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A STUDY OF FUNCTIONS ON THE TORUS AND MULTI-PERIODIC FUNCTIONS

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Abstract. In this paper, we are concerned with functions defined on the cube $Q^m = [-\pi, \pi]^m$ and functions defined on the torus $\mathbb{T}^m$. Especially, the harmonic analysis of Sobolev-type spaces is carefully studied. We analyze in particular periodic distributions and distributions on the torus. We introduce a space similar to H_0^1, for which we prove a Poincaré-Wirtinger inequality. We prove that the usual Rellich-Kondrachov result does not hold for these last space because of the lack of compactness. A result of absolute continuity and density of regular functions is then established and a theorem of traces is obtained.

1. Introduction

Functions which repeat themselves after a fixed length of their arguments, so-called period, are called the periodic functions. Common examples of the periodic functions are the trigonometric sine and cosine functions with period each. Geometrically, a periodic function is the one whose graph displays a translational symmetry. In particular, a function is periodic with period if the graph remains invariant under translation in the direction by a distance.

Periodic functions appear in many practical problems. In most of the cases, they are more complicated than the ordinary sine and cosine functions. Indeed, periodic functions are a vital part of all scientific, engineering, technological and mathematical processes. In all branches of mathematics, they have well-defined analogues. These functions are used in modeling many dynamical, physical, and biological processes. They have wide-range applications in different fields of science, mathematics and

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engineering to study and characterize phenomena like conduction of heat, mechanical vibrations, electric circuits and electromagnetic waves etc.

Based on the periodic functions, there exist their periodic extensions. In fact, periodic extensions of the periodic functions are another class of the functions which are used in modeling more complex physical, biological and more advanced systems. Most of the studies on the periodic functions and their extensions have been limited to a single period. Their transformation to the functions called the periodic extension are the half-wave rectification, full-wave rectifications in electrical engineering. The graph of such a function is obtained by periodic repetition of its graph in any interval of the length of its period. We call these periodic extensions as the multiperiodic functions. Multiperiodic functions are commonly used to model complex dynamic systems in applied and pure sciences such as population dynamics and weather forecasting.

Historically, the Fourier transform was first introduced by Joseph Fourier in his study of the heat equation. In this context, Fourier showed how a periodic function could be decomposed into a sum of sines and cosines which represent the frequencies of the function. From a modern point of view, the Fourier transform is a transformation which accepts a function and returns a new function, defined via the frequency data of the original function. As a bridge between the physical domain and the frequency domain, the Fourier transform seems to be the main tool of use in harmonic analysis.

Moreover, Sobolev spaces are the main tools in the modern theory of partial differential equations. They give a very natural functional analytical framework for the study of existence, regularity and qualitative properties of the boundary value problems. Indeed, Sobolev spaces are ubiquitous in harmonic analysis and Partial differential equations, where they appear naturally in problems about regularity of solutions or well-posedness. Tightly connected to these problems are certain embedding theorems that relate the norms of Lebesgue and Sobolev spaces for appropriate indices. The appeal of Sobolev spaces is due to the simplicity of their definition which captures both the regularity and size of a distribution. On the other hand, an efficient tool when dealing with Sobolev spaces and partial differential equations is the Poincaré-Wirtinger inequality that provides norm equivalences under appropriate assumptions. These inequalities usually provide Sobolev embeddings and compactness results (see Adams [1]).

Besides, the Rellich-Kondrachov compactness theorem, which gives compact embeddings of Sobolev spaces [2, Theorem 6.2], is fundamental for the study of elliptic boundary value problems.

On the other hand, due to its geometric and topological structure, the torus has shown an interesting means to study periodicity/quasi-periodicity of solutions in many applications. The interest of considering spaces of functions on the torus is because these can be identified with periodic functions, so it is natural to look for solutions of partial differential equations with periodic border conditions in these spaces. Now, boundary conditions are understood in the sense of the trace, which is related to the trace method and the trace spaces in the theory of interpolation.

In this paper, we aim to study the functions defined on the cube $[-\pi, \pi]^m$ and those defined on the torus $\mathbb{T}^m$. That means the behaviours of multiperiodic functions and functions on the torus are analyzed. We give a complete harmonic analysis of Sobolev-type spaces. The distributions on the torus and the periodic distributions are studied. Namely, we introduce a space similar to the classical space H_0^1 and we prove that on this introduced space, a Poincaré-Wirtinger inequality holds true while the known Rellich-Kondrachov is no longer valid due to the lack of compactness. We finally get a result on absolute continuity and density of regular functions and a theorem of traces.

2. Notations and Preliminary results

Throughout this manuscript, we denote by $\mathbb{N}^*$ or by $\mathbb{Z}_+$ the set of all positive integers and by $\mathbb{Z}$ the set of all integers. The set of real numbers is denoted by $\mathbb{R}$ and that of all complex numbers is denoted by $\mathbb{C}$.

We shall fix an integer $m \geq 2$ and a vector $\omega \in \mathbb{R}^m$ whose components are linearly independent on $\mathbb{Z}$ and are strictly positive. We denote also by Q^m the cube of dimension m, that is, $Q^m := [-\pi, \pi]^m$. The notation $\mathbb{T}^m$ shall be deserved for the torus of dimension m, that is $\mathbb{T}^m := \mathbb{R}^m/(2\pi\mathbb{Z})^m$.

For a vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, we denote by x_{-j} the vector of $\mathbb{R}^{m-1}$ defined by $x_{-j} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$. The couples (x_{-j}, x_j) and (x_j, x_{-j}) denote both x.

For $x, y \in \mathbb{K}^m$, we denote by $x \cdot y$ and by $|x|$ respectively the usual inner product of x and y and its associated euclidean norm:

$$x \cdot y = \sum_{j=1}^{m} x_j y_j, \quad |x| = \sqrt{x \cdot x}.$$

The notation $e_\nu : \mathbb{R}^m \to \mathbb{C}$, for $\nu \in \mathbb{R}$, is reserved for the function defined for all $x \in \mathbb{R}^m$ by $e_\nu(x) := \exp(i\nu \cdot x)$.

We shall denote by $\mathbb{Z}\langle u_1, \dots, u_p\rangle$ (u_i belongs to a linear space F over $\mathbb{R}$) [9, p. 81], $\mathbb{Z}$-modulus generated by $u_1, \dots u_p$, that is,

$$\mathbb{Z}\langle u_1, \dots, u_p\rangle := \Big\{ \sum_{j=1}^{p} k_j u_j : (k_j)_j \in \mathbb{Z}^p \Big\}.$$

We denote by $\tau_p(u)$, for a given function $u : \mathbb{R}^m \to \mathbb{E}$ and a given $p \in \mathbb{R}^m$, the translated function of u defined on $\mathbb{R}^m$ by $\tau_p(u)(x) := u(x + p)$.

If the space $\mathbb{E}$ is a topological space, we use the standard notation, for a given subset $A \subset \mathbb{E}$, $\mathrm{int}(A)$ the interior of the set A.

Let us consider a linear normed space X over $\mathbb{R}$ of finite dimension m.

2.1. **First notions.** We give here the following definition that can be found for example in [4, p. 55] or in [15, p. 64].

Definition 2.1. A function $F : X \to \mathbb{E}$ is said to be periodic if there exists a non zero $p \in X$ such that for all $x \in X$, we have $F(x+p) = F(x)$. Such a vector p, as well as 0, is called a period of F.

The set of all periods of a given function is an abelian subgroup of $(X, +)$, which becomes also a closed set if the function is continuous. We denote by $Per(F)$ the set of periods of a function F.

Proposition 2.1. *Let $F : X \to \mathbb{E}$ be a periodic function, Y be a normed linear space of dimension m and L be a linear isomorphism from Y into X. Let us define the function $G := F \circ L : Y \to \mathbb{E}$. Then, G is a periodic function, and $Per(G) = L^{-1}(Per(F))$.*

Proof. Take p one period of F and $y \in Y$. We have $G(z+L^{-1}(p)) = F \circ L(z+L^{-1}(p)) = F(L(z)+p) = F(L(z)) = G(z)$, which shows that G is a periodic function and that $L^{-1}(Per(F)) \subset Per(G)$. The converse inclusion can be obtained by interchanging F and G. □

2.2. **Lecture in a basis.** We consider the canonical basis of $X = \mathbb{R}^m$, denoted by $(e_j)_{1\leq j\leq m}$. We denote $(e_j^*)_{1\leq j\leq m}$ its dual basis and χ the isomoprhism from X into $(\mathbb{R}^m)^*$ defined by:

$$\chi(x) = (e_j^*(x))_{1\leq j\leq m}.$$

We denote also:

$$\chi^{-1}(x_1, \dots, x_m) = \sum_{i=1}^{m} x_i e_i.$$

We shall now analyze the link between the function F and the function $f := F \circ \chi^{-1}$ defined on $\mathbb{R}^m$.

Definition 2.2. The function f is said to be 2π-periodic in each variable if for all $j = 1, \dots, m$, $x_{-j} \in \mathbb{R}^{m-1}$, and for all $x_j \in \mathbb{R}$, we have $f(x_{-j}, x_j + 2\pi) = f(x_{-j}, x_j)$, which is also equivalent to say that for all $k \in \mathbb{Z}^m$, and for any $x \in \mathbb{R}^m$, we have $f(x + 2\pi k) = f(x)$.

Therefore, we have the following.

Proposition 2.2. *The following assertions are equivalent.*

(a) *F is periodic and $Per(F) \supset 2\pi\mathbb{Z}\langle e_1, \dots, e_m\rangle$.*
(b) *f is 2π-periodic in each variable.*

Proof. We refer to Proposition 2.3 which is more general. □

2.3. **Change of basis effects.** Let actually take another basis of $\mathbb{R}^m$, $(b_j)_{1\leq j\leq m}$. We denote by $(b_j^*)_{1\leq j\leq m}$ its dual basis and by χ_1 the isomorphism from X into $(\mathbb{R}^m)^*$ defined for all $x \in \mathbb{R}^m$ as:

$$\chi_1(x) = (b_j^*(x))_{1\leq j\leq m}.$$

Keeping the same previous notations, we set $f_1 := F \circ \chi_1^{-1}$.

Proposition 2.3. *The following assertions are equivalent.*

(a) *F is periodic and $Per(F) \supset 2\pi b_j$.*
(b) *For all $x_{-j} \in \mathbb{R}^{m-1}$, $f_1(x_{-j}, \cdot)$ is 2π-periodic from $\mathbb{R}$ into $\mathbb{E}$.*

Proof. (a) $\Rightarrow$ (b) Let us fix arbitrarily $x_{-j} \in \mathbb{R}^{m-1}$.
Then, for all $x_j \in \mathbb{R}$, we have:

$$\begin{aligned} f_1\left(x_{-j}, x_j + 2\pi\right) &= F \circ \chi_1^{-1}\left(x_{-j}, x_j + 2\pi\right) = F\left(2\pi b_j + \sum_{i=1}^{m} x_i b_i\right) \\ &= F\left(\sum_{i=1}^{m} x_i b_i\right) = F \circ \chi_1^{-1}\left(x_{-j}, x_j\right) = f_1\left(x_{-j}, x_j\right). \end{aligned}$$

Therefore, (a) implies (b).
(b) $\Rightarrow$ (a) Now, we fix an arbitrary $x \in \mathbb{R}^m$. We have

$$\begin{aligned} F(x + 2\pi b_j) &= F \circ \chi_1^{-1}\left(b^*_{-j}(x), b^*_j(x) + 2\pi\right) \\ &= f_1\left(b^*_{-j}(x), b^*_j(x) + 2\pi\right) = f_1\left(b^*_{-j}(x), b^*_j(x)\right) \\ &= F \circ \chi_1^{-1}\left(b^*_{-j}(x), b^*_j(x)\right) = F\left(x\right). \end{aligned}$$

This completes the proof. ☐

Notation 2.1. We introduce $Q^m \subset X$ as:

$$Q^m := \left\{\sum_{i=1}^{m} x_i e_i : \text{for all } i = 1, \dots, m,\ x_i \in [-\pi, \pi]\right\}.$$

We have $\chi(Q^m) = [-\pi, \pi]^m$. We denote by $K^m := \chi_1(Q^m)$.

2.4. **Change of basis and integrals.** We keep the previous notations, by assuming in addition that both basis are orthonormal.

We set, for $y_{-1} \in \mathbb{R}^{m-1}$,

$$K(y_{-1}) := \{y_1 \in \mathbb{R} : (y_1, y_{-1}) \in K^m\}$$

and

$$D := \left\{y_{-1} \in \mathbb{R}^{m-1} : K(y_{-1}) \neq \emptyset\right\}.$$

The theorem of change of variables consecutively with change of basis under the integrals allow us to get the following.

Proposition 2.4. *For any continuous function $f : \mathbb{R}^m \to \mathbb{E}$, we have:*

$$\int_{[-\pi,\pi]^m} f(x)dx = \int_D \left(\int_{K(y_{-1})} f_1(y)dy_1\right) dy_2 \cdots dy_m.$$

Proof. The functions f, f_1 and F are continuous. By Fubini's theorem, we have:

$$\int_{\chi_1(Q^m)} f_1(y)dy = \int_D \left(\int_{K(y_{-1})} f_1(y)dy_1\right) dy_2 \cdots dy_m.$$

Besides, since the isomorphisms χ and χ_1 are orthogonal, their determinant is in absolute value equal to 1, which gives the following identities by applying the change of variables formula:

$$\int_{\chi_1(Q^m)} f_1(y)dy = \int_{Q^m} F(x)dx = \int_{\chi(Q^m)} f(x)dx.$$

We deduce the result then by comparison of these inequalities and by using $\chi(Q^m) = [-\pi,\pi]^m$. □

Lemma 2.1. *For all $y_{-1} \in D$, $K(y_{-1})$ is a closed interval of $\mathbb{R}$ with diameter equal to $2\pi\sqrt{m}$.*

Proof. Let us fix $y_{-1} \in D$. $K(y_{-1})$ is closed convex set of $\mathbb{R}$, because it is the reciprocal image of the convex set K^m via the affine application $\phi(y_1) = (y_1, y_{-1})$. We deduce that $K(y_{-1})$ is a closed interval. Since $K(y_{-1}) \subset \chi_1(Q^m)$, we have:

$$\text{diam}[K(y_{-1})] \le \text{diam}[\chi_1(Q^m)].$$

Since χ_1 is orthogonal, we know, due to the formula of change of variable for the integrals, that:

$$\text{diam}[\chi_1(Q^m)] = \text{diam}[Q^m].$$

Now, if $x, y \in Q^m$, we have:

$$|x-y|^2 = \sum_{i=1}^{m}(x_i - y_i)^2 \le m(2\pi)^2,$$

the upper bound is reached (for example) for $x = (-\pi,\dots,-\pi)$ and $y = (\pi,\dots,\pi)$. We conclude that:

$$\text{diam}[Q^m] = 2\pi\sqrt{m},$$

which completes the proof of the lemma. □

2.5. **Change of basis and derivation.** $(e_i)_i$ denotes the canonical basis of X, and we denote by $(b_j)_j$ another orthonormal basis such that $b_1 = \omega/|\omega|$.

Let $U : \mathbb{R}^m \to \mathbb{E}$ be a function which is 2π-periodic in each variable, $F := U \circ \chi$ and $V := F \circ \chi_1^{-1}$.

Definition 2.3 ([8, p. 251])**.** Let $x \in \mathbb{R}^m$ and ϕ be a function defined on an open set U of $\mathbb{R}^m$ containing x and with values in $\mathbb{E}$. The function ϕ admits a directional derivative (called also *Gâteaux-variation*) in the direction v if

$$\frac{\phi(x+\theta v) - \phi(x)}{\theta}$$

has a limit when θ tends to 0.

This limit, denoted by $\vec{D}\phi(x,v)$, is called *directional derivative* (or also *Gâteaux-variation*) of ϕ in the direction v.

Definition 2.4. We define the Percival derivation operator (cf. [11,12]) for U differentiable in the direction of ω by:

$$d_\omega U(x) := \vec{D}U(x,\omega).$$

Remark 2.1. When U is in addition Fréchet-differentiable in x, we have

$$d_\omega U(x) = \sum_{i=1}^{m} \omega_i \frac{\partial U}{\partial x_i}(x).$$

The link between the notions of derivation is analyzed in the following proposition.

Proposition 2.5. *Let U be differentiable in the direction of ω. Then, V is differentiable with the respect to the first variable, and we have the relation*

$$d_\omega U(x_1,\dots,x_m) = |\omega| \frac{\partial V}{\partial y_1}(y_1,\dots,y_m).$$

Proof. Since $\omega = |\omega| b_1$, we have:

$$\vec{D}F(x,\omega) = |\omega|\vec{D}F(x,b_1).$$

Besides, we have $\vec{D}F(x,\omega) = d_\omega U(x)$ and $\vec{D}F(x,b_1) = \frac{\partial V}{\partial y_1}(y_1,\dots,y_m)$. This achieves the proof of the proposition. □

3. Functions on the Torus and Functions on Q^m

3.1. **Functions defined on the torus.** We give here a (non-geometric) definition of the functions defined on the torus $\mathbb{T}^m$ and we study how to extend a function defined on Q^m into a function defined on the torus. Here, the torus is not seen as a geometric object, but as a notation to specify the periodicity with respect to each variable of the functions involved.

For "regular" functions, we therefore set, when $k \in \mathbb{N} \cup \{+\infty\}$:

- $C^k(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k(\mathbb{R}^m,\mathbb{E})$ which are 2π-periodic in each variable;
- $C^k_\omega(\mathbb{T}^m,\mathbb{E})$ is the space of continuous functions, k times continuously differentiable in the direction of ω and 2π-periodic in each variable;
- $C^k_c(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k(\mathbb{T}^m,\mathbb{E})$ which vanish on an open neighborhood of ∂Q^m;
- $C^k_{c,\omega}(\mathbb{T}^m,\mathbb{E})$ is the space of functions of $C^k_\omega(\mathbb{T}^m,\mathbb{E})$ which vanish on an open neighborhood of ∂Q^m.

For the functions of Lebesgue spaces, we shall first define the notion of periodicity.

Definition 3.1. For a function u (strongly) measurable from $\mathbb{R}^m$ into $\mathbb{E}$, we call that u admits the vector p as a period if : $\tau_p u = u$. We call u is periodic if it has a non zero period and we denote

$$L^0(\mathbb{T}^m,\mathbb{E}) := \left\{ u \in L^0(\mathbb{R}^m,\mathbb{E}) : 2\pi\mathbb{Z}\langle (e_i)_i\rangle \in Per(u) \right\}.$$

The space $L^0(\mathbb{R}^m,\mathbb{E})$ denotes here the space of measurable functions.

For the other Lebesgue spaces L^α, we set, when $\alpha \in [1, +\infty]$:

$$L^\alpha(\mathbb{T}^m, \mathbb{E}) := \left\{u \in L^\alpha_{loc}(\mathbb{R}^m, \mathbb{E}) : 2\pi\mathbb{Z}\langle(e_i)_i)\rangle \in Per(u)\right\}.$$

We endow this space with the norm:

$$\|u\|_{L^\alpha} := \left(\int_{Q^m} |u(x)|^\alpha_{\mathbb{E}} dx\right)^{1/\alpha},$$

if α is finite, and when $\alpha = \infty$ with the norm:

$$\|u\|_\infty = \operatorname{ess\,sup} |u|_{\mathbb{E}} := \inf\left\{\alpha : |u|_{\mathbb{E}} \le \alpha \text{ a.e.}\right\}.$$

These spaces are Banach spaces. We remind that we note sup instead of ess sup.

Notation 3.1. In the sequel, we use the notation $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$.

Remark 3.1. For a given function $u \in L^1(\mathbb{T}^m, \mathbb{E})$, the integral $\int_{\mathbb{T}^m} u(x)dx$ denotes $\int_{Q^m} u(x)dx$. The relation between this integral and the integral with respect to the Haar measure on the torus is:

$$\int_{\mathbb{T}^m} u(x) d\mu_m(x) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} u(x)dx.$$

Moreover, we equip the space $L^2(\mathbb{T}^m, \mathbb{H})$ with the following inner product

$$\langle u; v\rangle := \int_{\mathbb{T}^m} u(x) \cdot_{\mathbb{H}} v(x)dx.$$

$L^2(\mathbb{T}^m, \mathbb{H})$ is then a Hilbert space.

We immediately have the following result.

Proposition 3.1. *For all $p \in \mathbb{R}^m$, the following assertions hold.*

1. *For all $k \in \mathbb{N} \cup \{+\infty\}$, $\tau_p(C^k(\mathbb{T}^m, \mathbb{E})) \subset C^k(\mathbb{T}^m, \mathbb{E})$.*
2. *For all $\alpha \in [1, +\infty]$, $\tau_p(L^\alpha(\mathbb{T}^m, \mathbb{E})) \subset L^\alpha(\mathbb{T}^m, \mathbb{E})$.*

3.2. **Extension theorems.** The extension of functions defined on Q^m into functions defined on the torus can be done using the following lemma.

Lemma 3.1. *The following statements hold true.*

1. *For all $x \in \mathbb{R}^m$, there exists $k \in \mathbb{Z}^m$, such that $x - 2\pi k \in Q^m$.*
2. *The boundary of Q^m is given by*

$$\partial Q^m = \left\{p \in Q^m : \textit{there exists } j \in \{1, \cdots, m\},\ p_j \in \{-\pi, \pi\}\right\}$$

and is Lebesgue-negligible in Q^m and in $\mathbb{R}^m$.
3. *If f is 2π-periodic in each variable, f satisfies the following condition at the boundary:*

(CF) *For all $i \in \{1, \dots, m\}$ and for all $x_{-i} \in \mathbb{R}^{m-1}$, $f(x_{-i}, -\pi) = f(x_{-i}, \pi)$, which can also be written as: for all $\xi, \zeta \in Q^m$, [for all $i \in \{1, \dots, m\}, \xi_i = \zeta_i$ or ($\xi_i \in \{-\pi, \pi\}$ and $\zeta_i \in \{-\pi, \pi\}$)], this implies that $f(\xi) = f(\zeta)$.*

Proof. **3.** Results from the definition of periodicity.

For the assertion **1**, given $x \in \mathbb{R}^m$, we set for each $j = 1, \dots, m$,

$$k_j = E\left(\frac{x_j + \pi}{2\pi}\right),$$

where E denotes the integer function. We verify that $k \in \mathbb{Z}^m$ and for all $j = 1, \dots, m$, we have $-\pi \leq x_j - 2\pi k_j < \pi$, and so $x - 2\pi k \in Q^m$.

Let us now prove **2.** We shall show that $\operatorname{Int} Q^m = (-\pi, \pi)^m$.

Firstly, we have $\operatorname{Int} Q^m \supset (-\pi, \pi)^m$ because $(-\pi, \pi)^m$ is an open set of $\mathbb{R}^m$ contained in Q^m. If the inclusion is strict, there is a $p \in \operatorname{Int} Q^m$ and a j_0 such that $p_{j_0} \in \{-\pi, \pi\}$. Without loss of generality, we assume that $p_{j_0} = \pi$.

The sequence $\left((p_j + \frac{1}{n})_{1 \leq j \leq m}\right)_n$ converges to p, but none of its elements are in Q^m. Therefore, $p \notin \operatorname{Int} Q^m$.

This ends the proof of this lemma. □

Let us start with the study of the extension in the case of Lebesgue spaces.

Proposition 3.2. *The map* $\mathfrak{J} : L^\alpha(\mathbb{T}^m, \mathbb{E}) \to L^\alpha(Q^m, \mathbb{E})$ *defined by:* $\mathfrak{J}(u) := u_{|Q^m}$ *is an isometric isomorphism of Banach spaces* (*and even Hilbert if* $\alpha = 2$ *and* $\mathbb{E} = \mathbb{H}$).

Proof. $\mathfrak{J}$ is obviously an isometric linear map.

We shall now prove that it is bijective.

Surjectivity. Let us take a function $f \in L^\alpha(Q^m, \mathbb{E})$. Even if it means modifying f on the boundary of Q^m (which is Lebesgue-negligible), we can assume that f is zero on ∂Q^m.

Let $x \in \mathbb{R}^m$. If there exists k, l in $\mathbb{Z}^m$, distinct, for which we simultaneously have $x - 2\pi k \in Q^m$ and $x - 2\pi l \in Q^m$ ans so for all i such that $k_i \neq l_i$, we have ($k_i = 0$ and $l_i = 2\pi$) or ($k_i = 2\pi$ and $l_i = 0$).

Therefore, by **(CF)**, $f(x - 2\pi k) = f(x - 2\pi l)$, and it is then possible to define $\tilde{f}(x) = f(x - 2\pi k)$ where k is arbitrarily chosen in $\mathbb{Z}^m$ so that $x - 2\pi k \in Q^m$.

Let us show now that the function $\tilde{f}$ previously defined is periodic. If $p \in \mathbb{Z}^m$ and $x \in \mathbb{R}^m$ are given, let us take $k \in \mathbb{Z}^m$ such that: $x - 2\pi k \in Q^m$. We have $(x + 2\pi p) - 2\pi(k + p) \in Q^m$. Hence, $\tilde{f}(x + 2\pi p) = f((x + 2\pi p) - 2\pi(k + p)) = f(x - 2\pi k) = \tilde{f}(x)$.

Finally, it remains to verify that it belongs to L^α. The restriction of $\tilde{f}$ to each $Q^m + 2\pi k$, where $k \in \mathbb{Z}^m$ has the form of $x \mapsto f(x + 2\pi k)$, and hence $\tilde{f} \in L^\alpha_{loc}(\mathbb{R}^m, \mathbb{E})$, which shows that $\tilde{f} \in L^\alpha(\mathbb{T}^m, \mathbb{E})$ and verifies $\mathfrak{J}(\tilde{f}) = f$.

Injectivity. Let $f \in L^\alpha(Q^m, \mathbb{E})$ and f_1 and f_2 be two functions such that $\mathfrak{J}(f_1) = \mathfrak{J}(f_2) = f$. We may suppose that the two functions f_i are equal on Q^m.

Let $x \in \mathbb{R}^m$. There exists $k \in \mathbb{Z}^m$ such that $x - 2\pi k \in Q^m$. We have then $f_1(x) = f_1(x - 2\pi k) = f_2(x - 2\pi k) = f_2(x)$.

The proposition is finally proved. □

We analyze now the case of continuous functions. We have precisely to study what is happening on the border.

Lemma 3.2. *Let k and l be two different elements of $\mathbb{Z}^m$ and let $p \in (Q^m + 2\pi k) \cap (Q^m + 2\pi l)$. Then,*

$$p \in \partial(Q^m + 2\pi k) \cap \partial(Q^m + 2\pi l) = (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l).$$

Proof. There exist $\xi, \zeta \in Q^m$ such that $p = \xi + 2\pi k = \zeta + 2\pi l$. Since $k \neq l$, there exists j such that $\xi_j \neq \zeta_j$. Besides, as $\xi_i + 2\pi k_i = \zeta_i + 2\pi l_i$ and $|\xi_i - \zeta_i| \leq 2\pi$, we have $|k_i - l_i| \leq 1$.

First case. $k_i = l_i$. Then, $\xi_i = \zeta_i$.

Second case. $k_i = l_i \pm 1$. Then, $\xi_i = \zeta_i \pm 2\pi$, that is, since $\xi, \zeta \in Q^m$, one of the two is equal to $-\pi$ and the other one is equal to π.

Therefore, we have shown that $i \in \{1, \dots, m\}$, $\xi_i = \zeta_i$ or $\xi_i, \zeta_i \in \{-\pi; \pi\}$ and there exists j such that $\xi_j \neq \zeta_j$. Finally, $p \in (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l)$ and the lemma is proven. □

Proposition 3.3. *Let $f \in C^0(Q^m, \mathbb{E})$. The following statements are equivalent.*

1. *There exists a unique $\tilde{f} \in C^0(\mathbb{T}^m, \mathbb{E})$ such that $\tilde{f}_{|Q^m} = f$.*
2. *f satisfies* **(CF)**.

Proof. The implication [**1. implies 2.**] is obvious.

We have to show the implication [**2. implies 1.**].

Existence. For $x \in Q^m + 2\pi k$, we set $f_k(x) = f(x - 2\pi k)$. When $(Q^m + 2\pi k) \cap (Q^m + 2\pi l) \neq \emptyset$, with $k \neq l$, we have $(Q^m + 2\pi k) \cap (Q^m + 2\pi l) = (\partial Q^m + 2\pi k) \cap (\partial Q^m + 2\pi l)$ in virtue of the lemma.

Hence, thanks to **(CF)**, we have $f_k(x) = f_l(x)$. Let us introduce $A_k := Q^m + 2\pi k$. The family $(A_k)_k$ forms a recovery of $\mathbb{R}^m$ such that if $A_k \cap A_l \neq \emptyset$, $f_k(x) = f_l(x)$. We can define the function $\tilde{f}$ as $\tilde{f}(x) = f_k(x)$ if $x \in A_k$. Since each f_k is continuous and as the recovery $(A_k)_k$ is closed and locally finite, we know that $\tilde{f}$ is continuous (cf. [13, p. 20]). Its periodicity is obvious. Therefore, the existence is shown.

Uniqueness. Two solutions take the same values on Q^m, and so that, they are equal on $\mathbb{R}^m$, by periodicity. □

Proposition 3.4. *For all function $f \in C_c^k(Q^m, \mathbb{E})$, there exists a unique $\tilde{f} \in C^k(\mathbb{T}^m, \mathbb{E})$ such that $\tilde{f}_{|Q^m} = f$.*

Proof. Uniqueness is acquired by Proposition 3.3. Moreover, Proposition 3.3 gives us, for all $j \leq k$, a unique $\tilde{f}_j \in C^0\left(\mathbb{T}^m, \mathcal{L}^j_{sym}\left((\mathbb{R}^m)^j; \mathbb{E}\right)\right)$ such that $\tilde{f}_{j_{|Q^m}} = f^{(j)}$. Let $\tilde{f} = \tilde{f}_0$. We aim to prove that this function belongs to $C^k(\mathbb{T}^m, \mathbb{E})$.

Let $x \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$ such that $x \in Q^m + 2\pi l$. We can distinguish two cases.

First case. $x \in \operatorname{Int}(Q^m + 2\pi l)$. In this case, near to x, $\tilde{f} = f \circ \tau_{-2\pi l}$ is of class C^k as composition of a map from C^k and an application from C^∞.

Second case. $x \in \partial(Q^m + 2\pi l)$. Let $\Lambda := \{\lambda \in \mathbb{Z}^m : x \in \partial Q^m + 2\pi\lambda\}$. Λ is a non empty finite set and since $\lambda \in \Lambda$, we have $\operatorname{supp}\left(\tilde{f}_{|Q^m + 2\pi\lambda}\right) = \operatorname{supp}(f) + 2\pi\lambda$ and since

$x \notin \operatorname{supp}\left(\tilde{f}_{|Q^m+2\pi\lambda}\right)$, we can consider

$$r := \min_{\lambda \in \Lambda} d(x; \operatorname{supp}(f) + 2\pi\lambda),$$

which is a strictly positive real and obtain then that $B(x;r)$, $\tilde{f} = 0$. Therefore, it belongs to C^k at the neighborhood of x.

This completes the proof. ☐

3.3. **Some other properties of the spaces of functions defined on the torus.**

Proposition 3.5. *Each function of $C^0(\mathbb{T}^m, \mathbb{E})$ is uniformly continuous on $\mathbb{R}^m$.*

Proof. Let r be a fixed positive real. The set $K := \{x \in \mathbb{R}^m : d(x, Q^m) \leq r\}$ is compact, so that due to Lemma of Heine: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, z \in K$, if $|x - z| \leq \delta$, then $|f(x) - f(z)|_{\mathbb{E}} \leq \varepsilon$.

We fix an arbitrary $\varepsilon > 0$ and a δ given by the previous inequality. We put $\delta' := \min\{r; \delta\}$. Let x, z be such that $|x - z| \leq \delta'$. There exists $k \in \mathbb{Z}^m$ such that $x - 2\pi k \in Q^m$ and so $z - 2\pi k \in K$.

So, we have: $|f(x) - f(z)|_{\mathbb{E}} = |f(x - 2\pi k) - f(z - 2\pi k)|_{\mathbb{E}} \leq \varepsilon$, which is exactly the uniform continuity. ☐

We shall prove now some density theorems. For this aim, we introduce the convolution product.

Proposition 3.6. *Let $j \in \mathbb{N} \cup \{+\infty\}$, $u \in C_c^j(\mathbb{T}^m, \mathcal{A})$ and $v \in L^\alpha(\mathbb{T}^m, \mathcal{B})$ with $\alpha \in [1, +\infty]$. So, $u * v \in C^j(\mathbb{T}^m, \mathcal{C})$.*

Proof. Since $v \in L^\alpha(\mathbb{T}^m, \mathcal{B})$, we get $v \in L^1_{loc}(\mathbb{R}^m, \mathcal{B})$ and the convolution product $u * v$ is given by:

$$u * v(z) = \int_{\mathbb{R}^m} u(x) \diamond v(z - x) dx.$$

Moreover, it is well defined on $\mathbb{R}^m$ and $u * v \in C^j(\mathbb{R}^m, \mathcal{C})$.

Let us verify that $2\pi\mathbb{Z}^m \subset Per(u * v)$, which will complete the demonstration.

Let $p \in 2\pi\mathbb{Z}^m$. p is a period of v, and

$$u * v(z + p) = \int_{\mathbb{R}^m} u(x) v(z + p - x) dx = \int_{\mathbb{R}^m} u(x) v(z - x) dx = u * v(z).$$

This is what had to be demonstrated. ☐

Proposition 3.7. *Let $j \in \mathbb{N} \cup \{+\infty\}$. $C^j(\mathbb{T}^m, \mathbb{E})$ and $C_c^j(\mathbb{T}^m, \mathbb{E})$ are dense in $L^2(\mathbb{T}^m, \mathbb{E})$.*

Proof. It suffices to prove this result on $C_c^j(\mathbb{T}^m, \mathbb{E})$.

Indeed, we recall that $C_c^\infty(\operatorname{Int}(Q^m), \mathbb{E})$ is dense in $L^2(Q^m, \mathbb{E})$ (the proof in Brezis's book [3, p. 71] can be adapted to Banach spaces). Fixing $u \in L^2(\mathbb{T}^m, \mathbb{E})$, we denote by $\underline{u}$ its restriction to Q^m. Hence, for a given $\varepsilon > 0$, there exists $w \in C_c^\infty(\operatorname{Int}(Q^m), \mathbb{E})$ such that

$$\int_{Q^m} |w(x) - \underline{u}(x)|^2_{\mathbb{E}} dx \leq \varepsilon^2.$$

Moreover, using Proposition 3.4, we can extend in a unique way w to an element $z \in C_c^\infty(\mathbb{T}^m, \mathbb{E})$. We get

$$\int_{\mathbb{T}^m} |z(x) - u(x)|_{\mathbb{E}}^2 dx = \int_{Q^m} |w(x) - \underline{u}(x)|_{\mathbb{E}}^2 dx \le \varepsilon^2,$$

which completes the proof of our proposition. □

Lemma 3.3. *Let $f \in L^1(\mathbb{T}^m, \mathbb{E})$ and $\beta \in \mathbb{R}^m$. So, we have*

$$\int_{Q^m} f(x+\beta)dx = \int_{Q^m} f(x)dx.$$

Proof. We will prove the result by induction on m.

For $m = 1$, we have successively

$$\begin{aligned}\int_{-\pi}^{\pi} f(t+\beta)dt = \int_{\beta-\pi}^{\beta+\pi} f &= \int_{\beta-\pi}^{-\pi} f + \int_{-\pi}^{\pi} f + \int_{\pi}^{\pi+\beta} f \\ &= -\int_{-\pi}^{\beta-\pi} f + \int_{-\pi}^{\pi} f + \int_{-\pi}^{\beta-\pi} f(t+2\pi)dt = \int_{-\pi}^{\pi} f.\end{aligned}$$

This is the desired result.

Now, suppose the result is true for 1 and $m-1$. We have

$$\int_{Q^m} f(x+\beta)dx = \int_{-\pi}^{\pi} \left[\int_{[-\pi,\pi]^{m-1}} f(x_1+\beta_1, x_{-1}+\beta_{-1})dx_{-1} \right] dx_1.$$

From the result at rank $m-1$, the right side is

$$\int_{-\pi}^{\pi} \left[\int_{[-\pi,\pi]^{m-1}} f(x_1+\beta_1, x_{-1})dx_{-1} \right] dx_1.$$

Due to Fubini's theorem then from the result to row 1, this integral is equal to $\int_{Q^m} f(x)dx$, which means that the proposition is proven. □

Proposition 3.8. *Let consider $u \in L^2(\mathbb{T}^m, \mathbb{E})$. The function from $\mathbb{R}^m$ into $L^2(\mathbb{T}^m, \mathbb{E})$ defined by $\beta \mapsto \tau_\beta u$ is uniformly continuous.*

Proof. Let us fix an $\varepsilon > 0$, and a function $v \in C_c^0(\mathbb{T}^m, \mathbb{E})$ such that $\|u - v\| \le \varepsilon/3$.

Since v is uniformly continuous on $\mathbb{R}^m$, $\beta \mapsto \tau_\beta$ is uniformly continuous from $\mathbb{R}^m$ into $C^0(\mathbb{T}^m, \mathbb{E})$. Hence, we can find $\eta > 0$ such that if γ and β are in $\mathbb{R}^m$ such that $|\gamma - \beta| \le \eta$, then

$$\sup_{x \in \mathbb{R}^m} |v(x+\gamma) - v(x+\beta)|_{\mathbb{E}} \le \frac{\varepsilon}{3(2\pi)^m}.$$

Let choose γ and β. We then get

$$\begin{aligned}\|u(\cdot+\beta) - u(\cdot+\gamma)\| \le& \|u(\cdot+\beta) - v(\cdot+\beta)\| + \|v(\cdot+\beta) - v(\cdot+\gamma)\| \\ &+ \|v(\cdot+\gamma) - u(\cdot+\gamma)\| \\ \le& 2\|u-v\| + \|v(\cdot+\beta) - v(\cdot+\gamma)\|,\end{aligned}$$

using Lemma 3.3. This last term being less than ε, the proof is achieved. □

4. Construction of Sobolev-type Spaces

We aim to present the construction of Sobolev type spaces adapted to our problems. We will start by introducing a notion of weak derivative of Percival for the elements of $L^2(\mathbb{T}^m, \mathbb{E})$ as infinite generator of a (semi-)group of contractions. The domain of this unbounded operator is the Sobolev type space that we build. We shall explain the relation between distributions on the torus and 2π-periodic distributions in each variable, and we shall show that the different ways of introducing the weak derivative of Percival coincide.

4.1. **Construction and first properties of the space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.** Due to Proposition 3.1, for each $u \in L^2(\mathbb{T}^m, \mathbb{E})$ and all $\beta \in \mathbb{R}^m$, we have $\tau_\beta u \in L^2(\mathbb{T}^m, \mathbb{E})$.

Thus, we can define, for all $t \in \mathbb{R}^+$, $T(t)$ from $L^2(\mathbb{T}^m, \mathbb{E})$ into $L^2(\mathbb{T}^m, \mathbb{E})$ by setting

$$T(t)u := \tau_{t\omega} u, \quad \text{for all } t \in \mathbb{R}^+, \text{ for all } u \in L^2(\mathbb{T}^m, \mathbb{R}^N).$$

It can be easily verified that $T(t)$ is a linear isometry of $L^2(\mathbb{T}^m, \mathbb{E})$.

Proposition 4.1. *The following statements hold.*

(1) *For all $s, t \in \mathbb{R}^+$, $T(s + t) = T(t) \circ T(s)$.*

(2) *$T(0) = id$.*

(3) *For all $u \in L^2(\mathbb{T}^m, \mathbb{E})$, $[t \mapsto T(t)u] \in C^0(\mathbb{R}^+, L^2(\mathbb{T}^m, \mathbb{E}))$.*

Proof. **(1)** We have for all $s, t \geq 0$,

$$\begin{aligned} T(t + s)u = \tau_{(s+t)\omega} u = u(\cdot + (s + t)\omega) = \tau_{s\omega} u(\cdot + t\omega) = T(s)[T(t)u] \\ =[T(s) \circ T(t)](u). \end{aligned}$$

(2) is obvious.

(3) $[t \mapsto t\omega]$ is continuous, which implies that this assertion is a consequence of Proposition 3.8. ☐

Hence, following [7, p. 614], the family $(T(t))_{t\in\mathbb{R}^+}$ is a strongly continuous semi-group of $\mathcal{L}(L^2(\mathbb{T}^m, \mathbb{E}); L^2(\mathbb{T}^m, \mathbb{E}))$.

We denote by ∇_ω the infinitesimal generator of this semi-group, and by $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ its domain. So, we have

$$H^1_\omega(\mathbb{T}^m, \mathbb{E}) := \left\{ u \in L^2(\mathbb{T}^m, \mathbb{E}) : \lim_{t\to 0+} \frac{T(t)u - u}{t} \text{ exists in } L^2(\mathbb{T}^m, \mathbb{E}) \right\}$$

and for $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, this limit is denoted by $\nabla_\omega u$.

We obtain from the theory of strongly continuous semi-groups, cf. [7].

Proposition 4.2. *The following assertions are true.*

1. *$H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is a linear subspace of $L^2(\mathbb{T}^m, \mathbb{E})$ and ∇_ω is a linear operator from $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ into $L^2(\mathbb{T}^m, \mathbb{E})$.*
2. *If $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and if $t \in \mathbb{R}^+$, then $\tau_{t\omega} u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and*

$$\frac{d}{dt}(\tau_{t\omega} u) = \nabla_\omega(\tau_{t\omega} u) = \tau_{t\omega}(\nabla_\omega u).$$

3. *If $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and if $0 \le s < t < +\infty$, then*

$$\tau_{t\omega}u - \tau_{s\omega}u = \int_s^t \tau_{r\omega}(\nabla_\omega u)dr.$$

4. *If $t \in \mathbb{R}^+$ and if $g \in L^1(\mathbb{R}, \mathbb{R})$ is continuous in t, then*

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} g(s)\tau_{s\omega}u ds = g(t)\tau_{t\omega}u.$$

5. *$H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is dense in $L^2(\mathbb{T}^m, \mathbb{E})$ and ∇_ω is of closed graph in $L^2(\mathbb{T}^m, \mathbb{E}) \times L^2(\mathbb{T}^m, \mathbb{E})$.*

Remark 4.1. The integrals considered in the previous proposition are integrals of continuous functions with values in the Banach space $L^2(\mathbb{T}^m, \mathbb{E})$.

The space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is endowed with the norm

$$\|u\|_{H^1_\omega(\mathbb{T}^m,\mathbb{E})} = \sqrt{\|u\|^2 + \|\nabla_\omega u\|^2},$$

which we denote also by $\|u\|_{1,\omega}$ if there is no ambiguity on $\mathbb{E}$.

If in addition $\mathbb{E} = \mathbb{H}$ is a Hilbert space, the space $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ is equipped with the following bi-linear form

$$\langle u; v\rangle_{H^1_\omega(\mathbb{T}^m,\mathbb{H})} := \langle u; v\rangle + \langle \nabla_\omega u; \nabla_\omega v\rangle.$$

Again, if there is no ambiguity on $\mathbb{H}$, we shall denote $\langle u; v\rangle_{1,\omega}$ instead of $\langle u; v\rangle_{H^1_\omega(\mathbb{T}^m,\mathbb{H})}$.

Proposition 4.3. *Equipped with the bi-linear form $\langle \cdot, \cdot\rangle_{1,\omega}$, $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ is a Hilbert space.*

Proof. We have only to prove the completeness of this space. Now, if $(u_n)_n$ is a Cauchy sequence in $H^1_\omega(\mathbb{T}^m, \mathbb{H})$, each of the sequences $(u_n)_n$ and $(\nabla_\omega u_n)_n$ is also of Cauchy in the complete space $L^2(\mathbb{T}^m, \mathbb{H})$, hence, they converge. We denote by u and v their respective limits. Since ∇_ω is of closed graph, we deduce that $v = \nabla_\omega u$, and then the sequence $(u_n)_n$ converges in $H^1_\omega(\mathbb{T}^m, \mathbb{H})$. □

We now verify that we correctly recover the usual notion for regular functions.

Proposition 4.4. *If $u \in C^1(\mathbb{T}^m, \mathbb{E})$, then $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, and $\nabla_\omega u(x) = u'(x) \cdot \omega$ for Lebesgue-almost all x.*

Proof. Since u' is continuous on $\mathbb{T}^m$, it is uniformly continuous, and so if we give and $\varepsilon > 0$, there exists $\eta > 0$ such that: for all $\xi, \zeta \in \mathbb{R}^m$, if $|\xi - \zeta| \le \eta$, we have $|u'(\zeta) - u'(\xi)|_{\mathbb{E}} \le \frac{\varepsilon}{|\omega|}$.

Let us fix a such η, $x \in \mathbb{R}^m$ and let $t \in (0, \eta/|\omega|)$. By the mean inequality applied to the function $y \mapsto u(y) - u'(x) \cdot y$ between x and $x + t\omega$, we get

$$|u(x + t\omega) - u(x) - u'(x) \cdot (t\omega)|_{\mathbb{E}} \le \frac{\varepsilon}{|\omega|}t|\omega|.$$

We divide by t, and we integrate the square of the inequality on Q^m. It comes, for all $t \in (0, \eta/|\omega|)$

$$\left\| \frac{\tau_{t\omega} u - u}{t} - u'(\cdot) \cdot \omega \right\| \leq \varepsilon (2\pi)^{m/2},$$

which completes the proof of our proposition. □

4.2. **Convolution and density theorems.** We call a regularizing sequence a sequence $(\rho_j)_{j\geq 0}$ of functions of $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ verifying

1. for all $x \in \mathbb{R}^m$, for all $j \in \mathbb{N}$, $\rho_j(x) \geq 0$;
2. for all $j \in \mathbb{N}$, $\int_{\mathbb{R}^m} \rho_j = 1$;
3. for all $j \in \mathbb{N}$, $\mathrm{supp}(\rho_j) \subset \mathrm{Int}(Q^m)$ and $\lim_{j\to+\infty} \mathrm{diam}[\mathrm{supp}(\rho_j)] = 0$.

Proposition 4.5. *Let* $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ *and* $\rho \in C^1_c(\mathbb{R}^m, \mathbb{K})$ *such that* $\mathrm{supp}(\rho) \subset \mathrm{Int}(Q^m)$*. We have* $(d_\omega \rho) * u = \rho * (\nabla_\omega u)$.

Proof. **First step.** Let $t > 0$. We firstly remark that

$$\int_{\mathbb{R}^m} \rho(x) \left[\frac{\tau_{t\omega} u - u}{t}\right](z - x)dx = \int_{\mathbb{R}^m} u(z - x) \left[\frac{\tau_{t\omega}\rho - \rho}{t}\right](x)dx.$$

Second step. Let us prove that, as t tends to 0, $\int_{\mathbb{R}^m} \rho(x) \left[\frac{\tau_{t\omega} u - u}{t}\right](z - x)dx$ tends to $\rho * (\nabla_\omega u)(z)$. Indeed, since $\int_{\mathbb{R}^m} \rho = 1$, the difference between this integral and $\rho * (\nabla_\omega u)(z)$ is equal to

$$\int_{\mathbb{R}^m} \rho(x) \left[\frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u\right](z - x)dx = \int_{Q^m} \rho(x) \left[\frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u\right](z - x)dx,$$

which is dominated by (if we denote by $I = \left[\int_{Q^m} \rho^2\right]^{1/2}$)

$$I \left[\int_{Q^m} \left|\frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u\right|^2_{\mathbb{E}} (z - x)dx\right]^{1/2} \leq I \left\|\frac{\tau_{t\omega} u - u}{t} - \nabla_\omega u\right\|$$

and this last term tends to 0 as t tends to 0.

Third step. We aim to prove that, if t tends to 0, $\int_{\mathbb{R}^m} u(z - x)[\frac{\tau_{t\omega}\rho - \rho}{t}](x)dx$ tends to $(d_\omega \rho) * u$. Since $\mathrm{supp}(\rho) \subset \mathrm{Int}\, Q^m$, there exists a real $r > 0$ such that if $t \leq r$, then $\mathrm{supp}(\tau_{t\omega}\rho) \subset \mathrm{Int}\, Q^m$.

Since $\mathrm{supp}(\rho')$ is compact, ρ' is uniformly continuous so by using the mean inequality, for all $\varepsilon > 0$, there exists $t' < r$ such that if $t \in (0; t')$, we have

$$\left|\frac{\tau_{t\omega}\rho - \rho}{t}(x) - d_\omega \rho(x)\right| \leq \varepsilon |\omega|.$$

We integrate $\mathbb{R}^m$ this inequality multiplied before by $u(z - x)$, and we get, taking into account $\mathrm{supp}\left[\frac{\tau_{t\omega}\rho - \rho}{t}\right] \subset Q^m$

$$\left|\int_{\mathbb{R}^m} \left(\frac{\tau_{t\omega}\rho - \rho}{t}(x) - d_\omega \rho(x)\right) u(z - x)dx\right|_{\mathbb{E}} \leq \int_{Q^m} \varepsilon |\omega| \cdot |u(z - x)|_{\mathbb{E}}.$$

Hence, the result of this step is valid.

Fourth step. Conclusion, we take the limit in the inequality of the first step to end the proof. □

Proposition 4.6. *Let* $(\rho_j)_j$ *regularizing sequence and* $u \in L^2(\mathbb{T}^m, \mathbb{E})$*. Then,*

$$\lim_{j\to+\infty} \|\rho_j * u - u\| = 0.$$

Proof. Since ρ_j is positive and with support contained in Q^m, we have for all $z \in \mathbb{R}^m$

$$|\rho_j * u(z)|_{\mathbb{E}} \le \int_{Q^m} \rho_j(x)|u(z-x)|_{\mathbb{E}} dx \le \sqrt{\int_{Q^m} \rho_j(x)|u(z-x)|^2_{\mathbb{E}} dx},$$

where the last inequality is obtained using the inequality of Cauchy-Schwarz. Hence, increasing to the square, integrating over $\mathbb{T}^m$ then using Fubini's theorem, we get

$$\|\rho_j * u - u\|^2 \le \int_{Q^m} \rho_j(x) \left(\int_{\mathbb{T}^m} |u(z-x)|^2_{\mathbb{E}} dz \right) dx.$$

But, due to Lemma 3.3, the inside integral is equal to $\|u\|^2$, and so that we get

$$\|\rho_j * u - u\|^2 \le \|u\|^2.$$

As $C^0_c(\operatorname{Int} Q^m, \mathbb{E})$ is dense in $L^2(Q^m, \mathbb{E})$, we know that $C^0_c(\mathbb{T}^m, \mathbb{E})$ is dense in $L^2(\mathbb{T}^m, \mathbb{E})$ and so that for a given $\varepsilon > 0$, there is $\varphi \in C^0_c(\mathbb{T}^m, \mathbb{E})$ such that $\|u - \varphi\| < \varepsilon$. Also, we have $\|\rho_j * u - \rho_j * \varphi\| \le \varepsilon$. We deduce then

$$\|\rho_j * u - u\| \le 2\varepsilon + \|\rho_j * \varphi - \varphi\|.$$

It remains to show that $\|\rho_j * \varphi - \varphi\| \le \varepsilon$ for j large enough. The uniform continuity of φ allows us to find an $\eta > 0$ such that if $|\xi - \zeta| \le \eta$, then $|\varphi(\xi) - \varphi(\zeta)|_{\mathbb{E}} \le \varepsilon$. Let j be large enough in order to get $z \in \operatorname{supp}(\rho_j)$ implies $|z| \le \eta$. Consider then an arbitrary $z \in \mathbb{R}^m$. So, from

$$\rho_j * \varphi(z) - \varphi(z) = \int_{\mathbb{R}^m} \rho_j(x) \left(\varphi(z-x) - \varphi(z)\right) dx,$$

as the integral only relates to $\operatorname{supp}(\rho_j)$, we deduce

$$|\rho_j * \varphi(z) - \varphi(z)|_{\mathbb{E}} \le \varepsilon \int_{\operatorname{supp}(\rho_j)} \rho_j(x) dx \le \varepsilon.$$

For j large enough, we obtain finally $\|\rho_j * u - u\| \le 3\varepsilon$, which achieves the proof. □

Proposition 4.7. *Let* $u \in L^2(\mathbb{T}^m, \mathbb{E})$ *and* $\rho \in C^1_c(\mathbb{R}^m, \mathbb{K})$*. Then*

$$d_\omega(\rho * u) = (d_\omega \rho) * u.$$

Proof. Thanks to (cf. [17, p. 122]), we know that

$$\frac{\partial}{\partial x_i}(\rho * u) = \left(\frac{\partial}{\partial x_i}\rho\right) * u.$$

We multiply by ω_i and add up over i to get the result. □

Proposition 4.8. *$C^1(\mathbb{T}^m, \mathbb{E})$ is dense in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.*

More precisely, if $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, then the sequence $(\rho_j * u)_j$ tends to u in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ for every regularizing sequence $(\rho_j)_j$.

Proof. By Proposition 3.6, we have $\rho_j * u \in C^1(\mathbb{T}^m, \mathbb{E})$ and $\rho_j * (\nabla_\omega u) \in C^1(\mathbb{T}^m, \mathbb{E})$. Now, by Proposition 4.6, we have

$$\lim_{j\to+\infty} \|\rho_j * u - u\| = 0 \quad \text{and} \quad \lim_{j\to+\infty} \|\rho_j * (\nabla_\omega u) - (\nabla_\omega u)\| = 0.$$

But, due to Propositions 4.5 and 4.7, we get $\rho_j * (\nabla_\omega u) = \nabla_\omega(\rho_j * u)$ and finally $\lim_{j\to+\infty} \|\rho_j * u - u\|_{1,\omega} = 0$. □

Proposition 4.9. *The following assertions are true.*

1. *For all $f \in C^1(\mathbb{T}^m, \mathbb{E})$, $\int_{\mathbb{T}^m} \partial_\omega f(x)dx = 0$.*
2. *For all $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, $\int_{\mathbb{T}^m} \nabla_\omega u(x)dx = 0$.*

Proof. **1.** By periodicity, we have for all i

$$\int_{\mathbb{T}^m} \frac{\partial f}{\partial x_i}(x)dx = 0,$$

from which we deduce the assertion 1. by linearity.

2. By density, we can find, for all $\varepsilon > 0$, a function $f \in C^1(\mathbb{T}^m, \mathbb{E})$ such that $\|f - u\|_{1,\omega} < \varepsilon$.

Therefore, $\|d_\omega f - \nabla_\omega u\| < \varepsilon$, and so that $\|d_\omega f - \nabla_\omega u\|_{L^1(\mathbb{T}^m,\mathbb{E})} < \varepsilon(2\pi)^m$. Hence, using **1**, it can be seen that $|\int_{\mathbb{T}^m} \nabla_\omega u(x)dx|_{\mathbb{E}} < \varepsilon(2\pi)^m$. □

Proposition 4.10. *Let $\varphi \in C^1(\mathbb{T}^m, \mathcal{A})$ and $u \in H^1_\omega(\mathbb{T}^m, \mathcal{B})$. Then, $\varphi \cdot u \in H^1_\omega(\mathbb{T}^m, \mathcal{C})$ and we have $\nabla_\omega(\varphi \diamond u) = (d_\omega\varphi) \diamond u + \varphi \diamond (\nabla_\omega u)$.*

Proof. **First case.** $(\mathcal{A}, \mathcal{B}) = (\mathbb{E}', \mathbb{E})$ (or $(\mathbb{E}, \mathbb{E}')$ which can be treated in the same manner).

First step in this first case. We show that

$$\lim_{t\to0} \left\| \frac{\tau_{t\omega}\varphi - \varphi}{t} \cdot_{\mathbb{E}'\times\mathbb{E}} \tau_{t\omega}u - (d_\omega\varphi) \cdot_{\mathbb{E}'\times\mathbb{E}} u \right\| = 0.$$

Let us fix an $\varepsilon > 0$. Thanks to the uniform continuity φ', there exists $t_0 > 0$ such that if $|t| < t_0$, for all $x \in \mathbb{R}^m$ and for all $\xi \in [x, x + t\omega]$, $|\varphi'(\xi) - \varphi'(x)|_{\mathbb{E}'} \le \varepsilon|\omega|^{-1}$.

Besides, using the mean inequality, we have

$$\left| \frac{\varphi(x + t\omega) - \varphi(x)}{t} \cdot_{\mathbb{E}'\times\mathbb{E}} u(x + t\omega) - (d_\omega\varphi(x)) \cdot_{\mathbb{E}'\times\mathbb{E}} u(x + t\omega) \right|$$
$$\le \sup_{\xi\in[x;x+t\omega]} |\varphi'(\xi) - \varphi'(x)|_{\mathbb{E}'}|\omega||u(x + t\omega)|_{\mathbb{E}}.$$

Therefore, if $|t| < t_0$, this term is less than $\varepsilon|u(x + t\omega)|_E$. Taking the square and integrating, due to Lemma 3.3, we deduce

$$\left\| \left[\frac{\tau_{t\omega}\varphi - \varphi}{t} - d_\omega\varphi \right] \cdot_{\mathbb{E}'\times\mathbb{E}} \tau_{t\omega}u \right\| \le \varepsilon\|u\|.$$

Thus, we have shown that

$$\lim_{t\to 0}\left\|\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}-d_\omega\varphi\right]\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u\right\|=0. \tag{4.1}$$

Moreover, we have

$$\|(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}(\tau_{t\omega}u-u)\|^2\le(2\pi)^m\|d_\omega\varphi\|_\infty^2\|\tau_{t\omega}u-u\|^2,$$

and by Proposition 3.8, this term tends to 0 as $t\to 0$.

Due to this result, we deduce that

$$\begin{aligned}&\left\|\frac{\tau_{t\omega}\varphi-\varphi}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u\right\|\\ &\le\left\|\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}-d_\omega\varphi\right]\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u\right\|+\|(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}(\tau_{t\omega}u-u)\|.\end{aligned}$$

The inequality (4.1) allows to deduce the result.

Second step of first case. We prove that

$$\lim_{t\to 0}\left\|\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\frac{\tau_{t\omega}u-u}{t}-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\nabla_\omega u\right\|=0.$$

In fact, the term considered is less than

$$(2\pi)^{m/2}\|\varphi\|_\infty\left\|\frac{\tau_{t\omega}u-u}{t}-\nabla_\omega u\right\|,$$

which tends to 0 as $t\to 0$.

Conclusion. We get

$$\begin{aligned}&\frac{\tau_{t\omega}(\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}u)-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}u}{t}-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}(\nabla_\omega u)\\ &=\left[\frac{\tau_{t\omega}\varphi-\varphi}{t}\cdot_{\mathbb{E}'\times\mathbb{E}}\tau_{t\omega}u-(d_\omega\varphi)\cdot_{\mathbb{E}'\times\mathbb{E}}u\right]+\left[\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\frac{\tau_{t\omega}u-u}{t}-\varphi\cdot_{\mathbb{E}'\times\mathbb{E}}\nabla_\omega u\right],\end{aligned}$$

and so that it results from the two first steps that the right hand side member tends to 0 when t tends to 0, and the assertion **1.** is consequently proven.

Second case. $(\mathcal{A},\mathcal{B})=(\mathbb{K},\mathbb{E})$ (or $(\mathbb{E},\mathbb{K})$ which is the same).

Let $e\in\mathbb{E}'$ and $\varphi_e(x):=\varphi(x)e$. Then, $\varphi_e\in H^1_\omega(\mathbb{T}^m,\mathbb{E}')$ and by assertion 1, we obtain

$$\nabla_\omega[(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u]=d_\omega(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u+(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}(\nabla_\omega u).$$

But, since e is constant, we have $\nabla_\omega[(\varphi e)\cdot_{\mathbb{E}'\times\mathbb{E}}u]=(\nabla_\omega\varphi)e\cdot_{\mathbb{E}'\times\mathbb{E}}u$ and $d_\omega(\varphi e)=(d_\omega\varphi)e$. Therefore, we obtain $e\cdot_{\mathbb{E}'\times\mathbb{E}}[\nabla_\omega(\varphi\cdot u)]=e\cdot_{\mathbb{E}'\times\mathbb{E}}[(d_\omega\varphi)u]+e\cdot_{\mathbb{E}'\times\mathbb{E}}[\varphi\cdot(\nabla_\omega u)]$. Since the relation is true for all $e\in\mathbb{E}'$, we conclude that

$$\nabla_\omega(\varphi\cdot u)=(d_\omega\varphi)u+\varphi\cdot(\nabla_\omega u).$$

□

Proposition 4.11. *We have the following integration formula by parts. For all* $\varphi\in C^1(\mathbb{T}^m,\mathcal{A})$ *and* $u\in H^1_\omega(\mathbb{T}^m,\mathcal{B})$, *we have*

$$\int_{\mathbb{T}^m}\varphi\diamond(\nabla_\omega u)=-\int_{\mathbb{T}^m}(d_\omega\varphi)\diamond u.$$

Proof. Using the previous proposition, we know that $\varphi \diamond u \in H^1_\omega(\mathbb{T}^m, \mathcal{C})$ and that we have

$$\nabla_\omega(\varphi \diamond u) = (d_\omega\varphi) \diamond u + \varphi \diamond (\nabla_\omega u).$$

Integrating this equality, as in Proposition 4.9, the integral of the left hand side is zero, we have

$$\int_{\mathbb{T}^m} \varphi \diamond (\nabla_\omega u) + \int_{\mathbb{T}^m} (d_\omega\varphi) \diamond u = 0,$$

and so that the proof is completed. □

4.3. **The space $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$.**

Definition 4.1. We define the space $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ as the closure of $C^1_c(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

Proposition 4.12. *Endowed with the norm of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$, $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is complete. If in addition the space $\mathbb{E} = \mathbb{H}$ is a Hilbert space, $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a Hilbert space.*

Proof. $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a closed linear subspace of the complete space $H^1_\omega(\mathbb{T}^m, \mathbb{E})$. Therefore, it is a complete space. □

Proposition 4.13. *$H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is also the closure of $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.*

Proof. Let $\tilde{H}$ be the closure of $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

The inclusion $\tilde{H} \subset H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$ is a consequence of $C^1_c(\mathbb{T}^m, \mathbb{E}) \subset C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$.

For the converse sens, we shall prove that the injection of $C^1_c(\mathbb{T}^m, \mathbb{E})$ in $C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ is dense for the norm of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$. Let take $\varphi \in C^1_{c,\omega}(\mathbb{T}^m, \mathbb{E})$ and $(\rho_n)_n$ a regularizing sequence. Since $\varphi \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$, from the Proposition 3.7 applied on φ and ρ_n, we get if $\psi_n := \rho_n * \varphi$, we have $\psi_n \in C^1(\mathbb{T}^m, \mathbb{E})$ and

$$\lim_{n\to+\infty} \|\psi_n - \varphi\|_{1,\omega} = 0.$$

Besides, we have

$$\mathrm{supp}(\psi_n) \subset \mathrm{supp}(\varphi) + \mathrm{supp}(\rho_n),$$

and as $\lim_{n\to+\infty} \mathrm{diam}(\mathrm{supp}(\rho_n)) = 0$, for n large enough, we have $\mathrm{supp}(\psi_n) \subset \mathrm{Int}\, Q^m$. We deduce that $\psi_n \in C^1_c(\mathbb{T}^m, \mathbb{E})$ for n large enough, which ends the proof. □

Proposition 4.14. *For all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$, we have the inequality of Poincaré-Wirtinger*

$$\|\nabla_\omega u\| \geq \frac{|\omega|}{\pi\sqrt{m}}\|u\|.$$

Moreover, the map $u \mapsto \|\nabla_\omega u\|$ is a norm $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$ equivalent to $H^1_\omega(\mathbb{T}^m, \mathbb{H})$.

We denote $\|\cdot\|_{1,\omega,0}$ the norm given in proposition, which means that for all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$,

$$\|u\|_{1,\omega,0} = \|\nabla_\omega u\|.$$

The proof of this proposition is based essentially on the verification of the inequality indicated for the regular functions, which is the main purpose of the following lemma.

Lemma 4.1. *We denote $\alpha = \frac{|\omega|}{\pi\sqrt{m}}$. Then, for all $u \in C_c^1(\mathbb{T}^m, \mathbb{H})$, we have*

$$\|d_\omega u\| \geq \alpha \|u\|.$$

Proof of the Proposition 4.14. Assume for the moment that Lemma 4.1 holds. Let $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$. Consider a sequence $(u_n)_n$ with values in $C_c^1(\mathbb{T}^m, \mathbb{H})$ tending to u and for which we apply the lemma. Hence, for all n, we have

$$\|\nabla_\omega u_n\| \geq \alpha \|u_n\|.$$

But, since

$$\lim_{n\to+\infty} \|u - u_n\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\nabla_\omega u - \nabla_\omega u_n\| = 0,$$

we can take the limit to obtain

$$\|\nabla_\omega u\| \geq \alpha \|u\|.$$

From this inequality, we get for all $u \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{H})$

$$\|u\|_{1,\omega,0} \leq \|u\|_{1,\omega} \leq \frac{\sqrt{1+\alpha^2}}{\alpha} \|u\|_{1,\omega,0}$$

which means the equivalence of the norms. This completes the proof of proposition. □

Now, let us prove the lemma.

Proof of Lemma 4.1. We use the results and notations of Section 2.1. Let $u \in C_c^1(\mathbb{T}^m, \mathbb{H})$ be a fixed function and $v = u \circ \chi \circ \chi_1^{-1}$. So, we have

$$\|\nabla_\omega u\| = |\omega|^2 \int_{K^m} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy = |\omega|^2 \int_D \left(\int_{K(y_{-1})} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy_1 \right) dy_{-1}.$$

Take $y_{-1} \in D$. We set $[a, b] := K(y_{-1})$ (we remind that $K(y_{-1})$ is a closed interval, cf. Lemma 2.1). We put also $\varphi(y_1) = v(y_1, \ldots, y_m)$. We notice that $\varphi \in C_c^1([a, b], \mathbb{H})$ and that

$$\varphi'(y_1) = \frac{\partial v}{\partial y_1}(y_1, \ldots, y_m).$$

It comes then

$$\int_{K(y_{-1})} \left| \frac{\partial v}{\partial y_1}(y) \right|_{\mathbb{H}}^2 dy_1 = \int_a^b |\varphi'(t)|_{\mathbb{H}}^2 \, dt.$$

But, since $\varphi(a) = 0$, we have

$$|\varphi(t)|_{\mathbb{H}}^2 = \int_a^t \frac{\varphi(s) \cdot_{\mathbb{H}} \varphi'(s)}{2} ds \leq \frac{\|\varphi\|_{L^2([a,b],\mathbb{H})} \cdot \|\varphi'\|_{L^2([a,b],\mathbb{H})}}{2},$$

where we have used the inequality of Cauchy-Schwarz and dominated each integral (of positive functions) by the integral on the integer segment. By integration on $[a, b]$, we deduce that

$$\|\varphi\|_{L^2([a,b],\mathbb{H})} \leq \frac{b-a}{2} \|\varphi'\|_{L^2([a,b],\mathbb{H})},$$

and since $b - a = \operatorname{diam} K(y_{-1}) \leq 2\pi\sqrt{m}$, we get

$$\|\varphi\|_{L^2([a,b],\mathbb{H})} \leq \pi\sqrt{m}\|\varphi'\|_{L^2([a,b],\mathbb{H})}.$$

Going back to v, we obtain

$$\int_{K(y_{-1})} \left|\frac{\partial v}{\partial y_1}(y)\right|_{\mathbb{H}}^2 dy \geq \frac{1}{m\pi^2}\int_{K(y_{-1})} |v(y)|_{\mathbb{H}}^2\, dy,$$

or, taking into account u

$$\|d_\omega u\| \geq \frac{|\omega|}{\pi\sqrt{m}}\|u\|.$$

This ends the proof of the lemma. □

Remark 4.2. The fact that a constant non zero function does not verify the relation of Poincaré-Wirtinger shows that $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ is different of $H^1_\omega(\mathbb{T}^m,\mathbb{H})$.

Notation 4.1. We denote by $\alpha_{PW}(m)$ (or α_{PW} if there is no ambiguity on m), the best Poincaré-Wirtinger constant, that is to say

$$\begin{aligned}\alpha_{PW}(m) :=& \inf_{u\in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})\setminus\{0\}} \frac{\|\nabla_\omega u\|}{\|u\|}\\ =& \sup\left\{\alpha > 0 : \text{for all } u \in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H}),\ \|\nabla_\omega u\| \geq \alpha\|u\|\right\}.\end{aligned}$$

We have then, for all m,

$$\alpha_{PW}(m) \geq \frac{|\omega|}{\pi\sqrt{m}}.$$

Proposition 4.15. *The canonical injection of $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ in $L^2(\mathbb{T}^m,\mathbb{H})$ is not compact.*

Remark 4.3. In other words, the space $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})$ does not verify a result of the type Rellich-Kondrachov. This lack of compactness makes it more difficult to obtain existence theorems in this space or in usual Sobolev's space $H^1_0(\Omega)$.

Proof. Given the characterization of strong compacts, i.e., for the topology of the norm of $L^2(\operatorname{Int} Q^m)$ (cf. [3, p. 74]), to deny Rellich-Kondrachov, it suffices to remark that: there exists $\varepsilon > 0$, there exists $\Omega \subset\subset \operatorname{Int} Q^m$, there exists $\delta_0 > 0$, such that for all $\delta \in (0;\delta_0)$, we can find $h \in \mathbb{R}^m$, and $u \in AB_{H^1_{\omega,0}(\mathbb{T}^m,\mathbb{H})}$, such that $|h| \leq \delta$ and $\|\tau_h u - u\| \geq \varepsilon$.

Before moving forward with the proof, let us make three remarks.

- It suffices to construct a counterexample with $\mathbb{H} = \mathbb{R}$.
- Of course, this does not contradict the continuity of translations in L^2 because u depends on δ.
- The fundamental idea to remember is that the absence of compactness is due to the absence of control of the derivatives which are not in the direction of ω.

We therefore take the case of $\mathbb{H} = \mathbb{R}$. Let (b_j) be an orthonormal basis such that $b_1 = \frac{\omega}{|\omega|}$. We put $A \subset \operatorname{Int} Q^m$, L_j, $j = 1, \ldots, m$, m strictly positive reals such that if

$$K := \left\{ A + \sum_{i=1}^{m} \lambda_i L_i b_i : \lambda = (\lambda_1, \ldots, \lambda_m) \in [0,1]^m \right\}$$

and

$$TK := \bigcup_{\alpha \in [0, \frac{2}{3} L_1]} \tau_\alpha K,$$

we have $TK \subset \operatorname{Int} Q^m$. We can then find an open set Ω containing TK and so that the closure is contained in $\operatorname{Int} Q^m$ (which means $\Omega \subset\subset \operatorname{Int} Q^m$). We set $\delta_0 := \frac{2}{3} L_1$ and $l = \frac{1}{3} L_1$.

First step. In this step (which we only do if $m \geq 3$), we just have to treat the case where $m = 2$.

Let $\phi \in C^0(\mathbb{R}^{m-2}, \mathbb{R}^+)$ be not identically zero, such that $\operatorname{supp}(\phi) \subset \prod_{j=3}^{m} [0, L_j]$. We denote

$$\mathcal{I} := \int_{\mathbb{R}^{m-2}} \phi^2,$$

which is a strictly positive real.

We shall look for $v = u \circ \chi \circ \chi_1^{-1}$ having the form

$$v(y_1, \ldots, y_m) = v_2(y_1, y_2) \phi(y_3, \ldots, y_m).$$

If $\operatorname{supp}(v) \subset K$ and if $h = \delta b_2$ with $\delta \in (0, \delta_0)$, we have $\|(\tau_h v - v)\chi_\Omega\| = \|\tau_h v - v\|$, and using Fubini's theorem we obtain the following identities

$$\|\tau_h u - u\|^2 = \|\tau_h v_2 - v_2\|^2_{L^2(\mathbb{T}^2)} \mathcal{I}$$

and

$$\|\nabla_\omega u\|^2 = |\omega|^2 \|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)} \mathcal{I},$$

and finally,

$$\frac{\|\tau_h u - u\|^2}{\|\nabla_\omega u\|^2} = |\omega|^2 \frac{\|\tau_h v_2 - v_2\|^2_{L^2(\mathbb{T}^2)}}{\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)}}.$$

We see from the previous equality that it is enough to build v_2, which amounts to doing the proof in the case $m = 2$.

Second step. We shall now construct v_2.

Let us fix $\delta \in (0, \delta_0)$. For $i, j \in \{0, 1, 2\}$, we denote

$$A_{i,j} = A + i \frac{L_1}{3} b_1 + j \frac{L_2}{3} \cdot \frac{\delta}{\delta_0} b_2.$$

For all $\lambda \in \mathbb{R}^+$, we define the function P_λ on $[0, l]$ by

$$P_\lambda(x) = 2 \frac{\lambda}{l^2} x^2 \chi_{[0, l/2]}(x) + \lambda \left(1 - 2 \left(1 - \frac{x}{l} \right)^2 \right) \chi_{[l/2, l]}(x).$$

So, we introduce f_λ as $f_\lambda(x) = P_\lambda(x)\chi_{[0,l]}(x) + \lambda\chi_{[l,2l]}(x) + P_\lambda(3l - x)\chi_{[0,l]}(x)$. The function f_λ is continuous and it is C^1 piecewise. We calculate

$$\|f'_\lambda\|_\infty = 2\frac{\lambda}{l}.$$

We set finally, $v_2(y_1, y_2) = f_{y_2/\delta}(y_1)\chi_{[0,\delta]}(y_2) + f_1(y_1)\chi_{[\delta,2\delta]}(y_2) + f_{3-y_2/\delta}(y_1)\chi_{[2\delta,3\delta]}(y_2)$. So, we have

$$\|\tau_{\delta b_2}v_2 - v_2\|^2_{L^2(\mathbb{T}^2)} \geq \int_{co(A_{1,1};A_{1,2};A_{2,1};A_{2,2})}(1-0)^2dy = \frac{l\delta}{2}$$

and

$$\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)} \leq 2\left[2\int_{co(A_{0,0};A_{1,0};A_{0,1};A_{1,1})}\frac{4.y_2^2}{l^2\delta^2}dy + \int_{co(A_{0,1};A_{1,1};A_{0,2};A_{2,2})}\frac{4}{l^2}dy\right] = \frac{17\delta}{4l}.$$

Finally, we have

$$\frac{\|\tau_{\delta b_2}v_2 - v_2\|^2_{L^2(\mathbb{T}^2)}}{\|\partial_1 v_2\|^2_{L^2(\mathbb{T}^2)}} \geq \frac{2l^2}{17},$$

hence we can take in the case arbitrary m

$$\varepsilon = \frac{l}{|\omega|}\sqrt{\frac{2}{17}}.$$

This achieves the proof. □

5. Fourier Analysis and Comparison of the Different Notions of Derivation

Remark 5.1. For convenience, we assume that $\mathbb{K} = \mathbb{C}$. In the real case, this consists in working in the complexification of $\mathbb{E}$, then in obtaining the Fourier coefficients of opposite indices.

We denote, for $u \in L^2(\mathbb{T}^m, \mathbb{E})$ and $\nu \in \mathbb{Z}^m$, $a(u;\nu)$ the element of $\mathbb{E}$

$$a(u;\nu) := \frac{1}{(2\pi)^m}\int_{Q^m} e_{-\nu}(x)u(x)dx.$$

We also denote

$$u \sim \sum_{\nu\in\mathbb{Z}^m} a(u;\nu)e_\nu.$$

We recall the following.

Remind 5.1. The map $u \mapsto (a(u;\nu))_{\nu\in\mathbb{Z}^m}$ is an isometric isomorphism from $L^2(\mathbb{T}^m, \mathbb{H})$ into $\ell^2(\mathbb{Z}^m;\mathbb{H})$.

Remark 5.2. The function e_ν is of class $C^1(\mathbb{T}^m, \mathbb{E})$ and

$$d_\omega e_\nu = i(\nu\cdot\omega)e_\nu.$$

Proposition 5.1. *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{E})$ and $\nu \in \mathbb{Z}^m$. We have*

$$a(\nabla_\omega u;\nu) = i(\nu\cdot\omega)a(u;\nu).$$

Proof. Fix $\nu \in \mathbb{Z}^m$. The application $a(\cdot;\nu)$ is linear continuous from $L^2(\mathbb{T}^m,\mathbb{E})$ into $\mathbb{E}$, hence

$$a(\nabla_\omega u;\nu) = \lim_{t\to 0} \frac{a(\tau_{t\omega}u;\nu) - a(u;\nu)}{t}.$$

By Lemma 3.3, we have $a(\tau_{t\omega}u;\nu) = e_\nu(t\omega)a(u;\nu)$. □

Proposition 5.2. *Let $u \in L^2(\mathbb{T}^m,\mathbb{H})$ such that*

$$\sum_{\nu\in\mathbb{Z}^m} (\nu\cdot\omega)^2 |a(u;\nu)|^2_{\mathbb{H}} < +\infty.$$

Then, $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$ and $\nabla_\omega u \sim \sum_\nu i(\nu\cdot\omega)a(u;\nu)e_\nu$.

Proof. We know (cf. (5.1)) that there exists $v \in L^2(\mathbb{T}^m,\mathbb{H})$ such that $v \sim \sum_\nu i(\nu\cdot\omega)a(u;\nu)e_\nu$. For $k \in \mathbb{N}^*$, then form the trigonometric polynomial $P_k(x) = \sum_{|\nu|\le k} a(u;\nu)e_\nu(x)$. Thus, $\lim_{k\to+\infty}\|u - P_k\| = 0$ and $\lim_{k\to+\infty}\|v - \nabla_\omega P_k\| = 0$.

Since $(P_k;\nabla_\omega P_k)$ is in the graph ∇_ω which is closed, so that we deduce that $v = \nabla_\omega u$ and then $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$. □

Proposition 5.3. *Let u and v be two elements of $L^2(\mathbb{T}^m,\mathbb{H})$ such that: for all $\varphi \in C^1(\mathbb{T}^m,\mathbb{C})$,*

$$\int_{\mathbb{T}^m} (d_\omega\varphi)\cdot_{\mathbb{H}} u = -\int_{\mathbb{T}^m} \varphi \cdot_{\mathbb{H}} v,$$

or equivalently, for all $\varphi \in C^1(\mathbb{T}^m,\mathbb{H})$,

$$\int_{\mathbb{T}^m} u \cdot_{\mathbb{H}} d_\omega\varphi = -\int_{\mathbb{T}^m} v \cdot_{\mathbb{H}} \varphi,$$

then $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$ and $\nabla_\omega u = v$.

Likewise, let u and v be two elements of $L^2(\mathbb{T}^m,\mathbb{C})$ such that: for all $\varphi \in C^1(\mathbb{T}^m,\mathbb{C})$,

$$\int_{\mathbb{T}^m} (d_\omega\varphi)u = -\int_{\mathbb{T}^m} \varphi v,$$

or for all $\varphi \in C^1(\mathbb{T}^m,\mathbb{H})$,

$$\int_{\mathbb{T}^m} u(d_\omega\varphi) = -\int_{\mathbb{T}^m} v\varphi,$$

then $u \in H^1_\omega(\mathbb{T}^m,\mathbb{C})$ and $\nabla_\omega u = v$.

Remark 5.3. This way of defining the derivative of an element of $L^2(\mathbb{T}^m,\mathbb{H})$ is analogous to that of Sobolev. Also, we will say that v is the weak derivative of Sobolev. This proposition thus shows that this weak derivative, when it exists, coincides with the notion already introduced. We will show the reciprocal later.

Proof of Proposition 5.3. Taking $\varphi = e_\nu$, we obtain that $a(v;\nu) = -i(\nu\cdot\omega)a(u;\nu)$. Since $v \in L^2(\mathbb{T}^m,\mathbb{H})$, we deduce that $((\nu\cdot\omega)a(u;\nu))_\nu \in \ell^2(\mathbb{Z}^m;\mathbb{H})$, and Proposition 5.2 allows us to conclude.

Let $h \in \mathbb{H}$ non zeros. Let $\varphi \in C^1(\mathbb{T}^m,\mathbb{C})$. We apply the hypothesis with the functions $\varphi_h(x) := \varphi(x)h$. We get the result.

It is a special case with $\mathbb{H} := \mathbb{C}$. □

6. Link with Periodic Distributions and Distributions on the Torus

6.1. **Preliminary on vector-valued distributions.** We denote by $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ the vector space of class functions C^∞ from $\mathbb{R}^m$ into $\mathbb{K}$ which vanish outside of a compact.

The space of distribution with values into $\mathbb{E}$ is by definition $\mathcal{L}(\mathcal{D}(\mathbb{R}^m, \mathbb{K}), \mathbb{E})$. We will note it $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$.

Proposition 6.1. *Each function $f \in L^p_{loc}(\mathbb{R}^m, \mathbb{E})$ defines a vector valued distribution T_f, for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$ by*

$$\langle T_f; \varphi \rangle := \int_{\mathbb{R}^m} \varphi(x) f(x) dx.$$

Proof. Since $f \in L^p_{loc}(\mathbb{R}^m, \mathbb{E})$, we have $f \in L^1_{loc}(\mathbb{R}^m, \mathbb{E})$ which shows in particular that for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$, $\varphi f \in L^1(\mathbb{R}^m, \mathbb{E})$ and so that $\int_{\mathbb{R}^m} \varphi f \in \mathbb{E}$. Besides, we have for all $e' \in \mathbb{E}'$

$$|e' \cdot_{\mathbb{E}' \times \mathbb{E}} f| \leq |e'|_{\mathbb{E}'} |f|_{\mathbb{E}},$$

and so $e' \cdot_{\mathbb{E}' \times \mathbb{E}} f \in L^1_{loc}(\mathbb{R}^m, \mathbb{R})$ for all $e' \in \mathbb{E}'$. Due to [14, Proposition 19, p. 66], we deduce that f define a vector distribution. □

6.2. **Periodification.** The main reference here is [18].

Proposition 6.2. *Let $\varphi : \mathbb{R}^m \to \mathbb{E}$ be a function with compact support. Then, for all $x \in \mathbb{R}^m$,*

$$\varpi(\varphi)(x) = \sum_{\lambda \in 2\pi\mathbb{Z}^m} \tau_\lambda \varphi(x)$$

is well defined, the function $\varpi(\varphi) : \mathbb{R}^m \to \mathbb{E}$ is periodic, and $2\pi\mathbb{Z}^m \subset Per(\varpi(\varphi))$.

Proof. **Existence.** We will verify that the sum defining $\varpi(\varphi)(x)$ is finite.

Let x be fixed. $\tau_\lambda \varphi(x) \neq 0$ implies $x + \lambda \in \text{supp}(\varphi)$ which gives $\lambda \in (\text{supp}(\varphi) - x) \cap 2\pi\mathbb{Z}^m$ and as this intersection is finite because the support of φ is bounded, we deduce that the sum defining $\varpi(\varphi)$ deals only with a finite number of terms, hence the existence of $\varpi(\varphi)$.

Periodicity. This property is a direct consequence of the fact that $2\pi\mathbb{Z}^m$ is a group. □

The operator ϖ extends to compactly supported distributions as follows. For $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, we set for all $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$

$$\langle \varpi T; \varphi \rangle := \langle T; \varpi\varphi \rangle.$$

Proposition 6.3. *ϖ applies continuously $\mathcal{D}(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$ and $\mathcal{E}'(\mathbb{R}^m, \mathbb{E})$ in $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$.*

Proof. Let K be a fixed compact of $\mathbb{R}^m$. ϖ applies continuously $\mathcal{D}_K(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$, hence $\mathcal{D}(\mathbb{R}^m, \mathbb{E})$ into $\mathcal{E}(\mathbb{R}^m, \mathbb{E})$. The expression defining ϖ for distributions allows to conclude at the end of the proposition, since ϖ is defined as being its transpose (with an abuse of notations). □

Proposition 6.4. *The following statements are true.*

1. *For all $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, ϖT is periodic and more exactly we have for all $\lambda \in 2\pi\mathbb{Z}^m$ $\varpi(\tau_\lambda T) = \tau_\lambda(\varpi T) = \varpi T$.*
2. *For all $F \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $\psi \in \mathcal{D}(\mathbb{R}^m, \mathbb{K})$, we have $\varpi(F\psi) = (\varpi\psi).F$.*
3. *For all $f \in C^\infty(\mathbb{T}^m, \mathbb{K})$ and $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, we have $\varpi(fT) = f.(\varpi T)$.*

Proof. We refer to [18, p. 62–63], where the proofs can be adapted without problems to the Banach framework as arrival space.

The assertion **1.** is immediate by transposition.

For the assertion **2.**, we have $\tau_\lambda(\psi F) = \tau_\lambda(\psi)\tau_\lambda(F) = \tau_\lambda(\psi)F$ as F is periodic. We can conclude by passing to the sum.

The last assertion can be done as the second one. □

Proposition 6.5. *The following statements hold true.*

1. *If $\varphi \in C_c^0(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^0(\mathbb{T}^m, \mathbb{E})$.*
2. *If $\varphi \in C_c^1(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^1(\mathbb{T}^m, \mathbb{E})$ and moreover, we have for all $i = 1, \dots, m$*
$$\varpi\left(\frac{\partial\varphi}{\partial x_i}\right) = \frac{\partial\varpi(\varphi)}{\partial x_i} \quad \textit{and} \quad \varpi(d_\omega\varphi) = d_\omega\varpi(\varphi).$$
3. *For all $k \in \mathbb{N} \cup \{+\infty\}$, if $\varphi \in C_c^k(\mathbb{R}^m, \mathbb{E})$, then $\varpi(\varphi) \in C^k(\mathbb{T}^m, \mathbb{E})$.*

Proof. We notice that **3.** is a consequence of **2.** by iteration. We also notice that $\operatorname{supp}(\tau_\lambda\varphi) = \operatorname{supp}(\varphi) - \lambda$.

Let fix an x. We shall prove that on a ball centered in x, we can choose a fix finite set of indexes λ for which the terms of the sum are non zero. The assertions of the proposition will follow immediately. Noticing K the compact $\operatorname{supp}(\varphi) - x$, we remark first that $d(x; \operatorname{supp}(\tau_\lambda\varphi)) = d(\lambda; K)$, and for some $r > 0$ being fixed, the set

$$Z := \{\lambda \in 2\pi\mathbb{Z}^m : d(x; \operatorname{supp}(\tau_\lambda\varphi)) < r\}$$

is finite, and since $Z = \{\lambda \in 2\pi\mathbb{Z}^m : \operatorname{Int} B(x,r) \cap \operatorname{supp}(\tau_\lambda\varphi) \neq \emptyset\}$, we have over $\operatorname{Int} B(x,r)$,$\varpi(\varphi) = \sum_{\lambda\in Z} \tau_\lambda\varphi$. Proposition is then proved. □

Remark 6.1. Previously, we have extended some functions $u : \operatorname{Int} Q^m \to \mathbb{E}$ to functions $\tilde{u} : \mathbb{T}^m \to \mathbb{E}$. Denoting by u_0 the extension of u to $\mathbb{R}^m$ by 0, we have $\tilde{u} = \varpi(u_0)$.

6.3. **Periodic distributions and distributions on the torus.** Here also, the main reference is [18] where the proofs can be adapted to the Banach framework. We begin by a lemma of periodic partition of the unit.

Lemma 6.1. *There exists a function $\theta \in \mathcal{D}(\mathbb{R}^m, \mathbb{R})$ such that $\varpi\theta = 1$.*

Proof. See [18, p. 63]. □

Lemma 6.2 (Surjectivity Lemma). **1.** *For all $f \in C^\infty(\mathbb{T}^m, \mathbb{E})$, there exists $\varphi \in \mathcal{D}(\mathbb{R}^m, \mathbb{E})$, such that $f = \varpi(\varphi)$.*

2. *For all $F \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, there exists $T \in \mathcal{E}'(\mathbb{R}^m, \mathbb{E})$, such that $F = \varpi(T)$.*

Proof. Taking into account (6.5), we can take $\varphi = \theta f$ and $T = \theta F$. □

Proposition 6.6. *The spaces* $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ *and* $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$*, equipped with the same dual topologies (strong ones or week ones), are algebraically and topologically isomorphic.*

Given the importance of this proposition, we demonstrate it in details.

Proof. **1.** ϖ applies continuously $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{E}(\mathbb{R}^m, \mathbb{K})$, and so does apply $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{E}(\mathbb{R}^m, \mathbb{K}) \cap \mathcal{D}'(\mathbb{T}^m, \mathbb{K}) = C^\infty(\mathbb{T}^m, \mathbb{K})$. Its transpose, denoted as ϖ^T and defined for all $L \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, and all $\varphi \in \mathcal{D}(\mathbb{T}^m, \mathbb{E})$ by

$$\langle \varpi^T L; \varphi \rangle = \langle L; \varpi\varphi \rangle_{\mathbb{T}^m}$$

applies then continuously $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$ in $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$. But, $\varpi^T L$ is a periodic distribution, and ϖ^T sends continuously $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$ to $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$.

2. Let θ given by lemma of periodic partition of the unit. The application $f \mapsto \theta f$ applies then continuously $\mathcal{E}(\mathbb{R}^m, \mathbb{K})$ in $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$. Let Θ be the restriction of this application to $C^\infty(\mathbb{T}^m, \mathbb{K})$. Θ applies also continuously $C^\infty(\mathbb{T}^m, \mathbb{K})$ in $\mathcal{D}(\mathbb{R}^m, \mathbb{K})$, and so its transpose continuously applies $\mathcal{D}'(\mathbb{R}^m, \mathbb{E})$ in $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$. Its restriction to $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$, again noted Θ^T, applies $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ continuously in $(C^\infty)'(\mathbb{T}^m, \mathbb{E})$.

3. We check by a simple calculation that ϖ^T and Θ^T are reciprocal. □

Remark 6.2. From now on, we will systematically do this identification.

Remark 6.3. We can explain this correspondence.

1. Given $F \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and T any distribution with compact verifying $\varpi T = F$, we have for all $f \in C^\infty(\mathbb{T}^m, \mathbb{R})$ $\langle F, f \rangle_{\mathbb{T}^m} = \langle T; f \rangle$.

2. If in addition F is locally integrable, we can take $T = \chi_{Q^m} F$, and

$$\langle F, f \rangle_{\mathbb{T}^m} = \int_{Q^m} F(x) f(x) dx.$$

6.4. **Link with the concepts previously introduced.** We define Percival operators for the distributions.

Definition 6.1. We define the following.

1. The operator ∂_ω on $\mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ in the following way. If $T \in (C^\infty)'(\mathbb{T}^m, \mathbb{E})$, we set for all $\varphi \in C^\infty(\mathbb{T}^m, \mathbb{K})$ $\langle \partial_\omega T; \varphi \rangle = -\langle T; d_\omega \varphi \rangle$.

2. For $T \in \mathcal{D}'(\operatorname{Int} Q^m, \mathbb{E})$, $D_\omega T$, for all $\varphi \in C^\infty(\operatorname{Int} Q^m, \mathbb{K})$ by $\langle D_\omega T; \varphi \rangle = -\langle T; d_\omega \varphi \rangle$.

Remark 6.4. The following assertions are true.

1. If $\varphi \in C^1(\mathbb{T}^m, \mathbb{E})$, then $\varphi \in \mathcal{D}'(\mathbb{T}^m, \mathbb{E})$ and $\partial_\omega \varphi = d_\omega \varphi$.

2. If $\varphi \in C^1_c(\operatorname{Int} Q^m, \mathbb{E})$, then $\varphi \in \mathcal{D}'(\operatorname{Int} Q^m, \mathbb{E})$ and $D_\omega \varphi = d_\omega \varphi$.

We now indicate a characterization of $H^1_\omega(\mathbb{T}^m, \mathbb{H})$ in terms of periodic distributions.

Proposition 6.7. *The following equality holds true*

$$H^1_\omega(\mathbb{T}^m,\mathbb{H}) = \left\{u \in L^2(\mathbb{T}^m,\mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{H})\right\},$$

and if $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$*, we have* $\nabla_\omega u = \partial_\omega u$*.*

Proof. We shall suppose that $\mathbb{K} = \mathbb{C}$ for sake of simplicity.

Inclusion $H^1_\omega(\mathbb{T}^m,\mathbb{H}) \supset \{u \in L^2(\mathbb{T}^m,\mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{H})\}$.

Let $u \in L^2(\mathbb{T}^m;\mathbb{H})$ such that $\partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{H})$. We remark first that we have for all $\varphi \in C^\infty(\mathbb{T}^m,\mathbb{K})$ the following relation on ∂_ω

$$\int_{\mathbb{T}^m} \varphi \cdot \partial_\omega u = -\int_{\mathbb{T}^m} d_\omega \varphi \cdot u.$$

By Proposition 5.3, we deduce that $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$ and $\partial_\omega u = \nabla_\omega u$.

Inclusion $H^1_\omega(\mathbb{T}^m,\mathbb{H}) \subset \{u \in L^2(\mathbb{T}^m,\mathbb{H}) : \partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{H})\}$.

From **4.** of Proposition 4.11, if $u \in H^1_\omega(\mathbb{T}^m,\mathbb{H})$, we have for all $\varphi \in C^\infty(\mathbb{T}^m,\mathbb{K})$

$$\int_{\mathbb{T}^m} \varphi \nabla_\omega u = -\int_{\mathbb{T}^m} d_\omega \varphi \cdot u.$$

But, this means that $\nabla_\omega u = \partial_\omega u$, and so $\partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{H})$. ☐

Each function $f \in L^1_{loc}(\mathbb{R}^m,\mathbb{E})$ presents a distribution noted by T_f. We denote D_i the distributional partial derivatives on $\mathcal{D}'(\mathbb{R}^m,\mathbb{E})$ and D_ω the operator $D_\omega = \sum_{i=1}^m \omega_i D_i$. We recall that $\partial_\omega u$ has been defined for $u \in H^1_\omega(\mathbb{T}^m,\mathbb{E})$.

Proposition 6.8. *Let* $u \in H^1_\omega(\mathbb{T}^m,\mathbb{E})$*. Then, we have on* $\mathcal{D}'(\mathbb{R}^m,\mathbb{E})$

$$D_\omega T_u = T_{\partial_\omega u},$$

that is, $D_\omega u$ *is represented by* $\partial_\omega u$*.*

To begin with, we notice that $T_{\partial_\omega u}$ is well defined because

$$\partial_\omega u \in L^2(\mathbb{T}^m,\mathbb{E}) \subset L^2_{loc}(\mathbb{R}^m,\mathbb{E}) \subset L^1_{loc}(\mathbb{R}^m,\mathbb{E}).$$

Proof. Let $\varphi \in C^\infty_c(\mathbb{R}^m,\mathbb{K})$ be a fixed function. There exist $\lambda_1,\dots,\lambda_p$ such that $\operatorname{supp}(\varphi) \subset \cup_{j=1}^p (Q^m + \lambda_j)$. Let $i \in \{1,\dots,m\}$. We have

$$\langle D_i T_u;\varphi\rangle = -\langle T_u, \frac{\partial\varphi}{\partial x_i}\rangle = -\int_{\mathbb{R}^m} \frac{\partial\varphi}{\partial x_i} u,$$

and since if $i \neq j$, $(Q^m + \lambda_i) \cap (Q^m + \lambda_j)$ is of zero measure, this integral is equal to

$$\begin{aligned}
-\sum_{j=1}^p \int_{(Q^m+\lambda_j)} \frac{\partial\varphi}{\partial x_i} u &= -\sum_{j=1}^p \int_{Q^m} \frac{\partial\varphi(x+\lambda_j)}{\partial x_i} u(x+\lambda_j)dx \\
&= -\sum_{j=1}^p \int_{Q^m} \frac{\partial\varphi(x+\lambda_j)}{\partial x_i} u(x)dx \\
&= -\int_{\operatorname{Int} Q^m} \left(\sum_{j=1}^p \tau_{\lambda_j} \frac{\partial\varphi}{\partial x_i}\right) u.
\end{aligned}$$

But, if $x \in \operatorname{Int} Q^m$ and λ is not a λ_j, $j = 1, \dots, p$, we have $\varphi(x+\lambda) = 0$ by definition of λ_j. Thus, the obtained integral is equal to

$$-\int_{\operatorname{Int} Q^m} \varpi \left(\frac{\partial \varphi}{\partial x_i} \right) u.$$

Therefore, we finally obtain

$$\langle D_\omega T_u; \varphi \rangle = - \int_{\operatorname{Int} Q^m} \varpi(d_\omega \varphi) u. \tag{6.1}$$

Moreover, arguing in the same way, we have the following identities

$$\begin{aligned} \int_{\mathbb{R}^m} \varphi \partial_\omega u &= \sum_{j=1}^p \int_{Q^m + \lambda_j} \varphi \partial_\omega u = \sum_{j=1}^p \int_{\operatorname{Int} Q^m} (\tau_{\lambda_j} \varphi) \partial_\omega u = \int_{\operatorname{Int} Q^m} \left(\sum_{j=1}^p \tau_{\lambda_j} \varphi \right) \partial_\omega u \\ &= \int_{\operatorname{Int} Q^m} (\partial_\omega u) \varpi(\varphi) = - \int_{\operatorname{Int} Q^m} u d_\omega \left(\varpi(\varphi) \right) \partial_\omega u = - \int_{\operatorname{Int} Q^m} \varpi \left(d_\omega \varphi \right) u. \end{aligned}$$

Thus, we have shown that

$$\int_{\mathbb{R}^m} \varphi \partial_\omega u = - \int_{\operatorname{Int} Q^m} \varpi \left(d_\omega(\varphi) \right) u.$$

Comparing this equality with the equality (6.1), we finally see that for all $\varphi \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$

$$\langle D_\omega T_u, \varphi \rangle = \int_{\operatorname{Int} Q^m} \varphi \cdot \partial_\omega u,$$

which ends the proof of proposition. □

7. Sobolev Spaces on $\operatorname{Int} Q^m$

Definition 7.1. We define

$$H^1_\omega(\operatorname{Int} Q^m, \mathbb{E}) := \left\{ u \in L^2(\operatorname{Int} Q^m, \mathbb{E}) : D_\omega u \in L^2(\operatorname{Int} Q^m, \mathbb{E}) \right\},$$

which we endow with the norm

$$\|u\|_\omega := \sqrt{\int_{\operatorname{Int} Q^m} |u|^2_{\mathbb{E}} + |D_\omega u|^2_{\mathbb{E}}}.$$

We define an inner product on $H^1_\omega(\operatorname{Int} Q^m, \mathbb{H})$ by setting, for all $u, v \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{H})$

$$(u; v)_\omega := \int_{\operatorname{Int} Q^m} u \cdot_{\mathbb{H}} v + D_\omega u \cdot_{\mathbb{H}} D_\omega v.$$

Proposition 7.1. *$H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$ is a Banach space (Hilbert space if $\mathbb{E} = \mathbb{H}$).*

Proof. Let $(u_n)_n$ be a Cauchy sequence with values in $H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$. Then, the two sequences $(u_n)_n$ and $(D_\omega u_n)_n$ are of Cauchy with values in the complete space $L^2(\operatorname{Int} Q^m, \mathbb{E})$, and so convergent to u and v respectively. Besides, the operator D_ω is continuous, and so that we can say $v = D_\omega u$, which proves that $u \in H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$. □

Definition 7.2. We define $H^1_{\omega,0}(\operatorname{Int} Q^m, \mathbb{E})$ as being the closure of $C^1_c(\operatorname{Int} Q^m, \mathbb{E})$ in $H^1_\omega(\operatorname{Int} Q^m, \mathbb{E})$.

Remark 7.1. We can define also $H^1_\omega(\mathbb{R}^m,\mathbb{E})$.

The following two propositions explain the links between Sobolev spaces on the torus and Sobolev spaces on the cube.

Proposition 7.2. *The following assertions hold true.*

1. *For all* $u \in H^1_\omega(\mathbb{T}^m,\mathbb{E})$, $u_{|\mathrm{Int}\,Q^m} \in H^1_\omega(\mathrm{Int}\,Q^m,\mathbb{E})$.

2. *For all* $u \in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{E})$, $u_{|\mathrm{Int}\,Q^m} \in H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$.

Proof. **1.** Let $u \in H^1_\omega(\mathbb{T}^m,\mathbb{E})$. We set $w = u_{|\mathrm{Int}\,Q^m}$ and $z = \nabla_\omega u_{|\mathrm{Int}\,Q^m}$. Let $\varphi \in C^\infty_c(\mathrm{Int}\,Q^m,\mathbb{K})$. There exists $\phi \in C^\infty_c(\mathbb{T}^m,\mathbb{K})$ such that $\phi_{|\mathrm{Int}\,Q^m} = \varphi$. We then successively have

$$\int_{\mathrm{Int}\,Q^m} \varphi z = \int_{\mathbb{T}^m} \phi\cdot(\nabla_\omega u) = -\int_{\mathbb{T}^m}(d_\omega\phi)u = -\int_{\mathrm{Int}\,Q^m}(d_\omega\varphi)w.$$

This shows that $z = D_\omega w$ and so $w \in H^1_\omega(\mathrm{Int}\,Q^m,\mathbb{E})$.

2. Let $u \in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{E})$. By **1**, $u_{|\mathrm{Int}\,Q^m} \in H^1_\omega(\mathrm{Int}\,Q^m,\mathbb{E})$. Let $(f_j)_j$ be a sequence of elements of $C^1_c(\mathbb{T}^m,\mathbb{E})$ converging to u in $H^1_\omega(\mathbb{T}^m,\mathbb{E})$. We denote by g_j the restriction of f_j to $\mathrm{Int}\,Q^m$ and $w = u_{|\mathrm{Int}\,Q^m}$. We have then : $\|w-g_j\|_\omega = \|u-f_j\|_{1,\omega}$ and so the term in the left tends to 0 as j tends to infinity, which means that : $w \in H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$. ☐

Proposition 7.3. *For all* $u \in H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$, *there exists a unique* $\tilde{u} \in H^1_{\omega,0}(\mathbb{T}^m,\mathbb{E})$ *such that* $\tilde{u}_{|Q^m} = u$. *Moreover,* $\nabla_\omega \tilde{u}_{|Q^m} = D_\omega u$.

Proof. Let $u \in H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$ and $(f_j)_j$ be a sequence of $C^1_c(\mathrm{Int}\,Q^m,\mathbb{E})$ converging to u in $H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$. So, there exists a sequence $(F_j)_j$ of $C^1_c(\mathbb{T}^m,\mathbb{E})$ such that the restriction of F_j to $\mathrm{Int}\,Q^m$ coincides with f_j. The sequence $(F_j)_j$ is of Cauchy in $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{E})$ and so converges to a function U. We denote v the restriction of U to $\mathrm{Int}\,Q^m$, which is a function of $H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$ due to the previous proposition.

Besides, we have $\|v-f_j\|_\omega = \|U-F_j\|_{1,\omega}$. Since the right hand side term tends to 0 as j goes to ∞, the term of right so is, and then by uniqueness of the limit, we have $v = u$, that is, $u = U_{|\mathrm{Int}\,Q^m}$ and so we obtain the existence of $\tilde{u}$.

Let us now prove the uniqueness. Let U_1 and U_2 be two candidates. We have

$$\int_{\mathbb{T}^m}|U_1-U_2|^2_{\mathbb{E}} = \int_{\mathrm{Int}\,Q^m}|U_1-U_2|^2_{\mathbb{E}} = \int_{\mathrm{Int}\,Q^m}|u-u|^2_{\mathbb{E}} = 0,$$

which ends the proof. ☐

Remark 7.2. The two preceding propositions show in particular that the application of $H^1_{\omega,0}(\mathbb{T}^m,\mathbb{E})$ to $H^1_{\omega,0}(\mathrm{Int}\,Q^m,\mathbb{E})$ which associates to u the value $u|_{\mathrm{Int}\,Q^m}$ is an isometric isomorphism. That allows to identify the two Hilbert spaces.

8. Higher Order Spaces

Let us quickly point out that we can of course define higher order Sobolev spaces.

Definition 8.1. Let $p \in \mathbb{N}^*$. We define the space $H^p_\omega(\mathbb{T}^m, \mathbb{E})$ as the space of $u \in L^2(\mathbb{T}^m, \mathbb{E})$ such that for all $j \leq p$, $\nabla^j_\omega u \in L^2(\mathbb{T}^m, \mathbb{E})$. It is endowed by the norm

$$\|u\|_{p,\omega} := \sqrt{\sum_{j=0}^{p} \|\nabla^j_\omega u\|^2},$$

and is a Hilbert space when $\mathbb{E} = \mathbb{H}$.

Similarly, we define of course $H^p_\omega(\operatorname{Int} Q^m, \mathbb{E})$. We can also define as well other spaces built using L^p where $p \neq 2$. Due to future needs, and since the study of $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ is already very detailed, we will not dwell more on these spaces.

9. On the Absolute Continuity of the Functions of $H^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$

We suppose in this paragraph that $\mathbb{E} = \mathbb{R}^N$.

Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, and let $g := u \circ \chi_1^{-1}$. Then, $g \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$ and $D_1 g \in \mathcal{D}'(\mathbb{R}^m, \mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$. Let now C^{m-1} be a convex set with non empty interior of $\mathbb{R}^{m-1}$, and $\Xi := \chi_1^{-1}(0 \times \operatorname{Int}(C^{m-1}))$.

Fix an $\varepsilon > 0$. We introduce

$$\Omega_n := (n - \varepsilon, n + 1 + \varepsilon) \times \operatorname{Int}(C^{m-1}),$$

this is an open convex subset of $\mathbb{R}^m$, $g_{|\Omega_n}$ and $D_1(g_{|\Omega_n})$ are into $L^2(\Omega_n, \mathbb{R}^N)$.

Set now

$$O_n := \Big\{ y_{-1} \in \operatorname{Int}(C^{m-1}) : [y_1 \mapsto g(y_1, y_{-1})] \in AC(\ (n - \varepsilon, n + 1 + \varepsilon)\ , \mathbb{R}^N) \Big\}.$$

Due to Necas [10, p. 61], for each integer n, O_n is of full measure in $\operatorname{Int}(C^{m-1})$. Since a countable union of negligible set is a negligible set, we deduce that $\bigcap_n O_n$ is also of full measure $\operatorname{Int}(C^{m-1})$. But

$$\bigcap_n O_n = \Big\{ y_{-1} \in \operatorname{Int}(C^{m-1}) : [y_1 \mapsto g(y_1, y_{-1})] \in AC_{loc}\Big(\mathbb{R}, \mathbb{R}^N\Big) \Big\}.$$

By remarking that $u(t\omega + \sum_{j=2}^m y_j b_j) = g(t|\omega|, y_{-1})$ and that χ_1^{-1} is a linear isometry, we have then established the following.

Lemma 9.1. *Let $u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$. Then*

$$\Xi' := \Big\{ \xi \in \Xi : [t \mapsto u(t\omega + \xi)] \in AC_{loc}(\mathbb{R}, \mathbb{R}^N) \Big\}$$

is of full measure in Ξ.

We shall now establish the following proposition.

Proposition 9.1. *Let $u \in H^p_\omega(\mathbb{T}^m, \mathbb{R}^N)$.*

1. *There exists Ξ_p of full measure in Ξ such that if $\xi \in \Xi_p$, for Lebesgue-almost every $t \in \mathbb{R}$, the function $t \mapsto u(t\omega + \xi)$ is differentiable, and*

$$\frac{d^j}{dt^j}[u(t\omega + \xi)] = (\nabla^j_\omega u)(t\omega + \xi), \quad j \in \{0, \dots, p\},\ [t \mapsto u(t\omega + \xi)] \in H^p_{loc}(\mathbb{R}, \mathbb{R}^N).$$

2. *If $u \in H^p_\omega(\mathbb{T}^m, \mathbb{R}^N) \cap L^\infty(\mathbb{T}^m, \mathbb{R}^N)$, then there exists Ξ'_p of full measure in Ξ such that if $\xi \in \Xi'_p$*

$$[t \mapsto u(t\omega + \xi)] \in H^p_{loc}(\mathbb{R}, \mathbb{R}^N) \quad \text{and} \quad \sup_{t\in\mathbb{R}} |u(t\omega + \xi)| \leq \|u\|_{L^\infty(\mathbb{T}^m,\mathbb{R}^N)}.$$

Proof. **First assertion when $p = 1$.**

1. For $y_{-1} \in \Xi'$, the function $y_1 \mapsto g(y_1, y_{-1})$ is locally absolutely continuous, and so almost everywhere differentiable, and

$$D_1 g(y_1, y_{-1}) = \frac{\partial g}{\partial y_1}(y_1, y_{-1}).$$

So, for almost every $t \in \mathbb{R}$, we have

$$D_1 g(t|\omega|, y_{-1}) = \frac{\partial g}{\partial y_1}(t|\omega|, y_{-1}).$$

Let t_0 arbitrary such that these two members exist. By composition, $t \mapsto g(t|\omega|, y_{-1})$ is differentiable in t_0, and we have in this point

$$\frac{d}{dt} g(t|\omega|, y_{-1}) = |\omega| \frac{\partial g}{\partial y_1}(t|\omega|, y_{-1}).$$

Besides, since we have

$$\frac{d}{dt} u\left(t\omega + \sum_{j=2}^m y_j b_j\right) = \frac{d}{dt} g(t|\omega|, y_{-1}),$$

we deduce $\frac{d}{dt} u(t\omega + \sum_{j=2}^m y_j b_j)$ exists t-almost everywhere, and then:

$$\frac{d}{dt} u\left(t\omega + \sum_{j=2}^m y_j b_j\right) = \nabla_\omega u\left(t\omega + \sum_{j=2}^m y_j b_j\right),$$

which we looked for.

2. We write $\xi = \sum_{j=2}^m y_j b_j$. Since χ_1^{-1} is a linear isometry and $\nabla_\omega u \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$, $D_1 g = \nabla_\omega u \circ \chi_1^{-1} \in L^2_{loc}(\mathbb{R}^m, \mathbb{R}^N)$. Thus, $|D_1 g|^2 \in L^1_{loc}(\mathbb{R}^m, \mathbb{R})$ and so by Fubini' theorem, $|D_1 g(\cdot, y_{-1})|^2 \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ whence $D_1 g(\cdot, y_{-1}) \in L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$. From the previous calculus, we get $[t \mapsto \frac{d}{dt} u(t\omega + \xi)] \in L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$, and then $[t \mapsto u(t\omega + \xi)] \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.

First assertion for any p. We shall proceed by induction. Let $p \geq 2$, and assume the assertion true for 1 and $p-1$. By induction hypothesis for $p-1$, there exists Ξ_{p-1} of full measure in Ξ such that for all $\xi \in \Xi_{p-1}$, $[t \mapsto u(t\omega + \xi)] \in H^{p-1}_{loc}(\mathbb{R}, \mathbb{R}^N)$ and

$$\frac{d^j}{dt^j}[u(t\omega + \xi)] = (\nabla^j_\omega u)(t\omega + \xi), \quad j \in \{0, \cdots, p-1\}.$$

Since $\nabla_\omega^{p-1}u \in H^1_\omega(\mathbb{T}^m, \mathbb{R}^N)$, by induction hypothesis for the rank 1, there exists Ξ^* of full measure in Ξ such that for all $\xi \in \Xi^*$, $[t \mapsto (\nabla_\omega^{p-1}u)(t\omega + \xi)] \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$ and :

$$\frac{d}{dt}[(\nabla_\omega^{p-1}u)(t\omega + \xi)] = (\nabla_\omega^{p}u)(t\omega + \xi), \quad j \in \{0, \dots, p-1\}$$

which gives the rank p as on the set $\Xi_p := \Xi_{p-1} \cap \Xi^*$ we have

$$\frac{d}{dt}[(\nabla_\omega^{p-1}u)(t\omega + \xi)] = \frac{d}{dt}\left[\frac{d^{p-1}}{dt^{p-1}}[u(t\omega + \xi)]\right] = \frac{d^p}{dt^p}[u(t\omega + \xi)].$$

Second assertion. Using the same technique as in the proof of the previous lemma and the positive version of Fubini's theorem, we see that all

$$\{\xi \in \Xi : |u(t\omega + \xi)| \leq \|u\|_\infty\}$$

is of full measure in Ξ, and so the assertion 2. results from the first one and taking into account the fact that the intersection of two sets of full measures is of full measure, too. □

10. Traces Theory

10.1. **Description of the boundary of Q^m.** The assertion **2.** of Lemma 3.1 describes the border of the cube. We can decompose it into parts of dimensions $k = 0$ to $m-1$. The part of dimension k is :

$$\{p \in \partial Q^m : \text{card}\{j : |p_j| = \pi\} = m - k\}.$$

We denote $\mathcal{F}^m$ the part (open faces) of dimension $m-1$. It corresponds to the regular border of Q^m (cf. [5, p. 77] and [6, p. 95]). Denoting, for $(i,j) \in \{1, \dots, m\} \times \{1, 2\}$

$$F^i_j := (-\pi, \pi)^{i-1} \times \{(2j-3)\pi\} \times (-\pi, \pi)^{m-i},$$

we then have

$$\mathcal{F}^m = \bigcup_{i,j} F^i_j.$$

We introduce the following notations.

- $R(\partial Q^m) := \partial Q^m + 2\pi\mathbb{Z}^m$ (network generated by ∂Q^m).
- If $p \in \mathcal{F}^m$, we denote $\omega(p) := \varepsilon(p)\frac{\omega}{|\omega|}$ where $\varepsilon(p)$ is equal to 1 or -1 so that $\omega(p)$ is returning at p in Q^m. F^i_j being a relative open, this has a good sense (If ω and $-\omega$ were simultaneously leaving (or returning) in p, ω would be tangent, which is contradicted by the freedom of its components). If $p, q \in F^i_j$, $\omega(p) = \omega(q)$, we note $\omega_{i,j}$ the common vector.
- We define an involution ρ on ∂Q^m by setting

$$\rho(-\pi, x_{-j}) = (\pi, x_{-j}) \quad \text{and} \quad \rho(\pi, x_{-j}) = (-\pi, x_{-j}).$$

  We remark that $\rho(F^i_j) = F^i_{3-j}$ and we call that these faces are opposite.

Remark 10.1. $\omega(\rho(p)) = -\omega(p)$ and so $\omega_{i,3-j} = -\omega_{i,j}$.

Lemma 10.1. *There exists $\gamma_0 > 0$ such that if $\gamma \in (0, \gamma_0]$, we have the following.*

- *At least one of the intersections is empty*
$$\left(p, p+\gamma\frac{\omega}{|\omega|}\right)\cap R(\partial Q^m), \quad \left(p, p-\gamma\frac{\omega}{|\omega|}\right)\cap R(\partial Q^m).$$
- *If* $(p, p+\gamma\omega(p))\cap R(\partial Q^m)\neq\emptyset$, *then* $(\rho(p), \rho(p)-\gamma\omega(p))\cap R(\partial Q^m)=\emptyset$.
- $co\{F^i_j; F^i_j+\gamma\omega_{i,j}\}\cap co\{F^i_{3-j}; F^i_{3-j}+\gamma\omega_{i,3-j}\}=\emptyset$, *where* $co\{A, B\} := \{\lambda a+(1-\lambda)b : (\lambda, a, b)\in[0,1]\times A\times B\}$.

Proof. Let $\tilde{\omega} := \frac{\omega}{|\omega|}$.

First step. Let
$$\gamma(p) := \sup\Big\{\gamma>0 : (p, p-\gamma\tilde{\omega})\cap R(\partial Q^m)=\emptyset \text{ or } (p, p+\gamma\tilde{\omega})\cap R(\partial Q^m)=\emptyset\Big\}$$
and $\gamma_1 := \inf_{p\in\mathcal{F}^m}\gamma(p)$. It is clear that $\gamma(p)$ is the upper bound of a nonempty set plus strictly positive reals, it is then into $\mathbb{R}_+\setminus\{0\}$. We will find a strictly positive real lowering all the $\gamma(p)$, which shows that $\gamma_1>0$.

For $\gamma(p)$, let us introduce
$$\gamma_j(p) := \sup\Big\{\gamma>0 : (p_j, p_j-\gamma\tilde{\omega}_j)\cap(\pi+2\pi\mathbb{Z})=\emptyset \text{ or } (p_j, p_j+\gamma\tilde{\omega}_j)\cap(\pi+2\pi\mathbb{Z})=\emptyset\Big\}.$$
Since $(p\in R(\partial Q^m))$ if and only if exists j, $p_j\in\pi+2\pi\mathbb{Z}$, we have $\gamma(p)\geq\min_j\gamma_j(p)$.

Let $p\in\mathcal{F}^m$ and assume without loss of generality that $p_1=\pi$. We calculate $\gamma_1(p)=\frac{\pi}{|\tilde{\omega}_1|}$ and if $j\geq 2$,
$$\gamma_j(p)=\frac{\max\{d(p_j;\pi+2\pi\mathbb{Z}); d(2\pi-p_j;\pi+2\pi\mathbb{Z})\}}{|\tilde{\omega}_j|}\geq\frac{\pi}{|\tilde{\omega}_j|}.$$
We conclude that
$$\gamma(p)\geq\min_{1\leq j\leq m}\frac{\pi}{|\tilde{\omega}_j|}.$$
Therefore, we have proved that $\gamma_1>0$, and all $\gamma_0\leq\gamma_1$ satisfying the first condition.

Second step. We shall show that $\gamma_0\leq\gamma_1$ satisfies the second condition.

In fact, the first condition being verified, if $\gamma\leq\gamma_1$ and if $(p, p+\gamma\omega(p))\cap R(\partial Q^m)$ is non empty, then $(\rho(p), \rho(p)-\gamma\omega(p))\cap R(\partial Q^m)$ is empty.

Third step. Let us fulfill the last condition.

We set $B^i_j(\gamma) := co\{F^i_j; F^i_j+\gamma\omega_{i,j}\}$. $B^i_j(\gamma)$ and $B^i_{3-j}(\gamma)$ are two parallel bands, of width less than $\gamma|\omega|$; they don't intersect if $2\gamma|\omega|<2\pi$. Setting $\gamma_2 := \frac{\pi}{2|\omega|}$, we ensure that
$$(\gamma\leq\gamma_2)\Rightarrow(B^i_j(\gamma)\cap B^i_{3-j}(\gamma)=\emptyset).$$
Thus, every $\gamma\leq\gamma_2$ permits to fulfill the third condition.

We may then set $\gamma_0 := \min\{\gamma_1;\gamma_2\}$ to conclude. ☐

Notations 10.1. We introduce now the following notations:

- $K^i_j := \{p\in F^i_j : (p, p+\gamma\omega_{i,j})\cap\partial Q^m=\emptyset\}$;

- $L_j^i := F_j^i \cap^c K_j^i$;
- $S_j^i := K_j^i + \omega_{i,j}[0, \gamma]$;
- $K := \cup_{i,j} K_j^i$;
- $L := \cup_{i,j} L_j^i$;
- $S := \cup_{i,j} S_j^i$.

Remark 10.2. The last condition of the lemma ensures that $\rho(L_j^i) \subset F_{3-j}^i$.

10.2. **Integration on the cube boundary.** For sake of simplicity, we introduce the following notations.

Notations 10.2. For $u \in C^0(\partial Q^m, \mathbb{E})$ and $(i,j) \in \mathbb{N}_m \times \mathbb{N}_2$, we put

- $I_j^i(u) := \int_{[-\pi,\pi]^{m-1}} u((2j-3)\pi, x_{-i}) dx_{-i}$;
- $\int_{\partial Q^m} u d\sigma_i := I_2^i(u) - I_1^i(u)$;
- $\int_{\partial Q^m} u d\sigma_\omega := \sum_{i=1}^m \omega_i \int_{\partial Q^m} u d\sigma_i$.

By density, these continuous linear forms extend to $L^1(\partial Q^m, \mathbb{E})$. We set finally, for $u \in L^2(\partial Q^m, \mathbb{E})$

$$\|u\|_{L^2(\partial Q^m, \mathbb{E})} := \sqrt{\sum_{1 \le i,j \le m} I_j^i(|u|_{\mathbb{E}}^2)}.$$

Lemma 10.2. *Let $f \in C^1(Q^m, \mathcal{A})$ and $g \in C^1(Q^m, \mathcal{B})$ with $\mathcal{C} = \mathbb{K}$. We have the following.*

1. *For all i*

$$\int_{Q^m} \frac{\partial f}{\partial x_i} \diamond g = -\int_{Q^m} f \diamond \frac{\partial g}{\partial x_i} + \int_{\partial Q^m} f \diamond g d\sigma_i.$$

2.

$$\int_{Q^m} (d_\omega f) \diamond g = -\int_{Q^m} f \diamond (d_\omega g) + \int_{\partial Q^m} f \diamond g d\sigma_\omega.$$

Proof. This is to use the Stokes formula for Q^m. This is allowed (cf. [16, p. 343] or [6, Chap. XXIV, n. 14]), and we have then, applying this formula to the differential form $f.g \bigwedge_{j \ne i} dx_j$

$$\int_{Q^m} \frac{\partial}{\partial x_i}(f \diamond g) = \int_{\partial Q^m} f \diamond g d\sigma_i,$$

and so that the first assertion is obtained by developing the derivative of product. From there, the second assertion is immediate by linearity. □

10.3. **Traces operators.** We introduce the operator of traces $\tilde{T}_0 : C^1(Q^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E})$, which canonically extends into an operator $T_0 : C^1(\mathbb{T}^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E})$.

10.3.1. *An intermediate estimate.*

Proposition 10.1. *There exists a constant $C > 0$ such that, for all $u \in C^1(\mathbb{T}^m, \mathbb{E})$, we have*

$$\|T_0(u)\|_{L^2(\partial Q^m,\mathbb{E})} \leq C\|u\|_{1,\omega}.$$

To prove this proposition, we will start by proving the following lemma.

Lemma 10.3. *There exists a constant $C_0 > 0$ such that, for all $u \in C^1(\mathbb{T}^m, \mathbb{E})$, we have :*

$$\int_K |T_0(u)|^2_{\mathbb{E}} d\sigma_\omega \leq C_0^2 \int_S \left[|u|^2_{\mathbb{E}} + |d_\omega u|^2_{\mathbb{E}}\right].$$

Proof. Fix u, and let i, j be given.

We choose on F^i_j a system of local coordinates (ξ, η), where $\xi \in \mathbb{R}^{m-1}$ is tangent to K^i_j and η is the coordinate following $\omega_{i,j}$, such that

$$K^i_j \subset \left\{(\xi, 0) : \xi \in \mathbb{R}^{m-1}\right\}.$$

For the sake of simplicity, noting by $u_\eta := \frac{\partial u}{\partial \eta}$, we have for $t \in [0; \gamma]$

$$u(\xi, 0) = \int_t^0 u_\eta(\xi, \eta) d\eta + u(\xi, t)$$

and so,

$$|u(\xi, 0)|^2_{\mathbb{E}} \leq 2\gamma \int_0^t |u_\eta(\xi, \eta)|^2_{\mathbb{E}} d\eta + 2|u(\xi, t)|^2_{\mathbb{E}},$$

and thus,

$$|u(\xi, 0)|^2_{\mathbb{E}} \leq 2\gamma \int_0^\gamma |u_\eta(\xi, \eta)|^2_{\mathbb{E}} d\eta + 2|u(\xi, t)|^2_{\mathbb{E}}.$$

We integrate over t between 0 and γ, to obtain

$$\gamma |u(\xi, 0)|^2_{\mathbb{E}} \leq 2 \int_0^\gamma \left(\gamma^2 |u_\eta(\xi, \eta)|^2_{\mathbb{E}} + |u(\xi, \eta)|^2_{\mathbb{E}}\right) d\eta.$$

Integrate overt ξ on K^i_j. It comes

$$\int_{K^i_j} |u(\xi, 0)|^2_E d\xi \leq \frac{2}{\gamma} \int_{K^i_j} \int_0^\gamma \left(\gamma^2 |u_\eta(\xi, \eta)|^2_{\mathbb{E}} + |u(\xi, \eta)|^2_{\mathbb{E}}\right) d\eta d\xi.$$

Let $\Delta_{i,j}$ be the absolute value of the Jacobian for the transformation $(\xi, \eta) \mapsto (x_1, \ldots, x_m)$. We have

$$\int_{K^i_j} |u(\xi, 0)|^2_{\mathbb{E}} d\xi \leq \frac{2}{\gamma} \Delta_{i,j} \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\} \int_{S^i_j} |u_\eta|^2_{\mathbb{E}} + |u|^2_{\mathbb{E}}.$$

Multiply by ω_j and sum over (i, j).

Denoting $\Delta := \max\limits_{(i,j)}\{\omega_i \Delta_{i,j}\}$ since each point of S is into at most $2m$ sets S^i_j, we have

$$\sum_{i,j} \omega_i \Delta_{i,j} \int_{S^i_j} \leq 2m\Delta \int_S,$$

and finally we get

$$\int_K |u|_{\mathbb{E}}^2 d\sigma_\omega \leq \frac{4}{\gamma} m\Delta \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\} \int_S |u_\eta|_{\mathbb{E}}^2 + |u|_{\mathbb{E}}^2.$$

We can then take

$$C_0 := \sqrt{\frac{4}{\gamma} m\Delta \max\left\{1, \frac{\gamma^2}{|\omega|^2}\right\}}. \qquad \square$$

Proof of Proposition 10.1. By periodicity of u and by Remark 10.2, we have

$$\int_L |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega \leq \int_K |T_0(u))|_{\mathbb{E}}^2 d\sigma_\omega.$$

Since in addition

$$\int_{\partial Q^m} |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega = \int_L |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega + \int_K |T_0(u)|_{\mathbb{E}}^2 d\sigma_\omega,$$

the periodicity of u and the lemma allow to conclude with $C = C_0\sqrt{2}$. $\square$

10.3.2. *Extension to* $H^1_\omega(\mathbb{T}^m, \mathbb{E})$.

Proposition 10.2. *The map* T_0 *can be extended to a linear continuous map*

$$\gamma_0 : H^1_\omega(\mathbb{T}^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E}).$$

Proof. By the previous proposition, the application T_0 is linear and continuous of $(C^1(\mathbb{T}^m, \mathbb{E}); \|\cdot\|_{H^1_\omega(\mathbb{T}^m,\mathbb{E})})$ into $L^2(\partial Q^m, \mathbb{E})$ and since $C^1(\mathbb{T}^m, \mathbb{E})$ is dense in $H^1_\omega(\mathbb{T}^m, \mathbb{E})$, T_0 can be extended in a unique way to a linear continuous application γ_0 from $H^1_\omega(\mathbb{T}^m, \mathbb{E})$ into $L^2(\partial Q^m, \mathbb{E})$. $\square$

Remark 10.3. $\tilde{T}_0$ extends

$$\tilde{\gamma}_0 : H^1_\omega(Q^m, \mathbb{E}) \to L^2(\partial Q^m, \mathbb{E}),$$

which is linear and continuous.

10.4. **Theorem of traces.** The main purpose here is to prove the theorem of traces, which gives

$$H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) = \operatorname{Ker} \gamma_0.$$

Let $u \in L^2(Q^m, \mathbb{E})$. u can be canonically extended to

- $\hat{u} \in L^2(\mathbb{T}^m, \mathbb{E})$;
- $\tilde{u} := \chi_{Q^m}.u$.

Lemma 10.4. *We have the following.*

1. *If* $u \in L^2(Q^m, \mathbb{E})$, *then* $\tilde{u} \in L^2(\mathbb{R}^m, \mathbb{E})$.

2. *If* $u \in H^1_\omega(Q^m, \mathbb{E})$ *and* $\tilde{\gamma}_0(u) = 0$, *then, for all* $\varphi \in C^1(Q^m, \mathbb{K})$, *we have*

$$\int_{Q^m} \varphi \cdot \nabla_\omega u = -\int_{Q^m} (d_\omega \varphi) \cdot u.$$

3. *If* $u \in H^1_\omega(Q^m, \mathbb{E})$ *and* $\tilde{\gamma}_0(u) = 0$, *then* $\tilde{u} \in H^1_\omega(\mathbb{R}^m, \mathbb{E})$ *and* $\nabla_\omega \tilde{u} = \widetilde{\nabla_\omega u}$.

Proof. **1.** Since u is measurable, $\tilde{u}$ is obviously measurable. We have

$$\int_{\mathbb{R}^m} |\tilde{u}|_{\mathbb{E}}^2 \leq \int_{Q^m} |\tilde{u}|_{\mathbb{E}}^2 \leq \int_{Q^m} |u|_{\mathbb{E}}^2 < +\infty,$$

and so that $\tilde{u} \in L^2(\mathbb{R}^m, \mathbb{E})$.

2. There exists $(u_n)_n$ with values into $C^1(Q^m, \mathbb{E})$ converging to u.

Since γ_0 is linear continuous, we see that $(\tilde{\gamma}_0(u_n))_n$ is of Cauchy in $L^2(Q^m, \mathbb{E})$ and so converges, and that the limit is $\tilde{\gamma}_0(u) = 0$.

Stokes formula immediately gives that

$$\int_{Q^m} \varphi d_\omega u_n = -\int_{Q^m} (d_\omega \varphi) u_n + \int_{\partial Q^m} \varphi \cdot \tilde{\gamma}_0(u_n) d\sigma_\omega.$$

Taking the limit, we get the desired result.

3. Let $\varphi \in C^1(Q^m, \mathbb{K})$. We have

$$\int_{R^m} \varphi \cdot \nabla_\omega \tilde{u} = -\int_{R^m} (d_\omega \varphi) \tilde{u},$$

due to derivation within the meaning of distributions; but the first term is

$$-\int_{Q^m} (d_\omega \varphi) u = \int_{Q^m} \varphi . (\nabla_\omega u) = \int_{Q^m} \varphi \cdot \widetilde{(\nabla_\omega u)},$$

because of **2.**

Thus, for all $\varphi \in C^1(Q^m, \mathbb{K})$, we have

$$\int_{R^m} \varphi \cdot \nabla_\omega \tilde{u} = \int_{Q^m} \varphi \cdot \widetilde{(\nabla_\omega u)},$$

which gives the assertion **3.** ☐

Lemma 10.5. *Let $u \in L^2(Q^m, \mathbb{E})$. We define for $\alpha > 1$, $\tilde{u}_\alpha : \mathbb{R}^m \to \mathbb{E}$ by*

$$\tilde{u}_\alpha(x) := \tilde{u}(\alpha x).$$

Then

1. $\tilde{u}_\alpha \in L^2(\mathbb{R}^m, \mathbb{E})$*;*
2. $\operatorname{supp}(\tilde{u}_\alpha) \subset \operatorname{Int} Q^m$*;*
3. $\lim_{\alpha \to 1^+} \|\tilde{u}_\alpha - \tilde{u}\|_{L^2(\mathbb{R}^m, \mathbb{E})} = 0$*;*
4. *If in addition $\tilde{u} \in H_\omega^1(\mathbb{R}^m, \mathbb{E})$, then $\tilde{u}_\alpha \in H_\omega^1(\mathbb{R}^m, \mathbb{E})$ and*

$$\lim_{\alpha \to 1^+} \|\tilde{u}_\alpha - \tilde{u}\|_{H_\omega^1(\mathbb{R}^m, \mathbb{E})} = 0.$$

Proof. **1.** It is a consequence of the previous lemma.

2. Since $\operatorname{supp}(\tilde{u}) \subset Q^m$, it comes that

$$\operatorname{supp}(\tilde{u}_\alpha) \subset \frac{1}{\alpha} Q^m \subset \operatorname{Int} Q^m.$$

3. Suppose, first, that u is in addition continuous.

There exists then in $\mathbb{R}$ the number $M := \sup_{x\in\mathbb{R}^m} |\tilde{u}(x)|_{\mathbb{E}}$. Besides,

$$\|\tilde{u}_\alpha - \tilde{u}\|^2_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \int_{Q^m} |\tilde{u}_\alpha(x) - \tilde{u}(x)|^2_{\mathbb{E}} dx.$$

For a fixed x, $|\tilde{u}_\alpha(x) - \tilde{u}(x)|^2_{\mathbb{E}}$ tends to 0 as α tends to 1, and this function is less than the constant $4M^2$, which is integrable on Q^m. Lebesgue's dominated convergence theorem allows us to conclude.

Now, let us move on to the general case. Let us fix $\varepsilon > 0$. By density, there exists $\varphi \in C^0(Q^m, \mathbb{E})$ such that $\|\tilde{u} - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/3$. Since φ is continuous, there exists $\alpha_0 > 1$ such that if $\alpha \in (1, \alpha_0)$, we have $\|\tilde{\varphi}_\alpha - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/3$. We have then if $\alpha \in (1, \alpha_0)$

$$\|\tilde{u}_\alpha - \tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \|\tilde{\varphi} - \tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|\tilde{\varphi}_\alpha - \tilde{\varphi}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|\tilde{\varphi}_\alpha - \tilde{u}_\alpha\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon.$$

4. Due to Lemma 10.4, we know that $\tilde{u}_\alpha \in H^1_\omega(\mathbb{R}^m, \mathbb{E})$. By **3**, it suffices to show that

$$\lim_{\alpha\to 1^+} \|\partial_\omega (\tilde{u}_\alpha - \tilde{u})\|_{L^2(\mathbb{R}^m,\mathbb{E})} = 0.$$

But, we have

$$\|\partial_\omega (\tilde{u}_\alpha - \tilde{u})\|_{L^2(\mathbb{R}^m,\mathbb{E})} \leq \|(\partial_\omega \tilde{u})_\alpha - \partial_\omega \tilde{u}\|_{L^2(\mathbb{R}^m,\mathbb{E})} + \|(\partial_\omega \tilde{u})_\alpha - \partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})}.$$

Besides, by **3**, the first term of the right hand side tends to 0. For the second, we may write

$$\|(\partial_\omega \tilde{u})_\alpha - \partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})} = (\alpha^m - 1)\|\partial_\omega(\tilde{u}_\alpha)\|_{L^2(\mathbb{R}^m,\mathbb{E})},$$

which is the product of a term tending to 0 by a term bounded at the neighborhood to the right of 1, so the limit is 0, which ends the proof of the lemma. ☐

Theorem 10.1 (Theorem of traces)**.** *We have* $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) = \operatorname{Ker}\gamma_0$.

Proof. We shall prove firstly that $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) \subset \operatorname{Ker}\gamma_0$.

Let $\hat{u} \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$, $u \in H^1_{\omega,0}(Q^m, \mathbb{E})$ associated to and $(\varphi_n)_n$ be a sequence of $C^1_c(\mathbb{T}^m, \mathbb{E})$ converging to u in $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$. We have for all integers n

$$\gamma_0(\varphi_n) = T_0(\varphi_n) = 0,$$

and so by continuity of γ_0, we have $\gamma_0(u) = 0$, and so that $\gamma_0(\hat{u}) = 0$ and the inclusion is therefore proven.

Conversely, we aim to prove that $H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E}) \supset \operatorname{Ker}\gamma_0$.

Let $\hat{u} \in \operatorname{Ker}\gamma_0$, $u \in H^1_\omega(Q^m, \mathbb{E})$ associated and $\varepsilon > 0$. By Lemma 10.5, there exists $\alpha_0 > 1$ such that $\|\tilde{u} - \tilde{u}_{\alpha_0}\|_{H^1_\omega(\mathbb{R}^m,\mathbb{E})} \leq \varepsilon/2$. Let $(\rho_n)_n$ be a regularizing sequence. Then, for all n, $\rho_n * \tilde{u}_{\alpha_0} \in C^1_{c,\omega}(\mathbb{R}^m, \mathbb{E})$.

Moreover, since $\operatorname{supp}(\tilde{u}_{\alpha_0}) \subset \operatorname{Int} Q^m$ and $\operatorname{diam}(\operatorname{supp}\rho_n)$ tends to 0 as n goes to $+\infty$, we know that for n large enough, $\operatorname{supp}(\rho_n * \tilde{u}_{\alpha_0}) \subset \operatorname{Int} Q^m$. There exists a $\varphi \in C^1_{c,\omega}(Q^m, \mathbb{E})$ such that $\|\varphi - \tilde{u}_{\alpha_0}\|_{1,\omega} \leq \varepsilon/2$. Finally, we get $\|\varphi - u\|_{1,\omega} \leq \varepsilon$, which proves that $u \in H^1_{\omega,0}(Q^m, \mathbb{E})$, i.e., $\hat{u} \in H^1_{\omega,0}(\mathbb{T}^m, \mathbb{E})$. ☐

11. Conclusion

In this work, we have completely studied the relations between functions on the torus and the functions defined on the m-dimensional cube $Q = [-\pi, \pi]^m$.

We have in particular presented the spaces derived from Percival's formalism and adapted to them the usual results. We have noticed that whether some results extend, some do not: for example, the Rellich-Kondrachov theorem is no longer valid here.

This study has a number of direct and indirect applications in the search for almost/quasi- periodic solutions of an ordinary differential equation and transforming it to the search of periodic solutions in each variable of a partial differential equation.

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