v) = -1$ and there is no vertex x adjacent to v with label 2. If $f(x) = 2$ and $f(v) = -1$, then assign w with label 1.

Let r contain two vertices say $\{w, y\}$ and $\{x, v, w, y\}$ is a path in T'. Set $f(w) = -1$ and $f(y) = 2$ if $f(v) = 1$. Also, set $f(w) = -1$ and $f(y) = 1$ if $f(v) = 2$ and there is no adjacent vertex x to v with $f(x) = -1$, otherwise set $f(w) = f(y) = 1$. Set $f(w) = 1 = f(y)$ if $f(v) = -1$.

The assigned values produces $PLSRD$-function f. Let V_2' denote the set of vertices in T' with label 2. Also for an arbitrary vertex $u \in T'$ with $f(u) = 2$, $|V_2'| \geq \frac{n'}{3}$, Now, for an arbitrary vertex $u \in T$ with $f(u) = 2$, we have

$$|V_2| \geq \frac{n'}{3} + s = \frac{n-l-s}{3} + s = \frac{n-l+2s}{3}.$$

Applying observation 2 above, we have $|D| = |V_2| \geq \frac{n-l+2s}{3}$. □

Theorem 2.2. *For any tree T of order $n \geq 2$ with l leaves and s support vertices,* $\gamma_{LSR}^P(T) \geq \frac{n+2l-s}{3}$.

Proof. Let T be the tree of order n and $f : V(T) \to \{-1, 1, 2\}$ be a $PLSRD$-function defined on T. The set V_2 is a minimum dominating set of T. Clearly, D is the locating dominating set for T, i.e. $\gamma_L(T) \leq |D|$. By Lemma 2.2, $|D| \geq \frac{n-l+2s}{3}$ which implies that $|V_2| \geq \frac{n-l+2s}{3}$. Also, $|V_2| = |V_{-1}|$ since f is perfect and locating dominating function. Let $x \in V(T)$ such that $f(x) = 1$, then by Lemma 2.1, $|V_1| \geq l - s$. Hence,

$$\begin{aligned}\gamma_{LSR}^P(T) =& |V_{-1}| + |V_1| + |V_2| = 2|V_2| + |V_1| \quad (\text{since } |V_2| = |V_{-1}|)\\ \geq& |V_2| + |V_1| \geq \frac{n-l+2s}{3} + l - s = \frac{n+2l-s}{3}.\end{aligned}$$ □

Corollary 2.1. *For any tree T of order $n \geq 2$,* $\gamma_{LSR}^P(T) \geq \frac{n}{3}$.

Proof. The proof follows from Theorem 2.2. □

Theorem 2.3. *If T is a tree of order $n \geq 4$, then $\gamma_{LSR}^P(T) \leq \frac{3}{4}n$, where T does not contain a star of order greater than 4.*

Proof. We proof the result by induction on the order n of the tree. If $n = 3$, then $\gamma_{LSR}^P(T) = 2 \leq \frac{3}{4}n$. Now, let $n \geq 4$, if T is a star graph S_n with maximum degree 3, then $\gamma_{LSR}^P(T) = 3 \leq \frac{3}{4}n$. Observation 1 applies if maximum degree in S_n is greater than 3. Assume that T^* and T are trees of order n^* and n respectively, with $n^* \geq 3$ and $n^* < n$. Then, $\gamma_{LSR}^P(T^*) \leq \frac{3}{4}n^*$.

Let the diam$(T) \geq 3$. Suppose diam$(T) = 3$, let T be a double star $S(r, t)$, where $r \geq t \geq 1$ with maximum degree 4. Let v, w be the vertices of T that are not leaves such that v and w has r and t leaf neighbors, respectively. The function f assign 2 to each vertices v and w, -1 to only one leaf neighbor of each vertex v and w, 1 to the remaining leaves in T is a $PLSRD$-function with weight $r + t$. So, $\gamma_{LSR}^P(T) = r + t \leq \frac{3}{4}(r + t + 2) = \frac{3}{4}n$. Hence, assume that diam$(T) \geq 4$.

Let v and w be two vertices in T with maximum distance apart. This implies that v and w are leaves and $d(v, w) = $ diam(T). Let root the tree at the vertex w and let $\{v, u, x, y, r, \ldots, w\}$ be a path in T. Note that if diam$(T) = 4$, then $r = w$; otherwise $r \neq w$. The remaining part of the theorem is split into the following claims.

Claim 1. If $d_T(u) \leq 4$, then $\gamma_{LSR}^P(T) \leq \frac{3}{4}n$.

Suppose $d_T(u) \leq 4$. Let T^* be the tree obtained from T by deleting vertex u and its children. Let T^* be of order n^*, then $n^* = n - d_T(u)$. Note that $n^* \geq 3$ since $\text{diam}(T) \geq 4$. Apply induction on tree T^*, $\gamma^P_{LSR}(T^*) \leq \frac{3}{4}n^* \leq \frac{3}{4}(n-3)$. Let f^* be a $\gamma^P_{LSR}(T^*)$-function. If $f^*(x) \in \{1,2\}$, then f^* can be extended to a PLSRD-function f of T by assigning 2 to vertex u, weights -1 and 1 to vertex v and other leaf neigbhor of u respectively. Furthermore, if $f^*(x) = -1$, this implies that there exist a neighbor vertex of x (say y) with weight 2. Then f can be obtained from f^* as follows:

If $d_{T^*}(y) = 1$, re-assigning weights $2, -1$ to vertices x, y respectively. If $d_{T^*}(y) \geq 2$ and $f^*(r) = 1$, re-assign $f^*(x), f^*(y), f^*(r)$ with weights $1, -1, 2$ respectively and leave the weight of the remaining vertices under f^* unchanged. If $d_{T^*}(y) \geq 2$ and $f^*(r) = 2$, re-assign $f^*(x)$ and $f^*(y)$ with 1 and leave the weights of the remaining vertices under f^* unchanged.

Also, if $d_{T^*}(y) \geq 2$ and y has a leaf neighbor, re-assign one of the leaf neighbor of y with -1 and re-assign vertex x with weight 1, leave the weight of the remaining vertices under f^* unchanged.

From the illustration above, $f^*(x) = -1$ has been reassign weight 1. Next, extend f^* to a $PLSRD$-function f as given above whenever $f(x) \in \{1,2\}$. Therefore, we have

$$\begin{aligned}
\gamma^P_{LSR}(T) &= w(f) \leq w(f^*) + d_T(u) - 1 \\
&\leq \frac{3}{4}(n - d_T(u)) + d_T(u) - 1 \\
&= \frac{3}{4}n + \frac{d_T(u)}{4} - 1 \\
&\leq \frac{3}{4}n.
\end{aligned}$$

Next, assume that every child of vertex x in T has at most degree 3. For $i = 1,2,3$, let q_i be the number of children of x with degree i. The leaf neighbor of vertex x is q_1. Note that $q_2 + q_3 \geq 1$ since vertex u has degree 2 or 3.

Claim 2. If $q_3 \geq 1$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose that $q_3 \geq 1$, let T^* be the tree obtained from T by deleting q_3 children of vertex x and their leaf neighbors. Let $n^* \geq 3$ be the order of tree T^* and $n^* = n - 3q_3$. Applying induction on T^*, $\gamma^P_{LSR}(T^*) \leq \frac{3}{4}n^* = \frac{3}{4}(n - 3q_3)$.

If $f^*(x) \in \{1,2\}$, then we can extend f^* to a PLSRD-function f of T by assigning weight 2 to each child of x with degree 3 and weight 1 and -1 to the leaf neighbors of each child of x. The resulting function f is a PLSRD-function of T since each vertex with weight -1 is adjacent to exactly one neighbor with weight 2, vertices with weight -1 do not have a common vertex with weight 2 and the sum of weights of each vertex and its neighbors is greater than 1. The weight $w(f) = w(f^*) + 2q_3 \leq \frac{3}{4}(n - 3q_3) + 2q_3 = \frac{3}{4}n - \frac{q_3}{4} \leq \frac{3}{4}n$.

If $f^*(x) = -1$, only one leaf of x can have weight 2 and each child of x with degree 3 and their leaf neigbhors will have weight 1. Furthermore, if $f^*(x) = -1$, one child of x

with degree 3 can be assign weight 2 and the remaining child of x of degree 3 and their leaves neighbors will have weight 1 each. This will produce another $\gamma^P_{LSR}(T^*)$-function that assign larger weight than when $f^*(x) \in \{1, 2\}$. Hence, vertex x cannot have weight -1.

Claim 3. If $q_3 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_3 = 0$, then every child of x is a support vertex of degree 2 or a leaf. Let T^* be the tree obtained from T by deleting the vertex x and all its descendants. Let n^* be the order of the tree T^* where $n^* = n - q_1 - 2q_2 - 1$. Note that $n^* \geq 2$ since $\{y, r\} \subseteq V(T^*)$.

Suppose $q_1 = 0$, then the tree T has the order $n = n^* + 2q_2 + 1$, assign $1, 2, -1$ to vertex x, support vertex adjacent to x and the leaf adjacent to the child of x, respectively. The assigned weight produces a $PLSRD$-function f of T of weight

$$w(f) = w(f^*) + q_2 + 1 = \frac{3}{4}(n - 2q_2 - 1) + q_2 + 1 = \frac{3}{4}n - \frac{q_2}{2} + \frac{1}{4} \leq \frac{3}{4}n.$$

Suppose $q_2 \geq q_1 \geq 1$, then the tree T has the order $n = n^* + 2q_2 + q_1 + 1$. Assign 2 to vertex x and all the support vertices adjacent to x. Also, assign -1 to only one leaf adjacent to x and the leaf adjacent to child of x. Assign 1 to the remaining leaf adjacent to x. The assigned weight produces a $PLSRD$-function f of T of weight

$$w(f) = w(f^*) + q_2 + q_1 = \frac{3}{4}(n - 2q_2 - q_1 - 1) + q_2 + q_1 = \frac{3}{4}n - \frac{q_2}{2} + \frac{q_1}{4} - \frac{3}{4} \leq \frac{3}{4}n.$$

Suppose $q_2 = 0$ and $0 < q_1 \leq 3$, then the tree T has the order $n = n^* + q_1 + 1$. Assign 2 to vertex x, -1 to only one leaf adjacent to x and 1 to the remaining leaf adjacent to x. The assigned weight produces a $PLSRD$-function f of T of weight

$$w(f) = w(f^*) + q_1 = \frac{3}{4}(n - q_1 - 1) + q_1 = \frac{3}{4}n + \frac{q_1}{4} - \frac{3}{4} \leq \frac{3}{4}n.$$

Claim 4. If $q_2 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_2 = 0$, then every child of x has a vertex with degree 3 or a leaf. Let T^* be the tree obtained from T by deleting the vertex x and all its descendants. Let n^* be the order of the tree T^* with $n^* = n - 3q_3 - q_1 - 1$. We consider the claim for $q_3 \geq q_1 \geq 1$. Assign 2 to vertex x and each child of x with degree 3. Also assign -1 to only one leaf adjacent to vertex x and one leaf adjacent to child of x with degree 3. Assign 1 to the remaining leaves adjacent to vertex x and the child of x with degree 3. The assigned weight produces a $PLSRD$-function f of T of weight

$$\begin{aligned} w(f) &= w(f^*) + 2q_3 + q_1 = \frac{3}{4}(n - 3q_3 - q_1 - 1) + 2q_3 + q_1 = \frac{3}{4}n - \frac{q_3}{4} + \frac{q_1}{4} - \frac{3}{4} \\ &\leq \frac{3}{4}n. \end{aligned}$$

Suppose $q_1 = 0$, then claim 2 holds. If $q_3 = 0$, claim 3 holds.

Claim 5. If $q_1 = 0$, then $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$.

Suppose $q_1 = 0$, then every child of x has vertices of degree 2 and 3. Let T^* with order n^* be the tree obtained from T by deleting the vertex x and its descendants.

Then $n^* = n - 3q_3 - 2q_2 - 1$. Assign 1 to vertex x and 2 to the child of x with degree 2 and degree 3. Also assign -1 to only one leaf adjacent to child of x and 1 to the remaining leaf adjacent to child of x. The assigned weights produced $PLSRD$-function f of T of weight

$$\begin{aligned} w(f) &= w(f^*) + q_2 + 2q_3 + 1 \\ &= \frac{3}{4}(n - 3q_3 - 2q_2 - 1) + 2q_3 + q_2 + 1 = \frac{3}{4}n - \frac{q_3}{4} - \frac{q_2}{2} - \frac{1}{4} \\ &\leq \frac{3}{4}n. \end{aligned}$$

Suppose $q_2 = 0$, then claim 2 holds. If $q_3 = 0$, then claim 3 holds.

In all these cases, $w(f) \leq \frac{3}{4}n$. Hence, for $n \geq 3$, $\gamma^P_{LSR}(T) \leq \frac{3}{4}n$. This complete the proof. □

3. Perfect Locating Signed Roman Domination of Cartesian Product Graph

In this section, we present an upper bond for the perfect locating signed Roman domination number of the Grid graph $G = P_n \Box P_m$. Let $i, 1 \leq i \leq n$ and $j, 1 \leq j \leq m$ denotes the rows and columns in the graph $P_n \Box P_m$ We denote the vertex in the row i and column j by u_{ij}.

Theorem 3.1. *Let $n > 5$ and $m \geq 2$. If $G = P_n \Box P_m$, then $\gamma^P_{LSR}(G) \leq \frac{3}{4}|G|$.*

Proof. Define the function $f : V(G) \to \{-1, 1, 2\}$ as shown in figure 1 as follows: For vertex $u_{ij} \in V(G)$, we have

$$f(u_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 6 \text{ and } j \text{ even}, \\ 1, & \text{if } i \equiv 3 \text{ or } 4 \pmod 6 \text{ and } j \text{ odd}, \\ -1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 6 \text{ and } j \text{ odd}, \\ -1, & \text{if } i \equiv 3 \text{ or } 4 \pmod 6 \text{ and } j \text{ even}, \\ 2, & \text{if } i \equiv 2 \pmod 3 \text{ for all } j. \end{cases}$$

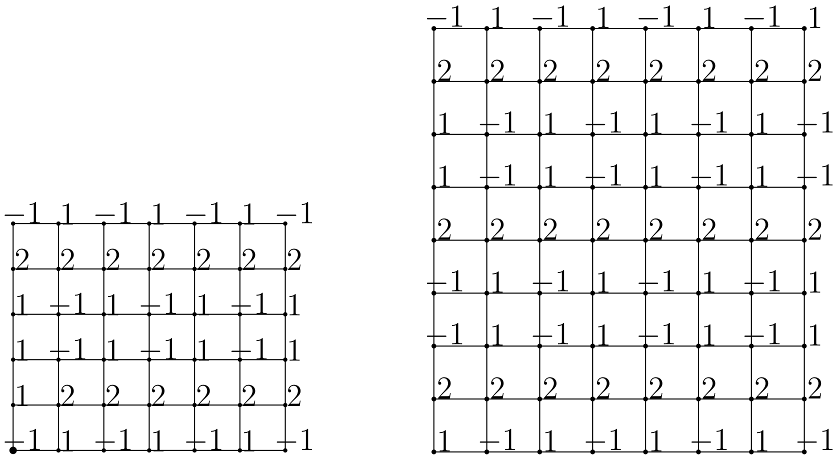
The function f has a pattern that reoccur at every six rows and every two columns. The above function f define on the vertices of G gives the $PLSRD$-function on G, since each vertex with label -1 is adjacent to only one vertex with label 2, any pair of vertices u_{ij}, v_{ij} with label -1 does not have a common neighbor vertex with label 2 and $f(u_{ij}) + \sum_{v_{ij} \in N(u_{ij})} f(v_{ij}) \geq 1$. The result will be prove in the following cases.

Case 1: when $n \equiv 0 \pmod 6$ and m a positive integer.

From the above function f, the total sum of labels on each column j is $\frac{4n}{6}$. Therefore, we have

$$w(f) = \frac{4nm}{6} \leq \frac{2}{3}nm + \frac{1}{12}nm = \frac{3}{4}nm = \frac{3}{4}|G|.$$

Case 2: when $n \equiv 1 \pmod 6$ and m a positive integer.



(a) $n \equiv 0 \pmod 6$ (b) $n \equiv 3 \pmod 6$

FIGURE 1. The function f, m odd in (a) and m even in (b)

Define a function $f^* : V(G) \to \{-1, 1, 2\}$ as follows:

$$f^*(u_{ij}) = \begin{cases} 1, & \text{if } i = n \text{ and } j \text{ odd}, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The above function f^* gives $PLSRD$-function which follows from the definition of $PLSRD$-function. From the function f^*, the total sum of the labels on each column j is $\frac{4(n-1)}{6} + 1$.

Hence, we have

$$w(f^*) = m\left(\frac{4(n-1)}{6} + 1\right) \leq \frac{2}{3}nm + \frac{1}{12}nm = \frac{3}{4}nm = \frac{3}{4}|G|.$$

Case 3: when $n \equiv 2 \pmod 6$ and m a positive integer.

Define a function $f^* : V(G) \to \{-1, 1, 2\}$ as follows:

$$f^*(u_{ij}) = \begin{cases} 1, & \text{if } i = n \text{ and j even}, \\ f(u_{ij}), & \text{otherwise.} \end{cases}$$

The above function f^* gives $PLSRD$-function. From the function f^*, the total sum of the labels on each odd column j is $\frac{4(n-2)}{6} + 1$ and $\frac{4(n-2)}{6} + 2$ on even column j.

If m is odd, we have

$$\begin{aligned} w(f^*) &= \frac{m+1}{2}\left(\frac{4(n-2)}{6}+1\right)+\frac{m-1}{2}\left(\frac{4(n-2)}{6}+2\right)=\frac{2}{3}nm+\frac{1}{6}m-\frac{1}{2} \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

If m is even, we have

$$\begin{aligned} w(f^*) &= \frac{m}{2}\left(\frac{4(n-2)}{6}+1\right)+\frac{m}{2}\left(\frac{4(n-2)}{6}+2\right)=\frac{2}{3}nm+\frac{1}{6}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Case 4: when $n \equiv 3 \pmod 6$ and m a positive integer.

From the above function f, the total sum of the labels on each column j is $\frac{4(n-3)}{6}+2$. Therefore, we have

$$w(f)=m\left(\frac{4(n-3)}{6}+2\right)\leq \frac{2}{3}nm+\frac{1}{12}nm=\frac{3}{4}nm=\frac{3}{4}|G|.$$

Case 5: when $n \equiv 4 \pmod 6$ and m a positive integer.

Define a function $f^*: V(G) \to \{-1,1,2\}$ as follows:

$$f^*(u_{ij})=\begin{cases} 1, & \text{if } i=n \text{ and } j \text{ even}, \\ f(u_{ij}), & \text{otherwise}. \end{cases}$$

The above function f^* gives $PLSRD$-function. From the function f^*, the sum of the labels on each column j is $\frac{4(n-4)}{6}+3$.

Hence, we have

$$\begin{aligned} w(f^*) &= m\left(\frac{4(n-4)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{3}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Case 6: when $n \equiv 5 \pmod 6$ and m a positive integer.

Define a function $f^*: V(G) \to \{-1,1,2\}$ as follows:

$$f^*(u_{ij})=\begin{cases} 1, & \text{if } i=n \text{ and } j \text{ odd}, \\ f(u_{ij}), & \text{otherwise}. \end{cases}$$

The above function f^* gives $PLSRD$-function which follows from the definition. From the function f^*, if j is odd, the sum of the labels on each column j is

$$\frac{4(n-5)}{6}+4 \quad \text{and} \quad \frac{4(n-5)}{6}+3.$$

for each even column j.

Therefore, if m is odd, we have

$$\begin{aligned} w(f^*) &= \frac{m+1}{2}\left(\frac{4(n-5)}{6}+4\right)+\frac{m-1}{2}\left(\frac{4(n-5)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{6}m+\frac{1}{2} \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Also, if m is even, we have

$$\begin{aligned} w(f^*) &= \frac{m}{2}\left(\frac{4(n-5)}{6}+4\right)+\frac{m}{2}\left(\frac{4(n-5)}{6}+3\right)=\frac{2}{3}nm+\frac{1}{6}m \\ &\leq \frac{2}{3}nm+\frac{1}{12}nm \\ &= \frac{3}{4}nm=\frac{3}{4}|G|. \end{aligned}$$

Hence, the result follows. □

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