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LIPSCHITZ p-APPROXIMATE SCHAUDER FRAMES

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Abstract. With the aim of representing subsets of Banach spaces as an infinite series using Lipschitz functions, we study a variant of metric frames which we call Lipschitz p-approximate Schauder frames (Lipschitz p-ASFs). We characterize Lipschitz p-ASFs and their duals completely using the canonical Schauder basis for classical sequence spaces. Similarity of Lipschitz p-ASF is introduced and characterized.

1. Introduction

Grochenig in 1991 introduced the notion of Banach frames [17] as a generalization of notion of frames for Hilbert spaces introduced by Duffin and Schaeffer in 1952 [11]. This notion originated from the study of atomic decompositions and coorbit spaces arising from square integrable representations of locally compact groups developed by Feichtinger and Grochenig in 1980's [13–15]. Casazza, Han and Larson in 2000 explored the connection between Banach frames and atomic decompositions and introduced the notion of (unconditional) Schauder frames [8]. In 2001, Aldroubi, Sun and Tang introduced the notion of p-frames and p-Riesz bases for Banach spaces, $1 \leq p < +\infty$ [1]. These notions have been generalized by Casazza, Christensen and Stoeva by introducing the notion of $\mathcal{X}_d$-frames [5, 9]. A slight variant notion of $\mathcal{X}_d$-frames for Banach spaces was given by Terekhin [24–26]. In 2014, Thomas, Freeman, Odell, Schlumprecht and Zsak [16, 27, 28] introduced the notion of approximate Schauder frames as a generalization of notion of Schauder frames by Casazza, Dilworth, Odell, Schlumprecht and Zsak [4] (also see [10]). In 2021, Krishna and Johnson characterized some classes of

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approximate Schauder frames [21]. In 2022, Krishna and Johnson introduced metric frames which have surprising connections with subsets of Banach spaces using Lipschitz-free Banach spaces [22]. We now ask the following question which is the main motivation for writing the paper.

(1.1) Can we represent a subset (which need not be a subspace) of a Banach space as an infinite series using Lipschitz maps and elements of the set?

Note that we can not demand linear functionals in the above problem as we are not considering subspaces. Motivated from 1.1 we study representation of subsets (need not be subspaces) of Banach spaces using Lipschitz functions.

The paper is organized as follows. We introduce the notion of Lipschitz p-approximate Schauder frame (Lipschitz p-ASF) for subsets of Banach spaces in Definition 2.1. Followed by interesting Examples 2.1, 2.2 and 2.3, factorization property of Lipschitz frame map is derived in Theorem 2.1. Lipschitz p-ASFs are characterized in Theorem 2.2. Next we introduce the notion of dual frames in Definition 2.2 and classify them in Theorem 2.4. Definition 2.3 introduces the notion of similarity and Theorem 2.5 gives an operator-theoretic characterization for similarity. Orthogonality of frames is introduced in Definition 2.4 and interpolation result is derived in Theorem 2.6. We end by formulating an open Problem in Section 3.

2. Lipschitz p-Approximate Schauder Frames

Let $\mathcal{X}$ be a real or complex Banach space and $\mathcal{M}$ be a non-empty subset of $\mathcal{X}$. The identity operator on $\mathcal{M}$ is denoted by $I_{\mathcal{M}}$. The set of all Lipschitz functions from $\mathcal{M}$ to $\mathcal{X}$ is denoted by $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$. For $1 \leq p < +\infty$, the canonical Schauder basis for $\ell^p(\mathbb{N})$ is denoted by $\{e_n\}_n$ and its coordinate functionals are denoted by $\{\zeta_n\}_n$. We introduce the following important notion as a first step in answering Motivation 1.1.

Definition 2.1. For $1 \leq p < +\infty$, let $\mathcal{X}$ be a Banach space and $\mathcal{M}$ be a subset (need not be a subspace) of $\mathcal{X}$. Let $\{\tau_n\}_n$ be a sequence in $\mathcal{M}$ and $\{f_n\}_n$ be a sequence in $\operatorname{Lip}(\mathcal{M}, \mathcal{X})$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a **Lipschitz p-approximate Schauder frame** (we write Lipschitz p-ASF) if the following conditions hold.

(i) The map (**analysis map**)
$$\theta_f : \mathcal{M} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N})$$
is a well-defined Lipschitz map.

(ii) The map (**synthesis operator**)
$$\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau \{a_n\}_n := \sum_{n=1}^{+\infty} a_n \tau_n \in \mathcal{X}$$
is a well-defined bounded linear operator.

(iii) The map (**Lipschitz frame map**)

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^{+\infty} f_n(x)\tau_n \in \mathcal{M}$$

is a well-defined invertible bi-Lipschitz map and

$$x = \sum_{n=1}^{+\infty} f_n(x) S_{f,\tau}^{-1}\tau_n, \quad \text{for all } x \in \mathcal{M}. \tag{2.1}$$

If $S_{f,\tau} = I_{\mathcal{M}}$, then we say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a **Lipschitz p-Schauder frame** (we write Lipschitz p-SF). If we do not impose the condition 'invertible bi-Lipschitz' and Equation (2.1) in (iii), then we say that $(\{f_n\}_n, \{\tau_n\}_n)$ is a **Lipschitz p-Bessel sequence** (we write Lipschitz p-BS) for $\mathcal{M}$.

Whenever $\mathcal{M} = \mathcal{X}$, and f_n's are all linear, Definition 2.1 reduces to definition of p-ASF given in [21]. It is important to note that the partial sums of series in (iii) of Definition 2.1 need not be inside $\mathcal{M}$ (which may not be as it is only a subset) but only demanding limit has to be inside $\mathcal{M}$. Definition 2.1 says that there are $a, b, c, d > 0$ satisfying following:

$$a\|x - y\| \le \left\|\sum_{n=1}^{+\infty}(f_n(x) - f_n(y))\tau_n\right\| \le b\|x - y\|, \quad \text{for all } x, y \in \mathcal{M},$$

$$\left(\sum_{n=1}^{+\infty}|f_n(x) - f_n(y)|^p\right)^{\frac{1}{p}} \le c\|x - y\|, \quad \text{for all } x, y \in \mathcal{M},$$

$$\left\|\sum_{n=1}^{+\infty} a_n\tau_n\right\| \le d\left(\sum_{n=1}^{+\infty}|a_n|^p\right)^{\frac{1}{p}}, \quad \text{for all } \{a_n\}_n \in \ell^p(\mathbb{N}).$$

We call a as lower Lipschitz frame bound, b as upper Lipschitz frame bound, c as Lipschitz analysis bound and d as Lipschitz synthesis bound. We give various interesting examples of Lipschitz p-ASFs.

Example 2.1. Let $\mathcal{X} := \mathbb{C}$, $p = 1$ and

$$\mathcal{M} := \left\{z \in \mathbb{C} : |z| \le \frac{1}{2}|z+1|\right\} = \left\{x + iy : x, y \in \mathbb{R}, \left(x - \frac{1}{3}\right)^2 + y^2 \le \left(\frac{2}{3}\right)^2\right\}.$$

For $n \in \mathbb{N}$, define

$$f_n : \mathcal{M} \ni z \mapsto f_n(z) := \left(\frac{z}{1+z}\right)^n \in \mathbb{C}, \quad \tau_n := 1 \in \mathcal{M}.$$

We first show that f_n is Lipschitz for all n. For $z \in \mathcal{M}$,

$$1 - |z+1| \le \Big|1 - |z+1|\Big| \le \Big|1 - (z+1)\Big| = |z| \le \frac{1}{2}|z+1|.$$

Hence,

$$|z+1| \geq \frac{2}{3}, \quad \text{for all } z \in \mathcal{M}.$$

Let $z, w \in \mathcal{M}$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned}|f_n(z) - f_n(w)| &= \left|\left(\frac{z}{1+z}\right)^n - \left(\frac{w}{1+w}\right)^n\right| \\ &= \left|\frac{z}{1+z} - \frac{w}{1+w}\right| \cdot \left|\left(\frac{z}{1+z}\right)^{n-1} + \cdots + \left(\frac{w}{1+w}\right)^{n-1}\right| \\ &\leq \frac{|z-w|}{|1+z| \cdot |1+w|} \cdot \frac{n}{2^{n-1}} \\ &\leq \frac{9}{4} \cdot \frac{n}{2^{n-1}}|z-w|.\end{aligned}$$

Therefore, each f_n is Lipschitz. Set

$$r \coloneqq \sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} < +\infty.$$

We then see that for $z, w \in \mathcal{M}$,

$$\begin{aligned}\|\theta_f z - \theta_f w\| &= \sum_{n=1}^{+\infty} |f_n(z) - f_n(w)| = \sum_{n=1}^{+\infty} \left|\left(\frac{z}{1+z}\right)^n - \left(\frac{w}{1+w}\right)^n\right| \\ &\leq \sum_{n=1}^{+\infty} \frac{9}{4} \cdot \frac{n}{2^{n-1}}|z-w| = \frac{9}{4}r|z-w|.\end{aligned}$$

Therefore, θ_f is Lipschitz. Clearly,

$$\theta_\tau : \ell^1(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{+\infty} a_n \cdot 1 \in \mathbb{C}$$

is a well-defined bounded linear operator. Finally, we observe that for $z \in \mathcal{M}$, we have $\frac{|z|}{|z+1|} < 1$ and hence

$$S_{f,\tau} z = \sum_{n=1}^{+\infty} f_n(z)\tau_n = \sum_{n=1}^{+\infty} \left(\frac{z}{1+z}\right)^n \cdot 1 = \frac{1}{1 - \frac{z}{1+z}} - 1 = z, \quad \text{for all } z \in \mathcal{M}.$$

Thus, we proved that $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz 1-SF for $\mathcal{M}$.

Example 2.2. Let $\mathcal{X} \coloneqq \mathbb{R}$, $p = 1$ and $\mathcal{M} \coloneqq [1, +\infty)$ For $n \in \mathbb{N} \cup \{0\}$, define $f_n : \mathcal{M} \to \mathbb{R}$ by

$$\begin{aligned}f_0(x) &\coloneqq 1, \quad \text{for all } x \in \mathcal{M}, \\ f_n(x) &\coloneqq \frac{(\log x)^n}{n!}, \quad \text{for all } x \in \mathcal{M}, \text{ for all } n \geq 1\end{aligned}$$

and $\tau_n \coloneqq 1 \in \mathcal{M}$. Then $f_n'(x) = \frac{(\log x)^{(n-1)}}{(n-1)!x}$, for all $x \in \mathcal{M}$, for all $n \geq 1$. Since f_n' is bounded on $\mathcal{M}$ for all $n \geq 1$, f_n is Lipschitz on $\mathcal{M}$ for all $n \geq 1$. For $x, y \in \mathcal{M}$, with $x < y$, we see that

$$\begin{aligned}\|\theta_f x - \theta_f y\| &= \sum_{n=0}^{+\infty} |f_n(x) - f_n(y)| = \sum_{n=0}^{+\infty} \frac{(\log y)^n}{n!} - \sum_{n=0}^{+\infty} \frac{(\log x)^n}{n!} \\ &= e^{\log y} - e^{\log x} = y - x = |x - y|.\end{aligned}$$

Therefore, θ_f is Lipschitz. It is clear that θ_τ is a well-defined bounded linear operator. For $x \in \mathcal{M}$,

$$S_{f,\tau}x = \sum_{n=1}^{+\infty} f_n(x)\tau_n = \sum_{n=0}^{+\infty} \frac{(\log x)^n}{n!} \cdot 1 = x.$$

Hence, $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz 1-SF for $\mathcal{M}$.

Example 2.3. For $1 \leq p < +\infty$, let $\mathcal{X}$ be a Banach space and $\mathcal{M}$ be a subset of $\mathcal{X}$. Assume that there is a Lipschitz map $U : \mathcal{M} \to \ell^p(\mathbb{N})$, a bounded linear operator $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, $VU : \mathcal{M} \to \mathcal{M}$ is an invertible bi-Lipschitz map and

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1}Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define

$$f_n \coloneqq \zeta_n U, \quad \tau_n \coloneqq Ve_n, \quad \text{for all } n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is Lipschitz p-ASF for $\mathcal{M}$. If $VU = I_{\mathcal{M}}$, then $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz p-SF for $\mathcal{M}$.

We show in the sequel that (in Theorem 2.2) every Lipschitz p-ASF can be written in the form of Example 2.3. Following theorem gives various fundamental factorization properties of Lipschitz p-ASFs whose proof is a direct calculation.

Theorem 2.1. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$. Then the following hold.*

(i) *We have*

$$x = \sum_{n=1}^{+\infty} (f_n S_{f,\tau}^{-1})(x)\tau_n, \quad \text{for all } x \in \mathcal{M}. \tag{2.2}$$

(ii) $(\{f_n S_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ *is a Lipschitz p-ASF for $\mathcal{M}$.*
(iii) *The analysis map θ_f is injective.*
(iv) *The synthesis operator θ_τ is surjective.*
(v) *Lipschitz frame map $S_{f,\tau}$ factors as $S_{f,\tau} = \theta_\tau\theta_f$.*
(vi) $P_{f,\tau} \coloneqq \theta_f S_{f,\tau}^{-1}\theta_\tau : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ *is a Lipschitz projection onto $\theta_f(\mathcal{M})$.*

Holub characterized frames for Hilbert spaces using standard orthonormal basis for the standard Hilbert space [20]. This result has been derived for Banach spaces in [21]. We show that such a result can be derived for Lipschitz p-ASFs.

Theorem 2.2. *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$ if and only if*

$$f_n = \zeta_n U, \quad \tau_n = Ve_n, \quad \text{for all } n \in \mathbb{N},$$

where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, $VU : \mathcal{M} \to \mathcal{M}$ is an invertible bi-Lipschitz map and

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1}Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

Proof. ($\Leftarrow$) Clearly θ_f is Lipschitz and θ_τ is a bounded linear operator. Now let $x \in \mathcal{M}$. Then

$$S_{f,\tau}x = \sum_{n=1}^{+\infty} f_n(x)\tau_n = \sum_{n=1}^{+\infty} \zeta_n(Ux)Ve_n = V\left(\sum_{n=1}^{+\infty} \zeta_n(Ux)e_n\right) = VUx. \tag{2.3}$$

Hence, $S_{f,\tau}$ is an invertible bi-Lipschitz map.

($\Rightarrow$) Define $U \coloneqq \theta_f$, $V \coloneqq \theta_\tau$. Then $(\zeta_n U)(x) = (\zeta_n\theta_f)(x) = \zeta_n(\{f_k(x)\}_k) = f_n(x)$, for all $x \in \mathcal{M}$, $Ve_n = \theta_\tau e_n = \tau_n$, for all $n \in \mathbb{N}$ and $VU = \theta_\tau\theta_f = S_{f,\tau}$ which is an invertible bi-Lipschitz map. □

Corollary 2.1. (i) *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz p-SF for $\mathcal{M} \subseteq \mathcal{X}$ if and only if $f_n = \zeta_n U, \tau_n = Ve_n$, for all $n \in \mathbb{N}$, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $VU = I_{\mathcal{M}}$.*

(ii) *A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is a Lipschitz p-BS for $\mathcal{M} \subseteq \mathcal{X}$ if and only if $f_n = \zeta_n U, \tau_n = Ve_n$, for all $n \in \mathbb{N}$, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator such that $VU(\mathcal{M}) \subseteq \mathcal{M}$ and $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$.*

Equations (2.1) and (2.2) lead us to define the notion of dual frame as follows.

Definition 2.2. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$. A Lipschitz p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$ is said to be a **dual** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$x = \sum_{n=1}^{+\infty} g_n(x)\tau_n = \sum_{n=1}^{+\infty} f_n(x)\omega_n, \quad \text{for all } x \in \mathcal{M}.$$

We can give a characterization of dual frames by using analysis map and synthesis operator.

Proposition 2.1. *Given two Lipschitz p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$, the following are equivalent:*

(a) $(\{g_n\}_n, \{\omega_n\}_n)$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$*;*
(b) $\theta_\tau\theta_g = \theta_\omega\theta_f = I_{\mathcal{M}}$.

Equations (2.1) and (2.2) show that the Lipschitz p-ASF $(\{f_n S_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ is a dual for $(\{f_n\}_n, \{\tau_n\}_n)$. We call $(\{f_n S_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ as the **canonical dual** for $(\{f_n\}_n, \{\tau_n\}_n)$. With this notion, the following theorem is evident.

Theorem 2.3. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$ *with frame bounds a and b. Then the following statements hold good.*

(a) *The canonical dual for the canonical dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *is itself.*
(b) $\frac{1}{b}, \frac{1}{a}$ *are frame bounds for the canonical dual for* $(\{f_n\}_n, \{\tau_n\}_n)$.
(c) *If* a, b *are optimal frame bounds for* $(\{f_n\}_n, \{\tau_n\}_n)$*, then* $\frac{1}{b}, \frac{1}{a}$ *are optimal frame bounds for its canonical dual.*

In 1995, Li derived a characterization of dual frames using standard orthonormal basis for $\ell^2(\mathbb{N})$ [23]. For Banach spaces, such a characterization using canonical Schauder basis for $\ell^p(\mathbb{N})$ is derived in [21]. Now we derive such characterization for Lipschitz p-ASF.

Lemma 2.1. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then a Lipschitz p-ASF* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M}$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *if and only if*

$$g_n = \zeta_n U, \quad \omega_n = V e_n, \quad \text{for all } n \in \mathbb{N},$$

where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz right-inverse of* θ_τ *and* $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a linear bounded left-inverse of* θ_f *such that* $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ *for all* $n \in \mathbb{N}$, VU *is an invertible bi-Lipschitz map and*

$$x = \sum_{n=1}^{+\infty} \zeta_n(Ux)(VU)^{-1}Ve_n, \quad \text{for all } x \in \mathcal{M}.$$

Proof. ($\Leftarrow$) Using the 'if' part of proof of Theorem 2.2, we get that $(\{g_n\}_n, \{\omega_n\}_n)$ is a Lipschitz p-ASF for $\mathcal{M}$. We check for duality of $(\{g_n\}_n, \{\omega_n\}_n)$: $\theta_\tau\theta_g = \theta_\tau U = I_{\mathcal{M}}$, $\theta_\omega\theta_f = V\theta_f = I_{\mathcal{M}}$.

($\Rightarrow$) Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a dual Lipschitz p-ASF for $(\{f_n\}_n, \{\tau_n\}_n)$. Then $\theta_\tau\theta_g = I_{\mathcal{M}}$, $\theta_\omega\theta_f = I_{\mathcal{M}}$. Define $U := \theta_g, V := \theta_\omega$. Then $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz right-inverse of θ_τ and $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a linear bounded left-inverse of θ_f such that the operator $VU = \theta_\omega\theta_g = S_{g,\omega}$ is invertible. Further,

$$(\zeta_n U)x = \zeta_n\left(\sum_{k=1}^{+\infty} g_k(x)e_k\right) = \sum_{k=1}^{+\infty} g_k(x)\zeta_n(e_k) = g_n(x), \quad \text{for all } x \in \mathcal{M},$$

and $Ve_n = \theta_\omega e_n = \omega_n$, for all $n \in \mathbb{N}$. □

Lemma 2.2. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then,*

(i) $R : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz right-inverse of* θ_τ *if and only if*

$$R = \theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U$$

where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz map;*

(ii) $L : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded left-inverse of* θ_f *if and only if*

$$L = S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right),$$

where $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded linear operator.*

Proof. (i) ($\Leftarrow$) $\theta_\tau\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U\right) = I_{\mathcal{M}} + \theta_\tau U - I_{\mathcal{M}}\theta_\tau U = I_{\mathcal{M}}$. Therefore, $\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U$ is a Lipschitz right-inverse of θ_τ.

($\Rightarrow$) Define $U \coloneqq R$. Then,

$$\begin{aligned}\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U &= \theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) R\\ &= \theta_f S_{f,\tau}^{-1} + R - \theta_f S_{f,\tau}^{-1} = R.\end{aligned}$$

(ii) ($\Leftarrow$) $\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\theta_f = I_{\mathcal{M}} + V\theta_f - V\theta_f I_{\mathcal{M}} = I_{\mathcal{M}}$. Therefore, $S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)$ is a bounded left-inverse of θ_f.

($\Rightarrow$) Define $V \coloneqq L$. Then,

$$\begin{aligned}S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) &= S_{f,\tau}^{-1}\theta_\tau + L\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\\ &= S_{f,\tau}^{-1}\theta_\tau + L - S_{f,\tau}^{-1}\theta_\tau = L.\end{aligned}$$

□

Theorem 2.4. *Let* $(\{f_n\}_n, \{\tau_n\}_n)$ *be a Lipschitz p-ASF for* $\mathcal{M} \subseteq \mathcal{X}$. *Then a Lipschitz p-ASF* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M}$ *is a dual for* $(\{f_n\}_n, \{\tau_n\}_n)$ *if and only if*

$$\begin{aligned}g_n &= f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,\\ \omega_n &= S_{f,\tau}^{-1}\tau_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

such that

$$S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U$$

is an invertible bi-Lipschitz map, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ *is a Lipschitz map,* $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ *is a bounded linear operator,* $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ *for all* $n \in \mathbb{N}$ *and*

$$\begin{aligned}&\sum_{n=1}^{+\infty}\zeta_n\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right) U\right) x\left[S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U\right]^{-1}\\ &\times\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right) e_n = x, \quad \text{for all } x \in \mathcal{M}.\end{aligned}$$

Proof. Lemmas 2.1 and 2.2 give the characterization of dual frame as

$$\begin{aligned}&g_n = \zeta_n\theta_f S_{f,\tau}^{-1} + \zeta_n U - \zeta_n\theta_f S_{f,\tau}^{-1}\theta_\tau U = f_n S_{f,\tau}^{-1} + \zeta_n U - f_n S_{f,\tau}^{-1}\theta_\tau U,\\ &\omega_n = S_{f,\tau}^{-1}\theta_\tau e_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\theta_\tau e_n = S_{f,\tau}^{-1}\tau_n + Ve_n - V\theta_f S_{f,\tau}^{-1}\tau_n, \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

such that

$$\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)$$

is an invertible bi-Lipschitz map, where $U : \mathcal{M} \to \ell^p(\mathbb{N})$ is a Lipschitz map, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ is a bounded linear operator, $VU(\mathcal{M}) \subseteq \mathcal{M}$, $Ve_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and for all $x \in \mathcal{M}$

$$\begin{aligned}&\sum_{n=1}^{+\infty} \zeta_n \left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)x\\ &\times W^{-1}\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)e_n = x,\end{aligned}$$

where

$$W \coloneqq \left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right).$$

Through expansion and simplification we get

$$\begin{aligned}&\left(S_{f,\tau}^{-1}\theta_\tau + V\left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)\right)\left(\theta_f S_{f,\tau}^{-1} + \left(I_{\ell^p(\mathbb{N})} - \theta_f S_{f,\tau}^{-1}\theta_\tau\right)U\right)\\ =&S_{f,\tau}^{-1} + VU - V\theta_f S_{f,\tau}^{-1}\theta_\tau U.\end{aligned}$$

□

Balan introduced the notion of similarity for frames for Hilbert space which gives an equivalence relation on frames [3]. It has been done for Banach spaces by Krishna and Johnson in [21]. We define the same for Lipschitz p-ASF as follows.

Definition 2.3. Two Lipschitz p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$ are said to be **similar** or **equivalent** if there exist invertible bi-Lipschitz map $T_{f,g} : \mathcal{M} \to \mathcal{M}$ and an invertible bounded linear operator $T_{\tau,\omega} : \mathcal{X} \to \mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M}) \subseteq \mathcal{M}$ and

$$g_n = f_n T_{f,g}, \quad \omega_n = T_{\tau,\omega}\tau_n, \qquad \text{for all } n \in \mathbb{N}.$$

Since maps giving similarity are invertible, similarity is an equivalence relation on the set $\{(\{f_n\}_n, \{\tau_n\}_n) : (\{f_n\}_n, \{\tau_n\}_n) \text{ is a Lipschitz } p\text{-ASF for } \mathcal{M}\}$. Observe that for every Lipschitz p-ASF $(\{f_n\}_n, \{\tau_n\}_n)$, both $(\{f_n S_{f,\tau}^{-1}\}_n, \{\tau_n\}_n)$ and $(\{f_n\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ are Lipschitz p-ASFs and are similar to $(\{f_n\}_n, \{\tau_n\}_n)$. Balan gave an operator algebraic characterization of similarity in Hilbert spaces [3] and it is extended to Banach spaces by Krishna and Johnson in [21]. We derive Lipschitz version in the following theorem.

Theorem 2.5. *For two Lipschitz p-ASFs $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M} \subseteq \mathcal{X}$, the following are equivalent:*

(a) $g_n = f_n T_{f,g}$, $\omega_n = T_{\tau,\omega}\tau_n$, *for all $n \in \mathbb{N}$, for some invertible bi-Lipschitz map $T_{f,g} : \mathcal{M} \to \mathcal{M}$, for some invertible linear map $T_{\tau,\omega} : \mathcal{X} \to \mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M}) \subseteq \mathcal{M}$;*

(b) $\theta_g = \theta_f T_{f,g}$, $\theta_\omega = T_{\tau,\omega}\theta_\tau$, *for some invertible bi-Lipschitz map $T_{f,g} : \mathcal{M} \to \mathcal{M}$, for some invertible linear map $T_{\tau,\omega} : \mathcal{X} \to \mathcal{X}$ such that $T_{\tau,\omega}(\mathcal{M}) \subseteq \mathcal{M}$;*

(c) $P_{g,\omega} = P_{f,\tau}$.

If one of the above conditions is satisfied, then invertible maps in (i) *and* (ii) *are unique and given by* $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g$, $T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. *In the case that* $(\{f_n\}_n, \{\tau_n\}_n)$ *is a Lipschitz p-SF, then* $(\{g_n\}_n, \{\omega_n\}_n)$ *is a Lipschitz p-SF if and only if* $T_{\tau,\omega}T_{f,g} = I_{\mathcal{M}}$ *if and only if* $T_{f,g}T_{\tau,\omega} = I_{\mathcal{M}}$.

Proof. (i) ⇒ (ii) Let $x \in \mathcal{M}$ and $\{a_n\}_n \in \ell^p(\mathbb{N})$. Then

$$\theta_g x = \{g_n(x)\}_n = \{f_n(T_{f,g}x)\}_n = \theta_f(T_{f,g}x),$$
$$\theta_\omega(\{a_n\}_n) = \sum_{n=1}^{+\infty} a_n\omega_n = \sum_{n=1}^{+\infty} a_n T_{\tau,\omega}\tau_n = T_{\tau,\omega}\theta_\tau\{a_n\}_n.$$

(ii) ⇒ (iii) $S_{g,\omega} = \theta_\omega\theta_g = T_{\tau,\omega}\theta_\tau\theta_f T_{f,g} = T_{\tau,\omega}S_{f,\tau}T_{f,g}$ and

$$P_{g,\omega} = \theta_g S_{g,\omega}^{-1}\theta_\omega = (\theta_f T_{f,g})(T_{\tau,\omega}S_{f,\tau}T_{f,g})^{-1}(T_{\tau,\omega}\theta_\tau) = P_{f,\tau}.$$

(ii) ⇒ (i) $\sum_{n=1}^{+\infty} g_n(x)e_n = \theta_g(x) = \theta_f(T_{f,g}x) = \sum_{n=1}^{+\infty} f_n(T_{f,g}x)e_n$, for all $x \in \mathcal{M}$. This gives (i).

(iii) ⇒ (ii) $\theta_g = P_{g,\omega}\theta_g = P_{f,\tau}\theta_g = \theta_f(S_{f,\tau}^{-1}\theta_\tau\theta_g)$ and $\theta_\omega = \theta_\omega P_{g,\omega} = \theta_\omega P_{f,\tau} = (\theta_\omega\theta_f S_{f,\tau}^{-1})\theta_\tau$. We show that $S_{f,\tau}^{-1}\theta_\tau\theta_g$ and $\theta_\omega\theta_f S_{f,\tau}^{-1}$ are invertible. For,

$$(S_{f,\tau}^{-1}\theta_\tau\theta_g)(S_{g,\omega}^{-1}\theta_\omega\theta_f) = S_{f,\tau}^{-1}\theta_\tau P_{g,\omega}\theta_f = S_{f,\tau}^{-1}\theta_\tau P_{f,\tau}\theta_f = I_{\mathcal{M}},$$
$$(S_{g,\omega}^{-1}\theta_\omega\theta_f)(S_{f,\tau}^{-1}\theta_\tau\theta_g) = S_{g,\omega}^{-1}\theta_\omega P_{f,\tau}\theta_g = S_{g,\omega}^{-1}\theta_\omega P_{g,\omega}\theta_g = I_{\mathcal{M}}$$

and

$$(\theta_\omega\theta_f S_{f,\tau}^{-1})(\theta_\tau\theta_g S_{g,\omega}^{-1}) = \theta_\omega P_{f,\tau}\theta_g S_{g,\omega}^{-1} = \theta_\omega P_{g,\omega}\theta_g S_{g,\omega}^{-1} = I_{\mathcal{M}},$$
$$(\theta_\tau\theta_g S_{g,\omega}^{-1})(\theta_\omega\theta_f S_{f,\tau}^{-1}) = \theta_\tau P_{g,\omega}\theta_f S_{f,\tau}^{-1} = \theta_\tau P_{f,\tau}\theta_f S_{f,\tau}^{-1} = I_{\mathcal{M}}.$$

Let $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ be invertible bi-Lipschitz maps and $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n$, for all $n \in \mathbb{N}$. Then $\theta_g = \theta_f T_{f,g}$ says that $\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g}$ which implies $T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g$. Similarly, $\theta_\omega = T_{\tau,\omega}\theta_\tau$ says that $\theta_\omega\theta_f = T_{\tau,\omega}\theta_\tau\theta_f = T_{\tau,\omega}S_{f,\tau}$. Hence, $T_{\tau,\omega} = \theta_\omega\theta_f S_{f,\tau}^{-1}$. □

In Definition 2.2 we defined the notion of dual frames [2,18,19] and for Banach spaces in [21]. We can define the orthogonality for Lipschitz p-ASFs as follows.

Definition 2.4. Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$. A Lipschitz p-ASF $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{M}$ is said to be **orthogonal** for $(\{f_n\}_n, \{\tau_n\}_n)$ if

$$0 = \sum_{n=1}^{+\infty} g_n(x)\tau_n = \sum_{n=1}^{+\infty} f_n(x)\omega_n, \quad \text{for all } x \in \mathcal{M}.$$

Similar to Proposition 2.1 we have the following proposition.

Proposition 2.2. *Given two Lipschitz p-ASFs* $(\{f_n\}_n, \{\tau_n\}_n)$ *and* $(\{g_n\}_n, \{\omega_n\}_n)$ *for* $\mathcal{M} \subseteq \mathcal{X}$*, the following are equivalent:*

(a) $(\{g_n\}_n, \{\omega_n\}_n)$ *is orthogonal for* $(\{f_n\}_n, \{\tau_n\}_n)$*;*

(b) $\theta_\tau\theta_g = \theta_\omega\theta_f = 0$.

Using orthogonality we derive following interpolation result. For the Hilbert space frames this is derived by Han and Larson in [19] and for Banach spaces in [21].

Theorem 2.6. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two Lipschitz p-SF for $\mathcal{M} \subseteq \mathcal{X}$ which are orthogonal. If $A, B : \mathcal{M} \to \mathcal{M}$ are bi-Lipschitz maps, $C, D : \mathcal{X} \to \mathcal{X}$ are bounded linear operators, $C(\mathcal{M}) \subseteq \mathcal{M}$, $D(\mathcal{M}) \subseteq \mathcal{M}$ and $CA + DB = I_{\mathcal{M}}$, then $(\{f_nA + g_nB\}_n, \{C\tau_n + D\omega_n\}_n)$ is a Lipschitz p-SF for $\mathcal{M}$. In particular, if scalars a, b, c, d satisfy $ca+db = 1$, then $(\{af_n+bg_n\}_n, \{c\tau_n+d\omega_n\}_n)$ is a Lipschitz p-SF for $\mathcal{M}$.*

Proof. We find

$$\begin{aligned}\theta_{fA+gB}x &= \{(f_nA + g_nB)(x)\}_n = \{f_n(Ax)\}_n + \{g_n(Bx)\}_n\\ &= \theta_f(Ax) + \theta_g(Bx), \quad \text{for all } x \in \mathcal{M}\end{aligned}$$

and

$$\begin{aligned}\theta_{C\tau+D\omega}\{a_n\}_n &= \sum_{n=1}^{+\infty} a_n(C\tau_n + D\omega_n)\\ &= C\theta_\tau\{a_n\}_n + D\theta_\omega\{a_n\}_n, \quad \text{for all } \{a_n\}_n \in \ell^p(\mathbb{N}).\end{aligned}$$

So

$$\begin{aligned}S_{fA+gB,C\tau+D\omega} &= \theta_{C\tau+D\omega}\theta_{fA+gB} = (C\theta_\tau + D\theta_\omega)(\theta_fA + \theta_gB)\\ &= C\theta_\tau\theta_fA + C\theta_\tau\theta_gB + D\theta_\omega\theta_fA + D\theta_\omega\theta_gB\\ &= CS_{f,\tau}A + 0 + 0 + DS_{g,\omega}B = CI_{\mathcal{M}}A + DI_{\mathcal{M}}B = I_{\mathcal{M}}. \qquad \square\end{aligned}$$

We use Theorem 2.5 to relate three notions duality, similarity and orthogonality.

Proposition 2.3. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$. Then the canonical dual $(\{f_nS_{f,\tau}^{-1}\}_n, \{S_{f,\tau}^{-1}\tau_n\}_n)$ is the only dual Lipschitz p-ASF that is similar to $(\{f_n\}_n, \{\tau_n\}_n)$.*

Proof. Let $(\{g_n\}_n, \{\omega_n\}_n)$ be a Lipschitz p-ASF for $\mathcal{M} \subseteq \mathcal{X}$ which is both similar and dual for $(\{f_n\}_n, \{\tau_n\}_n)$. Then there exist invertible bi-Lipschitz maps $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ such that $g_n = f_nT_{f,g}, \omega_n = T_{\tau,\omega}\tau_n$, for all $n \in \mathbb{N}$. Theorem 2.5 then gives

$$T_{f,g} = S_{f,\tau}^{-1}\theta_\tau\theta_g = S_{f,\tau}^{-1}I_{\mathcal{M}} = S_{f,\tau}^{-1} \text{ and } T_{\tau,\omega} = \theta_\omega\theta_fS_{f,\tau}^{-1} = I_{\mathcal{M}}S_{f,\tau}^{-1} = S_{f,\tau}^{-1}.$$

Hence, $(\{g_n\}_n, \{\omega_n\}_n)$ is the canonical dual for $(\{f_n\}_n, \{\tau_n\}_n)$. $\square$

Proposition 2.4. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two similar Lipschitz p-ASFs for $\mathcal{M} \subseteq \mathcal{X}$. Then $(\{f_n\}_n, \{\tau_n\}_n)$ is not orthogonal for $(\{g_n\}_n, \{\omega_n\}_n)$.*

Proof. Since $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ similar, there exist invertible bi-Lipschitz maps $T_{f,g}, T_{\tau,\omega} : \mathcal{M} \to \mathcal{M}$ such that $g_n = f_n T_{f,g}, \omega_n = T_{\tau,\omega}\tau_n,$ for all $n \in \mathbb{N}$. Theorem 2.5 then says $\theta_g = \theta_f T_{f,g}, \theta_\omega = T_{\tau,\omega}\theta_\tau$. Therefore,

$$\theta_\tau\theta_g = \theta_\tau\theta_f T_{f,g} = S_{f,\tau}T_{f,g} \neq 0.$$

Orthogonality condition demands $\theta_\tau\theta_g = 0$ whereas above equation says it is not true. □

Another use of orthogonal frames is to take direct sum. Given Lipschitz maps $f, g : \mathcal{M} \to \mathbb{K}$, we define $f \oplus g : \mathcal{M} \oplus \mathcal{M} \ni x \oplus y \mapsto f(x) + g(y) \in \mathbb{K}$.

Theorem 2.7. *Let $(\{f_n\}_n, \{\tau_n\}_n)$ and $(\{g_n\}_n, \{\omega_n\}_n)$ be two Lipschitz p-ASFs for $\mathcal{M} \subseteq \mathcal{X}$ which are orthogonal. Then $(\{f_n \oplus g_n\}_n, \{\tau_n \oplus \omega_n\}_n)$ is a Lipschitz p-ASF for $\mathcal{M} \oplus \mathcal{M} \subseteq \mathcal{X} \oplus \mathcal{X}$.*

Proof. Let $x \oplus y \in \mathcal{M} \oplus \mathcal{M}$. Then,

$$\begin{aligned} S_{f\oplus g,\tau\oplus\omega}(x \oplus y) &= \sum_{n=0}^{+\infty}(f_n \oplus g_n)(x \oplus y)(\tau_n \oplus \omega_n) \\ &= \left(\sum_{n=0}^{+\infty} f_n(x)\tau_n + \sum_{n=0}^{+\infty} g_n(x)\tau_n\right) \oplus \left(\sum_{n=0}^{+\infty} f_n(x)\omega_n + \sum_{n=0}^{+\infty} g_n(x)\omega_n\right) \\ &= (S_{f,\tau}x + 0) \oplus (0 + S_{g,\omega}y) = (S_{f,\tau} \oplus S_{g,\omega})(x \oplus y). \end{aligned}$$ □

3. An Open Problem

Motivated from the approximation properties of Banach spaces (Schauder basis problem) [6,12] and from the failure of atomic decompositions for (even separable) Banach spaces (see [7]), we formulate the following interesting (high-end) problem: Can anyone classify subsets of a Banach space having a Lipschitz p-ASF, for some $1 \leq p < +\infty$? In particular, does every subset of a Banach space have Lipschitz p-ASF, for some $1 \leq p < +\infty$?

4. Conclusions

In the literature, only frames coming from inner products and linear functionals are studied. The paper [22] is the first one to introduce and make a systematic study of frames for metric spaces by using Lipschitz functions. In this paper, we define a class of non-linear frames for subsets (need not be subspaces) of Banach spaces which can be characterized using standard Schauder basis and Lipschitz functions on sequence spaces. We derived Holub's theorem [20] in non-linear form. Duals and similar frames in non-linear form are also characterized.

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