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A LYAPUNOV TYPE INEQUALITY FOR A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH RIEMANN-LIOUVILLE DERIVATIVE

ANABELA S. SILVA[1]

Abstract. In this paper, a Lyapunov-type inequality is obtained for a class of fractional boundary value problems involving Riemann-Liouville fractional derivative of orders $\alpha \in (1,2)$ and $\beta \in (0,\alpha-1)$. The study is based on the construction of a Green's function and the obtaining of its corresponding maximum value.

1. Introduction

The fractional calculus is a field of mathematics that deals with generalizing the concept of differentiation and integration to non-integer orders. This definition departs from the traditional concept of derivative and integral in a differential and integral calculus. The concept dates back to 1695, in a famous correspondence between L'Hopital and Leibniz. However, it is only in the last decades that this area of mathematics has gained special prominence, mainly due to its proven applications in various sciences and engineering, such as mechanics and biology. In fact, fractional order operators have the characteristics of being non-local and with memory capacity associated to its kernel, which allows to create models more practical and realistic than those using integer derivatives.

The Lyapunov's inequality [8], named after its author, gives a necessary condition for the existence of non-trivial solutions for a boundary value problem with an ordinary second order differential equation. The original result states that if there is a nontrivial

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Key words and phrases. Fractional differential equations, Lyapunov inequality, Riemman-Liouville derivative, Green's function.

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solution of the boundary value problem

$$\begin{cases} x''(t) + q(t)x(t) = 0, & t \in [a, b], \\ x(a) = x(b) = 0, \end{cases}$$

and where $q : [a, b] \to \mathbb{R}$ is continuous, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}.$$

After that, several proofs and generalizations appeared in the literature (cf. e.g., [5, 7, 10, 11]). In recent years, some authors extended have obtained Lyapunov-type inequalities for boundary value problems involving the fractional derivatives. As a survey of results on Lyapunov-type inequalities for fractional differential equations, we recommend the work of Ntouyas et al. [9].

Inspired in the works of Dhar and Neugebauer [1] and Ferreira [2–4], in this paper we consider the fractional boundary value problem with (left) Riemann-Liouville fractional derivative

$$\begin{cases} (\mathcal{D}^{\alpha}_{a+}x)(t) + (\mathcal{D}^{\beta}_{a+}qx)(t) = 0, & t \in [a, b], \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

with $0 \le a < b$, $\alpha \in (1, 2)$, $\beta \in (0, \alpha - 1)$ and where q is a real and continuous function. To the best of author's knowledge, the theory presented in this paper has not been studied yet.

This paper is organized as follows: in the Section 2, some necessary definitions and results are presented; in Section 3 the problem is rewritten in the form of an integral equation, making use of Green's function. The maximum of this function is obtained here. It is then presented the main result of the paper, the Lyapunov's inequality for the problem under study. Finally, an example is presented applying the theory previously presented.

2. Preliminaries

In this section, we recall some important definitions and results.

Definition 2.1. Let $\alpha \in \mathbb{R}^+$. The (left) Riemann-Liouville fractional integral of order α of the function x on $[a, b]$ is defined by

$$I^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}x(s)ds,$$

where $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}\mathrm{d}t$ is the Euler Gamma function, provided the right-hand side is pointwise defined in $(a, +\infty)$.

Definition 2.2. The (left) Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x on $[a, b]$ is defined by

$$\mathcal{D}^{\alpha}_{a+}x(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a+}x)(t) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1}x(s)ds,$$

where n is the smallest integer greater or equal than α (provided the right-hand side is pointwise defined in $(a, +\infty)$).

The following results can be found in [6] and [12] and are essential thorough this paper.

Lemma 2.1. *For* $\alpha, \beta > 0$, *the Riemann-Liouville fractional integral satisfies the semigroup property*

$$I_{a+}^{\alpha}(I_{a+}^{\beta}x)(t) = I_{a+}^{\beta}(I_{a+}^{\alpha}x)(t) = (I_{a+}^{\alpha+\beta}x)(t), \tag{2.1}$$

at almost every point $t \in (a, b)$ *for* $x \in L^p(a, b)$, $1 \leq p \leq +\infty$. *If* $\alpha + \beta \geq 1$, *the relation holds for any point of* $[a, b]$.

Lemma 2.2. *Let* $x \in C([a, b]) \cap L^1([a, b])$ *and* $\alpha > 0$. *It follows that*

$$\mathcal{D}_{a+}^{\alpha} I_{a+}^{\alpha} x(t) = x(t)$$

and

$$I_{a+}^{\alpha} \mathcal{D}_{a+}^{\alpha} x(t) = x(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *where* n *is the smallest integer greater than or equal to* α.

Among the several properties of the Riemann-Liouville integral, we highlight the following.

Property 1. For $\alpha, \beta > 0$,

$$I_{a+}^{\alpha}[(s-a)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}.$$

3. Main Results

The aim of this section is to obtain a necessary condition for the existence of a nontrivial solution of the fractional boundary value problem under study. For that purpose, we start by rewriting the considered problem in terms of an integral equation.

Lemma 3.1. *Let* $1 < \alpha < 2$, $0 < \beta < \alpha - 1$ *and* $q \in C([a, b])$. *Then the solution* x *of* (1.1) *can be represented in the integral form as*

$$x(t) = \int_a^b G(t, s) q(s) x(s) ds,$$

where the Green's function $G(t, s)$ *is defined by*

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1}, & a \leq s \leq t \leq b. \end{cases} \tag{3.1}$$

Proof. Let us first observe that according to Proposition 2.1, we have that

$$(I^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}x)(t) = I^{\alpha-\beta}_{a+}(I^{\beta}_{a+}\mathcal{D}^{\beta}_{a+}qx)(t).$$

Thus, following Lemma 2.2 and applying Property 1 we get

$$(I^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}qx)(t) = (I^{\alpha-\beta}_{a+}qx)(t) + c_1 I^{\alpha-\beta}_{a+}(t-a)^{\beta-1} = (I^{\alpha-\beta}_{a+}qx)(t) + c_1\frac{\Gamma(\beta)}{\Gamma(\alpha)}(t-a)^{\alpha-1},$$

$c_1 \in \mathbb{R}$. In this sense, applying I^{α}_{a+} to both members of equation

$$(\mathcal{D}^{\alpha}_{a+}x)(t) + (\mathcal{D}^{\beta}_{a+}qx)(t) = 0,$$

we get

$$x(t) + d_1(t-a)^{\alpha-1} + d_2(t-a)^{\alpha-2} + (I^{\alpha-\beta}_{a+}qx)(t) + c(t-a)^{\alpha-1} = 0,$$

$d_1, d_2, c \in \mathbb{R}$. It follows that, for some constants $k_1, k_2 \in \mathbb{R}$,

$$x(t) = k_1(t-a)^{\alpha-1} + k_2(t-a)^{\alpha-2} - (I^{\alpha-\beta}_{a+}qx)(t).$$

From $x(a) = 0$, it follow that $k_2 = 0$. From $x(b) = 0$, we get that

$$k_1 = \frac{1}{(b-a)^{\alpha-1}\Gamma(\alpha-\beta)}\int_a^b (b-s)^{\alpha-\beta-1}q(s)x(s)ds.$$

Therefore,

$$\begin{aligned} x(t) =& \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma(\alpha-\beta)}\int_a^b (b-s)^{\alpha-\beta-1}q(s)x(s)ds \\ &- \frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}q(s)x(s)ds \\ =& \int_a^b G(t,s)q(s)x(s)ds, \end{aligned}$$

and the proof is complete. ☐

The following theorem concerns on the maximum of the Green's function (3.1) and constitutes an essential tool to obtain the main theorem of this paper.

Theorem 3.1. *For any* $(t,s) \in [a,b]\times[a,b]$*, the Green's function* (3.1) *satisfies the following property*

$$\max_{t,s\in[a,b]}|G(t,s)| = \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1},$$

with $\alpha \in (1,2)$ *and* $\beta \in (0, \alpha-1)$.

Proof. Let us consider

$$g_1(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}},$$

$$g_2(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-\beta-1},$$

and consequently, the Green's function can be rewritten as

$$G(t,s)=\frac{1}{\Gamma(\alpha-\beta)}\begin{cases} g_1(t,s), & a\le t\le s\le b,\\ g_2(t,s), & a\le s\le t\le b.\end{cases}$$

For any $(t,s)\in[a,b]\times[a,b]$ with $t\le s$, it is clear that $G(t,s)=\frac{g_1(t,s)}{\Gamma(\alpha-\beta)}\ge 0$. Moreover, fixing $t\in[a,b]$ and taking the derivative with respect to s

$$\frac{\partial}{\partial s}g_1(t,s)=-\frac{(\alpha-\beta-1)(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-1}}\le 0.$$

Thus, for $a\le t\le s\le b$, $G(t,s)$ is decreasing (on variable s), which implies that

$$G(t,s)=\frac{g_1(t,s)}{\Gamma(\alpha-\beta)}\le\frac{g_1(t,t)}{\Gamma(\alpha-\beta)}.$$

Let

$$h(t)=\frac{g_1(t,t)}{\Gamma(\alpha-\beta)}=\frac{(t-a)^{\alpha-1}(b-t)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(b-a)^{\alpha-1}}.$$

Since $h(a)=h(b)=0$, we can apply the Rolle's Theorem to conclude that there exists $c\in[a,b]$ such that $h(c)=\max_{t\in[a,b]}h(t)$. We have that

$$h'(t)=\frac{(t-a)^{\alpha-2}(b-t)^{\alpha-\beta-2}[(\alpha-1)(b-t)-(\alpha-\beta-1)(t-a)]}{\Gamma(\alpha-\beta)(b-a)^{\alpha-1}},$$

and hence, $h'(c)=0$ for

$$c=\frac{(\alpha-1)b+(\alpha-\beta-1)a}{2\alpha-\beta-2}.$$

Note that

$$c>\frac{(\alpha-1)a+(\alpha-\beta-1)a}{2\alpha-\beta-2}=a$$

and

$$c<\frac{(\alpha-1)b+(\alpha-\beta-1)b}{2\alpha-\beta-2}=b,$$

which shows that c is well defined. Consequently, we obtain that

$$G(c,c)=h(c)=\frac{1}{\Gamma(\alpha-\beta)}\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}(b-a)^{\alpha-\beta-1}. \tag{3.2}$$

Consider now the case $a\le s\le t\le b$, $(t,s)\in[a,b]\times[a,b]$. In that case, we have that

$$G(t,s)=\frac{g_2(t,s)}{\Gamma(\alpha-\beta)}.$$

Fixing $t \in [a,b]$, and taking the derivative with respect to s, we have

$$\begin{aligned}\frac{\partial}{\partial s} g_2(t,s) &= -\frac{(\alpha-\beta-1)(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-1}} + (\alpha-\beta-1)(t-s)^{\alpha-\beta-2} \\ &= (\alpha-\beta-1)(b-s)^{\alpha-\beta-2}\left[\left(\frac{t-s}{b-s}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}\right].\end{aligned}$$

Let

$$v(s) = \left(\frac{t-s}{b-s}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}.$$

Then, it follows that

$$v'(s) = (\alpha-\beta-2)\frac{t-b}{(b-s)^2}\left(\frac{t-s}{b-s}\right)^{\alpha-\beta-3},$$

which is positive since $\alpha-\beta-2<0$ and $\frac{t-b}{(b-s)^2} \le 0$ (with $t \le b$). Thus, we have that v is increasing, which means that

$$v(s) \ge v(a) = \left(\frac{t-a}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1} \ge 0,$$

since $0 \le \left(\frac{t-a}{b-a}\right) \le 1$ and $\alpha-\beta-2 < \alpha-1$. Thus, we can obtain that

$$\frac{\partial}{\partial s} g_2(t,s) \ge (\alpha-\beta-1)(b-s)^{\alpha-\beta-2}\left[\left(\frac{t-a}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-a}{b-a}\right)^{\alpha-1}\right] \ge 0,$$

and we conclude that $g_2(t,s)$ is increasing on variable s and thus

$$\frac{g_2(t,a)}{\Gamma(\alpha-\beta)} \le \frac{g_2(t,s)}{\Gamma(\alpha-\beta)} \le \frac{g_2(t,t)}{\Gamma(\alpha-\beta)},$$

for all $s \in [a,t]$. In this way, we obtain that, for $a \le s \le t \le b$,

$$\max_{t\in[s,b]}\left|\frac{g_2(t,s)}{\Gamma(\alpha-\beta)}\right| = \max_{t\in[a,b]}\left\{\left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right|, \left|\frac{g_2(t,t)}{\Gamma(\alpha-\beta)}\right|\right\}$$

Since $\frac{g_2(t,t)}{\Gamma(\alpha-\beta)} = \frac{g_1(t,t)}{\Gamma(\alpha-\beta)} \ge 0$, for any $t \in [a,b]$, we only need to consider $\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}$. Observing that

$$\frac{g_2(t,a)}{\Gamma(\alpha-\beta)} = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[\left(\frac{t-a}{b-a}\right)^{\beta} - 1\right] \le 0,$$

we can write that

$$\left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right| = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[1 - \left(\frac{t-a}{b-a}\right)^{\beta}\right].$$

Let

$$w(t) = \left|\frac{g_2(t,a)}{\Gamma(\alpha-\beta)}\right| = \frac{(t-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left[1 - \left(\frac{t-a}{b-a}\right)^{\beta}\right].$$

We can observe that $w(a) = 0$ and $w(b) = 0$. Since $w(t)$ is continuous and nonnegative, using again Rolle's Theorem, the maximum is achieved for a $c^* \in [a, b]$ such that $w'(c^*) = 0$. In this way, having in mind that $t \neq a$, we have

$$w'(t) = \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-\beta)} \left(\frac{\alpha-\beta-1}{(t-a)^{\beta}} - \frac{\alpha-1}{(b-a)^{\beta}} \right).$$

Consequently, $w'(c^*) = 0$ for

$$c^* = \sqrt[\beta]{\frac{\alpha-\beta-1}{\alpha-1}}(b-a) + a,$$

and as before, we can prove easily that $a < c^* < b$. We conclude

$$\begin{aligned} \frac{|g_2(c^*, a)|}{\Gamma(\alpha-\beta)} &= \frac{1}{\Gamma(\alpha-\beta)} \left(\left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-\beta-1}{\beta}} - \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-1}{\beta}} \right) (b-a)^{\alpha-\beta-1} \\ &= \frac{1}{\Gamma(\alpha-\beta)} \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1} \right) (b-a)^{\alpha-\beta-1}. \end{aligned} \tag{3.3}$$

Joining the results (3.2) and (3.3), we obtain that, for $\alpha \in (1, 2)$, $\beta \in (0, \alpha-1)$,

$$\max_{t,s\in[a,b]} |G(t,s)| = \frac{\max\left\{ \left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}, \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1}\right) \right\}}{\Gamma(\alpha-\beta)} (b-a)^{\alpha-\beta-1}. \tag{3.4}$$

To complete the proof, let us prove that

$$\left(\frac{\alpha-1}{2\alpha-\beta-2} \right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2} \right)^{\alpha-\beta-1} > \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-1}{\beta}} \left(\frac{\beta}{\alpha-\beta-1} \right).$$

To that purpose, we first observe that $\alpha-1 > \beta$ and $2\alpha-\beta-2 = 2(\alpha-1)-\beta < \alpha-1$. Moreover, since $\frac{\alpha-\beta-1}{\alpha-1} < 1$ and $\alpha-\beta-1 < \frac{\alpha-\beta-1}{\beta}$ (having in mind that $0 < \beta < 1$), we conclude that

$$\left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\alpha-\beta-1} > \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-\beta-1}{\beta}}.$$

Applying those results, we therefore obtain

$$\begin{aligned} \left(\frac{\alpha-1}{2\alpha-\beta-2} \right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{2\alpha-\beta-2} \right)^{\alpha-\beta-1} &> \left(\frac{\alpha-1}{\alpha-1} \right)^{\alpha-1} \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\alpha-\beta-1} \\ &= \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\alpha-\beta-1} \\ &> \left(\frac{\alpha-\beta-1}{\alpha-1} \right)^{\frac{\alpha-\beta-1}{\beta}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}}\left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{-1}\\
&= \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}}\left(\frac{\alpha-1}{\alpha-\beta-1}\right)\\
&> \left(\frac{\alpha-\beta-1}{\alpha-1}\right)^{\frac{\alpha-1}{\beta}}\left(\frac{\beta}{\alpha-\beta-1}\right).
\end{aligned}
$$

Thus, recalling (3.4), we conclude that

$$\max_{t,s\in[a,b]}|G(t,s)| = \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1},$$

which completes the proof. ☐

3.1. **A Lyapunov-type inequality for the problem under study.** The following theorem states a Lyapunov-type inequality for the fractional boundary value problem (1.1).

Theorem 3.2. *Let x be a nontrivial continuous solution of the fractional boundary value problem* (1.1). *Then*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha-\beta)(b-a)^{1+\beta-\alpha}}{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}.$$

Proof. Suppose that x is a solution of the fractional boundary value problem (1.1). Then, according to Lemma 3.1,

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds,$$

with G as defined in (3.1). Assume that x is a continuous and a nontrivial function on $[a,b]$. Thus, there exists an $m>0$ such that $\max_{t\in[a,b]}|x(t)| = m$. We have that

$$
\begin{aligned}
m &< m\max_{t\in[a,b]}\int_a^b |G(t,s)||q(s)|ds\\
&\le m\frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1}\int_a^b |q(s)|ds.
\end{aligned}
$$

Thus, we can conclude that

$$1 < \frac{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1}\int_a^b |q(s)|ds,$$

which gives the desired result. ☐

The following corollary is a natural consequence of Theorem 3.2.

Corollary 3.1. *If x is a continuous nontrivial solution of the fractional boundary value problem*

$$\begin{cases} (\mathcal{D}^{\alpha}_{a+}x)(t) + \lambda(\mathcal{D}^{\beta}_{a+}x)(t) = 0, & t \in [a,b], \\ x(a) = x(b) = 0, \end{cases}$$

with $\lambda \in \mathbb{R}\backslash\{0\}$, then

$$|\lambda| > \frac{\Gamma(\alpha-\beta)(b-a)^{\beta-\alpha}}{\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1}}.$$

4. Example

Consider the fractional boundary value problem

$$\begin{cases} (\mathcal{D}^{1.5}_{1+}x)(t) + (\mathcal{D}^{0.3}_{1+}ktx)(t) = 0, & t \in [1,2], \\ x(1) = x(2) = 0, \end{cases}$$

where k is a real number. In that case, for $\alpha = 1.5$ and $\beta = 0.3$, we have that

$$\left(\frac{\alpha-1}{2\alpha-\beta-2}\right)^{\alpha-1}\left(\frac{\alpha-\beta-1}{2\alpha-\beta-2}\right)^{\alpha-\beta-1} = \left(\frac{5}{7}\right)^{0.5}\left(\frac{2}{7}\right)^{0.2}.$$

Since

$$|k|\int_1^2 |t|\,dt = \frac{3}{4}|k|,$$

we have that if there is a nontrivial solution, then

$$|k| > \frac{4\Gamma(1.2)}{3\left(\frac{5}{7}\right)^{0.5}\left(\frac{2}{7}\right)^{0.2}} > 1.86.$$

5. Conclusion

In this paper, we obtain a Lyapunov type inequality for a boundary value problem with a fractional Riemann-Liouville derivatives. This result complements the existing results in the area. The novelty in this work is the fact that the problem considered uses different orders of the Riemann-Liouville derivative, which leads to different solutions and, consequently, different results. The inequality obtained is a necessary condition for the existence of a nontrivial continuous solutions of the problem under study.

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[1]CIDMA & DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF AVEIRO,
CAMPUS UNIVERSITÁRIO DE SANTIAGO, 3810-193 AVEIRO, PORTUGAL
Email address: `anabela.silva@ua.pt`

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COUPLED NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL INTEGRO-DIFFERENTIAL LANGEVIN SYSTEM VIA VARIABLE COEFFICIENT

LAHCEN IBNELAZYZ[1], KAMAL AIT TOUCHENT[2], AND KARIM GUIDA[3]

Abstract. In this paper, we aim to study a new coupled system of nonlinear fractional integro-differential Langevin equations with coupled multipoint boundary conditions. The existence and uniqueness of solution are investigated by using the Banach's and Krasnoselskii's fixed point theorems. The Ulam-Hyers stability of the mentioned equation is provided by applying the classical technique of functional analysis. Two examples are presented to verify our analysis.

1. Introduction

Fractional calculus has attained considerable interest due to their various applications in many scientific fields and the ability to describe the phenomena that have memory effects. In particular, fractional differential equations can be used to model a number of problems in physics, chemistry, biology and economy. As a result, many several authors have interested in it. For more details, one can go through in the books [1–4] and and the papers [5–16].

Langevin equations (introduced by Langevin in 1908) are used to model the evolution of physical phenomena in fluctuating environments (see [17]). Recently, the generalisation of the Langevin equations (fractional Langevin equations) has been considered by authors and researchers, for more details we give the following references [18–25].

Key words and phrases. Fractional integro-differential Langevin system, fractional derivatives and integrals, coupled nonlocal boundary value problems, Ulam-Hyers stability.

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On the other hand, coupled systems of fractional differential equations, including coupled nonlocal boundary conditions have been one of the important subjects in the field of fractional differential equations for their rich history, for more information see, [26–30].

In the last years, Ulam-Hyers stability has become of great importance to many researchers. It was introduced in 1940 by Ulam and then developed by Hyers. Many authors generalized the results obtained by Hyers for integer order differential equations. The mentioned stability for fractional differential equations are very important in many domains such as realistic problems, biology and economics. Recently, only a few authors have investigated in their work this type of Ulam Stabilities for coupled system of nonlinear fractional differential equations, see [31–35].

To our knowledge, coupled fractional integro-differential Langevin equations via variable coefficient involving coupled multipoint boundary conditions have not been extensively investigated yet. That's why, in the present article, we investigate a coupled system of fractional Langevin equations as follows:

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1}+\lambda_1(t))x(t)=f(t,x(t),y(t),\Phi y(t)), & t\in[0,1],\\ {}^cD^{\beta_2}({}^cD^{\alpha_2}+\lambda_2(t))y(t)=g(t,x(t),y(t),\Psi x(t)), & t\in[0,1], \end{cases}\tag{1.1}$$

subject to coupled multipoint boundary conditions

$$\begin{cases} x(0)=0, \quad x(a_1)=0, \quad x(1)=\sum_{i=1}^{n}\gamma_i y(s_i),\\ y(0)=0, \quad y(b_1)=0, \quad y(1)=\sum_{j=1}^{m}\delta_i x(u_j),\\ 0<a_1<b_1<s_1<s_2<\cdots<s_n<u_1<u_2<\cdots<u_m<1, \end{cases}\tag{1.2}$$

where $0<\alpha_k<1$, $1<\beta_k\leq 2$, for $k=1,2$, $\gamma_i,\delta_j\in\mathbb{R}^*$ for $i=1,\dots,n$, $j=1,2,\dots,m$, ${}^cD^{\beta_k}$, ${}^cD^{\alpha_k}$ are the Caputo's fractional derivatives, and $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\lambda_1,\lambda_2:[0,1]\to\mathbb{R}$ are a given continuous functions and $\Psi x(t)=\int_0^t\psi(t,s)y(s)ds$, $\Phi y(t)=\int_0^t\phi(t,s)y(s)ds$, where $\phi,\psi:[0,1]\times[0,1]\longrightarrow[0,+\infty)$, with $\lambda_0=\sup_{t\in[0,1]}|\int_0^t\phi(t,s)ds|<+\infty$, $\delta_0=\sup_{t\in[0,1]}|\int_0^t\psi(t,s)ds|<+\infty$.

This paper is arranged as follows: in the second section, we give some preliminaries and notations that will be useful throughout the work. In the third section, we establish the main results by using the fixed point theory. In the fourth section, we investigated that Problem (1.1)–(1.2) is Ulam-Hyers stability. The last section, we give some examples to illustrate the results.

2. PRELIMINARIES AND NOTATIONS

In this section, we introduce some notation, definitions and lemma that we use in our proofs later.

Definition 2.1 ([3]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2 ([3]). The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n \in \mathbb{N}$, $0 \leq n-1 < \alpha < n$, $t > 0$.

Theorem 2.1 ([36]). *Let M be a bounded, closed, convex and nonempty subset of a Banach space X. Let A and B be operators such that:*

- (i) *$Ax + By \in M$ whenever $x, y \in M$;*
- (ii) *A is compact and continuous;*
- (iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.1 ([3]). *Let $\alpha, \beta \geq 0$, then the following relation hold:*

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Lemma 2.2 ([3]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha\,c}D^\alpha f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}.$$

Lemma 2.3. *Let $x, y \in C([0,1], \mathbb{R})$ and $\Delta \neq 0$. Then the coupled system*

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x(t) = h_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y(t) = h_2(t), & t \in [0,1], \end{cases}$$

subject to the boundary conditions (1.2), *has a solution given by*

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1} h_1(s) ds - \frac{\displaystyle\int_0^t (t-s)^{\alpha_1-1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} \\ &+ B_1(t) \left[\frac{\displaystyle\int_0^1 (1-s)^{\alpha_1-1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} - \frac{\displaystyle\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2-1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)} \right. \\ &\left. + \frac{\displaystyle\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2+\beta_2-1} h_2(s) ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\displaystyle\int_0^1 (1-s)^{\alpha_1+\beta_1-1} h_1(s) ds}{\Gamma(\alpha_1+\beta_1)} \right] \end{aligned}$$

$$\begin{aligned}
&+B_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+B_3(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
&+B_4(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg],
\end{aligned}$$

$$\begin{aligned}
y(t)=&\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}h_2(s)ds-\frac{\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+C_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+C_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
&+C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg],
\end{aligned}$$

where

$$
\begin{aligned}
\Upsilon_1 =& \frac{(1-a_1)}{\Gamma(\alpha_1+2)}, \quad \Upsilon_2 = \frac{\sum\limits_{i=1}^{n} \gamma_i s_i^{\alpha_2}(b_1 - s_i)}{\Gamma(\alpha_2+2)}, \quad \Upsilon_4 = \frac{(1-b_1)}{\Gamma(\alpha_2+2)},\\
\Upsilon_3 =& \frac{\sum\limits_{j=1}^{m} \delta_j u_j^{\alpha_1}(a_1 - u_j)}{\Gamma(\alpha_1+2)}, \quad \Delta = \Upsilon_1\Upsilon_4 - \Upsilon_3\Upsilon_2, \quad U = -\frac{\Upsilon_4}{\Delta\Gamma(2+\alpha_1)},\\
V =& \frac{\Upsilon_2}{\Delta\Gamma(2+\alpha_1)}, \quad R = -\frac{\Upsilon_3}{\Delta\Gamma(2+\alpha_2)}, \quad T = -\frac{\Upsilon_1}{\Delta\Gamma(2+\alpha_2)},\\
B_1(t) =& Ut^{\alpha_1}(a_1 - t), \quad B_2(t) = Vt^{\alpha_1}(a_1 - t),\\
B_3(t) =& \frac{t^{\alpha_1}}{a_1^{\alpha_1}}\left[1 + (a_1 - t)\left(V\sum_{j=1}^{m} \delta_j u_j^{\alpha_1} - U\right)\right],\\
B_4(t) =& \frac{t^{\alpha_1}}{b_1^{\alpha_2}}\Big(a_1 - t\Big)\left(U\sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} - V\right),\\
C_1(t) =& Tt^{\alpha_2}(b_1 - t), \quad C_2(t) = Rt^{\alpha_2}(b_1 - t),\\
C_3(t) =& \frac{t^{\alpha_2}}{b_1^{\alpha_2}}\left[1 + (b_1 - t)\left(R\sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} - T\right)\right],\\
C_4(t) =& \frac{t^{\alpha_2}}{a_1^{\alpha_1}}\Big(b_1 - t\Big)\left(T\sum_{j=1}^{m} \delta_j u_j^{\alpha_1} - R\right).
\end{aligned}
$$

Proof. Using Lemma 2.2, we obtain $x(t) = I^{\alpha_1+\beta_1}h_1(t) + I^{\alpha_1}a_{01} + I^{\alpha_1}a_{11}t - I^{\alpha_1}\lambda_1(t)x(t) + a_{21}$, and $y(t) = I^{\alpha_2+\beta_2}h_2(t) + I^{\alpha_2}a_{02} + I^{\alpha_2}a_{12}t - I^{\alpha_2}\lambda_2(t)x(t) + a_{22}$, where $a_{01}, a_{11}, a_{21}, a_{02}, a_{12}, a_{22} \in \mathbb{R}$. According to the condition $x(0) = 0$, $y(0) = 0$, we get $a_{21} = a_{22} = 0$. Using the facts that $x(a_1) = y(b_1) = 0$, we obtain

$$
\begin{cases} a_{01} = \eta_1 + \theta_1 a_{11},\\ a_{02} = \eta_2 + \theta_2 a_{12}, \end{cases} \tag{2.1}
$$

where

$$
\begin{cases}
\eta_1 = \dfrac{\Gamma(\alpha_1+1)}{a_1^{\alpha_1}}\left(\dfrac{\int_0^{a_1}(a_1 - s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \dfrac{\int_0^{a_1}(a_1 - s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right),\\
\eta_2 = \dfrac{\Gamma(\alpha_2+1)}{b_1^{\alpha_2}}\left(\dfrac{\int_0^{b_1}(b_1 - s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \dfrac{\int_0^{b_1}(b_1 - s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\right),\\
\theta_1 = -\dfrac{a_1}{2+\alpha_1},\\
\theta_2 = \dfrac{b_1}{2+\alpha_2}.
\end{cases}
$$

By applying the conditions $x(1)=\sum_{i=1}^{n}\gamma_i y(s_i)$, $y(1)=\sum_{j=1}^{m}\delta_j x(u_j)$ and (2.1), we have

$$\begin{cases}\Upsilon_1 a_{11}+\Upsilon_2 a_{12}=\Lambda_1,\\ \Upsilon_3 a_{11}+\Upsilon_4 a_{12}=\Lambda_2,\end{cases}\tag{2.2}$$

where

$$\begin{aligned}
\Lambda_1 = & -\frac{1}{a_1^{\alpha_1}}\left(\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right)\\
& +\frac{\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{b_1^{\alpha_2}}\left(\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\right)\\
& +\frac{\int_0^{1}(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
& +\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^{1}(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)},\\
\Lambda_2 = & -\frac{1}{b_1^{\alpha_1}}\left(\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds\right)\\
& +\frac{\sum_{i=1}^{n}\delta_j u_j^{\alpha_1}}{a_1^{\alpha_1}}\left(\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right)\\
& +\frac{\int_0^{1}(1-s)^{\alpha_1-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
& +\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{1}(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds.
\end{aligned}$$

By solving the system (2.2), we get

$$\begin{aligned}
a_{11} &= \frac{1}{\Delta}\Big(\Lambda_1\Upsilon_4-\Lambda_2\Upsilon_2\Big),\\
a_{12} &= \frac{1}{\Delta}\Big(\Lambda_2\Upsilon_1-\Lambda_1\Upsilon_3\Big).
\end{aligned}$$

Substituting the values of a_{11} and a_{12} in (2.1), we get

$$\begin{aligned} a_{01} &= \eta_1 + \frac{\theta_1}{\Delta}\Big(\Lambda_1\Upsilon_4 - \Lambda_2\Upsilon_2\Big),\\ a_{02} &= \eta_2 + \frac{\theta_2}{\Delta}\Big(\Lambda_2\Upsilon_1 - \Lambda_1\Upsilon_3\Big). \end{aligned}$$

Substituting the value of a_{01}, a_{02}, a_{11} and a_{12}, we can deduce that

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}h_1(s)ds - \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\ &+ B_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\ &+ \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\ &+ B_2(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^m \delta_j \int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\ &+ \frac{\sum_{j=1}^m \delta_j \int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\ &+ B_3(t)\Bigg[\frac{\int_0^{a_1} (a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\ &+ B_4(t)\Bigg[\frac{\int_0^{b_1} (b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \end{aligned}$$

and

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}h_2(s)ds - \frac{\int_0^t (t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\ &+ C_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^m \delta_j \int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+C_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
&+C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg].
\end{aligned}
$$

By direct computation, it can easily be verified the converse of the lemma. ☐

3. Main Results

Let X be a Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with norm $\|x\|=\sup\{|x(t)|:t\in[0,1]\}$. Then, the product space $(X\times X,\|(x;y)\|)$ is also a Banach space equipped with the norm $\|(x;y)\|=\|x\|+\|y\|$. In view of Lemma 2.3, we define the operator $U:X\times X\to X\times X$ by $U(x,y)=(U_1(x,y),U_2(x,y))$. Here

$$
\begin{aligned}
&U_1(x,y)(t)\\
=&\frac{\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^t(t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+B_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\\
&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\Bigg]\\
&+B_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}\\
&-\left.\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]+B_3(t)\left[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\right.\\
&-\left.\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\right]+B_4(t)\\
&\times\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&U_2(x,y)(t)\\
=&\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds-\frac{1}{\Gamma(\alpha_2)}\\
&\times\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds+C_1(t)\left[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_1)}\right.\\
&\times(\sum_{j=1}^{m}\delta_j\int_0^{s_i}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds)+\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}\\
&\times f(s,x(s),y(s),\Phi y(s))ds-\left.\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]\\
&+C_2(t)\left[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\right.\\
&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds-\frac{1}{\Gamma(\alpha_1+\beta_1)}\\
&\times\left.\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\right]+C_3(t)\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\right.\\
&-\left.\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\right]+C_4(t)\left[\frac{1}{\Gamma(\alpha_1)}\right.
\end{aligned}
$$

$$\times\int_0^{a_1}(a_1-s)^{\alpha-1}\lambda_1(s)x(s)ds-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}\Bigg].$$

For computational convenience, we set

$$\begin{aligned}
r_{11}=&\max\Bigg\{\frac{\Big(1+B_1^*+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^*a_1^{\alpha_1+\beta_1}\Big)\sigma_1^*}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{(1+\delta_0)\sigma_2^*}{\Gamma(\alpha_2+\beta_2+1)}\Big(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\Big)\\
&+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\Big(B_1^*+1+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}+B_3^*a_1^{\alpha_1}\Big),\\
&\frac{1+B_1^*+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}(1+\lambda_0)\sigma_1^*+\frac{\sigma_2^*}{\Gamma(\alpha_2+\beta_2+1)}\\
&\times\Big(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\Big)+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\Big(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}+B_4^*b_1^{\alpha_2}\Big)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
r_{12}=&\max\Bigg\{\frac{C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\sigma_1^*+\frac{(1+\delta_0)\sigma_2^*}{\Gamma(\alpha_2+\beta_2+1)}\\
&\Big(1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}\Big)+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\Big(C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}+C_2^*+C_4^*a_1^{\alpha_1}\Big),\\
&\frac{C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\sigma_1^*(1+\lambda_0)\\
&+\frac{1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\sigma_2^*\\
&+\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\Big(1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}+C_3^*b_1^{\alpha_2}\Big)\Bigg\},
\end{aligned}$$

where $B_i^*=\sup\{B_i(t),\ t\in[0,1]\}$, $C_i^*=\sup\{C_i(t),\ t\in[0,1]\}$, $\lambda_j=\sup\{\lambda_j(t),t\in[0,1]\}$, $\sigma_j^*=\sup\{\sigma(t),t\in[0,1]\}$, for $i=1,2,3,4$ and $j=1,2$.

Before introducing the main results, we impose some assumptions.

(H_1) $f,g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions.

(H_2) There exist non negative functions $\sigma_1, \sigma_2 \in C\Big([0,1],[0,+\infty)\Big)$ such that for all $t \in [0,1]$ and $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, we have

$$\begin{aligned}|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \le& \sigma_1(t)\Big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\Big),\\ |g(t,x_1,y_1,z_1)-g(t,x_2,y_2,z_2)| \le& \sigma_2(t)\Big(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\Big),\end{aligned}$$

(H_3) $|f(t,x,y,z)| \le m_1(t)$, $|g(t,x,y,z)| \le m_2(t)$, for all $(t,x,y,z) \in [0,1]\times\mathbb{R}^3$, with $m_1, m_2 \in C([0,1];\mathbb{R}^+)$.

Theorem 3.1. *Let* $\Delta \neq 0$. *Suppose that* (H_1)-(H_2) *are satisfied. Then there exists a unique solution for System* (1.1)–(1.2) *provided that* $r_{11}+r_{12}<1$.

Proof. Define $\sup\limits_{0\le t\le 1}|f(t,0,0,0)| = A_1$, $\sup\limits_{0\le t\le 1}|g(t,0,0,0)| = A_2$.

Let $B_r = \{(x,y) \in X\times X : \|(x,y)\| \le r\}$, with $r \ge \frac{r_{21}+r_{22}}{1-(r_{11}+r_{12})}$, where

$$\begin{aligned} r_{21} =& \frac{B_1^* + B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}A_1 + \frac{B_2^* + B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}A_2,\\ r_{22} =& \frac{C_1^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}A_1 + \frac{1 + C_1^* + C_2^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}A_2.\end{aligned}$$

We prove that $TB_r \subseteq B_r$.

For $(x,y) \in B_r$, $t \in [0,1]$, we have:

$$\begin{aligned} |f(t,x(t),y(t),\Phi y(t))| \le& |f(t,x(t),y(t),\Phi y(t)) - f(t,0,0,0)| + |f(t,0,0,0)|\\ \le& \sigma_1(t)\Big(|x|+|y|+|\Phi y(t)|\Big) + A_1\\ \le& \sigma_1^*\Big(\|x\| + (1+\lambda_0)\|y\|\Big) + A_1,\\ |g(t,x(t),y(t),\Psi y(t))| \le& |g(t,x(t),y(t),\Psi y(t)) - g(t,0,0,0)| + |g(t,0,0,0)|\\ \le& \sigma_2(t)\Big(|x|+|y|+|\Psi y(t)|\Big) + A_2\\ \le& \sigma_2^*\Big(\|y\| + (1+\delta_0)\|x\|\Big) + A_2.\end{aligned}$$

Then,

$$\begin{aligned} &|U_1(x(t),y(t))|\\ \le& \Bigg[\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_1^*}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1+1}}{\Gamma(\alpha_1+\beta_1+1)}\\ &+ \frac{B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Bigg]\Big[\sigma_1^*\Big(\|x\| + (1+\lambda_0)\|y\| + A_1\Big] + \Bigg[\frac{B_2^*}{\Gamma(\alpha_2+\beta_2+1)}\end{aligned}$$

$$+\frac{B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}+\frac{B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\Bigg]\Big[\sigma_2^*\Big(\|y\|+(1+\delta_0)\|x\|+A_2\Big]$$

$$+\left(\frac{|\lambda_1|B_1^*}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{|\lambda_1|B_3^* a_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\|x\|$$

$$+\left(\frac{|\lambda_2|B_2^*}{\Gamma(\alpha_2+1)}+\frac{|\lambda_2|B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{|\lambda_2|B_4^* b_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\|y\|$$

$$\leq\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Big[\sigma_1^*\Big(\|x\|+(1+\lambda_0)\|y\|\Big)+A_1\Big]$$

$$+\frac{B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\times\Big[\sigma_2^*\Big(\|y\|+(1+\delta_0)\|x\|\Big)+A_2\Big]$$

$$+\frac{|\lambda_1|\left(B_1^*+1+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1}+B_3^* a_1^{\alpha_1}\right)}{\Gamma(\alpha_1+1)}\|x\|+\frac{|\lambda_2|\left(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}+B_4^* b_1^{\alpha_2}\right)}{\Gamma(\alpha_2+1)}\|y\|$$

$$\leq\Bigg[\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\sigma_1^*+\frac{B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}$$

$$\times(1+\delta_0)+\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\left(B_1^*+1+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1}+B_3^* a_1^{\alpha_1}\right)\Bigg]\|x\|$$

$$+\Bigg[\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}(1+\lambda_0)+\frac{\sigma_2^*}{\Gamma(\alpha_2+\beta_2+1)}$$

$$\times(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^* b_1^{\alpha_2+\beta_2})+\frac{|\lambda_2|\left(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}+B_4^* b_1^{\alpha_2}\right)}{\Gamma(\alpha_2+1)}\Bigg]\|y\|$$

$$+\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}A_1+\frac{B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}A_2.$$

Consequently, $\|U_1(x(t),y(t))\|\leq r_{11}r+r_{21}$.

In the same way, we obtain that $\|U_2(x(t),y(t))\|\leq r_{12}r+r_{22}$. Therefore, we have $\|U(x(t),y(t))\|=\|U_1(x,y)\|+\|U_2(x,y)\|\leq(r_{11}+r_{12})r+r_{21}+r_{22}\leq r$.

Now, for $(x_1,y_1),(x_2,y_2)\in X\times X$ and for $t\in[0,1]$, we get

$$
\begin{aligned}
&|U_1(x_1,y_1)(t)-U_1(x_2,y_2)(t)|\\
\leq&\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Big[\sigma_1^*\Big(\|x_1-x_2\|+(1+\lambda_0)\|y_1-y_2\|\Big)\Big]\\
&+\frac{B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\Big[\sigma_2^*\Big(\|y_1-y_2\|+(1+\delta_0)\|x_1-x_2\|\Big)\Big]\\
&+\frac{|\lambda_2|\left(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}+B_4^*b_1^{\alpha_2}\right)\|y_1-y_2\|}{\Gamma(\alpha_2+1)}\\
\leq&\left[\frac{1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\sigma_1^*+\frac{\left(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)}\right.\\
&\left.\times(1+\delta_0)+|\lambda_1|\frac{\left(B_1^*+1+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1}+B_3^*a_1^{\alpha_1}\right)}{\Gamma(\alpha_1+1)}\right]\|x_1-x_2\|+\left[\frac{(1+\lambda_0)}{\Gamma(\alpha_1+\beta_1+1)}\right.\\
&\times\left(1+B_1^*+B_2^*\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+B_3^*a_1^{\alpha_1+\beta_1}\right)+\frac{B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\sigma_2^*\\
&\left.+\frac{|\lambda_2|\left(B_2^*+B_1^*\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}+B_4^*b_1^{\alpha_2}\right)}{\Gamma(\alpha_2+1)}\right]\|y_1-y_2\|\\
\leq& r_{11}(\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}
$$

Analogously, we can also have $|U_2(x_1,y_1)(t)-U_2(x_2,y_2)(t)|\leq r_{12}(\|x_1-x_2\|+\|y_1-y_2\|)$, which leads to

$$\|U(x_1,y_1)-U(x_2,y_2)\|\leq(r_{11}+r_{12})(\|x_1-x_2\|+\|y_1-y_2\|).$$

As $r_{11}+r_{12}<1$, therefore the operator U is a contraction mapping. Then, we deduce that System (1.1)–(1.2) has a unique solution. □

Theorem 3.2. *Let $\Delta\neq0$. Assume that $(H_1),(H_3)$ hold. Then, System* (1.1)–(1.2) *has at least one solution on* $[0,1]$ *if $R<1$, where*

$$R=\max\left\{\frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\left(1+B_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}B_2^*+B_3^*a_1^{\alpha_1}+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}C_1^*+C_2^*+C_4^*a_1^{\alpha_1}\right),\right.$$

$$\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^*+B_4^*b_1^{\alpha_2}+1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^*+C_3^*b_1^{\alpha_2}\right)\Bigg\}.$$

Proof. We define a bounded closed and convex ball $B_{r'}=\{(x,y)\in X\times X:\|(x,y)\|\leq r'\}$ with $r'\geq\frac{r_2'}{1-R}$, where

$$\begin{aligned}r_2'=&\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Bigg(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}\\&+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}\Bigg)+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Bigg(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\\&+1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}\Bigg).\end{aligned}$$

Let us introduce the decomposition $U(x,y)(t)=W_1(x,y)(t)+W_2(x,y)(t)$, where $W_1(x,y)(t)=(T_1(x,y),R_1(x,y))(t)$, $W_2(x,y)(t)=(T_2(x,y),R_2(x,y))(t)$, with

$$\begin{aligned}T_1(x,y)(t)=&\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\\&+B_1(t)\left[\sum_{i=1}^{n}\gamma_i\frac{\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right.\\&\left.-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 f(s,x(s),y(s),\Phi y(s))(1-s)^{\alpha_1+\beta_1-1}ds\right]\\&+B_2(t)\left[\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}\right.\\&\left.-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]\\&-\frac{B_3(t)}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\\&-B_4(t)\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right],\end{aligned}$$

$$T_2(x,y)(t)=-\frac{\int_0^t(t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}+B_1(t)\left[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\right.$$

$$
\begin{aligned}
&-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\Bigg]+B_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\lambda_2(s)y(s)ds\\
&-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\Bigg]\\
&+B_3(t)\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+\frac{B_4(t)\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)},
\end{aligned}
$$

$$
\begin{aligned}
R_1(x,y)(t)=&\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\\
&+C_1(t)\Bigg[\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}\\
&-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\Bigg]\\
&+C_2(t)\Bigg[\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\\
&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\Bigg]\\
&-\frac{C_3(t)}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\\
&-\frac{C_4(t)}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
R_2(x,y)(t)=&-\frac{\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}+C_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&-\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ C_2(t)\Bigg[\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds\\
&- \frac{\lambda_2 \sum_{i=1}^{n}\gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}\Bigg]\\
&+ C_3(t)\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+ \frac{C_4(t)\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}.
\end{aligned}
$$

For $(x,y)\in B_{r'}$, we have

$$
\begin{aligned}
&|T_1(x,y)(t)+T_2(x,y)(t)|\\
\leq& \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_1^*\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{B_1^*\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\\
&+ \frac{B_2^*\|m_1\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_2^*\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} + \frac{B_3^* a_1^{\alpha_1+\beta_1}\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\\
&+ \frac{B_4^* b_1^{\alpha_2+\beta_2}\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\lambda_1|\cdot\|x\|}{\Gamma(\alpha_1+1)} + \frac{B_1^*|\lambda_1|\cdot\|x\|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_2|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}\|y\|B_1^*}{\Gamma(\alpha_2+1)}\\
&+ \frac{B_2^*|\lambda_2|\cdot\|y\|}{\Gamma(\alpha_2+1)} + \frac{|\lambda_1|\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}\|x\|B_2^*}{\Gamma(\alpha_1+1)} + \frac{B_3^*|\lambda_1|a_1^{\alpha_1}\|x\|}{\Gamma(\alpha_1+1)} + \frac{B_4^*|\lambda_2|b_1^{\alpha_2}\|y\|}{\Gamma(\alpha_2+1)}\\
\leq& \frac{|\lambda_1|\left(1+B_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}B_2^*+B_3^*a_1^{\alpha_1}\right)\|x\|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_2|\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^*+B_4^*b_1^{\alpha_2}\right)\|y\|}{\Gamma(\alpha_2+1)}\\
&+ \frac{\|m_1\|\left(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}\right)}{\Gamma(\alpha_1+\beta_1+1)}\\
&+ \frac{\|m_2\|\left(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

In a similar manner, we have

$$\begin{aligned}|R_1(x,y)(t)+R_2(x,y)(t)| \leq & \frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\left(\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}C_1^* + C_2^* + C_4^* a_1^{\alpha_1}\right)\|x\| \\ & + \frac{|\lambda_2|\left(1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^* + C_3^* b_1^{\alpha_2}\right)\|y\|}{\Gamma(\alpha_2+1)} \\ & + \frac{\|m_1\|\left(C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}\right)}{\Gamma(\alpha_1+\beta_1+1)} \\ & + \frac{\|m_2\|\left(1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)}.\end{aligned}$$

Further, we obtain

$$\|W_1(x_1,x_2)(t)+W_2(x_1,x_2)\| \leq Rr' + r_2' \leq r'.$$

Hence, $W_1(x_1,x_2)(t)+W_2(x_1,x_2)(t) \in B_{r'}$.

For $(x_1,y_1),(x_2,y_2) \in B_{r'}$ and $t \in [0,1]$, we have

$$\begin{aligned}|T_2(x_1,y_1)-T_2(x_2,y_2)| \leq & \frac{|\lambda_1|\left(1+B_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}B_2^* + B_3^* a_1^{\alpha_1}\right)\|x_1-x_2\|}{\Gamma(\alpha_1+1)} \\ & + \frac{|\lambda_2|\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^* + B_4^* b_1^{\alpha_2}\right)\|y_1-y_2\|}{\Gamma(\alpha_2+1)}, \\ |R_2(x_1,y_1)-R_2(x_2,y_2)| \leq & \frac{|\lambda_1|\left(\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}C_1^* + C_2^* + C_4^* a_1^{\alpha_1}\right)\|x_1-x_2\|}{\Gamma(\alpha_1+1)} \\ & + \frac{|\lambda_2|\left(1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^* + C_3^* b_1^{\alpha_2}\right)\|y_1-y_2\|}{\Gamma(\alpha_2+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|W_2(x_1,y_1)-W_2(x_2,y_2)\| \leq & R\|x_1-x_2\| + R\|y_1-y_2\| \\ \leq & R\|(x_1-x_2,y_1-y_2)\|.\end{aligned}$$

As $R<1$, then W_2 is a contraction.

Next, we prove that W_1 is compact and continuous. The continuity of f,g implies that the operator W_1 is continuous. Moreover, W_1 is uniformly bounded on $B_{r'}$.

Suppose that $0 \leq t_1 < t_2 \leq 1$. We have

$$\begin{aligned}
|T_1(x,y)(t_2) - T_1(x,y)(t_1)| \leq & \frac{\bigg| \int_0^{t_2} (t_2 - s)^{\alpha_1+\beta_1-1} f(s, x(s), y(s), \Phi y(s)) ds}{\Gamma(\alpha_1+\beta_1)} \\
& - \int_0^{t_1} (t_1 - s)^{\alpha_1+\beta_1-1} f(s, x(s), y(s), \Phi y(s)) ds \bigg| \\
& + |B_1(t_2) - B_1(t_1)| \\
& \times \left| \frac{\sum_{i=1}^{n} \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2+\beta_2-1} g(s, x(s), y(s), \Psi x(s)) ds}{\Gamma(\alpha_2+\beta_2)} \right. \\
& - \frac{\left. \int_0^{1} (1 - s)^{\alpha_1+\beta_1-1} f(s, x(s), y(s), \Phi y(s)) ds \right|}{\Gamma(\alpha_1+\beta_1)} \\
& + |B_2(t_2) - B_2(t_1)| \left| \sum_{j=1}^{m} \delta_j \right. \\
& \times \frac{\int_0^{u_j} (u_j - s)^{\alpha_1+\beta_1-1} f(s, x(s), y(s), \Phi y(s)) ds}{\Gamma(\alpha_1+\beta_1)} \\
& \left. - \frac{\int_0^{1} (1 - s)^{\alpha_2+\beta_2-1} g(s, x(s), y(s), \Psi x(s)) ds}{\Gamma(\alpha_2+\beta_2)} \right| \\
& + \frac{|B_3(t_2) - B_3(t_1)|}{\Gamma(\alpha_1+\beta_1)} \left| \int_0^{a_1} (a_1 - s)^{\alpha_1+\beta_1-1} \right. \\
& \left. \times f(s, x(s), y(s), \Phi y(s)) ds \right| + \frac{|B_4(t_2) - B_4(t_1)|}{\Gamma(\alpha_2+\beta_2)} \\
& \times \left| \int_0^{b_1} (b_1 - s)^{\alpha_2+\beta_2-1} g(s, x(s), y(s), \Psi x(s)) ds \right| \\
\leq & \frac{\|m_1\| (t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)} + |B_1(t_2) - B_1(t_1)| \\
& \times \left[\frac{\|m_2\| \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)} \right]
\end{aligned}$$

$$
\begin{aligned}
&+|B_2(t_2)-B_2(t_1)|\left[\frac{\|m_1\|\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\right.\\
&\left.+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\right]+\frac{|B_3(t_2)-B_3(t_1)|\cdot\|m_1\|a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{|B_4(t_2)-B_4(t_1)|\cdot\|m_2\|b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
|R_1(x,y)(t_2)-R_1(x,y)(t_1)|\leq&\frac{\|m_2\|(t_2^{\alpha_2+\beta_2}-t_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)}+|C_1(t_2)-C_1(t_1)|\\
&\times\left[\frac{\|m_1\|\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\right]\\
&+|C_2(t_2)-C_2(t_1)|\left[\frac{\|m_2\|\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\right.\\
&\left.+\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\right]+\frac{|C_3(t_2)-C_3(t_1)|\|m_2\|b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\\
&+\frac{|C_4(t_2)-C_4(t_1)|\|m_1\|a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}.
\end{aligned}
$$

Therefore, the operator W_1 is equicontinuous. Thus, W_1 is relatively compact on $B_{r'}$. Then by Arzela Ascoli theorem, the operator W_1 is compact on $B_{r'}$. In conclusion, all terms of Krasnoselskii's theorem have been applied perfectly. Hence, (1.1) and (1.2) has at least one solution on $B_{r'}$. ☐

4. Ulam-Hyers Stability

Definition 4.1. For some $\varepsilon_1,\ \varepsilon_2>0$, we consider the system of inequalities

$$
\left\{\begin{array}{ll}\left|{}^cD^{\beta_1}({}^cD^{\alpha_1}+\lambda_1(t))x^*(t)-f(t,x^*(t),y^*(t),\Phi y^*(t))\right|<\varepsilon_1, & t\in[0,1],\\ \left|{}^cD^{\beta_2}({}^cD^{\alpha_2}+\lambda_2(t))y^*(t)-g(t,x^*(t),y^*(t),\Psi x^*(t))\right|<\varepsilon_2, & t\in[0,1].\end{array}\right. \tag{4.1}
$$

Then System (1.1)–(1.2) is Ulam-Hyers stable if there exist $C_1,C_2>0$, such that there is a unique solution (x,y) of Problem (1.1)–(1.2), with

$$\|(x^*,y^*)-(x,y)\|\leq C_1\varepsilon_1+C_2\varepsilon_2.$$

Remark. (x^*, y^*) is a solution of system of inequalities (4.1) if we can find $\rho_1, \rho_2 \in \Big(C[0,1];\mathbb{R}\Big)$ such that $|\rho_1(t)| \leq \varepsilon_1$, $|\rho_2(t)| \leq \varepsilon_2$, $t \in [0,1]$ and

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x^*(t) = f(t, x^*(t), y^*(t), \Phi y^*(t)) + \rho_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y^*(t) = g(t, x^*(t), y^*(t), \Psi x^*(t)) + \rho_2(t), & t \in [0,1]. \end{cases} \tag{4.2}$$

Theorem 4.1. *If* (H_1), (H_2) *and* $r_{11} + r_{22} < 1$ *are satisfied, then Problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Proof. Let (x, y) be unique solution of System (1.1)-(1.2) and (x^*, y^*) be a solution of (4.1). Then we can find $\rho_1, \rho_2 \in \Big(C[0,1];\mathbb{R}\Big)$ such that

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x^*(t) = f(t, x^*(t), y^*(t), \Phi y^*(t)) + \rho_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y^*(t) = g(t, x^*(t), y^*(t), \Psi x^*(t)) + \rho_2(t), & t \in [0,1], \\ x(0) = 0, \quad x(a_1) = 0, \quad x(1) = \sum_{i=1}^{n} \gamma_i y(s_i), \\ y(0) = 0, \quad y(b_1) = 0, \quad y(1) = \sum_{j=1}^{m} \delta_i x(u_j), \\ 0 < a_1 < b_1 < s_1 < s_2 < \cdots < s_n < u_1 < u_2 < \cdots < u_m < 1. \end{cases} \tag{4.3}$$

By Lemma 2.3, we can obtain

$$\begin{aligned} x^*(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1}\Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds \\ &- \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} + B_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} \\ &- \frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} + \frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} \\ &\times \Big(g(s, x^*(s), y^*(s), \Psi x^*(s)) + \rho_2(s)\Big)ds - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \\ &\times \Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds\Bigg] + B_2(t)\Bigg[\frac{1}{\Gamma(\alpha_2)} \\ &\times \int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds - \frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} \\ &+ \frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}\Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds}{\Gamma(\alpha_1+\beta_1)} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\displaystyle\int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s))+\rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] + B_3(t)\\
&\times \Bigg[\frac{\displaystyle\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\Big(\rho_1(s)\\
&+ f(s,x^*(s),y^*(s),\Phi y^*(s))\Big)ds\Bigg] + B_4(t)\Bigg[\frac{\displaystyle\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}\\
&- \frac{\displaystyle\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s))+\rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg],\\
y^*(t) =&\frac{\displaystyle\int_0^{t}(t-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s))+\rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{1}{\Gamma(\alpha_2)}\\
&\times \int_0^t (t-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds + C_1(t)\Bigg[\frac{\displaystyle\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}\\
&- \frac{\displaystyle\sum_{j=1}^{m}\delta_j\int_0^{s_i}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_1+\beta_1)}\\
&\times \sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s))+\rho_1(s)\Big)\\
&- \frac{\displaystyle\int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s))+\rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] + C_2(t)\\
&\times \Bigg[\frac{\displaystyle\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\displaystyle\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}\\
&+ \frac{\displaystyle\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s))+\rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\\
&- \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s))+\rho_1(s)\Big)ds\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}\\
&\times \int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds\Bigg]\\
&+ C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{1}{\Gamma(\alpha_1+\beta_1)}\\
&\times \int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s)) + \rho_1(s)\Big)ds\Bigg].
\end{aligned}
$$

Using, $|\rho_1(t)| \leq \varepsilon_1$ and $|\rho_2(t)| \leq \varepsilon_2$, $t \in [0,1]$, we have

$$
\begin{aligned}
&\Bigg|x^*(t) - \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds\\
&- \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds + B_1(t)\Bigg[\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)}\\
&- \frac{1}{\Gamma(\alpha_2)}\sum_{i=1}^n \gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds + \frac{\sum_{i=1}^n \gamma_i}{\Gamma(\alpha_2+\beta_2)}\\
&\times \int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1} g(s,x^*(s),y^*(s),\Psi x^*(s))ds - \frac{1}{\Gamma(\alpha_1+\beta_1)}\\
&\times \int_0^1 (1-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] + B_2(t)\Bigg[\frac{1}{\Gamma(\alpha_2)}\\
&\times \int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds - \frac{\sum_{j=1}^m \delta_j \int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)}\\
&+ \frac{\sum_{j=1}^m \delta_j \int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds}{\Gamma(\alpha_1+\beta_1)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}\\
&\times \int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds\Bigg]\\
&+ B_3(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] + B_4(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&- \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg]\Bigg| \\
\le &\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Bigg(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}\Bigg) \\
&+ \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Bigg(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\Bigg), \\
&\Bigg|y^*(t)-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds-\frac{1}{\Gamma(\alpha_2)} \\
&\times\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds+C_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&- \frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^{m}\delta_j\int_0^{s_i}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds+\frac{1}{\Gamma(\alpha_1+\beta_1)} \\
&\times\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x^*(s),y^*(s),\Phi y^*(s)) \\
&- \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg]+C_2(t) \\
&\times\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&+ \frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds \\
&- \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] \\
&+ C_3(t)\Bigg[\frac{1}{\Gamma(\alpha_2)}\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds-\frac{1}{\Gamma(\alpha_2+\beta_2)} \\
&\times\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg]+C_4(t)\Bigg[\frac{1}{\Gamma(\alpha_1)} \\
&\times\int_0^{a_1}(a_1-s)^{\alpha-1}\lambda_1(s)x^*(s)ds-\frac{1}{\Gamma(\alpha_1+\beta_1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg]\Bigg|\\
\leq&\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\left(C_1^*\sum_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}\right)\\
&+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\left(1+C_1^*+C_2^*\sum_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}\right).
\end{aligned}
$$

By (H_2), we get

$$
\begin{aligned}
&|x^*(t)-x(t)|\\
\leq&\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\left(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}\right)\\
&+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\left(B_2^*+B_1^*\sum_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\right)\\
&+\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}\Big|f(s,x^*(s),y^*(s),\Phi y^*(s))\\
&-f(s,x(s),y(s),\Phi y(s))\Big|ds+\frac{|\lambda_1|}{\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}x^*(s)ds\\
&+|B_1(t)|\Bigg[\frac{|\lambda_1|}{\Gamma(\alpha_1)}\int_0^1(1-s)^{\alpha_1-1}|x^*(s)-x(s)|ds\\
&+\frac{|\lambda_2|\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}|y^*(s)-y(s)|ds}{\Gamma(\alpha_2)}+\frac{1}{\Gamma(\alpha_2+\beta_2)}\\
&\times\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}\Big|g(s,x^*(s),y^*(s),\Psi x^*(s))\\
&-g(s,x(s),y(s),\Psi x(s))\Big|ds\\
&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}\Big|f(s,x^*(s),y^*(s),\Phi y^*(s))\\
&-f(s,x(s),y(s),\Phi y(s))\Big|ds\Bigg]+|B_2(t)|\Bigg[\frac{|\lambda_2|\int_0^1(1-s)^{\alpha_2-1}|y^*(s)-y(s)|ds}{\Gamma(\alpha_2)}\\
&+\frac{|\lambda_1|}{\Gamma(\alpha_1)}\sum_{j=1}^m\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}|x^*(s)-x(s)|ds+\sum_{j=1}^m\delta_j\\
&\times\frac{\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}\Big|f(s,x^*(s),y^*(s),\Phi y^*(s))-f(s,x(s),y(s),\Phi y(s))\Big|ds}{\Gamma(\alpha_1+\beta_1)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big|g(s,x^*(s),y^*(s),\Psi x^*(s))\\
&-g(s,x(s),y(s),\Psi x(s))\Big|ds\Bigg]+|B_3(t)|\Bigg[\frac{|\lambda_1|\int_0^{a_1}(a_1-s)^{\alpha_1-1}|x^*(s)-x(s)|ds}{\Gamma(\alpha_1)}\\
&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}\Big|f(s,x^*(s),y^*(s),\Phi y^*(s))\\
&-f(s,x(s),y(s),\Phi y(s))\Big|ds\Bigg]+|B_4(t)|\\
&\Bigg[\frac{|\lambda_2|\int_0^{b_1}(b_1-s)^{\alpha_2-1}|y^*(s)-y(s)|ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}\\
&\times\Big|g(s,x^*(s),y^*(s),\Psi x^*(s))-g(s,x(s),y(s),\Psi x(s))\Big|ds\Bigg]\Bigg|\\
\le&\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\Bigg(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}\Bigg)\\
&+\frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\Bigg(B_2^*+B_1^*\sum_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\Bigg)\\
&+r_{11}\Bigg(\|x^*-x\|+\|y^*-y\|\Bigg).
\end{aligned}
$$

So, $(1-r_{11})\|x^*-x\|\le\Theta_1\varepsilon_1+\Theta_2\varepsilon_2+r_{11}\|y^*-y\|$, where

$$
\begin{aligned}
\Theta_1=&\frac{1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum\limits_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)},\\
\Theta_2=&\frac{B_2^*+B_1^*\sum\limits_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

In the same fashion, we have, $(1-r_{12})\|y^*-y\|\le\Theta_3\varepsilon_1+\Theta_4\varepsilon_2+r_{12}\|x^*-x\|$, where

$$
\begin{aligned}
\Theta_3=&\frac{C_1^*\sum\limits_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)},\\
\Theta_4=&\frac{1+C_1^*+C_2^*\sum\limits_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)},
\end{aligned}
$$

then, we get

$$\|x^*-x\| \leq \frac{\Theta_1(1-r_{12})+r_{11}\Theta_3}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\varepsilon_1+\frac{\Theta_2(1-r_{12})+r_{11}\Theta_4}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\varepsilon_2$$

and

$$\|y^*-y\| \leq \frac{\Theta_3(1-r_{11})+r_{12}\Theta_1}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\varepsilon_1+\frac{\Theta_4(1-r_{11})+r_{12}\Theta_2}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\varepsilon_2,$$

which implies that

$$\begin{aligned}\|x^*-x\|+\|y^*-y\| \leq{}& \frac{\Theta_1(1-r_{12})+r_{11}\Theta_3}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\varepsilon_1+\frac{\Big(\Theta_2(1-r_{12})+r_{11}\Theta_4\Big)\varepsilon_2}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}\\ &+\frac{\Big(\Theta_3(1-r_{11})+r_{12}\Theta_1\Big)\varepsilon_1}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}+\frac{\Big(\Theta_4(1-r_{11})+r_{12}\Theta_2\Big)\varepsilon_2}{(1-r_{11})(1-r_{12})-r_{11}r_{12}}.\end{aligned}$$

Hence, System (1.1)–(1.2) is Ulam-Hyers stable. □

5. Examples

Example 5.1. Consider the following system of fractional integro-differential Langevin equations:

(5.1)

$$\begin{cases} {}^cD^{\frac{12}{7}}\left({}^cD^{\frac{6}{7}}+\dfrac{t}{10^4}\right)x(t)=\dfrac{t^2}{3\times 10^4}\left(\dfrac{x(t)+y(t)}{4}+\dfrac{\int_0^t t^4s^3y(s)ds}{10^3}\right), \quad t\in[0,1],\\ {}^cD^{\frac{13}{8}}\left({}^cD^{\frac{7}{8}}+\dfrac{t}{10^4}\right)y(t)=\dfrac{\left(\sin(x(t))+\cos(y(t))+\dfrac{\int_0^t t^5s^4x(s)ds}{10^3}\right)}{4\times 10^4+t^2}, \quad t\in[0,1],\\ x(0)=0, \quad x\left(\frac{1}{1000}\right)=0, \quad x(1)=\dfrac{1}{3000}\Big(y(\tfrac{1}{50})+y(\tfrac{1}{40})+y(\tfrac{1}{30})\Big),\\ y(0)=0, \quad y\left(\frac{1}{100}\right)=0, \quad y(1)=\dfrac{1}{4000}\Big(x(\tfrac{1}{25})+x(\tfrac{1}{12})+x(\tfrac{1}{6})\Big), \end{cases}$$

where $\beta_1=\frac{12}{7}$, $\alpha_1=\frac{6}{7}$, $\beta_2=\frac{13}{8}$, $\alpha_2=\frac{7}{8}$, $\lambda_1=\lambda_2=\frac{1}{10000}$ and

$$\begin{aligned} f(t,x,y,z)=&\frac{t^2}{30000}\Big(\frac{x(t)+y(t)}{4}+z(t)\Big),\\ g(t,x,y,z)=&\frac{1}{40000+t^2}\Big(\sin(x(t))+\cos(y(t))+\frac{z(t)}{2}\Big),\\ \Phi y(t)=&\frac{1}{250}\int_0^t t^4s^3y(s)ds), \quad \Psi x(t)=\frac{1}{1000}\int_0^t t^5s^4x(s)ds, \end{aligned}$$

$a_1=\frac{1}{1000}$, $b_1=\frac{1}{100}$, $\gamma_1=\gamma_2=\gamma_3=\frac{1}{3000}$, $s_1=\frac{1}{50}$, $s_2=\frac{1}{40}$, $s_3=\frac{1}{30}$, $\delta_1=\delta_2=\delta_3=\dfrac{1}{4000}$, $u_1=\frac{1}{25}$, $u_2=\frac{1}{12}$, $u_3=\frac{1}{6}$. Clearly, $\delta_0=\dfrac{1}{5000}$, $\lambda_0=\dfrac{1}{1000}$ and $\sigma_1^*=\dfrac{1}{120000}$,

$\sigma_2^* = \dfrac{1}{40000}$. Furthermore, we have

$$r_{11} + r_{12} \approx 0.175 < 1.$$

Thus, by Theorem 3.1, System (5.1) has a unique solution.

Example 5.2. Consider the following problem:

$$(5.2)\quad \begin{cases} {}^cD^{\frac{14}{8}}\Big({}^cD^{\frac{6}{8}} + \dfrac{t}{2\times 10^4}\Big)x(t) = \dfrac{t\Big(\dfrac{x(t)+y(t)}{2} + \dfrac{1}{10^3}\displaystyle\int_0^t t^4 s^3 y(s)ds\Big)}{6\times 10^4}, & t\in[0,1],\\ {}^cD^{\frac{13}{7}}\Big({}^cD^{\frac{6}{7}} + \dfrac{t}{2\times 10^4}\Big)y(t) = \dfrac{t^2\Big(x(t)+y(t) + \dfrac{1}{10^3}\displaystyle\int_0^t t^5 s^4 x(s)ds\Big)}{4\times 10^4} & t\in[0,1],\\ x(0)=0,\quad x\big(\tfrac{1}{500}\big)=0,\quad x(1) = \dfrac{1}{6000}\big(y(\tfrac{1}{90}) + y(\tfrac{1}{70}) + y(\tfrac{1}{60})\big), \\ y(0)=0,\quad y\big(\tfrac{1}{300}\big)=0,\quad y(1) = \dfrac{1}{5000}\big(x(\tfrac{1}{50}) + x(\tfrac{1}{40}) + x(\tfrac{1}{10})\big), \end{cases}$$

where $\beta_1 = \frac{14}{8}$, $\alpha_1 = \frac{6}{8}$, $\beta_2 = \frac{13}{7}$, $\alpha_2 = \frac{6}{7}$, $\lambda_1 = \lambda_2 = \frac{1}{20000}$ and

$$f(t,x,y,z) = \frac{t}{60000}\Big(\frac{x(t)+y(t)}{2} + z(t)\Big),\quad g(t,x,y,z) = \frac{t^2}{40000}\Big(x(t)+y(t)+\frac{z(t)}{2}\Big),$$

$$\Phi y(t) = \frac{1}{250}\int_0^t t^4 s^3 y(s)ds,\quad \Psi x(t) = \frac{1}{1000}\int_0^t t^5 s^4 x(s)ds,$$

$a_1 = \frac{1}{500}$, $b_1 = \frac{1}{300}$, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{6000}$, $s_1 = \frac{1}{90}$, $s_2 = \frac{1}{70}$, $s_3 = \frac{1}{60}$, $\delta_1 = \delta_2 = \delta_3 = \frac{1}{5000}$, $u_1 = \frac{1}{50}$, $u_2 = \frac{1}{40}$, $u_3 = \frac{1}{10}$.

Clearly, $\delta_0 = \frac{1}{5000}$, $\lambda_0 = \frac{1}{2000}$ and $\sigma_1^* = \frac{1}{120000}$, $\sigma_2^* = \frac{1}{40000}$.

After calculating, we obtain $R \approx 0.0526 < 1$. So, by Theorem 3.2, Problem (5.2) has a least one solution.

6. CONCLUSION

In this paper, we suggested a new coupled fractional Langevin equation. More precisely, we have improved the existence and uniqueness results for a coupled system of nonlinear fractional Langevin equations via variable coefficient supplemented with multipoint boundary conditions by the application of the Banach contraction principle and Krasnoselskii's fixed point theorem. Further, we have established Ulam stability to the solution of mentioned system. Finally, we have presented two examples to demonstrate our results.

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[1] Laboratory of PDEs, Algebra and Spectral Geometry, Department of Mathematics, Faculty of Sciences,
ENSC,
Ibn Tofail University,
Kenitra, Morocco
Email address: `ibnelazyzlahcen@gmail.com`

[2]Department of Mathematics, Laboratory of Mathematics and Applications,
Abdelmalek Essaadi University,
Tangier, Morocco
Email address: `dr.aittouchent@gmail.com`

[3]Sultan Moulay Slimane University,
BP 523, 23000 Beni Mellal, Morocco
Email address: `guida.karim@gmail.com`

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ANALYSIS OF A WEAK GALERKIN MIXED FORMULATION FOR MAXWELL'S EQUATIONS

ABDELHAMID ZAGHDANI[1,2], ABDELHALIM HASNAOUI[1], AND SAYED SAYARI[3]

Abstract. In this paper we introduce and analyse a mixed weak Galekin finite element method for the Maxwell equations in the primary electric field-Lagrange multiplier. Our weak Galerkin method is equipped with stable finite elements composed of habitual polynomials of degree k for the electric field and polynomials of degree $k+1$ for the Lagrange multiplier. Optimal order error estimations for the proposed weak Galerkin mixed finite element formulation are demonstrated and are confirmed numerically on a two dimensional bounded domain.

1. Introduction

The idea of the weak Galerkin finite element method introduced by [13] consists in the approximation of the differential operators in the partial differential equation by weak forms as distributions over the space of discontinuous functions including boundary information. Compared to the discontinuous Galerkin methods [11, 16–19], the weak Galerkin methods also use discontinuous functions in the finite element procedure which gives a great flexibility to the WG-FEM in dealing with boundary conditions and different geometric complexities, while weak Galerkin methods require only weak continuity of variables through well-defined discrete differential operators and are absolutely stable when correctly constructed. Ever since it was introduced, the WG-method was used by several authors for the resolution of various partial differential equations such as linear parabolic problems [3, 20, 21], Helmoltz equations with large wave numbers [12] and elliptic interface problems [7, 8]. Recently, Lin

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Mu and his collaborators [9] construct a new WG-FEM which discretizes the second order elliptic equation in non-mixed form directly, and admit general finite element partitions consisting of arbitrary polytopal elements. The Weak Galerkin mixed finite element method is an extension of weak Galerkin finite element method [14] and it was used for the numerical resolution of partial differential equations [4, 6, 10, 15], In [14], a mixed WG-FEM has been introduced and analysed for the second order elliptic equation, in which the utilization of stabilization for the flux variable has an important role in the mixed formulation. In this paper, we are interested with the development of weak Galerkin mixed formulation for the Maxwell problem which consists in finding two unknown functions u and p such that

$$\nabla\times(\mu^{-1}\nabla\times u)-\varepsilon\nabla p = J,\quad \text{in } \Omega\subset\mathbb{R}^d, \tag{1.1}$$

$$\nabla\cdot(\varepsilon u) = 0,\quad \text{in } \Omega\subset\mathbb{R}^d, \tag{1.2}$$

$$n\times u = 0,\quad \text{on } \partial\Omega, \tag{1.3}$$

$$p = 0,\quad \text{on } \partial\Omega. \tag{1.4}$$

Here Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ or a bounded polyhedral domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. μ and ε denote the magnetic permeability and the electric permittivity of the medium and are assumed sufficiently smooth and in $L^\infty(\Omega)$. p is a Lagrange multiplier and u is related to the electric field E by the relation $E(x,t) = \mathrm{Re}(u(x)\exp(i\omega t))$ with a given non zero frequency ω. Our goal in this paper is to introduce and study a mixed weak Galerkin finite element method for (1.1)–(1.4) that is potent and sturdy by allowing the use of discontinuous functions on finite element partitions consisting of arbitrary elements with certain shape regularity.

The organization of this paper is as follows. In the next section, we recall some notations in Sobolev spaces and we describe in detail our mixed weak Galerkin discrete scheme. In Section 3, we study the properties of the bilinear forms given in the formulation while in Section 4, we analyse the convergence of the proposed numerical formulation and prove some optimal error estimations. Section 5 is done for studying some numerical examples for confirming the proven theoretical results.

2. Preliminaries and Notations

2.1. **Meshes.** In this work, we use the standard notations for Sobolev spaces and their norms [5], such as, $H^s(\mathcal{O})^d$, $d = 1, 2, 3$, $\|\cdot\|_{s,\mathcal{O}} = \|\cdot\|_{H^s(\mathcal{O})^d}$ for a domain $\mathcal{O}$ and a positive integer or fractional regularity exponent s. The space $H_0(\nabla\times,\Omega)$ is the space of vector-valued functions $u\in L^2(\Omega)^d$ such that $\nabla\times u\in L^2(\Omega)^d$ and $n\times u = 0$ on the boundary of Ω. The space $H(\nabla_\varepsilon\cdot,\Omega)$ is the space of vector-valued functions in $L^2(\Omega)^d$ where $\nabla\cdot\varepsilon u\in L^2(\Omega)$.

Consider $\mathcal{T}_h$ be a shape-regular partition of Ω which consists of tetrahedra in $\mathbb{R}^3$ or triangles in $\mathbb{R}^2$. We denote by $\mathcal{E}_h^I$ the set of all interior faces or edges, $\mathcal{E}_h^D$ the set of all exterior faces or edges of the triangulation and we set $\mathcal{E}_h = \mathcal{E}_h^I\cup\mathcal{E}_h^D$. For any T in $\mathcal{T}_h$, we denote by h_T its diameter and $h = \max_{T\in\mathcal{T}_h} h_T$ the mesh size of the partition

$\mathcal{T}_h$. For $d = 1, 2, 3$, we introduce the piecewise Sobolev spaces as

$$H^s(\mathcal{T}_h)^d := \left\{ v \in L^2(\Omega)^d : v_{|K} \in H^s(K)^d \text{ for all } K \in \mathcal{T}_h \right\}.$$

2.2. **Weak Galerkin formulation.** First, we introduce the following two finite element spaces $\mathcal{V}_h$ and $\mathcal{W}_h$ for $k \geq 0$:

$$\mathcal{V}_h := \left\{ v = \{v^0, v^b\} : v^0 \in [P_k(T)]^d,\ v^b \in [P_k(e)]^d,\ e \in \mathcal{E}_h,\ T \in \mathcal{T}_h \right\}$$

and

$$\mathcal{W}_h := \left\{ \psi \in L^2(\Omega) : \psi_{|T} \in P_{k+1}(T),\ \ T \in \mathcal{T}_h \right\}.$$

Next, introduce the subspace of $\mathcal{V}_h$ as

$$\mathcal{V}_h^0 := \left\{ v \in \mathcal{V}_h : v^b \times n = 0 \text{ on } \partial\Omega \right\}$$

and the subspace of $\mathcal{W}_h$ as

$$\mathcal{W}_h^0 := \left\{ \psi \in \mathcal{W}_h : \psi = 0 \text{ on } \partial\Omega \right\}.$$

2.2.1. *Weak differential operators.* Let T be any element in $\mathcal{T}_h$ and v any function in $\mathcal{V}_h$, a weak divergence $\nabla_w \cdot v \in P_k(T)$ is defined as the unique polynomial satisfying

$$(\nabla_w \cdot v, \psi)_T = -(v^0, \nabla\psi)_T + \langle v^b \cdot n, \psi \rangle_{\partial T}, \quad \text{for all } \psi \in P_k(T) \tag{2.1}$$

and a weak curl $\nabla_w \times v \in [P_k(T)]^d$ is defined as the only polynomial satisfying

$$(\nabla_w \times v, w)_T = (v^0, \nabla \times w)_T - \langle v^b \times n, w \rangle_{\partial T}, \quad \text{for all } w \in [P_k(T)]^d. \tag{2.2}$$

With these two definitions, one can naively formulate a finite element discretization of the problem (1.1)-(1.4) as: Find $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$ such that

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} (\mu^{-1}\nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T + \sum_{T\in\mathcal{T}_h} (p_h, \nabla_w \cdot \varepsilon v_h)_T \\ &= \sum_{T\in\mathcal{T}_h} (J, v_h^0)_T, \quad \text{for all } v_h \in \mathcal{V}_h^0, \\ &\sum_{T\in\mathcal{T}_h} (\nabla_w \cdot \varepsilon u_h, \psi_h)_T = 0, \quad \text{for all } \psi_h \in \mathcal{W}_h^0. \end{aligned}$$

This system does not have only one solution due to an insufficient enforcement of the components u_h^0 and u_h^b and we stabilize the bilinear form

$$\sum_{T\in\mathcal{T}_h} (\mu^{-1}\nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T,$$

by requiring some communications between u_h^0 and u_h^b. Hence, for $(u_h, v_h) \in \mathcal{V}_h^0 \times \mathcal{V}_h^0$ and $(v_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$, we define the following bilinear forms

$$\begin{aligned} a(u_h, v_h) &:= \sum_{T \in \mathcal{T}_h} (\mu^{-1} \nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T, \\ B(v_h, p_h) &:= \sum_{T \in \mathcal{T}_h} (p_h, \nabla_w \cdot \varepsilon v_h)_T, \\ s_T(u_h, v_h) &:= r \sum_{\partial T \in \mathcal{E}_h^I} h_T^{-1} \langle (\varepsilon u_h^0 - \varepsilon u_h^b) \cdot n, (\varepsilon v_h^0 - \varepsilon v_h^b) \cdot n \rangle_{\partial T} \\ &\quad + r \sum_{\partial T \in \mathcal{E}_h} h_T^{-1} \langle (u_h^0 - u_h^b) \times n, (v_h^0 - v_h^b) \times n \rangle_{\partial T}, \end{aligned}$$

where r is an arbitrary real parameter and assumed greater than zero. Next, for an approximate solution of (1.1)–(1.4), we find $u_h = \{u_h^0, u_h^b\} \in \mathcal{V}_h^0$, $p_h \in \mathcal{W}_h^0$ satisfying

$$A_s(u_h, v_h) + B(v_h, p_h) = \sum_{T \in \mathcal{T}_h} (J, v_h^0), \quad \text{for all } v_h \in \mathcal{V}_h^0, \tag{2.3}$$

$$B(u_h, \psi_h) = 0, \quad \text{for all } \psi_h \in \mathcal{W}_h^0, \tag{2.4}$$

where we have denoted by

$$A_s(u_h, v_h) := a(u_h, v_h) + s_T(u_h, v_h), \quad \text{for } (u_h, v_h) \in \mathcal{V}_h^0 \times \mathcal{V}_h^0.$$

Since our numerical scheme was given in (2.3)–(2.4), we first analyse its well posedness in the following theorem.

Theorem 2.1. *The mixed weak Galerkin scheme* (2.3)–(2.4) *is well posed and it has a unique solution* $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$.

Proof. Take $J = 0$ in (2.3)–(2.4), then we have to prove that $u_h = 0$ and $p_h = 0$. Substituting $v = u_h$ and $\psi = p_h$ in (2.3)–(2.4) and substracting the second equation from the first and obtain $A_s(u_h, u_h) = 0$. It follows from the definition of $A_s(\cdot, \cdot)$ that $\nabla_w \times u_h = \nabla_w \cdot \varepsilon u_h = 0$ on each element $T \in \mathcal{T}_h$ and $u^0 \times n = u^b \times n$, $\varepsilon u^0 \cdot n = \varepsilon u^b \cdot n$ on each edge $e \in \mathcal{E}_h$. Therefore, from the definition of the weak curl operator and $\nabla_w \times u_h = 0$, one can obtain for any $w \in P_k(T)^d$,

$$\begin{aligned} 0 = (\nabla_w \times u, w)_T &= (u^0, \nabla \times w)_T - \langle u^b \times n, w \rangle_{\partial T} \\ &= (\nabla \times u^0, w)_T - \langle (u^0 - u^b) \times n, w \rangle_{\partial T} \\ &= (\nabla \times u^0, w)_T, \end{aligned}$$

which gives $\nabla \times u^0 = 0$ on each $T \in \mathcal{T}_h$. From the fact that $u^0 \times n = u^b \times n$ on each edge $e \in \mathcal{E}_h$ and $v^b \times n = 0$ on $\partial\Omega$ we deduce that $u^0 \in H_0(\nabla \times, \Omega)$ with $\nabla \times u^0 = 0$ in Ω. Similarly, since $\nabla_w \cdot \varepsilon u = 0$ on each $T \in \mathcal{T}_h$ and $\varepsilon u^0 \cdot n = \varepsilon u^b \cdot n$ on each edge $e \in \mathcal{E}_h$ we conclude that $u^0 \in H(\nabla_\varepsilon \cdot, \Omega)$ with $\nabla \cdot \varepsilon u^0 = 0$ and it follows that $u^0 = 0$ in Ω. Then, $u^b \times n = \varepsilon u^b \cdot n = 0$ and therefore $u^b = 0$ in $\mathcal{T}_h$. Next, using the definition of the bilinear form B, the weak divergence operator and the first equation in (2.3)–(2.4) we deduce also that $p_h = 0$ and this end the proof. □

3. Error Estimations

Let us start by introducing the local projection operators. Define $\mathbb{Q}_0$ the projection from $(L^2(T))^d$ to $(P_k(T))^d$, $\mathbb{Q}_b$ the projection from $(L^2(e))^d$ to $(P_k(e))^d$ on each elements $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, respectively. We denote by $\mathbb{Q}_h$ the L^2-projection of $v = \{v^0, v^b\} \in \mathcal{V}_h^0$ defined as $\mathbb{Q}_h := \{\mathbb{Q}_0(v^0), \mathbb{Q}_b(v^b)\}$ and for $p \in \mathcal{W}_h^0$, we denote by $Q_h(p)$ the local projection from $L^2(T)$ onto $P_{k+1}(T)$. In the following lemma, we introduce and prove some essential equations and results which we need for proving some error equations that are essential for the study of error estimations.

Lemma 3.1. *Let* (u, p) *be the solution of* (1.1)–(1.4)*, then*

$$\begin{aligned}
\nabla_w \cdot (\mathbb{Q}_h(u))_T =& Q_h(\nabla \cdot u)_T,\\
\nabla_w \times (\mathbb{Q}_h(u))_T =& \mathbf{Q}_h(\nabla \times u)_T,\\
(\nabla_w \times v, \mathbb{Q}_h(\varphi))_T =& (v^0, \nabla \times \varphi)_T + \langle (v^b - v^0) \times n, \varphi - \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \varphi\rangle_{\partial T}.
\end{aligned}$$

Proof. From the definition of weak-divergence (2.1), $\mathbb{Q}_h$, Q_h, we have for $\psi \in P_k(T)$

$$\begin{aligned}
(\nabla_w \cdot \mathbb{Q}_h(u), \psi)_T =& -(\mathbb{Q}_0(u), \nabla\psi)_T + \langle \mathbb{Q}_b(u) \cdot n, \psi\rangle_{\partial T}\\
=& -(u, \nabla\psi)_T + \langle u \cdot n, \psi\rangle_{\partial T}\\
=& (\nabla \cdot u, \psi)_T - \langle u \cdot n, \psi\rangle_{\partial T} + \langle u \cdot n, \psi\rangle_{\partial T}\\
=& (\nabla \cdot u, \psi)_T = (Q_h(\nabla \cdot u), \psi)_T,
\end{aligned}$$

which means the first equation in the lemma and similarly we can prove the second equation. For the proof of the third assertion of the lemma, fix $v \in \mathcal{V}_h$ and φ sufficiently regular function, then from the definition of the weak curl operator (2.2) one can have

$$\begin{aligned}
(\nabla_w \times v, \mathbb{Q}_h(\varphi))_T =& (v^0, \nabla \times \mathbb{Q}_h(\varphi))_T - \langle v^b \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \mathbb{Q}_h(\varphi))_T + \langle v^0 \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \mathbb{Q}_h(\varphi))_T + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \varphi)_T + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (v^0, \nabla \times \varphi)_T - \langle v^0 \times n, \varphi\rangle_{\partial T} + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (v^0, \nabla \times \varphi)_T + \langle (v^b - v^0) \times n, \varphi - \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \varphi\rangle_{\partial T}. \square
\end{aligned}$$

In the following section, we derive some error equations which we need to establish optimal error estimates for the weak Galerkin mixed finite element scheme (2.3)–(2.4).

3.1. Error equations.

Let (u, p) be a sufficiently smooth solution of (1.1)–(1.4) and for the sake of simplicity, assume that the coefficients μ, ε are constants and to be equal to the identity. The use of Lemma 3.1, the definition of weak curl operator (2.2) and the usual integration by parts, implies

$$\begin{aligned}(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_T =&(\mathbf{Q}_h(\nabla \times u), \nabla_w \times v)_T\\ =&(v^0, \nabla \times \mathbf{Q}_h(\nabla \times u))_T - \langle v^b \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}\\ =&(\nabla \times v^0, \mathbf{Q}_h(\nabla \times u))_T + \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}.\end{aligned}$$

Therefore,

$$(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_T =(\nabla \times v^0, \nabla \times u)_T + \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}. \tag{3.1}$$

Also, the use of the definition of weak divergence operator (2.1), the usual integration by parts and the fact that $\sum_{T\in\mathfrak{T}_h} \langle v^b \cdot n, p\rangle_{\partial T} = 0$, gives

$$\begin{aligned}(\nabla_w \cdot v, Q_h(p))_\Omega =& - \sum_{T\in\mathfrak{T}_h} (v^0, \nabla(Q_h(p)))_T + \langle v^b \cdot n, Q_h(p)\rangle_{\partial T}\\ =& \sum_{T\in\mathfrak{T}_h} (\nabla \cdot v^0, Q_h(p))_T - \langle (v^0 - v^b) \cdot n, Q_h(p)\rangle_{\partial T}\\ =& \sum_{T\in\mathfrak{T}_h} (\nabla \cdot v^0, p)_T - \langle (v^0 - v^b) \cdot n, Q_h(p)\rangle_{\partial T}\\ =& - \sum_{T\in\mathfrak{T}_h} (v^0, \nabla p)_T + \langle v^0 \cdot n, p\rangle_{\partial T} - \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, Q_h(p)\rangle_{\partial T}\\ =& - \sum_{T\in\mathfrak{T}_h} (v^0, \nabla p)_T + \langle (v^0 - v^b) \cdot n, p\rangle_{\partial T} - \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, Q_h(p)\rangle_{\partial T}\\ =& - (v^0, \nabla p)_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T},\end{aligned}$$

which implies that

$$(v^0, \nabla p)_\Omega = -(\nabla_w \cdot v, Q_h(p))_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T}. \tag{3.2}$$

Now, testing the first equation in (1.1)–(1.4) by using v^0 in $v = \{v^0, v^b\} \in V_h^0$ and get

$$(\nabla \times \nabla \times u, v^0)_\Omega - (\nabla p, v^0)_\Omega = (J, v^0)_\Omega. \tag{3.3}$$

After an integration by parts and using the fact that $\sum_{T\in\mathfrak{T}_h} \langle v^b \times n, (\nabla \times u)\rangle_{\partial T} = 0$, one can arrive to

$$(\nabla \times \nabla \times u, v^0)_\Omega = \sum_{T\in\mathfrak{T}_h} (\nabla \times u, \nabla \times v^0)_T + \langle (v^0 - v^b) \times n, (\nabla \times u)\rangle_{\partial T}.$$

The use of this last equation together with (3.1) implies that

$$\begin{aligned}(\nabla\times\nabla\times u, v^0)_\Omega =&(\nabla_w\times(\mathbb{Q}_h(u)), \nabla_w\times v)_\Omega - \sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\times n, \mathbf{Q}_h(\nabla\times u)\rangle_{\partial T}\\ &+\sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\times n, (\nabla\times u)\rangle_{\partial T}\\ =&(\nabla_w\times(\mathbb{Q}_h(u)), \nabla_w\times v)_\Omega\\ &-\sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\times n, \mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}.\end{aligned}$$

Substituting the previous equation and (3.2) into (3.3) and get

$$\begin{aligned}(\nabla_w\times(\mathbb{Q}_h(u))_\Omega,&\nabla_w\times v)_\Omega + (\nabla_w\cdot v, Q_h(p))_\Omega\\ =&(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\cdot n, p-Q_h(p)\rangle_{\partial T}\\ &+\sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\times n, \mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}.\end{aligned}$$

As to the second equation in (1.1)–(1.4), we test it by a function $\nabla_w\cdot v$ and write

$$0=(\nabla\cdot u, \nabla\cdot v)_\Omega =(Q_h(\nabla\cdot u), \nabla_w\cdot v)_\Omega = (\nabla_w\cdot(\mathbb{Q}_h(u)), \nabla_w\cdot v)_\Omega.$$

The addition of these two last equations gives

$$\begin{aligned}(\nabla_w\times(\mathbb{Q}_h(u)),&\nabla_w\times v)_\Omega + (\nabla_w\cdot(\mathbb{Q}_h(u)), \nabla_w\cdot v)_\Omega + (\nabla_w\cdot v, Q_h(p))_\Omega\\ =&(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\cdot n, p-Q_h(p)\rangle_{\partial T}\\ &+\sum_{T\in\mathfrak{T}_h}\langle (v^0-v^b)\times n, \mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}.\end{aligned}\tag{3.4}$$

Now, it is the moment to introduce and prove the error equations, we have the following.

Lemma 3.2. *Let* $e_h := u_h - \mathbb{Q}_h(u)$ *and* $\epsilon_h := p_h - Q_h(p)$ *be the errors, then*

$$\begin{aligned}A_s(e_h, v) + B(v, \epsilon_h) =& \sum_{T\in\mathfrak{T}_h}\langle (v^b-v^0)\times n, \mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}\\ &+\sum_{T\in\mathfrak{T}_h}\langle (v^b-v^0)\cdot n, p-Q_h(p)\rangle_{\partial T} - s_T(\mathbb{Q}_h(u), v),\\ B(e_h,\psi) =&0.\end{aligned}$$

Proof. By adding $s_T(\mathbb{Q}_h(u), v)$ to the two sides of (3.4), one can obtain

$$\begin{aligned} &A_s(\mathbb{Q}_h(u),v) + B(v, Q_h(p)) \\ &\quad =(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T} \\ &\qquad + s_T(\mathbb{Q}_h(u), v) + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T}. \end{aligned}$$

Substract this equation from the first equation in (2.3)–(2.4), one can get

$$\begin{aligned} &A_s(e_h, v) + B(v, \epsilon_h) \\ &= \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T} - s_T(\mathbb{Q}_h(u), v) \\ &\quad + \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \cdot n, p - Q_h(p)\rangle_{\partial T}. \end{aligned}$$

Testing the second equation in (1.1)–(1.4) by a function ψ, then

$$0 = (\nabla \cdot u, \psi)_\Omega = (Q_h(\nabla \cdot u), \psi)_\Omega = (\nabla_w \cdot (\mathbb{Q}_h(u)), \psi)_\Omega,$$

which means that $B(\mathbb{Q}_h(u), \psi) = 0$. Substract the previous equation from the second equation in (2.3)–(2.4), we obtain $B(e_h, \psi) = 0$. □

The weak Galerkin mixed finite element formulation (2.3)–(2.4) is a typical saddle-point scheme which can be studied with the well known Babuška-Brezzi theory [1,2]. Thus, we have a great interest for studying the properties of the bilinear forms introduced in (2.3)–(2.4).

3.2. **Study of the bilinear forms.** First, we define the norms on the space $\mathcal{V}_h^0$ and $\mathcal{W}_h^0$. For $\psi \in \mathcal{W}_h^0$, we use $\|\psi\|$ the usual L^2-norm of ψ and we introduce a norm in $\mathcal{V}_h^0$ as

$$|||u|||^2 := A_s(u, u).$$

Note that from the proof of Theorem 2.1, we immediately deduce that $u = 0$ if $A_s(u, u) = 0$ and hence the triple-bar norm just introduced above define norm on the space $\mathcal{V}_h^0$. Also from this definition of norm, we remark that the coercivity of A_s follows directly. While, the continuity of the bilinear forms A_s and B can be demonstrated from classical techniques due to the Cauchy-Schwarz inequalities. Therefore, for an application of the Babuška-Brezzi theory, it remains to demonstrate an inf-sup condition for B. This is the objective of the following lemma.

Lemma 3.3. *There exists a constant β independent of h such that*

$$\inf_{\psi\in\mathcal{W}_h^0\setminus\{0\}} \sup_{v\in\mathcal{V}_h^0\setminus\{0\}} \frac{B(\varphi, v)}{|||v||| \cdot \|\varphi\|} \geq \beta > 0.$$

Proof. Let $\psi \in \mathcal{W}_h^0$, then ψ is in $L_0^2(\Omega)$ and it is well known that there exists $v \in H_0^1(\Omega)^d$ such that $(\nabla \cdot v, \psi) \geq C\|\psi\| \cdot \|v\|_1$. Choose $\tilde{v} = \mathbb{Q}_h(v)$ and let us prove that $|||\tilde{v}||| \leq C\|v\|_1$, we have

$$\begin{aligned}\sum_{T\in\mathfrak{T}_h} \|\nabla_w \times \tilde{v}\|^2 = \sum_{T\in\mathfrak{T}_h} \|\nabla_w \times \mathbb{Q}_h(v)\|^2 = \sum_{T\in\mathfrak{T}_h} \|\mathbf{Q}_h(\nabla \times v)\|^2 \\ \leq \sum_{T\in\mathfrak{T}_h} \|\nabla \times v\|^2 \leq \|v\|_1^2 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\sum_{T\in\mathfrak{T}_h} \|\nabla_w \cdot \tilde{v}\|^2 = \sum_{T\in\mathfrak{T}_h} \|\nabla_w \cdot \mathbb{Q}_h(v)\|^2 = \sum_{T\in\mathfrak{T}_h} \|Q_h(\nabla \cdot v)\|^2 \\ \leq \sum_{T\in\mathfrak{T}_h} \|\nabla \cdot v\|^2 \leq \|v\|_1^2. \end{aligned} \tag{3.6}$$

By selecting $\tilde{v} = \mathbb{Q}_h(v)$ in the definition of $s_T(\tilde{v}, \tilde{v})$, we need to estimate the following two terms

$$s_{NT}(\tilde{v}, \tilde{v}) := r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|(\mathbb{Q}_0(v - \mathbb{Q}_b(v)) \cdot n\|_{\partial T}^2$$

and

$$s_{TT}(\tilde{v}, \tilde{v}) := r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|(\mathbb{Q}_0(v - \mathbb{Q}_b(v)) \times n\|_{\partial T}^2.$$

For an estimation of the term $s_{NT}(\tilde{v}, \tilde{v})$, one can get

$$\begin{aligned} s_{NT}(\tilde{v}, \tilde{v}) \leq & 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|(\mathbb{Q}_0(v) - v) \cdot n\|_{\partial T}^2 + 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|(\mathbb{Q}_b(v) - v) \cdot n\|_{\partial T}^2 \\ \leq & 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|\mathbb{Q}_0(v) - v\|_{\partial T}^2 + 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|\mathbb{Q}_0(v) - v\|_{\partial T}^2 \\ \leq & Cr \sum_{\partial T\in\mathcal{E}_h} h_T^{-1} \|\mathbb{Q}_0(v) - v\|_{\partial T}^2 \\ \leq & Cr \sum_{T\in\mathfrak{T}_h} h_T^{-2} \|\mathbb{Q}_0(v) - v\|_T^2 + \|\nabla(\mathbb{Q}_0(v) - v)\|_T^2 \\ \leq & Cr \sum_{T\in\mathfrak{T}_h} h_T^{-2} \|\mathbb{Q}_0(v) - v\|_T^2 + \|\nabla(\mathbb{Q}_0(v) - v)\|_T^2 \\ \leq & C\|\nabla v\|^2, \end{aligned} \tag{3.7}$$

and similarly, one can obtain

$$s_{TT}(\tilde{v}, \tilde{v}) \leq C\|\nabla v\|^2. \tag{3.8}$$

It follows from (3.5), (3.6), (3.7) and (3.8) that

$$|||\tilde{v}||| \leq C\|v\|_1,$$

and the use of Lemma 3.1, the definition of Q_h, means

$$B(\tilde{v}, \psi) = (\nabla_w \cdot \mathbb{Q}_h(v), \psi) = (Q_h(\nabla \cdot v), \psi) = (\nabla \cdot v, \psi) \geq C\|v\|_1\|\psi\| \geq \beta|||\tilde{v}||| \cdot \|\psi\|,$$

which ends the proof. □

In the next subsection, we shall demonstrate optimal order error estimates for the electrostatic field u_h in a norm which is equivalent to the standard $H_0(\nabla\times,\Omega)\cap H(\nabla\cdot,\Omega)$ norm, and for the Lagrange multiplier p_h in the usual L^2 norm. Moreover, we give an error estimate result for the electrostatic field u_h in the L^2 norm.

3.3. **Error estimations.** Let us start by introducing the following lemma which we need for a rigourous proof of error convergence results.

Lemma 3.4. (a) *Given $u \in H^{k+2}(\Omega)$ and $s \in [0, k+1]$, then*

$$\sum_{T\in\mathcal{T}_h} \|u - \mathbb{Q}_h(u)\|_T^2 + h_T^2\|\nabla(u - \mathbb{Q}_h(u))\|_T^2 \leq h^{2(s+1)}\|u\|_{s+1}^2, \tag{3.9}$$

$$\sum_{T\in\mathcal{T}_h} \|\nabla(u - \mathbb{Q}_h(u))\|_T^2 \leq h^{2s}\|u\|_{s+1}^2. \tag{3.10}$$

(b) *For any $\theta \in H^1(T)$, $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$,*

$$\|\theta\|_e^2 \leq C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\theta|_T^2\right). \tag{3.11}$$

Proof. See the equalities (4.3), (4.2) and the inequality (A.1) in [14]. ☐

One of our main result in this paper, which demonstrate clearly the optimal convergence of the mixed weak Galerkin formulation (2.3)–(2.4) is given and proven in the following theorem.

Theorem 3.1. *Let $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$ be the approximate solution of* (2.3)–(2.4), *(u, p) the exact solution of* (1.1)–(1.4) *and suppose that $(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ with $k \geq 0$, then, we have the following two convergence results*

$$|\!|\!|\mathbb{Q}_h(u) - u_h|\!|\!| + \|Q_h(p) - p_h\| \leq Ch^{s+1}\Big(\|u\|_{s+2} + \|p\|_{s+1}\Big) \tag{3.12}$$

and

$$\|\mathbb{Q}_0(u) - u_0\| \leq Ch^{k+2}\Big(\|u\|_{k+2} + \|p\|_{k+1}\Big). \tag{3.13}$$

Proof. Define

$$\begin{aligned}
T_1(u, v) :=& \sum_{T\in\mathcal{T}_h} \langle (v^b - v^0) \times n, \mathbf{Q}_h(\nabla\times u) - \nabla\times u\rangle_{\partial T},\\
T_2(u, v) :=& s_T(\mathbb{Q}_h(u), v),\\
=& r\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mathbb{Q}_0(u) - \mathbb{Q}_b(u)) \times n, (v^0 - v^b) \times n\rangle_{\partial T}\\
&+ r\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle(\mathbb{Q}_0(u) - \mathbb{Q}_b(u)) \cdot n, (v^0 - v^b) \cdot n\rangle_{\partial T},\\
T_3(p, v) :=& \sum_{T\in\mathcal{T}_h} \langle (v^b - v^0) \cdot n, p - Q_h(p)\rangle_{\partial T}
\end{aligned}$$

and

$$\ell(v) := T_1(u,v) - T_2(u,v) + T_3(p,v).$$

Then the error equations in Lemma 3.2 can be written as

$$A_s(e_h,v) + B(v,\varepsilon_h) = \ell(v), \quad B(e_h,\psi) = 0,$$

and we deduce from the general theory of Babuška and Brezzi that

$$|||e_h||| + \|\varepsilon_h\| \leq C\|\ell\|_{\mathcal{V}_h^{0\prime}}.$$

Then, it is sufficient to find a bound of $\|\ell\|_{\mathcal{V}_h^{0\prime}}$. Let us start by estimating the term $\|u - \mathbb{Q}_b(u)\|_{\partial T}$ which we need for estimating $T_2(u,v)$. We have

$$\begin{aligned}\|u - \mathbb{Q}_b(u)\|_{\partial T}^2 =& \langle u - \mathbb{Q}_b(u), u - \mathbb{Q}_b(u)\rangle_{\partial T}\\ =& \langle u - \mathbb{Q}_b(u), u - \mathbb{Q}_0(u)\rangle_{\partial T}\\ \leq& \|u - \mathbb{Q}_b(u)\|_{\partial T}\|u - \mathbb{Q}_0(u)\|_{\partial T}\end{aligned}$$

and then,

$$\|u - \mathbb{Q}_b(u)\|_{\partial T} \leq \|u - \mathbb{Q}_0(u)\|_{\partial T}. \tag{3.14}$$

Now, the use of the definition of $\mathbb{Q}_0$, $\mathbb{Q}_b$, Cauchy Schwarz inequality, (3.14), (3.11) and (3.9), (3.10) imply that

$$\begin{aligned}& r\sum_{T\in\mathfrak{T}_h} |h_T^{-1}\langle(\mathbb{Q}_b(u) - \mathbb{Q}_0(u)) \times n, (v^0 - v^b) \times n\rangle_{\partial T}|\\ =& r\sum_{T\in\mathfrak{T}_h} |h_T^{-1}\langle(u - \mathbb{Q}_0(u)) \times n - (u - \mathbb{Q}_b(u)) \times n, (v^0 - v^b) \times n\rangle_{\partial T}|\\ \leq& \left(r\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|(v^0 - v^b) \times n\|_{\partial T}^2\right)^{\frac{1}{2}}\\ &\times\left(r\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|(u - \mathbb{Q}_0(u) - (u - \mathbb{Q}_b(u))) \times n\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|u - \mathbb{Q}_0(u)\|_{\partial T}^2 + h_T^{-1}\|u - \mathbb{Q}_b(u)\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|u - \mathbb{Q}_0(u)\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h}\left(h_T^{-2}\|u - \mathbb{Q}_0(u)\|_T^2 + \|\nabla(u - \mathbb{Q}_0(u)\|_T^2\right)\right)^{\frac{1}{2}}\\ \leq& Ch^{s+1}|||v||| \cdot \|u\|_{s+2}.\end{aligned}$$

With a similar, one can obtain

$$r\sum_{T\in\mathfrak{T}_h}|h_T^{-1}\langle(\mathbb{Q}_b(u)-\mathbb{Q}_0(u))\cdot n,(v^0-v^b)\cdot n\rangle_{\partial T}|\leq Ch^{s+1}|||v|||\cdot\|u\|_{s+2}.$$

and deduce that

$$|T_2(u,v)|\leq Ch^{s+1}\|u\|_{s+2}|||v|||.$$

For finding an estimation of $T_1(u,v)$, we use the Cauchy Schwarz inequality, the definition of $|||\cdot|||$ and the trace inequality (3.11) and write

$$\begin{aligned}|T_1(u,v)|&\leq\sum_{T\in\mathfrak{T}_h}\left|\langle(v^b-v^0)\times n,\mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}\right|\\&\leq\sum_{T\in\mathfrak{T}_h}h_T^{-\frac{1}{2}}\|(v^b-v^0)\times n\|_{0,\partial T}h_T^{\frac{1}{2}}\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}\\&\leq\Bigg(\sum_{T\in\mathfrak{T}_h}h_T^{-1}\|(v^b-v^0)\times n\|_{0,\partial T}^2\Bigg)^{\frac{1}{2}}\\&\quad\times\Bigg(\sum_{T\in\mathfrak{T}_h}h_T\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}^2\Bigg)^{\frac{1}{2}}\\&\leq\Bigg(\sum_{T\in\mathfrak{T}_h}h_T\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}^2\Bigg)^{\frac{1}{2}}|||v|||\\&\leq\Bigg(\sum_{T\in\mathfrak{T}_h}\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_T^2\\&\quad+\sum_{T\in\mathfrak{T}_h}h_T^2\|\nabla(\mathbf{Q}_h(\nabla\times u)-\nabla\times u)\|_T^2\Bigg)^{\frac{1}{2}}|||v|||.\end{aligned}$$

Using (3.9) for $\nabla\times u$, we get

$$|T_1(u,v)|\leq h^{s+1}\|\nabla\times u\|_{s+1}|||v|||\leq h^{s+1}\|u\|_{s+2}|||v|||.$$

The same technique applied for $T_1(u,v)$ can also be applied for estimating $T_3(p,v)$ and we obtain $|T_3(p,v)|\leq Ch^{s+1}\|p\|_{s+1}|||v|||$. The inequality (3.12) follows immediately from the previous inequalities and for the proof of (3.13), we can use a similar technique to the given in [6, 14] for the second order Laplacien operator. □

4. NUMERICAL TESTS

In this paragraph, two numerical examples are tested for the two dimensional Maxwell equations (1.1)–(1.4) with constant coefficients $\mu(x)=1$ and $\varepsilon(x)=1$ on a domain $\Omega=(0,1)^2$. The parameter r which appears in (2.3)–(2.4) is chosen as $r=1$ and can be taken as any strictly positive real number. The approximate solution (u_h,p_h) is discritised with the lowest order (i.e., $k=0$) on the space $\mathcal{V}_h^0\times\mathcal{W}_h^0$. The numerical experiments indicate that the weak Galerkin methods are accurate and easy to implement and the numerical convergence results obtained on the two examples confirm perfectly the estimations proven in theorem 3.1.

Example 4.1. In this example, we consider the Maxwell equations and Lagrange multiplier together with boundary conditions (1.1)–(1.4) on the unit square $\Omega = (0,1)^2$. We assume that the true solutions are given by $u(x,y) = \begin{pmatrix} y(y-1)\cos(y) \\ x(x-1)\cos(x) \end{pmatrix}$ and $p(x,y) = x(x-1)y(y-1)\cos(x+y)$. The numerical experiments of the algorithm are presented in Table 1. We see that these results show the $O(h)$ error for the electrostatic field in the $|||\cdot|||$-norm and $O(h^2)$ error of the Lagrange multiplier in the L^2-norm. The convergence rate with respect $O(h^2)$ for the electric field u in the L^2-norm is also observed, which confirms the proven estimations (3.12) and (3.13).

Table 1. Numerical results for Example 1.

h	$\|\|\|e_h\|\|\|$	rate	$\|\|\varepsilon_h\|\|_{1,h}$	rate	$\|\|u_h^0 - \mathbb{Q}_h(u^0)\|\|$	rate
$\frac{1}{2}$	6.2592e-01	-	1.7009e-02	-	1.7249e-01	-
$\frac{1}{4}$	3.5102e-01	8.3443e-01	2.6726e-03	2.6700	3.5422e-02	2.2838e
$\frac{1}{8}$	1.8526e-01	9.2201e-01	6.0702e-04	2.1384	8.9524e-03	1.9843
$\frac{1}{16}$	9.3879e-02	9.8066e-01	1.4837e-04	2.0325	2.2470e-03	1.9943
$\frac{1}{32}$	4.7096e-02	9.9518e-01	3.6887e-05	2.0080	5.6233e-04	1.9985
$\frac{1}{64}$	2.3568e-02	9.9880e-01	9.2090e-06	2.0020	1.4062e-04	1.9996
$\frac{1}{128}$	1.1786e-02	9.9970e-01	2.3014e-06	2.0005	3.5157e-05	1.9999

Example 4.2. In this numerical example, we shall consider the 2-dimensional Maxwell problem with Lagrange multiplier (1.1)–(1.4). Consider $\Omega = (0,1) \times (0,1)$ and the right-hand side function J be chosen such that the functions $u(x,y) = (e^y \sin(y^2 - y), e^x \sin(x^2 - x))^T$ and $p(x,y) = e^{x+y} \sin((x^2 - x)(y^2 - y))$ are the true solutions of the problem (1.1)–(1.4). The convergence results and error profiles are presented in Table 2. It can be observed $|||\cdot|||$-error, L^2-error for the electric field u, and L^2-error for the Lagrange multiplier p converge, respectively, with respect to $O(h)$, $O(h^2)$, and $O(h^2)$, which confirms the theoretical estimations (3.12) and (3.13).

5. Conclusions and Remarks

In this paper, we analysed the new formulation of weak Galerkin mixed finite element method for solving numerically the Maxwell equations with Lagrange multiplier. The well posedness as well as the optimal convergence of the numerical scheme was shown, established and tested numerically. The results obtained in this paper are powerful and encourage applications to other systems of partial differential equations.

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Table 2. Numerical results for Example 2.

h	$\lvert\lvert\lvert e_h \rvert\rvert\rvert$	rate	$\lVert \varepsilon_h \rVert_{1,h}$	rate	$\lVert u_h^0 - \mathbb{Q}_h(u^0) \rVert$	rate
$\frac{1}{2}$	1.3330e+00	-	4.1335e-02	-	3.9249e-01	-
$\frac{1}{4}$	6.7970e-01	9.7174e-01	1.1172e-02	1.8874	7.2042e-02	2.4457
$\frac{1}{8}$	3.5995e-01	9.1709e-01	2.9750e-03	1.9090	1.8123e-02	1.9910
$\frac{1}{16}$	1.8272e-01	9.7819e-01	7.5563e-04	1.9771	4.5461e-03	1.9951
$\frac{1}{32}$	9.1709e-02	9.9449e-01	1.8965e-04	1.9943	1.1376e-03	1.9986
$\frac{1}{64}$	4.5898e-02	9.9862e-01	4.7460e-05	1.9986	2.8447e-04	1.9996
$\frac{1}{128}$	2.2955e-02	9.9966e-01	1.1868e-05	1.9996	7.1123e-05	1.9999

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[1]Northern Border University,
Faculty of Arts and Science, Rafha. P.O 840, Saudi Arabia.

[2]Department of Mathematics,
University of Tunis,
Boulevard du 9 avril 1939 Tunis. Ensit, Taha Hussein Avenue, Montfleury. Tunis, Tunisia.
Email address: `abduhameed.zadany@nbu.edu.sa & hamido20042002@yahoo.fr`
Email address: `Abdllhalim.hasanawa@nbu.edu.sa & halim.hasnaoui1@gmail.com`

[3]Department of Mathematics,
Carthage University,
Isteub, 2 Rue de l'Artisana, Charguia 2, 2035 Tunis, Tunisia.
Email address: `sayed.sayari@isteub.ucar.tn, sayari.sayed@gmail.com`

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A QUALITATIVE STUDY ON FRACTIONAL LOGISTIC INTEGRO-DIFFERENTIAL EQUATIONS IN AN ARBITRARY TIME SCALE

NIMAI SARKAR[1], MAUSUMI SEN[2], DIPANKAR SAHA[3], AND BIPAN HAZARIKA[4,*]

Abstract. This manuscript deals with the investigation related to uniqueness and existence of solution of fractional order nonlinear pantograph integro-differential equation in arbitrary time scale. The fractional derivatives are defined in Riemann-Liouville sense, the primary tools are taken as Banach contraction principle and Schauder's fixed point theory to establish the theoretical outcomes. Finally, we give examples to show the efficiency of our results.

1. Introduction

The theory on time scale calculus is a new field of interest for the mathematicians. This particular branch was first encountered by Aulbach and Hilger [20] in the year 1990. To unify the calculus for both discrete and continuous problems, time scale theory was introduced. Due to the unification nature the topic "time scale", frequently appears in numerous physical modelling problems, where the discrete and continuous data are simultaneously involved. In current time, it is a topic of vigorous research in diverse areas, such as economics, control theory, robotics, biology, quantum calculus and many other fields. Different problems in engineering and natural phenomenons are extensively modelled into fractional equations. See, for example [13, 16, 17, 27, 28, 33–35, 38] and references cited therein. Over the past few years different researchers have qualitatively studied fractional integro-differential equations and time scales integro-differential equations separately, see [1, 4, 5, 11, 12, 21, 32] and references cited

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therein. But as per our knowledge, a less number of work has been reported in the combined treatment for time scale fractional integro-differential equations.

In this study, our main objective is to provide a theoretical platform to investigate existence and uniqueness of solution for the considered types of equations. With this motivation, we introduce the following nonlinear equation

$$(1.1)\quad {}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}w(\theta)=\mu w(\theta)\Big(u(\theta)-a(\theta)w(\theta)-b(\theta)\int_{\theta_0}^{\Theta}k_*(s)\mathcal{N}(w(s))\Delta s\Big),\quad \theta\in(\theta_0,\Theta),$$

we treat $\gamma\in(0,1)$ as the order of fractional derivative, μ is nonzero constant; ${}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}$ is the left differentiation operator of order γ in Riemann-Liouville sense (**L**-RLFD); ${}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\xi}_{\theta}$ is the left integral operator of order ξ in Riemann-Liouville fractional sense (**L**-RLFI). Moreover, $\mathbb{T}$ represents arbitrary time scale; $u(\theta),a(\theta),b(\theta)$ are continuous, nonnegative and bounded functions; $k_*:[\theta_0,\Theta]_{\mathbb{T}}\to[0,\infty)$ is a smooth function on $[\theta_0,\Theta]_{\mathbb{T}}$. This manuscript deals with Equation (1.1) subject to the initial condition defined as

$$(1.2)\qquad {}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\xi}_{\theta}w(\theta_0)=0,\quad \text{for all } \xi\in(0,1).$$

One can relate Equation (1.1) and Condition (1.2) as model problem for mix of stop-start phenomenon, such as insect populations that are smooth during the incubating season and die out in winter. If we include toxic effect in the insect population model, then the considered equation perfectly support the concerned physical event. Consequently, $u(\theta)$ denotes the birth rate, $a(\theta)$ is crowding coefficient, $b(\theta)$ is the toxicity coefficient. $\mu w(\theta)\int_{\theta_0}^{\Theta}k_*(s)\mathcal{N}(w(s))\Delta s$ represents the effect of toxin accumulation on the species. $w(\theta)$ denotes insect population at time θ. However, the initial condition is taken as homogeneous (die out season) to avoid the complexity. Although, this condition is often nonzero in real life situation.

By construction, Equation (1.1) together with condition (1.2) takes the form of a logistic integro-differential equation of fractional order in an arbitrary time scale $\mathbb{T}$. In literature, the particular cases have been discussed; if $\mathbb{T}=\mathbb{R}$ then Equation (1.1) and condition (1.2) leads to classical fractional order logistic integro-differential equation [15, 25, 26, 36]. If $\mathbb{T}=\mathbb{Z}$, that leads to logistic integro-difference equation of fractional order [9, 22, 24]. In either case, if one concentrate upon integer order operations only, then that leads to classical pantograph integro-differential [23, 29] and integro-difference [6, 37] equations, respectively. For the above cases the theoretical investigation related to existence and uniqueness of solution have been carried out by J. M. Cushing [14], A. Aghajani et al. [2], K. Balachandran et al. [7], Sarkar et al. [31], Saha et al. [30], Gupta et al. [19], S. Etemad et al. [8], B. Ahmad et al. [3] and many other researchers. However, all the previously published works are primarily focused on separate cases only. In this paper, we establish a unified theory on an arbitrary time scale. Banach contraction principle and Schauder's fixed point theory are adopted along with some fundamental results from time scale theory.

The paper is organized in the following manner. Section 2 deals with the fundamental concepts related to the present work, Section 3 covers the main results, in Section

4 we validate our findings through some examples and finally, in Section 5 we draw some concluding remarks.

2. Fundamental Concepts

This section is entirely dedicated to the recapitulation of basic concepts and definitions that are frequently used throughout the article.

As per the suggestion of Benkhettou et al. [10] we proceed with the notion of **L**-RLFI and **L**-RLFD operators on arbitrary measure chain. For local treatment to non integer order calculus on any measure chain we recommend the works of Benkhettou et al. [10]. In the present study, we only focus on delta derivative over $\mathbb{T}$, parallel investigations are gradually applicable for ∇-fractional case [35].

Definition 2.1 ([10]). Consider $[p,q]$ an interval from the arbitrary time scale $\mathbb{T}$. Then the **L**-RLFI on $\mathbb{T}$, for $f:\mathbb{T}\to\mathbb{R}$ is defined as

$$ {}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{\xi}f(\theta)=\int_{p}^{\theta}\frac{(\theta-s)^{\xi-1}}{\Gamma(\xi)}f(s)\Delta s, $$

with Γ as the stranded gamma function, $\xi\in(0,1)$.

Definition 2.2 ([10]). Consider $[p,q]$ an interval from the arbitrary time scale $\mathbb{T}$. Then the **L**-RLFD on $\mathbb{T}$, for $f:\mathbb{T}\to\mathbb{R}$ is defined as

$$ {}_{p}^{\mathbb{T}}\mathfrak{D}_{\theta}^{\gamma}f(\theta)=\Big(\int_{p}^{\theta}\frac{(\theta-s)^{-\gamma}}{\Gamma(1-\gamma)}f(s)\Delta s\Big)^{\Delta}, $$

with $\gamma\in(0,1)$.

Remark 2.1. If one choose real line as time scale, then from previous two definitions, we have the classical RL fractional operators.

Proposition 2.1 ([10]). *For $f:\mathbb{T}\to\mathbb{R}$ and $\gamma\in(0,1)$ the following estimate holds*

$$ {}_{p}^{\mathbb{T}}\mathfrak{D}_{\theta}^{\gamma}f=\Delta\circ_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{1-\gamma}f. $$

Proposition 2.2 ([10]). *For $\gamma>0$ and $f\in C([p,q])$, the following estimate holds*

$$ {}_{p}^{\mathbb{T}}\mathfrak{D}_{\theta}^{\gamma}\circ_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{\gamma}f=f. $$

Proposition 2.3 ([10]). *Under the initial condition ${}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{1-\gamma}w(p)=0$, $f\in C([p,q])$ and $\gamma\in(0,1)$ leads to the following estimate*

$$ {}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{\gamma}\circ_{p}^{\mathbb{T}}\mathfrak{D}_{\theta}^{\gamma}f=f. $$

Theorem 2.1 ([10]). *Consider $f\in C([p,q])$, $\gamma>0$, and ${}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{\gamma}([p,q])$ be the space of functions that can be expressed as the Riemann-Liouville $\Delta-$integral with order γ. Then, $f\in_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{\gamma}([p,q])$ if and only if ${}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{1-\gamma}([p,q])f\in C^{1}([p,q])$ and ${}_{p}^{\mathbb{T}}\mathfrak{I}_{\theta}^{1-\gamma}f(p)=0$.*

Proposition 2.4 ([4])**.** *Let us consider an arbitrary measure chain* $\mathbb{T}$, *and an non-decreasing smooth function* f *on* $[p,q]$. *Suppose,* ϕ *is the extension of* f *in* $[p,q]$ *defined by*

$$\phi(s)=\begin{cases} f(s), & \text{if } s\in\mathbb{T},\\ f(\theta), & \text{if } s\in(\theta,\sigma(\theta))\not\subset\mathbb{T}.\end{cases}$$

Then,

$$\int_p^q f(\theta)\Delta\theta\le\int_p^q\phi(\theta)d\theta,$$

where $\sigma(\theta)=\inf\{s\in\mathbb{T}:s>\theta\}$ *is the forward jump operator on the consider measure chain.*

To avoid complexity, throughout the article $\theta=0$ is considered, the space of all continuous function on $[0,\Theta]$ is denoted by $C([0,\Theta])$, and $\|x\|_\infty=\sup_{\theta\in[0,\Theta]}\{|x(\theta)|\}$. Then, $\mathcal{X}=(C([0,\Theta]),\|\cdot\|_\infty)$ is a Banach space.

Theorem 2.2 ([4])**.** *A function* Ψ *is said to be primitive of* $\psi:\mathbb{T}\to\mathbb{R}$ *provided* $\Psi^\Delta(x)=\psi(x)$ *for each* $x\in\mathbb{T}$, *then the* Δ*-integral is given by*

$$\textstyle\int_{x_0}^x\psi(\tilde{x})\Delta\tilde{x}=\Psi(x)-\Psi(x_0).$$

A function $\psi:\mathbb{T}\to\mathbb{R}$ is called a rd-continuous on $\mathbb{T}$, if ψ is continuous at $x\in\mathbb{T}$ with $\sigma(x)=x$ and has finite left-sided limits at points $x\in\mathbb{T}$ with

$$\sup\{\tau\in\mathbb{T}:\tau<x\}=x,$$

and the set of all rd-continuous functions $\psi:\mathbb{T}\to\mathbb{R}$ is represented by $C_{rd}(\mathbb{T},\mathbb{R})$.

Theorem 2.3 (Schauder fixed point theorem)**.** *Let* Ω *be a nonempty closed, bounded, convex subset of a Banach space* S *and* $\Phi:\Omega\to\Omega$ *be a continuous compact operator. Then* Φ *has a fixed point in* Ω.

Theorem 2.4 (Arzela-Ascoli theorem)**.** *Let* S *be a compact Hausdroff metric space. Then* $Y\subset M(S)$ *is relatively compact if and only if* Y *is uniformly bounded and uniformly equi-continuous.*

3. Main Results

In this section, we present and illustrate the theoretical findings elaborately. We start with the corresponding integral representation of equation (1.1) with condition (1.2) and subsequently derive the main results.

Lemma 3.1. *Let* $0<\gamma<1$. *Then, Equation* (1.1) *is equivalent to*

(3.1)

$$w(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_{\theta_0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s.$$

Proof. By definition we have

$$\begin{aligned}
&{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}w(\theta)\\
=&\frac{\mu}{\Gamma(\gamma)}\Big\{\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(\theta)\Big(u(s)-a(s)w(s)-b(s)\int_{\theta_0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big\}^{\Delta}\\
=&\Big(\int_{\theta_0}^{\mathbb{T}}\mathcal{J}^{1-\gamma}_{\theta}w(\theta)\Big)^{\Delta}=\Big(\Delta\circ^{\mathbb{T}}_{\theta_0}\mathcal{J}^{1-\gamma}_{\theta}\Big)w(\theta).
\end{aligned}$$

The conclusion follows from Proposition 2.3 that ${}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{\gamma}_{\theta}\circ({}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{\gamma}_{\theta}w(\theta))=w(\theta)$. ☐

To avoid complexity, we set $\theta_0=0$. It is convenient to observe that Equation (1.1) posses a solution w if and only if w turns out to be a fixed point of the operator $\Lambda:\mathcal{X}\to\mathcal{X}$, defined as

(3.2)
$$\Lambda w(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_{\theta_0}^{\theta}(\theta-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_{0}^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s.$$

We consider the following two assumptions.

($\mathcal{A}1$) The nonlinear function $\mathcal{N}:\mathbb{R}^+\to\mathbb{R}^+$ is Lipschitz continuous function with Lipschitz constant $L_{\mathcal{N}}$.

($\mathcal{A}2$) The functions $u,a,b,k_*:\mathbb{R}^+\to\mathbb{R}^+$ are such that

$$\sup_{\theta\in\mathbb{R}^+}|u(\theta)|=U,\quad \sup_{\theta\in\mathbb{R}^+}|a(\theta)|=A,\quad \sup_{\theta\in\mathbb{R}^+}|b(\theta)|=B,\quad \sup_{\theta\in\mathbb{R}^+}|k_*(\theta)|=K.$$

With all these constructions, now we present our main result as follows.

Theorem 3.1. *If* ($\mathcal{A}1$) *and* ($\mathcal{A}2$) *are satisfied by Equation* (1.1) *together with condition* (1.2)*, then it admits at least one solution in* $\mathcal{X}$ *for all* $\mu>0$.

Proof. To prove the proposed claim, we proceed with the following three steps.

Step I. We take a convergent sequence w_n with limit $w\in\mathcal{X}$. Then,

$$\begin{aligned}
|\Lambda w_n(\theta)-\Lambda w(\theta)|\leq&\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|u(s)||w_n(s)-w(s)|\Delta s\\
&+\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|a(s)||(w_n(s))^2-(w(s))^2|\Delta s\\
&+\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|b(s)|\Big|w_n(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w_n(x))\Delta x\\
&-w(s)\int_0^{\Theta}k_*\mathcal{N}(w(x))\Delta x\Big|\Delta s.
\end{aligned}$$

This leads to the following estimate

$$|\Lambda w_n(\theta)-\Lambda w(\theta)|\leq J^1(\theta)+J^2(\theta)+J^3(\theta), \tag{3.3}$$

where

$$J^1(\theta)=\frac{\mu}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}|u(s)||w_n(s)-w(s)|\Delta s,$$

$$\begin{aligned}J^2(\theta) =&\frac{\mu}{\Gamma(\gamma)}\int_0^\theta (\theta-s)^{\gamma-1}|a(s)||(w_n(s))^2-(w(s))^2|\Delta s,\\ J^3(\theta) =&\frac{\mu}{\Gamma(\gamma)}\int_0^\theta (\theta-s)^{\gamma-1}|b(s)|\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w_n(x))\Delta x\\ &-w(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s.\end{aligned}$$

Now we separately consider all the right-hand terms of (3.3), and obtain the following inequalities:

$$\begin{aligned}J^1(\theta) &\le \frac{\mu U}{\Gamma(\gamma)}\int_0^\theta (\theta-s)^{\gamma-1}|w_n(s)-w(s)|\Delta s\\ &\le \frac{\mu U}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s\\ &\le \frac{\mu U}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}ds,\end{aligned}$$

as $(\theta-s)^{\gamma-1}$ is nondecreasing. Therefore,

$$J^1(\theta)\le\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.4}$$

This implies

$$\begin{aligned}J^2(\theta) &\le \frac{\mu A}{\Gamma(\gamma)}\int_0^\theta (\theta-s)^{\gamma-1}|w_n(s)-w(s)||w_n(s)+w(s)|\Delta s\\ &\le \frac{\mu A}{\Gamma(\gamma)}\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}(|w_n(s)|+|w(s)|)\Delta s,\end{aligned}$$

since $w_n\to w$ in $\mathcal{X}$, it is bounded by some constant W. Thus we have

$$J^2(\theta)\le\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.5}$$

Therefore,

$$\begin{aligned}&J^3(\theta)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w_n(x))\Delta x-w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\\ &+\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x-w(s)\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\Big)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(\int_0^\theta(\theta-s)^{\gamma-1}\Big|w_n(s)\Big(\int_0^\Theta k_*(x)\Big(\mathcal{N}(w_n(x))-\mathcal{N}(w(x))\Big)\Delta x\Big)\Big|\Delta s\\ &+\int_0^\theta(\theta-s)^{\gamma-1}\Big|(w_n(s)-w(s))\int_0^\Theta k_*(x)\mathcal{N}(w(x))\Delta x\Big|\Delta s\Big)\\ \le&\frac{\mu B}{\Gamma(\gamma)}\Big(WKL_{\mathcal{N}}\int_0^\theta(\theta-s)^{\gamma-1}\Big(\int_0^\Theta|w_n(x)-w(x)|\Delta x\Big)\Delta s\end{aligned}$$

$$
\begin{aligned}
&+KN\int_0^\theta(\theta-s)^{\gamma-1}|w_n(s)-w(s)|\Big(\int_0^\Theta\Delta x\Big)\Delta s\Big)\\
\le&\frac{\mu B}{\Gamma(\gamma)}\Big(WKL_{\mathcal{N}}\Theta\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s+KN\Theta\|w_n-w\|_\infty\int_0^\theta(\theta-s)^{\gamma-1}\Delta s\Big)\\
\le&\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\|w_n-w\|_\infty.
\end{aligned}
$$

It follows that

$$J^3(\theta)\le\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\|w_n-w\|_\infty. \tag{3.6}$$

Bringing inequalities (3.4)–(3.6) in (3.3), we have

$$|\Lambda w_n(\theta)-\Lambda w(\theta)|\le\Big(\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\Big)\|w_n-w\|_\infty.$$

Then,

$$
\begin{aligned}
&\|\Lambda w_n(\theta)-\Lambda w(\theta)\|_\infty\\
&\le\Big(\frac{\mu U\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^\gamma}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\Big)\|w_n-w\|_\infty.
\end{aligned}\tag{3.7}
$$

Hence, the right side of (3.7) tends to 0 as $w_n\to w$. Therefore, $\Lambda w_n\to\Lambda w$. This concludes that the operator Λ is continuous.

Step II. In this step, our primary goal is to show that Λ preserves boundedness. For this purpose, we take $\Omega=[0,\Theta]$. We claim that for all $R>0$, $L>0$ and for all $w\in\mathbb{B}_R=\{w\in C(\Omega,\mathbb{R}):\|w\|_\infty\le R\}$ we have $\|\Lambda w\|_\infty\le L$. Consider $\tau\in\Omega$ and $w\in\mathbb{B}_R$. Then

$$
\begin{aligned}
|\Lambda w(\tau)|\le&\frac{\mu}{\Gamma(\gamma)}\Big(U\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|\Delta s+A\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|^2\Delta s\\
&+BKN\int_0^\tau(\tau-s)^{\gamma-1}|w(s)|\Big(\int_0^\Theta\Delta x\Big)\Delta s\Big)\\
\le&\frac{\mu}{\Gamma(\gamma)}\Big(UR+AR^2+BKNR\Theta\Big)\int_0^\tau(\tau-s)^{\gamma-1}\Delta s\\
\le&\frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\Theta^\gamma.
\end{aligned}
$$

Further, if we consider the supremum over τ, then the following result holds

$$\|\Lambda w\|_\infty\le\frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\Theta^\gamma. \tag{3.8}$$

This implies, Λw is bounded.

Step III. Our claim is the equi-continuity of Λ. We proceed by considering $\tau_1,\tau_2\in\Omega$ so that $0\le\tau_1<\tau_2\le\Theta$, $\mathbb{B}_R$ is a bounded set of $C(\Omega,\mathbb{R})$. For $w\in\mathbb{B}_R$, we get

$$|\Lambda w(\tau_2)-\Lambda w(\tau_1)|$$

$$
\begin{aligned}
&\le \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}(\tau_2-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\\
&\quad-\int_0^{\tau_1}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big|\\
&\le \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}+(\tau_1-s)^{\gamma-1}\Big)\\
&\quad\times w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\\
&\quad-\int_0^{\tau_1}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big|\\
&\le \frac{\mu}{\Gamma(\gamma)}\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)\\
&\quad-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\\
&\quad+\int_{\tau_1}^{\tau_2}(\tau_1-s)^{\gamma-1}w(s)\Big(u(s)-a(s)w(s)-b(s)\int_0^{\Theta}k_*(x)\mathcal{N}(w(x))\Delta x\Big)\Delta s\Big|\\
&\le \frac{\mu}{\Gamma(\gamma)}\Big(UR+AR^2+BKNR\Theta\Big)\Big|\int_0^{\tau_2}\Big((\tau_2-s)^{\gamma-1}-(\tau_1-s)^{\gamma-1}\Big)\Delta s\\
&\quad+\int_{\tau_1}^{\tau_2}(\tau_1-s)^{\gamma-1}\Delta s\Big|\\
&\le \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_1^{\gamma}-\tau_2^{\gamma}+(\tau_1-\tau_2)^{\gamma}-(\tau_1-\tau_2)^{\gamma}\right|\\
&\le \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_2^{\gamma}-\tau_1^{\gamma}\right|.
\end{aligned}
$$

Thus, we get

$$
|\Lambda w(\tau_2)-\Lambda w(\tau_1)| \le \frac{\mu}{\Gamma(\gamma+1)}\Big(UR+AR^2+BKNR\Theta\Big)\left|\tau_2^{\gamma}-\tau_1^{\gamma}\right|. \tag{3.9}
$$

Inequality (3.9) is independent of w and approaches to 0, when $\tau_2 \to \tau_1$. Therefore, $\Lambda(\widetilde{\mathbb{B}}_R)$ is relatively compact. By Arzela-Ascoli theorem, it is compact. Subsequently, since Λ is continuous, the result follows from Schauder's fixed point theorem. This completes the proof. ☐

Theorem 3.2. *Under the considered assumptions, Equation* (1.1) *admits an unique solution if*

$$
0<\mu<\frac{\Gamma(\gamma+1)}{(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta)\,\Theta^{\gamma}}.
$$

Proof. Consider ζ and ζ_1 as two solutions of considered problem. Then, from (3.7), we have

$$
\|\Lambda\zeta-\Lambda\zeta_1\|_{\infty} \le \left(\frac{\mu U\Theta^{\gamma}}{\Gamma(\gamma+1)}+\frac{2\mu AW\Theta^{\gamma}}{\Gamma(\gamma+1)}+\frac{\mu B\left(WL_{\mathcal{N}}+N\right)K\Theta^{\gamma+1}}{\Gamma(\gamma+1)}\right)\|\zeta-\zeta_1\|_{\infty}. \tag{3.10}
$$

If one choose μ as

$$0<\mu<\frac{\Gamma(\gamma+1)}{\Big(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta\Big)\Theta^{\gamma}},$$

then (3.10) suggests that $\Lambda:\mathcal{X}\to\mathcal{X}$ is a contractive operator. Hence, by the Banach principle Λ has one and only one fixed point. That implies $\zeta=\zeta_1$, is the unique solution for (1.1)–(1.2). ☐

4. Numerical Examples

In this section two fractional order real time scale problems are studied to validate our theoretical outcomes.

Example 4.1. Consider the logistic equation

(4.1)

$$\begin{aligned}
{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{0.5}_{\theta}w(\theta)=&\frac{1}{100}w(\theta)\Big(2+\sin\left(\frac{\pi\theta}{2}\right)-w(\theta)-(2-\cos(\pi\theta))\int_{\theta_0}^{\Theta}2e^{-2s}w(s)\Delta s\Big),\\
{}^{\mathbb{T}}_{\theta_0}\mathcal{J}^{0.5}_{\theta}w(\theta_0)=&0.
\end{aligned}$$

Here $\gamma=0.5$, $u(\theta)=2+\sin(\frac{\pi\theta}{2})$, $a(\theta)=1$, $b(\theta)=2-\cos(\pi\theta)$, $k_*(s)=2$ and $\mathcal{N}(w(s))=e^{-2s}w(s)$. Since, in this problem $\mathbb{T}=\mathbb{R}$, therefore we take $[\theta_0,\Theta]\equiv[0,1]$.

Then we have the following estimates, $|u(\theta)|=|2+\sin(\frac{\pi\theta}{2})|\leq 2+1=3$, thus $U=3$. Following the same way, we have $B=3$, and $A=1,K=2$. We consider $W=4$, for this particular problem $N=4$. One can observe that

$$|\mathcal{N}(w(s))-\mathcal{N}(w_1(s))|=e^{-2s}|w(s)-w_1(s)|,$$

that implies for all $s\in[0,1]$, $\|\mathcal{N}(w(s))-\mathcal{N}(w_1(s))\|\leq\|w(s)-w_1(s)\|$. Consequently, $L_{\mathcal{N}}=1$. Thus, assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_2)$ are satisfied.

Finally, we calculate

$$\frac{\Gamma(\gamma+1)}{\Big(U+2AW+B(WL_{\mathcal{N}}+N)K\Theta\Big)\Theta^{\gamma}}=\frac{\Gamma(0.5+1)}{\Big(3+2.1.1+3.(4.1+4).2.1\Big)\cdot(1)^{0.5}}=0.0167,$$

thus, we have $\mu=0.01<0.0167$. Consequently, by Theorem 3.2, Equation (4.1) admits an unique solution.

Example 4.2. Consider the following equation on $[0,0.5]$, on the general time scale $\mathbb{T}$

$$\begin{aligned}
{}^{\mathbb{T}}_{\theta_0}\mathcal{D}^{0.25}_{\theta}w(\theta)=&w(\theta)\Big(\frac{1+\cos(\theta)}{25}-\frac{w(\theta)}{40e^{\theta^2+3}}-\frac{\theta}{20}\int_{\theta_0}^{\Theta}\frac{s}{s+e^s}\cdot\frac{\sin(w(s))}{e^{s^2}+5}\Delta s\Big),\\
{}^{\mathbb{T}}_{\theta}\mathcal{J}^{0.25}_{\theta}w(\theta_0)=&0.
\end{aligned}\tag{4.2}$$

Here, $u(\theta)=\frac{1+\cos(\theta)}{25}$, $a(\theta)=\frac{1}{40e^{\theta^2+3}}$, $b(\theta)=\frac{\theta}{20}$, $k_*(s)=\frac{s}{s+e^s}$, $\mathcal{N}(w(s))=\frac{\sin(w(s))}{e^{s^2}+5}$, $\gamma=0.25$ and $\mu=1$.

Thus we have, $U = \frac{2}{25}$, $A = \frac{1}{40}$, $B = \frac{1}{20}$, $K = 1$, $L_{\mathbb{N}} = \frac{1}{e^5}$. One can easily check that the concerned assumptions are fulfilled. For $W = 2$ and $N = 1$ we have the following estimates

$$\frac{\Gamma(\gamma+1)}{(U+2AW+B(WL_{\mathbb{N}}+N)K\Theta)\Theta^{\gamma}} = \frac{\Gamma(0.25+1)}{\left(\frac{2}{25}+2\frac{1}{40}\cdot 2+\frac{1}{20}(2\frac{1}{e^5}+1)\cdot 1\cdot \frac{1}{2}\right)(\frac{1}{2})^{0.25}} = 5.594,$$

thus we have $\mu = 1 < 5.594$. Consequently, by Theorem 3.2, the Equation (4.2) admits an unique solution.

5. Conclusion

This article deals with the investigation of existence and uniqueness of solution for the fractional order logistic integro differential equation in an arbitrary time scale. As per our knowledge, the considered class of equation has not yet been studied in theoretical point of view. The Banach contraction principle and Schauder's fixed point theory have been adopted to develop the theoretical findings. Two examples are considered to validate the main results.

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[1]Department of Mathematics,
VIT-AP University,
Inavolu, Amaravati, 522237, India
Email address: `nimaisarkar298@gmail.com`

[2]Department of Mathematics,
National Institute of Technology,
Silchar, Assam 788010, India
Email address: `mausumi@math.nits.ac.in`

[3]Department of Mathematics,
Royal Global University,
Betkuchi, Guwahati 781035, Assam, India
Email address: `nayan0507@gmail.com`

[4]Department of Mathematics,
Gauhati University,
Guwahati 781014, Assam, India
Email address: `bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in`

[*]Corresponding Author

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DOUBLE TOTAL DOMINATION NUMBER ON SOME CHEMICAL NANOTUBES

ANA KLOBUČAR BARIŠIĆ[1] AND ANTOANETA KLOBUČAR[2]

Abstract. Suppose G is a graph with the vertex set $V(G)$. A set $D \subseteq V(G)$ is a total k-dominating set if every vertex $v \in V(G)$ has at least k neighbours in D. The total k-domination number $\gamma_{kt}(G)$ is the size of the smallest total k-dominating set. When $k = 2$ the total 2-dominating set is referred to as a double total dominating set. In this work we compute the exact values for double total domination number on H-phenylenic nanotubes $HPH(m, n)$, $m, n \geq 2$ and H-naphtalenic nanotubes $HN(m, n)$, $n = 2k$, $m, n \geq 2$. As all vertices have a degree 2 or 3, there is no total k-domination for $k \geq 3$ for H-phenylenic and H-naphtalenic nanotubes, and the double total domination is the maximum possible.

1. Introduction

Graph dominations hold significance due to their presence in diverse applications like dominating queens, computer networks, school bus route planning, social network issues, and chemistry [2, 6, 8, 9, 13–17]. In representing chemical structures as graphs, atoms correspond to vertices and chemical bonds to edges. Owing to this resemblance, numerous physical and chemical attributes of molecules are linked to graph-theoretical constants. The total (double) domination number serves as an example of such an invariant [2–4, 6–8, 11, 14, 15].

We explore double total dominations on H-phenylenic nanotube $HPH(m, n)$, $m, n \geq 2$ and H-naphtalenic nanotube $HN(m, n)$, $n = 2k$, $m, n \geq 2$. Furthermore, we give exact values for the double total domination number on mentioned graphs.

Key words and phrases. Total domination, double total domination, hexagonal systems, molecular graph, H-phenylenic nanotube, H-naphtalenic nanotube.

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H-phenylenic $HPH(m,n)$, $m,n \geq 2$ and H-naphtalenic $HN(m,n)$, $n = 2k, m,n \geq 2$ are carbon nanotubes [16]. Carbon nanotubes are molecular cylinders used for fabrication of nanoscale devices by providing molecular probes, pipes, wires, bearings and springs. Because of their substantiality and stiffness, they have many potential applications in different technologies.

Currently, there are only a limited number of publications on total and double total domination on chemical graphs [2,6,10,12,14,15]. This work is in a close relationship with our previous papers [10,12], in which we also study double total domination, but on a hexagonal grid and pyrene network.

In addition to this introduction, the paper is structured as follows. Section 2 provides an overview of the total and double domination, dominating sets, and hexagonal systems. Section 3 provides the double total domination number γ_{2t} on H-phenylenic nanotube $HPH(m,n)$, $m,n \geq 2$. Section 4 provides the double total domination number on H-naphtalenic nanotube $HN(m,n)$, $n = 2k$, $m,n \geq 2$.

2. Preliminaries

Consider a graph G with vertex set V(G). A set $D \subset V(G)$ is a dominating set of G if every vertex y in $V(G) \setminus D$ has a neighbour in D. The domination number $\gamma(G)$ is the size of the smallest dominating set. Total domination is the stronger version of domination, where a set $D \subset V(G)$ is a total dominating set of G if every vertex y in $V(G)$ has a neighbour in D. The total domination number $\gamma_t(G)$ is the size of the smallest total dominating set.

A set $D \subseteq V(G)$ is a k-dominating set, if every vertex $v \in V(G) \setminus D$ has at least k neighbours in D. The k-domination number $\gamma_k(G)$ is the size of the smallest k-dominating set. A set $D \subseteq V(G)$ is a total k-dominating set if every vertex $v \in V(G)$ has at least k neighbours in D. In such case, it must be $k \leq \delta(G)$ where $\delta(G)$ is the minimum degree of vertices in G and $|D| \geq k+1$. The total k-domination number $\gamma_{kt}(G)$ is the size of the smallest total k-dominating set. A double total dominating set is also called the total 2-dominating set.

Each vertex in H-phenylenic nanotube and H-naphtalenic nanotube is either of degree 2 or of degree 3. As a result, there is no total k-domination for $k \geq 3$ on H-phenylenic and H-naphtalenic nanotubes.

3. Double Total Domination Number of H-phenylenic Nanotubes

H-phenylenic nanotubes $HPH(m,n)$ are molecular graphs that are covered by C_6, C_4 and C_8 [1]. We denote by $HPH(m,n)$ H-phenylenic nanotube with m hexagonal rows and n hexagonal columns. The number of vertices in H-phenylenic nanotube $HPH(m,n)$ is $6mn$. See Figure 1 and Figure 2.

Lemma 3.1. $\gamma_{2t}(HPH(2,2)) = 20$.

Proof. Since we are considering double total domination, every vertex adjacent to a vertex with degree 2 must be included in any double total dominating set.

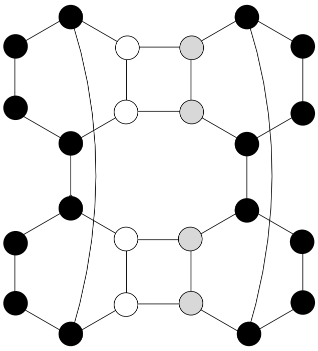

FIGURE 1. A double total dominating set in $HPH(2,2)$

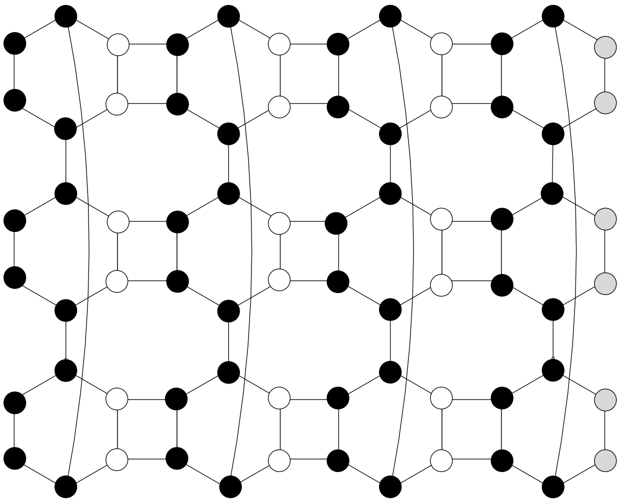

FIGURE 2. A double total dominating set in $HPH(3,4)$

Let T be double total dominating set on $HPH(2,2)$. All vertices from $HPH(2,2)$ which are not on square must be in T because they are adjacent to at least one vertex with degree 2. See Figure 1. Mentioned vertices are in black color. There is 16 such vertices on $HPH(2,2)$, 4 on each hexagonal ring.

It follows that $\gamma_{2t}(HPH(2,2)) \geq 16$. If there were only this 16 vertices in the double total dominating set T, vertices on both squares would be total dominated only once. To double total dominate vertices on one square we need at least 2 vertices at each square. See Figure 1. Additional vertices are in gray color. Thus,

$$\gamma_{2t}(HPH(2,2)) \geq 16 + 4 = 20.$$

But, it can be easily checked that 20 vertices can double total dominate all vertices on $HPH(2,2)$, hence $\gamma_{2t}(HPH(2,2)) \leq 20$. □

The following theorem is well known see [5].

Theorem 3.1. *Let* $k \in \mathbb{N}$ *and* $G = (V, E)$ *be a graph of order* n *with minimum degree* $\delta(G) \geq k$. *Then* $\gamma_{kt}(G) \geq \frac{kn}{\Delta(G)}$ *where* $\Delta(G)$ *is maximum degree.*

Theorem 3.2. *For H-phenylenic nanotube* $HPH(m, n)$, $m, n \geq 2$ *it holds*

$$\gamma_{2t}(HPH(m, n)) = 4mn + 2m.$$

Proof. From each hexagonal column of $HPH(m, n)$ we will take 4 vertices form each hexagon and denote these vertices with T_1. See Figure 2. Vertices belonging to T_1 are in black color. $|T_1| = 4mn$ as there are n hexagonal columns with m hexagons.

Set T_1 double total dominate all vertices on $HPH(m, n)$, except gray vertices on the last column see Figure 2. Gray vertices are total dominated only once. Also, gray vertices are adjacent to some vertex of degree 2. It follows that all of them must be in the double total dominating set. There are m rows, each containg 2 gray vertices, so we need at least $2m$ vertices to double total dominate all vertices on $HPH(m, n)$. It follows $\gamma_{2t}(HPH(m, n)) \leq 4mn + 2m$.

From Theorem 3.1. follows that $\gamma_{2t}(HPH(m, n)) \geq \frac{2 \cdot 6mn}{3} = 4mn$. But $|T_1| = 4mn$ and its vertices double total dominate all vertices except vertices from the last column. Moreover, the dominated vertices are double total dominated with only 2 vertices from T_1 which is minimal. See Figure 2. It follows that we need at least $2m$ more vertices to double total dominate remaining undominated vertices. Hence, $\gamma_{2t}(HPH(m, n)) \geq 4mn + 2m$. ☐

4. DOUBLE TOTAL DOMINATION NUMBER OF H-NAPHTALENIC NANOTUBES

H-naphtalenic nanotubes are molecular graphs that are obtained by the sequence C_6, C_6, C_4, C_6 and $C_6, \ldots, C_6, C_6, C_4, C_6, C_6$ and the repeat unit C_6, C_6, C_4 [18]. See Figure 3 and Figure 4. We denote by $HN(m, n)$, $n = 2k$ H-naphtalenic nanotube with m hexagonal rows and n hexagonal columns. The number of vertices in H-naphtalenic nanotube $HPH(m, n)$ is $5mn$.

A zigzag line in HN(m, n) that does not contain vertical edges is referred to as a horizontal zigzag line The horizontal zigzag line of $HN(m, n)$ are denoted by L_i, $1 \leq i \leq 2m$. For all zigzag lines on $HN(m, n)$ it holds $|L_i| = 5n/2$. See Figure 3.

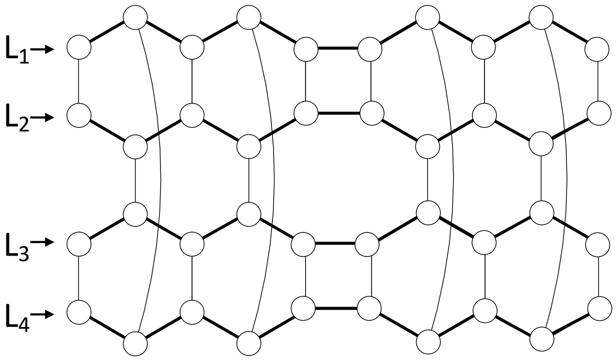

FIGURE 3. Zig-zag lines on $HN(2,4)$

Theorem 4.1. *For H-naphtalenic nanotubes $HN(m,n), n=2k$ it holds*

$$\gamma_{2t}(HN(m,n)) = \begin{cases} 2m\left(2\left\lfloor\frac{5n}{6}\right\rfloor+2\right), & \textit{if } n\equiv 2 \bmod 3, n\equiv 1 \bmod 3,\\ 2m\left(2\left\lfloor\frac{5n}{6}\right\rfloor+1\right), & \textit{if } n\equiv 0 \bmod 3. \end{cases}$$

Proof. We will consider three cases.

(a) Case $n\equiv 2 \bmod 3, n=2k$.

Let T_i be the subset of the double total dominating set T on L_i of $HN(m,n)$, $1\leq i\leq 2m$. For each $i, 1\leq i\leq 2m$, it holds

$$T_i=\left\{v_{i,1+3j}, v_{i,2+3j}, j=0,\dots,\left\lfloor\frac{5n}{6}\right\rfloor-1\right\}\cup\left(i,\frac{5n}{2}-1\right),\left(i,\frac{5n}{2}\right).$$

Then

$$|T_i|=2\left\lfloor\frac{5n}{6}\right\rfloor+2 \quad\text{and}\quad |T|=2m\left(2\left\lfloor\frac{5n}{6}\right\rfloor+2\right).$$

T double total dominates all vertices on $HN(m,n)$. See Figure 4 for the double total dominating set on $HN(3,8)$.

Thus for $n\equiv 2 \bmod 3, n=2k$ follows

$$\gamma_{2t}(HN(m,n))\leq|T|.$$

But also, each vertex on $HN(m,n)$ is double dominated by exactly 2 vertices, which is minimal. So,

$$\gamma_{2t}(HN(m,n))\geq|T|.$$

(b) Case Let $n\equiv 1 \bmod 3, n=2k$.

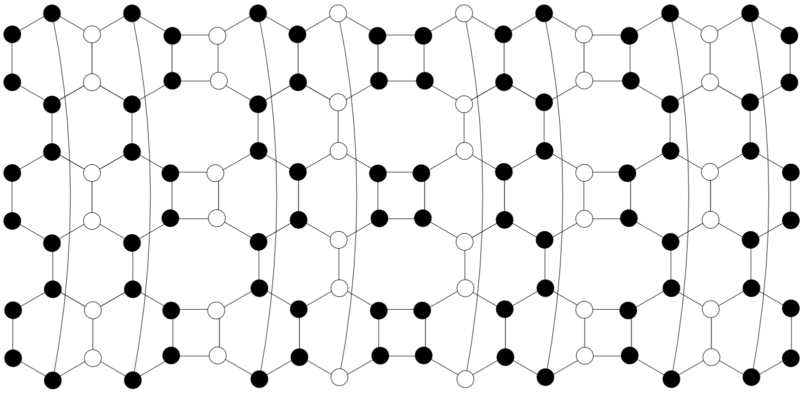

Figure 4. A double total dominating set in $HN(3,8)$

Again, let T_i be the subset of the double total dominating set T on the L_i of $HN(m,n)$, $1 \leq i \leq 2m$. For each $i, 1 \leq i \leq 2m$, it holds

$$T_i = \left\{v_{i,1+3j}, v_{i,2+3j}, j = 0, \dots, \left\lfloor \frac{5n}{6} \right\rfloor - 1\right\} \cup \left(i, \frac{5n}{2} - 1\right), \left(i, \frac{5n}{2}\right).$$

Then

$$|T_i| = 2\left\lfloor \frac{5n}{6} \right\rfloor + 2 \quad \text{and} \quad |T| = 2m\left(2\left\lfloor \frac{5n}{6} \right\rfloor + 2\right).$$

T double total dominates all vertices on $HN(m,n)$. See Figure 5 for the double total dominating set on $HN(2,10)$.

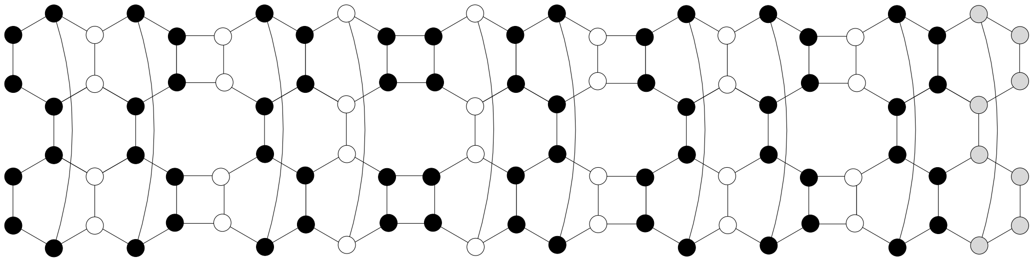

Figure 5. A double total dominating set in $HN(2,10)$

Hence, for $n \equiv 1 \bmod 3, n = 2k$ it follows

$$\gamma_{2t}(HN(m,n)) \leq |T|.$$

Further, note that

$$T_i \setminus \left\{\left(i, \frac{5n}{2} - 1\right), \left(i, \frac{5n}{2}\right)\right\}, \quad 1 \leq i \leq 2m,$$

double total dominate all verices on L_i except $(i, 5n/2-1), (i, 5n/2)$, $1 \leq i \leq 2m$ and each vertex is double total dominated by two vertices, which is minimal. In order to double total dominate also $(i, 5n/2-1), (i, 5n/2)$, $1 \leq i \leq 2m$ they must be in any double total dominatig set as they are adjacent to the some vertex of degree 2.

So, $\gamma_{2t}(HN(m,n)) \geq |T|$.

(c) Case $n \equiv 0 \bmod 3, n = 2k$

Again, let T_i be the subset of the double total dominating set T on the L_i of $HN(m,n)$, $1 \leq i \leq 2m$. For each $i, 1 \leq i \leq 2m$, it holds

$$T_i = \left\{v_{i,1+3j}, v_{i,2+3j}, j = 0, \dots, \left\lfloor \frac{5n}{6} \right\rfloor - 1\right\} \cup \left(i, \frac{5n}{2}\right).$$

Then

$$|T_i| = 2\left\lfloor \frac{5n}{6} \right\rfloor + 1 \quad \text{and} \quad |T| = 2m\left(2\left\lfloor \frac{5n}{6} \right\rfloor + 1\right).$$

T double total dominates all vertices on $HN(m,n)$. See Figure 6 for the double total dominating set on $HN(2,6)$.

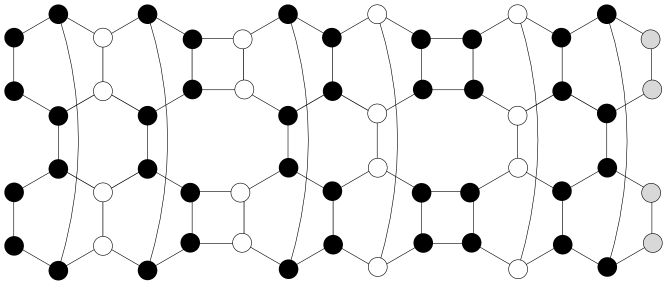

FIGURE 6. A double total dominating set in $HN(2,6)$

Therefore, for $n \equiv 0 \bmod 3, n = 2k$ follows

$$\gamma_{2t}(HN(m,n)) \leq |T|.$$

Further note that

$$T_i \setminus \left(i, \frac{5n}{2}\right), \quad 1 \leq i \leq 2m,$$

double total dominate all vertices on L_i except $(i, 5n/2), 1 \leq i \leq 2m$ and each vertex is double total dominated by two vertices, which is minimal. In order to double total dominate also $(i, 5n/2), 1 \leq i \leq 2m$ they must be included in any double total dominating set as they are adjacent to some vertex of degree 2.

So, $\gamma_{2t}(HN(m,n)) \geq |T|$. □

5. Conclusions

We have determined the exact values for the double total domination number on the H-phenylenic nanotube $HPH(m,n)$, $m,n \geq 2$ and the H-naphtalenic nanotube $HN(m,n)$, $n = 2k$, $m,n \geq 2$. In our future work, we plan to study the total and the double total domination on some other chemical graphs.

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[1]Faculty of Mechanical Engineering and Naval Architecture,
University of Zagreb,
Ivana Lučića 5, 10000 Zagreb, Croatia
Email address: `aklobucar@fsb.hr`

[2]Faculty of Economics,
Josip Juraj Strossmayer University of Osijek,
Trg Ljudevita Gaja 7, 31000 Osijek, Croatia
Email address: `aneta@efos.hr`

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ON THE GENERALIZED LEONARDO QUATERNIONS AND ASSOCIATED SPINORS

MUNESH KUMARI[1,2], KALIKA PRASAD[1,2*], HRISHIKESH MAHATO[1],
AND PAULA MARIA MACHADO CRUZ CATARINO[3]

ABSTRACT. In this paper, we introduce and study a new family of sequences called the generalized Leonardo spinors by defining a linear correspondence between the generalized Leonardo quaternions and spinors. We start with defining the generalized Leonardo quaternions and then present their some important properties such as Binet type formula, Catalan's identity, d'Ocagne's identity, series sums, etc. We give some interrelations of these quaternions with the Fibonacci and Lucas quaternions. Then, we present the generating functions, sum formulae, various well-known identities, etc. for the Leonardo spinors and show their connection with the Fibonacci and Lucas spinors.

1. INTRODUCTION

At the beginning of the 13th century, Leonardo of Pisa solved the famous rabbit growth problem based on idealized assumptions and that solution became a fascinating recursive integer sequence famed as the Fibonacci sequence [11]. For $n \geq 0$, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given as $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$, $F_1 = 1$.

Recently, Catarino and Borges [2] studied the recurrence relations and various properties for the Leonardo numbers, in continuation Alp and Koçer [22] investigated their interesting properties. Kuhapatanakul and Juthamas [12] extended this study to the generalized Leonardo numbers along with their matrix representation and also in [20] the authors studied the matrix representation of Leonardo numbers. Karatas [10] defined the complex Leonardo numbers and studied their combinatorial properties.

Key words and phrases. Leonardo number, generalized Leonardo quaternions, generalized Leonardo spinors, Binet's formula, generating function, series sum.

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İşbilir et al. [23] investigated the Pauli-Leonardo quaternions. Some recent developments on Leonardo numbers, their generalizations and interesting properties can be seen in [3, 5, 14–19, 22]. Here we restate some of them.

For $k \in \mathbb{Z}^+$, the generalized Leonardo numbers $\{\mathcal{L}_{k,n}\}$ are defined [12] by the recurrence relation

$$\mathcal{L}_{k,n+2} = \mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k, \quad n \geq 0, \quad \text{with } \mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1.$$

In negative subscript, these numbers are given as $\mathcal{L}_{k,-n} = (-1)^n(\mathcal{L}_{k,n-2} + k) - k$.

The Binet type formula for the generalized Leonardo numbers is

$$\mathcal{L}_{k,n} = (k+1)\Big(\frac{\lambda^{n+1} - \xi^{n+1}}{\lambda - \xi}\Big) - k, \tag{1.1}$$

where $\lambda = (1+\sqrt{5})/2$ and $\xi = (1-\sqrt{5})/2$.

In non-homogeneous form, the generalized Leonardo numbers satisfy the third order recurrence relation:

$$\mathcal{L}_{k,n+1} = 2\mathcal{L}_{k,n} - \mathcal{L}_{k,n-2}.$$

These numbers are associated with the Fibonacci numbers by the relation

$$\mathcal{L}_{k,n} = (k+1)F_{n+1} - k.$$

In 1963, Horadam [8] defined quaternion sequences with components as Fibonacci and Lucas numbers. The Fibonacci quaternion Q_n is defined as

$$Q_n = F_n e_0 + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = (F_n, F_{n+1}, F_{n+2}, F_{n+3}), \quad n \geq 0, \tag{1.2}$$

and Lucas quaternion T_n as

$$T_n = L_n e_0 + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = (L_n, L_{n+1}, L_{n+2}, L_{n+3}), \quad n \geq 0,$$

where $\{e_0 = 1, e_1, e_2, e_3\}$ is the quaternion basis satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2.$$

Iyer [9] studied the relations between Fibonacci and Lucas quaternions. Further, Halici [7] obtained Binet's formula, generating functions and finite sum of these quaternions. The Binet's formulae for the Fibonacci and Lucas quaternions are given, respectively, by

$$Q_n = \frac{\lambda^*\lambda^n - \xi^*\xi^n}{\sqrt{5}} \quad \text{and} \quad T_n = \lambda^*\lambda^n + \xi^*\xi^n,$$

where $\lambda^* = 1 + \lambda e_1 + \lambda^2 e_2 + \lambda^3 e_3$ and $\xi^* = 1 + \xi e_1 + \xi^2 e_2 + \xi^3 e_3$.

For Fibonacci quaternions, Cassini's identity is given by

$$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n(2Q_1 - 3e_3), \tag{1.3}$$

and Catalan's identity is

$$Q_{n-r}Q_{n+r} - Q_n^2 = (-1)^{n-r+1}(2F_rQ_r - 3F_{2r}e_3). \tag{1.4}$$

Theorem 1.1 (Sum formulae). *For $n \geq 0$, we have*

(a) $\sum_{j=0}^{n} Q_j = Q_{n+2} - Q_1$;
(b) $\sum_{j=0}^{n} Q_{2j} = Q_{2n+1} - (1,0,1,1)$;
(c) $\sum_{j=0}^{n} Q_{2j+1} = Q_{2n} - Q_0$.

This paper will relate a new sequence of the generalized Leonardo quaternions and spinors that was motivated by a recent study of spinors with the Fibonacci numbers by Erişir and Güngör [6] and with the k-Fibonacci numbers by Kumari et al. [13].

This paper is structured as follows. In Section 2 we present a new quaternions sequence - the generalized Leonardo quaternions and study some properties of them. Section 3 is dedicated to the introduction of the generalized Leonardo spinors by defining a correspondence between the generalized Leonardo quaternions and spinors. We start the section recalling the important results involving spinors and we finish showing the relationship among the generalized Leonardo spinors and Fibonacci and Lucas spinors.

2. The Generalized Leonardo Quaternions

In this section, we first define the generalized Leonardo quaternions and then investigate their algebraic properties which we need later to prove some identities for spinors.

Definition 2.1. For $n \geq 0$, nth generalized Leonardo quaternion $\mathcal{QL}_{k,n}$ is defined as

$$\mathcal{QL}_{k,n} = \mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+1}e_1 + \mathcal{L}_{k,n+2}e_2 + \mathcal{L}_{k,n+3}e_3.$$

And the conjugate $\overline{\mathcal{QL}}_{k,n}$ is defined as

$$\overline{\mathcal{QL}}_{k,n} = \mathcal{L}_{k,n}e_0 - \mathcal{L}_{k,n+1}e_1 - \mathcal{L}_{k,n+2}e_2 - \mathcal{L}_{k,n+3}e_3. \tag{2.1}$$

The above defined generalized Leonardo quaternions can be written in recurrence form as

$$\mathcal{QL}_{k,n+2} = \mathcal{QL}_{k,n+1} + \mathcal{QL}_{k,n} + k\gamma, \quad \text{where } \gamma = e_0 + e_1 + e_2 + e_3. \tag{2.2}$$

From Definition 2.1 and (2.1), we get

$$\mathcal{QL}_{k,n}\overline{\mathcal{QL}}_{k,n} = \overline{\mathcal{QL}}_{k,n}\mathcal{QL}_{k,n} = (k+1)[3(k+1)F_{2n+5} - 2kL_{n+4}] + 4k^2.$$

Similar to the generalized Leonardo numbers, the generalized Leonardo quaternions can also be extended in negative indices given in the following definition.

Definition 2.2. For $n > 0$, the generalized Leonardo quaternions with negative subscript $\mathcal{QL}_{k,-n}$ are defined as

$$\mathcal{QL}_{k,-n} = (-1)^n \sum_{r=0}^{3} (-1)^r (\mathcal{L}_{k,n-2-r} + k)e_r - k\gamma.$$

The conjugate of $\mathcal{QL}_{k,-n}$ is given as

$$\overline{\mathcal{QL}}_{k,-n} = (-1)^n(\mathcal{L}_{k,n-2} + k) + (-1)^{n+1} \sum_{r=1}^{3} (-1)^r (\mathcal{L}_{k,n-2-r} + k)e_r - k\overline{\gamma},$$

where $\overline{\gamma} = e_0 - e_1 - e_2 - e_3$.

Now we give some relations among generalized Leonardo quaternion, Fibonacci quaternion and Lucas quaternion in the next theorem.

Theorem 2.1. *For $n \geq 0$, the following identities are verified:*

(a) $\mathcal{QL}_{k,n+3} = 2\mathcal{QL}_{k,n+2} - \mathcal{QL}_{k,n}$;
(b) $\mathcal{QL}_{k,n} = (k+1)Q_{n+1} - k\gamma$;
(c) $\mathcal{QL}_{k,-n} = 2\mathcal{QL}_{k,-n+2} - \mathcal{QL}_{k,-n+3}$;
(d) $\mathcal{QL}_{k,-n} = (k+1)Q_{-n+1} - k\gamma$;
(e) $\mathcal{QL}_{k,n} + \mathcal{QL}_{k,-n} = \begin{cases} (k+1)F_nT_1 - 2k\gamma, & n \text{ is odd,} \\ (k+1)L_nQ_1 - 2k\gamma, & n \text{ is even;} \end{cases}$
(f) $\mathcal{QL}_{k,-n} + \overline{\mathcal{QL}}_{k,-n} = 2\mathcal{L}_{k,-n}$.

Proof. The first identity follows from expression (2.2) and the second identity uses $\mathcal{L}_{k,n} = (k+1)F_{n+1} - k$ and (1.2). To prove third and fourth statements, we use Definition 2.2 and definition of the Fibonacci quaternions, respectively. Fifth and sixth identities follow from the definitions of $\mathcal{QL}_{k,-n}$ and its conjugate. ☐

In the next theorem, we present the Binet type formula for the generalized Leonardo quaternions and with the help of that we investigate some well known identities and properties of these quaternions.

Theorem 2.2 (Binet type formula)**.** *For $n \geq 0$, we have*

$$\mathcal{QL}_{k,n} = \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*\Big) - k\gamma,$$

where $\lambda^* = 1 + \lambda e_1 + \lambda^2 e_2 + \lambda^3 e_3$, $\xi^* = 1 + \xi e_1 + \xi^2 e_2 + \xi^3 e_3$ and $\gamma = e_0 + e_1 + e_2 + e_3$.

Proof. Using (2.1) and Binet's formula (1.1), we write

$$\begin{aligned}\mathcal{QL}_{k,n} &= \sum_{r=0}^{3} \mathcal{L}_{k,n+r}e_r,\\ &= \sum_{r=0}^{3} \frac{(k+1)(\lambda^{n+r+1} - \xi^{n+r+1}) - \sqrt{5}k}{\sqrt{5}}e_r,\\ &= (k+1)\sum_{r=0}^{3} \frac{(\lambda^{n+r+1} - \xi^{n+r+1})}{\sqrt{5}}e_r - k\sum_{r=0}^{3} e_r\\ &= \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\sum_{r=0}^{3}\lambda^r e_r - \xi^{n+1}\sum_{r=0}^{3}\xi^r e_r\Big) - k\sum_{r=0}^{3} e_r,\\ &= \frac{k+1}{\sqrt{5}}\Big(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*\Big) - k\gamma.\end{aligned}$$

☐

Theorem 2.3. *For $n \geq 0$, the following identities are verified:*

(a) $\mathcal{QL}_{k,n} = \frac{k+1}{2}\Big(T_{n+2} - Q_{n+2}\Big) - k\gamma$;

(b) $\mathcal{QL}_{k,n} = \frac{k+1}{5}\Big(T_n + T_{n+2}\Big) - k\gamma$.

Proof. (a) Using the Binet's formula of Fibonacci and Lucas quaternions, we have

$$\begin{aligned} T_{n+2} - Q_{n+2} &= \Big(\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}\Big) - \Big(\frac{\lambda^*\lambda^{n+2} - \xi^*\xi^{n+2}}{\sqrt{5}}\Big) \\ &= \frac{1}{\sqrt{5}}(\lambda^*\lambda^{n+2}(\sqrt{5}-1) + \xi^*\xi^{n+2}(\sqrt{5}+1)) \\ &= \frac{1}{\sqrt{5}}(-2\xi\lambda^*\lambda^{n+2} + 2\lambda\xi^*\xi^{n+2}) \\ &= \frac{2}{\sqrt{5}}(-\lambda\xi(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*)) \\ &= \frac{2}{\sqrt{5}}(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*) \\ &= 2\Big(\frac{\mathcal{QL}_{k,n} + k\gamma}{k+1}\Big). \end{aligned}$$

Thus, on simplification

$$\mathcal{QL}_{k,n} = \frac{k+1}{2}\Big(T_{n+2} - Q_{n+2}\Big) - k\gamma.$$

(b) Similar to (a), we have

$$\begin{aligned} T_n + T_{n+2} &= (\lambda^*\lambda^n + \xi^*\xi^n) + (\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}) \\ &= \lambda\xi\Big(\frac{\lambda^*\lambda^n + \xi^*\xi^n}{\lambda\xi}\Big) + (\lambda^*\lambda^{n+2} + \xi^*\xi^{n+2}) \\ &= -\lambda^{n+1}\lambda^*(\xi - \lambda) + \xi^{n+1}\xi^*(\xi - \lambda) \\ &= \sqrt{5}(\lambda^{n+1}\lambda^* - \xi^{n+1}\xi^*) \\ &= \sqrt{5}\Big(\frac{\sqrt{5}}{k+1}(\mathcal{QL}_{k,n} + k\gamma)\Big), \end{aligned}$$

as required. □

Theorem 2.4 (Catalan's identity)**.** *For $n, r \in \mathbb{N}$ such that $n \geq r$, we have*

$$\begin{aligned} \mathcal{QL}_{k,n-r}\mathcal{QL}_{k,n+r} - \mathcal{QL}_{k,n}^2 =& (k+1)^2[(-1)^{n-r+2}(2F_rQ_r - 3F_{2r}e_3)] \\ &+ k(k+1)(Q_{n+1} - Q_{n+1-r})\gamma \\ &+ k(k+1)\gamma(Q_{n+1} - Q_{n+1+r}). \end{aligned}$$

Proof. Using (b) of Theorem 2.1 and identity of (1.4), we have

$$\begin{aligned} &\mathcal{QL}_{k,n-r}\mathcal{QL}_{k,n+r} - \mathcal{QL}_{k,n}^2 \\ =&\Big((k+1)Q_{n-r+1} - k\gamma\Big)\Big((k+1)Q_{n+r+1} - k\gamma\Big) - \Big((k+1)Q_{n+1} - k\gamma\Big)^2 \end{aligned}$$

$$
\begin{aligned}
=&\Big((k+1)^2Q_{n-r+1}Q_{n+r+1}-(k+1)kQ_{n-r+1}\gamma-k(k+1)\gamma Q_{n+r+1}\\
&+k^2(\gamma)^2\Big)-\Big((k+1)^2Q_{n+1}^2+k^2(\gamma)^2-(k+1)kQ_{n+1}\gamma-(k+1)k\gamma Q_{n+1}\Big)\\
=&(k+1)^2(Q_{n+1-r}Q_{n+1+r}-Q_{n+1}^2)+k(k+1)(Q_{n+1}-Q_{n+1-r})\gamma\\
&+k(k+1)\gamma(Q_{n+1}-Q_{n+1+r})\\
=&(k+1)^2[(-1)^{n-r+2}(2F_rQ_r-3F_{2r}e_3)]\\
&+k[(\mathcal{QL}_{k,n}-\mathcal{QL}_{k,n-r})\gamma-\gamma(\mathcal{QL}_{k,n}-\mathcal{QL}_{k,n+r})]. \qquad \square
\end{aligned}
$$

If we substitute $r=1$ in the above identity, the Cassini's identity for Leonardo quaternions $\mathcal{QL}_{k,n}$ is obtained which is stated in the next theorem.

Theorem 2.5 (Cassini's identity). *For any natural number n, we have*

$$
\begin{aligned}
&\mathcal{QL}_{k,n-1}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,n}^2\\
=&(k+1)^2[(-1)^{n+1}(2Q_1-3e_3)]+k[\mathcal{QL}_{k,n-2}\gamma+\gamma\mathcal{QL}_{k,n-1}+2k\overline{\gamma}].
\end{aligned}
$$

Theorem 2.6 (d'Ocagne's identity). *For $n,r\in\mathbb{N}$ such that $n\geq r$, we have*

$$
\begin{aligned}
&\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}\\
=&(k+1)^2[(-1)^{r+1}(F_{n-r}T_0+L_{n-r}(Q_0-3e_3))]+k(k+1)[Q_r\gamma-\gamma Q_n].
\end{aligned}
$$

Proof. Using identity (2) of Theorem 2.1, we have

$$
\begin{aligned}
\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}=&\Big((k+1)Q_{r+1}-k\gamma\Big)\Big((k+1)Q_{n+2}-k\gamma\Big)\\
&-\Big((k+1)Q_{r+2}-k\gamma\Big)\Big((k+1)Q_{n+1}-k\gamma\Big)\\
=&(k+1)^2(Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1})\\
&-k(k+1)[(Q_{r+1}-Q_{r+2})\gamma+\gamma(Q_{n+2}-Q_{n+1})]\\
=&(k+1)^2(Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1})\\
&+k(k+1)[Q_r\gamma-\gamma Q_n].
\end{aligned}
$$

Now, using the d'Ocagne's identity for the Fibonacci quaternions, i.e., $Q_{r+1}Q_{n+2}-Q_{r+2}Q_{n+1}=(-1)^{r+1}[F_{n-r}T_0+L_{n-r}(Q_0-3e_3)]$, we get

$$
\begin{aligned}
\mathcal{QL}_{k,r}\mathcal{QL}_{k,n+1}-\mathcal{QL}_{k,r+1}\mathcal{QL}_{k,n}=&(k+1)^2[(-1)^{r+1}(F_{n-r}T_0+L_{n-r}(Q_0-3e_3))]\\
&+k(k+1)[Q_r\gamma-\gamma Q_n]. \qquad \square
\end{aligned}
$$

Theorem 2.7. *The generating function for the generalized Leonardo quaternions is*

$$
\phi(t)=\frac{\mathcal{QL}_{k,0}-\mathcal{QL}_{k,-2}t-\mathcal{QL}_{k,-1}t^2}{1-2t+t^3}.
$$

Proof. Let the generating function for sequence $\{\mathcal{QL}_{k,n}\}_{n=0}^{+\infty}$ is $\phi(t)=\sum_{n=0}^{+\infty}\mathcal{QL}_{k,n}t^n$.

Now using relation (2) of Theorem 2.1, we write

$$\phi(t)=\sum_{n=0}^{+\infty}((k+1)Q_{n+1}-k\gamma)t^n=(k+1)\sum_{n=0}^{+\infty}Q_{n+1}t^n-k\gamma\sum_{n=0}^{+\infty}t^n.$$

Taking into account the generating function of the Fibonacci quaternions, i.e., $\sum_{n=0}^{+\infty}Q_{n+1}t^n=\frac{Q_1+Q_0t}{1-t-t^2}$, we have

$$\begin{aligned}\phi(t)&=(k+1)\frac{Q_0+Q_0t}{1-t-t^2}-k\gamma\frac{1}{1-t}\\&=\frac{(k+1)[Q_1-(Q_1-Q_0)t-Q_0t^2]-k\gamma(1-t-t^2)}{(1-t-t^2)(1-t)}\\&=\frac{\mathcal{QL}_{k,0}-\mathcal{QL}_{k,-2}t-\mathcal{QL}_{k,-1}t^2}{1-2t+t^3}.\end{aligned}$$ □

In the next theorem, we give the sum of finite terms of $\mathcal{QL}_{k,n}$ and also with even and odd subscripts.

Theorem 2.8 (Finite sum formulae). *For any positive integer n, we have*

(a) $\sum_{r=0}^{n}\mathcal{QL}_{k,r}=\mathcal{QL}_{k,n+2}-\mathcal{QL}_{k,1}-k(n+1)\gamma;$
(b) $\sum_{r=0}^{n}\mathcal{QL}_{k,2r}=\mathcal{QL}_{k,2n+1}-\mathcal{QL}_{k,-1}-k(n+1)\gamma;$
(c) $\sum_{r=1}^{n}\mathcal{QL}_{k,2r-1}=\mathcal{QL}_{k,2n}-\mathcal{QL}_{k,0}-kn\gamma.$

Proof. Using (b) of Theorem 2.1 and sum identity (a) of Theorem 1.1, we have

$$\begin{aligned}\sum_{r=0}^{n}\mathcal{QL}_{k,r}&=\sum_{r=0}^{n}((k+1)Q_{r+1}-k\gamma)\\&=(k+1)\sum_{r=0}^{n}Q_{r+1}-k\sum_{r=0}^{n}\gamma\\&=(k+1)(Q_{n+3}-Q_1-Q_0)-k\gamma(n+1)\\&=\mathcal{QL}_{k,n+2}-\mathcal{QL}_{k,1}-k(n+1)\gamma.\end{aligned}$$

Proof of identities (b) and (c) are similar using the finite sum formulae for even (b) and odd (c) of Theorem 1.1, respectively. □

Theorem 2.9. *For $n\in\mathbb{N}$, the following identity is verified.*

$$\sum_{r=0}^{n}\binom{n}{r}\mathcal{QL}_{k,r}=\mathcal{QL}_{k,2n}-k(2^n-1)\gamma. \tag{2.3}$$

Proof. The identity can be easily proved by making use of relation (2) of Theorem 2.1 and identity $\sum_{r=0}^{n}\binom{n}{r}Q_r=Q_{2n}$ for the Fibonacci quaternions. □

3. The Generalized Leonardo Spinors

Spinors are vectorial objects without their multilinear features for the mathematicians. A French Mathematician, Elie Cartan discovered the spinors first time. In 3-dimensional Euclidean space, there are many possible approaches to spinor theory like spinor ring algebra, Cartan's isotropic vectors, Clifford algebra, stereographic projection etc.

Here, we restate the definition given by the E. Cartan [1]. Consider a 3-dimensional space $\mathbb{C}^3$. Let $(x, y, z) \in \mathbb{C}^3$ be an isotropic vector. Then this vector can be associated with two numbers Ψ_1 and Ψ_2 given by

$$x = \Psi_1^2 - \Psi_2^2, \quad y = i(\Psi_1^2 + \Psi_2^2), \quad z = -2\Psi_1\Psi_2.$$

And solutions of these equations are

$$\Psi_1 = \pm\sqrt{\frac{x - iy}{2}} \quad \text{and} \quad \Psi_2 = \pm\sqrt{\frac{-x - iy}{2}}.$$

Thus spinor is the two-dimensional complex vectors described as

$$\Psi = (\Psi_1, \Psi_2) \equiv \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}.$$

A different approach to spinors derived from Euler's theorem was presented by Vivarelli [21] in 1984. He studied quaternions and one-index spinors by defining a linear and injective correspondence between them.

The correspondence $\Phi : \mathbb{H} \to \mathbb{S}$ between the set of quaternions $\mathbb{H}$ and spinors $\mathbb{S}$ is defined as

$$\Phi(a + be_1 + ce_2 + de_3) = \begin{bmatrix} d + ia \\ b + ic \end{bmatrix} \equiv Q, \quad a + be_1 + ce_2 + de_3 = p \in \mathbb{H}.$$

And, the product of two quaternions (qp) associated to a spinor-matrix product is given by $qp \mapsto -i\hat{Q}P$, where P is the spinor corresponding to the quaternion q and $\hat{Q}$ is the square(complex) unitary matrix given as

$$\begin{bmatrix} d + ia & b - ic \\ b + ic & -d + ia \end{bmatrix}. \tag{3.1}$$

E. Cartan [1] introduced the spinor conjugate to Ψ given as

$$\tilde{\Psi} = iA\overline{\Psi},$$

where $\overline{\Psi}$ is complex conjugate of Ψ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Castillo [4] defined the mate of spinor Ψ as

$$\check{\Psi} = -A\overline{\Psi},$$

Recently, Erişir and Güngör [6] studied the spinors with Fibonacci and Lucas numbers components and obtained their properties. They defined the Fibonacci spinors sequence $\{S_n\}_{n\geq 0}$ as

$$S_0 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}, \quad S_1 = \begin{bmatrix} 3+i \\ 1+2i \end{bmatrix}, \quad S_{n+2} = S_{n+1} + S_n,$$

and Lucas spinors sequence $\{S_n^{'}\}_{n\geq 0}$ as

$$S_0^{'} = \begin{bmatrix} 2+2i \\ 1+i \end{bmatrix}, \quad S_1^{'} = \begin{bmatrix} 4+i \\ 1+3i \end{bmatrix}, \quad S_{n+2}^{'} = S_{n+1}^{'} + S_n^{'}.$$

The Binet type formulae for the Fibonacci and Lucas spinors are, respectively,

$$S_n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^n - \frac{1}{\sqrt{5}} \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^n$$

and

$$S_n^{'} = \begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix} \lambda^n + \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix} \xi^n.$$

Motivated by the work of Erişir and Güngör [6], we are extending this study to the generalized Leonardo numbers and introducing a new sequence of the generalized Leonardo spinors.

Let $\mathbb{L}$ and $\mathbb{S}$ denote the set of generalized Leonardo quaternions and set of spinors, respectively. Then the generalized Leonardo spinor $\mathfrak{L}_n$ given by a linear and injective correspondence $\Phi : \mathbb{L} \to \mathbb{S}$ is defined as

$$\Phi(\mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+1}e_1 + \mathcal{L}_{k,n+2}e_2 + \mathcal{L}_{k,n+3}e_3) = \begin{bmatrix} \mathcal{L}_{k,n+3} + i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} + i\mathcal{L}_{k,n+2} \end{bmatrix} \equiv \mathfrak{L}_n.$$

Spinor $\mathfrak{L}_n^*$ corresponding to the conjugate quaternion $\overline{\mathfrak{Q}\mathcal{L}_{k,n}}$ is given as

$$\mathfrak{L}_n^* = \begin{bmatrix} -\mathcal{L}_{k,n+3} + i\mathcal{L}_{k,n} \\ -\mathcal{L}_{k,n+1} - i\mathcal{L}_{k,n+2} \end{bmatrix}.$$

Definition 3.1. For $n \geq 0$, the sequence of generalized Leonardo spinors $\{\mathfrak{L}_n\}_{n\geq 0}$ is defined recursively as $\mathfrak{L}_{n+2} = \mathfrak{L}_{n+1} + \mathfrak{L}_n + k\mathfrak{J}$, where $\mathfrak{L}_0 = \begin{bmatrix} (2k+3)+i \\ 1+i(k+2) \end{bmatrix}$, $\mathfrak{L}_1 = \begin{bmatrix} (4k+5)+i \\ (k+2)+i(2k+3) \end{bmatrix}$ and $\mathfrak{J} = \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}$.

Lemma 3.1. *For generalized Leonardo spinors, the following identities are verified.*

$$\mathfrak{L}_{n+3} = 2\mathfrak{L}_{n+2} - \mathfrak{L}_n \quad \textit{and} \quad \mathfrak{L}_n = (k+1)S_{n+1} - k\mathfrak{J}. \tag{3.2}$$

In the next lemma, we present conjugates and mate of the generalized Leonardo spinors.

Lemma 3.2. *For generalized Leonardo spinors $\mathfrak{L}_n$, we have*

(a) *Complex Conjugate:* $\overline{\mathfrak{L}}_n = \begin{bmatrix} \mathcal{L}_{k,n+3} - i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} - i\mathcal{L}_{k,n+2} \end{bmatrix}$;

(b) *Spinor Conjugate:* $\widetilde{\mathfrak{L}}_n = \begin{bmatrix} \mathcal{L}_{k,n+2} + i\mathcal{L}_{k,n+1} \\ -\mathcal{L}_{k,n} - i\mathcal{L}_{k,n+3} \end{bmatrix}$;

(c) *Mate of Spinor:* $\check{\mathfrak{L}}_{k,n} = \begin{bmatrix} -\mathcal{L}_{k,n+1} + i\mathcal{L}_{k,n+2} \\ \mathcal{L}_{k,n+3} - i\mathcal{L}_{k,n} \end{bmatrix}$.

Proof. Using the definition of spinor conjugate and mate of spinor, above results can be easily established. □

Theorem 3.1 (Binet type formula)**.** *For $n \geq 0$, we have*

$$\mathfrak{L}_n = \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{n+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{n+1}\right) - k\mathfrak{J}. \tag{3.3}$$

Proof. We prove this theorem by induction on n. For $n = 1$, the R.H.S of (3.3) is

$$\mathfrak{L}_1 = (k+1)S_2 - k\mathfrak{J} = \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} (\lambda^5 - \xi^5) + i(\lambda^2 - \xi^2) \\ (\lambda^3 - \xi^3) + i(\lambda^4 - \xi^4) \end{bmatrix}\right) - k\mathfrak{J}.$$

Assume expression (3.3) is true for $n = m$, i.e.,

$$\mathfrak{L}_m = \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{m+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{m+1}\right) - k\mathfrak{J}.$$

By Definition 3.1 and taking into account the fact $\lambda^2 = \lambda + 1$ and $\xi^2 = \xi + 1$, we get

$$\begin{aligned}
\mathfrak{L}_{m+1} &= \mathfrak{L}_m + \mathfrak{L}_{m-1} + k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{m+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{m+1}\right) - k\mathfrak{J} \\
&\quad + \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{m} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{m}\right) - k\mathfrak{J} + k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}(\lambda^{m+1} + \lambda^m) - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}(\xi^{m+1} + \xi^m)\right) - k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{m}(\lambda + 1) - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{m}(\xi + 1)\right) - k\mathfrak{J} \\
&= \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{m+2} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{m+2}\right) - k\mathfrak{J}.
\end{aligned}$$

This completes the proof. □

By replacing n with $-n$ in Binet type formula (3.3), we extend the generalized Leonardo spinors in negative direction. Thus,

$$\mathfrak{L}_{-n} = \frac{k+1}{\sqrt{5}}\left(\begin{bmatrix} \lambda^3 + i \\ \lambda + i\lambda^2 \end{bmatrix}\lambda^{-n+1} - \begin{bmatrix} \xi^3 + i \\ \xi + i\xi^2 \end{bmatrix}\xi^{-n+1}\right) - k\mathfrak{J} = (k+1)S_{-n+1} - k\mathfrak{J}.$$

In next theorem, we state some relations among $\mathfrak{L}_n$, $\mathfrak{L}_n^*, \mathfrak{L}_{-n}$ and $\mathfrak{L}_{-n}^*$ by omitting their proofs, as proofs of these identities can be seen easily using their definitions.

Theorem 3.2. *For the generalized Leonardo spinors, we have*

(a) $\mathfrak{L}_{-n} = 2\mathfrak{L}_{-n+2} - \mathfrak{L}_{-n+3}$;

(b) $\mathfrak{L}_{-n} = (k+1)S_{-n+1} - k\mathfrak{J}$;

(c) $\mathfrak{L}_n + \mathfrak{L}_{-n} = \begin{cases} (k+1)L_nS_1 - 2k\mathfrak{J}, & n \text{ is even,} \\ (k+1)F_nS_1^{'} - 2\mathfrak{J}, & n \text{ is odd;} \end{cases}$

(d) $\mathfrak{L}_n + \mathfrak{L}_n^* = \begin{bmatrix} i2\mathcal{L}_{k,n} \\ 0 \end{bmatrix}$;

(e) $\mathfrak{L}_{-n} + \mathfrak{L}_{-n}^* = \begin{bmatrix} i2\mathcal{L}_{k,-n} \\ 0 \end{bmatrix}$.

Theorem 3.3. *For $n \geq 0$, the following relations among the generalized Leonardo, Fibonacci and Lucas spinors hold*

$$\mathfrak{L}_n = \frac{k+1}{2}\Big(S_{n+2}^{'} - S_{n+2}\Big) - k\mathfrak{J} \tag{3.4}$$

and

$$\mathfrak{L}_n = \frac{k+1}{5}\Big(S_n^{'} + S_{n+2}^{'}\Big) - k\mathfrak{J}. \tag{3.5}$$

Proof. By using relation $S_n^{'} = S_{n-1} + S_{n+1}$ in the R.H.S. of expression (3.4), we write

$$\frac{k+1}{2}\Big(S_{n+2}^{'} - S_{n+2}\Big) - k\mathfrak{J} = \frac{k+1}{2}\Big(2S_{n+1}\Big) - k\mathfrak{J} = \mathfrak{L}_n.$$

Similarly, (3.5) follows from the identity $5S_n = S_{n-1}^{'} + S_{n+1}^{'}$. □

Theorem 3.4. *The generating function for the generalized Leonardo spinors is*

$$f(t) = \frac{-1}{1-2t+t^2}\begin{bmatrix} [(k+2)t^2 + t - (2k+3)] - i[kt^2 - t + 1] \\ [t^2 - kt - 1] + i[t^2 + t - (k+2)] \end{bmatrix}.$$

Proof. Let $f(t) = \sum_{n=0}^{+\infty} \mathfrak{L}_n t^n$ be the ordinary generating function. Now consider the recurrence relation $\mathfrak{L}_{n+3} = 2\mathfrak{L}_{n+2} - \mathfrak{L}_n$. Then multiplying it by t^{n+3} and taking summation, we have

$$\begin{aligned} &\sum_{n=0}^{+\infty} \mathfrak{L}_{n+3}t^{n+3} - 2\sum_{n=0}^{+\infty} \mathfrak{L}_{n+2}t^{n+3} + \sum_{n=0}^{+\infty} \mathfrak{L}_n t^{n+3} = 0 \\ \Longrightarrow\ & (f(t) - \mathfrak{L}_0 - \mathfrak{L}_1 t - \mathfrak{L}_2 t^2) - 2t(f(t) - \mathfrak{L}_0 - \mathfrak{L}_1 t) + t^2 f(t) = 0 \\ \Longrightarrow\ & f(t)(1-2t+t^2) = \mathfrak{L}_0 + t(\mathfrak{L}_1 - 2\mathfrak{L}_0) + t^2(\mathfrak{L}_2 - 2\mathfrak{L}_1) \\ \Longrightarrow\ & f(t) = \frac{\mathfrak{L}_0 + t(\mathfrak{L}_1 - 2\mathfrak{L}_0) + t^2(\mathfrak{L}_2 - 2\mathfrak{L}_1)}{1-2t+t^2} \\ \Longrightarrow\ & f(t) = \frac{-1}{1-2t+t^2}\begin{bmatrix} [(k+2)t^2 + t - (2k+3)] - i[kt^2 - t + 1] \\ [t^2 - kt - 1] + i[t^2 + t - (k+2)] \end{bmatrix}. \end{aligned}$$

□

Theorem 3.5 (Finite sum formulae)**.** *We have the following.*

Sum of first $n+1$ terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,r} = \mathfrak{L}_{k,n+2} - (k+1)S_2 - nk\mathfrak{J}. \tag{3.6}$$

Sum of first $n+1$ even indexed terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,2r} = \mathfrak{L}_{k,2n+1} - (k+1)S_0 - nk\mathfrak{J}. \tag{3.7}$$

Sum of first $n+1$ odd indexed terms

$$\sum_{r=0}^{n} \mathfrak{L}_{k,2r+1} = \mathfrak{L}_{k,2n+2} - (k+1)S_1 - nk\mathfrak{J}. \tag{3.8}$$

Proof. Using relation (3.2) and sum identity $\sum_{r=1}^{n} S_r = S_{n+2} - S_2$, we have

$$\begin{aligned}\sum_{r=0}^{n} \mathfrak{L}_{k,r} &= \sum_{r=0}^{n}[(k+1)S_{r+1} - k\mathfrak{J}] = (k+1)\sum_{r=0}^{n} S_{r+1} - \sum_{r=0}^{n} k\mathfrak{J}\\ &= (k+1)[S_{n+3} - S_2] - (n+1)k\mathfrak{J}\\ &= \mathfrak{L}_{k,n+2} - (k+1)S_2 - nk\mathfrak{J}.\end{aligned}$$

This proves expression (3.6).

The rest of the two expressions (3.7) and (3.8) follow directly from the identities $\sum_{r=1}^{n} S_{2r} = S_{2n+1} - S_1$ and $\sum_{r=1}^{n} S_{2r-1} = S_{2n} - S_0$, respectively. ☐

4. CONCLUSION

In summary, we defined and studied the generalized Leonardo quaternions and a new sequence of spinors by considering a linear and injective correspondence between the set of quaternions and the set of spinors. For generalized Leonardo quaternions, we obtained various identities, interrelations with the Fibonacci and Lucas quaternions, Catalan's identity, d'Ocagne's identity, generating functions, finite sum formulas, etc. For spinors, we presented the closed form formula, several identities, generating functions, finite sums with odd and even indexed terms, etc.

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[1]DEPARTMENT OF MATHEMATICS,
CENTRAL UNIVERSITY OF JHARKHAND,
RANCHI, 835205, INDIA.
Email address: `muneshnasir94@gmail.com`
Email address: `klkaprsd@gmail.com`
Email address: `hrishikesh.mahato@cuj.ac.in`

[2]DEPARTMENT OF MATHEMATICS,
GOVERNMENT ENGINEERING COLLEGE BHOJPUR,
BIHAR, 802301, INDIA.

[3]DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF TRÁS-OS-MONTES E ALTO DOURO,
5001-801, VILA REAL, PORTUGAL.
Email address: `pcatarin@utad.pt`

*CORRESPONDING AUTHOR

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A STUDY ON SOME CONFORMABLE FRACTIONAL IMPLICIT HYBRID DIFFERENTIAL EQUATIONS WITH DELAY

ABDELKRIM SALIM[1,2], SALIM KRIM[3,2], JAMAL EDDINE LAZREG[1],
AND MOUFFAK BENCHOHRA[1]

Abstract. This paper deals with some existence results for a class of conformable implicit fractional differential Hybrid equations with delay. The results are based on some suitable fixed point theorems. In the last section, we provide different examples to illustrate our obtained results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). Its versatility has made it a crucial tool in the field. In the previous decades more and more researchers have paid their attentions to fractional calculus, since they found that the fractional order integrals and derivatives were more suitable for the description of the phenomena in the real world, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on. For some details and recent publication on the subject, see the monographs [1,5–8,21,32,34–36] and the papers [2–4]. The study of implicit differential equations has received great attention in the last years; see [1,13,26–28].

As models of equations, functional differential equations with delay are commonly used. Several authors studied differential equations with delay [14,15,17,19]. For more details, see the papers which are concerned with finite delay [29,30], infinite delay [1,10,14,18], and state-dependent delay [1,17].

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The fractional derivative of an unknown function hybrid with nonlinearity is used in hybrid differential equations. This class of equations derives from several fields of practical mathematics and physics, such as the deflection of a curved beam with a constant or variable cross-section, a three-layer beam, electromagnetic waves, or gravity-driven flows, and so forth. For more details on the subject, we recommend readers to the publications [13, 25, 27, 31].

The authors of [33] studied the nonlinear fractional differential hybrid system with periodic boundary conditions, given by:

$$\begin{cases} {}^{C}\mathfrak{D}_{a^+}^{\varrho,\Psi}\left(v(\vartheta)\mathrm{g}_1(\vartheta, v(\vartheta))\right) = \mathrm{g}_2(\vartheta, v(\vartheta)), \quad \varrho \in [a,b], \\ v(a) = v(b), \end{cases}$$

where $\vartheta \in [a,b]$, ${}^{C}\mathfrak{D}_{a^+}^{\varrho,\Psi}$ is the Ψ-Caputo fractional derivative, $g_1 : [a,b] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $\mathrm{g}_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Their arguments are based on Dhage's fixed point theorem.

In [20], Khalil et al. provided a unique concept of fractional derivative, which is a natural extension of the traditional first derivative. The conformable fractional derivative is natural, and it contains most of the features of the classical integral derivative, such as product rule, quotient rule, linearity, chain rule, and power rule, and it is very useful for modelling different physical problems. Indeed, several publications have been produced since that time, and various equations have been solved using that notion [9, 12, 24].

In [22], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}_{\zeta_\jmath}^{\vartheta} x(\zeta) = f\left(\zeta, x_\zeta, \mathcal{T}_\jmath^{\vartheta} x(\zeta)\right), \quad \zeta \in \Omega_\jmath, \quad \jmath = 0, 1, \dots, \beta, \\ \Delta x|_{\zeta = \zeta_\jmath} = \Upsilon_\jmath(x_{\zeta_\jmath^-}), \quad \jmath = 1, 2, \dots, \beta, \\ x(\zeta) = \mu(\zeta), \quad \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \le \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}_{\zeta_\jmath}^{\vartheta} x(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $f : \Omega \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$, $\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]$, $\jmath = 1, 2, \dots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

In this paper, first we investigate the following class of conformable fractional differential Hybrid equation with finite delay:

$$\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right) = f\left(t, x_t, \mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)\right), \quad t \in \Theta := (\varepsilon, \beta], \tag{1.1}$$

$$x(t) = \zeta(t), \quad t \in (\varepsilon - \kappa, \varepsilon], \tag{1.2}$$

where $\mathcal{T}_{\varepsilon}^{\varsigma} x(t)$ is the conformable fractional derivative starting from the initial time ε of the function f of order $\varsigma \in (0,1)$, $f : \Theta \times C([\varepsilon - \kappa, \varepsilon], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\zeta \in C((\varepsilon - \kappa, \beta], \mathbb{R})$, $\Phi \in C\left(\Theta, \mathbb{R}\backslash\{0\}\right)$, $\varepsilon < \beta < +\infty$ and $\kappa > 0$ is the time delay. For any $t \in \Theta$, we give x_t by

$$x_t(\vartheta) = x(t + \vartheta), \quad \text{for } \vartheta \in [\varepsilon - \kappa, \varepsilon].$$

Next, we consider the following infinite delay problem:

$$\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)=f\left(t,x_t,\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)\right),\quad t\in\Theta, \tag{1.3}$$

$$x(t)=\zeta(t),\quad t\in(-\infty,\varepsilon], \tag{1.4}$$

where $f:\Theta\times\mathcal{G}\times\mathbb{R}\to\mathbb{R}$, $\Phi\in C\left(\Theta,\mathbb{R}\backslash\{0\}\right)$, $\zeta:(-\infty,\varepsilon]\to\mathbb{R}$, $\varepsilon<\beta<+\infty$ are given continuous functions, and $\mathcal{G}$ is called a phase space that will be determined later.

For any $t\in\Theta$, we define $x_t\in\mathcal{G}$ by $x_t(\vartheta)=x(t+\vartheta)$ for $\vartheta\in(-\infty,\varepsilon]$. In the next segment, we look into the following state-dependent finite delay problem:

$$\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)=f\left(t,x_{\rho(t,x_t)},\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)\right),\quad t\in\Theta, \tag{1.5}$$

$$x(t)=\zeta(t),\quad t\in(\varepsilon-\kappa,\varepsilon], \tag{1.6}$$

where $f\in C((\varepsilon,\beta],\mathbb{R})$, $\Phi\in C\left(\Theta,\mathbb{R}\backslash\{0\}\right)$, $\zeta\in C((\varepsilon-\kappa,\beta],\mathbb{R})$, $\varepsilon<\beta<+\infty$ are given continuous functions.

Finally, we study the following problem with state dependent infinite delay:

$$\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)=f\left(t,x_{\rho(t,x_t)},\mathcal{T}_{\varepsilon}^{\varsigma}\left(\Phi(t)x(t)\right)\right),\quad t\in\Theta, \tag{1.7}$$

$$x(t)=\zeta(t),\quad t\in(-\infty,\varepsilon], \tag{1.8}$$

where $f:\Theta\times\mathcal{G}\times\mathbb{R}\to\mathbb{R}$, $\Phi\in C\left(\Theta,\mathbb{R}\backslash\{0\}\right)$, $\zeta:(-\infty,\varepsilon]\to\mathbb{R}$, $\varepsilon<\beta<+\infty$ are given continuous functions.

2. Preliminaries

Let $C(\Theta)$ be the Banach space of all real continuous functions on Θ with the norm

$$\|x\|_{\infty}=\sup_{t\in\Theta}|x(t)|.$$

Let $\mathcal{C}:=C([\varepsilon-\kappa,\beta])$ be a Banach space with the norm

$$\|x\|_{C}:=\sup_{t\in[\varepsilon-\kappa,\beta]}|x(t)|.$$

By $L^1(\Theta)$ we denote the Banach space of measurable functions $x:\Theta\to\mathbb{R}$ which are Lebesgue integrable, equipped with the norm

$$\|x\|_{L^1}=\int_{\varepsilon}^{\beta}|x(t)|dt.$$

Definition 2.1 (The conformable fractional derivative [9])**.** Let $f:[0,+\infty)\to\mathbb{R}$ be a given function, the conformable fractional derivative of f of order ς is defined by

$$\mathcal{T}^{\varsigma}(f)(t)=\lim_{\alpha\to 0}\frac{f\left(t+\alpha t^{1-\varsigma}\right)-f(t)}{\alpha},$$

for $t>0$ and $\varsigma\in(0,1]$. If f is ς-differentiable in some $(0,\varepsilon)$, $\varepsilon>0$, and $\lim_{t\to 0+}\mathcal{T}^{\varsigma}(f)(t)$ exists, then define $\mathcal{T}^{\varsigma}(f)(0)=\lim_{t\to 0+}\mathcal{T}^{\varsigma}(f)(t)$. If the conformable fractional derivative of f of order ς exists, then we simply say that f is ς-differentiable. It is easy to see that if f is differentiable, then $\mathcal{T}^{\varsigma}(f)(t)=t^{1-\varsigma}f'(t)$.

Definition 2.2 (The conformable fractional integral [9])**.** The conformable fractional integral starting from ε of the function $f : [\varepsilon, +\infty) \to \mathbb{R}$ of order $\varsigma \in (0, 1]$ is defined as

$$I^{r}_{\varepsilon} f(t) = \int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{1-\varsigma} f(t) d\vartheta.$$

Lemma 2.1 ([9])**.** *Let* $\varsigma \in (0, 1]$ *and* $f : [\varepsilon, +\infty) \to \mathbb{R}$ *be a continuous function. Then, for all* $t > \varepsilon$*,*

$$\mathcal{T}^{r}_{\varepsilon} I^{r}_{\varepsilon} f(t) = f(t).$$

Further, if f *is differentiable on* $(\varepsilon, +\infty)$*, then, for all* $t > \varepsilon$*,*

$$I^{r}_{\varepsilon} \mathcal{T}^{r}_{\varepsilon} f(t) = f(t) - f(\varepsilon).$$

By following the same approach as in the paper [9], we may obtain the following result.

Lemma 2.2 ([9])**.** *A function* x *is a solution of problem* (1.1)–(1.2)*, if and only if* x *satisfies the following integral equation*

$$x(t) = \begin{cases} \dfrac{1}{\Phi(t)} \left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{\varsigma-1} \widehat{f}(\vartheta) d\vartheta \right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in [\varepsilon - \kappa, \varepsilon], \end{cases} \tag{2.1}$$

where $\widehat{f} \in C(\Theta)$*, with* $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$*.*

For our purpose we will need the following fixed point theorems.

3. Existence of Solutions with Finite Delay

In this section, we are concerned with the existence results of the problem (1.1)–(1.2).

Let us introduce the following hypothesis.

(H_1) There exist constants $\omega_1, \mathcal{M} > 0,\ 0 < \omega_2 < 1$ such that:

$$|f(t, x_1, \Im_1) - f(t, x_2, \Im_2)| \leq \omega_1 \|x_1 - x_2\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2 |\Im_1 - \Im_2|,$$

and

$$|\Phi(t)| \geq \mathcal{M},$$

for any $x_1, x_2 \in \mathcal{C},\ \Im_1, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$.

Remark 3.1. We note that by taking: $\varpi_1 = \omega_1$, $\varpi_2 = \omega_2$ and $\varpi_3 = f^*$, where $f^* = \sup\limits_{t\in[\varepsilon,\beta]} f(t, 0, 0)$, hypothesis (H_1) implies that

$$|f(t, x, \Im)| \leq \varpi_1 \|x\|_{[\varepsilon-\kappa,\varepsilon]} + \varpi_2 |\Im| + \varpi_3.$$

Now, we will give our first existence and uniqueness result that is based on Banach's fixed point theorem.

Theorem 3.1. *Assume that hypothesis* (H_1) *holds. If*

$$\ell := \frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)} < 1, \tag{3.1}$$

then the problem (1.1)–(1.2) *has a unique solution on* $[\varepsilon-\kappa, \beta]$.

Proof. Consider the operator $\mathcal{H} : C(\Theta) \to C(\Theta)$ such that,

$$(\mathcal{H}x)(t) = \begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in [\varepsilon-\kappa, \varepsilon], \end{cases} \tag{3.2}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

Let $x, \Im \in C(\Theta)$. Then, for each $t \in [\varepsilon-\kappa, \varepsilon]$, we get

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| = 0,$$

and for each $t \in \Theta$, we obtain

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta) - \Upsilon(\vartheta)|d\vartheta\right],$$

where $\widehat{f}, \Upsilon \in C(\Theta)$ such that

$$\widehat{f}(t) = f(t, x_t, \widehat{f}(t)) \quad \text{and} \quad \Upsilon(t) = f(t, \Im_t, \Upsilon(t)).$$

From (H_1), we have

$$\begin{aligned} |\widehat{f}(t) - \Upsilon(t)| &= |f(t, x_t, \widehat{f}(t)) - f(t, \Im_t, \Upsilon(t))| \\ &\leq \omega_1\|x_t - \Im_t\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2|\widehat{f}(t) - \Upsilon(t)| \\ &\leq \omega_1\|x_t - \Im_t\|_{[\varepsilon-\kappa,\varepsilon]} + \omega_2|\widehat{f}(t) - \Upsilon(t)|. \end{aligned}$$

Thus,

$$\|\widehat{f} - \Upsilon\|_{\infty} \leq \frac{\omega_1}{1-\omega_2}\|(x - \Im)\|_C.$$

Then, for each $t \in \Theta$, we get

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)}\|x - \Im\|_C \leq \ell\|x - \Im\|_C.$$

Hence, we get

$$\|\mathcal{H}(x) - \mathcal{H}(\Im)\|_C \leq \ell\|x - \Im\|_C.$$

Consequently, by Banach's fixed point theorem, the operator $\mathcal{H}$ has a unique fixed point, which is the unique solution of our problem (1.1)–(1.2) on $[\varepsilon-\kappa, \beta]$. ☐

Theorem 3.2. *If* (H_1) *holds, and*

$$\frac{(\beta-\varepsilon)^{\varsigma}\varpi_1}{\mathcal{M}\varsigma(1-\varpi_2)} < 1,$$

then problem (1.1)–(1.2) *has at least one solution on* $[\varepsilon-\kappa, \beta]$.

Proof. Consider $\mathcal{H}: C(\Theta) \to C(\Theta)$ defined in (3.2). Let $\delta > 0$ such that

$$\delta \geq \max\left\{ \|\zeta\|_{C([\varepsilon-\kappa,\beta])}, \frac{\frac{|\Phi(\varepsilon)\zeta(\varepsilon)|}{\mathcal{M}} + \frac{(\beta-\varepsilon)^{\varsigma}\varpi_3}{\mathcal{M}\varsigma(1-\varpi_2)}}{1 - \frac{(\beta-\varepsilon)^{\varsigma}\varpi_1}{\mathcal{M}\varsigma(1-\varpi_2)}} \right\}. \tag{3.3}$$

Define the ball $\Omega_\delta = \{\xi \in C(\Theta) : \|\xi\|_C \leq \delta\}$.

Step 1. $\mathcal{H}$ is continuous.

Let $\{x_n\}_n$ be a sequence such that $x_n \to x$ on Ω_δ. For each $t \in [\varepsilon - \kappa, \varepsilon]$, we have

$$|(\mathcal{H}x_n)(t) - (\mathcal{H}x)(t)| = 0,$$

and for each $t \in \Theta$, we have

$$|(\mathcal{H}x)(t) - (\mathcal{H}\Im)(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}_n(\vartheta) - \widehat{f}(\vartheta)|d\vartheta\right], \tag{3.4}$$

where $\widehat{f}_n, \widehat{f} \in C(\Theta)$ such that

$$\widehat{f}_n(t) = f(t, x_{nt}, \widehat{f}_n(t)) \quad \text{and} \quad \widehat{f}(t) = f(t, x_t, \widehat{f}(t)).$$

Since

$$\|x_n - x\|_C \to 0, \qquad \text{as } n \to +\infty,$$

and $f, \widehat{f}$ and $\widehat{f}_n$ are continuous, then by Lebesgue dominated convergence theorem, we deduce that

$$\|\mathcal{H}(x_n) - \mathcal{H}(x)\|_C \to 0 \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{H}$ is continuous.

Step 2. $\mathcal{H}(\Omega_\delta) \subset \Omega_\delta$.

Let $x \in \Omega_\delta$. If $t \in [\varepsilon - \kappa, \varepsilon]$, then $|(\mathcal{H}x)(t)| \leq \|\zeta\|_C \leq \delta$. From Remark 3.1, for each $t \in \Theta$, we have

$$\begin{aligned}|\widehat{f}(t)| \leq |f(t, x_t, \widehat{f}(t))| &\leq \varpi_1\|x_t\|_{[\varepsilon-\kappa,\beta]} + \varpi_2|\widehat{f}(t)| + \varpi_3\\ &\leq \varpi_1\|x\|_C + \varpi_2\|\widehat{f}\|_\infty + \varpi_3 \leq \varpi_1\delta + \varpi_2\|\widehat{f}\|_\infty + \varpi_3.\end{aligned}$$

Then,

$$\|\widehat{f}\|_\infty \leq \frac{\delta\varpi_1 + \varpi_3}{1 - \varpi_2}.$$

Thus,

$$\begin{aligned}|(\mathcal{H}x)(t)| &\leq \frac{1}{|\Phi(t)|}\left|\Phi(\varepsilon)\zeta(\varepsilon) + \int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right|\\ &\leq \frac{1}{|\Phi(t)|}\left[|\Phi(\varepsilon)\zeta(\varepsilon)| + \int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right]\\ &\leq \frac{1}{\mathcal{M}}\left[|\Phi(\varepsilon)\zeta(\varepsilon)| + \frac{(\beta - \varepsilon)^{\varsigma}(\delta\varpi_1 + \varpi_3)}{\varsigma(1 - \varpi_2)}\right]\\ &\leq \delta.\end{aligned}$$

Hence, $\|\mathcal{H}(x)\|_C \leq \delta$. Consequently, $\mathcal{H}(\Omega_\delta) \subset \Omega_\delta$.

Step 3. $\mathcal{H}(\Omega_\delta)$ is equicontinuous.

For $\varepsilon \leq t_1 \leq t_2 \leq \beta$, and $x \in \Omega_\delta$, we get

$$\begin{aligned}|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| &\leq \frac{1}{|\Phi(t)|}\left|\int_\varepsilon^{t_1}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta - \int_\varepsilon^{t_2}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right|\\ &\leq \frac{\delta\varpi_1+\varpi_3}{\mathcal{M}\varsigma(1-\varpi_2)}\left|(t_2-\varepsilon)^\varsigma - (t_1-\varepsilon)^\varsigma\right|.\end{aligned}$$

As $t_2 \to t_1$ then $|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| \to 0$. We deduce that $\mathcal{H}(\Omega_\delta)$ is equicontinuous.

Consequently, Arzelá-Ascoli theorem implies that $\mathcal{H}$ is continuous and compact. Thus, by Schauder's fixed point theorem [37], we deduce that $\mathcal{H}$ has at least a fixed point which is a solution of (1.1)–(1.2). ☐

4. Existence of Solutions with Infinite Delay

In this section, we are concerned with the existence results of (1.3)–(1.4). Let the space $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ is a seminormed linear space of functions mapping $(-\infty,\varepsilon]$ into $\mathbb{R}$, and verifying the following axioms which were derived from Hale and Kato's originals [14].

(Ax_1) If $x : (-\infty,\beta] \to \mathbb{R}$, and $x_0 \in \mathcal{G}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $t \in \Theta$; we have:

(i) x_t is in $\mathcal{G}$;

(ii) $\|x_t\|_{\mathcal{G}} \leq \xi_1\|x_1\|_{\mathcal{G}} + \xi_2 \sup_{\vartheta\in[\varepsilon,t]}|x(\vartheta)|$;

(iii) $\|x(t)\| \leq \xi_3\|x_t\|_{\mathcal{G}}$.

(Ax_2) For the function $x(\cdot)$ in (Ax_1), y_t is a $\mathcal{G}$-valued continuous function on Θ.

(Ax_3) The space $\mathcal{G}$ is complete.

Consider the space $\Omega = \{x : (-\infty,\beta] \to \mathbb{R},\ x|_{(-\infty,\varepsilon]} \in \mathcal{G},\ x|_\Theta \in C([\varepsilon-\kappa,\beta],\mathbb{R})\}$.

Definition 4.1. By a solution of problem (1.3)–(1.4), we mean a function $x \in \Omega$ such that

$$x(t) = \begin{cases}\dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_\varepsilon^t(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t\in[\varepsilon,\beta],\\ \zeta(t), & t\in(-\infty,\varepsilon],\end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

The following hypothesis will be used in the sequel.

(H_2) The functions f and Φ verify:

$$|f(t,x_1,\Im_1) - f(t,x_2,\Im_2)| \leq b_1\|x_1-x_2\|_{\mathcal{G}} + b_2|\Im_1 - x_2|$$

and

$$|\Phi(t)| \geq \mathcal{M},$$

for any $x_1, \Im_1 \in \mathcal{G}$, $x_2, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$, where $b_1, \mathcal{M} > 0$ and $0 < b_2 < 1$.

Remark 4.1. We note that by taking: $B_1 = b_1$, $B_2 = b_2$ and $B_3 = f^*$, where $f^* = \sup\limits_{t\in[\varepsilon,\beta]} f(t,0,0)$, hypothesis (H_2) implies that

$$|f(t,x,\Im)| \leq B_1\|x\|_{\mathcal{G}} + B_2|\Im| + B_3,$$

for any $x \in \mathcal{G}$, $\Im \in \mathbb{R}$, and each $t \in \Theta$.

Theorem 4.1. *Assume that the hypothesis* (H_2) *holds. If*

$$\lambda := \frac{(\beta-\varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1-b_2)} < 1, \tag{4.1}$$

then the problem (1.3)–(1.4) *has a unique solution on* $(-\infty, \beta]$.

Proof. Consider the operator $N_1 : \Omega \to \Omega$ such that,

$$(N_1 x)(t) = \begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in (-\infty, \varepsilon], \end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_t, \widehat{f}(t))$.

Let $\varkappa_1 : (-\infty, \beta] \to \mathbb{R}$ be a function given by

$$\varkappa_1(t) = \begin{cases} \zeta(t), & t \in (-\infty, \varepsilon], \\ \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon)\right], & t \in \Theta. \end{cases}$$

For each $\varkappa_2 \in C(\Theta)$, with $\varkappa_2(0) = 0$, we denote by $\overline{\varkappa_2}$ the function defined by

$$\overline{\varkappa_2} = \begin{cases} 0, & t \in (-\infty, \varepsilon], \\ \varkappa_2(t), & t \in \Theta. \end{cases}$$

If $x(\cdot)$ satisfies the integral equation

$$x(t) = \frac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon) + \int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right],$$

we can decompose $x(\cdot)$ as $x(t) = \overline{\varkappa_2}(t) + \varkappa_1(t)$, for $t \in \Theta$, which implies that $x_t = \overline{\varkappa_2}_t + \varkappa_{1t}$ for every $t \in \Theta$, and the function $\varkappa_2(\cdot)$ satisfies

$$\varkappa_2(t) = \frac{1}{\Phi(t)}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right],$$

where $\widehat{f}(t) = f(t, \overline{\varkappa_2}_t + \varkappa_{1t}, \widehat{f}(t))$, $t \in \Theta$. Set

$$C_0 = \{\varkappa_2 \in C(\Theta) : \varkappa_{2\varepsilon} = 0\},$$

and let $\|\cdot\|_T$ be the norm in C_0 defined by

$$\|\varkappa_2\|_T = \|\varkappa_{2\varepsilon}\|_{\mathcal{G}} + \sup_{t \in \Theta}|\varkappa_2(t)| = \sup_{t \in \Theta}|\varkappa_2(t)|, \quad \varkappa_2 \in C_0,$$

where C_0 is a Banach space with norm $\|\cdot\|_T$. Define the operator $\mathcal{K} : C_0 \to C_0$ by

$$(\mathcal{K}\varkappa_2)(t) = \frac{1}{\Phi(t)}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], \tag{4.2}$$

where $\widehat{f}(t) = f(t, \overline{\varkappa_2}_t + \varkappa_{1t}, \widehat{f}(t))$, $t \in \Theta$. We shall show that $\mathcal{K} : C_0 \to C_0$ is a contraction map. Let $\varkappa_2, \varkappa_2' \in C_0$, then we have for each $t \in \Theta$

$$|\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2')(t)| \le \frac{1}{|\Phi(t)|}\left[\int_{\varepsilon}^{t} (\vartheta-\varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta) - \Upsilon(\vartheta)|d\vartheta\right], \tag{4.3}$$

where $\widehat{f}, \Upsilon \in C(\Theta)$ such that

$$\widehat{f}(t) = f(t, \overline{\varkappa_{2}}_{t} + \varkappa_{1t}, \widehat{f}(t)) \quad \text{and} \quad \Upsilon(t) = f(t, \overline{\varkappa_{2}'}_{t} + \varkappa_{1t}, \Upsilon(t)).$$

Since, for each $t \in \Theta$, we have

$$|\widehat{f}(t) - \Upsilon(t)| \leq \frac{b_1}{1 - b_2} \|\overline{\varkappa_{2}}_{t} - \overline{\varkappa_{2}'}_{t}\|_{\mathcal{G}}.$$

Then, for each $t \in \Theta$, we get

$$\begin{aligned}|\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2')(t)| &\leq \frac{(\beta - \varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1 - b_2)} \|\overline{\varkappa_{2}}_{t} - \overline{\varkappa_{2}'}_{t}\|_{\mathcal{G}} \leq \frac{(\beta - \varepsilon)^{\varsigma} b_1}{\mathcal{M}\varsigma(1 - b_2)} \|\overline{\varkappa_2} - \overline{\varkappa_2'}\|_{\beta} \\ &= \lambda \|\overline{\varkappa_2} - \overline{\varkappa_2'}\|_{\beta}.\end{aligned}$$

Thus, we get $\|\mathcal{K}(\varkappa_2)(t) - \mathcal{K}(\varkappa_2')(t)\|_T \leq \lambda \|\overline{\varkappa_2} - \overline{\varkappa_2'}\|_{\beta}$. Hence, from the Banach contraction principle, $\mathcal{K}$ admit a unique fixed point which is the unique solution of (1.3)–(1.4). □

Now, we demonstrate an existence result for problem (1.3)–(1.4) by using Scheafer's fixed point theorem [16].

Theorem 4.2. *Suppose that* (H_2) *holds. Then,* (1.3)–(1.4) *admit at least one solution on* $(-\infty, \beta]$.

Proof. Let $\mathcal{K} : C_0 \to C_0$ defined as in (4.2), For each given $\delta > 0$, we define the ball

$$\Omega_{\delta} = \{\varkappa_1 \in C_0 : \ \|\varkappa_1\|_{\beta} \leq \delta\}.$$

Step 1. $\mathcal{K}$ is continuous.

Let $\varkappa_{2n}$ be a sequence where $\varkappa_{2n} \to \varkappa_2$ in C_0. For each $t \in \Theta$, we have

$$|(\mathcal{K}\varkappa_{2n})(t) - (\mathcal{K}\varkappa_2)(t)| \leq \frac{1}{|\Phi(t)|} \left[\int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{\varsigma - 1} |\widehat{f}_n(\vartheta) - \widehat{f}(\vartheta)| d\vartheta\right], \tag{4.4}$$

where $\widehat{f}_n, \widehat{f} \in C(\Theta)$ such that

$$\widehat{f}_n(t) = f(t, \overline{\varkappa_{2}}_{nt} + \varkappa_{1t}, \widehat{f}_n(t)) \quad \text{and} \quad \widehat{f}(t) = f(t, \overline{\varkappa_{2}}_{t} + \varkappa_{1t}, \widehat{f}(t)).$$

Since $\|\varkappa_{2n} - \varkappa_2\|_{\beta} \to 0$, as $n \to \infty$ and $f, \widehat{f}$ and $\widehat{f}_n$ are continuous, then

$$\|\mathcal{K}(x_n) - \mathcal{K}(x)\|_{\beta} \to 0, \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{K}$ is continuous.

Step 2. $\mathcal{K}(\Omega_{\delta})$ is bounded.

Let $\varkappa_2 \in \Omega_{\delta}$, for $t \in \Theta$, we have

$$\begin{aligned}|\widehat{f}(t)| &\leq |f(t, \overline{\varkappa_{2}}_{t} + \varkappa_{1t}, \widehat{f}(t))| \\ &\leq B_1 \|\overline{\varkappa_{2}}_{t} + \varkappa_{1t}\|_{\mathcal{G}} + B_2 |\widehat{f}(t)| + B_3 \\ &\leq B_1 \Big[\|\overline{\varkappa_{2}}_{t}\|_{\mathcal{G}} + \|\varkappa_{1t}\|_{\mathcal{G}}\Big] + B_2 \|\widehat{f}\|_{\infty} + B_3 \\ &\leq B_1 \xi_2 \delta + B_1 \xi_1 \|\varphi\|_{\mathcal{G}} + B_2 \|\widehat{f}\|_{\infty} + B_3.\end{aligned}$$

Then,

$$\|\widehat{f}\|_\infty \leq \frac{B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{1 - B_2}.$$

Thus,

$$\begin{aligned}|(\mathcal{K}\varkappa_2)(t)| &\leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right]\\ &\leq \frac{(\beta - \varepsilon)^{\varsigma}(B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3)}{\mathcal{M}\varsigma(1 - B_2)} := \tilde{\ell}.\end{aligned}$$

Hence, $\|\mathcal{K}(\varkappa_2)\|_\beta \leq \tilde{\ell}$. Consequently, $\mathcal{K}$ maps bounded sets into bounded sets in C_0.

Step 3. $\mathcal{K}(\Omega_\delta)$ is equicontinuous.

For $\varepsilon \leq t_1 \leq t_2 \leq \beta$, and $\varkappa_2 \in \Omega_\delta$, we have

$$\begin{aligned}|\mathcal{K}(x)(t_1) - \mathcal{K}(x)(t_2)| &\leq \frac{1}{|\Phi(t)|}\left|\int_\varepsilon^{t_1} (\vartheta - \varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta - \int_\varepsilon^{t_2} (\vartheta - \varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right|\\ &\leq \frac{B_1\xi_2\delta + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{\mathcal{M}\varsigma(1 - B_2)}\left|(t_2 - \varepsilon)^{\varsigma} - (t_1 - \varepsilon)^{\varsigma}\right|.\end{aligned}$$

As $t_2 \to t_1$ we have that $|\mathcal{H}(x)(t_1) - \mathcal{H}(x)(t_2)| \to 0$. We deduce that $\mathcal{K}$ maps bounded sets into equicontinuous sets in C_0. Thus, $\mathcal{K} : C_0 \to C_0$ is completely continuous.

Step 4. The priori bounds.

We prove that the set

$$\mathcal{E} = \{x \in C_0 : \Im = \lambda\mathcal{K}(x), \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Let $\varkappa_2, u \in C_0$ such that $\varkappa_2 = \lambda\mathcal{K}(\varkappa_2)$, for some $\lambda \in (0, 1)$. Then, for each $t \in \Theta$, we have

$$\varkappa_2(t) = \lambda(\mathcal{K}\varkappa_2)(t) = \lambda\frac{1}{\Phi(t)}\left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right].$$

By Remark 4.1, we have

$$\begin{aligned}|\widehat{f}(t)| &\leq |f(t, \overline{\varkappa_2}_t + \varkappa_{1t}, \widehat{f}(t))|\\ &\leq B_1\|\overline{\varkappa_2}_t + \varkappa_{1t}\|_{\mathcal{G}} + B_2|\widehat{f}(t)| + B_3\\ &\leq B_1\left[\|\overline{\varkappa_2}_t\|_{\mathcal{G}} + \|\varkappa_{1t}\|_{\mathcal{G}}\right] + B_2\|\widehat{f}\|_\infty + B_3\\ &\leq B_1\xi_2\|\varkappa_2\|_T + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_2\|\widehat{f}\|_\infty + B_3.\end{aligned}$$

This gives

$$\|\widehat{f}\|_\infty \leq \frac{B_1\xi_2\|\varkappa_2\|_T + B_1\xi_1\|\varphi\|_{\mathcal{G}} + B_3}{1 - B_2} := \eta.$$

Thus, for each $t \in \Theta$, we obtain

$$|\varkappa_2(t)| \leq \frac{1}{|\Phi(t)|}\left[\int_\varepsilon^t (\vartheta - \varepsilon)^{\varsigma-1}|\widehat{f}(\vartheta)|d\vartheta\right] \leq \frac{\eta(\beta - \varepsilon)^{\varsigma}}{\mathcal{M}\varsigma} := \eta'.$$

Hence, $\|\varkappa_2\|_\beta \leq \eta'$. This shows that the set $\mathcal{E}$ is bounded. Thus, by Scheafer's fixed point theorem [16], $\mathcal{K}$ has a fixed point which is a solution of problem (1.3)–(1.4). ☐

5. Existence Results with State-Dependent Delay

5.1. **The Finite Delay Case.** We now consider the problem (1.5)-(1.6).

Definition 5.1. By a solution of problem (1.5)–(1.6), we mean a function $x \in C([\varepsilon-\kappa,\beta],\mathbb{R})$ such that

$$x(t)=\begin{cases} \dfrac{1}{\Phi(t)}\left[\Phi(\varepsilon)\zeta(\varepsilon)+\displaystyle\int_{\varepsilon}^{t}(\vartheta-\varepsilon)^{\varsigma-1}\widehat{f}(\vartheta)d\vartheta\right], & t\in[\varepsilon,\beta],\\ \zeta(t), & t\in[\varepsilon-\kappa,\varepsilon],\end{cases}$$

where $\widehat{f}\in C(\Theta)$, with $\widehat{f}(t)=f(t,x_{\rho(t,x_t)},\widehat{f}(t))$.

For the next result we will make use of the following hypothesis.
(H_3) The functions f and Φ verify:

$$|f(t,x_1,\Im_1)-f(t,x_2,\Im_2)|\le\omega_3\|x_1-x_2\|_{[\varepsilon-\kappa,\varepsilon]}+\omega_4|\Im_1-\Im_2|$$

and

$$|\Phi(t)|\ge\mathcal{M},$$

for any $x_1,\Im_1\in C([\varepsilon-\kappa,\varepsilon],\mathbb{R}),\ x_2,\Im_2\in\mathbb{R}$, and each $t\in\Theta$, where $\omega_3,\mathcal{M}>0$ and $0<\omega_4<1$.

Remark 5.1. We note that by taking:

$$A_1=\omega_3,\quad A_2=\omega_4\quad\text{and}\quad A_3=f^*,$$

where $f^*=\sup\limits_{t\in[\varepsilon,\beta]}f(t,0,0)$. Then, hypothesis (H_3) implies that

$$|f(t,x,\Im)|\le A_1\|x\|_{[\varepsilon-\kappa,\varepsilon]}+A_2|\Im|+A_3,$$

for any $x\in C([\varepsilon-\kappa,\varepsilon],\mathbb{R}),\ v\in\Im$, and each $t\in\Theta$.

As in Theorems 3.1 and 3.2, we have the following results.

Theorem 5.1. *Assume that the hypothesis* (H_3) *holds. If*

$$\frac{(\beta-\varepsilon)^{\varsigma}\omega_3}{\mathcal{M}\varsigma(1-\omega_4)}<1,$$

then problem (1.5)–(1.6) *has a unique solution on* $[\varepsilon-\kappa,\beta]$.

Theorem 5.2. *Suppose that* (H_3) *holds. If*

$$\frac{(\beta-\varepsilon)^{\varsigma}A_1}{\mathcal{M}\varsigma(1-A_2)}<1,$$

then problem (1.5)–(1.6) *has at least one solution on* $[\varepsilon-\kappa,\beta]$.

5.2. **The Infinite Delay Case.** In this part, we present the results concerning the last problem (1.7)–(1.8).

Definition 5.2. By a solution of (1.7)-(1.8), we mean a function $x \in \Omega$ such that

$$x(t) = \begin{cases} \dfrac{1}{\Phi(t)} \left[\Phi(\varepsilon)\zeta(\varepsilon) + \displaystyle\int_{\varepsilon}^{t} (\vartheta - \varepsilon)^{\varsigma-1} \widehat{f}(\vartheta) d\vartheta\right], & t \in [\varepsilon, \beta], \\ \zeta(t), & t \in (-\infty, \varepsilon], \end{cases}$$

where $\widehat{f} \in C(\Theta)$, with $\widehat{f}(t) = f(t, x_{\rho(t,x_t)}, \widehat{f}(t))$.

Set $\delta' := \delta'_{\rho^-} = \{\rho(t,x) : t \in \Theta,\ x \in \mathcal{G}\ \rho(t,x) < 0\}$. We suppose that $\rho : \Theta \times \mathcal{G} \to \mathbb{R}$ is continuous and $t \to x_t$ is continuous from δ' into $\mathcal{G}$.

(H_η) There exists a continuous bounded function $\varpi : \delta'_{\rho^-} \to (0, +\infty)$ where

$$\|\eta_t\|_{\mathcal{G}} \leq \varpi(t)\|\eta\|_{\mathcal{G}}, \quad \text{for any } t \in \delta'.$$

Lemma 5.1. *If* $x \in \Omega$, *then*

$$\|x_t\|_{\mathcal{G}} = (\xi_2 + \varpi')\|\eta\|_{\mathcal{G}} + \xi_1 \sup_{\tau \in [0, \max\{0,t\}]} \|x(\tau)\|,$$

where $\varpi' = \sup_{t \in \delta'} \varpi(t)$.

The following hypothesis will be used in the sequel.

(H_4) The functions f and Φ verify:

$$|f(t, x_1, \Im_1) - f(t, x_2, \Im_2)| \leq b_3\|x_1 - x_2\|_{\mathcal{G}} + b_4|\Im_1 - \Im_2|$$

and $|\Phi(t)| \geq \mathcal{M}$, for any $x_1, \Im_1 \in \mathcal{G}$, $x_2, \Im_2 \in \mathbb{R}$, and each $t \in \Theta$, where $b_3, \mathcal{M} > 0$ and $0 < b_4 < 1$.

Remark 5.2. We note that by taking: $B_4 = b_3$, $B_5 = b_4$ and $B_6 = f^*$, where $f^* = \sup_{t \in [\varepsilon, \beta]} f(t, 0, 0)$. Then, hypothesis (H_4) implies that

$$|f(t, x, \Im)| \leq B_4\|x\|_{\mathcal{G}} + B_5|\Im| + B_6,$$

for any $x \in \mathcal{G}$, $\Im \in \mathbb{R}$, and $t \in \Theta$.

As in Theorems 4.1 and 4.2, we have the following results.

Theorem 5.3. *Suppose that* (H_4) *holds. If*

$$\frac{(\beta - \varepsilon)^{\varsigma} b_3}{\mathcal{M}\varsigma(1 - b_4)} < 1,$$

then the problem (1.7)–(1.8) *has a unique solution on* $(-\infty, \beta]$.

Theorem 5.4. *Suppose that* (H_ζ) *and* (H_4) *hold. Then,* (1.7)–(1.8) *admit at least one solution on* $(-\infty, \beta]$.

6. Examples

We give now some examples that illustrate our obtained results throughout the paper.

Example 6.1. Consider the following problem

$$\begin{cases} (\mathcal{T}_0^{1/2}\Phi x)(t) = \dfrac{1}{90\,(1+\|x_t\|)} + \dfrac{1}{30\left(1+|(\mathcal{T}_0^{1/2}\Phi x)(t)|\right)}, & t\in[0,1],\\ x(t) = 1+t^2, & t\in[-1,0]. \end{cases} \tag{6.1}$$

Set

$$f(t,x,\Im) = \frac{1}{90\,(1+\|x\|)} + \frac{1}{30\,(1+|\Im|)}$$

and

$$\Phi(t) = \frac{\sqrt{2}}{3}\left(t^2+3|\sin(t)|+1\right),$$

where $t\in[0,1]$, $x\in\mathcal{C}$, $\Im\in\mathbb{R}$. Thus, f is continuous. For $x,\widetilde{x}\in\mathcal{C}$, $x,\widetilde{x}\in\mathbb{R}$, and $t\in[0,1]$, we have

$$|f(t,x,\Im)-f(t,\widetilde{x},\widetilde{\Im})| \le \frac{1}{90}\|x-\widetilde{x}\|_{[-1,0]} + \frac{1}{30}|\Im-\widetilde{\Im}|.$$

Hence, hypothesis (H_1) is satisfied with $\omega_1 = \frac{1}{90}$, $\mathcal{M} = \frac{\sqrt{2}}{3}$ and $\omega_2 = \frac{1}{30}$. Next, the condition (3.1) is verified with $\beta = 1$ and $\varsigma = \frac{1}{2}$. Indeed,

$$\frac{(\beta-\varepsilon)^{\varsigma}\omega_1}{\mathcal{M}\varsigma(1-\omega_2)} = \frac{\frac{1}{90}}{\frac{\sqrt{2}}{3}(1-\frac{1}{30})\frac{1}{2}} \approx 0.0487659849094171 < 1.$$

Some calculations indicate that all of the requirements of Theorem 3.1 are verified. Thus, the problem (6.1) has a unique solution defined on $[-1,1]$.

Example 6.2. Consider now the following problem

$$\begin{cases} (\mathcal{T}_0^{1/2}\Phi x)(t) = \dfrac{x_t e^{-\gamma t+t}}{180\,(e^t-e^{-t})\,(1+\|x_t\|)} \\ \qquad\qquad\qquad + \dfrac{x(t)e^{-\gamma t+t}}{60\,(e^t-e^{-t})\left(1+|(\mathcal{T}_0^{1/2}\Phi x)(t)|\right)}, & t\in[0,1],\\ x(t) = t+1, & t\in(-\infty,0], \end{cases} \tag{6.2}$$

where Φ is the function defined in the first example. Let γ be a positive real constant and

$$B_\gamma = \left\{x\in C((-\infty,1],\mathbb{R},) : \lim_{\tau\to-\infty} e^{\gamma\tau}x(\tau) \text{ exists in } \mathbb{R}\right\}. \tag{6.3}$$

The norm of B_γ is given by

$$\|x\|_\gamma = \sup_{\tau\in(-\infty,1]} e^{\gamma\tau}|x(\tau)|.$$

Let $x : (-\infty, 0] \to \mathbb{R}$ be such that $x_0 \in B_\gamma$. Then

$$\begin{aligned}\lim_{\tau\to-\infty} e^{\gamma\tau} x_t(\tau) &= \lim_{\tau\to-\infty} e^{\gamma\tau} x(t+\tau-1) = \lim_{\tau\to-\infty} e^{\gamma(\tau-t+1)} x(\tau)\\ &= e^{\gamma(-t+1)} \lim_{\tau\to-\infty} e^{\gamma(\tau)} x_1(\tau) < +\infty.\end{aligned}$$

Hence, $x_t \in B_\gamma$. Finally, we prove that

$$\|x_t\|_\gamma \leq \xi_1 \|x_1\|_\gamma + \xi_2 \sup_{\vartheta\in[0,t]} |x(\vartheta)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have $\|x_t(\tau)\| = |x(t+\tau)|$. If $t + \tau \leq 1$, we get

$$\|x_t(\xi)\| \leq \sup_{\vartheta\in(-\infty,0]} |x(\vartheta)|.$$

For $t + \tau \geq 0$, then we have

$$\|x_t(\xi)\| \leq \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

Thus, for all $t + \tau \in \Theta$, we get

$$\|x_t(\xi)\| \leq \sup_{\vartheta\in(-\infty,0]} |x(\vartheta)| + \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

Then,

$$\|x_t\|_\gamma \leq \|x_0\|_\gamma + \sup_{\vartheta\in[0,t]} |x(\vartheta)|.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ is a phase space.

Set

$$f(t, x, \Im) = \frac{e^{-\gamma t+t}}{180\left(e^t - e^{-t}\right)\left(1 + \|x\|_{B_\gamma}\right)} + \frac{e^{-\gamma t+t}}{60\left(e^t - e^{-t}\right)\left(1 + |\Im|\right)},$$

where $t \in [0, 1],\ x \in B_\gamma,\ \Im \in \mathbb{R}$.

For any $x, \tilde{x} \in B_\gamma,\ \Im, \tilde{\Im} \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$|f(t, x, \Im) - f(t, \tilde{x}, \tilde{\Im})| \leq \frac{1}{180}\|x - \tilde{x}\|_{B_\gamma} + \frac{1}{60}|\Im - \tilde{\Im}|.$$

Hence, hypothesis (H_2) is satisfied with $b_1 = \frac{1}{180}$, $\mathcal{M} = \frac{\sqrt{2}}{3}$ and $b_2 = \frac{1}{60}$. All requirements of Theorem 4.2 are met. Then, the problem (6.2) has at least one solution defined on $(-\infty, 1]$.

Example 6.3. We consider the following problem

(6.4)

$$\begin{cases}(\mathcal{T}_0^{1/2}\Phi x)(t) = \dfrac{1}{90(1 + |x(t - \sigma(x(t)))|)} + \dfrac{1}{30\left(1 + |(\mathcal{T}_0^{1/2}\Phi x)(t)|\right)}, & t \in [0, 1],\\ x(t) = 1 + t^2, & t \in [-1, 0],\end{cases}$$

where Φ is the function defined in the first example and $\sigma \in C(\mathbb{R}, [0, 1])$. Set

$$\rho(t, \zeta) = t - \sigma(\zeta(0)), \quad (t, \zeta) \in [0, e] \times C([-1, 0], \mathbb{R}),$$

$$f(t,x,\Im)=\frac{1}{90(1+|x(t-\sigma(x(t)))|)}+\frac{1}{30\,(1+|\Im(t)|)},\quad t\in[0,1],\ x\in\mathcal{C},\ \Im\in\mathbb{R}.$$

Obviously, f is jointly continuous. For any $x,\widetilde{x}\in\mathcal{C},\ \Im,\widetilde{\Im}\in\mathbb{R}$ and $t\in[0,1]$, we have

$$|f(t,x,\Im)-f(t,\widetilde{x},\widetilde{\Im})|\le\frac{1}{90}\|x-\widetilde{x}\|_{[-1,0]}+\frac{1}{30}|\Im-\widetilde{\Im}|.$$

Hence, hypothesis (H_3) is satisfied with $\omega_3=\frac{1}{90}$, $\mathcal{M}=\frac{\sqrt{2}}{3}$ and $\omega_4=\frac{1}{30}$. All requirements of Theorem 5.1 are verified. Thus, the problem (6.4) has a unique solution defined on $[-1,1]$.

Example 6.4. Consider now the problem

$$\begin{cases}(\mathcal{T}_0^{1/2}\Phi x)(t)=\dfrac{x(t-\lambda(x(t)))e^{-\gamma t+t}}{180\,(e^t-e^{-t})\,(1+|x(t-\sigma(x(t)))|)}\\ \qquad\qquad\qquad+\dfrac{x(t)e^{-\gamma t+t}}{60\,(e^t-e^{-t})\left(1+|(\mathcal{T}_0^{1/2}\Phi x)(t)|\right)}, & t\in[0,2],\\ x(t)=t, & t\in(-\infty,0],\end{cases}\tag{6.5}$$

where Φ is the function defined in the first example.

Let $\gamma>0$ and B_γ be given in Example 2.

Let $\rho(t,\zeta)=t-\lambda(\zeta(0))$, $(t,\zeta)\in[0,2]\times B_\gamma$, and set

$$f(t,x,\Im)=\frac{e^{-\gamma t+t}}{180\,(e^t-e^{-t})\left(1+\|x\|_{B_\gamma}\right)}+\frac{e^{-\gamma t+t}}{60\,(e^t-e^{-t})\,(1+|\Im|)},$$

where $t\in[0,2],\ x\in B_\gamma,\ \Im\in\mathbb{R}$.

We can demonstrate that all conditions of Theorem 4.2 are verified. Then, the problem (6.5) has at least one solution defined on $(-\infty,2]$.

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[1]Faculty of Technology,
Hassiba Benbouali University of Chlef,
P.O. Box 151 Chlef 02000, Algeria
Email address: `salim.abdelkrim@yahoo.com`

[2]Laboratory of Mathematics,
Djillali Liabes University of Sidi Bel-Abbes,
P.O. Box 89, Sidi Bel-Abbes 22000, Algeria
Email address: `lazregjamal@yahoo.fr`
Email address: `benchohra@yahoo.com`

[3]Ecole National Supérieure d'Oran,
BP 1063 SAIM MOHAMED, Oran 31003, Algeria
Email address: `salimsalimkrim@gmail.com`

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STABILITY AND ULTIMATE BOUNDEDNESS OF SOLUTIONS OF CERTAIN THIRD ORDER NONLINEAR RECTANGULAR MATRIX DIFFERENTIAL EQUATIONS

A. L. OLUTIMO[1] AND M. O. OMEIKE[2]

ABSTRACT. We present in this paper the qualitative study of solutions of certain third order nonlinear matrix differential equations where the unknown function X is matrix-valued. The properties of solutions were investigated using Lyapunov's direct method by employing the use of suitable Lyapunov functionals obtained from the differential equations describing the system satisfying certain requirements for establishing the stability and boundedness of solutions of the system considered. An example is given to demonstrate the significance of the results obtained as well as analysis through geometric graphs describing the dynamics of the system's solutions. The results obtained are novel and will significantly enhance and extend the results of those mentioned in the literature.

1. INTRODUCTION

We investigate the stability and boundedness of solutions of matrix differential equations

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

where $X : \mathbb{R} \to \widetilde{M}$ is the unknown function, $A \in \mathcal{N}$ is a symmetric matrix with constant values, $\Psi, H : \widetilde{M} \to \widetilde{M}$ and $P : \mathbb{R} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \widetilde{M}$; $\widetilde{M}$ is an $n \times m$ and $\mathcal{N}$ an $n \times n$ matrices, $\mathbb{R}$ the real line $-\infty < t < +\infty$.

The study of characteristics of solutions to differential equations is majorly about deducting essential qualities of solutions of the differential equations without actually

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solving them. For many years, the characteristics of solutions to nonlinear differential equations of the third order have been studied by many mathematicians and have gotten several interesting results for some various and special cases of $n = m = 1$ and $m = 1$ in equation (1.1) (see [1–5, 7–15, 17, 18, 20, 21, 23–26, 28–32], respectively.

In the relevant literature, we observe that works in the area of **matrix** differential equation are not as active as they were in scalar and vector differential equations. Hence, results for the nonlinear differential equation in which the unknown function X is **matrix-valued** (so that $X : \mathbb{R} \to \mathbb{R}^{n\times n}$) are relatively scarce (see [19, 22] and [27]). For example, in 1976, Tejumola [27] discussed the asymptotic stability, ultimate boundedness and presence of periodic solutions of second order matrix differential equation here $\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X})$, where A is an $n \times n$ symmetric matrix with constant values. X, $H(X)$ and $P(t, X, \dot{X})$ being continuous $n \times n$ matrices in the real domain. He introduced some standard matrix notations which were widely used. That is, the continuous $n \times n$ matrix function $H(X)$ with $n^2 \times n^2$ generalized Jacobian matrix denoted by $JH(X)$ and the constant $n \times n$ matrix A. He also proved two lemmas which are vital to the proof of the stated theorems. The obtained results are a generalization of an earlier result of [10]. If $X \in \mathbb{R}^n$, the special case for which $n = m$ and $\Psi(\dot{X}) = B(\dot{X})$ in (1.1) for the equation $\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X})$, where A, B are $n \times n$ symmetric matrices with constant values, with X, $H(X)$ and $P(t, X, \dot{X})$ being continuous $n \times n$ matrices in the real domain has been studied by Omeike [19] for the ultimate boundedness of solutions of a certain third order nonlinear matrix differential equations. In the same vein, Omeike and Afuwape [22] proved the ultimate boundedness results of the same equation under some specified conditions on the nonlinear terms. The result obtained here is a rectangular matrix analogue of the results obtained in [19, 22] and an extension of the matrix result achieved in [27]. This means that if $n = m$ in (1.1), the result obtained in this study reduces to the results obtained in [19] and [22] which are square matrix equations and which themselves are matrix analogues of the vector equations in [3] and [12].

The investigations in Olutimo [16] are related to [27] and provided the extensions of some of the results of [27] to (1.1), where X is a rectangular matrix (i.e., $X : \mathbb{R} \to \mathbb{R}^{n\times m}$) and A, B are $n \times n$ symmetric matrices with constant values. X, $H(X)$, and $P(t, X, \dot{X}, \ddot{X})$ being continuous $n \times m$ matrices in the real domain. The present investigation is based on [16] where $X : \mathbb{R} \to \mathbb{R}^{n\times m}$, $n \neq m$ and $B\dot{X} = \Psi(\dot{X})$ in (1.1). Based on our review of the literature, no research derived from [16] was discovered. Moreover, the results for which the unknown function X is not a square matrix were left open in [27]. Tejumola in [27] remarked: *"Our present investigation is of explanatory nature, efforts are being made to expand its scope to cover the situation for which the unknown function X is not necessarily a square matrix. Our results in this direction will be announced elsewhere."* To our knowledge, results in this path do not exist. In this case, we shall give augmentation of some results of [16] to certain third order matrix differential equations (1.1). In addition, matrix differential equations contribute appreciably to the study and plan of complex dynamic systems across various fields, giving valuable insights into the dynamic behaviour of interconnected elements. Systems of this type occur in response and stability of electrical and coupled circuits (see [6] and [27]). In particular, the intuitive idea of qualitative properties of solutions of rectangular matrix differential equations is of practical importance in analyzing the layout of control systems, models of cross interactions between competing species, and spread of diseases in a community with different traits as well in image compression and processing of tasks. Also, the results obtained in this work will be comparable in generality to the results obtained in [3,12,16,22,26,27] and some results existing in the literature. A numerical instance is provided to demonstrate the importance and relevance of the results achieved as well as provide a graphical analysis to corroborate our discoveries regarding the behaviour of solutions of the rectangular matrix equation (1.1).

2. Representation and Definition

We shall use the following standard matrix representation in this study. For $X \in \widetilde{M}$, X^T and x_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, represent the transpose and the elements of X, respectively with $(x_{ij})(y_{jk})$, $k = 1, 2, \ldots, n$, the matrix product XY^T of $X, Y \in \widetilde{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ with $X^j = col(x_{1j}, x_{i2}, \ldots, x_{nj})$ signify the ith row and jth column of X, respectively, and $\underline{X} = (X_1, X_2, \ldots, X_n)^T$ being nm column vector having n rows of X. Now let us represent $\mathbf{JH(X)}$ the $nm \times nm$ generalized the matrix representing the partial derivatives is the matrix linked to the Jacobian determinant $\frac{\partial(H_1, H_2, \ldots, H_n)}{\partial(X_1, X_2, \ldots, X_n)}$ at X when using the function $H : \widetilde{M} \to \widetilde{M}$. Also, $\mathbf{J}\Psi(X)$ the $nm \times nm$ generalized matrix representing the partial derivatives is the matrix linked to the Jacobian determinant $\frac{\partial(\Psi_1, \Psi_2, \ldots, \Psi_n)}{\partial(X_1, X_2, \ldots, X_n)}$ at X when using the function $\Psi : \widetilde{M} \to \widetilde{M}$.

For matrix $A \in \mathcal{N}$ with constant values, we assign an $nm \times nm$ matrix $\widehat{\mathbf{A}}$ having nm diagonal $m \times m$ matrices $(a_{ij} I_m)$ (I_m is the unit $m \times m$ matrix) and so that $(a_{ij} I_m)$ belongs to the *ith*-n row and jth-n column of $\widehat{\mathbf{A}}$. $\widehat{\mathbf{A}}$ is a $l \times l$ matrix where $l = mn$. In the particular instance in which A is a 2×2 matrix, X is a 2×3 matrix, $\widehat{\mathbf{A}}$ is the 6×6 matrix

$$\begin{pmatrix} a_{11} I_3 & a_{12} I_3 \\ a_{21} I_3 & a_{22} I_3 \end{pmatrix}.$$

For any given $X, Y \in \mathcal{M}$, $\langle X, Y \rangle = \operatorname{trace} XY^T$. $\|X - Y\|^2 = \langle X - Y, X - Y \rangle$ defines a norm on $\mathcal{M}$. $\|X\| = |\underline{X}|_{nm}$, where $| \cdot |_{nm}$ refers to the standard Euclidean norm in $\mathbb{R}^{nm}$ and $\underline{X} \in \mathbb{R}^{nm}$ is defined as mentioned earlier.

3. Preliminary Results

We shall use the following results to prove our theorems.

Lemma 3.1. *Assume that matrices* $\widehat{\boldsymbol{A}}$ *and* $\mathbf{JH(X)}$ *are symmetric and commute with respect to* $X \in \widetilde{M}$ *and* $H(0) = 0$*. Then,*

$$\langle H(X), AX\rangle = \int_0^1 \underline{X}^T \widehat{\boldsymbol{A}}\boldsymbol{JH}(\sigma X)\underline{X}d\sigma. \tag{3.1}$$

Proof. Since $H(0) = 0$ and each $h_{ij} \in \mathcal{C}'(\widetilde{M})$, $i, j = 1, 2, \ldots, n$, we have the following:

$$h_{ij}(X) = \int_0^1 \frac{d}{d\sigma} h_{ij}(\xi)d\sigma = \int_0^1 \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial(\sigma x)_{kl}} x_{kl} d\sigma, \quad \xi = \sigma X. \tag{3.2}$$

But, by definition

$$\langle H(X), AX\rangle = \text{trace}\left\{ h_{ij}(X) \left(\sum_{r=1}^{n} a_{ir}x_{rj} \right)^T \right\},$$

so that, in the light of Equation (3.2),

$$\langle H(X), AX\rangle = \int_0^1 \sum_{i,j=1}^{n,m} \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl} \sum_{k=1}^{n} a_{ik}x_{kj} d\sigma.$$

The representation (3.1) follows from the definitions of $\widehat{\mathbf{A}}$ and $\mathbf{JH(X)}$ and the fact that $\widehat{\mathbf{A}}$ is symmetric. ☐

Lemma 3.2. *Consider* $\mathbf{JH(X)}$ *being symmetric for any* $X \in \widetilde{M}$ *with* $H(0) = 0$*. Then,*

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \langle H(X), \dot{X}\rangle, \quad \textit{for all } X \in \widetilde{M}. \tag{3.3}$$

Proof. We know that

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \int_0^1 \langle H(\sigma X), \dot{X}\rangle d\sigma + \int_0^1 \left\langle \frac{d}{dt} H(\sigma X), X \right\rangle d\sigma. \tag{3.4}$$

Observe from equation (3.2) that:

$$\frac{d}{dt} H(\sigma X) = \left(\sigma \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} \dot{x}_{kl} \right), \quad \text{where } \xi = \sigma X,$$

from which it follows, by the definition of the inner product, that

$$\left\langle \frac{d}{dt} H(\sigma X), X \right\rangle = \sigma \sum_{i,j=1}^{n,m} \left(\sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} \dot{x}_{kl} \right) x_{ij} = \sigma \sum_{i,j=1}^{n,m} \left(\sum_{k,l=1}^{n,m} \frac{\partial h_{kl}(\xi)}{\partial x_{ij}} \dot{x}_{kl} \right) x_{ij},$$

since $\mathbf{JH(X)}$ is symmetric. Therefore, by interchanging the order of summation and replacing k, l by i and j, respectively, we have that:

$$\left\langle \frac{d}{dt} H(\sigma X), X \right\rangle = \sigma \sum_{i,j=1}^{n,m} \left(\sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl} \right) \dot{x}_{ij} = \left\langle \sigma \frac{d}{d\sigma} H(\sigma X), \dot{X} \right\rangle. \tag{3.5}$$

Since

$$\frac{d}{d\sigma}h_{ij}(\xi) = \sum_{k,l=1}^{n,m} \frac{\partial h_{ij}(\xi)}{\partial x_{kl}} x_{kl}, \quad \text{by (3.2)},$$

integrating (3.5) by parts, we have

$$\int_0^1 \sigma \frac{d}{d\sigma} H(\sigma X) d\sigma = H(X) - \int_0^1 H(\sigma X) d\sigma.$$

The integral (3.5) equals

$$\langle H(X), \dot{X} \rangle - \int_0^1 \langle H(\sigma X) d\sigma, \dot{X} \rangle,$$

and substituting into (3.4), the result (3.3) is obtained. □

Remark 3.1. Lemmas 1 and 2 respectively of [27] is now included in Lemma 3.1 and Lemma 3.2 if $n = m$.

Lemma 3.3. *Set* $\Psi(0) = 0$ *and presume that* $\boldsymbol{J}\Psi(Y)$ *is symmetric for any* $Y \in \widetilde{M}$. *Then,*

$$\langle \Psi(Y), Y \rangle = \int_0^1 \{\underline{Y}^T [\boldsymbol{J}\Psi(\tau Y)] \underline{Y}\} d\tau.$$

Proof. The proof proceeds by making use of the result

$$\Psi(Y) = \int_0^1 \mathbf{J}\Psi(\tau Y) d\tau,$$

for $Y \in \widetilde{M}$, which is obtained by integrating the equality

$$\frac{d}{d\sigma}\Psi(\tau Y) = \mathbf{J}\Psi(\tau Y)Y,$$

that is,

$$\psi_{ij}(y) = \int_0^1 \frac{d}{d\sigma}\psi_{ij}(\rho) d\sigma = \int_0^1 \sum_{k,l=1}^{n,m} \frac{\partial \psi_{ij}(\rho)}{\partial (\sigma y)_{kl}} y_{kl} d\sigma, \quad \rho = \sigma y,$$

with respect to σ, taking into account that $\Psi(0) = 0$ and each $\psi_{ij} \in \mathcal{C}'(\widetilde{M})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. □

We express (1.1) as

$$\begin{aligned} \dot{X} &= Y, \\ \dot{Y} &= Z, \\ \dot{Z} &= -AZ - \Psi(Y) - H(X) + P(t, X, Y, Z), \end{aligned} \tag{3.6}$$

where an $n \times m$ matrix X is the unknown function, A is an $n \times n$ symmetric matrix with constant values, $\Psi(Y)$, $H(X)$ and P are continuous $n \times m$ matrices in the real domain.

4. Stability of Solutions

Here, we investigate the stability of solutions of equation (1.1), where $P = 0$ in equation (3.6).

The following result will establish the stability of solutions of (1.1).

Theorem 4.1. *Let us assume H satisfies a condition for the existence and uniqueness of solutions of* (3.6) *with $H(0) = 0$ and for any $X, Y \in \widetilde{M}$.*

(i) *The matrices $\widehat{\boldsymbol{A}}$, $\boldsymbol{J}\Psi(Y)$, and $\boldsymbol{JH}(\boldsymbol{X})$ exhibit symmetry and are positively definite. $\widehat{\boldsymbol{A}}$, $\boldsymbol{J}\Psi(Y)$, and $\boldsymbol{JH}(\boldsymbol{X})$ commute pairwise and are associative.*

(ii) *The eigenvalues $\lambda_i(\widehat{\boldsymbol{A}})$ of $\widehat{\boldsymbol{A}}$, $\lambda_i(\boldsymbol{J}\Psi(Y))$ of $\boldsymbol{J}\Psi(Y)$ and $\lambda_i(\boldsymbol{JH}(\boldsymbol{X}))$ of $\boldsymbol{JH}(\boldsymbol{X})$, $i = 1, 2, \dots, nm$, satisfy:*

$$\begin{aligned} 0 &< \delta_a \le \lambda_i(\widehat{\boldsymbol{A}}) \le \triangle_a, \\ 0 &< \delta_\psi \le \lambda_i(\boldsymbol{J}\Psi(Y)) \le \triangle_\psi, \\ 0 &< \delta_h \le \lambda_i(\boldsymbol{JH(X)}) \le \triangle_h, \end{aligned}$$

with δ_a, δ_ψ, δ_h, $\triangle_a$, $\triangle_\psi$, $\triangle_h$ being finite constants.

Then, every solution of equation (3.6) *satisfies*

$$\|X(t)\|^2 \to 0, \quad \|Y(t)\|^2 \to 0 \quad \text{and} \quad \|Z(t)\|^2 \to 0, \quad \text{as} \quad t \to +\infty.$$

Proof. For the proof of Theorem 4.1, we use the following function $V = V(X, Y, Z)$ as specified by

$$2V = 2V_a + 2V_b, \tag{4.1}$$

where

$$\begin{aligned} 2V_a =& 2\int_0^1 \langle H(\xi X), X\rangle d\xi + 2\varrho \int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + \varrho\langle Z, Z\rangle \\ &+ 2\varrho\langle Y, H(X)\rangle + 2\langle Y, Z\rangle + \langle AY, Y\rangle \end{aligned}$$

and

$$\begin{aligned} 2V_b =& 2\delta_a \int_0^1 \langle H(\xi X), X\rangle d\xi + 2\int_0^1 \langle \Psi(\tau Y), Y\rangle d\tau + \langle AY, AY\rangle + \langle Z, Z\rangle \\ &+ \eta\delta_a\delta_\psi^2\langle X, X\rangle + 2\eta\delta_a^2\delta_\psi\langle X, Y\rangle + 2\eta\delta_a\delta_\psi\langle X, Z\rangle + 2\delta_a\langle Y, Z\rangle \\ &+ 2\langle Y, H(X)\rangle - \eta\delta_a\delta_\psi\langle Y, Y\rangle, \end{aligned}$$

where

$$\frac{1}{\delta_a} < \varrho < \frac{\delta_\psi}{\triangle_h} \tag{4.2}$$

and

$$\begin{aligned} \eta < \min\Big\{ &\frac{1}{\delta_a}, \frac{\delta_a}{\delta_\psi}, \frac{2(1+\delta_a)\delta_\psi - 2(1+\varrho)\triangle_h - \delta_a^2(\triangle_a - \delta_a)^2}{2[(1+\varrho)\delta_a - (1+\delta_a)][\delta_a^2\delta_\psi + \delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi - \delta_\psi)^2]}, \\ &\frac{(1+\varrho)\delta_a - (1+\delta_a)}{2\delta_a\delta_\psi\delta_h^{-1}(\triangle_a - \delta_a)^2}\Big\}. \end{aligned} \tag{4.3}$$

It is clear from (4.1), that $V(0,0,0)=0$.

V_a can be re-written as

$$\begin{aligned}2V_a =&2\varrho\int_0^1\langle\Psi(\tau Y),Y\rangle d\tau+2\varrho\langle Y,H(X)\rangle+\langle Y,AY\rangle+2\int_0^1\langle H(\xi X),X\rangle d\xi\\&+\varrho\langle Z+\varrho^{-1}Y,Z+\varrho^{-1}Y\rangle-\varrho^{-1}\langle Y,Y\rangle.\end{aligned}$$

For each term of the above expression, it is clear that

$$2\varrho\int_0^1\langle\Psi(\tau Y),Y\rangle d\tau+2\varrho\langle Y,H(X)\rangle.$$

By Lemma 3.3,

$$2\int_0^1\langle\Psi(\tau Y),Y\rangle d\tau=2\int_0^1\int_0^1\tau_1\langle\mathbf{J}\Psi(\tau_1\tau_2Y)\underline{Y},\underline{Y}\rangle d\tau_1 d\tau_2$$

and

$$2\langle Y,H(X)\rangle=4\int_0^1\int_0^1\tau_1\langle .Y,H(X)\rangle d\tau_1 d\tau_2.$$

So, that

$$\begin{aligned}&2\varrho\int_0^1\langle\Psi(\tau Y),Y\rangle d\tau+2\varrho\langle Y,H(X)\rangle\\=&2\varrho\int_0^1\int_0^1\tau_1\left\{\langle\mathbf{J}\Psi(\tau_1\tau_2Y)\underline{Y},\underline{Y}\rangle+\langle Y,H(X)\rangle\right\}d\tau_1 d\tau_2.\end{aligned}$$

It should be noted that matrix $\mathbf{J}\Psi$ is as assumed in condition (i) of Theorem 4.1. Hence, $\mathbf{J}\Psi^{\frac{1}{2}}$ and $\mathbf{J}\Psi^{-\frac{1}{2}}$ do exist which are non-singular and symmetric for all $Y\in\widetilde{M}$. So, we have

(4.4)
$$\varrho\langle\mathbf{J}\Psi Y,Y\rangle d\tau+\varrho\langle Y,H(X)\rangle=\sum_{i=1}^m\varrho|\mathbf{J}\Psi^{\frac{1}{2}}Y^i+\mathbf{J}\Psi^{-\frac{1}{2}}H(X^i)|_n^2-\varrho\{\underline{X}^T[\mathbf{J}\Psi^{-1}\mathbf{J}\mathbf{H}^2]\underline{X}\},$$

where $\mathbf{J}\Psi$ stands for $\mathbf{J}\Psi(\tau_1\tau_2Y)$ and $\mathbf{JH}$ for $\mathbf{JH(X)}$. Thus,

$$\begin{aligned}2V_a =&2\int_0^1\langle H(\xi X),X\rangle d\tau\\&-2\varrho\int_0^1\tau_1\int_0^1\{\underline{X}^T[\mathbf{J}\Psi^{-1}(\tau_1\tau_2Y)\mathbf{JH}(\tau_1X)]\mathbf{JH}(\tau_1\tau_2X)\underline{X}\}d\tau_1\tau_2\\&+\varrho\langle Z+\varrho^{-1}Y,Z+\varrho^{-1}Y\rangle+\langle AY,Y\rangle-\varrho^{-1}\langle Y,Y\rangle\\&+2\int_0^1\tau_1\int_0^1\varrho\sum_{i=1}^m|\mathbf{J}\Psi^{\frac{1}{2}}Y^i+\mathbf{J}\Psi^{-\frac{1}{2}}H(X^i)|_n^2 d\tau_1\tau_2.\end{aligned}\tag{4.5}$$

From (4.5), the expression

$$\begin{aligned}&2\int_0^1\langle H(\tau X),X\rangle d\tau-2\varrho\int_0^1\tau_1\int_0^1\{\underline{X}^T[\mathbf{J}\Psi^{-1}(\tau_1\tau_2Y)\mathbf{JH}(\tau_1X)]\mathbf{JH}(\tau_1\tau_2X)\underline{X}\}d\tau_1\tau_2\\=&2\int_0^1\tau_1\int_0^1\{\underline{X}^T[\widehat{\mathbf{I}}-\varrho\mathbf{J}\Psi^{-1}(\tau_1\tau_2Y)\mathbf{JH}(\tau_1X)]\mathbf{JH}(\tau_1\tau_2X)\underline{X}\}d\tau_1\tau_2\end{aligned}$$

$$\geq (1-\varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|_{nm}^2.$$

Also, we give the estimate for this expression in equation (4.5)

$$\varrho\langle Z+\varrho^{-1}Y, Z+\varrho^{-1}Y\rangle = \sum_{i=1}^{m}|Z^i+\varrho^{-1}Y^i|_n^2.$$

Also,

$$\langle AY, Y\rangle - \varrho^{-1}\langle Y, Y\rangle = \langle(\widehat{\mathbf{A}}-\varrho^{-1}\widehat{\mathbf{I}})\underline{Y}, \underline{Y}\rangle = \{\underline{Y}^T(\widehat{\mathbf{A}}-\varrho^{-1}\widehat{\mathbf{I}})\underline{Y}\} \geq (\delta_a-\varrho^{-1})|\underline{Y}|_{nm}^2.$$

Combining all the estimates of V_a we obtain

$$\begin{aligned} 2V_a &\geq (1-\varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|_{nm}^2 + (\delta_a-\varrho^{-1})|\underline{Y}|_{nm}^2 + \sum_{i=1}^{m}|Z^i+\varrho^{-1}Y^i|_n^2 \\ &\geq (1-\varrho\delta_\psi^{-1}\triangle_h)\delta_h\|X\|^2 + (\delta_a-\varrho^{-1})\|Y\|^2 + \|Z+\varrho^{-1}Y\|^2. \end{aligned} \tag{4.6}$$

Similarly, we re-arrange $2V_b$ to get

$$\begin{aligned} 2V_b =& \sum_{i=1}^{m}|Z^i+\delta_aY^i+\eta\delta_a\delta_\psi X^i|_n^2 + \langle AY, AY\rangle \\ &+ 2\int_0^1\langle\Psi(\tau Y), Y\rangle d\tau - \delta_\psi\langle Y, Y\rangle + \eta\delta_a\delta_\psi^2(1-\eta\delta_a)\langle X, X\rangle \\ &+ 2\delta_a\int_0^1\langle H(\tau X), X\rangle d\tau - \delta_\psi^{-1}\langle H(X), H(X)\rangle \\ &+ \delta_a(\delta_a-\eta\delta_\psi)\langle Y, Y\rangle + \delta_\psi\langle Y+\delta_\psi^{-1}H(X), Y+\delta_\psi^{-1}H(X)\rangle. \end{aligned}$$

For this function it is easy to see term by term that

$$\begin{aligned} \langle AY, AY\rangle - \delta_a^2\langle Y, Y\rangle = \langle(\widehat{\mathbf{A}}^2-\delta_a^2\widehat{\mathbf{I}})\underline{Y}, \underline{Y}\rangle = \underline{Y}^T(\widehat{\mathbf{A}}^2-\delta_a^2\widehat{\mathbf{I}})\underline{Y} &= (\delta_a^2-\delta_a^2)|\underline{Y}|_{nm}^2 \\ = (\delta_a^2-\delta_a^2)\|Y\|^2 > 0. \end{aligned}$$

By Lemma 3.3, we obtain for the following expression

$$\begin{aligned} 2\int_0^1\tau\int_0^1\langle[\mathbf{J}\Psi(\tau Y)-\delta_\psi\widehat{\mathbf{I}}]\underline{Y}, \underline{Y}\rangle d\tau d\sigma =& 2\int_0^1\int_0^1\tau\{\underline{Y}^T[\mathbf{J}\Psi(\tau Y)-\delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}d\tau d\sigma \\ =& \underline{Y}^T(\delta_\psi\widehat{\mathbf{I}}-\delta_\psi\widehat{\mathbf{I}})\underline{Y} \\ \geq& (\delta_\psi-\delta_\psi)|\underline{Y}|_{nm}^2 \\ =& (\delta_\psi-\delta_\psi)\|Y\|^2 \\ \geq& 0. \end{aligned}$$

Moreover, the expression gives

$$\eta\delta_a\delta_\psi^2(1-\eta\delta_a)\langle X, X\rangle = \underline{X}^T(\eta\delta_a\delta_\psi^2(1-\eta\delta_a)\underline{X} \geq \eta\delta_a\delta_\psi^2(1-\eta\delta_a)|\underline{X}|_{nm}^2.$$

Also, we get the estimate for this

$$\delta_a(\delta_a-\eta\delta_\psi)\langle Y, Y\rangle = \underline{Y}^T\delta_a(\delta_a-\eta\delta_\psi)\underline{Y} \geq \delta_a(\delta_a-\eta\delta_\psi)|\underline{Y}|_{nm}^2.$$

Furthermore, this expression yields

$$\begin{aligned}
&2\delta_a \int_0^1 \langle H(\tau X), X\rangle d\tau - \delta_\psi^{-1}\langle H(X), H(X)\rangle \\
=&2\int_0^1 \tau_1 \int_0^1 \{\underline{X}^T[\delta_a \widehat{\mathbf{I}} - \delta_\psi^{-1}\mathbf{JH}(\tau_1 X)]\mathbf{JH}(\tau_1\tau_2 X)\underline{X}\} d\tau_1\tau_2 \\
\geq&(\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm}
\end{aligned}$$

and

$$\delta_\psi\langle Y + \delta_\psi^{-1}H(X), Y + \delta_\psi^{-1}H(X)\rangle = \delta_\psi \sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n.$$

Combining the estimates of V_b, we have:

$$\begin{aligned}
2V_b \geq&\eta\delta_a\delta_\psi^2(1-\eta\delta_a)|\underline{X}|^2_{nm} + (\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} \\
&+ \delta_a(\delta_a - \eta\delta_\psi)|\underline{Y}|^2_{nm} + \delta_\psi \sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n \\
&+ \sum_{i=1}^m |Z^i + \delta_a Y^i + \eta\delta_a\delta_\psi X^i|^2_n.
\end{aligned} \tag{4.7}$$

Thus, combining estimates (4.6)–(4.7) in Equation (4.1), we obtain

$$\begin{aligned}
2V \geq&(1 - \varrho\delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} + \eta\delta_a\delta_\psi^2(1-\eta\delta_a)|\underline{X}|^2_{nm} + (\delta_a - \delta_\psi^{-1}\triangle_h)\delta_h|\underline{X}|^2_{nm} \\
&+ (\delta_a - \varrho^{-1})|\underline{Y}|^2_{nm} + \delta_a(\delta_a - \eta\delta_\psi)|\underline{Y}|^2_{nm} + \sum_{i=1}^m |Z^i + \varrho^{-1}Y^i|^2_n \\
&+ \delta_\psi \sum_{i=1}^m |Y^i + \delta_\psi^{-1}H(X^i)|^2_n + \sum_{i=1}^m |Z^i + \delta_a Y^i + \eta\delta_a\delta_\psi X^i|^2_n.
\end{aligned} \tag{4.8}$$

That is,

$$\begin{aligned}
2V \geq\ & \delta_h(1 - \varrho\delta_\psi^{-1}\triangle_h)\|X\|^2 + \eta\delta_a\delta_\psi^2(1-\eta\delta_a)\|X\|^2 + \delta_h(\delta_a - \delta_\psi^{-1}\triangle_h)\|X\|^2 \\
&+ (\delta_a - \varrho^{-1})\|Y\|^2 + \delta_a(\delta_a - \eta\delta_\psi)\|Y\|^2 + \|Z + \varrho^{-1}Y\|^2 \\
&+ \delta_\psi\|Y + \delta_\psi^{-1}H(X)\|^2 + \|Z + \delta_a Y + \eta\delta_a\delta_\psi X\|^2,
\end{aligned}$$

where

$$(\delta_\psi - \triangle_h\delta_a^{-1}) > 0 \quad \text{and} \quad \triangle_h\, \delta_\psi^{-1}(\delta_a - \triangle_h\delta_\psi^{-1}) > 0, \quad \text{by (4.2)}.$$

Thus, it is very clear from the terms in equation (4.8) there is a constant $D_1 > 0$ very small so that:

$$V \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{4.9}$$

for every $X, Y, Z \in \widetilde{M}$. The above estimates are valid since

$$\sum_{i=1}^m |X^i|^2_n = \sum_{i=1}^m |X_i|^2_n = |\underline{X}|^2_{nm} = \|X\|^2, \quad \text{for any } X \in \widetilde{M}.$$

Consider $(X(t), Y(t), Z(t))$ as arbitrary solutions of the system in (3.6). We now differentiate the function $V(t) = (X(t), Y(t), Z(t))$ defined in (4.1) with respect to t along the system (3.6) and using Lemma 3.2, yields

$$\begin{aligned}\dot{V}(t) = &-\frac{1}{2}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X}d\tau - \{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)\\ &-(1+\varrho)\mathbf{JH(X)} - \eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}\\ &-\frac{1}{2}\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\}\\ &-\frac{1}{4}\eta\delta_a\delta_\psi\left\{\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})X, Z\rangle\right\}d\tau\\ &-\frac{1}{4}\eta\delta_a\delta_\psi\left\{\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\mathbf{J}\Psi(\tau Y) - \delta_\psi\widehat{\mathbf{I}})X, Y\rangle\right\}d\tau\\ &-\frac{1}{2}\left\{\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\} + 2\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y, Z\rangle\right\}.\end{aligned}$$

Following the same reasoning in (4.4), it can be observed that

$$\begin{aligned}&\underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})X, Z\rangle\\ =&\sum_{i=1}^m |\mathbf{JH}^{\frac{1}{2}}X^i + 2\mathbf{JH}^{-\frac{1}{2}}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})Z^i|_n^2 - \{\underline{Z}^T[2(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Z}\}.\end{aligned}$$

Also,

$$\begin{aligned}&\underline{X}^T\mathbf{JH}(\tau X)\underline{X} + 4\langle(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})X, Y\rangle\\ =&\sum_{i=1}^m |\mathbf{JH}^{\frac{1}{2}}X^i + 2\mathbf{JH}^{-\frac{1}{2}}(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})Y^i|_n^2 - \{\underline{Y}^T[2(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Y}\},\end{aligned}$$

where $\mathbf{JH} = \mathbf{JH}(X)$ and

$$\begin{aligned}&\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z} + 2\langle(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y, Z\rangle\\ =&\sum_{i=1}^m \left|[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{\frac{1}{2}}Z^i + [(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{-\frac{1}{2}}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\widehat{\mathbf{A}}Y^i\right|_n^2\\ &-\{\underline{Y}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2\underline{Y}\}.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{V}(t) \leq &-\frac{1}{2}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{JH}(\tau X)\underline{X}d\tau - \{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)\\ &-(1+\varrho)\mathbf{JH(X)} - \eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}]\underline{Y}\}\\ &-\frac{1}{2}\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}} - (1+\delta_a)\widehat{\mathbf{I}}]\underline{Z}\}\\ &+\frac{1}{4}\eta\delta_a\delta_\psi\int_0^1 \{\underline{Z}^T[2(\widehat{\mathbf{A}} - \delta_a\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Z}\}d\tau\\ &+\frac{1}{4}\eta\delta_a\delta_\psi\int_0^1 \underline{Y}^T[2(\mathbf{J}\Psi(Y) - \delta_\psi\widehat{\mathbf{I}})\mathbf{JH}^{-\frac{1}{2}}]^2\underline{Y}d\tau\end{aligned}$$

$$+\frac{1}{2}\{\underline{Y}^T[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2\underline{Y}\}.$$

Note that,

$$\int_0^1 \{\underline{Z}^T[2(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})\mathbf{J}\mathbf{H}^{-\frac{1}{2}}]^2\underline{Z}\}d\tau = 4\int_0^1 \{\underline{Z}^T[\mathbf{J}\mathbf{H}^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2]\underline{Z}\}d\tau$$

and

$$\int_0^1 \underline{Y}^T[2(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})\mathbf{J}\mathbf{H}^{-\frac{1}{2}}]^2\underline{Y}d\tau = 4\int_0^1 \underline{Y}^T[\mathbf{J}\mathbf{H}^{-1}(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})^2]\underline{Y}d\tau.$$

It follows that

$$\begin{aligned}
\dot{V}(t) \leq & -2^{-1}\eta\delta_a\delta_\psi\int_0^1 \underline{X}^T\mathbf{J}\mathbf{H}(\tau X)\underline{X}d\tau \\
& -\int_0^1 \Big\{\underline{Y}^T[(1+\delta_a)\mathbf{J}\Psi(Y)-(1+\varrho)\mathbf{J}\mathbf{H}(\mathbf{X}) \\
& -\eta\delta_a^2\delta_\psi\widehat{\mathbf{I}}-\eta\delta_a\delta_\psi\mathbf{J}\mathbf{H}^{-1}(\mathbf{J}\Psi(Y)-\delta_\psi\widehat{\mathbf{I}})^2 \\
& -2^{-1}[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}]^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2\widehat{\mathbf{A}}^2]\underline{Y}\Big\}d\tau \\
& -2^{-1}\int_0^1 \Big\{\underline{Z}^T[(1+\varrho)\widehat{\mathbf{A}}-(1+\delta_a)\widehat{\mathbf{I}}-2\eta\delta_a\delta_\psi\mathbf{J}\mathbf{H}^{-1}(\widehat{\mathbf{A}}-\delta_a\widehat{\mathbf{I}})^2]\underline{Z}\Big\}\,d\tau.
\end{aligned}$$

Using the hypothesis (ii) of Theorem 4.1 and following the same reasoning in [24, Lemma 1] and [26, Lemma 2], to get

$$\begin{aligned}
\dot{V}(t) \leq & -2^{-1}\eta\delta_a\delta_\psi\delta_h|\underline{X}|_{nm}^2 \\
& -\Big[(1+\delta_a)\delta_\psi-(1+\varrho)\triangle_h-\eta\delta_a^2\delta_\psi-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi-\delta_\psi)^2 \\
& -2^{-1}\left[((1+\varrho)\delta_a-(1+\delta_a))^{-1}(\triangle_a-\delta_a)^2\delta_a^2\right]\Big]|\underline{Y}|_{nm}^2 \\
& -2^{-1}\left[(1+\varrho)\delta_a-(1+\delta_a)-2\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_a-\delta_a)^2\right]|\underline{Z}|_{nm}^2.
\end{aligned}$$

If we choose η, such that it satisfies (4.3), then we obtain

$$\dot{V}(t) \leq -\delta_1|\underline{X}|_{nm}^2-\delta_2|\underline{Y}|_{nm}^2-\delta_3|\underline{Z}|_{nm}^2, \tag{4.10}$$

where

$$\begin{aligned}
\delta_1 =& 2^{-1}\eta\delta_a\delta_\psi\delta_h, \\
\delta_2 =& (1+\delta_a)\delta_\psi-(1+\varrho)\triangle_h-\eta\delta_a^2\delta_\psi-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_\psi-\delta_\psi)^2 \\
& -2^{-1}\left[((1+\varrho)\delta_a-(1+\delta_a))^{-1}(\triangle_a-\delta_a)^2\delta_a^2\right], \\
\delta_3 =& 2^{-1}\left[(1+\varrho)\delta_a-(1+\delta_a)-\eta\delta_a\delta_\psi\delta_h^{-1}(\triangle_a-\delta_a)^2\right].
\end{aligned}$$

The above estimates are valid since

$$\sum_{i=1}^m |X^i|_n^2 = \sum_{i=1}^m |X_i|_n^2 = |\underline{X}|_{nm}^2 = \|X\|^2, \quad \text{for any } X\in\widetilde{M}.$$

Thus, $\dot{V}(t) \leq 0$. Now, using $\frac{d}{dt}V(X,Y,Z) = 0$ with (3.6), it is evident that $X = Y = Z = 0$. Thus, the conditions stated in Theorem 4.1 are met.

Therefore, the trivial solution of (3.6) exhibits asymptotic stability. □

4.1. **Numerical Example.** Let us consider (1.1):

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = 0, \quad X \in \widetilde{M}, \tag{4.11}$$

$\widetilde{M}$ being the set of all matrices with dimensions $n \times m$ over the real numbers.

We consider the equivalent system of equation (4.11) in equation (3.6). Let us take for $n = 2$ and $m = 3$ with

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \Psi(Y) = \begin{pmatrix} 4y_1 + \frac{0.01y_1}{1+y_1^2} & 5y_2 + \frac{0.1y_2}{1+y_2^2} & 5y_3 + \frac{0.01y_3}{1+y_3^2} \\ 6y_4 + \frac{0.1y_4}{1+y_4^2} & 4y_5 + \frac{y_5}{1+y_5^2} & 5y_6 + \frac{0.001y_6}{1+y_6^2} \end{pmatrix}$$

and

$$H(X) = \begin{pmatrix} 0.1\tan^{-1}x_1 + 0.01x_1 & 0.1x_2 & 0.01\tan^{-1}x_3 + 0.1x_3 \\ 0.2x_4 & \tan^{-1}x_5 + 0.1x_5 & 0.11x_6 \end{pmatrix}.$$

Thus,

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}.$$

By the notation,

$$\widehat{\mathbf{A}} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbf{JH(X)} = \begin{pmatrix} \frac{0.1}{1+x_1^2} + 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{0.01}{1+x_3^2} + 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1+x_5^2} + 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.11 \end{pmatrix}$$

and

$$\mathbf{J}\Psi(Y) = \mathbf{J}\Psi(Y_p) + \mathbf{J}\Psi(Y_q),$$

where

$$\mathbf{J}\Psi(Y_p)=\begin{pmatrix} 4-\frac{0.02y_1^2}{(1+y_1^2)^2}+\frac{0.01}{1+y_1^2} & 0 & 0 & 0 & 0 & 0\\ 0 & 5-\frac{0.2y_2^2}{(1+y_2^2)^2}+\frac{0.1}{1+y_2^2} & 0 & 0 & 0 & 0\\ 0 & 0 & 5-\frac{0.02y_3^2}{(1+y_3^2)^2}+\frac{0.01}{1+y_3^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\mathbf{J}\Psi(Y_q)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 6-\frac{0.2y_4^2}{(1+y_4^2)^2}+\frac{0.1}{1+y_4^2} & 0 & 0\\ 0 & 0 & 0 & 0 & 4-\frac{2y_5^2}{(1+y_5^2)^2}+\frac{1}{1+y_5^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 5-\frac{0.002y_6^2}{(1+y_6^2)^2}+\frac{0.001}{1+y_6^2} \end{pmatrix}.$$

Clearly, $\widehat{\mathbf{A}}$, $\mathbf{J}\Psi(Y)$ are symmetric. $\widehat{\mathbf{A}}$, $\mathbf{J}\Psi(Y)$ and $\mathbf{JH(X)}$ are associative and commute pairwise.

A simple calculation (with the earlier notations), it is clear that:

$$\delta_a = 2 \le \lambda_i(\widehat{\mathbf{A}}) \le 4 = \triangle_a, \quad i = 1, 2, 3, 4, 5, 6.$$

Thus,

$$\delta_\psi = 4 \le \lambda_i(\mathbf{J}\Psi(Y)) \le 6.1 = \triangle_\psi, \quad i = 1, 2, 3, 4, 5, 6,$$

and

$$\delta_h = 0.1 \le \lambda_i(\mathbf{JH(X)}) \le 1.1 = \triangle_h \quad i = 1, 2, 3, 4, 5, 6,$$

and since, by (4.2),

$$\frac{1}{2} < \varrho < \frac{20}{11},$$

we choose $\varrho = \frac{6}{5}$ so that

$$\eta < \min\{0.5, 1, 0.0009, 0.002\},$$

with the fulfillment of the conditions of Theorem 4.1, the solutions of (3.6) exhibit asymptotic stability.

Remark 4.1. For the case $m = 1$ (that is in $\mathbb{R}^n$), Theorem 4.1 reduces to Corollary 1 in [10] and [27], with obvious modifications.

Remark 4.2. If specialized to case $n = m = 1$ (that is in $\mathbb{R}$) and $h(x) = cx$, equation (1.1) reduces to the scalar differential equation with constant coefficients:

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0.$$

5. Boundedness of Solutions

Here, we investigate the boundedness of solutions of (1.1), where $P \neq 0$ in the equivalent system (3.6).

The following is our boundedness result for (1.1).

Theorem 5.1. *Assuming all the conditions of Theorem* 4.1 *are met and* P *satisfies:*

$$\begin{aligned} \|P(t,X,Y,Z)\| \leq& \theta_1(t) + \theta_2(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{\nu}{2}} \\ &+ \delta_0(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}, \end{aligned} \tag{5.1}$$

for all $t > 0$, *uniformly in* (X,Y,Z), *where* ν, $0 \leq \nu < 1$, *and* $\delta_0 \geq 0$ *are constants and the continuous functions* $\theta_1(t)$, $\theta_2(t)$.

There are constants $\triangle_0$, $\triangle_1$, *such that if* $\delta_0 \leq \triangle_0$, *then every solution* $X(t)$ *of* (1.1) *ultimately satisfies*

$$\|X(t)\|^2 \leq \triangle_1, \quad \|\dot{X}(t)\|^2 \leq \triangle_1, \quad \|\ddot{X}(t)\|^2 \leq \triangle_1,$$

for all sufficiently large t.

To prove Theorem 5.1 we use the matrix scalar function defined in (4.1).

The result below readily follows from (4.1).

Lemma 5.1. *Assuming the satisfaction of all conditions of Theorem* 4.1, *there exist constants* D_1 *and* D_2 *such that:*

$$D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq V(X,Y,Z) \leq D_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{5.2}$$

for any given $X, Y, Z \in \widetilde{M}$.

Proof. It should be noted that $V_a + V_b$ is now the expression in (4.8).

The left side of (5.2) in Lemma 5.1 is established in (4.9) if we can find $D_1 \geq 0$ very small so that:

$$V \geq D_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \quad \text{for } X, Y, Z \in \widetilde{M}.$$

Also the right side of (5.2) of Lemma 5.1 follows by the same reasoning in [3, 10, 24] and [26] if we choose

$$\begin{aligned} D_2 = \max\{&2 \triangle_h + \delta_a \triangle_h + \eta\delta_a\delta_\psi^2 + \eta\delta_a^2\delta_\psi + \eta\delta_a\delta_\psi, \\ &2\delta_a + \varrho \triangle_\psi + 1 + \varrho \triangle_h + \delta_a^2 + \triangle_\psi + \eta\delta_a^2\delta_\psi + \triangle_h + \eta\delta_a\delta_\psi, 2 + \varrho + \eta\delta_a\delta_\psi\}. \end{aligned}$$

The above estimates are valid since

$$\sum_{i=1}^{m} |X^i|_n^2 = \sum_{i=1}^{m} |X_i|_n^2 = |\underline{X}|_{nm}^2 = \|X\|^2, \quad \text{for any } X \in \widetilde{M}.$$

The proof of Lemma 5.1 is now concluded. ☐

We also require the following lemma.

Lemma 5.2. *Assuming the satisfaction of all the conditions in Theorem* 4.1*, consider solutions* $X(t), Y(t), Z(t)$ *be solutions of* (3.6) *with* $V(t) = V(X(t), Y(t), Z(t))$*. Constants* $\triangle_0$*,* D_3 *and* D_4 *exist such that if* δ_0 *in* (5.1) *satisfies* $\delta_0 \leq \triangle_0$*, then*

$$\dot{V}(t) \leq -D_3 Q^2 + D_4(\theta_1(t)Q + \theta_2(t)Q^{1+\nu}), \quad Q \equiv (\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}. \tag{5.3}$$

Proof. Given $\dot{V}(t)_{(3.6)}$, for $P = 0$ in (4.10), now for $P \neq 0$ in (1.1), along any solutions of (3.6) we have

$$\begin{aligned}\dot{V}(t) \leq & -\delta_1|\underline{X}|^2_{nm} - \delta_2|\underline{Y}|^2_{nm} - \delta_3|\underline{Z}|^2_{nm} \\ & + \langle \eta\delta_a^2\delta_\psi X + (1+\delta_a)Y + (1+\varrho)Z, P(t,X,Y,Z)\rangle.\end{aligned}$$

That is,

$$\begin{aligned}\dot{V}(t) \leq & -\delta_1|\underline{X}|^2_{nm} - \delta_2|\underline{Y}|^2_{nm} - \delta_3|\underline{Z}|^2_{nm} \\ & + \{\eta\delta_a^2\delta_\psi\|X\| + (1+\delta_a)\|Y\| + (1+\varrho)\|Z\|\}\|P(t,X,Y,Z)\|.\end{aligned}$$

If $P(t,X,Y,Z)$ satisfies (5.1), we get

$$\begin{aligned}\dot{V}(t) \leq & -\delta_1\|X\|^2 - \delta_2\|Y\|^2 - \delta_3\|Z\|^2 \\ & + (\eta\delta_a^2\delta_\psi\|X\| + (1+\delta_a)\|Y\| + (1+\varrho)\|Z\|)\Big[(\theta_1(t)+ \\ & + \theta_2(t)\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{\nu}{2}} + \delta_0\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right)^{\frac{1}{2}}\Big].\end{aligned}$$

Thus,

$$\dot{V}(t) \leq -\delta_1\|X\|^2 - \delta_2\|Y\|^2 - \delta_3\|Z\|^2 + \delta_4 Q\theta_1(t) + \delta_4\theta_2(t)Q^{\nu+1} + \delta_4\delta_0 Q^2,$$

where $\delta_4 = \max\{\eta\delta_a^2\delta_\psi, (1+\delta_a), (1+\varrho)\}$.

Let $\triangle_0$ be now fixed as

$$\triangle_0 = \frac{1}{2}\delta_4^{-1}\min\{\delta_1, \delta_2, \delta_3\} > 0.$$

Then, for $\delta_0 \leq \triangle_0$, we shall have from the above inequality for V that

$$\dot{V}(t) \leq -\delta_5 Q^2 + \delta_4\{\theta_1(t)Q + \theta_2(t)Q^{\nu+1}\},$$

where $\delta_5 = \delta_0\triangle_0$. Thus, we obtain (5.3) with $D_3 = \delta_5$ and $D_4 = \delta_4$. The estimates above are valid since

$$\sum_{i=1}^{m}|X^i|^2_n = \sum_{i=1}^{m}|X_i|^2_n = |\underline{X}|^2_{nm} = \|X\|^2,$$

for any $X \in \widetilde{M}$.

We conclude the proof of Theorem 5.1 by using inequalities (5.2) and (5.3) and by adapting the reasoning presented in [16], we can easily conclude this part of the proof, therefore, we skip it. Hence, Theorem 5.1 then follows as pointed out earlier. □

5.1. **Numerical Example.** We consider the non-homogeneous form of (4.11) in Example 4.1 as

$$\dddot{X} + A\ddot{X} + \Psi(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}), \quad X \in \widetilde{M}.$$

Suppose we choose

$$P(t,X,Y,Z) = \begin{pmatrix} \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} \\ \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} & \frac{1}{1+t^2+x^2+y^2+z^2} \end{pmatrix}.$$

We have that

$$\begin{aligned} \|P(t,X,Y,Z)\| &= \frac{6}{1+t^2}\left(\sum_{i=1}^{3}|X_i|_n^2 + \sum_{i=1}^{3}|Y_i|_n^2 + \sum_{i=1}^{3}|Z_i|_n^2\right) \\ &\leq 6\left(|\underline{X}|_6^2 + |\underline{Y}|_6^2 + |\underline{Z}|_6^2\right) \leq 6\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right) \leq 6. \end{aligned}$$

Remark 5.1. If $n = m$ (that is $\mathbb{R}^{n\times n}$) and $\Psi(\dot{X}) = B(\dot{X})$ in (1.1), Theorem 5.1 reduces to Theorem 1 in [22]. That is, a direct generalization of [22] and [19].

Remark 5.2. For the case $m = 1$ (that is in $\mathbb{R}^n$) in equation (1.1), this result is a matrix analogue of a result of [3, 12] and [26] with obvious modifications.

6. CONCLUSION

This study gives an insight into the qualitative behaviour of solutions of third order rectangular matrix differential equations. The use of Lyapunov's direct method provides an effective approach to analyze and establish sufficient conditions on stability and ultimate boundedness of solutions of rectangular matrix differential equations as well as provides a valuable tool for the wider study of dynamical systems whose state variables are valued in rectangular array. Numerical simulations and analysis were given in system (4.11) which satisfies all the conditions of Theorem 4.1, Theorem 5.1 and inequalities (4.2) and (4.3). These new results significantly improve those present in existing literature as well as contribute to the qualitative aspects of the theory of matrix differential equations thus providing for the development of more general formulations.

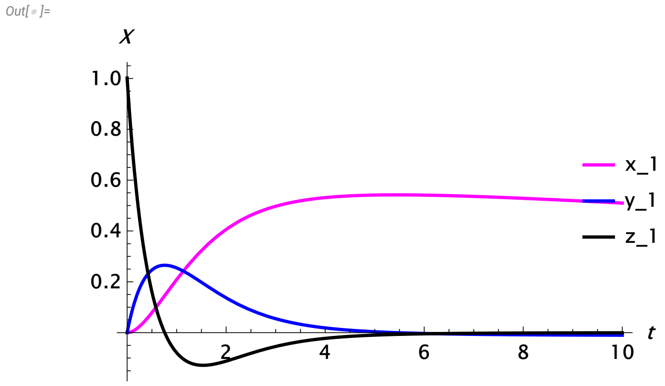

FIGURE 1. The plot of triple $(x_1(t), y_1(t), z_1(t))$ where $x_1(t)$ (in pink), $y_1(t)$ (in blue) and $z_1(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

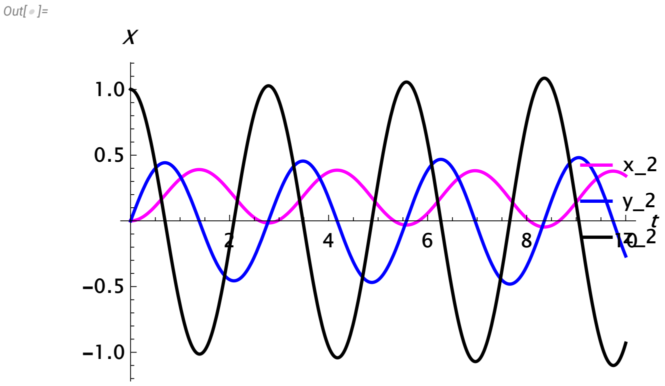

FIGURE 2. The plot of triple $(x_2(t), y_2(t), z_2(t))$ where $x_2(t)$ (in pink), $y_2(t)$ (in blue) and $z_2(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1 and Theorem 5.1

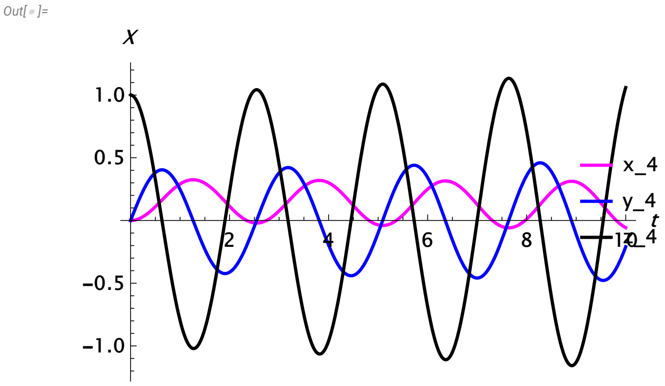

FIGURE 4. The plot of triple $(x_4(t), y_4(t), z_4(t))$ where $x_4(t)$ (in pink), $y_4(t)$ (in blue) and $z_4(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

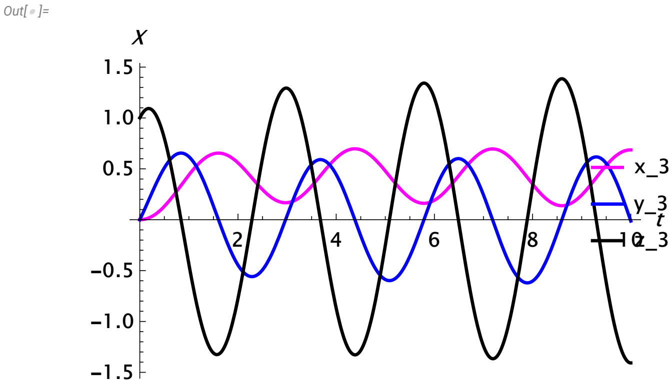

Figure 3. The plot of triple $(x_3(t), y_3(t), z_3(t))$ where $x_3(t)$ (in pink), $y_3(t)$ (in blue) and $z_3(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

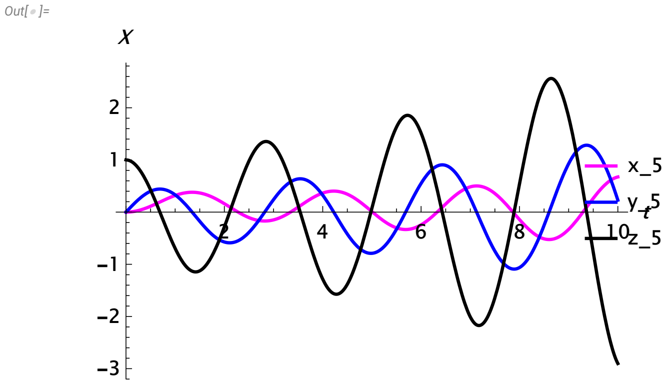

Figure 5. The plot of triple $(x_5(t), y_5(t), z_5(t))$ where $x_5(t)$ (in pink), $y_5(t)$ (in blue) and $z_5(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

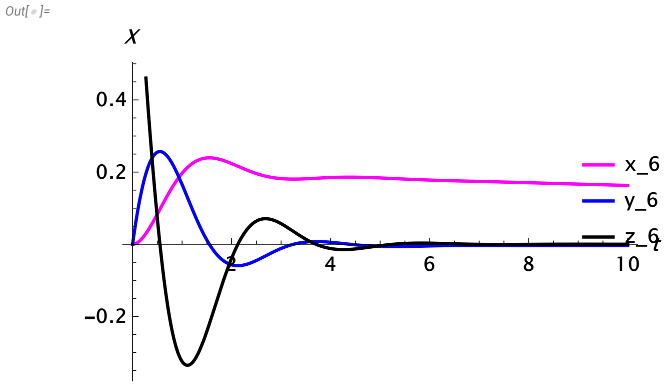

FIGURE 6. The plot of triple $(x_6(t), y_6(t), z_6(t))$ where $x_6(t)$ (in pink), $y_6(t)$ (in blue) and $z_6(t)$ (black) respectively of system (4.11) meeting the conditions of Theorem 4.1

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[1]Department of Mathematics,
Lagos State University,
Ojo, Lagos Nigeria.
Email address: `akinwale.olutimo@lasu.edu.ng`

[2]Department of Mathematics,
Federal University of Agriculture,
Abeokuta, Ogun State, Nigeria.
Email address: `moomeike@yahoo.com`

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A STUDY OF THE SCATTERING PROPERTIES OF EIGENPARAMETER-DEPENDENT MATRIX DIFFERENCE OPERATOR WITH TRANSMISSION CONDITION

GÜLER BAŞAK ÖZNUR[1] AND YELDA AYGAR[2]

Abstract. In this paper, we set a transmission boundary value problem for a matrix valued difference equation on the semi axis. The main purpose of this study is to examine the properties of scattering solutions and scattering functions of this problem. Firstly, by giving the Jost solution and scattering solutions of this problem, we obtain the Jost function and the scattering function of the problem. We also investigate eigenvalues, spectral singularities, resolvent operator and continuous spectrum of this problem.

1. Introduction

In daily life, boundary value or initial value problems are used in the functional analysis, applied mathematics, spectral analysis and scattering analysis modeling of many problems encountered in the fields of physics, mathematics and engineering. For solving these problems in spectral and scattering theory, operator theory is an important tool. For many years, many scientists have used it to analyse the spectral and scattering properties of differential and difference operators in physics, quantum mechanics and applied mathematics. The Sturm-Liouville operator, which is a one-dimensional Schrödinger operator, has an important one in the literature [23, 25, 27, 31] for this analysis. On the other hand, the state of the process can suddenly change during some physical and chemical events, including natural problems. Both differential equations and difference equations theory could not answer this situation.

Key words and phrases. Transmission condition, difference equation, eigenvalues, Jost function, spectral singularity, resolvent operator, scattering function.

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Therefore, a new theory was needed. Sudden and sharp changes can be encountered at same stages of scientific processes. Compared to the whole process, the duration of this sudden and sharp change is negligible, but the functioning of this system still changes. These short-term effects are called impulse effects, and to deal with these effects, the conditions called transmission condition, point interaction, impulsive condition, jump condition and interface condition are applied to the value problem [1, 21, 26, 28, 29]. Non-stationary biological systems such as heart rhythm beats, blood flows, population dynamics; physical phenomena with variable structure such as theoretical physics, atomic physics, radiophysics, phormacokinetics, and many other such as mathematical economy, chemical technology, electrical technology, metallurgy, ecology, industrial robotics, medicine contain impulse effects. Therefore, as a natural response to the developing technology, interest in differential equations with transmission condition has increased and these equations have been the subject of both theoretical and experimental researches. The problems for the differential equation systems with transmission condition were examined in detail by Samoilenko and Perestyuk, Perestyuk et al. and Lakshmikantham et al. and important results were obtained [22, 32, 33]. There are many studies in the literature examining the spectral and scattering analysis of transmission boundary value problems [7, 10–15, 18, 34]. On the other hand, although there are many studies investigating the spectral and scattering theory of various matrix-valued operators without transmission condition [2–5,9,17,30], there are few studies examining the spectral and scattering theory of transmission boundary value problem with matrix coefficients [6, 8, 16]. In this study, our aim is to examine some spectral and scattering properties of a matrix difference operator with transmission conditions. The difference from [8] is that the spectral parameter λ is included in both the matrix coefficient difference equation and the boundary condition. This gives a different perspective to the problem and so this paper becomes the general form of [8].

Let $\mathcal{L}$ denote the matrix difference operator generated in the Hilbert space $l_2\left(\mathbb{N},\mathbb{C}^{\mu}\right)$ given by

$$l_2\left(\mathbb{N},\mathbb{C}^{\mu}\right) := \left\{ Y=\left\{Y_n\right\}_{n\in\mathbb{N}}, Y_n\in\mathbb{C}^{\mu}, ||Y||^2=\sum_{n\in\mathbb{N}}||Y_n||^2<+\infty \right\},$$

where $\mathbb{C}^{\mu}$ is a μ-dimensional ($\mu<\infty$) Euclidian space, $||\cdot||$ denotes the matrix norm in $\mathbb{C}^{\mu}$. We shall consider that the operator $\mathcal{L}$ is created by the following difference expression

$$Y_{n-1}+D_nY_n+Y_{n+1}=\lambda Y_n, \quad n\in\mathbb{N}\backslash\{m_0-1,m_0,m_0+1\}, \tag{1.1}$$

with the boundary condition

$$\left(\gamma_0+\gamma_1\lambda\right)Y_1+\left(\nu_0+\nu_1\lambda\right)Y_0=0, \quad \gamma_0\nu_1-\gamma_1\nu_0\neq 0, \tag{1.2}$$

and the transmission conditions

$$\begin{cases} Y_{m_0+1} = \widetilde{K} Y_{m_0-1}, \\ Y_{m_0+2} = \widetilde{M} Y_{m_0-2}, \end{cases} \tag{1.3}$$

where $\lambda = 2\cos z$ is a spectral parameter, for $i = 0, 1$, γ_i, ν_i are real numbers, $D := \{D_n\}_{n\in\mathbb{N}}$ is a selfadjoint matrix acting in $\mathbb{C}^\mu$ satisfying

$$\sum_{n\in\mathbb{N}} n\, ||D_n|| < +\infty, \tag{1.4}$$

and m_0 is an arbitrary natural number. Throughout this paper, we assume that $\widetilde{K}$ and $\widetilde{M}$ are selfadjoint diagonal matrices in $\mathbb{C}^\mu$ such that all eigenvalues of $\widetilde{K}$ and $\widetilde{M}$ are different and nonzero. Since D is a selfadjoint matrix, it is clear that if $Y_n(z)$ is a solution of (1.1), then $Y_n^T(z)$ is a solution of (1.1), where "T“ is the transpose operator.

The set of this paper is summarized as follows. In Section 2, we give the basic solutions and properties of equation of (1.1) without the transmission condition. In Section 3, we obtain basic results and theorems for Jost solution, Jost function and scattering function of this problem. In Section 4, we find resolvent operator and Green function of the operator $\mathcal{L}$. We also get the sets of eigenvalues and spectral singularities of this problem. Then, we obtain the asymptotic representation of the Jost function and continuous spectrum of (1.1)–(1.3).

2. Preliminaries And Auxiliary Results

In this section, we first give useful information and results for matrix difference equation with a general boundary condition that we use throughout the study. We remark that Wronskian of any two solutions $U = \{U_n(z)\}$ and $V = \{V_n(z)\}$ of the equation (1.1) is known as

$$W\left[U, V^T\right](n) = V_{n-1}^T U_n - V_n^T U_{n-1}. \tag{2.1}$$

Now, let us define two semi-strips

$$B := \left\{z \in \mathbb{C} : z = x + iy, y > 0, -\frac{\pi}{2} \le x \le \frac{3\pi}{2}\right\}, \quad B_0 := B \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Assume that $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (1.1) for $z \in B_0$ and $n = 0, 1, \dots, m_0 - 1$, fulfilling the initial conditions

$$P_0(z) = 0, \quad P_1(z) = I,$$

$$Q_0(z) = I, \quad Q_1(z) = 0.$$

The solutions $P_n(z)$ and $Q_n(z)$ are entire functions of z.

Furthermore, for $z \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im} z \ge 0\}$, the bounded solution $E(z) = \{E_n(z)\}$ of (1.1) which is represented by

$$E_n(z) = e^{inz}\left[I + \sum_{m=1}^{+\infty} K_{nm} e^{imz}\right], \quad n = m_0 + 1, m_0 + 2, \dots,$$

where K_{nm} is expressed in terms of $\{D_n\}$. $E(z)$ is called the Jost solution of the equation (1.1) and provides the following asymptotic equalities for $z \in \overline{\mathbb{C}}_+$ [20]

$$\begin{aligned} E_n(z) =& e^{inz}\left[I + o(1)\right], \quad n \to +\infty, \\ E_n(z) =& e^{inz}\left[I + o(1)\right], \quad \operatorname{Im} z \to +\infty. \end{aligned} \tag{2.2}$$

Additionally, equation (1.1) has an unbounded solution, denoted by $\widehat{E}(z) = \left\{\widehat{E}_n(z)\right\}$, which satisfies the following asymptotic equation

$$\widehat{E}_n(z) = e^{-inz}\left[I + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad n \to +\infty.$$

3. JOST SOLUTION, JOST FUNCTION AND SCATTERING MATRIX

For $z \in B_0$, let us define the following solution of (1.1)–(1.3) by using $P(z)$, $Q(z)$ and $E(z)$

$$J_n(z) = \begin{cases} P_n(z)\theta_1(z) + Q_n(z)\theta_2(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ E_n(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}, \end{cases}$$

here θ_1 and θ_2 are z-dependent coefficients. By the help of (1.3), we can obtain the following equalities

$$\widetilde{K}^{-1}E_{m_0+1}(z) = P_{m_0-1}(z)\theta_1(z) + Q_{m_0-1}(z)\theta_2(z) \tag{3.1}$$

and

$$\widetilde{M}^{-1}E_{m_0+1}(z) = P_{m_0-2}(z)\theta_1(z) + Q_{m_0-2}(z)\theta_2(z). \tag{3.2}$$

From (2.1), it can be easily found that $W\left[P(z), P^T(z)\right] = 0$, $W\left[Q(z), Q^T(z)\right] = 0$ and $W\left[P(z), Q^T(z)\right] = I$ for all $z \in \overline{\mathbb{C}}_+$. Using these Wronskian equalities, (3.1) and (3.2), $\theta_1(z)$ and $\theta_2(z)$ must be as follows:

$$\begin{aligned} \theta_1(z) &= \widetilde{K}^{-1}\widetilde{M}^{-1}\left[\widetilde{M}Q^T_{m_0-2}(z)E_{m_0+1}(z) - \widetilde{K}Q^T_{m_0-1}(z)E_{m_0+2}(z)\right], \\ \theta_2(z) &= \widetilde{K}^{-1}\widetilde{M}^{-1}\left[\widetilde{K}P^T_{m_0-1}(z)E_{m_0+2}(z) - \widetilde{M}P^T_{m_0-2}(z)E_{m_0+1}(z)\right], \end{aligned}$$

respectively. The function $J_n(z)$ is called the Jost solution of (1.1)–(1.3). We define the Jost function of (1.1)–(1.3) by applying the boundary condition (1.2) to the Jost solution $J_n(z)$ of the operator $\mathcal{L}$

$$\widetilde{J}(z) = (\gamma_0 + \gamma_1\lambda)\,J_1(z) + (\nu_0 + \nu_1\lambda)\,J_0(z) = (\gamma_0 + \gamma_1\lambda)\,\theta_1(z) + (\nu_0 + \nu_1\lambda)\,\theta_0(z).$$

It is easily seen that the function $\widetilde{J}$ is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

For $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, (1.1) has another solution $F(z) := \{F_n(z)\}$ represented by

$$F_n(z) = \begin{cases} \psi_n(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ E_n(z)\theta_3(z) + E_n(-z)\theta_4(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

By using the transmission conditions (1.3), it is easy to write

$$E_{m_0+1}(z)\theta_3(z) + E_{m_0+1}(-z)\theta_4(z) = \widetilde{K}\psi_{m_0-1}(z) \tag{3.3}$$

and

$$E_{m_0+2}(z)\theta_3(z) + E_{m_0+2}(-z)\theta_4(z) = \widetilde{M}\psi_{m_0-2}(z). \tag{3.4}$$

Since $W\left[E(z), E^T(z)\right] = 0$ and $W\left[E(-z), E^T(z)\right] = -2\sin z$, by making some calculations in equations (3.3) and (3.4), we find

$$\begin{aligned}\theta_3(z) &= -\frac{1}{2i\sin z}\left[\widetilde{K}E^T_{m_0+2}(-z)\psi_{m_0-1}(z) - \widetilde{M}E^T_{m_0+1}(-z)\psi_{m_0-2}(z)\right],\\ \theta_4(z) &= \frac{1}{2i\sin z}\left[\widetilde{K}E^T_{m_0+2}(z)\psi_{m_0-1}(z) - \widetilde{M}E^T_{m_0+1}(z)\psi_{m_0-2}(z)\right],\end{aligned}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}$.

Corollary 3.1. *The coefficients θ_3 and θ_4 have the following relation between the Jost function $\widetilde{J}$*

$$\theta_4^T(z) = \theta_3^T(-z) = -\frac{\widetilde{K}\widetilde{M}}{2i\sin z}\widetilde{J}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}. \tag{3.5}$$

Theorem 3.1. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}$, $\det \widetilde{J}(z) \neq 0$.*

Proof. We assume that there exists a $z_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}$, such that $\det \widetilde{J}(z_0) = 0$. In accordance with (3.5), we get

$$\det \theta_4^T(z_0) = \det \theta_3^T(-z_0) = \frac{1}{4\sin^2 z}\det \widetilde{K}\det \widetilde{M}\det \widetilde{J}(z)$$

and

$$\det \theta_4(z_0) = \det \theta_3(z_0) = 0.$$

It follows from that $F_n(z_0) = 0$, that is, F is a trivial solution of (1.1)–(1.3). This gives a contradiction with our assumption, i.e., for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}$, $\det \widetilde{J}(z) \neq 0$. The proof is completed. ☐

Theorem 3.1 says that the inverse of the function $\widetilde{J}$ exists and we give the following definition.

Definition 3.1. The matrix function

$$S(z) = \widetilde{J}^{-1}(z)\widetilde{J}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\},$$

is called the scattering matrix of (1.1)–(1.3).

Theorem 3.2. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash\{0, \pi\}$, the matrix function $S(z)$ satisfies*

$$S(-z) = S^{-1}(z) = S^*(z),$$

and it is an uniter matrix, where ”$*$“ denotes the adjoint operator.

Proof. By the help of definition of scattering matrix, for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, we obtain

$$S(-z) = \widetilde{J}^{-1}(-z)\widetilde{J}(z),$$

and it concludes

$$S(z)S(-z) = S(-z)S(z) = I, \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}.$$

From the last equality, we find

$$S(-z) = S^{-1}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}.$$

Now, let us consider the solutions $J_n(z)$, $J_n(-z)$ and $F_n(z)$, to prove $S^*(z) = S(-z)$. Hence, we write

$$\begin{aligned} F_n(z) =& J_n(z)\eta + J_n(-z)\alpha, \\ F_{n+1}(z) =& J_{n+1}(z)\eta + J_{n+1}(-z)\alpha, \end{aligned} \tag{3.6}$$

where η and α are matrices not depending on n. By making some calculations in (3.6), η and α are obtained as follows:

$$\eta = W^{-1}\left[\widetilde{J}(z), \widetilde{J}^*(z)\right]\left\{J^*_{n+1}(z)F_n(z) - J^*_n(z)F_{n+1}(z)\right\}$$

and

$$\alpha = W^{-1}\left[\widetilde{J}(-z), \widetilde{J}^*(-z)\right]\left\{J^*_{n+1}(-z)F_n(z) - J^*_n(-z)F_{n+1}(z)\right\},$$

respectively. Because of the characteristic features of the transmission conditional equations, we find that $W^{-1}\left[J(z), J^*(z)\right] = -W^{-1}\left[J(-z), J^*(-z)\right]$. Then, letting $n = 0$ in η and α, the following expressions are obtained

$$\eta = W^{-1}\left[J(z), J^*(z)\right]J^*(z), \quad \alpha = -W^{-1}\left[J(z), J^*(z)\right]J^*(-z).$$

When we substitute η and α in (3.6), we get

$$F_n(z) = W^{-1}\left[J(z), J^*(z)\right]\left\{J_n(z)J^*(z) - J_n(-z)J^*(-z)\right\}.$$

By taking $n = 0$ and $n = 1$ in last equation, we find the following equations

$$(\gamma_0 + \gamma_1\lambda) = W^{-1}\left[J(z), J^*(z)\right]\left\{J_0(z)J^*(z) - J_0(-z)J^*(-z)\right\}, \tag{3.7}$$

$$(\nu_0 + \nu_1\lambda) = -W^{-1}\left[J(z), J^*(z)\right]\left\{J_1(z)J^*(z) - J_1(-z)J^*(-z)\right\}. \tag{3.8}$$

By making some calculations in (3.7) and (3.8), we obtain

$$\widetilde{J}(z)\widetilde{J}^*(z) = \widetilde{J}(-z)\widetilde{J}^*(-z). \tag{3.9}$$

Using (3.9), we easily find

$$\widetilde{J}^*(z) = \widetilde{J}^{-1}(z)\widetilde{J}(-z)\widetilde{J}^*(-z)$$

and

$$\widetilde{J}^*(z)\left[\widetilde{J}^*(-z)\right]^{-1} = \widetilde{J}^{-1}(z)\widetilde{J}(-z).$$

Finally, it is clear that $S^*S = SS^* = I$, $||S|| = I$, i.e., S is unitary. □

Lemma 3.1. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$*, the following equation holds*

$$W[J(z), F^T(z)](n) = \begin{cases} \widetilde{J}(z), & \textit{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ -\widetilde{K}\widetilde{M}\widetilde{J}(z), & \textit{if } n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

Proof. From (2.1), we obtain

$$W\left[J(z), F^T(z)\right](n) = F_0^T(z)J_1(z) - F_1^T(z)J_0(z),$$

for $n = 0, 1, \dots, m_0 - 1$. Since it is known that $P_0(z) = 0$, $P_1(z) = I$, $Q_0(z) = I$ and $Q_1(z) = 0$, the following Wronskian is easily found

$$W[J(z), F^T(z)](n) = \widetilde{J}(z), \quad n = 0, 1, \dots, m_0 - 1.$$

Similarly, for $n = m_0 + 1, m_0 + 2, \dots$, we find $W[J(z), F^T(z)](n) = 2i \sin z \theta_4^T(z)$. In view of (3.5), the Wronskian can be arranged

$$W[J(z), F^T(z)](n) = -\widetilde{K}\widetilde{M}\widetilde{J}(z), \quad n = m_0 + 1, m_0 + 2, \dots$$

The proof is completed. □

4. Resolvent Operator, Eigenvalues, Spectral Singularities And Continuous Spectrum

In the following, we will define the other solution of (1.1)–(1.3) for all $z \in B_0$

$$G_n(z) = \begin{cases} \psi_n(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ E_n(z)\theta_5(z) + \widehat{E}_n(z)\theta_6(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

By using the transmission condition (1.3) to $G_n(z)$, we get

$$E_{m_0+1}(z)\theta_5(z) + \widehat{E}_{m_0+1}(z)\theta_6(z) = \widetilde{K}\psi_{m_0-1}(z),$$
$$E_{m_0+2}(z)\theta_5(z) + \widehat{E}_{m_0+2}(z)\theta_6(z) = \widetilde{M}\psi_{m_0-}(z).$$

To get the coefficients $\theta_5(z)$ and $\theta_6(z)$, we will use same way as finding $\theta_1(z)$ and $\theta_2(z)$. Since

$$W\left[E(z), E^T(z)\right] = 0, \quad W\left[\widehat{E}(z), E^T(z)\right] = -2i \sin z$$

and

$$W\left[\widehat{E}(z), \widehat{E}^T(z)\right] = 0, \quad W\left[E(z), \widehat{E}^T(z)\right] = 2i \sin z,$$

$\theta_5(z)$ and $\theta_6(z)$ must be as follows:

$$\theta_5(z) = \frac{1}{2i \sin z}\left[\widetilde{K}\widehat{E}_{m_0+2}^T(z)\psi_{m_0-1}(z) - \widetilde{M}\widehat{E}_{m_0+1}^T(z)\psi_{m_0-2}(z)\right]$$

and

$$\theta_6(z) = \frac{1}{2i \sin z}\left[\widetilde{K}E_{m_0+2}^T(z)\psi_{m_0-1}(z) - \widetilde{M}E_{m_0+1}^T(z)\psi_{m_0-2}(z)\right].$$

Note that

$$\theta_6(z) = -\frac{\widetilde{K}\widetilde{M}}{2i \sin z}\widetilde{J}^T(z).$$

Similar to Lemma 3.1, the following Wronskian equation is obtained

$$\widetilde{C}(z) := W[J(z), G^T(z)](n) = \begin{cases} \widetilde{J}(z), & \text{if } n \in \{0, 1, \dots, m_0 - 1\}, \\ -\widetilde{K}\widetilde{M}\widetilde{J}(z), & \text{if } n \in \{m_0 + 1, m_0 + 2, \dots\}, \end{cases}$$

for $z \in B_0$.

Theorem 4.1. *The resolvent operator of* $\mathcal{L}$ *has the representation*

$$(\mathcal{R}_\lambda(\mathcal{L})\varphi)_n := \sum_{k=0}^{\infty} \mathcal{H}_{n,k}(z)\varphi(k), \quad \varphi := \{\varphi_k\} \in l_2\left(\mathbb{N}, \mathbb{C}^h\right),$$

where

$$\mathcal{H}_{n,k} = \begin{cases} J_n(z)\widetilde{C}^{-1}(z)G_k^T(z), & \text{if } k < n, \\ G_n(z)\left[\widetilde{C}^{-1}(z)\right]^T J_k^T(z), & \text{if } k \geq n, \end{cases}$$

is the Green function of $\mathcal{L}$ *for* $z \in B_0$ *and* $k, n \neq m_0$.

Proof. To obtain the resolvent operator and Green function of $\mathcal{L}$, we need to find the solutions of the following equation

$$\nabla(\triangle Y_n) + M_n Y_n - \lambda Y_n = \psi_n, \tag{4.1}$$

where $M_n = 2I_n + D_n$. Using $J(z)$ and $G(z)$, we can write the general solution of (4.1) as

$$Y_n(z) = J_n(z)R_n + G_n(z)T_n,$$

where $R := \{R_n\}_{n\in\mathbb{N}}$ and $T := \{T_n\}_{n\in\mathbb{N}}$ are self-adjoint diagonal matrices in $\mathbb{C}^\mu$. By the help of the method of variation of parameters, the coefficients R and T can be written

$$R_n = R_0 + \sum_{k=1}^{n} \frac{G_k^T(z)\varphi_k(z)}{\widetilde{C}(z)}, \quad T_n = \zeta + \sum_{k=n+1}^{\infty} \frac{J_k^T(z)\varphi_k(z)}{\widetilde{C}^T(z)},$$

where R_0 and ζ are self-adjoint diagonal matrices in $\mathbb{C}^\mu$. Since the solution $Y_n(z)$ in $l_2(\mathbb{N}, \mathbb{C}^\mu)$, ζ is zero. By the help of the boundary condition (1.2), we find that R_0 is equal to zero. It completes the proof of Theorem 4.1. □

Now, from Theorem 4.1, we define the sets of eigenvalues and spectral singularities of $\mathcal{L}$ as follows:

$$\sigma_d(\mathcal{L}) = \left\{\lambda = 2\cos z : z \in D, \det\widetilde{J}(z) = 0\right\},$$
$$\sigma_{ss}(\mathcal{L}) = \left\{\lambda = 2\cos z : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}, \det\widetilde{J}(z) = 0\right\},$$

respectively.

Theorem 4.2. *Assume* (1.4). *Then the Jost function* $\widetilde{J}$ *satisfies the following asymptotic equation*

$$\widetilde{J}(z) = \nu_1\left(\widetilde{K}\widetilde{M}\right)^{-1}\left(\widetilde{K} - \widetilde{M}\right)[I + o(1)]\left(e^{5iz} + e^{3iz}\right), \quad z \in B_0, \ |z| \to +\infty.$$

Proof. Since the polynomial function $P_n(z)$ is of $(n-1)$. degree and polynomial function $Q_n(z)$ is of $(n-2)$. degree with respect to λ, we get

$$(\nu_0 + \nu_1\lambda) P_n^T(z)e^{i(n-1)z} = \nu_1 [I + o(1)], \quad |z| \to +\infty, \, z \in B_0. \tag{4.2}$$

It is clear that

$$\begin{aligned}\widetilde{J}(z) =& \widetilde{K}^{-1}\widetilde{M}^{-1} (\nu_0 + \nu_1\lambda) \Big[\widetilde{K}P_{m_0-1}^T(z)e^{i(m_0-2)z}e^{-i(m_0-2)z}E_{m_0+2}(z)e^{-i(m_0+2)z}e^{i(m_0+2)z} \\ &- \widetilde{M}P_{m_0-2}^T(z)e^{i(m_0-3)z}e^{-i(m_0-3)z}E_{m_0+1}(z)e^{-i(m_0+1)z}e^{i(m_0+1)z}\Big].\end{aligned}$$

By using (2.2) and (4.2), we write the following asymptotic equation

$$\widetilde{J}(z) = \nu_1 \left(\widetilde{K}\widetilde{M}\right)^{-1} \left(\widetilde{K} - \widetilde{M}\right) [I + o(1)] \left(e^{5iz} + e^{3iz}\right), \quad z \in B_0, \, |z| \to +\infty.$$

Theorem 4.3. *If the condition* (1.4) *satisfies, then* $\sigma_c(\mathcal{L}) = [-2, 2]$, *where* $\sigma_c(\mathcal{L})$ *denotes the continuous spectrum of* $\mathcal{L}$.

Proof. Let us introduce the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ generated by the following difference expression in $l_2(\mathbb{N}, \mathbb{C}^\mu)$ with (1.2) and (1.3)

$$\begin{aligned}(\mathcal{L}_0 y)_n =& Y_{n-1} + Y_{n+1}, \quad n \in \mathbb{N}\backslash\{m_0 - 1, m_0 + 1\}, \\ (\mathcal{L}_1 Y)_n =& D_n Y_n, \quad n \in \mathbb{N}\backslash\{m_0\},\end{aligned}$$

respectively. Under the condition (1.4), it is clear to see the compactness of $\mathcal{L}_1$ [24]. On the other hand, we write $\mathcal{L} = \mathcal{L}_0^1 + \mathcal{L}_0^2 + \mathcal{L}_1$, where L_0^1 is a selfadjoint operator with $\sigma_c(\mathcal{L}_0^1) = [-2, 2]$ and L_0^2 is a finite dimensional operator in $l_2(\mathbb{N}, \mathbb{C}^\mu)$. Then, by the help of Weyl theorem of a compact perturbation [19], we find the continuous spectrum of $\mathcal{L}$. ☐

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[1]DEPARTMENT OF MATHEMATICS,
GAZI UNIVERSITY,
TEKNIK OKULLAR, ANKARA, 06500, TURKEY
Email address: `basakoznur@gazi.edu.tr`

[1]DEPARTMENT OF MATHEMATICS,
ANKARA UNIVERSITY,
TANDOĞAN, 06100, TURKEY
Email address: `yaygar@ankara.edu.tr`

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ON HADAMARD-CAPUTO IMPLICIT FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY FRACTIONAL CONDITIONS

AHMED A. HAMOUD[1*], CHAHRA KECHAR[2], ABDELOUAHEB ARDJOUNI[2],
HOMAN EMADIFAR[3,4,5], ATUL KUMAR[6], AND LAITH ABUALIGAH[7,8,9,10,11,12]

Abstract. The purpose of this paper is to investigate the existence and uniqueness of solutions for nonlinear fractional implicit integro-differential equations of Hadamard-Caputo type with fractional boundary conditions. The reasoning is inspired by diverse classical fixed point theory, such as the Schauder and Banach fixed point theorems. The theoretical findings are illustrated through an example.

1. Introduction

In mathematical analysis, fractional calculus (FC) is a subject that studies different approaches of defining non-integer order derivatives (i.e., fractional differential calculus (FDC)) and integrals (i.e., fractional integral calculus (FIC)). Fractional calculus is widely and efficiently used to describe many phenomena arising in physics, engineering, bioengineering and biomedical sciences, finance, viscoelasticity, control theory, stochastic processes and economy. Recently, fractional differential equations (FDEs) have attracted many authors (see for example [1–3, 8, 13] and references therein).

By flipping the differential and integral sections of the Hadamard derivative, a novel method known as the Hadamard-Caputo derivative is created. The primary distinction between the Hadamard fractional derivative and the Hadamard-Caputo fractional derivative, notwithstanding the various demands placed on the function itself, is that the Hadamard-Caputo derivative of a constant is zero [24]. The most

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significant benefit of the Hadamard-Caputo derivative is that it gave rise to a new concept that can be used to establish the integer order beginning conditions for fractional.

For more details and properties of Hadamard-Caputo derivative and Hadamard fractional derivatives, integrals see [10,11,26].

The implicit fractional differential equations (IFDEs) are a very important class of fractional differential equations. This type of equation is derived from the implicit ordinary differential equation (IODE) of the following form

$$H\left(\varrho, G(\varrho), G'(\varrho), \ldots, G^{(n-1)}(\varrho)\right) = 0,$$

with different kind of initial or boundary conditions, for more details see [6,14,15,23].

Benchohra et al. [7,9] and Nieto et al. [28,30] have initiated the study of implicit fractional differential equations (IFDEs) of the form

$$D^{\alpha} G(\varrho) = H\left(\varrho, G(\varrho), D^{\alpha} G(\varrho)\right),$$

with different kind of initial or boundary conditions.This sort of equation is crucial in many different fields of science and engineering [34].

In [33], Vivek et al. showed that a class of boundary value systems for nonlinear IFDEs with complex order have a solution and are stable

$$\begin{aligned} &{}^{c}D^{\theta} G(\varrho) = H\left(\varrho, G(\varrho), {}^{c}D^{\theta} G(\varrho)\right), \quad \theta = m + i\alpha, \quad \varrho \in Y := [0, \chi], \\ &aG(0) + bG(\chi) = c, \end{aligned}$$

where $\theta \in \mathbb{C}$, ${}^{c}D^{\theta}$ is the Caputo fractional derivative. Suppose $m \in (0,1]$, $\alpha \in \mathbb{R}_{+}$, $0 < \alpha < 1$, $H : Y \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Further $a, b, c \in \mathbb{R}$ with $a + b \neq 0$. The results are based upon the Schaefer's fixed point theorem and Banach contraction principle.

In [27] Karthikeyan and Arul stydied the uniqueness of integral BVP for IFDEs involving Hadamard-Caputo fractional derivative

$$\begin{aligned} {}^{CH}D^{\theta} \zeta(\varrho) &= H\left(\varrho, \zeta(\varrho), {}^{CH}D^{\theta} \zeta(\varrho)\right), \quad \varrho \in \xi := [n, \chi], \\ \zeta(n) = 0, \quad \zeta(\chi) &= \Omega \int_{n}^{\sigma} \zeta(\omega)\, d\omega, \quad n < \sigma < \chi, \end{aligned}$$

where $\Omega \in \mathbb{R}$, $n < \sigma < \chi$, $1 < \theta \leq 2$, ${}^{CH}D^{\theta}$ is the Hadamard-Caputo fractional derivative, and $H : \xi \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

In [12] N. Derdar established the uniqueness of solutions for the system:

$$\begin{aligned} {}_{H}^{C}D^{r} \upsilon(\varrho) &= W\left(\varrho, \upsilon(\varrho), {}_{H}^{C}D^{r} \upsilon(\varrho)\right), \\ \upsilon(1) = 0, \quad \alpha\, {}_{H}I^{q} \upsilon(\eta) &+ \beta\, {}_{H}^{C}D^{\gamma} \upsilon(\Psi) = \lambda, \end{aligned}$$

by using different fixed point theorem.

Recently, many authors focus on the development of techniques for discussing the solutions of FIDEs.

Balachandran and Trujillo [5], investigated the existence of a unique solution for FIDEs with boundary value conditions.

$$^{C}\mathfrak{D}^{\alpha} f(\upsilon) = W(\upsilon, f(\upsilon)) + \int_{\upsilon_0}^{\upsilon} K(\upsilon, \varrho, f(\varrho))\, d\varrho, \quad 0 < \alpha \leq 1.$$

In [32] the authors investigated the uniqueness of solution for iterative integro-differential system:

$$\begin{aligned} \mathfrak{D}^{\alpha} w(\upsilon) &= Q(\upsilon) + \int_0^{\varrho} H(\upsilon, s)\, w(\lambda w(s))\, ds, \\ w(0) &= w_0. \end{aligned}$$

In [17] A. A. Hamoud established the uniqueness and stability for fractional nonlinear Fredholm–Volterra system:

$$\begin{aligned} &^{C}\mathfrak{D}^{\alpha} w(\varrho) = H(\varrho) + \int_0^{\varrho} \vartheta(\varrho, s)\, w(w(s))\, ds + \int_0^{\Psi} \theta(\varrho, s)\, w(w(s))\, ds, \\ &a w(0) + b w(\Psi) = c, \quad a, b, c, \in \mathbb{R}. \end{aligned}$$

For some other results on FIDE, see [4, 5, 16, 18–22, 25, 29].

Motived by the above papers and the reference [12], we study the theoretical analysis of solutions for a class of system for nonlinear implicit FIDEs of Hadamard-Caputo type with fractional boundary conditions

$$\begin{aligned} &c{}^{C}_{H}D^{r} \upsilon(\varrho) = f\left(\varrho, \upsilon(\varrho), {}^{C}_{H}D^{r} \upsilon(\varrho), \int_1^{\varrho} K(\varrho, s, \upsilon(s))\, ds\right), \\ &\upsilon(1) = 0, \quad \alpha\ {}_{H}I^{q} \upsilon(\eta) + \beta\ {}^{C}_{H}D^{\gamma} \upsilon(\Psi) = \lambda, \end{aligned} \tag{1.1}$$

where ${}_{H}I^{q}$ is the standard Hadamard fractional integral, ${}^{C}_{H}D^{r}$ is the Hadamard-Caputo fractional derivative, $f : \xi \times \mathbb{R}^3 \to \mathbb{R}$, $K : \xi \times \xi \times \mathbb{R} \to \mathbb{R}$ are given functions, $\eta \in \xi =: (1, \Psi], \Psi > 1$ and α, λ, β are real constants.

2. Preliminaries and Background Materials

Let us introduce some necessary notations and definitions which will be utilised throughout the entire process [28, 29, 31, 33–35].

We represent by the symbol $C(\xi, \mathbb{R})$ the space of all continuous functions $\upsilon : \xi \to \mathbb{R}$ in Banach space with the supremum norm

$$\|\upsilon\|_{\infty} = \sup\{|\upsilon(\varrho)| : \varrho \in \xi\}.$$

Let now $[b, c]$, $-\infty < b < c < +\infty$, is finite interval and we suppose $AC([b, c], \mathbb{R})$ is the space of functions $\psi : [b, c] \to \mathbb{R}$ that are absolutely continuous.

Assume $\delta = \varrho \frac{d}{dt}\ d\varrho$ is the Hadamard derivative, $\delta^n = \delta(\delta^{n-1})$, we consider the set of functions:

$$AC^n_{\delta}([b, c], \mathbb{R}) = \left\{\psi : [b, c] \to \mathbb{R} : \delta^{n-1} \psi(\varrho) \in AC([b, c], \mathbb{R})\right\}.$$

Definition 2.1 ([28])**.** The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $\psi : [1, +\infty) \to \mathbb{R}$ is given by

$$ {}_H I_1^{\alpha} \psi(\varrho) = \frac{1}{\Gamma(\alpha)} \int_1^{\varrho} \left(\log \frac{\varrho}{s} \right)^{\alpha - 1} \psi(s) \frac{ds}{s}, $$

where $\log(\cdot) = \log_e(\cdot)$ and $\Gamma(\cdot)$ is Gamma function.

Definition 2.2 ([31])**.** For a function $\psi \in AC_{\delta}^{n}([b, c], \mathbb{R})$, the Hadamard-Caputo fractional derivative of order α is given by

$$ {}_H^C D_1^{\alpha} \psi(\varrho) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt} \right)^n \int_1^{\varrho} \left(\log \frac{\varrho}{s} \right)^{n - \alpha - 1} \psi(s) \frac{ds}{s}, \quad n - 1 < \alpha < n, $$

where $\delta^n = \left(\varrho \frac{d}{d\varrho} \right)^n$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Lemma 2.1 ([24])**.** *Let $\psi \in AC_{\delta}^{n}[b, c]$ or $\psi \in C_{\delta}^{n}[b, c]$ and $\alpha \in \mathbb{C}$. Then,*

$$ {}_H I_b^{\alpha} \left({}_H^C D_b^{\alpha} \psi \right)(\varrho) = \psi(\varrho) - \sum_{k=0}^{n-1} \frac{\delta^{(k)} \psi(b)}{k!} \left(\log \frac{\varrho}{b} \right)^k . $$

Proposition 2.1 ([24])**.** *Let $\alpha > 0$, $\beta > 0$, $n = [\alpha] + 1$, and $b > 0$. Then,*

$$ \begin{aligned} \left({}_H I_{b^+}^{\alpha} \left(\log \frac{\upsilon}{b} \right)^{\beta - 1} \right)(\upsilon) &= \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} \left(\log \frac{\upsilon}{b} \right)^{\beta + \alpha - 1}, \\ \left({}_H^C D_{b^+}^{\alpha} \left(\log \frac{\upsilon}{b} \right)^{\beta - 1} \right)(\upsilon) &= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} \left(\log \frac{\upsilon}{b} \right)^{\beta - \alpha - 1}, \quad \alpha < \beta. \end{aligned} $$

Theorem 2.1. [24] *Let $\upsilon(\varrho) \in AC_{\delta}^{n}[b, c]$, $0 < b < c < +\infty$ and $\alpha \geq 0$, $\beta \geq 0$. Then,*

$$ {}^C D_b^{\alpha} \left(I^{\beta} \upsilon \right)(\varrho) = \left(I^{\beta - \alpha} \upsilon \right)(\varrho), $$

$$ {}^C D^{\alpha} \left({}^C D^{\beta} \right)(\varrho) = {}^C D^{\alpha + \beta}(\varrho). $$

Theorem 2.2 (Schauder's fixed point [35])**.** *Suppose that E be a Banach space, and P be a nonempty, convex and closed subset of E. Assume that $\mathcal{A} : P \to P$ be a continuous mapping and $\mathcal{A}(P)$ is a relatively compact subset of E. Then $\mathcal{A}$ admits at least one fixed point in P.*

3. Main Results

Definition 3.1. A function $\upsilon \in AC_{\delta}^{2}(\xi, \mathbb{R})$ is said to be a solution of the system (1.1) if υ satisfies the equation ${}_H^C D^{\kappa} \upsilon(\varrho) = f\left(\varrho, \upsilon(\varrho), {}_H^C D^{\kappa} \upsilon(\varrho), \int_1^{\varrho} K(\varrho, s, \upsilon(s))\, ds \right)$, and satisfies the conditions $\upsilon(1) = 0$, $\alpha\ {}_H I^{q} \upsilon(\eta) + \beta\ {}_H^C D^{\gamma} \upsilon(\Psi) = \lambda$.

In what follows, we present the following lemma to show the existence of solutions of the system (1.1).

Lemma 3.1 ([12]). *Suppose that $h : [0, +\infty) \to \mathbb{R}$ is a continuous. A function υ is a solution of the following system*

$$\begin{aligned}\upsilon(\varrho) =& \frac{1}{\Gamma(\kappa)} \int_1^{\varrho} \left(\log \frac{\varrho}{s}\right)^{\kappa-1} h(s) \frac{ds}{s} \\ &+ \frac{\log \varrho}{\Lambda} \left[\lambda - \frac{\alpha}{\Gamma(\kappa+q)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\kappa+q-1} h(s) \frac{ds}{s} \right. \\ &\left. - \frac{\beta}{\Gamma(\kappa-\gamma)} \int_1^{\Psi} \left(\log \frac{\Psi}{s}\right)^{\kappa-\gamma-1} h(s) \frac{ds}{s}\right],\end{aligned}$$

where

$$\Lambda = \frac{\alpha (\log \eta)^{q+1}}{\Gamma(q+2)} + \frac{\beta (\log \Psi)^{1-\gamma}}{\Gamma(2-\gamma)},$$

is equivalent to υ is a solution of the following problem

$$\begin{aligned}&{}_H^C D^{\kappa} \upsilon(\varrho) = h(\varrho) \\ &\upsilon(1) = 0, \quad \alpha\, {}_H I^q \upsilon(\eta) + \beta\, {}_H^C D^{\gamma} \upsilon(\Psi) = \lambda, \quad q, \gamma \in [0,1].\end{aligned}$$

Now, we prove the existence of a solution of the system (1.1).

Our hypotheses are as follows.

$(H1)$ $f : \xi \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

$(H2)$ There exist three constants $L_1 > 0$, $0 < L_2 < 1$ and $L_3 > 0$ as follows

$$|f(\varrho, \varsigma, \varphi, w) - f(\varrho, \overline{\varsigma}, \overline{\varphi}, \overline{w})| \leq L_1 |\varsigma - \overline{\varsigma}| + L_2 |\varphi - \overline{\varphi}| + L_3 |w - \overline{w}|,$$

for each ς, φ, w, $\overline{\varsigma}$, $\overline{\varphi}$ and $\overline{w} \in \mathbb{R}$ for a.e. $\varrho \in \xi$.

$(H3)$ There exists a function $k(\varrho, s) \in C[0,1]$, as follows:

$$|K(\varrho, s, \upsilon(s)) - K(\varrho, s, y(s))| \leq k(\varrho, s) |\upsilon(s) - y(s)|.$$

Also, we denote

$$\begin{aligned}\sigma_k &= \sup_{\varrho \in \xi} \int_1^{\varrho} |K(\varrho, s, 0)| \, ds, \\ \sigma_k^* &= \sup_{\varrho \in \xi} \sigma_k(\varrho), \\ \beta_k &= \sup_{\varrho \in \xi} \int_1^{\varrho} |k(\varrho, s)| \, ds.\end{aligned}$$

Theorem 3.1. *Let the assumptions $(H1)$-$(H3)$ be true. If*

$$\rho := \frac{L_1 + \beta_k L_3}{1 - L_2} \left[\frac{(\log \Psi)^{\kappa}}{\Gamma(\kappa+1)} + \frac{|\alpha| (\log \Psi)(\log \eta)^{\kappa+q}}{|\Lambda| \Gamma(\kappa+q+1)} + \frac{|\beta| (\log \Psi)^{\kappa-\gamma+1}}{|\Lambda| \Gamma(\kappa-\gamma+1)}\right] < 1, \tag{3.1}$$

then the system (1.1) has a unique solution $\upsilon \in AC_{\delta}^2(\xi, \mathbb{R})$ on ξ.

Proof. Let $F : C(\xi, \mathbb{R}) \to C(\xi, \mathbb{R})$ be defined as

$$F\upsilon(\varrho) = \frac{1}{\Gamma(\kappa)} \int_1^{\varrho} \left(\log \frac{\varrho}{s}\right)^{\kappa-1} \sigma_{\upsilon}(s) \frac{ds}{s}$$

$$
\begin{aligned}
&+\frac{\log\varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_{\upsilon}(s)\frac{ds}{s}\right.\\
&\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_{\upsilon}(s)\frac{ds}{s}\right],
\end{aligned}
\tag{3.2}
$$

where

$$
\sigma_{\upsilon}(s)=f\left(s,\upsilon(s),D^{\kappa}\upsilon(s),\int_1^{s}K(s,\tau,\upsilon(\tau))\,d\tau\right).
$$

Clearly, the fixed points of F are solutions of the system (1.1).

Let $\upsilon, y\in AC_{\delta}^{2}(\xi,\mathbb{R})$. Then for each $\varrho\in\xi$ we obtain

$$
\begin{aligned}
|(F\upsilon)(\varrho)-(Fy)(\varrho)|=&\left|\frac{1}{\Gamma(\kappa)}\int_1^{\varrho}\left(\log\frac{t}{s}\right)^{\kappa-1}\sigma_{\upsilon}(s)\frac{ds}{s}\right.\\
&+\frac{\log\varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_{\upsilon}(s)\frac{ds}{s}\right.\\
&\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_{\upsilon}(s)\frac{ds}{s}\right]\\
&-\frac{1}{\Gamma(\kappa)}\int_1^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\sigma_{y}(s)\frac{ds}{s}\\
&-\frac{\log\varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_{y}(s)\frac{ds}{s}\right.\\
&\left.\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_{y}(s)\frac{ds}{s}\right]\right|\\
&\leq\frac{1}{\Gamma(\kappa)}\int_1^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}|\sigma_{\upsilon}(s)-\sigma_{y}(s)|\frac{ds}{s}\\
&+\frac{|\alpha|\log\varrho}{|\Lambda|\,\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}|\sigma_{\upsilon}(s)-\sigma_{y}(s)|\frac{ds}{s}\\
&+\frac{|\beta|\log\varrho}{|\Lambda|\,\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}|\sigma_{\upsilon}(s)-\sigma_{y}(s)|\frac{ds}{s},
\end{aligned}
\tag{3.3}
$$

with

$$
\sigma_{\upsilon}(\varrho)=f\left(\varrho,\upsilon(\varrho),\sigma_{\upsilon}(\varrho),\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right)
$$

and

$$
\sigma_{y}(\varrho)=f\left(\varrho,y(\varrho),\sigma_{y}(\varrho),\int_1^{\varrho}K(\varrho,s,y(s))\,ds\right).
$$

By using $(H2)$, we find

$$
\begin{aligned}
\left|\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds-\int_1^{\varrho}K(\varrho,s,y(s))\,ds\right|&\leq\int_1^{\varrho}k(\varrho,s)\,|\upsilon(\varrho)-y(\varrho)|\,ds\\
&\leq\beta_k\,\|\upsilon-y\|
\end{aligned}
$$

and

$$\begin{aligned}\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right| =& \Big\| f\left(\varrho,\upsilon(\varrho),\sigma_{\upsilon}(\varrho),\int_{1}^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right)\\ &\quad - f\left(t,y(\varrho),\sigma_{y}(\varrho),\int_{1}^{\varrho}K(t,s,y(s))\,ds\right)\Big\|\\ \leq& L_1\left|\upsilon(\varrho)-y(\varrho)\right| + L_2\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right| + L_3\beta_k\left\|\upsilon-y\right\|.\end{aligned}$$

Thus,

$$\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right| \leq \frac{L_1+\beta_k L_3}{1-L_2}\left\|\upsilon-y\right\|. \tag{3.4}$$

Replacing (3.4) in (3.3), we obtain

$$\begin{aligned}&\left|(F\upsilon)(\varrho)-(Fy)(\varrho)\right|\\ \leq& \frac{1}{\Gamma(\kappa)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ &+\frac{|\alpha|\log\varrho}{|\Lambda|\,\Gamma(\kappa+q)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ &+\frac{|\beta|\log\varrho}{|\Lambda|\,\Gamma(\kappa-\gamma)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ \leq& \left[\frac{1}{\Gamma(\kappa)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\frac{ds}{s}\right.\\ &+\frac{|\alpha|\log\varrho}{|\Lambda|\,\Gamma(\kappa+q)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\frac{ds}{s}\\ &\left.+\frac{|\beta|\log\varrho}{|\Lambda|\,\Gamma(\kappa-\gamma)}\left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\int_{1}^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\frac{ds}{s}\right]\left|\upsilon(s)-y(s)\right|\\ \leq& \left(\frac{L_1+\beta_k L_3}{1-L_2}\right)\\ &\times\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\,\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\,\Gamma(\kappa-\gamma+1)}\right]\left|\upsilon(s)-y(s)\right|.\end{aligned}$$

Hence,

$$\left\|(F\upsilon)(\varrho)-(Fy)(\varrho)\right\|_{\infty}\leq\rho\left\|\upsilon-y\right\|_{\infty},$$

for $\upsilon, y \in AC_{\delta}^{2}(\xi,\mathbb{R})$, where

$$\rho := \frac{L_1+\beta_k L_3}{1-L_2}\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\,\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\,\Gamma(\kappa-\gamma+1)}\right].$$

Consequently by (3.1), F is a contraction. As a consequence of Banach's contraction principle, we conclude that F admits a unique fixed point which is the solution of the system (1.1). □

Next, we study the second result, by using the fixed point theorem of Schauder.

$(H4)$ There exist $p, \varphi, w, z \in C(\xi, \mathbb{R}_+)$, with $z^* = \sup_{\varrho\in\xi} z(\varrho) < 1$, $\omega^* = \sup_{\varrho\in\xi} \omega(\varrho) < 1$, $\varphi^* = \sup_{\varrho\in\xi} \varphi(\varrho) < 1$ and $p^* = \sup_{\varrho\in\xi} p(\varrho) < 1$, such that

$$f(\varrho, \varsigma, \varphi, w) \leq p(\varrho) + \varphi(\varrho)|\varsigma| + \omega(\varrho)|\varphi| + z(\varrho)|w|,$$

for any $\varsigma, \varphi, w \in \mathbb{R}$ for a.e. $\varrho \in \xi$.

Theorem 3.2. *Assume that* $(H1)$, $(H3)$ *and* $(H4)$ *are true. Moreover if*

$$\omega^* + M(v^* + \beta_k z^*) < 1, \tag{3.5}$$

with

$$M := \frac{(\log\Psi)^\kappa}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)},$$

then the system (1.1) *admits at least one solution.*

Proof. Let

$$R \geq \frac{M(p^* + \sigma_k^* z^*) + \frac{|\lambda|(1-\omega^*)\log\Psi}{|\Lambda|}}{1 - (\omega^* + M(\varphi^* + \beta_k z^*))},$$

and consider $\Delta_R = \{v \in C(\xi, \mathbb{R}) : \|v\|_\infty \leq R\}$. It is clear that the subset Δ_R is closed, convex and bounded. We will use Schauder's fixed point theorem to demonstrate that F defined by (3.2) admits a fixed point.

This could be proved through three steps.

Step 1: F is a continuous mapping.

Let $\{v_n\}$ be a sequence as follows $v_n \to v$ in $AC_\delta^2(\xi, \mathbb{R})$. Then for any $\varrho \in \xi$

$$\begin{aligned}
&|(Fv_n)(\varrho) - (Fv)(\varrho)| \\
\leq&\frac{1}{\Gamma(\kappa)}\int_1^\varrho \left(\log\frac{\varrho}{s}\right)^{\kappa-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
&+ \frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\kappa+q-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
&+ \frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\int_1^\Psi \left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
\leq& \left[\frac{(\log\Psi)^\kappa}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right] |\psi_n(s) - \psi(s)|,
\end{aligned}$$

where $\psi, \psi_n \in C(\xi, \mathbb{R})$ are

$$\psi(\varrho) = f\left(\varrho, v(\varrho), \psi(\varrho), \int_1^\varrho K(\varrho, s, v(s))\,ds\right),$$

$$\psi_n(\varrho) = f\left(\varrho, v(\varrho), \psi_n(\varrho), \int_1^\varrho K(\varrho, s, v(s))\,ds\right).$$

Since ψ is a continuous functions (i.e., f is continuous), then from the Lebesgue theorem of dominated convergence, we get

$$\left\|F\left(\upsilon_n\right)(\varrho)-(F\upsilon)(\varrho)\right\|_{\infty}\rightarrow 0, \qquad \text{as } n\rightarrow +\infty$$

Hence, $F\left(\upsilon_n\right)(\varrho)\rightarrow (F\upsilon)(\varrho)$ as $n\rightarrow +\infty$ which implies that F is continuous.

Step 2: $F\left(\Delta_R\right)\subset\Delta_R$.

Let $\upsilon\in\Delta_R$. We demonstrate $F(\upsilon)\in\Delta_R$, for all $\varrho\in\xi$, we find

$$\begin{aligned}\left|F\upsilon(\varrho)\right|\leq&\frac{1}{\Gamma(\kappa)}\int_1^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\left|\psi(s)\right|\frac{ds}{s}\\&+\frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\left|\psi(s)\right|\frac{ds}{s}\\&+\frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\left|\psi(s)\right|\frac{ds}{s}+\frac{|\lambda|\log\varrho}{|\Lambda|},\end{aligned}\tag{3.6}$$

where $\psi\in C(\xi,\mathbb{R})$ is

$$\psi(\varrho)=f\left(\varrho,\upsilon(\varrho),\psi(\varrho),\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right).$$

From $(H3)$, we get

$$\begin{aligned}&\left|\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds-\int_1^{\varrho}K(\varrho,s,0)\,ds+\int_1^{\varrho}K(\varrho,s,0)\,ds\right|\\\leq&\int_1^{\varrho}\left|k(\varrho,s)\right|\left|\upsilon(\varrho)\right|ds+\int_1^{\varrho}\left|K(\varrho,s,0)\right|ds\\\leq&\beta_k\left\|\upsilon\right\|_{\infty}+\sigma_k(\varrho).\end{aligned}$$

From $(H4)$, for $\varrho\in\xi$ we get

$$\begin{aligned}\left|\psi(\varrho)\right|&=\left|f\left(\varrho,\upsilon(\varrho),\psi(\varrho),\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right)\right|\\&\leq p(\varrho)+\varphi(\varrho)\left|\upsilon(\varrho)\right|+\omega(\varrho)\left|\psi(\varrho)\right|+\beta_k z(\varrho)\left\|\upsilon\right\|_{\infty}+\sigma_k(\varrho)z(\varrho)\\&\leq p^*+\sigma_k^*z^*+\left(\varphi^*+\beta_k z^*\right)\left\|\upsilon\right\|_{\infty}+\omega^*\left|\psi(\varrho)\right|.\end{aligned}$$

Hence,

$$\left|\psi(\varrho)\right|\leq\frac{p^*+\sigma_k^*z^*+\left(\varphi^*+\beta_k z^*\right)R}{1-\omega^*}.\tag{3.7}$$

In the inequality (3.6), we obtain by substituting (3.7)

$$\begin{aligned}\left|F\upsilon(\varrho)\right|\leq&\frac{1}{\Gamma(\kappa)}\int_1^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\left|\psi(s)\right|\frac{ds}{s}+\frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\left|\psi(s)\right|\frac{ds}{s}\\&+\frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\int_1^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\left|\psi(s)\right|\frac{ds}{s}+\frac{|\lambda|\log\varrho}{|\Lambda|}\\\leq&\left(\frac{p^*+\sigma_k^*z^*+\left(\varphi^*+\beta_k z^*\right)R}{1-\omega^*}\right)\end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right]+\frac{|\lambda|\log\Psi}{|\Lambda|}\\
=&\frac{M\left(p^{*}+\sigma_{k}^{*}z^{*}+\left(\varphi^{*}+\beta_{k}z^{*}\right)R\right)}{1-\omega^{*}}+\frac{|\lambda|\log\Psi}{|\Lambda|}\\
\leq&R.
\end{aligned}
$$

Step 3: We demonstrate that the expression $F(\Delta_R)$ is equicontinuous.

It is clear from step 2 that $F(\Delta_R)\subset\Delta_R$ is bounded. For the $F(\Delta_R)$ equicontinuity. Let $\mu_1,\mu_2\in(1,\Psi]$, $\mu_1<\mu_2$ and let $\upsilon\in\Delta_R$. Then

$$
\begin{aligned}
&\left|(F\upsilon)(\mu_2)-(F\upsilon)(\mu_1)\right|\\
=&\left|\frac{1}{\Gamma(\kappa)}\int_1^{\mu_1}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}-\frac{1}{\Gamma(\kappa)}\int_1^{\mu_2}\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}\right|\\
=&\left|\frac{1}{\Gamma(\kappa)}\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\psi(s)\frac{ds}{s}\right.\\
&\left.+\frac{1}{\Gamma(\kappa)}\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}\right|\\
\leq&\frac{|\psi(s)|}{\Gamma(\kappa)}\left|\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\frac{ds}{s}\right|\\
&+\frac{|\psi(s)|}{\Gamma(\kappa)}\left|\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\frac{ds}{s}\right|\\
\leq&\frac{p^{*}+\sigma_{k}^{*}z^{*}+\left(\varphi^{*}+\beta_{k}z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa)}\left|\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\frac{ds}{s}\right|\\
&+\frac{p^{*}+\sigma_{k}^{*}z^{*}+\left(\varphi^{*}+\beta_{k}z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa)}\left|\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\frac{ds}{s}\right|\\
\leq&\frac{p^{*}+\sigma_{k}^{*}z^{*}+\left(\varphi^{*}+\beta_{k}z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa+1)}\left[\left|(\log\mu_1)^{\kappa}+\left(\log\frac{\mu_2}{\mu_1}\right)^{\kappa}-(\log\mu_2)^{\kappa}\right|+\left|\left(\log\frac{\mu_2}{\mu_1}\right)^{\kappa}\right|\right]\\
\leq&\frac{p^{*}+\sigma_{k}^{*}z^{*}+\left(\varphi^{*}+\beta_{k}z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa+1)}\left|(\log\mu_1)^{\kappa}-(\log\mu_2)^{\kappa}\right|\\
\to&0,\quad\text{as }\mu_1\longrightarrow\mu_2.
\end{aligned}
$$

The Arzela-Ascoli theorem shows that F is relatively compact in both scenarios, and Schauder's fixed point theorem states that F has a fixed point. Then, F is a solution of the system (1.1). □

4. Applications

Example 4.1. Assume that the nonlinear system is,

$$
{}_{H}^{C}D^{\frac{3}{2}}\upsilon(\varrho)=f\left(\varrho,\upsilon(\varrho),{}_{H}^{C}D^{\frac{7}{5}}\upsilon(\varrho),\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right),\quad\varrho\in[1,e],
$$

$$\upsilon(1) = 0, \quad \alpha\ {}_{H}I^{\frac{2}{3}}\upsilon(\eta) + \beta\ {}_{H}^{C}D^{\frac{3}{7}}\upsilon(\Psi) = \lambda.$$

We see that $\kappa = \frac{3}{2}$, $q = \frac{2}{3}$, $\gamma = \frac{3}{7}$, $\eta = 2$, $\alpha = \frac{1}{3}$, $\beta = 3$, $\lambda = \frac{4}{5}$, $\Psi = e$, and

$$f(\varrho, \varsigma, \varphi, w) = \frac{\varrho^4}{11+\varrho^2} + \frac{|\varsigma|}{7\varrho^4(|\varsigma|+3)} + \frac{3}{7\varrho^2}\cos\varphi + \frac{1}{\varrho^2}w, \quad \varrho \in [1,e], \varsigma, \varphi, w \in \mathbb{R}.$$

Clearly, f is a continuous, and for $\varrho \in [1,e]$ and $w, \varphi, \varsigma, \overline{w}, \overline{\varsigma}, \overline{\varphi} \in \mathbb{R}$, we get

$$|f(\varrho, \varsigma, \varphi, w) - f(\varrho, \overline{\varsigma}, \overline{\varphi}, \overline{w})| \le L_1|\varsigma - \overline{\varsigma}| + L_2|\varphi - \overline{\varphi}| + L_3|w - \overline{w}|,$$

with $L_1 = \frac{1}{21}$, $L_2 = \frac{3}{7}$, $L_3 = 1$ and

$$|K(\varrho, s, \upsilon) - K(\varrho, s, y)| \le \frac{1}{3s}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}|\upsilon - y|,$$

$$\beta_k = \sup\int_1^{\varrho} k(\varrho, s)\,ds = \frac{1}{3}\int_1^{\varrho}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}\frac{ds}{s} = \frac{(\log\varrho)^{\kappa}}{3\kappa} \le \frac{1}{3\kappa} = \frac{2}{9}.$$

Hence, conditions $(H2)$ and $(H3)$ are satisfied. Moreover,

$$\Lambda = \frac{\alpha(\log\eta)^{q+1}}{\Gamma(q+2)} + \frac{\beta(\log\Psi)^{1-\gamma}}{\Gamma(2-\gamma)} = 3.4887.$$

Thus,

$$\rho := \frac{L_1 + \beta_k L_3}{1 - L_2}\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right] = 0.75728 < 1.$$

Thus, the given system has a unique solution $\upsilon \in C_{\delta}^{n}([1,e], \mathbb{R})$, according to Theorem 3.1,

$$|f(\varrho, \varsigma, \varphi, w)| \le \frac{\varrho^4}{11+\varrho^2} + \frac{|\varsigma|}{7\varrho^4(|\varsigma|+3)} + \frac{3}{7\varrho^2}|\cos\varphi| + \frac{1}{3s}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}|\sin w|,$$

so condition $(H4)$ is satisfied with

$$p(\varrho) = \frac{\varrho^4}{11+\varrho^2}, \quad \varphi(\varrho) = \frac{1}{21\varrho^4}, \quad \omega(\varrho) = \frac{3}{7\varrho^2}, \quad z(\varrho) = \frac{1}{3s}\log\left(\frac{t}{s}\right)^{\kappa-1}$$

and

$$\varphi^* = \frac{1}{21}, \quad \omega^* = \frac{3}{7}, \quad z^* = 1.$$

We have

$$M := \frac{(\log\Psi)^{\kappa}}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)} = 1.9712.$$

We will demonstrate that condition (3.5) is true for $\Psi = e$. Indeed,

$$\omega^* + M(\varphi^* + \beta_k z^*) = 0.96048 < 1.$$

Simple calculations demonstrate that conditions of Theorem 3.2 are all satisfied. Example 4.1 must thus have at least one solution specified on $[1, e]$.

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[1]DEPARTMENT OF MATHEMATICS,
TAIZ UNIVERSITY,
TAIZ P.O. BOX 6803, YEMEN.
Email address: `ahmed.hamoud@taiz.edu.ye`

[2]DEPARTMENT OF MATHEMATICS AND INFORMATICS,
UNIVERSITY OF SOUK AHRAS,
P.O. BOX 1553, SOUK AHRAS, 41000, ALGERIA.
Email address: `c.kechar@univ-soukahras.dz`

Email address: `abd-ardjouni@yahoo.fr`

[3]DEPARTMENT OF MATHEMATICS,
SAVEETHA SCHOOL OF ENGINEERING,
SAVEETHA INSTITUTE OF MEDICAL AND TECHNICAL SCIENCES,
SAVEETHA UNIVERSITY,
CHENNAI-602 105,TAMIL NADU, INDIA.

[4]MEU RESEARCH UNIT,
MIDDLE EAST UNIVERSITY,
AMMAN 11831, JORDAN.

[5]Department of Mathematics,
Hamedan Branch, Islamic Azad University,
Hamedan, Iran.
Email address: `homan_emadi@yahoo.com`

[6]Department of Mathematics,
Dayalbagh Educational Institute,
India.
Email address: `atul1230123@gmail.com`

[7]Artificial Intelligence and Sensing Technologies (AIST) Research Center,
University of Tabuk,
Tabuk 71491, Saudi Arabia

[8]Computer Science Department,
Al al-Bayt University,
Mafraq 25113, Jordan.

[9]Hourani Center for Applied Scientific Research,
Al-Ahliyya Amman University,
Amman 19328, Jordan.

[10]Department of Electrical and Computer Engineering,
Lebanese American University,
Byblos 13-5053, Lebanon.

[11]School of Engineering and Technology,
Sunway University Malaysia,
Petaling Jaya 27500, Malaysia.

[12]Applied science research center,
Applied science private university,
Amman 11931, Jordan.
Email address: `aligah.2020@gmail.com`

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