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TOPOLOGICAL DEGREE METHOD FOR A CLASS OF Ψ-CAPUTO FRACTIONAL DIFFERENTIAL LANGEVIN EQUATION

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Abstract. This paper deals with the existence and uniqueness of solution for a new class of Ψ-Caputo fractional differential Langevin equation. The suggested study is based on some basic definitions of topological degree theory and fractional calculus. We established the existence result by using the topological degree method for condensing maps, and by means of Banach's fixed point theorem we obtained the uniqueness result. As application, we give an illustrative example to demonstrate our theoretical result.

1. Introduction

Newly, fractional differential equations have attracted the interest of many mathematicians, because it can represent and verified to be effective modeling of many phenomena in several fields of science as physics, mechanics, biology, chemistry, and control theory, and other domains for exemple, see [8, 13, 16, 19, 27, 33].

In 1908 Paul Langevin, introduced the Langevin equation of the form $m\frac{d^2w}{d\tau^2} = -\lambda\frac{dw}{d\tau} + \eta(\tau)$ where, $\frac{dw}{d\tau}$ is the velocity of the particle, and m is its mass and a noise term $\eta(\tau)$ representing the effect of the collisions with the molecules of the fluid. For the removal of the noise term, mathematicians used fractional order differential equations, for this reason it is very important to study Langevin equations via fractional derivatives, for more details see [3, 4, 22–25, 28, 31].

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There are several definitions of fractional integrals and derivatives, the popular definitions are the Riemann-Liouville and the Caputo fractional derivatives, in [15], Almeida introduce the generalization of these derivatives under the name of Ψ-Caputo fractional derivative, for more details for Ψ-Caputo fractional derivative, we direct readers to the papers [2, 17, 20, 21, 29, 30]. Furthermore, distinct version of fixed point theorems are commonly utilized to prove the existence and uniqueness of solutions for various classes of fractional differential equations, Isaia [12] proved a new fixed theorem that was obtained via coincidence degree theory for condensing operators. This fixed point theorem due to Isaia was utilized by researchers to establish the existence of solutions for several classes of nonlinear differential equations [1, 5, 11, 14, 18].

Recently, Baitiche et al. [6], discussed the existence and uniqueness of solutions to some nonlinear fractional differential equations involving the Ψ-Caputo fractional derivative with multi-point boundary conditions based on the technique of topological degree theory for condensing maps and the Banach contraction principle. Faree and Panchal [9], investigated the existence and uniqueness of solutions to boundary value problems involving the Caputo fractional derivative in Banach space by topological structures with some appropriate conditions. Hilal et al. [10], discussed the existence and uniqueness of solution for a boundary value problem for the Langevin equation and inclusion, based on Krasnoselskii's fixed point theorem, Banach's contraction principle and Leray-Schauder's alternative. Rizwan [26], considered a non local boundary value problem of nonlinear fractional Langevin equation with non-instantaneous impulses. Baitiche et al. [32], proved the Ulam-Hyers stability of solutions for a new form of nonlinear fractional Langevin differential equations involving two fractional orders in the Ψ-Caputo sense.

Motivated by the mentioned works, and by using topological degree methods we investigate the existence and uniqueness result for the following problem

$$\tag{1.1} \begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}\left[{}^{\mathfrak{C}}\mathfrak{D}^{q;\Psi}_{a^+}+\lambda\right]w(\tau)=\varphi(\tau,w(\tau)), \quad \tau\in\Upsilon:=[a,b],\\ w(a)=0, \quad w'(a)=0, \quad w(b)=\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\beta_i;\Psi}_{a^+}w(\kappa_i). \end{cases}$$

The originality of this work is studing a new and a challenging case of fractional derivative named the Ψ-Caputo fractional derivative [15], this kind of fractional derivative generalize the well-known fractional derivatives, for different values of function Ψ such as the following.

- ⋆ If $\Psi(\tau)=\tau$, then Problem (1.1) reduces to Caputo-type fractional derivative.
- ⋆ If $\Psi(\tau)=\log(\tau)$, then Problem (1.1) reduces to Caputo-Hadamard-type fractional derivative.
- ⋆ If $\Psi(\tau)=\tau^{\rho}$, then Problem (1.1) reduces to Caputo-Katugampola-type fractional derivative.

The rest of the paper is organized as follows. In Section 2, we recall some theorems, notations, lemmas, and definitions from fractional calculus and important results of topological degree method that will be used throughout this study. In Section 3, based on the application of topological degree method, we discuss the existence result for the problem (1.1), and by making use of Banach's contraction principle we prove the uniqueness of solution. In Section 4, we give an example to support the main result.

2. Preliminaries

In this section, we introduce some definitions, lemmas and useful notations that we can used throughout this paper.

Denote by X a Banach space and Γ_X the class of non-empty and bounded subsets of X. $\mathcal{C}(\Upsilon, \mathbb{R})$ denote the Banach space of all continuous functions from Υ into $\mathbb{R}$ with the norm defined by $\|\varphi\| = \sup_{\tau\in\Upsilon}\{|\varphi(\tau)|\}$. We denote by $\mathcal{C}^n(\Upsilon, \mathbb{R})$ the n-times absolutely continuous functions given by $\mathcal{C}^n(\Upsilon, \mathbb{R}) = \left\{\varphi : \Upsilon \to \mathbb{R} : \varphi^{(n-1)} \in \mathcal{C}(\Upsilon, \mathbb{R})\right\}$. $\mathcal{B}_\rho(0)$ denote the closed ball centered at 0 with radius ρ.

Definition 2.1 ([15]). For $p > 0$, $\varphi \in \mathbb{L}^1(\Upsilon, \mathbb{R})$ and $\Psi \in \mathcal{C}^n(\Upsilon, \mathbb{R})$, with $\Psi'(\tau) > 0$, for all $\tau \in \Upsilon$, the Ψ-Riemann-Liouville fractional integral of order p of a function φ is defined by

$$\mathfrak{I}^{p;\Psi}_{a^+}\varphi(\tau) = \frac{1}{\Gamma(p)}\int_a^\tau \Psi'(\tau)(\Psi(\tau) - \Psi(s))^{p-1}\varphi(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ represents the gamma function.

Definition 2.2 ([15]). For $p > 0$, $\varphi \in \mathcal{C}^{n-1}(\Upsilon, \mathbb{R})$ and $\Psi \in \mathcal{C}^n(\Upsilon, \mathbb{R})$, with $\Psi'(\tau) > 0$, for all $\tau \in \Upsilon$, the Ψ-Caputo fractional derivative of order p of a function φ is defined by

$$^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}\varphi(\tau) = \mathfrak{I}^{n-p;\Psi}_{a^+}\varphi^{[k]}_\Psi(\tau) = \frac{1}{\Gamma(n-p)}\int_a^\tau \Psi'(\tau)(\Psi(\tau) - \Psi(s))^{n-p-1}\varphi^{[k]}_\Psi(s)ds,$$

where $\varphi^{[k]}_\Psi(\tau) = \left(\frac{1}{\Psi'(\tau)}\cdot\frac{d}{d\tau}\right)^n$, $n - 1 < p < n$, $n = [p] + 1$ and $[p]$ denotes the integer part of the real number p.

Lemma 2.1 ([15]). *Let $p, q > 0$. Then we have the following semigroup property given by*

$$\mathfrak{I}^{p;\Psi}_{a^+}\mathfrak{I}^{q;\Psi}_{a^+}\varphi(\tau) = \mathfrak{I}^{p+q;\Psi}_{a^+}\varphi(\tau), \quad \tau > a. \tag{2.2}$$

Proposition 2.1 ([15]). *Let $p > 0$, $\upsilon > 0$ and $\tau \in \Upsilon$. Then*

(i) $\mathfrak{I}^{p;\Psi}_{a^+}(\Psi\tau) - \Psi(a))^{\upsilon-1} = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+p)}(\Psi(\tau) - \Psi(a))^{\upsilon+p-1}$;

(ii) $^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}(\Psi(\tau) - \Psi(a))^{\upsilon-1} = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon-p)}(\Psi(\tau) - \Psi(a))^{\upsilon-p-1}$;

(iii) $^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}(\Psi(\tau) - \Psi(a))^k = 0$, *for all* $k < n \in \mathbb{N}$.

Lemma 2.2 ([15]). *If $\varphi \in \mathcal{C}^n(\Upsilon, \mathbb{R})$, $n - 1 < p < n$, then*

$$\mathfrak{I}^{p;\Psi}_{a^+}(^{\mathfrak{C}}\mathfrak{D}^{p;\Psi}_{a^+}\varphi)(\tau) = \varphi(\tau) - \sum_{k=0}^{n-1}\frac{\varphi^{[k]}_\Psi}{k!}(\Psi(\tau) - \Psi(a))^k, \tag{2.3}$$

for all $\tau \in \Upsilon$, *where* $\varphi_{\Psi}^{[k]}(\tau) := \left(\frac{1}{\Psi'(\tau)} \cdot \frac{d}{d\tau}\right)^k \varphi(\tau)$.

Definition 2.3 ([7])**.** The Kuratowski measure of non-compactness is the mapping $\vartheta : \Gamma_X \to \mathbb{R}_+$ defined by $\vartheta(B) = \inf\{\xi > 0 :$ B can be covered by finitely many sets with diameter less or equal to $\xi\}$.

Proposition 2.2 ([7])**.** *The Kuratowski measure of noncompactness* ϑ *satisfies the following properties*

1. $\vartheta(A) = 0$ *if and only if* A *is relatively compact;*
2. $A \subset B \Rightarrow \vartheta(A) \leq \vartheta(B)$*;*
3. $\vartheta(A) = \vartheta(\overline{A}) = \vartheta(\mathrm{conv}(A))$, *where* $\overline{A}$ *and* $\mathrm{conv}(A)$ *denote the closure and the convex hull of* A*, respectively;*
4. $\vartheta(A + B) \leq \vartheta(A) + \vartheta(B)$*;*
5. $\vartheta(kA) = |k|\vartheta(A)$, $k \in \mathbb{R}$.

Definition 2.4. Let $\mathcal{F} : A \to X$ be a continuous bounded map. The operator $\mathcal{F}$ is said to be ϑ-Lipschitz if there exists $l \geq 0$ such that

$$\vartheta(\mathcal{F}(B)) < l\vartheta(B), \quad \text{for every } B \subset A. \tag{2.4}$$

Furthermore, if $l < 1$, then $\mathcal{F}$ is a strict ϑ-contraction.

Definition 2.5. $\mathcal{F} : A \to X$ is called ϑ-condensing if

$$\vartheta(\mathcal{F}(B)) < \vartheta(B), \tag{2.5}$$

for every bounded and nonprecompact subset B of A, with $\vartheta(B) > 0$.

Definition 2.6. We say that the function $\mathcal{F} : A \to X$ is Lipschitz if there exists $l > 0$ such that

$$\|\mathcal{F}(w) - \mathcal{F}(v)\| \leq l\|u - v\|, \quad \text{for all } w, v \in A, \tag{2.6}$$

Furthermore, if $l < 1$, then $\mathcal{F}$ is a strict contraction.

Proposition 2.3 ([7,12])**.** *If* $\mathcal{F}, \mathcal{Y} : A \to X$ *are* ϑ*-Lipschitz mapping with constants* l_1 *and* l_2 *respectively, then* $\mathcal{F} + \mathcal{Y} : A \to X$ *is* ϑ*-Lipschitz mapping with constant* $l_1 + l_2$*.*

Proposition 2.4 ([7,12])**.** *If* $\mathcal{F} : A \to X$*, is compact, then* $\mathcal{F}$ *is* ϑ*-Lipschitz mapping with constant* $l = 0$*.*

Proposition 2.5 ([7,12])**.** *If* $\mathcal{F} : A \to X$ *is Lipschitz mapping with constant* l*, then* $\mathcal{F}$ *is* ϑ*-Lipschitz mapping with the same constant* l*.*

Theorem 2.1 ([12])**.** *Let* $\mathcal{W} : A \to X$ *be* ϑ*-condensing and*

$$\Pi_\epsilon = \{w \in X : w = \epsilon\mathcal{W}w, \ \textit{for some } 0 \leq \epsilon \leq 1\}. \tag{2.7}$$

If Π_ϵ *is a bounded set in* X*, so there exists* $r > 0$*, such that* $\Pi_\epsilon \in \mathcal{B}_r(0)$*, then the degree*

$$\deg\left(I - \epsilon\mathcal{W}, \mathcal{B}_r(0), 0\right) = 1, \quad \textit{for all } \epsilon \in [0, 1]. \tag{2.8}$$

Consequently, $\mathcal{W}$ *has at least one fixed point and the set of the fixed points of* $\mathcal{W}$ *lies in* $\mathcal{B}_r(0)$.

3. Main Result

Definition 3.1. A function $w \in \mathcal{C}(\Upsilon, \mathbb{R})$ is said to be a solution of Problem (1.1), if w satisfies the equation ${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{p;\Psi}\left[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} + \lambda\right] w(\tau) = \varphi(t, w(\tau))$, a.e. on Υ with the conditions $w(a) = 0$, $w'(a) = 0$, $w(b) = \sum\limits_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{\beta_i;\Psi} w(\kappa_i)$.

Lemma 3.1. *Let* $a \geq 0$, $0 < p \leq 1$, $0 < q \leq 2$ *and* $h \in \mathcal{C}(\Upsilon, \mathbb{R})$. *Then the function* w *is a solution of the following boundary value problem*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{p;\Psi}\left[{}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} + \lambda\right] w(\tau) = h(\tau), \quad \tau \in \Upsilon := [a, b], \\ w(a) = 0, \quad w'(a) = 0, \quad w(b) = \sum\limits_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{\beta_i;\Psi} w(\kappa_i), \quad a < \kappa_i < b, \end{cases} \tag{3.1}$$

if and only if

$$\begin{aligned} w(\tau) =& \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + \frac{(\Psi(\tau) - \Psi(a))^q}{\Delta\Gamma(q+1)} \Big(\mathfrak{I}_{a^+}^{p+q;\Psi} h(b) \\ & - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(b) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} h(\kappa_i) + \lambda \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{q+\beta_i;\Psi} w(\kappa_i) \Big), \end{aligned} \tag{3.2}$$

where

$$\Delta = \sum_{i=1}^{n} \iota_i \frac{(\Psi(\kappa_i) - \Psi(a))^{q+\beta_i}}{\Gamma(q + \beta_i + 1)} - \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} \neq 0. \tag{3.3}$$

Proof. Applying the Ψ-Riemann-Liouville fractional integral of order p to both sides of (3.1) and by using Lemma 2.2 we get

$${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{q;\Psi} w(\tau) + \lambda w(\tau) = \mathfrak{I}_{a^+}^{p;\Psi} h(\tau) + d_0, \quad \tau \in \Upsilon, \tag{3.4}$$

where d_0 is a constant, applying the Ψ-Riemann-Liouville fractional integral of order q to both sides of (3.4) we obtain by using Lemma 2.2

$$w(\tau) = \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + d_0 \frac{(\Psi(\tau) - \Psi(a))^q}{\Gamma(q+1)} + d_1 + d_2(\Psi(\tau) - \Psi(a)), \tag{3.5}$$

where d_1 and d_2 are constants, next by using the boundary condition $w(a) = 0$ in (3.5) we obtain that $d_1 = 0$. Then, we get

$$w(\tau) = \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + d_0 \frac{(\Psi(\tau) - \Psi(a))^q}{\Gamma(q+1)} + d_2(\Psi(\tau) - \Psi(a)). \tag{3.6}$$

It follows that

$$w'(\tau) = \left(\mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau)\right)' - \lambda \left(\mathfrak{I}_{a^+}^{q;\Psi} w(\tau)\right)' + d_0 \frac{q\Psi'(\tau)(\Psi(\tau) - \Psi(a))^{q-1}}{\Gamma(q+1)} + d_2 \Psi'(\tau), \tag{3.7}$$

by using $w'(a) = 0$, in (3.7) we find $d_2 = 0$ (Definition 2.2: $\Psi'(\tau) > 0$, for all $\tau \in \Upsilon$). Then,

$$w(\tau) = \mathfrak{I}_{a^+}^{p+q;\Psi} h(\tau) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + d_0 \frac{(\Psi(\tau) - \Psi(a))^q}{\Gamma(q+1)}. \tag{3.8}$$

By making use of the boundary condition $w(b) = \sum\limits_{i=1}^{n} \iota_i w(\kappa_i)$, in (3.8) we find

$$d_0 = \frac{1}{\Delta}\left(\mathfrak{I}_{a^+}^{p+q;\Psi} h(b) - \lambda \mathfrak{I}_{a^+}^{q;\Psi} w(b) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} h(\kappa_i) + \lambda \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{q+\beta_i;\Psi} w(\kappa_i)\right). \tag{3.9}$$

Substituting the value of d_0 in (3.8) we obtain the integral equation in (3.2). The converse follows by direct computation. $\square$

In this part, we deal with the existence and uniqueness of solution for the problem (1.1), for that to simplify the computations, we use the following notation

$$\begin{aligned}\Lambda_1 =& \frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)}\\ &+ \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)}\left(\frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)}\right),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Lambda_2 =& |\lambda| \Bigg\{ \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)}\\ &+ \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)}\left[\frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right]\Bigg\}.\end{aligned} \tag{3.11}$$

Assume that the following hold.

(H_1) There exists a constant $\mathcal{L}_\varphi > 0$ such that

$$|\varphi(\tau, w) - \varphi(\tau, v)| \leq \mathcal{L}_\varphi |w - v|, \quad \text{for each } \tau \in \Upsilon \text{ and } w, v \in \mathcal{C}(\Upsilon, \mathbb{R}). \tag{3.12}$$

(H_2) There exist two constants $\mathcal{K}_\varphi, \mathcal{N}_\varphi > 0$ and $\alpha \in (0, 1)$ such that

$$|\varphi(\tau, w)| \leq \mathcal{K}_\varphi |w|^\alpha + \mathcal{N}_\varphi, \quad \text{for each } \tau \in \Upsilon \text{ and } w \in \mathcal{C}(\Upsilon, \mathbb{R}). \tag{3.13}$$

From Lemma 3.1 we define the operators $\mathcal{F}, \mathcal{Y} : \mathcal{C}(\Upsilon, \mathbb{R}) \to \mathcal{C}(\Upsilon, \mathbb{R})$ by

$$\begin{aligned}\mathcal{F}w(\tau) =& \mathfrak{I}_{a^+}^{p+q;\Psi} \varphi(\tau, w(\tau)) + \frac{(\Psi(\tau) - \Psi(a))^q}{\Delta\Gamma(q+1)}\\ &\times \left(\mathfrak{I}_{a^+}^{p+q;\Psi} \varphi(b, w(b)) - \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi} \varphi(\kappa_i, w(\kappa_i))\right), \quad \tau \in \Upsilon,\end{aligned} \tag{3.14}$$

and

$$\mathcal{Y}w(\tau) = -\lambda \mathfrak{I}_{a^+}^{q;\Psi} w(\tau) + \frac{(\Psi(\tau) - \Psi(a))^q}{\Delta\Gamma(q+1)}\left(-\lambda \mathfrak{I}_{a^+}^{q;\Psi} w(b) + \lambda \sum_{i=1}^{n} \iota_i \mathfrak{I}_{a^+}^{q+\beta_i;\Psi} w(\kappa_i)\right), \tag{3.15}$$

then, the fractional integral equation (3.2) can be written as follows

$$\mathcal{W}w(\tau) = \mathcal{F}w(\tau) + \mathcal{Y}w(\tau), \quad \tau \in \Upsilon. \tag{3.16}$$

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *are satisfied, then Problem* (1.1) *has at least one solution* $w \in \mathcal{C}(\Upsilon, \mathbb{R})$ *as long as* $\Lambda_2 < 1$. *Moreover, the set of the solution of Problem* (1.1) *is bounded in* $\mathcal{C}(\Upsilon, \mathbb{R})$.

As a way to prove Theorem 3.1, we will demonstrate it in several lemmas.

Lemma 3.2. $\mathcal{Y}$ *is* ϑ*-Lipschitz with the constant* Λ_2, *where* Λ_2 *is given by* (3.11).

Proof. Let $w, v \in \mathcal{C}(\Upsilon, \mathbb{R})$, then we get

$$\begin{aligned}
|\mathcal{Y}w(\tau) - \mathcal{Y}v(\tau)| \leq & |\lambda| \mathfrak{I}_{a^+}^{q;\Psi} |w(\tau) - v(\tau)| + \frac{(\Psi(\tau) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \Big(|\lambda| \mathfrak{I}_{a^+}^{q;\Psi} |w(b) - v(b)| \\
& + |\lambda| \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{q+\beta_i;\Psi} |w(\kappa_i) - v(\kappa_i)| \Big) \\
\leq & |\lambda| \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} |w(\tau) - v(\tau)| + \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\
& \times \left(|\lambda| \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} |w(b) - v(b)| \right. \\
& \left. + |\lambda| \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)} |w(\kappa_i) - v(\kappa_i)| \right) \\
\leq & |\lambda| \left\{ \frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} + \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \right. \\
& \left. \times \left(\frac{(\Psi(b) - \Psi(a))^q}{\Gamma(q+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right) \right\} \|w - v\|, \\
\leq & \Lambda_2 \|w - v\|,
\end{aligned}$$

where Λ_2 is given by (3.11). Taking the supremum over τ, we obtain

$$\|\mathcal{Y}w - \mathcal{Y}v\| \leq \Lambda_2 \|w - v\|.$$

Then, $\mathcal{Y}$ is Lipschitz with the constant Λ_2 and by Proposition 2.5, $\mathcal{Y}$ is ϑ-Lipschitz with the same constant Λ_2. □

Lemma 3.3. $\mathcal{F}$ *is continuous and satisfies the inequality given below*

$$\|\mathcal{F}w\| \leq \Lambda_1 \Big(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi \Big), \tag{3.17}$$

where Λ_1 *is given by* (3.10).

Proof. Let $w_n, w \in \mathcal{C}(\Upsilon, \mathbb{R})$ such that w_n converging to w in $\mathcal{C}(\Upsilon, \mathbb{R})$, implies that there exists $\mu > 0$ such that $\|w_n\| \leq \mu$ for all $n \geq 1$, in addition by taking limits, we get $\|w\| \leq \mu$. By using the fact that φ is continuous and (H_2), for every $\tau \in \Upsilon$ we have

$$|\varphi(\tau, w_n(\tau)) - \varphi(\tau, w(\tau))| \leq |\varphi(\tau, w_n(\tau))| + |\varphi(\tau, w(\tau)| \leq 2(\mathcal{K}_\varphi \mu^\alpha + \mathcal{N}_\varphi).$$

The function $s \mapsto 2(\mathcal{K}_\varphi \mu^\alpha + \mathcal{N}_\varphi)$ is integrable for $s \in [0, \tau]$, $\tau \in \Upsilon$ by making use of Lebesgue dominated convergence theorem we get

$$\begin{aligned}|\mathcal{F}w_n(\tau) - \mathcal{F}w(\tau)| \leq & \mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(\tau, w_n(\tau)) - \varphi(\tau, w(\tau))| + \frac{(\Psi(\tau) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ & \times \Big(\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(b, w_n(b)) - \varphi(b, w(b))| \\ & + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}|\varphi(\kappa_i, w_n(\kappa_i)) - \varphi(\kappa_i, w(\kappa_i))| \Big) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Then, $\|\mathcal{F}w_n - \mathcal{F}w\| \to 0$ as $n \to \infty$, implies that $\mathcal{F}$ is continuous.

In addition, for every $\tau \in \Upsilon$ we get

$$\begin{aligned}|\mathcal{F}w(\tau)| \leq & \mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(\tau, w(\tau))| + \frac{(\Psi(\tau) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ & \times \left(\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(b, w(b))| + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}|\varphi(\kappa_i, w(\kappa_i)| \right), \\ \leq & \Bigg\{ \frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \frac{(\Psi(b) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ & \times \left(\frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \sum_{i=1}^{n} |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)} \right) \Bigg\} \Big(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi \Big),\end{aligned}$$

implies that $\|\mathcal{F}w\| \leq \Lambda_1 \Big(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi \Big)$. □

Lemma 3.4. *$\mathcal{F}$ is compact, as a consequence $\mathcal{F}$ is ϑ-Lipschitz with zero constant.*

Proof. To prove that $\mathcal{F}$ is compact, let $\mathcal{M}$ be a bounded set, such that $\mathcal{M} \subset \mathcal{B}_\rho$. It remain to prove that $\mathcal{F}(\mathcal{M})$ is relatively compact in $\mathcal{C}(\Upsilon, \mathbb{R})$. For this reason let $w \in \mathcal{M} \subset \mathcal{B}_\rho$ and by making use of (3.17), we obtain

$$\|\mathcal{F}w\| \leq \Lambda_1 \Big(\mathcal{K}_\varphi \rho^\alpha + \mathcal{N}_\varphi \Big) := \upsilon. \tag{3.18}$$

Then, $\mathcal{F}(\mathcal{M}) \subset \mathcal{B}_\upsilon$, as a consequence, $\mathcal{F}(\mathcal{M})$ is bounded.

For the equicontinuity of $\mathcal{F}$, let $\tau_1, \tau_2 \in \Upsilon$ with $\tau_1 < \tau_2$ and for $w \in \mathcal{M}$ we have

$$\begin{aligned}& |\mathcal{F}w(\tau_2) - \mathcal{F}w(\tau_1)| \\ \leq & \mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(\tau_2, w(\tau_2)) - \varphi(\tau_1, w(\tau 1))| + \frac{(\Psi(\tau_2) - \Psi(a))^q - (\Psi(\tau_1) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\ & \times \left(\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(b, w(b))| + \sum_{i=1}^{n} |\iota_i| \mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}|\varphi(\kappa_i, w(\kappa_i)| \right)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\left(\mathcal{K}_\varphi \rho^\alpha + \mathcal{N}_\varphi\right)}{\Gamma(p+q)} \Big| \int_a^{\tau_1} \Psi'(s)\Big((\Psi(\tau_2) - \Psi(s))^{p+q-1} \\
&\quad - (\Psi(\tau_1) - \Psi(s))^{p+q-1}\Big) ds + \int_{\tau_1}^{\tau_2} \Psi'(s)(\Psi(\tau_2) - \Psi(s))^{p+q-1} ds \Big| \\
&\quad + \frac{(\Psi(\tau_2) - \Psi(a))^q - \Psi(\tau_1) - \Psi(a))^q}{|\Delta|\Gamma(q+1)} \\
&\quad \times \left(\frac{(\Psi(b) - \Psi(a))^{p+q}}{\Gamma(p+q+1)} + \sum_{i=1}^n |\iota_i| \frac{(\Psi(\kappa_i) - \Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)} \right) \left(\mathcal{K}_\varphi \rho^\alpha + \mathcal{N}_\varphi\right),
\end{aligned}
$$

By using the continuity of the function Ψ, the right hand side of the above inequality tends to 0 as τ_2 tends to τ_1, this implies that $\mathcal{F}(\mathcal{M})$ is equicontinuous. It follows by using Arzelá-Ascoli's theorem that $\mathcal{F}$ is compact as a consequence of Proposition 2.4, $\mathcal{F}$ is ϑ-Lipschitz with zero constant. □

Since all the conditions are satisfied we demonstrate the validity of our main result as Theorem 3.1.

Proof of Theorem 3.1. Let $\mathcal{F}$, $\mathcal{Y}$ and $\mathcal{W}$, be the operators given by (3.14), (3.15), (3.16), respectively. These operators are continuous and bounded. Furthermore, by making use of Lemma 3.2 , $\mathcal{Y}$ is is ϑ-Lipschitz with constant Λ_2, and by using Lemma 3.4, $\mathcal{F}$ is ϑ-Lipschitz with constant zero, hence $\mathcal{W}$ is a strict ϑ-contraction with constant Λ_2, finally $\mathcal{W}$ is ϑ-condensing because $\Lambda_2 < 1$.

Next, let us consider the following set

$$\Pi_\epsilon = \{w \in X : w = \epsilon \mathcal{W} w, \text{ for some } 0 \leq \epsilon \leq 1\}. \tag{3.19}$$

It remains to show that the set Π_ϵ is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$, for that let $w \in \Pi_\epsilon$ then we have $w = \epsilon \mathcal{W} w = \epsilon(\mathcal{F}w + \mathcal{Y}w)$. It follows, by using Lemma 3.3 and 3.2,

$$
\begin{aligned}
&\|w\| = \epsilon \|\mathcal{F}w + \mathcal{Y}w\|, \\
&\quad \leq \|\mathcal{F}w\| + \|\mathcal{Y}w\| \leq \Lambda_1\left(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi\right) + \Lambda_2 \|w\| \leq \frac{\Lambda_1\left(\mathcal{K}_\varphi \|w\|^\alpha + \mathcal{N}_\varphi\right)}{1 - \Lambda_2},
\end{aligned}
$$

where Λ_1 and Λ_2 are given by (3.10) and (3.11), respectively. Then, the set Π_ϵ is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$. If the set Π_ϵ is not bounded, then we suppose that $\chi := \|w\| \to +\infty$ and by using the above inequality we get

$$1 \leq \lim_{\chi \to +\infty} \frac{\Lambda_1\left(\mathcal{K}_\varphi \chi^\alpha + \mathcal{N}_\varphi\right)}{\chi(1 - \Lambda_2)} = 0, \tag{3.20}$$

which is a contradiction. Thus by using Theorem 2.1, $\mathcal{W}$ has at least one fixed point which is the solution of Problem (1.1). Moreover, the set of solution of Problem (1.1) is bounded in $\mathcal{C}(\Upsilon, \mathbb{R})$. □

To deal with the uniqueness of solution for Problem (1.1), we use Banach's contraction principle.

Theorem 3.2. *Assume that* (H_1) *hold. If* $\mathcal{L}_\varphi\Lambda_1+\Lambda_2<1$*, then Problem* (1.1) *has a unique solution on* $\mathcal{C}(\Upsilon,\mathbb{R})$.

Proof. For every $w,v\in\mathcal{C}(\Upsilon,\mathbb{R})$ and $\tau\in\Upsilon$ we have

$$\begin{aligned}
&|\mathcal{W}w(\tau)-\mathcal{W}v(\tau)|\\
\leq&|\mathcal{F}w(\tau)-\mathcal{F}v(\tau)+\mathcal{Y}w(\tau)-\mathcal{Y}v(\tau)|\\
\leq&|\mathcal{F}w(\tau)-\mathcal{F}v(\tau)|+|\mathcal{Y}w(\tau)-\mathcal{Y}v(\tau)|\\
\leq&\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(\tau,w(\tau))-\varphi(\tau,v(\tau))|+\frac{(\Psi(\tau)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\Bigg(\mathfrak{I}_{a^+}^{p+q;\Psi}|\varphi(b,w(b))-\varphi(b,v(b))|\\
&+\sum_{i=1}^n|\iota_i|\mathfrak{I}_{a^+}^{p+q+\beta_i;\Psi}|\varphi(\kappa_i,w(\kappa_i))-\varphi(\kappa_i,v(\kappa_i))|\Bigg)\\
&+|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(\tau)-v(\tau)|+\frac{(\Psi(\tau)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\Bigg[|\lambda|\mathfrak{I}_{a^+}^{q;\Psi}|w(b)-v(b)|\\
&+|\lambda|\sum_{i=1}^n|\iota_i|\mathfrak{I}_{a^+}^{q+\beta_i;\Psi}|w(\kappa_i)-v(\kappa_i)|\Bigg]\\
\leq&\Bigg\{\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\left(\frac{(\Psi(b)-\Psi(a))^{p+q}}{\Gamma(p+q+1)}+\sum_{i=1}^n|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{p+q+\beta_i}}{\Gamma(p+q+\beta_i+1)}\right)\Bigg\}\mathcal{L}_\varphi|w(\tau)-v(\tau)|\\
&+|\lambda|\Bigg\{\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\frac{(\Psi(b)-\Psi(a))^q}{|\Delta|\Gamma(q+1)}\\
&\times\left(\frac{(\Psi(b)-\Psi(a))^q}{\Gamma(q+1)}+\sum_{i=1}^n|\iota_i|\frac{(\Psi(\kappa_i)-\Psi(a))^{q+\beta_i}}{\Gamma(q+\beta_i+1)}\right)\Bigg\}|w(\tau)-v(\tau)|\\
\leq&(\mathcal{L}_\varphi\Lambda_1+\Lambda_2)\|w-v\|,
\end{aligned}$$

where Λ_1 and Λ_2 are given by (3.10) and (3.11), respectively. Then, by taking the supremum over τ, we get $\|\mathcal{W}w-\mathcal{W}v\|\leq(\mathcal{L}_\varphi\Lambda_1+\Lambda_2)\|w-v\|$. Using the fact that $\mathcal{L}_\varphi\Lambda_1+\Lambda_2<1$, it follows that $\mathcal{W}$ is a contraction. Finally, by the Banach fixed point theorem, $\mathcal{W}$ has a unique fixed point which is a unique solution of Problem (1.1). ☐

4. Example

Consider the following problem

$$\begin{cases}{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac12;e^\tau}\left({}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac32;e^\tau}+\frac15\right)w(\tau)=\frac{e^{-\tau}}{e^\tau+10}\left(1+\frac{|w(\tau)|}{1+|w(\tau)|}\right), & 0\leq\tau\leq1,\\ w(0)=0,\quad w'(0)=0,\quad w(1)=\frac35\mathfrak{I}_{0^+}^{\frac27;e^\tau}w\left(\frac14\right)+\frac45\mathfrak{I}_{0^+}^{\frac25;e^\tau}w\left(\frac12\right),\end{cases}\tag{4.1}$$

where $p = \frac{1}{2}$, $q = \frac{3}{2}$, $\lambda = \frac{1}{5}$, $a = 0$, $b = 1$, $\Upsilon = [0, 1]$, $n = 2$, $\iota_1 = \frac{3}{5}$, $\iota_2 = \frac{4}{5}$, $\beta_1 = \frac{2}{7}$, $\beta_2 = \frac{2}{5}$, $\kappa_1 = \frac{1}{4}$, $\kappa_2 = \frac{1}{2}$ and $\Psi(\tau) = e^{\tau}$.

For $(\tau, w) \in \Upsilon \times \mathbb{R}_+$, we define $\varphi(\tau, w) = \frac{e^{-\tau}}{e^{\tau}+10}\Big(1 + \frac{w(\tau)}{1+w(\tau)}\Big)$. Function φ is a continuous function, in addition for every $\tau \in \Upsilon$ and for every $w, v \in \mathbb{R}_+$ we have

$$|\varphi(\tau, w) - \varphi(\tau, v)| \le \left|\frac{e^{-\tau}}{e^{\tau} + 10}\right| \cdot \left|\frac{w - v}{(1 + w)(1 + v)}\right| \le \frac{1}{11}|w - v|.$$

Then, Hypotheses (H_1) holds with $\mathcal{L}_{\varphi} = \frac{1}{11} > 0$. In addition, for every $\tau \in \Upsilon$ and $w, v \in \mathbb{R}_+$ we have

$$|\varphi(\tau, w)| \le \left|\frac{e^{-\tau}}{e^{\tau} + 10}\right|\Big(1 + |w|\Big) \le \frac{1}{11}\Big(1 + |w|\Big).$$

Then, Hypotheses (H_2) holds with $\mathcal{K}_{\varphi} = \mathcal{N}_{\varphi} = \frac{1}{11} > 0$ and $\alpha = 1$ moreover $\Lambda_2 = 0.70263036 < 1$. Finally, all the conditions of Theorem 3.1 are satisfied, consequently Problem (4.1) has at least one solution defined on $[0, 1]$.

To deal with the uniqueness we use the data given above, we get, $|\Delta| \simeq 1.74138859$, $\Lambda_1 = 2.9745821$, $\Lambda_2 = 0.70263036 < 1$, and $\mathcal{L}_{\varphi} = \frac{1}{11} = 0.09$. Then, $\mathcal{L}_{\varphi}\Lambda_1 + \Lambda_2 = 0.09 \times 2.9745821 + 0.70263036 = 0.97034275 < 1$.

Accordingly, by Theorem 3.2, Problem (4.1) has a unique solution on $[0, 1]$.

5. Conclusion

In this article, we have studied and discussed the existence and uniqueness of solution for a class of Ψ-Caputo fractional differential Langevin equation. The suggested study is based on some basic definitions of fractional calculus and topological degree theory. The novelty of this work is that it is more general than the works based on the well-known fractional derivatives such as (Caputo fractional derivative, Caputo-Hadamard fractional derivative and Caputo-Katugampola fractional derivative) for different values of function Ψ. Additionally, as a scope of future direction, by studying this specific case of fractional derivative, it can be used as an overview to studying the general case known by the Ψ-Hilfer fractional derivative. In this paper we proved the existence and uniqueness results for Problem (1.1), by using the topological degree method and Banach's fixed point theorem. Finally, a numerical example is presented to clarify the theoretical result.

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