Kragujevac Journal of Mathematics
Volume 50(2) (2026), Pages 205–215.

CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY LINEAR MULTIPLIER FRACTIONAL q-DIFFERENTIAL OPERATOR

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Abstract. This paper introduces a novel subclass of analytic and bi-univalent functions that are linked to a linear multiplier fractional q-differential operator, defined in the open unit disk $\mathbb{D}$. The authors establish the upper bounds for the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to this new subclass and its subclasses.

1. Introduction and preliminaries

Let the class of functions $\mathcal{A}$ be of the form:

$$\eta(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1.1}$$

which are analytic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let S indicates the functions of all subclasses in $\mathcal{A}$, which are univalent in $\mathbb{D}$. Since univalent functions are one-to-one, they are invertible. Although the inverse functions of single-valued functions are inverse functions, they do not need to be defined for the entire unit disk $\mathbb{D}$. Certainly, according to Koebe's quarter theorem [1], the disk with radius $\frac{1}{4}$ is in the image $\mathbb{D}$. Thus, every univalent function η has an inverse η^{-1} that satisfies $\eta^{-1}(\eta(z)) = z$, $z \in \mathbb{D}$, and $\zeta(w) = \eta^{-1}(\eta(w)) = w$, $|w| < r_0(\eta)$, $r_0(\eta) \geq \frac{1}{4}$, where

$$\eta^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

Key words and phrases. Analytic function, univalent function, bi-univalent function, starlike function, convex function, q-derivative operator.

2020 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30C50.

DOI 10.46793/KgJMat2602.205K

Received: April 12, 2023.

Accepted: August 23, 2023.

A function $\eta \in \mathcal{A}$ is bi-univalent in $\mathbb{D}$ if both $\eta(z)$ and $\eta^{-1}(z)$ are univalent $\mathbb{D}$. Let Σ be the class of bi-univalent functions on $\mathbb{D}$ given by (1.1). Example of functions in the class Σ are

$$\frac{z}{1-z}, \quad \log\frac{1}{1-z}, \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathbb{D}$ such as

$$\frac{2z-z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ.

The widely-cited by Srivastava et al. [2] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (see, for example, [3–23]).

If $|q| < 1$, the q-shifted factorial, also known as the q-Pochhammer symbol, is defined for all $n \in \mathbb{N}$ by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$

where a and q are complex numbers. When $n = +\infty$, the product becomes

$$(a;q)_{+\infty} = \prod_{k=0}^{+\infty}(1-aq^k).$$

If $|q| < 1$, then the product converges absolutely, and we can define the q-shifted factorial for $n = +\infty$ as the limit of the sequence of partial products

$$(a;q)_{+\infty} = \lim_{n\to+\infty}(a;q)_n = \lim_{n\to+\infty}\prod_{k=0}^{n-1}(1-aq^k).$$

Therefore, when $|q| < 1$, the q-shifted factorial remains meaningful for $n = +\infty$ as a convergent infinite product.

The q-gamma function is a q-analogue of the gamma function, defined by the recurrence relation $\Gamma_q(y+1) = [y]_q\Gamma_q(y)$, where $[y]_q = \frac{(1-q^y)}{(1-q)}$ is the q-analogue of y.

Jackson's [24] q-derivative and q-integral of a function η defined on a subset of $\mathbb{C}$ are given by

$$D_q^a\eta(x) = \frac{\eta(q^a x)-\eta(x)}{(1-q^a)x-x}, \quad I_q^a\eta(x) = (1-q^a)x\sum_{n=0}^{+\infty}q^{an}\eta(q^n x),$$

where $a \in \mathbb{C}$ is a fixed parameter. These operators are also known as the q-difference and q-integral operators, respectively. The theory of q-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, as well as geometric function theory of complex analysis. The application of q-calculus was initiated by Jackson [24]. Recently, many researchers studied q-calculus such as Srivastava et al.

[25], Muhammad and Darus [26], Kanas and Răducanu [27], (see also, [28–33]) and also the reference cited therein.

Definition 1.1 ([34])**.** The fractional integral operator $I^{\delta}_{q,z}$ of order $\delta > 0$, for the function $\eta(z)$ is defined by

$$I^{\delta}_{q,z} = D^{-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z - rq)_{1-\delta}\eta(r)d_q r,$$

where $\eta(z)$ is the analytic of the simply connected regions of the z plane containing the origin. Here, the term $(z - rq)_{\delta-1}$ is a q-binomial function defined by

$$(z - rq)_{\delta-1} = z^{\delta-1}\prod_{k=0}^{+\infty}\left[\frac{1 - (\frac{rq}{z})q^k}{1 - (\frac{rq}{z})q^{\delta} + k - 1}\right] = z^{\delta}\ {}_1\phi_0\left[q^{-\delta+1}; -; q, \frac{rq^{\delta}}{z}\right].$$

Definition 1.2. The fractional q-derivative operator $D^{\delta}_{q,z}$ of a $\eta(z)$ of order $0 \le \delta < 1$, is defined by

$$D^{\delta}_{q,z}\eta(z) = D_{q,z}I^{1-\delta}_{q,z}\eta(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z - rq)_{-\delta}\eta(r)d_q r,$$

where $\eta(z)$ is suitably constrained and the multiplicity of $(z - rq)_{-\delta}$ is removed as in Definition 1.1 above.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional q-derivative for the function $\eta(z)$ of order δ is defined by

$$D^{\delta}_{q,z}\eta(z) = D^{n}_{q,z}I^{n-\delta}_{q,z}\eta(z),$$

where $n - 1 \le \delta < n$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.4 ([35])**.** The definition of the fractional q-differintegral operator $\Omega^{\delta}_{q,z}$ is as follows. For a function $\eta(z)$ of the form (1.1), we define

$$\Omega^{\delta}_{q}\eta(z) = \Gamma_q(2-\delta)z^{\delta}D^{\delta}_{q,z}\eta(z),$$

where $D^{\delta}_{q,z}$ denotes the fractional δ order of the q-integral $\eta(z)$ when $-\infty < \delta < 0$ and the fractional δ order q-derivative of $\eta(z)$ if $0 < \delta < 2$.

The expression for $\Omega^{\delta}_{q}\eta(z)$ in terms of the coefficients a_k of the power series expansion of $\eta(z)$ is given by

$$\Omega^{\delta}_{q}\eta(z) = z + \sum_{k=2}^{+\infty}\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}a_k z^k.$$

Definition 1.5 ([34]). A linear multiplier fractional q-differintegral operator is defined as

$$\begin{aligned}
\mathcal{L}_{q,\lambda}^{\delta,0}\eta(z) =&\eta(z),\\
\mathcal{L}_{q,\lambda}^{\delta,1}\eta(z) =&(1-\lambda)\Omega_q^{\delta}\eta(z)+\lambda z\mathcal{L}_q\left(\Omega_q^{\delta}\eta(z)\right),\\
\mathcal{L}_{q,\lambda}^{\delta,2}\eta(z) =&\mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,1}\eta(z)\right),\\
&\vdots\\
\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z) =&\mathcal{L}_{q,\lambda}^{\delta,1}\left(\mathcal{L}_{q,\lambda}^{\delta,n-1}\eta(z)\right).
\end{aligned}\tag{1.3}$$

We note that if $f\in\mathcal{A}$ is given by (1.1), then by (1.3), we have

$$\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)=z+\sum_{k=2}^{+\infty}C\left(k,\delta,\lambda,n,q\right)a_kz^k,$$

where

$$C\left(k,\delta,\lambda,n,q\right)=\left(\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}\Big[\left([k]_q-1\right)\lambda+1\Big]\right)^n.$$

We define two new subclasses of the function class Σ by utilizing the linear multiplier fractional q-differential operator of a function $\eta\in\mathcal{A}$. Then, we provide coefficient estimates for $|a_2|$ and $|a_3|$ for functions belonging to these new subclasses of the function class Σ.

First, we have to follow the lemma to get the main results.

Lemma 1.1 ([36]). *Let $\mathcal{H}$ be the family of all functions $\mathfrak{h}$ that are analytic in the open unit disk $\mathbb{D}$ and satisfy $\mathfrak{h}(0)=1$ and $\Re(\mathfrak{h}(z))>0$ for all $z\in\mathbb{D}$. If a function $\mathfrak{h}\in\mathcal{H}$ is given by $\mathfrak{h}(z)=1+d_1z+d_2z^2+\cdots$ for $z\in\mathbb{D}$, then $|d_k|\leq 2$ for all $k\in\mathbb{N}$.*

2. Coefficient Bounds for the Function Class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$

Definition 2.1. A function $\eta(z)$ given by (1.1) is said to be in the class $M_\Sigma\left(q,\alpha,\tau,\delta,\lambda,n\right)$ if the following conditions are satisfied: $\eta\in\Sigma$ and

$$\left|\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right|<\frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $z\in\mathbb{D}$, and

$$\left|\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right|<\frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$, $0\leq\tau<1$, $\delta\leq 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $w\in\mathbb{D}$ and function ζ is given by

$$\zeta(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.\tag{2.1}$$

We note that the following hold.

(a) When we set $\delta = 0$, $\lambda = 1$, and $q \to 1^-$, the class $M_\Sigma(q,\alpha,\tau,\delta,\lambda,n)$ reduces to the class $S_\Sigma^{n,\tau}(\alpha)$, where $0 < \alpha \leq 1$, $0 \leq \tau < 1$, and $n \in \mathbb{N}_0$. This class was previously introduced and studied by Jothibasu [37].

(b) If we set $\delta = 0$, $\lambda = 1$, $q \to 1^-$, $n = 0$, and $\tau = 0$ in the class $M_\Sigma(q,\alpha,\tau,\delta,\lambda,n)$, it reduces to the class of strongly bi-starlike functions $S_\Sigma^\star(\alpha)$ of order α introduced and studied by Brannan and Taha [38], where $0 < \alpha \leq 1$.

Theorem 2.1. *Let $\eta(z)$ given by* (1.1) *be in the class $M_\Sigma(q,\alpha,\tau,\delta,\lambda,n)$, $0 < \alpha \leq 1$, $0 \leq \tau < 1$, $\delta \leq 2$, $\lambda > 0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha Y q(q+1)(1-\tau) - 2X^2\alpha q(1-\tau)\left[\tau q+1\right] + X^2(1-\alpha)^2(1-\tau)^2}} \tag{2.2}$$

and

$$|a_3| \leq \frac{4\alpha^2}{X^2q^2(1-\tau)^2} + \frac{2\alpha}{Yq(q+1)(1-\tau)}, \tag{2.3}$$

where $X = C(2,\delta,\lambda,n,q)$ and $Y = C(3,\delta,\lambda,n,q)$.

Proof. It follows from the Definition 2.1

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)} = \left[s(z)\right]^\alpha \tag{2.4}$$

and

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right) + (1-\tau)\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)} = \left[t(w)\right]^\alpha, \tag{2.5}$$

respectively, where $s(z)$ and $t(w)$ satisfy the following inequalities: $\mathfrak{R}(s(z)) > 0$, $z \in \mathbb{D}$, and $\mathfrak{R}(t(w)) > 0$, $w \in \mathbb{D}$.

Furthermore, the functions $s(z)$ and $t(w)$ have the forms

$$s(z) = 1 + s_1z + s_2z^2 + s_3z^3 + \cdots, \tag{2.6}$$

$$t(w) = 1 + t_1w + t_2w^2 + t_3w^3 + \cdots. \tag{2.7}$$

Now, equating the coefficients in (2.4) and (2.5), we get

$$a_2Xq(1-\tau) = \alpha s_1, \tag{2.8}$$

$$a_3Yq(q+1)(1-\tau) - a_2^2X^2q(1-\tau)[\tau q+1] = \alpha s_2 + \frac{\alpha(\alpha-1)}{2}s_1^2, \tag{2.9}$$

$$-a_2Xq(1-\tau) = \alpha t_1 \tag{2.10}$$

and

$$\begin{aligned} &- a_3Yq(q+1)(1-\tau) + 2a_2^2Yq(q+1)(1-\tau) - a_2^2X^2q(1-\tau)[\tau q+1] \\ &= \alpha t_2 + \frac{\alpha(\alpha-1)}{2}t_1^2. \end{aligned} \tag{2.11}$$

From (2.8) and (2.10), we get

$$s_1 = -t_1 \tag{2.12}$$

and

$$2a_2^2 X^2 q^2 (1-\tau)^2 = \alpha^2 \left(s_1^2 + t_1^2\right). \tag{2.13}$$

From (2.9), (2.11) and (2.13), we obtain

$$a_2^2 = \frac{\alpha^2 (s_2 + t_2)}{2\alpha Y q(q+1)(1-\tau) - 2X^2 \alpha q (1-\tau)\left[\tau q + 1\right] + X^2 (1-\alpha) q^2 (1-\tau)^2}.$$

Applying Lemma 1.1 to the coefficients s_2 and t_2, we immediately get

$$|a_2| \le \frac{2\alpha}{\sqrt{2\alpha Y q(q+1)(1-\tau) - 2X^2 \alpha q (1-\tau)\left[\tau q + 1\right] + X^2 (1-\alpha) q^2 (1-\tau)^2}}.$$

This gives the value of $|a_2|$ as shown in (2.2)

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.9), we get

$$\begin{aligned} &2a_3 Y q(q+1)(1-\tau) - 2a_2^2 Y q(q+1)(1-\tau) \\ =&\alpha (s_2 - t_2) + \frac{\alpha(\alpha-1)}{2}\left(s_1^2 - t_1^2\right). \end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$|a_3| = \frac{\alpha^2 (s_1^2 + t_1^2)}{2X^2 q^2 (1-\tau)^2} + \frac{\alpha (s_2 - t_2)}{2Y q(q+1)(1-\tau)}.$$

Applying Lemma 1.1 again to the coefficients s_1, s_2, t_1 and t_2, we easily get

$$|a_3| \le \frac{4\alpha^2}{X^2 q^2 (1-\tau)^2} + \frac{2\alpha}{Y q(q+1)(1-\tau)}.$$

This end the proof of Theorem 2.1. □

Utilizing the parameters setting of Definition 2.1 in the Theorem 2.1, we get the following corollaries.

Corollary 2.1. *If $\eta(z)$ given by (1.1) be in the class $S_\Sigma^{n,\tau}(\alpha)$, $0 < \alpha \le 1$, $0 \le \tau < 1$ and $n \in \mathbb{N}_0$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\tau)3^n + \left[2\alpha(\tau^2 - 1) - (\alpha - 1)(1-\tau)^2\right]2^{2n}}}$$

and

$$|a_3| \le \frac{\alpha}{3^n (1-\tau)} + \frac{4\alpha^2}{2^{2n}(1-\tau)^2}.$$

Corollary 2.2. *If $\eta(z)$ given by (1.1) and in the class $S_\Sigma^\star(\alpha)$, $0 < \alpha \le 1$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \quad \textit{and} \quad |a_3| \le 4\alpha^2 + \alpha.$$

3. Coefficient Bounds for the Function Class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$

Definition 3.1. A function $\eta(z)$ given by (1.1) is said to be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ if the following conditions are satisfied: $\eta\in\Sigma$ and

$$\Re\left(\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}\right)>\gamma,$$

where $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $z\in\mathbb{D}$, and

$$\Re\left(\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}\right)>\gamma,$$

where $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$, $n\in\mathbb{N}_0$, $w\in\mathbb{D}$.

The function ζ is defined as given in equation (2.1).

(a) If we set $\delta=0$, $\lambda=1$, and $q\to 1^-$ in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, it reduces to the class $S_\Sigma^{n,\tau}(\gamma)$ introduced and studied by Jothibasu [37], where $0\le\gamma<1$, $0\le\tau<1$ and $n\in\mathbb{N}_0$.

(b) When $\delta=0$, $\lambda=1$, $q\to 1^-$, $n=0$ and $\tau=0$, the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$ simplifies to the class of strongly bi-starlike functions $S_\Sigma^\star(\gamma)$ of order γ introduced and studied by Brannan and Taha [38].

Theorem 3.1. *Let $\eta(z)$ given by (1.1) be in the class $B_\Sigma(q,\gamma,\tau,\delta,\lambda,n)$, $0\le\gamma<1$, $0\le\tau<1$, $\delta\le 2$, $\lambda>0$. Then*

$$|a_2|\le\sqrt{\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)-X^2q(1-\tau)[\tau q+1]}}\tag{3.1}$$

and

$$|a_3|\le\frac{4(1-\gamma)^2}{X^2q^2(1-\tau)^2}+\frac{2(1-\gamma)}{Yq(q+1)(1-\tau)},\tag{3.2}$$

where $X=C(2,\delta,\lambda,n,q)$ and $Y=C(3,\delta,\lambda,n,q)$.

Proof. It follows from the Definition 3.1 that there exist $s(z)$ and $t(w)\in\mathcal{H}$ such that

$$\frac{zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)}{\tau zD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\eta(z)}=\gamma+(1-\gamma)\,s(z),\tag{3.3}$$

$$\frac{wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)}{\tau wD_q\left(\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)\right)+(1-\tau)\,\mathcal{L}_{q,\lambda}^{\delta,n}\zeta(w)}=\gamma+(1-\gamma)\,t(w),\tag{3.4}$$

where $s(z)$ and $t(w)$ in $\mathcal{H}$ and have the forms (2.6) and (2.7), respectively.

Equating the coefficients in (3.3) and (3.4) yields

$$a_2 X q\,(1-\tau) = (1-\gamma)\,s_1, \tag{3.5}$$

$$a_3 Y q(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\,s_2, \tag{3.6}$$

$$-a_2 X q\,(1-\tau) = (1-\gamma)\,t_1 \tag{3.7}$$

and

$$\begin{aligned} & -a_3 Y q(q+1)\,(1-\tau) + 2a_2^2 Y q(q+1)\,(1-\tau) - a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] \\ & = (1-\gamma)\,t_2. \end{aligned} \tag{3.8}$$

From (3.5) and (3.7), we get $s_1 = -t_1$ and

$$2a_2^2 X^2 q^2\,(1-\tau)^2 = (1-\gamma)^2\left(s_1^2 + t_1^2\right). \tag{3.9}$$

Also, from (3.6) and (3.8), we find that

$$2a_2^2 Y q(q+1)\,(1-\tau) - 2a_2^2 X^2 q\,(1-\tau)\,[\tau q+1] = (1-\gamma)\,(s_2+t_2)\,.$$

Applying Lemma 1.1 to the coefficients s_2 and t_2, we immediately get

$$|a_2| \le \sqrt{\frac{2\,(1-\gamma)}{Yq(q+1)\,(1-\tau) - X^2 q\,(1-\tau)\,[\tau q+1]}},$$

which is the bound on $|a_2|$ as given in (3.1). Then, to get the limit of $|a_3|$ by subtracting (3.8) from (3.6),

$$2a_3 Y q(q+1)\,(1-\tau) - 2a_2^2 Y q(q+1)\,(1-\tau) = (1-\gamma)\,(s_2-t_2)\,,$$

or, equivalently

$$a_3 = a_2^2 + \frac{(1-\gamma)\,(s_2-t_2)}{2Yq(q+1)\,(1-\tau)}.$$

Substituting the values of a_2^2 into (3.9), we get

$$a_3 = \frac{(1-\gamma)^2\,(s_1^2+t_1^2)}{2X^2q^2\,(1-\tau)^2} + \frac{(1-\gamma)\,(s_2-t_2)}{2Yq(q+1)\,(1-\tau)}.$$

After applying Lemma 1.1 to the coefficients s_1, s_2, t_1 and t_2, we get

$$|a_3| \le \frac{4\,(1-\gamma)^2}{X^2q^2\,(1-\tau)^2} + \frac{2\,(1-\gamma)}{Yq(q+1)\,(1-\tau)}.$$

This completes the proof of Theorem 3.1. □

Utilizing the parameters setting of Definition 3.1 in the Theorem 3.1, we get the following corollaries.

Corollary 3.1. *If $\eta(z)$ given by (1.1) is in the class $S_\Sigma^{n,\tau}(\gamma)$, $0 \le \gamma < 1$, $0 \le \tau < 1$ and $n \in \mathbb{N}_0$, then*

$$|a_2| \le \sqrt{\frac{2\,(1-\gamma)}{2^{2n}\,(\tau^2-1) + 2\,(1-\tau)\,3^n}}$$

and

$$|a_3| \leq \frac{4\left(1-\gamma\right)^2}{2^{2n}\left(1-\tau\right)^2} + \frac{\left(1-\gamma\right)}{3^n\left(1-\tau\right)}.$$

Corollary 3.2. *If $\eta(z)$ given by* (1.1) *and in the class $S^{\star}_{\Sigma}(\gamma)$, $0 \leq \gamma < 1$, then*

$$|a_2| \leq \sqrt{2\left(1-\gamma\right)} \quad \text{and} \quad |a_3| \leq 4\left(1-\gamma\right)^2 + \left(1-\gamma\right).$$

4. Conclusions

The main contribution of this paper is the introduction of new subclasses of bi-univalent functions defined by the linear multiplier fractional q-differential operator. Additionally, we provide upper bounds for the coefficients $|a_2|$ and $|a_3|$ for functions belonging to this new subclass and its subclasses.

References

[1] P. Duren, *Geometric Function Theory, Linear and Complex Analysis Problem*, Book 3, 1994, 383–422.

[2] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23**(10) (2010), 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009

[3] I. Aldawish, T. Al-Hawary and B. A. Frasin, *Subclasses of bi-univalent functions defined by Frasin differential operator*, Mathematics **8**(5) (2020), 1–11. https://doi.org/10.3390/math8050783

[4] S. Altinkaya and S. Yalcin, *Fekete-Szegö inequalities for certain classes of biunivalent functions*, Int. Sch. Res. Notices **1726** (2016), Paper ID 020078. https://doi.org/10.1063/1.4945904

[5] A. Amourah, B. A. Frasin, G. Murugusundaramoorthy and T. Al-Hawary, *Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with $(p;q)$-Lucas polynomials*, AIMS Math. **6**(5) (2021), 4296–4305. https://doi.org/10.3934/math.2021254

[6] A. Amourah, B. A. Frasin, M. Ahmad and F. Yousef, *Exploiting the Pascal distribution series and Gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions*, Symmetry **14**(147) (2022), 8 pages. https://doi.org/10.3390/sym14010147

[7] A. Alsoboh, M. Darus, A. Amourah and W. G. Atshan, *A certain subclass of harmonic meromorphic functions with respect to k-symmetric points*, International Journal of Open Problems in Complex Analysis **15**(1) (2023), 1–16.

[8] B. Alshlool, A. abu Alasal, A. Mannaâa, A. Alsoboh and A. Amourah, *Consolidate a certain class of $(p;q)$-Lucas polynomial based bi-univalent functions with a specific discrete probability distribution*, International Journal of Open Problems in Complex Analysis **15** (1) (2023), 26–37.

[9] A. O. Mostafa and Z. M. Saleh, *Coefficient bounds for a class of bi-univalent functions defined by Chebyshev polynomials*, International Journal of Open Problems in Complex Analysis **13** (3) (2021), 1–10.

[10] F. Yousef, A. Amourah, B. A. Frasin and T. Bulboaca, *An avant-Garde construction for subclasses of analytic bi-univalent functions*, Axioms **11**(6) (2022), Paper ID 267. https://doi.org/10.3390/axioms11060267

[11] L. I. Cotirla, *New classes of analytic and bi-univalent functions*, AIMS Math. **6**(10) (2021), 10642–10651. https://doi.org/10.3934/math.2021618

[12] L. I. Cotirla and A. K. Wanas, *Coefficient-related studies and Fekete-Szegö inequalities for new classes of bi-starlike and bi-convex functions*, Symmetry **14**(11) (2022), Paper ID 2263. https://doi.org/10.3390/sym14112263

[13] B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24**(9) (2011) 1569–1573. https://doi.org/10.1016/j.aml.2011.03.048

[14] B. A. Frasin, S. R. Swamy and J. Nirmala, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to k-Fibonacci numbers involving modified Sigmoid activation function*, Afr. Mat. **32** (2020), 631–643. https://doi.org/10.1007/s13370-020-00850-w

[15] Q-H. Xu, Y-C. Gui and H. M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett. **25**(6) (2012), 990–994. https://doi.org/10.1016/j.aml.2011.11.013

[16] Q-H. Xu, H-G. Xiao and H. M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput. **218**(23) (2012), 11461–11465. https://doi.org/10.1016/j.amc.2012.05.034

[17] G. Murugusundaramoorthy, N. Magesh and V. Prameela, *Coefficient bounds for certain subclasses of bi-univalent function*, Abst. Appl. Anal. **2013** (2013). https://doi.org/10.1155/2013/573017

[18] Z. Peng, G. Murugusundaramoorthy and T. Janani, *Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator*, J. Complex Anal. **2014** (2014). https://doi.org/10.1155/2014/693908

[19] F. M. Sakar and S. M. Aydogan, *Inequalities of bi-starlike functions involving Sigmoid function and Bernoulli lemniscate by subordination*, International Journal of Open Problems in Computer Science and Mathematics **16** (2023), 71–82.

[20] S. R. Swamy and A. K. Wanas, *A comprehensive family of bi-univalent functions defined by (m, n)-Lucas polynomials*, Boletin De La Sociedad Matematica Mexicana **28** (2022). https://doi.org/10.1007/s40590-022-00411-0

[21] A. K. Wanas, F. M. Sakar and A. Alb Lupaş, *Applications Laguerre polynomials for families of bi-univalent functions defined with (p, q)-Wanas operator*, Axioms **12**(5) (2023), Paper ID 430. https://doi.org/10.3390/axioms12050430

[22] F. Yousef, S. Alroud and M. Illafe, *A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind*, Boletin De La Sociedad Matematica Mexicana **26** (2019), 329–339. https://doi.org/10.1007/s40590-019-00245-3

[23] F. Yousef, S. Alroud and M. Illafe, *New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems*, Anal. Math. Phys. **11**(58) (2021), Article ID 58. https://doi.org/10.1007/s13324-021-00491-7

[24] F. H. Jackson, *On q-functions and a certain difference operator*, Earth and Environmental Science Transactions of the Royal Society of Edinburgh **46**(2) (1909), 253–281. https://doi.org/10.1017/S0080456800002751

[25] R. Srivastava and H. M. Zayed, *Subclasses of analytic functions of complex order defined by q-derivative operator*, Stud. Univ. Babeş-Bolyai Math. **64** (2019), 71–80. https://doi.org/10.24193/subbmath.2019.1.07

[26] A. Mohammed and M. Darus, *A generalized operator involving the q-hypergeometric function*, Mat. Vesnik **65**(4) (2013), 454–465.

[27] S. Kanas and D. Răducanu, *Some subclass of analytic functions related to conic domains*, Math. Slovaca **64**(5) (2014), 1183–1196. https://doi.org/10.2478/s12175-014-0268-9

[28] C. Ramachandran, S. Annamalai and B. A. Frasin, *The q-difference operator associated with the multivalent function bounded by conical sections*, Bol. Soc. Parana. Mat. **39**(1) (2021), 133–146. https://doi.org/10.5269/bspm.32913

[29] C. Ramachandran, T. Soupramanien and B. A. Frasin, *New subclasses of analytic function associated with q-difference operator*, Eur. J. Pure Appl. **10**(2) (2017), 348–362.

[30] F. M. Sakar, M. Naeem, S. Khan and S. Hussain, *Hankel determinant for class of analytic functions involving q-derivative operator*, J. Adv. Math. Stud. **14** (2021), 265–278.

[31] T. M. Seoudy and M. K. Aouf, *Convolution properties for certain classes of analytic functions defined by q-derivative operator*, Abstr. Appl. Anal. **2014** (2014). `https://doi.org/10.1155/2014/846719`
[32] T. M. Seoudy and M. K. Aouf, *Coefficient estimates of new classes of q-starlike and q-convex functions of complex order*, J. Math. Inequal. **10** (2016), 135–145. `https://doi.org/10.7153/jmi-10-11`
[33] A. Amourah, B. A. Frasin and T. Al-Hawary, *Coefficient estimates for a subclass of bi-univalent functions associated with symmetric q-derivative operator by means of the Gegenbauer polynomials*, Kyungpook Math. J. **62**(2) (2022), 257–269.
[34] S. D. Purohit and R. K. Raina, *Fractional q-calculus and certain subclass of univalent analytic functions*, Mathematica **55**(78) (2013).
[35] S. D. Purohit and R. K. Raina, *Certain subclasses of analytic functions associated with fractional q-calculus operators*, Math. Scand. **109** (2011), 55–70. `https://doi.org/10.7146/math.scand.a-15177`
[36] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, 1975.
[37] J. Jothibasu, *Certain subclasses of bi-univalent functions defined by salagean operator*, Electron. J. Math. Anal. Appl. **3**(1) (2015), 150–157.
[38] D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, Math. Anal. Appl. (1988), 53–60. `https://doi.org/10.1016/B978-0-08-031636-9.50012-7`

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