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STUDY OF A STOCHASTIC DIFFERENTIAL SYSTEM OF ARBITRARY ORDER UNDER G-BROWNIAN MOTION

EL-HACÈNE CHALABI[1], AMAR OUAOUA[2], AND SALIM MESBAHI[3]

Abstract. In this paper, we study the existence and uniqueness of the solution for a class of stochastic differential systems of arbitrary order driven by G-Brownian motion. We prove under certain suitable conditions that our system has a unique solution. We also prove a stability theorem for our system.

1. Introduction

The theory of nonlinear expectations are a generalization of the classical mathematical concept of expectation. Unlike the classical expectation, which is linear, the nonlinear expectation allows for nonlinearity, which makes it a useful tool in modeling situations involving uncertainty and risk. Nonlinear expectations have found many applications in finance, where they are used to model and measure risk. In particular, they are used in the context of super-hedging, which is a risk management strategy used to minimize the potential losses of an investment portfolio. By using them, investors can account for the possibility of extreme market events, which may not be captured by traditional linear models. They also have applications in other fields, such as decision theory, statistics, and machine learning. They are used to model situations where the outcome depends on a combination of factors, rather than just a single factor, and where there is uncertainty about the relationship between these factors and the outcome. We find models and applications in different fields in Denis et al. [2], Y. Lin [9], Peng [13], Ren et al. [15, 16], Soumana-Hima [17], Yang [18] and

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corresponding references therein; where we also find techniques and methods used to discuss such problems.

Peng [14] (for more details see Peng [11]-[13]) introduced the theory of nonlinear expectation, G-Brownian motion and defined the related stochastic calculus, especially stochastic integrals of Itô's type with respect to G-Brownian motion and derived the related Itô's formula. Also, the notion of G-normal distribution plays the same important role in the theory of nonlinear expectation as that of normal distribution with the classical probability. Gao [5] studied pathwise properties and homeomorphic property with respect to the initial values for stochastic differential equations driven by G-Brownian motion. Later Faizullah et al. extended this theory, see for example [3] and [4].

The existence and uniqueness theorem for some stochastic differential equations under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients was developed by Peng and Gao. This theorem is established by using the stated method under the Lipschitz and the linear growth conditions.

$$
\begin{aligned}
X(t) = & X(0) + \int_0^t f(s, X(s))\, ds + \sum_{i,j=1}^{d} \int_0^t g_{i,j}(s, X(s))\, d\left\langle B^i, B^j \right\rangle (s) \\
& + \sum_{j=1}^{d} \int_0^t h_j(s, X(s))\, dB^j(s), \quad t \in [0, T],
\end{aligned} \tag{1.1}
$$

where T is a positive constant.

The existence and uniqueness of the solution $X(t)$ for G-SDEs (1.1) under different conditions was proved by Bai and Y. Lin [1], Faizullah [3,4], Gao [5], Q. Lin [7], Y. Lin [8], Peng and Falei [10], Ren et al. [16], Zhang and Chen [19]. In this paper, we study the existence, uniqueness and stability of the solution for the following stochastic differential system driven by G-Brownian motion (SG-SDEs)

$$
\left\{
\begin{aligned}
X_1(t) = \; & X_1(0) + \textstyle\int_0^t f_1(s, X_1(s), \ldots, X_n(s))\, ds \\
& + \sum_{i,j=1}^{d} \textstyle\int_0^t f_{1,i,j}(s, X_1(s), \ldots, X_n(s))\, d\langle B^i, B^j \rangle (s) \\
& + \sum_{j=1}^{d} \textstyle\int_0^t f_{1,j}(s, X_1(s), \ldots, X_n(s))\, dB^j(s), \\
& \vdots \\
X_n(t) = \; & X_n(0) + \textstyle\int_0^t f_n(s, X_1(s), \ldots, X_n(s))\, ds \\
& + \sum_{i,j=1}^{d} \textstyle\int_0^t f_{n,i,j}(s, X_1(s), \ldots, X_n(s))\, d\langle B^i, B^j \rangle (s) \\
& + \sum_{j=1}^{d} \textstyle\int_0^t f_{n,j}(s, X_1(s), \ldots, X_n(s))\, dB^j(s),
\end{aligned}
\right. \tag{1.2}
$$

where $(X_1(0), \ldots, X_n(0))$ is a given initial condition, $(\langle B^i, B^j \rangle (t))_{t \geq 0}$ is the quadratic variation process of the G-Brownian motion $(B(t))_{t \geq 0}$ and all $f_k(t, x_1, \ldots, x_n)$,

$f_{k,i,j}(t, x_1, \ldots, x_n)$, $f_{k,j}(t, x_1, \ldots, x_n)$ for $t \in [0, T]$, $k = 1, 2, \ldots, n$ and $i, j = 1, 2, \ldots, d$ are the integral-Lipschitz coefficients with respect to $(x_1, \ldots, x_n)$.

The paper is organized as follows. In the following section, we provide some definitions, remarks and lemmas necessary to fully understand the content of this work. The third section is devoted to our first contribution, where we prove the existence and uniqueness of the solution of System (1.2). The last section is devoted to our second contribution, where we prove another important result on the stability of solutions.

2. Preliminaries

In this section, we recall some of the basic concepts, definitions, and lemmas that we will use in this work. More details can be found in Gao [5], Hu and Li [6].

Let Ω be a given non-empty set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant is an element of $\mathcal{H}$, and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 2.1 ([14])**.** A functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if for all X, Y in $\mathcal{H}$, c in $\mathbb{R}$ and $\lambda \geq 0$, the following properties are satisfied:
(i) (Monotonicity): if $X \leq Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$;
(ii) (Constant preserving): $\mathbb{E}[c] = c$;
(iii) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$;
(iv) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.
The triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is called sublinear expectation space.

We assume that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$, then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ is the space of linear functions φ defined as follows, for all $x, y \in \mathbb{R}^n$

$$C_{l,\mathrm{Lip}}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R} : |\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|\},$$

where C is a positive constant and $m \in \mathbb{N}^\star$ dependent only on φ.

Definition 2.2 ([13])**.** Let X, Y be two n-dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called identically distributed, denoted by $X \overset{d}{=} Y$, if $\mathbb{E}_2[\varphi(Y)] = \mathbb{E}_1[\varphi(X)]$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$.

Definition 2.3 ([6])**.** In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ if

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}], \quad \text{for all } \varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

$\tilde{X}$ is called an independent copy of X, if $\tilde{X} \overset{d}{=} X$ and $\tilde{X}$ is independent from X.

Remark 2.1. Under a sublinear expectation space, Y is independent from X means that the distributional uncertainty of Y does not change after the realization of $X = x$. Or, in other words, the conditional sublinear expectation of Y knowing X is

$\mathbb{E}\left[\varphi\left(x,Y\right)\right]_{x=X}$. In the case of linear expectation, this notion of independence is just the classical one.

Remark 2.2. It is important to note that Y is independent from X does not imply that X is independent from Y.

Let S^d be the space of $d\times d$ symmetric matrices. Γ is a given non-empty, bounded and closed subset of S^d. For $A=\left(A_{i,j}\right)_{i,j=1}^{d}\in S^d$ given, we define $G:S^d\to\mathbb{R}$ by

$$G\left(A\right)=\frac{1}{2}\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\gamma\gamma^{Tr}A\right),$$

where γ^{Tr} is the transpose matrix of γ, and $\operatorname{tr}\left(\gamma\gamma^{Tr}A\right)$ is the trace of a matrix $\left(\gamma\gamma^{Tr}A\right)$.

Definition 2.4 ([8])**.** In a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$, a d-dimensional vector of random variables $X\in\mathcal{H}^d$ is G-normal distributed, if for each $\varphi\in C_{l,Lip}(\mathbb{R}^d)$, the function $u\left(t,x\right)=\mathbb{E}\left(\varphi\left(x+\sqrt{t}X\right)\right)$ is the unique viscosity solution of the following parabolic equation called the G-heat equation

$$\frac{\partial u}{\partial t}=G\left(D^2u\right),\qquad\text{with } u\left(0,x\right)=\varphi\left(x\right),\left(t,x\right)\in\mathbb{R}_+\times\mathbb{R}^d,$$

where $D^2u=\left(\partial^2_{x_ix_j}u\right)_{i,j}^{d}$ is the Hessian matrix of u.

Remark 2.3. In fact, if $d=1$ we have $G\left(\alpha\right)=\frac{1}{2}\left(\overline{\sigma}^2\alpha^+-\underline{\sigma}^2\alpha^-\right)$, where $\overline{\sigma}^2=\mathbb{E}[X^2]$, $\underline{\sigma}^2=-\mathbb{E}[-X^2]$, $\alpha^+=\max\left\{\alpha,0\right\}$ and $\alpha^-=\max\left\{-\alpha,0\right\}$. We write $X\sim\mathcal{N}\left(0;[\underline{\sigma}^2,\overline{\sigma}^2]\right)$.

Definition 2.5 ([14])**.** A process $\left(B\left(t\right)\right)_{t\geq0}$ in a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied:

(i) $B\left(0\right)=0$;

(ii) for each $t,s\geq0$, the increment $B\left(t+s\right)-B\left(t\right)$ is $\mathcal{N}(0;[\underline{\sigma}^2s,\overline{\sigma}^2s])$-distributed and is independent from $\left(B\left(t_1\right),\ldots,B\left(t_n\right)\right)$ for each $n\in\mathbb{N}$ and $0\leq t_1\leq\cdots\leq t_n\leq t$.

Remark 2.4. For any $a\in\mathbb{R}^d$, $B^a\left(t\right):=\sum\limits_{k=1}^{d}a_kB^k\left(t\right)$ is a one-dimensional G_a-Brownian motion, where

$$G_a\left(\beta\right)=\frac{1}{2}\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\beta\gamma\gamma^{Tr}aa^{Tr}\right)=\frac{1}{2}\left(\sigma_{aa^{Tr}}\beta^++\sigma_{-aa^{Tr}}\beta^-\right),\quad\beta\in\mathbb{R},$$

and

$$\sigma_{aa^{Tr}}=\sup_{\gamma\in\Gamma}\operatorname{tr}\left(\gamma\gamma^{Tr}aa^{Tr}\right),\quad\sigma_{-aa^{Tr}}=-\sup_{\gamma\in\Gamma}\operatorname{tr}\left(-\gamma\gamma^{Tr}aa^{Tr}\right).$$

We denote by $\Omega=C_0(\mathbb{R})$ the space of all $\mathbb{R}$-valued continuous functions ω defined on $\mathbb{R}_+$ such that $\omega\left(0\right)=0$, equipped with the distance

$$\rho(\omega_1,\omega_2)=\sum_{k=1}^{+\infty}\frac{1}{2^k}\max_{t\in[0,k]}\left[\left|\left(\omega_1\left(t\right)-\omega_2\left(t\right)\right)\right|\wedge1\right].$$

For each fixed $T > 0$, let $\Omega_T = \{\omega\left(._{\wedge T}\right) : \omega \in \Omega\}$,

$\mathrm{Lip}(\Omega_T) = \{\varphi(B\left(t_1\right), \ldots, B\left(t_m\right)) : m \geq 1,\ t_1, \ldots, t_m \in [0, T],\ \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^m)\}$,

and let

$$\mathrm{Lip}(\Omega) = \bigcup_{n=1}^{+\infty} \mathrm{Lip}\left(\Omega_n\right).$$

Peng [13] constructs a sublinear expectation $\mathbb{E}$ on $(\Omega, \mathrm{Lip}(\Omega))$ under which the canonical process $\left(B\left(t\right)\right)_{t\geq 0}$ (i.e., $B(t, \omega) = \omega\left(t\right)$) is a G-Brownian motion. In what follows, we consider this G-Brownian motion.

We denote by $L^p_G(\Omega_T)$, $p \geq 1$, the completion of $\mathrm{Lip}(\Omega_T)$ under the norm $||X||_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$. Similarly, we denote $L^p_G(\Omega)$ the completion of $\mathrm{Lip}(\Omega)$. We can represent this sublinear expectation by the following theorem.

Theorem 2.1 ([13]). *In a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$, *a sublinear expectation* $\mathbb{E}\left[\cdot\right]$ *has the following representation: there exist a family of probability measures* $\mathcal{P}$ *on* Ω *such that*

$$\mathbb{E}\left(X\right) = \sup_{P\in\mathcal{P}} E^P\left[X\right], \quad \text{for } X \in L^1_G\left(\Omega\right),$$

where E^P *stands for the linear expectation under the probability* P.

For a finite partition of $[0, T]$, $\pi_T = \{t_0, t_1, \ldots, t_N\}$, we set

$$\mu(\pi_T) = \max\{|t_{k+1} - t_k| : k = 0, 1, \ldots, N - 1\}.$$

Consider the collection $M^{p,0}_G\left(0, T\right)$ of simple processes defined by

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t),$$

where $\xi_k \in L^p_G(\Omega_{t_k})$, $k = 0, 1, \ldots, N - 1$, and $p \geq 1$.

The completion of $M^{p,0}_G\left(0, T\right)$ under the norm

$$\|\eta\| = \left(\frac{1}{T} \int_0^T \mathbb{E}\left(|\eta_t|^p\right) dt\right)^{\frac{1}{p}}$$

is denoted by $M^p_G(0, T)$. Note that $M^q_G(0, T) \subset M^p_G(0, T)$ for $1 \leq p \leq q$.

Definition 2.6. For each $\eta \in M^{2,0}_G(0, T)$, the G-Itô's integral is defined by

$$I(\eta) = \int_0^T \eta\left(s\right) dB^a\left(s\right) := \sum_{k=0}^{N-1} \xi_k(B^a\left(t_{k+1}\right) - B^a\left(t_k\right)).$$

The mapping $\eta \mapsto I(\eta)$ can be extended continuously to $M^2_G(0, T)$.

Definition 2.7. The increasing continuous process $\left(\langle B^a\rangle\left(t\right)\right)_{t\geq 0}$, with $\langle B^a\rangle\left(0\right) = 0$ defined by

$$\langle B^a\rangle\left(t\right) := \lim_{\mu\left(\pi^N_t\right)\to 0} \sum_{k=0}^{N-1} \left(B^a\left(t^N_{k+1}\right) - B^a\left(t^N_k\right)\right)^2 = \left(B^a\left(t\right)\right)^2 - 2\int_0^t B^a\left(s\right) dB^a\left(s\right)$$

is called the quadratic variation process of $(B^a(t))_{t\geq 0}$.

Definition 2.8. Define a mapping $Q_{0,T} : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ as follows

$$Q_{0,T}(\eta) = \int_0^T \eta(s)\, d\langle B^a\rangle(s) := \sum_{k=0}^{N-1} \xi_k(\langle B^a\rangle(t_{k+1}) - \langle B^a\rangle(t_k)).$$

Then $Q_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$Q_{0,T}(\eta) = \int_0^T \eta(s)\, d\langle B^a\rangle(s), \quad \eta \in M_G^1(0,T).$$

Burkholder-Davis-Gundy (BDG) inequalities play an important role in the study of G-stochastic differential equations. There has been an increased interest in the following lemmas, see Gao [5].

Lemma 2.1 ([5])**.** *Let $p \geq 1$, $\eta \in M_G^p(0,T)$, a, $\overline{a} \in \mathbb{R}^d$ and $0 \leq s \leq t \leq T$. Then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, d\left\langle B^a, B^{\overline{a}}\right\rangle(r)\right|^p\right]\\ \leq &\left(\frac{\sigma_{(a+\overline{a})(a+\overline{a})^{Tr}} + \sigma_{(a-\overline{a})(a-\overline{a})^{Tr}}}{4}\right)^p (t-s)^{p-1}\int_s^t \mathbb{E}\left[|\eta(u)|^p\right] du.\end{aligned}$$

Lemma 2.2 ([5])**.** *Let $p \geq 2$, $\eta \in M_G^p(0,T)$, $a \in \mathbb{R}^d$ and $0 \leq s \leq t \leq T$. Then*

$$\mathbb{E}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, dB^a(r)\right|^p\right] \leq C_p \sigma_{aa^{Tr}}^{\frac{p}{2}} |t-s|^{\frac{p}{2}-1}\int_s^t \mathbb{E}\left[|\eta(u)|^p\right] du,$$

where $C_p > 0$ is a constant independent of η and a.

In the following, we also need the following two important lemmas.

Lemma 2.3 (Bihari's inequality, [1])**.** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and increasing function that $\varphi(0^+) = 0$ and $\int_0^1 \frac{ds}{\varphi(s)} = +\infty$. Let u be a measurable and nonnegative function defined on $\mathbb{R}_+$ that satisfies*

$$u(t) \leq a + \int_0^t \alpha(s)\varphi(u(s))\, ds,$$

where $a \in \mathbb{R}^+$ and α is a positive function and Lebesgue integrable. We have the following.

i) If $a = 0$, then $u(t) = 0$, $t \in \mathbb{R}^+$.

ii) If $a > 0$, then

$$u(t) \leq v^{-1}\left(v(a) + \int_0^t \alpha(s)\, ds\right),$$

where

$$v(t) := \int_0^t \frac{ds}{\varphi(s)}, \quad t \in \mathbb{R}^+.$$

Lemma 2.4 ([11])**.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing concave function, then for each $X \in L_G^1(\Omega)$ and $t \geq 0$, we have the following Jensen inequality holds*

$$\varphi(\mathbb{E}[X \mid \Omega_t]) \geq \mathbb{E}[\varphi(X) \mid \Omega_t].$$

3. Existence and Uniqueness Result

In this section, we will present our first contribution to this paper, which results from the study of the existence and uniqueness of the solution of SG-SDEs (1.2), where $(X_1(0), \ldots, X_n(0)) \in \left(\mathbb{R}^d\right)^n$, and for $k = 1, \ldots, n$ and $i, j = 1, \ldots, d$, $f_k, f_{k,i,j}, f_{k,j} \in M_G^2\left(0, T; \mathbb{R}^d\right)$ the completion of the collection $M_G^2\left(0, T; \mathbb{R}^d\right)$ of simple processes defined by

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}[}(t), \quad \omega \in \Omega^d,$$

under the norm

$$\|\eta\| = \left(\frac{1}{T} \int_0^T \mathbb{E}\left(|\eta_t|^2\right) dt\right)^{\frac{1}{2}},$$

where $\xi_k \in L_G^2(\Omega_{t_k})$, $k = 0, 1, \ldots, N-1$.

We assume the following assumptions (A1) and (A2) about $J = f_k, f_{k,j}$ or $f_{k,i,j}$, $k = 1, \ldots, n$ and $i, j = 1, \ldots, d$.

(A1)

$$|J(t, x_1, x_2, \ldots, x_n)|^2 \leq |\alpha_1(t)|^2 + \alpha_2^2 \left(\sum_{k=1}^n |x_k|^2\right),$$

for each $x_1, \ldots, x_n \in \mathbb{R}^d$, $t \in [0, T]$, $\alpha_1 \in M_G^2(0, T)$ and $\alpha_2 \in \mathbb{R}_+$.

(A2)

$$|J(t, x_1, \ldots, x_n) - J(t, y_1, \ldots, y_n)|^2 \leq |\alpha(t)|^2 \varphi\left(\sum_{k=1}^n |x_k - y_k|^2\right),$$

for each $x_1, y_1, \ldots, x_n, y_n \in \mathbb{R}^d$ and $t \in [0, T]$, α is a positive function square integrable on $[0, T]$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, increasing and concave function satisfying

$$\varphi\left(0^+\right) = 0, \quad \int_0^1 \frac{ds}{\varphi(s)} = +\infty.$$

The space of processes in $\left(M_G^2\left(0, T; \mathbb{R}^d\right)\right)^n$ equipped with the norm

$$||(X_1, \ldots, X_n)|| = \mathbb{E}^{\frac{1}{2}}\left[\sup_{0 \leq t \leq T}\left(\sum_{k=1}^n |x_k(t)|^2\right)\right]$$

is a Banach space.

Now we can state our first contribution of this work, it is the following theorem.

Theorem 3.1. *Under assumptions* (A1) *and* (A2), *System* (1.2) *has a unique solution*

$$(X_1(t), \ldots, X_n(t)) \in \left(M_G^2\left(0, T; \mathbb{R}^d\right)\right)^n.$$

Proof. We will prove the theorem in four steps.

Step 1. Suppose that $(X_1(t), \ldots, X_n(t))$ and $(Y_1(t), \ldots, Y_n(t))$ are two solutions of System (1.2) with initial conditions $(X_1(0), \ldots, X_n(0))$ and $(Y_1(0), \ldots, Y_n(0))$, respectively. Then, we have for $1 \leq k \leq n$

$$\begin{aligned}
&X_k(t) - Y_k(t) \\
=&X_k(0) - Y_k(0) + \int_0^t [f_k(s, X_1(s), \ldots, X_n(s)) - f_k(s, Y_1(s), \ldots, Y_n(s))]\, ds \\
&+ \sum_{i,j=1}^{d} \int_0^t [f_{k,i,j}(s, X_1(s), \ldots, X_n(s)) - f_{k,i,j}(s, Y_1(s), \ldots, Y_n(s))]\, d\left\langle B^i, B^j \right\rangle(s) \\
&+ \sum_{j=1}^{d} \int_0^t [f_{k,j}(s, X_1(s), \ldots, X_n(s)) - f_{k,j}(s, Y_1(s), \ldots, Y_n(s))]\, dB^j(s).
\end{aligned}$$

By using the inequality, $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, we obtain

$$\begin{aligned}
&|X_k(t) - Y_k(t)|^2 \\
\leq&4|X_k(0) - Y_k(0)|^2 + 4\left|\int_0^t [f_k(s, X_1(s), \ldots, X_n(s)) - f_k(s, Y_1(s), \ldots, Y_n(s))]\, ds\right|^2 \\
&+ 4\left|\sum_{i,j=1}^{d} \int_0^t [f_{k,i,j}(s, X_1(s), \ldots, X_n(s)) - f_{k,i,j}(s, Y_1(s), \ldots, Y_n(s))]\, d\left\langle B^i, B^j \right\rangle_s\right|^2 \\
&+ 4\left|\sum_{j=1}^{d} \int_0^t [f_{k,j}(s, X_1(s), \ldots, X_n(s)) - f_{k,j}(s, Y_1(s), \ldots, Y_n(s))]\, dB^j_s\right|^2.
\end{aligned}$$

We use the fact that $\left(\sum\limits_{i=1}^{d} a_i\right)^2 \leq d \sum\limits_{i=1}^{d} a_i^2$, for each positive constants $a_i,\ i = 1, \ldots, d$, we have

$$\begin{aligned}
&|X_k(t) - Y_k(t)|^2 \\
\leq&4|X_k(0) - Y_k(0)|^2 + 4\left|\int_0^t [f_k(s, X_1(s), \ldots, X_n(s)) - f_k(s, Y_1(s), \ldots, Y_n(s))]\, ds\right|^2 \\
&+4d^2 \sum_{i,j=1}^{d} \left|\int_0^t [f_{k,i,j}(s, X_1(s), \ldots, X_n(s)) - f_{i,j,j}(s, Y_1(s), \ldots, Y_n(s))]\, d\left\langle B^i, B^j \right\rangle(s)\right|^2 \\
&+ 4d\sum_{j=1}^{d} \left|\int_0^t [f_{k,j}(s, X_1(s), \ldots, X_n(s)) - f_{k,j}(s, Y_1(s), \ldots, Y_n(s))]\, dB^j(s)\right|^2.
\end{aligned}$$

Taking the supremum and the G-expectation, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0 \leq s \leq t} |X_k(s) - Y_k(s)|^2\right] \\
\leq&4|X_k(0) - Y_k(0)|^2
\end{aligned}$$

$$
\begin{aligned}
&+4\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s\left[f_k\left(r,X_1\left(r\right),\dots,X_n\left(r\right)\right)-f_k\left(r,Y_1\left(r\right),\dots,Y_n\left(r\right)\right)\right]dr\right|^2\right]\\
&+4d^2\sum_{i,j=1}^{d}\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s\left[f_{k,i,j}\left(r,X_1\left(r\right),\dots,X_n\left(r\right)\right)-f_{k,i,j}\left(r,Y_1\left(r\right),\dots,Y_n\left(r\right)\right)\right]\\
&d\left\langle B^i,B^j\right\rangle\left(r\right)\Big|^2\Big]\\
&+4d\sum_{j=1}^{d}\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s\left[f_{k,j}\left(r,X_1\left(r\right),\dots,X_n\left(r\right)\right)-f_{k,j}\left(r,Y_1\left(r\right),\dots,Y_n\left(r\right)\right)\right]dB^j\left(r\right)\right|^2\right]
\end{aligned}
$$

By the Hölder inequality and Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right]\\
\le&4\left|X_k\left(0\right)-Y_k\left(0\right)\right|^2\\
&+4T\int_0^t\mathbb{E}\left[\left|f_k\left(s,X_1\left(s\right),\dots,X_n\left(s\right)\right)-f_k\left(s,Y_1\left(s\right),\dots,Y_n\left(s\right)\right)\right|^2ds\right]\\
&+4C_1Td^2\sum_{i,j=1}^{d}\int_0^t\mathbb{E}\left[\left|f_{k,i,j}\left(s,X_1\left(s\right),\dots,X_n\left(s\right)\right)-f_{k,i,j}\left(s,Y_1\left(s\right),\dots,Y_n\left(s\right)\right)\right|^2ds\right]\\
&+4C_2d\sum_{j=1}^{d}\int_0^t\mathbb{E}\left[\left|f_{k,j}\left(s,X_1\left(s\right),\dots,X_n\left(s\right)\right)-f_{k,j}\left(s,Y_1\left(s\right),\dots,Y_n\left(s\right)\right)\right|^2ds\right],
\end{aligned}
$$

and by assumption (A2), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right]\\
\le&4\left|X_k\left(0\right)-Y_k\left(0\right)\right|^2+4T\int_0^t\left|\alpha\left(s\right)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right)\right]ds\\
&+4C_1Td^2\sum_{i,j=1}^{d}\int_0^t\left|\alpha\left(s\right)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right)\right]ds\\
&+4C_2d\sum_{j=1}^{d}\int_0^t\left|\alpha\left(s\right)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right)\right]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right]\\
\le&4\left|X_k\left(0\right)-Y_k\left(0\right)\right|^2+4\left(T+C_1Td^4+C_2d^2\right)\int_0^t\left|\alpha\left(s\right)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k\left(s\right)-Y_k\left(s\right)\right|^2\right)\right]ds
\end{aligned}
$$

and

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right] \leq & \sum_{k=1}^{n}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_k(s)-Y_k(s)\right|^2\right] \\ \leq & 4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2+4\left(T+C_1Td^4+C_2d^2\right)n \\ & \times\int_0^t\left|\alpha(s)\right|^2\mathbb{E}\left[\varphi\left(\sum_{k=1}^{n}\left|X_k(s)-Y_k(s)\right|^2\right)\right]ds.\end{aligned}$$

Since, φ is a concave function, by Lemma 2.4, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right] \leq & 4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2+4\left(T+C_1Td^4+C_2d^2\right)n \\ & \times\int_0^t\left|\alpha(s)\right|^2\varphi\left(\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k(s)-Y_k(s)\right|^2\right)\right)ds.\end{aligned}$$

Taking $(X_1(0),\ldots,X_n(0))=(Y_1(0),\ldots,Y_n(0))$, we get

$$4\sum_{k=1}^{n}\left|X_k(0)-Y_k(0)\right|^2=0.$$

By Lemma 2.3, we see that for $t\in[0,T]$

$$\mathbb{E}\left[\left(\sum_{k=1}^{n}\left|X_k(t)-Y_k(t)\right|^2\right)\right]=0,$$

which implies

$$(X_1(t),\ldots,X_n(t))=(Y_1(t),\ldots,Y_n(t)),\qquad\text{for each } t\in[0,T].$$

Step 2. We define a Picard sequence $(X_1^m(\cdot),\ldots,X_n^m(\cdot))_{m\in\mathbb{N}}$ in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ by the following

$$\left(X_1^0(t),\ldots,X_n^0(t)\right)=(x_1,\ldots,x_n),$$

and for each integer $k=1,\ldots,n$ and $t\in[0,T]$

$$\begin{aligned}X_k^{m+1}(t)= & x_k+\int_0^t f_k\left(s,X_1^m(s),\ldots,X_n^m(s)\right)ds \\ & +\sum_{i,j=1}^{d}\int_0^t f_{k,i,j}\left(s,X_1^m(s),\ldots,X_n^m(s)\right)d\left\langle B^i,B^j\right\rangle(s) \\ & +\sum_{j=1}^{d}\int_0^t f_{k,j}\left(s,X_1^m(s),\ldots,X_n^m(s)\right)dB^j(s).\end{aligned}$$

We will prove that is a Cauchy sequence for each $t\in[0,T]$. First, we prove an a priory estimate for $\left(\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k^m(t)\right|^2\right]\right)_{m\in\mathbb{N}}$. By the same arguments, we have for each

$m \in \mathbb{N}$,

$$\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] \leq 4|x_k|^2 + 4\left(T + C_1 T d^2 + C_2 d\right)\left(\int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right] ds + \alpha_2^2 \int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right] ds\right),$$

then

$$\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] \leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right] ds + \alpha_2^2 \int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right] ds\right),$$

where $C = \max\{4, 4(T + C_1 T d^2 + C_2 d)\}$.

Let

$$q(t) = Ce^{C\alpha_2^2 t}\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right] ds\right),$$

then $q(\cdot)$ is a solution of the following ordinary differential equation

$$q(t) = C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[|\alpha_1(s)|^2\right] ds + \alpha_2^2 \int_0^t p(s)\, ds\right).$$

By induction, it is easy that for each $m \in \mathbb{N}$

$$\mathbb{E}\left[\sum_{k=1}^n |X_k^m(t)|^2\right] \leq nq(t).$$

Suppose for each m, $\mathbb{E}\left[|X_k^m(t)|^2\right] \leq q(t)$, then

$$\begin{aligned}\mathbb{E}\left[\left|X_k^{m+1}(t)\right|^2\right] &\leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[\alpha_1^2(s)\right] ds + \alpha_2^2 \int_0^t \mathbb{E}\left[|X_k^m(s)|^2\right] ds\right)\\ &\leq C\left(|x_k|^2 + \int_0^t \mathbb{E}\left[\alpha_1^2(s)\right] ds + \alpha_2^2 \int_0^t p(s)\, ds\right) = q(t)\end{aligned}$$

and

$$\mathbb{E}\left[\sum_{k=1}^n |X_k^m(t)|^2\right] \leq \sum_{k=1}^n \mathbb{E}\left[|X_k^m(t)|^2\right] \leq nq(t).$$

Step 3. For each $l, m \in \mathbb{N}$, by the definition of $(X_1^m(.), \ldots, X_n^m(\cdot))$, we have

$$\begin{aligned}&X_k^{l+1+m}(t) - X_k^{l+1}(t)\\ =&\int_0^t \left[f_k\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_k\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] ds\\ &+ \sum_{i,j=1}^d \int_0^t \left[f_{k,i,j}\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,i,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] d\left\langle B^i, B^j\right\rangle(s)\\ &+ \sum_{j=1}^d \int_0^t \left[f_{k,j}\left(s, X^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] dB^j(s).\end{aligned}$$

Using the inequality, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we obtain

$$\left|X_k^{l+1+m}(t) - X_k^{l+1}(t)\right|^2$$

$$\leq 3\left|\int_0^t \left[f_k\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_k\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] ds\right|^2$$

$$+3\left|\sum_{i,j=1}^{d}\int_0^t \left[f_{k,i,j}\left(s, X_1^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,i,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] d\left\langle B^i, B^j\right\rangle(s)\right|^2$$

$$+3\left|\sum_{j=1}^{d}\int_0^t \left[f_{k,j}\left(s, X^{l+m}(s), \ldots, X_n^{l+m}(s)\right) - f_{k,j}\left(s, X_1^l(s), \ldots, X_n^l(s)\right)\right] dB^j(s)\right|^2$$

Taking the supremum and the G-expectation, by using the Hölder inequality and lemmas 2.1, 2.2, 2.4 and assumption (A2), we obtain

$$\mathbb{E}\left(\sup_{0\leq s\leq t}\left[\left|X_k^{l+1+m}(s) - X_k^{l+1}(s)\right|^2\right]\right)$$
$$\leq 3\left(T + C_1 T d^4 + C_2 d^2\right)\int_0^t |\alpha(s)|^2 \varphi\left(\sum_{k=1}^{n}\left|X_k^{l+m}(s) - X_k^l(s)\right|^2\right) ds.$$

Let

$$h_{n,l}(t) = \sup_{m\in\mathbb{N}}\left[\mathbb{E}\left(\sup_{0\leq s\leq t}\sum_{k=1}^{n}\left|X_k^{l+m}(s) - X_k^l(s)\right|^2\right)\right], \quad 0\leq t\leq T,$$

$$0 \leq h_{n,l+1}(t) \leq C_3\int_0^t |\alpha(s)|^2 \varphi\left(h_{n,l}(s)\right) ds.$$

We define

$$g(t) := \lim_{l\to+\infty}\sup_{0\leq t\leq T} h_{n,l}(t), \tag{3.1}$$

which is uniformly bounded by $4nq(t)$. By using the Fatou-Lebesgue theorem to (3.1), we deduce

$$0 \leq g(t) \leq C_3\int_0^t |\alpha(s)|^2 \varphi\left(g(s)\right) ds.$$

By Lemma 2.3, we obtain

$$g(t) = 0, \quad 0\leq t\leq T,$$

which implies that $\left(X_1^m(\cdot), \ldots, X_n^m(\cdot)\right)_{m\in\mathbb{N}}$ is a Cauchy sequence under the norm

$$\sup_{0\leq t\leq T}\left[\mathbb{E}\left(\sum_{k=1}^{n}|X_k(\cdot)|^2\right)\right]^{\frac{1}{2}}.$$

Step 4. We will prove that the limite $(X_1(t), \ldots, X_n(t))$ in $\left(M_G^2\left(0, T; \mathbb{R}^d\right)\right)^n$ of $\left(X_1^m(t), \ldots, X_n^m(t)\right)$ is the solution of system (1.2). By the same arguments as those used in Step 1, we have for each $m\in\mathbb{N}$

$$\sup_{0\leq t\leq T}\left[\mathbb{E}\left(\sum_{k=1}^{n}|X_k^m(t) - X_k(t)|^2\right)\right] \leq C_3\int_0^T |\alpha(t)|^2 \varphi\left(\mathbb{E}\left(\sum_{k=1}^{n}|X_k^m(t) - X_k(t)|^2\right)\right) dt$$
$$\leq C_3\int_0^T |\alpha(t)|^2 \varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^{n}|X_k^m(t) - X_k(t)|^2\right)\right) dt$$

$$\leq C\varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right).$$

By the continuity of φ and $\varphi(0^+)=0$, we know that

$$\varphi\left(\sup_{0\leq t\leq T}\mathbb{E}\left(\sum_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right)\to 0,$$

and $\sup\limits_{0\leq t\leq T}\left[\mathbb{E}\left(\sum\limits_{k=1}^{n}\left|X_k^m(t)-X_k(t)\right|^2\right)\right]$ converge to 0. Thus, $\left(X_1^m(\cdot),\ldots,X_n^m(\cdot)\right)_{m\in\mathbb{N}}$ is a successive approximation to $(X_1(t),\ldots,X_n(t))$, which is a solution to SG-SDEs (1.2) in $\left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$. □

4. Stability Theorem

In this section, we prove another important result on the stability of the solutions of (1.2). We consider the following perturbed SG-SDEs (4.1) whith a parameter $\epsilon\geq 0$, for $0\leq t\leq T$

$$\tag{4.1}\begin{cases} X_1^\epsilon(t)= & X_1^\epsilon(0)+\int_0^t f_1^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)ds\\ & +\sum\limits_{i,j=1}^{d}\int_0^t f_{1,i,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)d\left\langle B^i,B^j\right\rangle(s)\\ & +\sum\limits_{j=1}^{d}\int_0^t f_{1,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)dB^j(s),\\ & \vdots\\ X_n^\epsilon(t)= & X_n^\epsilon(0)+\int_0^t f_n^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)ds\\ & +\sum\limits_{i,j=1}^{d}\int_0^t f_{n,i,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)d\left\langle B^i,B^j\right\rangle(s)\\ & +\sum\limits_{j=1}^{d}\int_0^t f_{n,j}^\epsilon\left(s,X_1^\epsilon(s),\ldots,X_n^\epsilon(s)\right)dB^j(s),\end{cases}$$

where $\left(X_1^\epsilon(0),\ldots,X_n^\epsilon(0)\right)\in\left(\mathbb{R}^d\right)^n$ and $f_k,f_{k,i,j},f_{k,j}\in M_G^2\left(0,T;\mathbb{R}^d\right)$.

Now, we make the following assumptions.

For any $\epsilon\geq 0$, $x_k\in\mathbb{R}^d$, $J^\epsilon=f_k^\epsilon,f_{k,i,j}^\epsilon$ or $f_{k,j}^\epsilon\in M_G^2\left(0,T;\mathbb{R}^d\right)$, $X_k(0)\in\mathbb{R}^d$, $1\leq k\leq n$ and $1\leq i,j\leq d$

(B1)

$$\left|J^\epsilon(t,x_1,\ldots,x_n)\right|^2\leq\left|\alpha_1(t)\right|^2+\alpha_2^2\left(\sum_{k=1}^{n}\left|x_k\right|^2\right),$$

for each $x_1,\ldots,x_n\in\mathbb{R}^d$, where $\alpha_1\in M_G^2(0,T)$ and $\alpha_2\in\mathbb{R}_+$.

(B2)

$$\left|J^\epsilon(t,x_1,\ldots,x_n)-J^\epsilon(t,y_1,\ldots,y_n)\right|^2\leq\left|\alpha(t)\right|^2\varphi\left(\sum_{k=1}^{n}\left|x_k-y_k\right|^2\right),$$

for each $x_1, y_1, \ldots, x_n, y_n \in \mathbb{R}^d$, where α is a positive and square integrable function on $[0,T]$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, increasing and concave function satisfying $\varphi(0^+) = 0$, $\int_0^1 \frac{ds}{\varphi(s)} = +\infty$.

(B3)

(i) For all $t \in [0,\ T]$,

$$\lim_{\epsilon\to 0}\int_0^t \mathbb{E}\left[\left|J^\epsilon\left(s, X_1^0(s), \ldots, X_n^0(s)\right) - J^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right|^2\right] ds = 0.$$

(ii) $\lim_{\epsilon\to 0}\left(X_1^\epsilon(0), \ldots, X_n^\epsilon(0)\right) = \left(X_1^0(0), \ldots, X_n^0(0)\right)$.

Remark 4.1. Assumptions (B1) and (B2) guarantee, for any $\epsilon \geq 0$, the existence of unique solution $\left(X_1^\epsilon(t), \ldots, X_n^\epsilon(t)\right) \in \left(M_G^2\left(0,T;\mathbb{R}^d\right)\right)^n$ of our system while assumption (B3) will allows us to deduce the following stability theorem for the system.

Theorem 4.1. *Under assumptions* (B1), (B2) *and* (B3) *we have*

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\sum_{k=1}^n\left|X_k^\epsilon(t) - X_k^0(t)\right|^2\right] = 0, \quad \textit{for all } t \in [0,T].$$

Proof. For all $1 \leq k \leq n$, we have

$$\begin{cases} X_k^\epsilon(t) = & X_k^\epsilon(0) + \int_0^t f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) ds \\ & + \sum_{i,j=1}^d \int_0^t f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) d\left\langle B^i, B^j\right\rangle(s) \\ & + \sum_{j=1}^d \int_0^t f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) dB_s^j(s), \\ & \vdots \\ X_k^0(t) = & X_k^0(0) + \int_0^t f_k^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) ds \\ & + \sum_{i,j=1}^d \int_0^t f_{k,i,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) d\left\langle B^i, B^j\right\rangle(s) \\ & + \sum_{j=1}^d \int_0^t f_{k,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right) dB^j(s), \end{cases}$$

and

$$\begin{aligned} & X_k^\epsilon(t) - X_k^0(t) \\ =& X_k^\epsilon(0) - X_k^0(0) + \int_0^t \left[f_k^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_k^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] ds \\ & + \sum_{i,j=1}^d \int_0^t \left[f_{k,i,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_{k,i,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] d\left\langle B^i, B^j\right\rangle_s \\ & + \sum_{j=1}^d \int_0^t \left[f_{k,j}^\epsilon\left(s, X_1^\epsilon(s), \ldots, X_n^\epsilon(s)\right) - f_{k,j}^0\left(s, X_1^0(s), \ldots, X_n^0(s)\right)\right] dB^j s. \end{aligned}$$

We have

$$\left|X_k^\epsilon(t) - X_k^0(t)\right|^2$$

$$\le 4\left|X_k^\epsilon(0)-X_k^0(0)\right|^2+4\left|\int_0^t\left[f_k^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_k^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]ds\right|^2$$
$$+4\left|\sum_{i,j=1}^d\int_0^t\left[f_{k,i,j}^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_{k,i,j}^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]d\left\langle B^i,B^j\right\rangle(s)\right|^2$$
$$+4\sum_{j=1}^d\left|\int_0^t\left[f_{k,j}^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_{k,j}^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]dB^j(s)\right|^2$$

and

$$\left|X_k^\epsilon(t)-X_k^0(t)\right|^2$$
$$\le 4\left|X_k^\epsilon(0)-X_k^0(0)\right|^2+4\left|\int_0^t\left[f_k^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_k^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]ds\right|^2$$
$$+4d^2\sum_{i,j=1}^d\left|\int_0^t\left[f_{k,i,j}^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_{k,i,j}^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]d\left\langle B^i,B^j\right\rangle(s)\right|^2$$
$$+4d\sum_{j=1}^d\left|\int_0^t\left[f_{k,j}^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_{k,j}^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]dB^j(s)\right|^2.$$

Taking the supremum and the G-expectation, we have

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k^\epsilon(s)-X_k^0(s)\right|^2\right]$$
$$\le 4\mathbb{E}\left[\left|X_k^\epsilon(0)-X_k^0(0)\right|^2\right]$$
$$+4\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\left[f_k^\epsilon(r,X_1^\epsilon(r),\ldots,\ X_n^\epsilon(r))-f_k^\epsilon\left(r,X_1^0(r),\ldots,X_n^0(r)\right)\right]dr\right|^2$$
$$+4d^2\sum_{i,j=1}^d\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\left[f_{k,i,j}^\epsilon(r,X_1^\epsilon(r),\ldots,\ X_n^\epsilon(r))\right.\right.$$
$$\left.\left.-f_{k,i,j}^\epsilon\left(r,X_1^0(r),\ldots,X_n^0(r)\right)\right]d\left\langle B^i,B^j\right\rangle(r)\right|^2$$
$$+4d\sum_{j=1}^d\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s\left[f_{k,j}^\epsilon(r,X_1^\epsilon(r),\ldots,\ X_n^\epsilon(r))\left|-f_{k,j}^\epsilon\left(r,X_1^0(r),\ldots,X_n^0(r)\right)\right]dB^j(r)\right|\right|^2.$$

By lemmas 2.1, 2.2 and Hölder's inequality, we have

$$\mathbb{E}\left[\sup_{0\le s\le t}\left|X_k^\epsilon(s)-X_k^0(s)\right|^2\right]$$
$$\le 4\mathbb{E}\left[\left|X_k^\epsilon(0)-X_k^0(0)\right|^2\right]$$
$$+8T\int_0^t\mathbb{E}\left|\left[f_k^\epsilon(s,X_1^\epsilon(s),\ldots,\ X_n^\epsilon(s))-f_k^\epsilon\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]\right|^2ds$$
$$+8T\int_0^t\mathbb{E}\left|\left[f_k^\epsilon\left(s,X_1^0(s),\ldots,\ X_n^0(s)\right)-f_k^0\left(s,X_1^0(s),\ldots,X_n^0(s)\right)\right]\right|^2ds$$

$$+ 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^{\epsilon} (s), \dots, \ X_n^{\epsilon} (s) \right) - f_{k,i,j}^{\epsilon} \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds$$
$$+ 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^{\epsilon} (s), \dots, \ X_n^0 (s) \right) - f_{k,i,j}^0 \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds$$
$$+ 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^{\epsilon} (s), \dots, \ X_n^{\epsilon} (s) \right) - f_{k,j}^{\epsilon} \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds$$
$$+ 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^{\epsilon} (s), \dots, \ X_n^0 (s) \right) - f_{k,j}^{\epsilon} \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds,$$

then by assumption (B2) we obtained

$$\mathbb{E} \left[\sup_{0 \le s \le t} \left| X_k^{\epsilon} (s) - X_k^0 (s) \right|^2 \right] \le C_{k,\epsilon} (T) + C_d (T) \int_0^t |\alpha (s)|^2 \times \mathbb{E} \left[\varphi \left(\sum_{k=1}^{n} \left| X_k^{\epsilon} (s) - X_k^0 (s) \right|^2 \right) \right] ds,$$

where

$$\begin{aligned} & C_{k,\epsilon} (t) \\ = & 4\mathbb{E} \left[\left| X_k^{\epsilon} (0) - X_k^0 (0) \right|^2 \right] \\ & + 8T \int_0^t \mathbb{E} \left| f_k^{\epsilon} \left(s, X_1^0 (s), \dots, \ X_n^0 (s) \right) - f_k^0 \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds \\ & + 8C_1Td^2 \sum_{i,j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,i,j}^{\epsilon} \left(s, X_1^0 (s), \dots, \ X_n^0 (s) \right) - f_{k,i,j}^0 \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds \\ & + 8C_2d \sum_{j=1}^{d} \mathbb{E} \int_0^t \left| f_{k,j}^{\epsilon} \left(s, X_1^0 (s), \dots, \ X_n^0 (s) \right) - f_{k,j}^{\epsilon} \left(s, X_1^0 (s), \dots, X_n^0 (s) \right) \right|^2 ds, \end{aligned}$$

and $C_d (t) = 8 \left(t + C_1td^2 + C_2d \right)$, then

$$\begin{aligned} & \mathbb{E} \left[\sum_{k=1}^{n} \left| X_k^{\epsilon} (t) - X_k^0 (t) \right|^2 \right] \\ \le & \sum_{k=1}^{n} \mathbb{E} \left| X_k^{\epsilon} (t) - X_k^0 (t) \right|^2 \le \sum_{k=1}^{n} \mathbb{E} \left[\sup_{0 \le s \le t} \left| X_k^{\epsilon} (s) - X_k^0 (s) \right|^2 \right] \\ \le & C_{n,\epsilon} (T) + nC_d (T) \int_0^t |\alpha (s)|^2 \, \mathbb{E} \left[\varphi \left(\sum_{k=1}^{n} \left| X_k^{\epsilon} (s) - X_k^0 (s) \right|^2 \right) \right] ds, \end{aligned}$$

where $C_{n,\epsilon} (T) = \sum\limits_{k=1}^{n} C_{k,\epsilon} (T)$.

Since φ is a continuous concave function, by Lemma 2.4, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k^{\epsilon}(t)-X_k^{0}(t)\right|^2\right]\\ \leq&C_{n,\epsilon}(T)+nC_d(T)\int_0^t|\alpha(s)|^2\varphi\left(\mathbb{E}\left[\sum_{k=1}^{n}\left|X_k^{\epsilon}(s)-X_k^{0}(s)\right|^2\right]\right)ds.\end{aligned}$$

Since $C_{n,\epsilon}(T)\to 0$ when $\epsilon\to 0$, we get by Lemma 2.3

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\sum_{j=1}^{n}\left|X_j^{\epsilon}(t)-X_j^{0}(t)\right|^2\right]=0,\quad \text{for all } t\in[0,T],$$

that is what we want to prove. □

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[1]Department of Pharmacy,
Faculty of Medicine,
Salah Boubnider University,
Constantine, Algeria
Email address: elhacene25@gmail.com

[2]Department of Sciences and Technology,
Faculty of Technology,
20 August 1955 University,
Skikda, Algeria
Email address: ouaouaama21@gmail.com

[3]Department of Mathematics,
Faculty of Sciences,
Ferhat Abbas University,
Setif, Algeria
Email address: salimbra@gmail.com