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GENERALISATION OF COMPANION OF OSTROWSKI'S TYPE INEQUALITY VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL FOR MAPPINGS WHOSE 1^{st} DERIVATIVES ARE BOUNDED WITH APPLICATIONS

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Abstract. We apply Riemann-Liouville fractional integral to get generalisation of companion of Ostrowski's type integral inequality for differentiable mappings whose 1st derivatives are bounded. The present article recapture all results of M. W. Alomari's article and also for one more article of different authors. Applications are also deduced for numerical integration, probability theory and special means.

1. Introduction

In the development of mathematics, inequalities are one of the most powerful tools. From two decades back, scholars researched on fractional calculus because of its importance in inequalities.

We quote from [4]: "The subject of fractional calculus (that is, calculus of integrals and derivatives of an arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers in. In recent years, the fractional calculus has played a significant role in many areas of science and engineering."

Due to worth of fractional integral inequalities, many scholars have mentioned certain generalisations of fractional integral inequalities (see [3, 17–19]).

Key words and phrases. Fractional Calculus, Riemann-Liouville fractional integral operator, Ostrowski's inequality, Grüss inequality, Differentiable mapping, Bounded mapping, Numerical integration, Probability density function, Special means.

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In 1938, A. M. Ostrowski gave an inequality (see [16]). Now-a-days this inequality is called Ostrowski inequality and this result had obtained by applying the Montgomery identity. For more discussion about Ostrowski inequality (see [9–12]).

Here, we present an inequality from article [6] that is given below. Throughout the article $I \subset \mathbb{R}$ and I° is the interior of the interval I.

Proposition 1.1. *Suppose $g : I \to \mathbb{R}$ is a differentiable mapping in the interval I^o such that $g' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|g'(\theta)| \leq M$ for all $\theta \in (a,b)$, where $M > 0$ is constant. Then*

$$\left| g(\theta) - \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(\theta - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].$$

The value $\frac{1}{4}$ is the best possible constant that this can not be replaced by the smallest one.

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss inequality [9, 14].

Proposition 1.2. *Let $f, g : [a,b] \to \mathbb{R}$ be both integrable functions such that $m_1 \leq f(\tau) \leq M_1$ and $m_2 \leq g(\tau) \leq M_2$ for all $\tau \in [a,b]$, where m_1, M_1, m_2, M_2 are real constants, then*

$$\left| \frac{1}{b-a}\int_a^b f(\tau)g(\tau)d\tau - \frac{1}{b-a}\int_a^b f(\tau)d\tau \cdot \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

In [7], S. S. Dragomir has derived the following companion of the Ostrowski inequality.

Proposition 1.3. *Let $g : I \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, where $a, b \in I$. Then we have the inequalities*

$$\begin{aligned} &\left| \frac{g(\theta) + g(a+b-\theta)}{2} - \frac{1}{b-a}\int_a^b g(\tau)d\tau \right| \\ &\leq \begin{cases} \left[\frac{1}{8} + 2\left(\frac{\theta - \frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|g'\|_{\infty}, \quad g' \in L_{\infty}[a,b], \\ \\ \frac{2^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{\theta-a}{b-a}\right)^{q+1} + \left(\frac{\frac{a+b}{2}-\theta}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|g'\|_{[a,b],p}\ , \\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \text{and } g' \in L_p[a,b], \\ \\ \left[\frac{1}{4} + \left|\frac{\theta - \frac{3a+b}{4}}{b-a}\right|\right]\|g'\|_{[a,b],1}, \end{cases} \end{aligned} \tag{1.1}$$

for all $\theta \in [a, \frac{a+b}{2}]$.

In 2011, M. W. Alomari has proved the following result about a companion inequality for differentiable functions whose derivatives are bounded (see [1]).

Proposition 1.4. *Let $g : I \to \mathbb{R}$ be a differentiable function in the interval I° and let $a, b \in I$ with $a < b$. If $g' \in L^1[a,b]$ and $m_2 \le g'(\theta) \le M_2$, for all $\theta \in [a,b]$, then the following inequality holds*

$$\left| \frac{g(\theta) + g(a+b-\theta)}{2} - \frac{1}{b-a} \int_a^b g(\tau) d\tau \right| \le (b-a) \left[\frac{1}{16} + \left(\frac{\theta - \frac{3a+b}{4}}{b-a} \right)^2 \right] (M_2 - m_2),$$

for all $\theta \in [a, \frac{a+b}{2}]$.

We need here to define Riemann-Liouville fractional integral (RLFI) (see [8]) for proving our next main result in the second section.

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\gamma > 0$ is stated as

$$J_a^\gamma g(\theta) = \frac{1}{\Gamma(\gamma)} \int_a^\theta (\theta - \tau)^{\gamma - 1} g(\tau) d\tau, \quad J_a^0 g(\theta) = g(\theta),$$

where gamma function $\Gamma(\gamma)$ is defined as

$$\Gamma(\gamma) = \int_0^\infty \theta^{\gamma-1} e^{-\theta} d\theta.$$

In 2009, Z. Liu [13] introduced some companions of an Ostrowski type inequality for functions whose second derivatives are absolutely continuous. In 2009, Barnett et. al [5] have derived some companions for Ostrowski inequality and the generalised trapezoid inequality. In 2012, M. W. Alomari [2] obtained a companion inequality of Ostrowski's type using Grüss result with applications. Recently, authors [15] gave a companion of weighted Ostrowski's type inequality using Grüss result with application.

In the present article we would prove a companion of weighted Fractional Ostrowski's type inequality by applying Grüss result and then we would give its applications.

2. Generalisation of Companion of Ostrowski's Type Inequality Via Riemann-Liouville Fractional Integral

Under present section we would give our results about companion of Ostrowski's type inequality which are as follow.

Theorem 2.1. *Suppose $g : [a,b] \to \mathbb{R}$ is a differentiable mapping in the interval (a,b) and $a < b$ and $w : [a,b] \to \mathbb{R}$ is a probability density function. If $g' \in L^1[a,b]$ and*

$m_2 \le g'(\tau) \le M_2$, for all $\tau \in [a,b]$, then

$$\begin{aligned}
&\Bigg| g(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&\quad - (b-\theta)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b)) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)g(b))\Bigg| \\
&\le \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2) \qquad (2.1)
\end{aligned}$$

holds for all $\theta \in [a, \frac{a+b}{2}]$.

Proof. For the sake of proof we state the weighted kernel as

$$P(\theta,\tau) = (b-\theta)^{1-\gamma}\Gamma(\gamma)\begin{cases} \displaystyle\int_a^{\tau} w(u)du, & \text{if } \tau \in [a,\theta], \\ \displaystyle\int_{\frac{a+b}{2}}^{\tau} w(u)du, & \text{if } \tau \in (\theta, a+b-\theta], \\ \displaystyle\int_b^{\tau} w(u)du, & \text{if } \tau \in (a+b-\theta, b], \end{cases} \qquad (2.2)$$

for all $\theta \in [a, \frac{a+b}{2}]$.

Applying RLFI operator and by parts formula of integration, obtain

$$\begin{aligned}
J_a^{\gamma}(P(\theta,b)g(b)) =& \frac{1}{\Gamma(\gamma)}\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau \\
=& g(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&- (b-\theta)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b)) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)g(b)).
\end{aligned}$$

It is clear that for all $\tau \in [a,b]$ and $\theta \in [a, \frac{a+b}{2}]$, we have

$$\theta - \frac{a+b}{2} \le P(\theta,\tau) \le \theta - a.$$

Applying Proposition (1.2) to the mappings $P(\theta,\cdot)$ and $(b-\cdot)^{\gamma-1}g'(\cdot)$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\gamma)}\left(\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau - \int_a^b P(\theta,\tau)d\tau \cdot \frac{1}{b-a}\int_a^b (b-\tau)^{\gamma-1}g'(\tau)d\tau\right)\right| \\
\le& \frac{1}{4\Gamma(\gamma)}\left(\theta - a - \left(\theta - \frac{a+b}{2}\right)\right) = \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2),
\end{aligned} \qquad (2.3)$$

for all $\theta \in [a, \frac{a+b}{2}]$. Since $\int_a^b P(\theta,\tau)d\tau = 0$, then (2.3) implies

$$\left|\frac{1}{\Gamma(\gamma)}\int_a^b (b-\tau)^{\gamma-1}P(\theta,\tau)g'(\tau)d\tau\right| \le \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2). \qquad (2.4)$$

Finally, we obtain desired result (2.1) from (2.4). □

Remark 2.1. If we put $\gamma = 1$ and $w = \frac{1}{b-a}$ in Theorem 2.1, then we recapture the Theorem 2.1 of [2].

Remark 2.2. If we put $\gamma = 1$ in Theorem 2.1, then we recapture the result of Theorem 2.1 of [15].

Corollary 2.1. *In the inequality* (2.1)*, select*
(i) $\theta = \frac{a+b}{2}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{a+b}{2}\right)\int_a^b w(\tau)d\tau - \left(\frac{b-a}{2}\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))\right.\\&\left.+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{a+b}{2},b\right)g(b)\right)\right|\\\leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2);\end{aligned}$$

(ii) $\theta = \frac{3a+b}{4}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{3a+b}{4}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + 3^{1-\gamma}g(\frac{a+3b}{4})\int_{\frac{a+b}{2}}^b w(\tau)d\tau\right.\\&\left.-\left(\frac{3}{4}(b-a)\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{3a+b}{4},b\right)g(b)\right)\right|\\\leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2);\end{aligned}\tag{2.5}$$

(iii) $\theta = \frac{2a+b}{3}$ *to obtain the following:*

$$\begin{aligned}&\left| g\left(\frac{2a+b}{3}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + 2^{1-\gamma}g(\frac{a+2b}{3})\int_{\frac{a+b}{2}}^b w(\tau)d\tau\right.\\&\left.-\left(\frac{2}{3}(b-a)\right)^{1-\gamma}\Gamma(\gamma)J_a^{\gamma}(w(b)g(b))+(\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{2a+b}{3},b\right)g(b)\right)\right|\\\leq&\frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2).\end{aligned}$$

In the following we present special case of (iii) of Corollary 2.1.
Special Case. If put $w = \frac{1}{b-a}$ and $\gamma = 1$ in (iii) of Corollary 2.1, then we get

$$\left|\frac{g(\frac{2a+b}{3})+g(\frac{a+2b}{3})}{2} - \frac{1}{b-a}\int_a^b g(\tau)d\tau\right| \leq \frac{1}{8}(b-a)(M_2-m_2).$$

Remark 2.3. (i) First by putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Theorem 2.1 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.1 (a) of [2].

(ii) By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in (i) of Corollary 2.1, we recapture Corollary 2.1 (c) of [2].

(iii) By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in (ii) of Corollary 2.1, we recapture Corollary 2.1 (b) of [2].

Remark 2.4. (i) First by putting $\gamma = 1$ in Theorem 2.1 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.3 (i) of [15].

(ii) By putting $\gamma = 1$ in (i) of Corollary 2.1, we recapture Corollary 2.3 (ii) of [15].

(iii) By putting $\gamma = 1$ in (ii) of Corollary 2.1, we recapture Corollary 2.3 (iii) of [15].

(iv) By putting $\gamma = 1$ in (iii) of Corollary 2.1, we recapture Corollary 2.3 (iv) of [15].

Ostrowski's type inequality can be defined in the form of following corollary.

Corollary 2.2. *Let the suppositions of Theorem* 2.1 *be valid. Further, if g is symmetric about the θ-axis, i.e., $g(a+b-\theta) = g(\theta)$, then*

$$\begin{aligned}
&\Bigg| g(\theta) \int_a^{\frac{a+b}{2}} w(\tau)d\tau + g(\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1} \int_{\frac{a+b}{2}}^b w(\tau)d\tau \\
&\quad - (b-\theta)^{1-\gamma}\Gamma(\gamma) J_a^{\gamma}(w(b)g(b)) + (\gamma-1) J_a^{\gamma-1}(P(\theta,b)g(b)) \Bigg| \\
&\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2 - m_2)
\end{aligned} \tag{2.6}$$

holds.

Remark 2.5. First by putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 2.2 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.2 of [2].

Remark 2.6. First by putting $\gamma = 1$ in Corollary 2.2 and then put $\theta = a$ in obtained inequality, we recapture Corollary 2.5 of [15].

3. Application to Numerical Integration

Let $I_n : a = \theta_0 < \theta_1 < \cdots < \theta_n = b$ be division of interval $[a,b]$ and $h_i = \theta_{i+1} - \theta_i$, $i = 0, 1, 2, \ldots, n-1$.

Consider the quadrature formula

$$\begin{aligned}
Q_n(I_n, g) := \sum_{i=0}^{n-1} \Bigg[& g(\frac{3\theta_i + \theta_{i+1}}{4}) \int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i + 3\theta_{i+1}}{4}\right) \\
& \times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma-1) J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4}, \theta_{i+1}\right) g(\theta_{i+1})\right) \Bigg].
\end{aligned}$$

We give following result.

Theorem 3.1. *Suppose $g : I \to \mathbb{R}$ is a differentiable mapping in interval I° and $w : [a,b] \to \mathbb{R}$ is a probability density function, where $a, b \in I$ with $a < b$. If $g' \in L^1[a,b]$ and $m_2 \le g'(\theta) \le M_2$, for all $\theta \in [a,b]$, then the following holds*

$$\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma} J^{\gamma}_{\theta_i}(w(\theta_{i+1})g(\theta_{i+1})) = Q_n(I_n,g) + R_n(I_n,g), \tag{3.1}$$

where $Q_n(I_n,g)$ is stated as above and the following remainder $R_n(I_n,g)$ satisfies the estimates

$$|R_n(I_n,g)| \le \frac{1}{8\Gamma(\gamma)}(M_2 - m_2)h_i. \tag{3.2}$$

Proof. Applying inequality (2.5) on the intervals $[\theta_i, \theta_{i+1}]$, we get

(3.3)
$$\begin{aligned} R_i(I_i,g) =& \Gamma(\gamma)\left(\frac{3}{4}h_i\right)^{1-\gamma} J^{\gamma}_{\theta_i}(w(\theta_{i+1})g(\theta_{i+1})) \\ & - \left[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau \right. \\ & \left. + (\gamma-1)J^{\gamma-1}_{\theta_i}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\right]. \end{aligned}$$

Summing (3.3) over i from 0 to $n-1$, then

$$\begin{aligned} R_n(I_n,g) =& \Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma} J^{\gamma}_{\theta_i}(w(\theta_{i+1})g(\theta_{i+1})) \\ & - \sum_{i=0}^{n-1}\left[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma} g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\right. \\ & \left. \times \int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau + (\gamma-1)J^{\gamma-1}_{\theta_i}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\right], \end{aligned}$$

which follows the form of (2.5), i.e.,

$$\begin{aligned}|R_n(I_n,g)| =&\Bigg|\Gamma(\gamma)\sum_{i=0}^{n-1}\left(\frac{3}{4}h_i\right)^{1-\gamma} J_{\theta_i}^{\gamma}(w(\theta_{i+1})g(\theta_{i+1}))\\ &-\sum_{i=0}^{n-1}\left[g\left(\frac{3\theta_i+\theta_{i+1}}{4}\right)\int_{\theta_i}^{\frac{\theta_i+\theta_{i+1}}{2}} w(\tau)d\tau + 3^{1-\gamma}g\left(\frac{\theta_i+3\theta_{i+1}}{4}\right)\right.\\ &\times\left.\int_{\frac{\theta_i+\theta_{i+1}}{2}}^{\theta_{i+1}} w(\tau)d\tau+(\gamma-1)J_{\theta_i}^{\gamma-1}\left(P\left(\frac{3\theta_i+\theta_{i+1}}{4},\theta_{i+1}\right)g(\theta_{i+1})\right)\right]\Bigg|\\ \le&\frac{1}{8\Gamma(\gamma)}(M_2-m_2)\sum_{i=0}^{n-1}h_i.\end{aligned}$$

This completes the required proof. □

Remark 3.1. By putting $\gamma=1$ and $w=\frac{1}{b-a}$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [2].

Remark 3.2. By putting $\gamma=1$ in Theorem 3.1, we recapture the result of Theorem 3.1 of [15].

4. Applications to Probability Theory

Throughout this section we consider $w:[a,b]\to[0,1]$. Suppose Y is a random variable taking values in the finite interval $[a,b]$ with probability density function $g:[a,b]\to[0,1]$ and with cumulative distribution function $G:[a,b]\to[0,1]$ is introduced and defined by us, i.e.,

$$G(\theta)=P(Y\le\theta)=\Gamma(\gamma)J_a^{\gamma}(w(\theta)g(\theta))=\int_a^{\theta}(\theta-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau,\quad a\le\theta\le\frac{a+b}{2},$$

and

$$\begin{aligned}E(Y)=\int_a^b \tau g(\tau)d\tau,\quad E_w(Y)&=\int_a^b \tau w(\tau)g(\tau)d\tau,\\ E_{wf}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau,\\ E_{wf1}(Y)=&\Gamma(\gamma)J_a^{\gamma-1}(bw(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-2}w(\tau)g(\tau)d\tau,\\ E_{wf2}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw'(b)g(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w'(\tau)g(\tau)d\tau,\\ E_{wf3}(Y)=&\Gamma(\gamma)J_a^{\gamma}(bw(b)g'(b))=\int_a^b\tau(b-\tau)^{\gamma-1}w(\tau)g'(\tau)d\tau,\end{aligned}$$

are the expectation, weighted expectation and weighted fractional expectation of random variable 'Y' in interval $[a,b]$, respectively. Then we can write the following theorem.

Theorem 4.1. *Suppose $g : [a,b] \to \mathbb{R}$ is a differentiable mapping in the interval (a,b) and $a < b$. If $g' \in L^1[a,b]$ and $m_2 \le g'(\tau) \le M_2$, for all $\tau \in [a,b]$. Further, suppose that function w is differentiable, then*

(4.1)
$$\Bigg| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(a+b-\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau$$
$$- (b-\theta)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)G(b))\Bigg|$$
$$\le \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2)$$

holds for all $\theta \in [a, \frac{a+b}{2}]$.

Proof. Select $g = G$, we obtain (4.1), by applying the identity

$$\Gamma(\gamma)J_a^{\gamma}(w(b)g(b)) = \int_a^b (b-\tau)^{\gamma-1}w(\tau)g(\tau)d\tau$$
$$= (\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y).$$

Since $G(a) = 0$ and $G(b) = 1$.

We left the details to research scholars. □

Corollary 4.1. *Select $\gamma = 1$ in Theorem 4.1. Then get the following*

$$\Bigg| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(a+b-\theta)\int_{\frac{a+b}{2}}^b w(\tau)d\tau + E_w(Y) + \int_a^b \tau w'(\tau)G(\tau)d\tau - bw(b)\Bigg|$$
$$\le \frac{1}{8}(b-a)(M_2-m_2)$$

holds for all $\theta \in [a, \frac{a+b}{2}]$, where $E_w(Y)$ is the weighted expectation of Y.

Remark 4.1. If we put $w = \frac{1}{b-a}$ in Corollary 4.1 and taking the expectation $E(Y) = \int_a^b \tau G(\tau)d\tau = b - \int_a^b G(\tau)d\tau$, we recapture Theorem 4.1 of [2].

Corollary 4.2. *Select $\theta = \frac{3a+b}{4}$ in Theorem 4.1, we get*

$$\Bigg| G\left(\frac{3a+b}{4}\right)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G\left(\frac{a+3b}{4}\right)\left(\frac{3}{4}(b-a)\right)^{1-\gamma}\left(\frac{b-a}{4}\right)^{\gamma-1}\int_{\frac{a+b}{2}}^b w(\tau)d\tau$$
$$- \left(\frac{3}{4}(b-a)\right)^{1-\gamma}\Big((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Big)$$
$$+ (\gamma-1)J_a^{\gamma-1}\left(P\left(\frac{3a+b}{4}, b\right)G(b)\right)\Bigg|$$
$$\le \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2).$$

Remark 4.2. By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 4.2, we recapture Corollary 4.1 of [2].

Corollary 4.3. *In Theorem* 4.1*, if* G *is symmetric about the* $\theta-$*axis, i.e.,* $G(a+b-\theta) = G(\theta)$*, then*

$$\Bigg| G(\theta)\int_a^{\frac{a+b}{2}} w(\tau)d\tau + G(\theta)(b-\theta)^{1-\gamma}(\theta-a)^{\gamma-1}\int_{\frac{a+b}{2}}^{b} w(\tau)d\tau$$

$$-(b-\theta)^{1-\gamma}\Bigg((\gamma-1)E_{wf1}(Y) - E_{wf2}(Y) - E_{wf3}(Y)\Bigg) + (\gamma-1)J_a^{\gamma-1}(P(\theta,b)G(b))\Bigg|$$

$$\leq \frac{1}{8\Gamma(\gamma)}(b-a)(M_2-m_2)$$

holds for all $\theta \in [a, \frac{a+b}{2}]$.

Remark 4.3. By putting $\gamma = 1$ and $w = \frac{1}{b-a}$ in Corollary 4.3, we recapture Corollary 4.2 of [2].

Before application to special means, we would present some special means and these means will apply in the 5th section.

Special Means. These means can be found in [20].

(a) The Arithmetic Mean

$$A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0.$$

(b) The Geometric Mean

$$G = G(a,b) = \sqrt{ab}, \quad a,b \geq 0.$$

(c) The Harmonic Mean

$$H = H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b > 0.$$

(d) The Logarithmic Mean

$$L = L(a,b) = \begin{cases} a, & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b, \end{cases} \quad a,b > 0.$$

(e) Identric Mean

$$I = I(a,b) = \begin{cases} a, & \text{if } a = b, \\ \ln\left(\frac{\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}}{e}\right), & \text{if } a \neq b, \end{cases} \quad a,b > 0.$$

(f) p-Logarithmic Mean

$$L_p = L_p(a,b) = \begin{cases} a, & \text{if } a = b, \\ \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}$, $a, b > 0$. It is known that L_p monotonically increasing over $p \in \mathbb{R}$, $L_0 = I$ and $L_{-1} = L$.

5. Application to Special Means

Example 5.1. Consider $\gamma = 1$, $g(\theta) = \theta^p$, $p \in \mathbb{R}\backslash\{-1,0\}$. Then for $a < b$, we have

$$\frac{1}{(b-a)}\int_a^b g(\tau)d\tau = L_p^p(a,b), \quad \frac{g(a)+g(b)}{2} = A(a^p, b^p),$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left|\frac{\theta^p + (2A-\theta)^p}{2} - L_p^p(a,b)\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|A(a^p, b^p) - L_p^p(a,b)\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

Example 5.2. Consider $\gamma = 1$, $g(\theta) = \frac{1}{\theta}$, $\theta \neq 0$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = L^{-1}(a,b), \quad \frac{g(a)+g(b)}{2} = \frac{A}{G^2},$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}] \subset (0,\infty)$.

Therefore, (2.1) becomes

$$\left|\frac{A}{\theta(a+b-\theta)} - L^{-1}(a,b)\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|\frac{A}{G^2} - L^{-1}(a,b)\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

Example 5.3. Consider $\gamma = 1$, $g(\theta) = \ln\theta$, $\theta \in (0,\infty)$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = \ln(I(a,b)), \quad \frac{g(a)+g(b)}{2} = \ln G,$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}] \subset (0,\infty)$. Therefore, (2.1) becomes

$$\left|\ln\left[\frac{[\theta(2A-\theta)]^{\frac{1}{2}}}{I(a,b)}\right]\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|\ln\left[\frac{G}{I(a,b)}\right]\right| \le \frac{1}{8}(b-a)(M_2 - m_2).$$

Example 5.4. Consider $\gamma = 1$, $g(\theta) = e^\theta$, $\theta \in (-\infty, \infty)$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = \frac{e^b - e^a}{b-a}, \quad \frac{g(a)+g(b)}{2} = A(e^a, e^b),$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left|\frac{e^\theta + e^{(2A-\theta)}}{2} - \frac{e^b - e^a}{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|A(e^a, e^b) - \frac{e^b - e^a}{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

Example 5.5. Consider $\gamma = 1$, $g(\theta) = \tan\theta$, $\theta \neq \frac{\pi}{2} \pm n\pi$. Then

$$\frac{1}{b-a}\int_a^b g(\tau)d\tau = \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}, \quad \frac{g(a)+g(b)}{2} = A(\tan a, \tan b),$$

and $\frac{a+b}{2} = A(a,b)$, where $\theta \in [a, \frac{a+b}{2}]$. Therefore, (2.1) becomes

$$\left|\frac{\tan\theta + \tan(2A-\theta)}{2} - \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

If we choose $\theta = a$ (or $\theta = b$) in (2.1), we get

$$\left|A(\tan a, \tan b) - \ln\left[\frac{\sec b}{\sec a}\right]^{b-a}\right| \leq \frac{1}{8}(b-a)(M_2 - m_2).$$

6. Conclusion

In this article our target was to generalise the results of [2] and [15]. We have obtained generalisation of companion of Ostrowski's type integral inequality for differentiable mappings whose 1st derivatives are bounded by using the Riemann-Liouville fractional integral. By applying suitable substitutions we have recaptured all results of M. W. Alomari's article [2] and also recaptured all results of one more article [15] of different authors. Moreover, we have given applications to numerical integration, probability theory and special means.

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