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A FINITE DIFFERENCE TECHNIQUE FOR NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM IN ODES OF ORDER THREE

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Abstract. In the article, we study the approximate numerical solution to the boundary value problem in ordinary differential equations. In the present article, a third-order two-point boundary value problem is considered for discussion. We developed a second order accurate finite difference method for the approximate numerical solution of the considered problem. We took a special boundary condition; we did not find this boundary condition in the literature. We have discussed the standard convergence analysis of the proposed method. Numerical experiments on linear, nonlinear, and obstacle problems approve the order of accuracy and efficiency of the method.

1. Introduction

The present article is aimed at finite difference method for the numerical solution of the third-order boundary value problems of the following form

$$u'''(x) = f(x, u), \quad a < x < b, \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u'(b) = \beta \quad \text{and} \quad u''(b) = \gamma,$$

where α, β and γ are real constants.

The importance of third order boundary value problems is well-established in the physical and natural sciences. The analytical solution to such problems is subject

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to a variety of boundary conditions, and restricted forcing function $f(x, u)$ has been studied by many researchers. But for an arbitrary forcing function $f(x, u)$, it is difficult to find closed-form analytical solution.

The theory on the existence and uniqueness of the solution of higher order boundary value problems can be found in [1]. The existence and uniqueness of the solution, especially for the third order boundary value problem (1.1) in detail are discussed in [2–5] and references therein. So, we have assumed the existence and uniqueness of the solution for problem (1.1) with the considered boundary conditions.

There are a variety of approximation techniques available in the literature for numerical solutions to third-order two-point boundary value problems. But not many researchers considered the problem with the boundary conditions as described in this article. For instance, among a substantial number of works, we refer to work reported by [6, 7] and references therein for numerical approximation of the solution using the finite difference method with different boundary conditions.

Based on the idea in [8], the purpose of the present article is to develop an algorithm using the finite difference method to deal with the numerical solution of the third-order boundary value problems that is accurate, inexpensive, and simple in its computational efforts. We hope the present technique will supplement the existing literature on the solution of third-order boundary value problems.

In this article, we have organised our work as follows. In Section 2, we have derived our finite difference method. In Section 3, we have discussed and analysed the standard convergence of the proposed method. The computational work presented in Section 4 and a discussion on the computational performance of the proposed method is presented in Section 5.

2. The Difference Method

We define $a \leq x_0 < x_1 < x_2 < \cdots < x_N \leq b$, $N - 1$ number of nodal points in the domain $[a, b]$ of the problem (1.1). In this domain we wish to determine an approximate numerical solution of the problem (1.1), using uniform step length h such that $x_i = a + ih$, $i = 0, 1, 2, \ldots, N$. We wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1.1) at these discrete nodal points x_i, $i = 1, 2, \ldots, N$. We denote the numerical approximation of $u(x)$ by u_i and source function $f(x, u(x))$ by f_i at nodes $x = x_i$, $i = 1, 2, \ldots, N$. Thus, the boundary value problem (1.1) at node $x = x_i$ may be written as

$$u_i''' = f_i, \quad a \leq x_i \leq b, \tag{2.1}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_N' = \beta \quad \text{and} \quad u_N'' = \gamma.$$

Let we define nodes $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}$, $i = 1, 2, \ldots, N - 1$, and denote the solution of the problem (1.1) at these nodes as $u_{i\pm\frac{1}{2}}$. Following the idea in [8] and using method of undetermined coefficients and Taylor series expansion, we discretize problem (2.1) at these nodes in $[a, b]$ as follows

$$\begin{aligned}
(2.2)\quad & 15u_{i-\frac{1}{2}} - 10u_{i+\frac{1}{2}} + 3u_{i+\frac{3}{2}} = 8u_{i-1} + \frac{h^3}{16}(15f_{i-\frac{1}{2}} + 25f_{i+\frac{1}{2}}) + T_i, \quad i = 1,\\
& -u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} = \frac{h^3}{2}(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}) + T_i, \quad 2 \le i \le N-2,\\
& -u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} = -h^2u''_{i+1} - \frac{h^3}{24}(25f_{i-\frac{3}{2}} - 61f_{i-\frac{1}{2}}) + T_i, \quad i = N-1,\\
& -u_{i-\frac{3}{2}} + u_{i-\frac{1}{2}} = hu'_i - h^2u''_i - \frac{h^3}{24}(36f_{i-\frac{3}{2}} - 49f_{i-\frac{1}{2}}) + T_i, \quad i = N,
\end{aligned}$$

where T_i, $i = 1, 2, \ldots, N$ are truncating terms. Also, in discretization we have used boundary conditions in a natural way.

After neglecting the terms T_i in (2.2), at nodal points $x_{i-\frac{1}{2}}$, $i = 1, 2, \ldots, N$ we will obtain the N linear or nonlinear system of equations in N unknown namely $u_{i-\frac{1}{2}}$ which depends on the source function $f(x, u)$. We have applied Gauss Seidel and Newton-Raphson iterative method to solve system of linear and system of nonlinear equations, respectively.

We compute numerical value of u_i, $i = 1, 2, \ldots, N$ by using following second order approximation

$$u_i = \begin{cases} \frac{1}{2}(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}), & 1 \le i \le N-1,\\ \frac{1}{2}(3u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}}), & i = N. \end{cases} \tag{2.3}$$

3. Convergence Analysis

We will consider following linear test equation for convergence analysis of the proposed method (2.2)

$$u'''(x) = f(x, u), \quad a < x < b, \tag{3.1}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u'_N = \beta \quad \text{and} \quad u''_N = \gamma.$$

Let $u_{i-\frac{1}{2}}$ and $U_{i-\frac{1}{2}}$ for $i = 1, 2, \ldots, N$ are, respectively approximate and exact solution of (2.2). Let us define

$$F_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}), \quad i = 1, 2, \ldots, N,$$

and error that occur in approximate solution

$$\epsilon_{i-\frac{1}{2}} = U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}, \quad i = 1, 2, \ldots, N.$$

Let we linearize source function $f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}})$ by application of Taylor series expansion, i.e.,

$$f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}) - f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}) = (U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}) \left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}}, \quad i = 1, 2, \ldots, N.$$

Thus, using these above definitions and boundary condition, we can write an equation in (2.2) as follows

$$15\epsilon_{i-\frac{1}{2}} - 10\epsilon_{i+\frac{1}{2}} + 3\epsilon_{i+\frac{3}{2}} = \frac{h^3}{16}\left(128\left(\frac{\partial f}{\partial u}\right)_{i-\frac{1}{2}} + 25\left(\frac{\partial f}{\partial u}\right)_{i+\frac{1}{2}}\right) + T_i, \quad i = 1.$$

Similarly we can write remaining equations in (2.2). Thus, we can write proposed method (2.2) in the matrix form as

$$\mathbf{J}\mathbf{E} = \mathbf{T}, \tag{3.2}$$

where $\mathbf{J}$, $\mathbf{E}$ and $\mathbf{T}$ are matrices. These matrices are defined as

$$\mathbf{J} = \mathbf{A} + \mathbf{L},$$

$$\mathbf{A} = \begin{pmatrix} 15 & -10 & 3 & & & 0 \\ -1 & 3 & -3 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 3 & -3 & 1 \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 1 \end{pmatrix}_{N\times N},$$

$$\mathbf{L} = -\frac{h^3}{48}\begin{pmatrix} 384\delta_{\frac{1}{2}} & 75\delta_{\frac{3}{2}} & & & 0 \\ & 24\delta_{\frac{3}{2}} & 24\delta_{\frac{5}{2}} & & \\ & \ddots & \ddots & & \\ & & 24\delta_{N-\frac{5}{2}} & 24\delta_{N-\frac{3}{2}} & \\ & & -50\delta_{N-\frac{5}{2}} & 122\delta_{N-\frac{3}{2}} & \\ 0 & & & -72\delta_{N-\frac{3}{2}} & 98\delta_{N-\frac{1}{2}} \end{pmatrix}_{N\times N},$$

where $\delta = \frac{\partial f}{\partial u}$,

$$\mathbf{E} = \left(\epsilon_{\frac{1}{2}}, \epsilon_{\frac{3}{2}}, \ldots, \epsilon_{N-\frac{3}{2}}, \epsilon_{N-\frac{1}{2}}\right)^T \quad \text{and} \quad \mathbf{T} = (T_1, T_2, \ldots, T_{N-1}, T_N)^T,$$

where

$$T_i = \begin{cases} -\frac{19}{40}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = 1, \\ \frac{1}{240}h^7 u^{(7)}_{i-\frac{1}{2}}, & 2 \le i \le N-2, \\ -\frac{13}{12}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = N-1, \\ -\frac{27}{40}h^5 u^{(5)}_{i-\frac{1}{2}}, & i = N. \end{cases}$$

Thus, we note from (3.2) that the convergence of the difference method (2.2) depends on matrix $\mathbf{J}$. So, we have determined $\mathbf{A}^{-1} = (a_{i,j})$ explicitly where

$$a_{i,j} = \begin{cases} \frac{(4j-1)+4(i-1)(2j-i)}{8}, & i < j < N, \\ \frac{2i-1}{2}, & i \leq j = N, \\ \frac{1}{8}, & 1 = j \leq i \leq N, \\ \frac{4j^2-1}{8}, & 1 < j < i \leq N, \\ \frac{4j^2-1}{8}, & 1 < j = i < N. \end{cases} \tag{3.3}$$

We observed that $a_{i,j} > 0$ for all i and j. Also, we have calculated the row sum of $\mathbf{A}^{-1}$ which are given as

$$R_i = \frac{4i^3 - 12i^2 + 32i - 18}{24} + \frac{(N-1)(2i-1)(2N-2i+1)}{8}.$$

Thus, we have obtained

$$R_N = \max_{1 \leq i \leq N} |R_i| = \frac{4N^3 - N^2 + 23N - 15}{24}. \tag{3.4}$$

Hence, it is easy from (3.4) to prove that

$$||\mathbf{A}^{-1}|| < \frac{(b-a)^3}{6h^3}. \tag{3.5}$$

Let square matrix $\mathbf{M}$ and identity matrix $\mathbf{I}$ have the same order and $||\mathbf{M}|| < 1$. Then square matrix $(\mathbf{I} + \mathbf{M})$ is invertible [9–11] and

$$||(\mathbf{I} + \mathbf{M})^{-1}|| < \frac{1}{1 - ||\mathbf{M}||}.$$

Let us assume $||\mathbf{A}^{-1}\mathbf{L}|| < 1$. Thus, from (3.2), we have

$$||\mathbf{E}|| < \frac{1}{1 - ||\mathbf{A}^{-1}\mathbf{L}||} ||\mathbf{A}^{-1}|| \cdot ||\mathbf{T}||. \tag{3.6}$$

Let $V = \max_{x \in [a,b]} |u^{(5)}(x)|$, $v = \max_{x \in [a,b]} \delta_{i-\frac{1}{2}}$ and $v > 0$. Thus, from (3.5) and (3.6) we obtained

$$||\mathbf{E}|| < \frac{52(b-a)^3 V h^2}{9(32 - 51v(b-a)^3)}. \tag{3.7}$$

From equation (3.7), we conclude $||\mathbf{E}||$ is bounded above and as $h \to 0$ implies $||\mathbf{E}|| \to 0$. Thus, we have established the convergence of our proposed method (2.2). The order of convergence of the proposed method is at least quadratic.

4. Numerical Results

To test the computational efficiency and validity of the theoretical development of the proposed method (2.2), we have considered two linear, a nonlinear, and an obstacle model problem. In each model problem, we took a uniform step size h. In Table 1–4, we have shown the maximum absolute error MAE in the computed solution $u(x)$ of the problem for different values of N. We have used the following formula in the computation of MAE

$$MAE = \max_{1 \le i \le N} |U(x_i) - u_i|,$$

where $U(x)$ is the exact solution of the problem. All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compilers (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iterations reached 10^6.

Problem 4.1. The linear model problem in [12] with different boundary conditions is given by

$$u'''(x) = xu(x) + (x^3 - 2x^2 - 5x - 3)\exp(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 1 \quad \text{and} \quad u''(1) = -4\exp(1).$$

The analytical solution of the problem is $u(x) = x(1-x)\exp(x)$. The MAE computed by method (2.2) for different values of N are presented in Table 1.

Table 1. Maximum absolute error in solution of Problem 1.1.

	N			
	128	256	512	1024
MAE	.21027867e-3	.61334344e-4	.15507685e-4	.38828002e-5

Problem 4.2. The linear model problem in [13] with different boundary conditions is given by

$$u'''(x) = -u(x) + (7 - x^2)\cos(x) + (x^2 - x - 1)\sin(x), \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 2\sin(1) \quad \text{and} \quad u''(1) = 2\sin(1) + 4\cos(1).$$

The analytical solution of the problem is $u(x) = (x^2 - 1)\sin(x)$. The MAE computed by method (2.2) for different values of N are presented in Table 2.

TABLE 2. Maximum absolute error in solution of Problem 2.1.

	N			
	128	256	512	1024
MAE	.98107150e-4	.22081891e-4	.55033015e-5	.34570694e-5

Problem 4.3. The nonlinear model problem in [14] with different boundary conditions is given by

$$u'''(x) = -2\exp(-3u(x)) + 4(1+x)^{-3}, \quad 0 < x < 1,$$

subject to boundary conditions

$$u(0) = 0 \quad , \quad u'(1) = \frac{1}{2} \quad \text{and} \quad u''(1) = -\frac{1}{4}.$$

The analytical solution of the problem is $u(x) = \ln(1+x)$. The MAE computed by method (2.2) for different values of N are presented in Table 3.

TABLE 3. Maximum absolute error in solution of Problem 2.2.

	N			
	32	64	128	256
MAE	.70750713e-4	.24229288e-4	.68414956e-5	.19595027e-5

Problem 4.4. Let consider the following third-order obstacle problems [15]

$$u'''(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ u(x) - 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} \le x \le 1, \end{cases}$$

subject to boundary conditions

$$u(0) = 0, \quad u'(1) = 0 \quad \text{and} \quad u''(1) = a_5.$$

The analytical solution of the problem is

$$u(x) = \begin{cases} \frac{1}{2}a_1 x^2, & 0 \le x \le \frac{1}{4}, \\ 1 + a_2\exp(x) + \exp(\frac{-x}{2})\left(a_3\cos\left(\frac{\sqrt{3}}{2}x\right) + a_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right), & \frac{1}{4} \le x \le \frac{3}{4}, \\ \frac{1}{2}a_5 x(x-2) + a_6, & \frac{3}{4} \le x \le 1, \end{cases}$$

where a_i, $i = 1, 2, \ldots, 6$ are constants. To determine these constants, we apply a continuity condition to the solution, first and second derivatives of the solution of the problem. Hence, we shall get a system of linear equations and solve the system of equations in variable a_i, $i = 1, 2, \ldots, 6$. The maximum absolute error in domain $D_1 = [0, \frac{1}{4}]$, $D_2 = [\frac{1}{4}, \frac{3}{4}]$ and $D_3 = [\frac{3}{4}, 1]$ in computed solution are presented in Table 4, by the proposed method (2.2) for the different values of N. Hence, we presented the maximum absolute error in the computed solution in the considered domain $D = [0, 1]$ of the problem in Table 4.

TABLE 4. The maximum absolute error in solution of Problem 4.4.

N	Maximum Absolute Error in			
	D_1	D_2	D_3	D
16	.10749675e -3	.97104762e -4	.35469564e -3	.35469564e -3
32	.28237705e -4	.27046958e -4	.88673911e -4	.88673911e -4
64	.72459166e -5	.71096102e -5	.22168478e -4	.22168478e -4
128	.18359854e -5	.18398562e -5	.55421195e -5	.55421195e -5
256	.46230928e -6	.48803843e -6	.13855249e -5	.13855249e -5
512	.14240736e -6	.14589006e -6	.34638247e -6	.34638247e -6

We have tested our finite difference method for the approximate numerical solution of linear and nonlinear model problems. Observing the numerical result in the tables, we found error in the computed solution decreases with a decrease in step size h in each considered model problem. The order of accuracy in computed solution of Problem 4.1, 4.2 is quadratic, and the order of accuracy computed solution of Problem 4.3 is non quadratic. We observed from the tabulated result for Problem 4.4, that the order of accuracy in the computed solution in the domain is quadratic. Hence, the maximum absolute error in the computed solution is in the domain D_3. We have noted in numerical experiments that our method is efficient, convergent, and consistent with theoretical development.

5. Conclusion

We considered a third-order two-point boundary value problem in ordinary differential equations for the approximate numerical solution. There are numerous techniques for the approximate solution in the literature of numerical analysis. Hence, we have developed an algorithm of quadratic order exact using the finite difference method for the approximate numerical solution of third order boundary value problems. The main concern in the present article is the boundary conditions. Some work with these boundary conditions has been reported in the literature for a closed analytical solution of the problem, but no algorithm or technique has been developed for an approximate solution of the problem. We converted a differential equation, a continuous problem, into a difference equation, a discrete problem, i.e., we discretized the problem at discrete nodal points in the domain of the considered problem. Hence, we have obtained a system of algebraic equations (2.2) and the solution of system of equations (2.2) is an approximate numerical solution of the considered problem (1.1). We considered four model problems, including an obstacle problem, to test the efficiency and accuracy of the proposed method (2.2). The numerical experiments produced a good approximate numerical solution for model problems. The numerical experiments approved the theoretical discussion on the order of accuracy and efficiency of the proposed method (2.2). Thus, we arrived at the conclusion that our method is computationally efficient and the order of accuracy is quadratic. The idea presented in this article is simple and leads to the possibility of developing higher order finite difference methods. Work in these directions and areas is in progress.

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