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ON HADAMARD-CAPUTO IMPLICIT FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY FRACTIONAL CONDITIONS

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of solutions for nonlinear fractional implicit integro-differential equations of Hadamard-Caputo type with fractional boundary conditions. The reasoning is inspired by diverse classical fixed point theory, such as the Schauder and Banach fixed point theorems. The theoretical findings are illustrated through an example.

1. Introduction

In mathematical analysis, fractional calculus (FC) is a subject that studies different approaches of defining non-integer order derivatives (i.e., fractional differential calculus (FDC)) and integrals (i.e., fractional integral calculus (FIC)). Fractional calculus is widely and efficiently used to describe many phenomena arising in physics, engineering, bioengineering and biomedical sciences, finance, viscoelasticity, control theory, stochastic processes and economy. Recently, fractional differential equations (FDEs) have attracted many authors (see for example [1–3, 8, 13] and references therein).

By flipping the differential and integral sections of the Hadamard derivative, a novel method known as the Hadamard-Caputo derivative is created. The primary distinction between the Hadamard fractional derivative and the Hadamard-Caputo fractional derivative, notwithstanding the various demands placed on the function itself, is that the Hadamard-Caputo derivative of a constant is zero [24]. The most

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significant benefit of the Hadamard-Caputo derivative is that it gave rise to a new concept that can be used to establish the integer order beginning conditions for fractional.

For more details and properties of Hadamard-Caputo derivative and Hadamard fractional derivatives, integrals see [10, 11, 26].

The implicit fractional differential equations (IFDEs) are a very important class of fractional differential equations. This type of equation is derived from the implicit ordinary differential equation (IODE) of the following form

$$H\left(\varrho, G(\varrho), G'(\varrho), \ldots, G^{(n-1)}(\varrho)\right) = 0,$$

with different kind of initial or boundary conditions, for more details see [6, 14, 15, 23].

Benchohra et al. [7, 9] and Nieto et al. [28, 30] have initiated the study of implicit fractional differential equations (IFDEs) of the form

$$D^{\alpha} G(\varrho) = H(\varrho, G(\varrho), D^{\alpha} G(\varrho)),$$

with different kind of initial or boundary conditions.This sort of equation is crucial in many different fields of science and engineering [34].

In [33], Vivek et al. showed that a class of boundary value systems for nonlinear IFDEs with complex order have a solution and are stable

$$\begin{aligned} &{}^{c}D^{\theta} G(\varrho) = H\left(\varrho, G(\varrho), {}^{c}D^{\theta} G(\varrho)\right), \quad \theta = m + i\alpha, \quad \varrho \in Y := [0, \chi], \\ &aG(0) + bG(\chi) = c, \end{aligned}$$

where $\theta \in \mathbb{C}$, ${}^{c}D^{\theta}$ is the Caputo fractional derivative. Suppose $m \in (0, 1]$, $\alpha \in \mathbb{R}_{+}$, $0 < \alpha < 1$, $H : Y \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Further $a, b, c \in \mathbb{R}$ with $a + b \neq 0$. The results are based upon the Schaefer's fixed point theorem and Banach contraction principle.

In [27] Karthikeyan and Arul stydied the uniqueness of integral BVP for IFDEs involving Hadamard-Caputo fractional derivative

$$\begin{aligned} {}^{CH}D^{\theta} \zeta(\varrho) &= H\left(\varrho, \zeta(\varrho), {}^{CH}D^{\theta} \zeta(\varrho)\right), \quad \varrho \in \xi := [n, \chi], \\ \zeta(n) = 0, \quad \zeta(\chi) &= \Omega \int_{n}^{\sigma} \zeta(\omega)\, d\omega, \quad n < \sigma < \chi, \end{aligned}$$

where $\Omega \in \mathbb{R}$, $n < \sigma < \chi$, $1 < \theta \leq 2$, ${}^{CH}D^{\theta}$ is the Hadamard-Caputo fractional derivative, and $H : \xi \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

In [12] N. Derdar established the uniqueness of solutions for the system:

$$\begin{aligned} &{}^{C}_{H}D^{r} \upsilon(\varrho) = W\left(\varrho, \upsilon(\varrho), {}^{C}_{H}D^{r} \upsilon(\varrho)\right), \\ &\upsilon(1) = 0, \quad \alpha\, {}_{H}I^{q} \upsilon(\eta) + \beta\, {}^{C}_{H}D^{\gamma} \upsilon(\Psi) = \lambda, \end{aligned}$$

by using different fixed point theorem.

Recently, many authors focus on the development of techniques for discussing the solutions of FIDEs.

Balachandran and Trujillo [5], investigated the existence of a unique solution for FIDEs with boundary value conditions.

$$ {}^{C}\mathfrak{D}^{\alpha} f(\upsilon) = W(\upsilon, f(\upsilon)) + \int_{\upsilon_0}^{\upsilon} K(\upsilon, \varrho, f(\varrho))\, d\varrho, \quad 0 < \alpha \leq 1. $$

In [32] the authors investigated the uniqueness of solution for iterative integro-differential system:

$$ \begin{aligned} \mathfrak{D}^{\alpha} w(\upsilon) &= Q(\upsilon) + \int_{0}^{\varrho} H(\upsilon, s)\, w(\lambda w(s))\, ds, \\ w(0) &= w_0. \end{aligned} $$

In [17] A. A. Hamoud established the uniqueness and stability for fractional nonlinear Fredholm–Volterra system:

$$ \begin{aligned} &{}^{C}\mathfrak{D}^{\alpha} w(\varrho) = H(\varrho) + \int_{0}^{\varrho} \vartheta(\varrho, s)\, w(w(s))\, ds + \int_{0}^{\Psi} \theta(\varrho, s)\, w(w(s))\, ds, \\ &a w(0) + b w(\Psi) = c, \quad a, b, c, \in \mathbb{R}. \end{aligned} $$

For some other results on FIDE, see [4, 5, 16, 18–22, 25, 29].

Motived by the above papers and the reference [12], we study the theoretical analysis of solutions for a class of system for nonlinear implicit FIDEs of Hadamard-Caputo type with fractional boundary conditions

$$ \begin{aligned} c{}^{C}_{H}D^{r}\upsilon(\varrho) &= f\left(\varrho, \upsilon(\varrho), {}^{C}_{H}D^{r}\upsilon(\varrho), \int_{1}^{\varrho} K(\varrho, s, \upsilon(s))\, ds\right), \\ \upsilon(1) &= 0, \quad \alpha\; {}_{H}I^{q}\upsilon(\eta) + \beta\; {}^{C}_{H}D^{\gamma}\upsilon(\Psi) = \lambda, \end{aligned} \tag{1.1} $$

where ${}_{H}I^{q}$ is the standard Hadamard fractional integral, ${}^{C}_{H}D^{r}$ is the Hadamard-Caputo fractional derivative, $f : \xi \times \mathbb{R}^3 \to \mathbb{R}$, $K : \xi \times \xi \times \mathbb{R} \to \mathbb{R}$ are given functions, $\eta \in \xi =: (1, \Psi]$, $\Psi > 1$ and α, λ, β are real constants.

2. Preliminaries and Background Materials

Let us introduce some necessary notations and definitions which will be utilised throughout the entire process [28, 29, 31, 33–35].

We represent by the symbol $C(\xi, \mathbb{R})$ the space of all continuous functions $\upsilon : \xi \to \mathbb{R}$ in Banach space with the supremum norm

$$ \|\upsilon\|_{\infty} = \sup\{|\upsilon(\varrho)| : \varrho \in \xi\}. $$

Let now $[b, c]$, $-\infty < b < c < +\infty$, is finite interval and we suppose $AC([b, c], \mathbb{R})$ is the space of functions $\psi : [b, c] \to \mathbb{R}$ that are absolutely continuous.

Assume $\delta = \varrho \frac{d}{dt}\, d\varrho$ is the Hadamard derivative, $\delta^n = \delta(\delta^{n-1})$, we consider the set of functions:

$$ AC^{n}_{\delta}([b, c], \mathbb{R}) = \left\{\psi : [b, c] \to \mathbb{R} : \delta^{n-1}\psi(\varrho) \in AC([b, c], \mathbb{R})\right\}. $$

Definition 2.1 ([28])**.** The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $\psi : [1, +\infty) \to \mathbb{R}$ is given by

$$ {}_{H}I_{1}^{\alpha}\psi\left(\varrho\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_{1}^{\varrho} \left(\log \frac{\varrho}{s}\right)^{\alpha-1} \psi\left(s\right) \frac{ds}{s}, $$

where $\log\left(\cdot\right) = \log_{e}\left(\cdot\right)$ and $\Gamma\left(\cdot\right)$ is Gamma function.

Definition 2.2 ([31])**.** For a function $\psi \in AC_{\delta}^{n}\left(\left[b, c\right], \mathbb{R}\right)$, the Hadamard-Caputo fractional derivative of order α is given by

$$ {}_{H}^{C}D_{1}^{\alpha}\psi\left(\varrho\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \left(t\frac{d}{dt}\right)^{n} \int_{1}^{\varrho} \left(\log \frac{\varrho}{s}\right)^{n-\alpha-1} \psi\left(s\right) \frac{ds}{s}, \quad n-1 < \alpha < n, $$

where $\delta^{n} = \left(\varrho \frac{d}{d\varrho}\right)^{n}$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Lemma 2.1 ([24])**.** *Let $\psi \in AC_{\delta}^{n}\left[b, c\right]$ or $\psi \in C_{\delta}^{n}\left[b, c\right]$ and $\alpha \in \mathbb{C}$. Then,*

$$ {}_{H}I_{b}^{\alpha}\left({}_{H}^{C}D_{b}^{\alpha}\psi\right)\left(\varrho\right) = \psi\left(\varrho\right) - \sum_{k=0}^{n-1} \frac{\delta^{(k)}\psi\left(b\right)}{k!} \left(\log \frac{\varrho}{b}\right)^{k}. $$

Proposition 2.1 ([24])**.** *Let $\alpha > 0$, $\beta > 0$, $n = [\alpha] + 1$, and $b > 0$. Then,*

$$ \left({}_{H}I_{b^{+}}^{\alpha}\left(\log \frac{\upsilon}{b}\right)^{\beta-1}\right)\left(\upsilon\right) = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta+\alpha\right)} \left(\log \frac{\upsilon}{b}\right)^{\beta+\alpha-1}, $$

$$ \left({}_{H}^{C}D_{b^{+}}^{\alpha}\left(\log \frac{\upsilon}{b}\right)^{\beta-1}\right)\left(\upsilon\right) = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha\right)} \left(\log \frac{\upsilon}{b}\right)^{\beta-\alpha-1}, \quad \alpha < \beta. $$

Theorem 2.1. [24] *Let $\upsilon\left(\varrho\right) \in AC_{\delta}^{n}\left[b, c\right]$, $0 < b < c < +\infty$ and $\alpha \geq 0$, $\beta \geq 0$. Then,*

$$ {}^{C}D_{b}^{\alpha}\left(I^{\beta}\upsilon\right)\left(\varrho\right) = \left(I^{\beta-\alpha}\upsilon\right)\left(\varrho\right), $$

$$ {}^{C}D^{\alpha}\left({}^{C}D^{\beta}\right)\left(\varrho\right) = {}^{C}D^{\alpha+\beta}\left(\varrho\right). $$

Theorem 2.2 (Schauder's fixed point [35])**.** *Suppose that E be a Banach space, and P be a nonempty, convex and closed subset of E. Assume that $\mathcal{A} : P \to P$ be a continuous mapping and $\mathcal{A}\left(P\right)$ is a relatively compact subset of E. Then $\mathcal{A}$ admits at least one fixed point in P.*

3. Main Results

Definition 3.1. A function $\upsilon \in AC_{\delta}^{2}\left(\xi, \mathbb{R}\right)$ is said to be a solution of the system (1.1) if υ satisfies the equation ${}_{H}^{C}D^{\kappa}\upsilon\left(\varrho\right) = f\left(\varrho, \upsilon\left(\varrho\right), {}_{H}^{C}D^{\kappa}\upsilon\left(\varrho\right), \int_{1}^{\varrho} K\left(\varrho, s, \upsilon\left(s\right)\right) ds\right)$, and satisfies the conditions $\upsilon\left(1\right) = 0$, $\alpha\ {}_{H}I^{q}\upsilon\left(\eta\right) + \beta\ {}_{H}^{C}D^{\gamma}\upsilon\left(\Psi\right) = \lambda$.

In what follows, we present the following lemma to show the existence of solutions of the system (1.1).

Lemma 3.1 ([12]). *Suppose that $h : [0, +\infty) \to \mathbb{R}$ is a continuous. A function υ is a solution of the following system*

$$\begin{aligned}\upsilon(\varrho) =& \frac{1}{\Gamma(\kappa)} \int_1^{\varrho} \left(\log \frac{\varrho}{s}\right)^{\kappa-1} h(s) \frac{ds}{s} \\ &+ \frac{\log \varrho}{\Lambda} \left[\lambda - \frac{\alpha}{\Gamma(\kappa+q)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\kappa+q-1} h(s) \frac{ds}{s} \right. \\ &\left. - \frac{\beta}{\Gamma(\kappa-\gamma)} \int_1^{\Psi} \left(\log \frac{\Psi}{s}\right)^{\kappa-\gamma-1} h(s) \frac{ds}{s}\right],\end{aligned}$$

where

$$\Lambda = \frac{\alpha (\log \eta)^{q+1}}{\Gamma(q+2)} + \frac{\beta (\log \Psi)^{1-\gamma}}{\Gamma(2-\gamma)},$$

is equivalent to υ is a solution of the following problem

$$\begin{aligned}&{}_H^C D^{\kappa} \upsilon(\varrho) = h(\varrho) \\ &\upsilon(1) = 0, \quad \alpha \, {}_H I^q \upsilon(\eta) + \beta \, {}_H^C D^{\gamma} \upsilon(\Psi) = \lambda, \quad q, \gamma \in [0, 1].\end{aligned}$$

Now, we prove the existence of a solution of the system (1.1).

Our hypotheses are as follows.

$(H1)$ $f : \xi \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

$(H2)$ There exist three constants $L_1 > 0$, $0 < L_2 < 1$ and $L_3 > 0$ as follows

$$|f(\varrho, \varsigma, \varphi, w) - f(\varrho, \overline{\varsigma}, \overline{\varphi}, \overline{w})| \leq L_1 |\varsigma - \overline{\varsigma}| + L_2 |\varphi - \overline{\varphi}| + L_3 |w - \overline{w}|,$$

for each ς, φ, w, $\overline{\varsigma}$, $\overline{\varphi}$ and $\overline{w} \in \mathbb{R}$ for a.e. $\varrho \in \xi$.

$(H3)$ There exists a function $k(\varrho, s) \in C[0, 1]$, as follows:

$$|K(\varrho, s, \upsilon(s)) - K(\varrho, s, y(s))| \leq k(\varrho, s) |\upsilon(s) - y(s)|.$$

Also, we denote

$$\begin{aligned}\sigma_k &= \sup_{\varrho \in \xi} \int_1^{\varrho} |K(\varrho, s, 0)| \, ds, \\ \sigma_k^* &= \sup_{\varrho \in \xi} \sigma_k(\varrho), \\ \beta_k &= \sup_{\varrho \in \xi} \int_1^{\varrho} |k(\varrho, s)| \, ds.\end{aligned}$$

Theorem 3.1. *Let the assumptions $(H1)$-$(H3)$ be true. If*

$$\rho := \frac{L_1 + \beta_k L_3}{1 - L_2} \left[\frac{(\log \Psi)^{\kappa}}{\Gamma(\kappa+1)} + \frac{|\alpha| (\log \Psi)(\log \eta)^{\kappa+q}}{|\Lambda| \Gamma(\kappa+q+1)} + \frac{|\beta| (\log \Psi)^{\kappa-\gamma+1}}{|\Lambda| \Gamma(\kappa-\gamma+1)}\right] < 1, \tag{3.1}$$

then the system (1.1) has a unique solution $\upsilon \in AC_{\delta}^2(\xi, \mathbb{R})$ on ξ.

Proof. Let $F : C(\xi, \mathbb{R}) \to C(\xi, \mathbb{R})$ be defined as

$$F\upsilon(\varrho) = \frac{1}{\Gamma(\kappa)} \int_1^{\varrho} \left(\log \frac{\varrho}{s}\right)^{\kappa-1} \sigma_{\upsilon}(s) \frac{ds}{s}$$

$$+\frac{\log \varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_v(s)\frac{ds}{s}\right.$$

$$\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^\Psi\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_v(s)\frac{ds}{s}\right], \tag{3.2}$$

where

$$\sigma_v(s)=f\left(s,v(s),D^\kappa v(s),\int_1^s K(s,\tau,v(\tau))\,d\tau\right).$$

Clearly, the fixed points of F are solutions of the system (1.1).

Let $v,y\in AC_\delta^2(\xi,\mathbb{R})$. Then for each $\varrho\in\xi$ we obtain

$$\begin{aligned}
|(Fv)(\varrho)-(Fy)(\varrho)| =& \left|\frac{1}{\Gamma(\kappa)}\int_1^\varrho\left(\log\frac{t}{s}\right)^{\kappa-1}\sigma_v(s)\frac{ds}{s}\right.\\
&+\frac{\log\varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_v(s)\frac{ds}{s}\right.\\
&\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^\Psi\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_v(s)\frac{ds}{s}\right]\\
&-\frac{1}{\Gamma(\kappa)}\int_1^\varrho\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\sigma_y(s)\frac{ds}{s}\\
&-\frac{\log\varrho}{\Lambda}\left[\lambda-\frac{\alpha}{\Gamma(\kappa+q)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\sigma_y(s)\frac{ds}{s}\right.\\
&\left.\left.-\frac{\beta}{\Gamma(\kappa-\gamma)}\int_1^\Psi\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\sigma_y(s)\frac{ds}{s}\right]\right|\\
\leq& \frac{1}{\Gamma(\kappa)}\int_1^\varrho\left(\log\frac{\varrho}{s}\right)^{\kappa-1}|\sigma_v(s)-\sigma_y(s)|\frac{ds}{s}\\
&+\frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}|\sigma_v(s)-\sigma_y(s)|\frac{ds}{s}\\
&+\frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\int_1^\Psi\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}|\sigma_v(s)-\sigma_y(s)|\frac{ds}{s},
\end{aligned} \tag{3.3}$$

with

$$\sigma_v(\varrho)=f\left(\varrho,v(\varrho),\sigma_v(\varrho),\int_1^\varrho K(\varrho,s,v(s))\,ds\right)$$

and

$$\sigma_y(\varrho)=f\left(\varrho,y(\varrho),\sigma_y(\varrho),\int_1^\varrho K(\varrho,s,y(s))\,ds\right).$$

By using $(H2)$, we find

$$\begin{aligned}
\left|\int_1^\varrho K(\varrho,s,v(s))\,ds-\int_1^\varrho K(\varrho,s,y(s))\,ds\right| \leq& \int_1^\varrho k(\varrho,s)\,|v(\varrho)-y(\varrho)|\,ds\\
\leq& \beta_k\,\|v-y\|
\end{aligned}$$

and

$$\begin{aligned}\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right| =&\left\| f\left(\varrho,\upsilon(\varrho),\sigma_{\upsilon}(\varrho),\int_{1}^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right)\right.\\ &\left.-f\left(t,y(\varrho),\sigma_{y}(\varrho),\int_{1}^{\varrho}K(t,s,y(s))\,ds\right)\right\|\\ \leq& L_{1}\left|\upsilon(\varrho)-y(\varrho)\right|+L_{2}\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right|+L_{3}\beta_{k}\left\|\upsilon-y\right\|.\end{aligned}$$

Thus,

$$\left|\sigma_{\upsilon}(\varrho)-\sigma_{y}(\varrho)\right|\leq\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\left\|\upsilon-y\right\|. \tag{3.4}$$

Replacing (3.4) in (3.3), we obtain

$$\begin{aligned}&\left|(F\upsilon)(\varrho)-(Fy)(\varrho)\right|\\ \leq&\frac{1}{\Gamma(\kappa)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ &+\frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ &+\frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\left|\upsilon(s)-y(s)\right|\frac{ds}{s}\\ \leq&\left[\frac{1}{\Gamma(\kappa)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\varrho}\left(\log\frac{\varrho}{s}\right)^{\kappa-1}\frac{ds}{s}\right.\\ &+\frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\eta}\left(\log\frac{\eta}{s}\right)^{\kappa+q-1}\frac{ds}{s}\\ &\left.+\frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)}\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\int_{1}^{\Psi}\left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1}\frac{ds}{s}\right]\left|\upsilon(s)-y(s)\right|\\ \leq&\left(\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\right)\\ &\times\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right]\left|\upsilon(s)-y(s)\right|.\end{aligned}$$

Hence,

$$\left\|(F\upsilon)(\varrho)-(Fy)(\varrho)\right\|_{\infty}\leq\rho\left\|\upsilon-y\right\|_{\infty},$$

for $\upsilon,y\in AC_{\delta}^{2}(\xi,\mathbb{R})$, where

$$\rho:=\frac{L_{1}+\beta_{k}L_{3}}{1-L_{2}}\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right].$$

Consequently by (3.1), F is a contraction. As a consequence of Banach's contraction principle, we conclude that F admits a unique fixed point which is the solution of the system (1.1). $\square$

Next, we study the second result, by using the fixed point theorem of Schauder.

$(H4)$ There exist $p, \varphi, w, z \in C(\xi, \mathbb{R}_+)$, with $z^* = \sup_{\varrho\in\xi} z(\varrho) < 1$, $\omega^* = \sup_{\varrho\in\xi} \omega(\varrho) < 1$, $\varphi^* = \sup_{\varrho\in\xi} \varphi(\varrho) < 1$ and $p^* = \sup_{\varrho\in\xi} p(\varrho) < 1$, such that

$$f(\varrho, \varsigma, \varphi, w) \leq p(\varrho) + \varphi(\varrho)|\varsigma| + \omega(\varrho)|\varphi| + z(\varrho)|w|,$$

for any $\varsigma, \varphi, w \in \mathbb{R}$ for a.e. $\varrho \in \xi$.

Theorem 3.2. *Assume that* $(H1)$, $(H3)$ *and* $(H4)$ *are true. Moreover if*

$$\omega^* + M(v^* + \beta_k z^*) < 1, \tag{3.5}$$

with

$$M := \frac{(\log\Psi)^{\kappa}}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)},$$

then the system (1.1) *admits at least one solution.*

Proof. Let

$$R \geq \frac{M(p^* + \sigma_k^* z^*) + \frac{|\lambda|(1-\omega^*)\log\Psi}{|\Lambda|}}{1 - (\omega^* + M(\varphi^* + \beta_k z^*))},$$

and consider $\Delta_R = \{v \in C(\xi, \mathbb{R}) : \|v\|_\infty \leq R\}$. It is clear that the subset Δ_R is closed, convex and bounded. We will use Schauder's fixed point theorem to demonstrate that F defined by (3.2) admits a fixed point.

This could be proved through three steps.

Step 1: F is a continuous mapping.

Let $\{v_n\}$ be a sequence as follows $v_n \to v$ in $AC_\delta^2(\xi, \mathbb{R})$. Then for any $\varrho \in \xi$

$$\begin{aligned}
&|(Fv_n)(\varrho) - (Fv)(\varrho)| \\
\leq& \frac{1}{\Gamma(\kappa)} \int_1^{\varrho} \left(\log\frac{\varrho}{s}\right)^{\kappa-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
&+ \frac{|\alpha|\log\varrho}{|\Lambda|\Gamma(\kappa+q)} \int_1^{\eta} \left(\log\frac{\eta}{s}\right)^{\kappa+q-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
&+ \frac{|\beta|\log\varrho}{|\Lambda|\Gamma(\kappa-\gamma)} \int_1^{\Psi} \left(\log\frac{\Psi}{s}\right)^{\kappa-\gamma-1} |\psi_n(s) - \psi(s)| \frac{ds}{s} \\
\leq& \left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\alpha+1)} + \frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)} + \frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right] |\psi_n(s) - \psi(s)|,
\end{aligned}$$

where $\psi, \psi_n \in C(\xi, \mathbb{R})$ are

$$\psi(\varrho) = f\left(\varrho, v(\varrho), \psi(\varrho), \int_1^{\varrho} K(\varrho, s, v(s))\,ds\right),$$

$$\psi_n(\varrho) = f\left(\varrho, v(\varrho), \psi_n(\varrho), \int_1^{\varrho} K(\varrho, s, v(s))\,ds\right).$$

Since ψ is a continuous functions (i.e., f is continuous), then from the Lebesgue theorem of dominated convergence, we get

$$\left\| F\left(\upsilon_n\right)(\varrho) - (F\upsilon)(\varrho)\right\|_\infty \to 0, \qquad \text{as } n \to +\infty$$

Hence, $F\left(\upsilon_n\right)(\varrho) \to (F\upsilon)(\varrho)$ as $n \to +\infty$ which implies that F is continuous.

Step 2: $F\left(\Delta_R\right) \subset \Delta_R$.

Let $\upsilon \in \Delta_R$. We demonstrate $F(\upsilon) \in \Delta_R$, for all $\varrho \in \xi$, we find

$$\begin{aligned}\left|F\upsilon(\varrho)\right| \leq & \frac{1}{\Gamma(\kappa)} \int_1^\varrho \left(\log \frac{\varrho}{s}\right)^{\kappa-1} |\psi(s)| \frac{ds}{s} \\ & + \frac{|\alpha| \log \varrho}{|\Lambda| \Gamma(\kappa+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\kappa+q-1} |\psi(s)| \frac{ds}{s} \\ & + \frac{|\beta| \log \varrho}{|\Lambda| \Gamma(\kappa-\gamma)} \int_1^\Psi \left(\log \frac{\Psi}{s}\right)^{\kappa-\gamma-1} |\psi(s)| \frac{ds}{s} + \frac{|\lambda| \log \varrho}{|\Lambda|},\end{aligned} \tag{3.6}$$

where $\psi \in C(\xi, \mathbb{R})$ is

$$\psi(\varrho) = f\left(\varrho, \upsilon(\varrho), \psi(\varrho), \int_1^\varrho K(\varrho, s, \upsilon(s))\, ds\right).$$

From $(H3)$, we get

$$\begin{aligned} & \left| \int_1^\varrho K(\varrho, s, \upsilon(s))\, ds - \int_1^\varrho K(\varrho, s, 0)\, ds + \int_1^\varrho K(\varrho, s, 0)\, ds \right| \\ \leq & \int_1^\varrho |k(\varrho, s)| \, |\upsilon(\varrho)|\, ds + \int_1^\varrho |K(\varrho, s, 0)|\, ds \\ \leq & \beta_k \|\upsilon\|_\infty + \sigma_k(\varrho).\end{aligned}$$

From $(H4)$, for $\varrho \in \xi$ we get

$$\begin{aligned} |\psi(\varrho)| = & \left| f\left(\varrho, \upsilon(\varrho), \psi(\varrho), \int_1^\varrho K(\varrho, s, \upsilon(s))\, ds\right) \right| \\ & \leq p(\varrho) + \varphi(\varrho) |\upsilon(\varrho)| + \omega(\varrho) |\psi(\varrho)| + \beta_k z(\varrho) \|\upsilon\|_\infty + \sigma_k(\varrho) z(\varrho) \\ & \leq p^* + \sigma_k^* z^* + \left(\varphi^* + \beta_k z^*\right) \|\upsilon\|_\infty + \omega^* |\psi(\varrho)|.\end{aligned}$$

Hence,

$$|\psi(\varrho)| \leq \frac{p^* + \sigma_k^* z^* + \left(\varphi^* + \beta_k z^*\right) R}{1 - \omega^*}. \tag{3.7}$$

In the inequality (3.6), we obtain by substituting (3.7)

$$\begin{aligned} \left|F\upsilon(\varrho)\right| \leq & \frac{1}{\Gamma(\kappa)} \int_1^\varrho \left(\log \frac{\varrho}{s}\right)^{\kappa-1} |\psi(s)| \frac{ds}{s} + \frac{|\alpha| \log \varrho}{|\Lambda| \Gamma(\kappa+q)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\kappa+q-1} |\psi(s)| \frac{ds}{s} \\ & + \frac{|\beta| \log \varrho}{|\Lambda| \Gamma(\kappa-\gamma)} \int_1^\Psi \left(\log \frac{\Psi}{s}\right)^{\kappa-\gamma-1} |\psi(s)| \frac{ds}{s} + \frac{|\lambda| \log \varrho}{|\Lambda|} \\ \leq & \left(\frac{p^* + \sigma_k^* z^* + \left(\varphi^* + \beta_k z^*\right) R}{1 - \omega^*}\right)\end{aligned}$$

$$\begin{aligned}
&\times\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right]+\frac{|\lambda|\log\Psi}{|\Lambda|}\\
=&\frac{M\left(p^{*}+\sigma_k^{*}z^{*}+\left(\varphi^{*}+\beta_k z^{*}\right)R\right)}{1-\omega^{*}}+\frac{|\lambda|\log\Psi}{|\Lambda|}\\
\leq&R.
\end{aligned}$$

Step 3: We demonstrate that the expression $F(\Delta_R)$ is equicontinuous.

It is clear from step 2 that $F(\Delta_R)\subset\Delta_R$ is bounded. For the $F(\Delta_R)$ equicontinuity. Let $\mu_1,\mu_2\in(1,\Psi]$, $\mu_1<\mu_2$ and let $\upsilon\in\Delta_R$. Then

$$\begin{aligned}
&\left|(F\upsilon)(\mu_2)-(F\upsilon)(\mu_1)\right|\\
=&\left|\frac{1}{\Gamma(\kappa)}\int_1^{\mu_1}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}-\frac{1}{\Gamma(\kappa)}\int_1^{\mu_2}\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}\right|\\
=&\left|\frac{1}{\Gamma(\kappa)}\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\psi(s)\frac{ds}{s}\right.\\
&\left.+\frac{1}{\Gamma(\kappa)}\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\psi(s)\frac{ds}{s}\right|\\
\leq&\frac{|\psi(s)|}{\Gamma(\kappa)}\left|\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\frac{ds}{s}\right|\\
&+\frac{|\psi(s)|}{\Gamma(\kappa)}\left|\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\frac{ds}{s}\right|\\
\leq&\frac{p^{*}+\sigma_k^{*}z^{*}+\left(\varphi^{*}+\beta_k z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa)}\left|\int_1^{\mu_1}\left[\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}-\left(\log\frac{\mu_1}{s}\right)^{\kappa-1}\right]\frac{ds}{s}\right|\\
&+\frac{p^{*}+\sigma_k^{*}z^{*}+\left(\varphi^{*}+\beta_k z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa)}\left|\int_{\mu_1}^{\mu_2}\left(\log\frac{\mu_2}{s}\right)^{\kappa-1}\frac{ds}{s}\right|\\
\leq&\frac{p^{*}+\sigma_k^{*}z^{*}+\left(\varphi^{*}+\beta_k z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa+1)}\left[\left|(\log\mu_1)^{\kappa}+\left(\log\frac{\mu_2}{\mu_1}\right)^{\kappa}-(\log\mu_2)^{\kappa}\right|+\left|\left(\log\frac{\mu_2}{\mu_1}\right)^{\kappa}\right|\right]\\
\leq&\frac{p^{*}+\sigma_k^{*}z^{*}+\left(\varphi^{*}+\beta_k z^{*}\right)R}{\left(1-\omega^{*}\right)\Gamma(\kappa+1)}\left|(\log\mu_1)^{\kappa}-(\log\mu_2)^{\kappa}\right|\\
\to&0,\quad\text{as }\mu_1\longrightarrow\mu_2.
\end{aligned}$$

The Arzela-Ascoli theorem shows that F is relatively compact in both scenarios, and Schauder's fixed point theorem states that F has a fixed point. Then, F is a solution of the system (1.1). □

4. Applications

Example 4.1. Assume that the nonlinear system is,

$${}_H^C D^{\frac{3}{2}}\upsilon(\varrho)=f\left(\varrho,\upsilon(\varrho),{}_H^C D^{\frac{7}{5}}\upsilon(\varrho),\int_1^{\varrho}K(\varrho,s,\upsilon(s))\,ds\right),\quad\varrho\in[1,e],$$

$$\upsilon(1)=0, \quad \alpha \;_{H}I^{\frac{2}{3}}\upsilon(\eta)+\beta \;_{H}^{C}D^{\frac{3}{7}}\upsilon(\Psi)=\lambda.$$

We see that $\kappa=\frac{3}{2}$, $q=\frac{2}{3}$, $\gamma=\frac{3}{7}$, $\eta=2$, $\alpha=\frac{1}{3}$, $\beta=3$, $\lambda=\frac{4}{5}$, $\Psi=e$, and

$$f(\varrho,\varsigma,\varphi,w)=\frac{\varrho^4}{11+\varrho^2}+\frac{|\varsigma|}{7\varrho^4(|\varsigma|+3)}+\frac{3}{7\varrho^2}\cos\varphi+\frac{1}{\varrho^2}w, \quad \varrho\in[1,e], \varsigma,\varphi,w\in\mathbb{R}.$$

Clearly, f is a continuous, and for $\varrho\in[1,e]$ and $w,\varphi,\varsigma,\overline{w},\overline{\varsigma},\overline{\varphi}\in\mathbb{R}$, we get

$$|f(\varrho,\varsigma,\varphi,w)-f(\varrho,\overline{\varsigma},\overline{\varphi},\overline{w})|\le L_1|\varsigma-\overline{\varsigma}|+L_2|\varphi-\overline{\varphi}|+L_3|w-\overline{w}|,$$

with $L_1=\frac{1}{21}$, $L_2=\frac{3}{7}$, $L_3=1$ and

$$|K(\varrho,s,\upsilon)-K(\varrho,s,y)|\le\frac{1}{3s}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}|\upsilon-y|,$$

$$\beta_k=\sup\int_1^{\varrho}k(\varrho,s)\,ds=\frac{1}{3}\int_1^{\varrho}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}\frac{ds}{s}=\frac{(\log\varrho)^{\kappa}}{3\kappa}\le\frac{1}{3\kappa}=\frac{2}{9}.$$

Hence, conditions $(H2)$ and $(H3)$ are satisfied. Moreover,

$$\Lambda=\frac{\alpha(\log\eta)^{q+1}}{\Gamma(q+2)}+\frac{\beta(\log\Psi)^{1-\gamma}}{\Gamma(2-\gamma)}=3.4887.$$

Thus,

$$\rho:=\frac{L_1+\beta_kL_3}{1-L_2}\left[\frac{(\log\Psi)^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}\right]=0.75728<1.$$

Thus, the given system has a unique solution $\upsilon\in C_{\delta}^{n}([1,e],\mathbb{R})$, according to Theorem 3.1,

$$|f(\varrho,\varsigma,\varphi,w)|\le\frac{\varrho^4}{11+\varrho^2}+\frac{|\varsigma|}{7\varrho^4(|\varsigma|+3)}+\frac{3}{7\varrho^2}|\cos\varphi|+\frac{1}{3s}\log\left(\frac{\varrho}{s}\right)^{\kappa-1}|\sin w|,$$

so condition $(H4)$ is satisfied with

$$p(\varrho)=\frac{\varrho^4}{11+\varrho^2}, \quad \varphi(\varrho)=\frac{1}{21\varrho^4}, \quad \omega(\varrho)=\frac{3}{7\varrho^2}, \quad z(\varrho)=\frac{1}{3s}\log\left(\frac{t}{s}\right)^{\kappa-1}$$

and

$$\varphi^*=\frac{1}{21}, \quad \omega^*=\frac{3}{7}, \quad z^*=1.$$

We have

$$M:=\frac{(\log\Psi)^{\kappa}}{\Gamma(\alpha+1)}+\frac{|\alpha|(\log\Psi)(\log\eta)^{\kappa+q}}{|\Lambda|\Gamma(\kappa+q+1)}+\frac{|\beta|(\log\Psi)^{\kappa-\gamma+1}}{|\Lambda|\Gamma(\kappa-\gamma+1)}=1.9712.$$

We will demonstrate that condition (3.5) is true for $\Psi=e$. Indeed,

$$\omega^*+M(\varphi^*+\beta_kz^*)=0.96048<1.$$

Simple calculations demonstrate that conditions of Theorem 3.2 are all satisfied. Example 4.1 must thus have at least one solution specified on $[1,e]$.

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