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COUPLED NONLOCAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL INTEGRO-DIFFERENTIAL LANGEVIN SYSTEM VIA VARIABLE COEFFICIENT

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Abstract. In this paper, we aim to study a new coupled system of nonlinear fractional integro-differential Langevin equations with coupled multipoint boundary conditions. The existence and uniqueness of solution are investigated by using the Banach's and Krasnoselskii's fixed point theorems. The Ulam-Hyers stability of the mentioned equation is provided by applying the classical technique of functional analysis. Two examples are presented to verify our analysis.

1. Introduction

Fractional calculus has attained considerable interest due to their various applications in many scientific fields and the ability to describe the phenomena that have memory effects. In particular, fractional differential equations can be used to model a number of problems in physics, chemistry, biology and economy. As a result, many several authors have interested in it. For more details, one can go through in the books [1–4] and and the papers [5–16].

Langevin equations (introduced by Langevin in 1908) are used to model the evolution of physical phenomena in fluctuating environments (see [17]). Recently, the generalisation of the Langevin equations (fractional Langevin equations) has been considered by authors and researchers, for more details we give the following references [18–25].

Key words and phrases. Fractional integro-differential Langevin system, fractional derivatives and integrals, coupled nonlocal boundary value problems, Ulam-Hyers stability.

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On the other hand, coupled systems of fractional differential equations, including coupled nonlocal boundary conditions have been one of the important subjects in the field of fractional differential equations for their rich history, for more information see, [26–30].

In the last years, Ulam-Hyers stability has become of great importance to many researchers. It was introduced in 1940 by Ulam and then developed by Hyers. Many authors generalized the results obtained by Hyers for integer order differential equations. The mentioned stability for fractional differential equations are very important in many domains such as realistic problems, biology and economics. Recently, only a few authors have investigated in their work this type of Ulam Stabilities for coupled system of nonlinear fractional differential equations, see [31–35].

To our knowledge, coupled fractional integro-differential Langevin equations via variable coefficient involving coupled multipoint boundary conditions have not been extensively investigated yet. That's why, in the present article, we investigate a coupled system of fractional Langevin equations as follows:

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x(t) = f(t, x(t), y(t), \Phi y(t)), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y(t) = g(t, x(t), y(t), \Psi x(t)), & t \in [0,1], \end{cases} \tag{1.1}$$

subject to coupled multipoint boundary conditions

$$\begin{cases} x(0) = 0, \quad x(a_1) = 0, \quad x(1) = \sum_{i=1}^{n} \gamma_i y(s_i), \\ y(0) = 0, \quad y(b_1) = 0, \quad y(1) = \sum_{j=1}^{m} \delta_i x(u_j), \\ 0 < a_1 < b_1 < s_1 < s_2 < \cdots < s_n < u_1 < u_2 < \cdots < u_m < 1, \end{cases} \tag{1.2}$$

where $0 < \alpha_k < 1$, $1 < \beta_k \leq 2$, for $k = 1, 2$, $\gamma_i, \delta_j \in \mathbb{R}^*$ for $i = 1, \ldots, n$, $j = 1, 2, \ldots, m$, ${}^cD^{\beta_k}$, ${}^cD^{\alpha_k}$ are the Caputo's fractional derivatives, and $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\lambda_1, \lambda_2 : [0,1] \to \mathbb{R}$ are a given continuous functions and $\Psi x(t) = \int_0^t \psi(t,s)y(s)ds$, $\Phi y(t) = \int_0^t \phi(t,s)y(s)ds$, where $\phi, \psi : [0,1] \times [0,1] \longrightarrow [0, +\infty)$, with $\lambda_0 = \sup_{t\in[0,1]} |\int_0^t \phi(t,s)ds| < +\infty$, $\delta_0 = \sup_{t\in[0,1]} |\int_0^t \psi(t,s)ds| < +\infty$.

This paper is arranged as follows: in the second section, we give some preliminaries and notations that will be useful throughout the work. In the third section, we establish the main results by using the fixed point theory. In the fourth section, we investigated that Problem (1.1)–(1.2) is Ulam-Hyers stability. The last section, we give some examples to illustrate the results.

2. Preliminaries and Notations

In this section, we introduce some notation, definitions and lemma that we use in our proofs later.

Definition 2.1 ([3]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2 ([3]). The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n \in \mathbb{N}$, $0 \leq n-1 < \alpha < n$, $t > 0$.

Theorem 2.1 ([36]). *Let M be a bounded, closed, convex and nonempty subset of a Banach space X. Let A and B be operators such that:*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous;*
(iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.1 ([3]). *Let $\alpha, \beta \geq 0$, then the following relation hold:*

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Lemma 2.2 ([3]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha}\, {}^cD^{\alpha} f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}.$$

Lemma 2.3. *Let $x, y \in C([0,1], \mathbb{R})$ and $\Delta \neq 0$. Then the coupled system*

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x(t) = h_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y(t) = h_2(t), & t \in [0,1], \end{cases}$$

subject to the boundary conditions (1.2)*, has a solution given by*

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1} h_1(s) ds - \frac{\int_0^t (t-s)^{\alpha_1-1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} \\ &+ B_1(t) \left[\frac{\int_0^1 (1-s)^{\alpha_1-1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2-1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)} \right. \\ &+ \left. \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2+\beta_2-1} h_2(s) ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1} h_1(s) ds}{\Gamma(\alpha_1+\beta_1)} \right] \end{aligned}$$

$$
\begin{aligned}
&+ B_2(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+ \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+ B_3(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
&+ B_4(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
y(t) =&\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}h_2(s)ds - \frac{\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+ C_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+ \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+ C_2(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+ \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]\\
&+ C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]\\
&+ C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon_1 =&\frac{(1-a_1)}{\Gamma(\alpha_1+2)}, \quad \Upsilon_2 = \frac{\sum\limits_{i=1}^{n}\gamma_i s_i^{\alpha_2}(b_1-s_i)}{\Gamma(\alpha_2+2)}, \quad \Upsilon_4 = \frac{(1-b_1)}{\Gamma(\alpha_2+2)},\\
\Upsilon_3 =&\frac{\sum\limits_{j=1}^{m}\delta_j u_j^{\alpha_1}(a_1-u_j)}{\Gamma(\alpha_1+2)}, \quad \Delta = \Upsilon_1\Upsilon_4 - \Upsilon_3\Upsilon_2, \quad U = -\frac{\Upsilon_4}{\Delta\Gamma(2+\alpha_1)},\\
V =&\frac{\Upsilon_2}{\Delta\Gamma(2+\alpha_1)}, \quad R = -\frac{\Upsilon_3}{\Delta\Gamma(2+\alpha_2)}, \quad T = -\frac{\Upsilon_1}{\Delta\Gamma(2+\alpha_2)},\\
B_1(t) =&Ut^{\alpha_1}(a_1-t), \quad B_2(t) = Vt^{\alpha_1}(a_1-t),\\
B_3(t) =&\frac{t^{\alpha_1}}{a_1^{\alpha_1}}\left[1+(a_1-t)\left(V\sum_{j=1}^{m}\delta_j u_j^{\alpha_1} - U\right)\right],\\
B_4(t) =&\frac{t^{\alpha_1}}{b_1^{\alpha_2}}\Big(a_1-t\Big)\left(U\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} - V\right),\\
C_1(t) =&Tt^{\alpha_2}(b_1-t), \quad C_2(t) = Rt^{\alpha_2}(b_1-t),\\
C_3(t) =&\frac{t^{\alpha_2}}{b_1^{\alpha_2}}\left[1+(b_1-t)\left(R\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2} - T\right)\right],\\
C_4(t) =&\frac{t^{\alpha_2}}{a_1^{\alpha_1}}\Big(b_1-t\Big)\left(T\sum_{j=1}^{m}\delta_j u_j^{\alpha_1} - R\right).
\end{aligned}
$$

Proof. Using Lemma 2.2, we obtain $x(t) = I^{\alpha_1+\beta_1}h_1(t)+I^{\alpha_1}a_{01}+I^{\alpha_1}a_{11}t-I^{\alpha_1}\lambda_1(t)x(t)+a_{21}$, and $y(t) = I^{\alpha_2+\beta_2}h_2(t)+I^{\alpha_2}a_{02}+I^{\alpha_2}a_{12}t-I^{\alpha_2}\lambda_2(t)x(t)+a_{22}$, where $a_{01}, a_{11}, a_{21}, a_{02}, a_{12}, a_{22} \in \mathbb{R}$. According to the condition $x(0) = 0$, $y(0) = 0$, we get $a_{21} = a_{22} = 0$. Using the facts that $x(a_1) = y(b_1) = 0$, we obtain

$$
\begin{cases} a_{01} = \eta_1 + \theta_1 a_{11}, \\ a_{02} = \eta_2 + \theta_2 a_{12}, \end{cases} \tag{2.1}
$$

where

$$
\begin{cases}
\eta_1 = \dfrac{\Gamma(\alpha_1+1)}{a_1^{\alpha_1}}\left(\dfrac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \dfrac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right),\\
\eta_2 = \dfrac{\Gamma(\alpha_2+1)}{b_1^{\alpha_2}}\left(\dfrac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \dfrac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\right),\\
\theta_1 = -\dfrac{a_1}{2+\alpha_1},\\
\theta_2 = \dfrac{b_1}{2+\alpha_2}.
\end{cases}
$$

By applying the conditions $x(1) = \sum_{i=1}^{n} \gamma_i y(s_i)$, $y(1) = \sum_{j=1}^{m} \delta_j x(u_j)$ and (2.1), we have

$$\begin{cases} \Upsilon_1 a_{11} + \Upsilon_2 a_{12} = \Lambda_1, \\ \Upsilon_3 a_{11} + \Upsilon_4 a_{12} = \Lambda_2, \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
\Lambda_1 =& -\frac{1}{a_1^{\alpha_1}}\left(\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right)\\
&+\frac{\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}}{b_1^{\alpha_2}}\left(\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\right)\\
&+\frac{\int_0^{1}(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^{1}(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)},\\
\Lambda_2 =& -\frac{1}{b_1^{\alpha_1}}\left(\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds\right)\\
&+\frac{\sum_{i=1}^{n}\delta_j u_j^{\alpha_1}}{a_1^{\alpha_1}}\left(\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\right)\\
&+\frac{\int_0^{1}(1-s)^{\alpha_1-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{1}(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds.
\end{aligned}$$

By solving the system (2.2), we get

$$\begin{aligned}
a_{11} &= \frac{1}{\Delta}\Big(\Lambda_1\Upsilon_4 - \Lambda_2\Upsilon_2\Big),\\
a_{12} &= \frac{1}{\Delta}\Big(\Lambda_2\Upsilon_1 - \Lambda_1\Upsilon_3\Big).
\end{aligned}$$

Substituting the values of a_{11} and a_{12} in (2.1), we get

$$a_{01} = \eta_1 + \frac{\theta_1}{\Delta}\Big(\Lambda_1\Upsilon_4 - \Lambda_2\Upsilon_2\Big),$$
$$a_{02} = \eta_2 + \frac{\theta_2}{\Delta}\Big(\Lambda_2\Upsilon_1 - \Lambda_1\Upsilon_3\Big).$$

Substituting the value of a_{01}, a_{02}, a_{11} and a_{12}, we can deduce that

$$\begin{aligned}
x(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}h_1(s)ds - \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} \\
&+ B_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} \\
&+ \frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\
&+ B_2(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} \\
&+ \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)} - \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] \\
&+ B_3(t)\Bigg[\frac{\int_0^{a_1} (a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg] \\
&+ B_4(t)\Bigg[\frac{\int_0^{b_1} (b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
y(t) =& \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}h_2(s)ds - \frac{\int_0^t (t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} \\
&+ C_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)} - \frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}
\end{aligned}$$

$$+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+C_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}$$

$$+\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}-\frac{\int_0^1(1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg]$$

$$+C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}h_2(s)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg]$$

$$+C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}h_1(s)ds}{\Gamma(\alpha_1+\beta_1)}\Bigg].$$

By direct computation, it can easily be verified the converse of the lemma. ☐

3. Main Results

Let X be a Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with norm $\|x\|=\sup\{|x(t)|:t\in[0,1]\}$. Then, the product space $(X\times X,\|(x;y)\|)$ is also a Banach space equipped with the norm $\|(x;y)\|=\|x\|+\|y\|$. In view of Lemma 2.3, we define the operator $U:X\times X\to X\times X$ by $U(x,y)=(U_1(x,y),U_2(x,y))$. Here

$$\begin{aligned}
&U_1(x,y)(t)\\
=&\frac{\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}-\frac{\int_0^t(t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\\
&+B_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\\
&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\Bigg]\\
&+B_2(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds
\end{aligned}$$

$$+\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}$$
$$-\left.\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]+B_3(t)\left[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\right.$$
$$\left.-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\right]+B_4(t)$$
$$\times\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$

and

$$U_2(x,y)(t)$$
$$=\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds-\frac{1}{\Gamma(\alpha_2)}$$
$$\times\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y(s)ds+C_1(t)\left[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}-\frac{1}{\Gamma(\alpha_1)}\right.$$
$$\times(\sum_{j=1}^{m}\delta_j\int_0^{s_i}(u_j-s)^{\alpha_1-1}\lambda_1(s)x(s)ds)+\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}$$
$$\left.\times f(s,x(s),y(s),\Phi y(s))ds-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]$$
$$+C_2(t)\left[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}-\frac{\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\right.$$
$$+\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds-\frac{1}{\Gamma(\alpha_1+\beta_1)}$$
$$\left.\times\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\right]+C_3(t)\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\right.$$
$$\left.-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds\right]+C_4(t)\left[\frac{1}{\Gamma(\alpha_1)}\right.$$

$$\times \int_0^{a_1} (a_1 - s)^{\alpha-1} \lambda_1(s)x(s)ds - \frac{\int_0^{a_1} (a_1 - s)^{\alpha_1+\beta_1-1} f(s, x(s), y(s), \Phi y(s))ds}{\Gamma(\alpha_1 + \beta_1)} \Bigg].$$

For computational convenience, we set

$$\begin{aligned} r_{11} =& \max \Bigg\{ \frac{\left(1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}\right)\sigma_1^*}{\Gamma(\alpha_1 + \beta_1 + 1)} \\ &+ \frac{(1 + \delta_0)\sigma_2^*}{\Gamma(\alpha_2 + \beta_2 + 1)} \left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}\right) \\ &+ \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \left(B_1^* + 1 + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} + B_3^* a_1^{\alpha_1}\right), \\ &\frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} (1 + \lambda_0)\sigma_1^* + \frac{\sigma_2^*}{\Gamma(\alpha_2 + \beta_2 + 1)} \\ &\times \left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}\right) + \frac{|\lambda_2|}{\Gamma(\alpha_2 + 1)} \left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2}\right) \Bigg\} \end{aligned}$$

$$\begin{aligned} r_{12} =& \max \Bigg\{ \frac{C_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} \sigma_1^* + \frac{(1 + \delta_0)\sigma_2^*}{\Gamma(\alpha_2 + \beta_2 + 1)} \\ &\left(1 + C_1^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}\right) + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \left(C_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} + C_2^* + C_4^* a_1^{\alpha_1}\right), \\ &\frac{C_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} \sigma_1^*(1 + \lambda_0) \\ &+ \frac{1 + C_1^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2 + \beta_2 + 1)} \sigma_2^* \\ &+ \frac{|\lambda_2|}{\Gamma(\alpha_2 + 1)} \left(1 + C_1^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + C_3^* b_1^{\alpha_2}\right) \Bigg\}, \end{aligned}$$

where $B_i^* = \sup\{B_i(t),\ t \in [0, 1]\}$, $C_i^* = \sup\{C_i(t),\ t \in [0, 1]\}$, $\lambda_j = \sup\{\lambda_j(t), t \in [0, 1]\}$, $\sigma_j^* = \sup\{\sigma(t), t \in [0, 1]\}$, for $i = 1, 2, 3, 4$ and $j = 1, 2$.

Before introducing the main results, we impose some assumptions.

(H_1) $f, g : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

(H_2) There exist non negative functions $\sigma_1, \sigma_2 \in C\Big([0,1],[0,+\infty)\Big)$ such that for all $t \in [0,1]$ and $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, we have

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \le \sigma_1(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big),$$
$$|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)| \le \sigma_2(t)\Big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|\Big),$$

(H_3) $|f(t,x,y,z)| \le m_1(t)$, $|g(t,x,y,z)| \le m_2(t)$, for all $(t,x,y,z) \in [0,1]\times\mathbb{R}^3$, with $m_1, m_2 \in C([0,1];\mathbb{R}^+)$.

Theorem 3.1. *Let $\Delta \neq 0$. Suppose that (H_1)-(H_2) are satisfied. Then there exists a unique solution for System (1.1)–(1.2) provided that $r_{11} + r_{12} < 1$.*

Proof. Define $\sup_{0\le t\le 1} |f(t,0,0,0)| = A_1$, $\sup_{0\le t\le 1} |g(t,0,0,0)| = A_2$.

Let $B_r = \{(x,y) \in X \times X : \|(x,y)\| \le r\}$, with $r \ge \frac{r_{21}+r_{22}}{1-(r_{11}+r_{12})}$, where

$$r_{21} = \frac{B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} A_1 + \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} A_2,$$
$$r_{22} = \frac{C_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} A_1 + \frac{1 + C_1^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} A_2.$$

We prove that $TB_r \subseteq B_r$.

For $(x,y) \in B_r$, $t \in [0,1]$, we have:

$$\begin{aligned} |f(t,x(t),y(t),\Phi y(t))| &\le |f(t,x(t),y(t),\Phi y(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\ &\le \sigma_1(t)\Big(|x| + |y| + |\Phi y(t)|\Big) + A_1 \\ &\le \sigma_1^*\Big(\|x\| + (1+\lambda_0)\|y\|\Big) + A_1, \\ |g(t,x(t),y(t),\Psi y(t))| &\le |g(t,x(t),y(t),\Psi y(t)) - g(t,0,0,0)| + |g(t,0,0,0)| \\ &\le \sigma_2(t)\Big(|x| + |y| + |\Psi y(t)|\Big) + A_2 \\ &\le \sigma_2^*\Big(\|y\| + (1+\delta_0)\|x\|\Big) + A_2. \end{aligned}$$

Then,

$$\begin{aligned} &|U_1(x(t),y(t))| \\ \le &\Bigg[\frac{1}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_1^*}{\Gamma(\alpha_1+\beta_1+1)} + \frac{B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1+1}}{\Gamma(\alpha_1+\beta_1+1)} \\ &+ \frac{B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Bigg]\Big[\sigma_1^*\Big(\|x\| + (1+\lambda_0)\|y\| + A_1\Big] + \Bigg[\frac{B_2^*}{\Gamma(\alpha_2+\beta_2+1)} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\Bigg]\Big[\sigma_2^*\Big(\|y\| + (1+\delta_0)\|x\| + A_2\Big]\\
&+ \left(\frac{|\lambda_1| B_1^*}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1|}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1| B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{|\lambda_1| B_3^* a_1^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\|x\|\\
&+ \left(\frac{|\lambda_2| B_2^*}{\Gamma(\alpha_2+1)} + \frac{|\lambda_2| B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{|\lambda_2| B_4^* b_1^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\|y\|\\
\leq & \frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\Big[\sigma_1^*\Big(\|x\| + (1+\lambda_0)\|y\|\Big) + A_1\Big]\\
&+ \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} \times \Big[\sigma_2^*\Big(\|y\| + (1+\delta_0)\|x\|\Big) + A_2\Big]\\
&+ \frac{|\lambda_1|\left(B_1^* + 1 + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} + B_3^* a_1^{\alpha_1}\right)}{\Gamma(\alpha_1+1)}\|x\| + \frac{|\lambda_2|\left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2}\right)}{\Gamma(\alpha_2+1)}\|y\|\\
\leq & \Bigg[\frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}\sigma_1^* + \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}\\
&\times (1+\delta_0) + \frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\left(B_1^* + 1 + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} + B_3^* a_1^{\alpha_1}\right)\Bigg]\|x\|\\
&+ \Bigg[\frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}(1+\lambda_0) + \frac{\sigma_2^*}{\Gamma(\alpha_2+\beta_2+1)}\\
&\times (B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}) + \frac{|\lambda_2|\left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2}\right)}{\Gamma(\alpha_2+1)}\Bigg]\|y\|\\
&+ \frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}A_1 + \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}A_2.
\end{aligned}
$$

Consequently, $\|U_1(x(t), y(t))\| \leq r_{11} r + r_{21}$.

In the same way, we obtain that $\|U_2(x(t), y(t))\| \leq r_{12} r + r_{22}$. Therefore, we have $\|U(x(t), y(t))\| = \|U_1(x, y)\| + \|U_2(x, y)\| \leq (r_{11} + r_{12}) r + r_{21} + r_{22} \leq r$.

Now, for $(x_1, y_1), (x_2, y_2) \in X \times X$ and for $t \in [0, 1]$, we get

$$
\begin{aligned}
&|U_1(x_1, y_1)(t) - U_1(x_2, y_2)(t)| \\
\leq & \frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} \Big[\sigma_1^* \Big(\|x_1 - x_2\| + (1 + \lambda_0)\|y_1 - y_2\|\Big)\Big] \\
&+ \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2 + \beta_2 + 1)} \Big[\sigma_2^* \Big(\|y_1 - y_2\| + (1 + \delta_0)\|x_1 - x_2\|\Big)\Big] \\
&+ \frac{|\lambda_2| \Big(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2}\Big) \|y_1 - y_2\|}{\Gamma(\alpha_2 + 1)} \\
\leq & \Bigg[\frac{1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} \sigma_1^* + \frac{\Big(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}\Big)}{\Gamma(\alpha_2 + \beta_2 + 1)} \\
&\times (1 + \delta_0) + |\lambda_1| \frac{\Big(B_1^* + 1 + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} + B_3^* a_1^{\alpha_1}\Big)}{\Gamma(\alpha_1 + 1)}\Bigg] \|x_1 - x_2\| + \Bigg[\frac{(1 + \lambda_0)}{\Gamma(\alpha_1 + \beta_1 + 1)} \\
&\times \Big(1 + B_1^* + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + B_3^* a_1^{\alpha_1+\beta_1}\Big) + \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2 + \beta_2 + 1)} \sigma_2^* \\
&+ \frac{|\lambda_2| \Big(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2} + B_4^* b_1^{\alpha_2}\Big)}{\Gamma(\alpha_2 + 1)}\Bigg] \|y_1 - y_2\| \\
\leq & r_{11}(\|x_1 - x_2\| + \|y_1 - y_2\|).
\end{aligned}
$$

Analogously, we can also have $|U_2(x_1, y_1)(t) - U_2(x_2, y_2)(t)| \leq r_{12}(\|x_1 - x_2\| + \|y_1 - y_2\|)$, which leads to

$$\|U(x_1, y_1) - U(x_2, y_2)\| \leq (r_{11} + r_{12})(\|x_1 - x_2\| + \|y_1 - y_2\|).$$

As $r_{11} + r_{12} < 1$, therefore the operator U is a contraction mapping. Then, we deduce that System (1.1)–(1.2) has a unique solution. □

Theorem 3.2. *Let $\Delta \neq 0$. Assume that $(H_1), (H_3)$ hold. Then, System* (1.1)–(1.2) *has at least one solution on $[0, 1]$ if $R < 1$, where*

$$R = \max\left\{\frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)}\left(1 + B_1^* + \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} B_2^* + B_3^* a_1^{\alpha_1} + \sum_{j=1}^{m} \delta_j u_j^{\alpha_1} C_1^* + C_2^* + C_4^* a_1^{\alpha_1}\right),\right.$$

$$\frac{|\lambda_2|}{\Gamma(\alpha_2+1)}\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^*+B_4^*b_1^{\alpha_2}+1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^*+C_3^*b_1^{\alpha_2}\right)\Bigg\}.$$

Proof. We define a bounded closed and convex ball $B_{r'}=\{(x,y)\in X\times X:\|(x,y)\|\leq r'\}$ with $r'\geq\frac{r_2'}{1-R}$, where

$$\begin{aligned}r_2'=&\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\Bigg(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}\\&+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}\Bigg)+\frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}\Bigg(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\\&+1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}\Bigg).\end{aligned}$$

Let us introduce the decomposition $U(x,y)(t)=W_1(x,y)(t)+W_2(x,y)(t)$, where $W_1(x,y)(t)=(T_1(x,y),R_1(x,y))(t)$, $W_2(x,y)(t)=(T_2(x,y),R_2(x,y))(t)$, with

$$\begin{aligned}T_1(x,y)(t)=&\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\\&+B_1(t)\left[\sum_{i=1}^{n}\gamma_i\frac{\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right.\\&\left.-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 f(s,x(s),y(s),\Phi y(s))(1-s)^{\alpha_1+\beta_1-1}ds\right]\\&+B_2(t)\left[\frac{\sum_{j=1}^{m}\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)}\right.\\&\left.-\frac{\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right]\\&-\frac{B_3(t)}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s),\Phi y(s))ds\\&-B_4(t)\left[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)}\right],\end{aligned}$$

$$T_2(x,y)(t)=-\frac{\int_0^t(t-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}+B_1(t)\left[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}\right.$$

$$
\begin{aligned}
&- \frac{\sum_{i=1}^{n} \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2 - 1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)} \Bigg] + B_2(t) \Bigg[\frac{\int_0^1 (1-s)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \lambda_2(s) y(s) ds \\
&- \frac{\sum_{j=1}^{m} \delta_j \int_0^{u_j} (u_j - s)^{\alpha_1 - 1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} \Bigg] \\
&+ B_3(t) \frac{\int_0^{a_1} (a_1 - s)^{\alpha_1 - 1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} \\
&+ \frac{B_4(t) \int_0^{b_1} (b_1 - s)^{\alpha_2 - 1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)},
\end{aligned}
$$

$$
\begin{aligned}
R_1(x,y)(t) =& \frac{1}{\Gamma(\alpha_2 + \beta_2)} \int_0^t (t-s)^{\alpha_2 + \beta_2 - 1} g(s, x(s), y(s), \Psi x(s)) ds \\
&+ C_1(t) \Bigg[\frac{\sum_{j=1}^{m} \delta_j \int_0^{u_j} (u_j - s)^{\alpha_1 + \beta_1 - 1} f(s, x(s), y(s), \Phi y(s)) ds}{\Gamma(\alpha_1 + \beta_1)} \\
&- \frac{1}{\Gamma(\alpha_2 + \beta_2)} \int_0^1 (1-s)^{\alpha_2 + \beta_2 - 1} g(s, x(s), y(s), \Psi x(s)) ds \Bigg] \\
&+ C_2(t) \Bigg[\frac{\sum_{i=1}^{n} \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2 + \beta_2 - 1} g(s, x(s), y(s), \Psi x(s)) ds}{\Gamma(\alpha_2 + \beta_2)} \\
&- \frac{1}{\Gamma(\alpha_1 + \beta_1)} \int_0^1 (1-s)^{\alpha_1 + \beta_1 - 1} f(s, x(s), y(s), \Phi y(s)) ds \Bigg] \\
&- \frac{C_3(t)}{\Gamma(\alpha_2 + \beta_2)} \int_0^{b_1} (b_1 - s)^{\alpha_2 + \beta_2 - 1} g(s, x(s), y(s), \Psi x(s)) ds \\
&- \frac{C_4(t)}{\Gamma(\alpha_1 + \beta_1)} \int_0^{a_1} (a_1 - s)^{\alpha_1 + \beta_1 - 1} f(s, x(s), y(s), \Phi y(s)) ds,
\end{aligned}
$$

$$
\begin{aligned}
R_2(x,y)(t) =& - \frac{\int_0^t (t-s)^{\alpha_2 - 1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)} + C_1(t) \Bigg[\frac{\int_0^1 (1-s)^{\alpha_2 - 1} \lambda_2(s) y(s) ds}{\Gamma(\alpha_2)} \\
&- \frac{\sum_{j=1}^{m} \delta_j \int_0^{u_j} (u_j - s)^{\alpha_1 - 1} \lambda_1(s) x(s) ds}{\Gamma(\alpha_1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ C_2(t)\left[\frac{1}{\Gamma(\alpha_1)}\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds\right.\\
&\left.- \frac{\lambda_2\sum_{i=1}^{n}\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}y(s)ds}{\Gamma(\alpha_2)}\right]\\
&+ C_3(t)\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y(s)ds}{\Gamma(\alpha_2)}\\
&+ \frac{C_4(t)\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x(s)ds}{\Gamma(\alpha_1)}.
\end{aligned}
$$

For $(x,y)\in B_{r'}$, we have

$$
\begin{aligned}
&|T_1(x,y)(t)+T_2(x,y)(t)|\\
\leq&\frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}+\frac{B_1^*\|m_2\|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}+\frac{B_1^*\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{B_2^*\|m_1\|\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{B_2^*\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}+\frac{B_3^* a_1^{\alpha_1+\beta_1}\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{B_4^* b_1^{\alpha_2+\beta_2}\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)}+\frac{|\lambda_1|\cdot\|x\|}{\Gamma(\alpha_1+1)}+\frac{B_1^*|\lambda_1|\cdot\|x\|}{\Gamma(\alpha_1+1)}+\frac{|\lambda_2|\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}\|y\|B_1^*}{\Gamma(\alpha_2+1)}\\
&+\frac{B_2^*|\lambda_2|\cdot\|y\|}{\Gamma(\alpha_2+1)}+\frac{|\lambda_1|\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}\|x\|B_2^*}{\Gamma(\alpha_1+1)}+\frac{B_3^*|\lambda_1|a_1^{\alpha_1}\|x\|}{\Gamma(\alpha_1+1)}+\frac{B_4^*|\lambda_2|b_1^{\alpha_2}\|y\|}{\Gamma(\alpha_2+1)}\\
\leq&\frac{|\lambda_1|\left(1+B_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}B_2^*+B_3^*a_1^{\alpha_1}\right)\|x\|}{\Gamma(\alpha_1+1)}+\frac{|\lambda_2|\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^*+B_4^*b_1^{\alpha_2}\right)\|y\|}{\Gamma(\alpha_2+1)}\\
&+\frac{\|m_1\|\left(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}\right)}{\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{\|m_2\|\left(B_2^*+B_1^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

In a similar manner, we have

$$
\begin{aligned}
|R_1(x,y)(t)+R_2(x,y)(t)| \leq & \frac{|\lambda_1|}{\Gamma(\alpha_1+1)}\left(\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}C_1^*+C_2^*+C_4^*a_1^{\alpha_1}\right)\|x\| \\
& + \frac{|\lambda_2|\left(1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^*+C_3^*b_1^{\alpha_2}\right)\|y\|}{\Gamma(\alpha_2+1)} \\
& + \frac{\|m_1\|\left(C_1^*\sum_{j=1}^{m}\delta_j u_j^{\alpha_1+\beta_1}+C_2^*+C_4^*a_1^{\alpha_1+\beta_1}\right)}{\Gamma(\alpha_1+\beta_1+1)} \\
& + \frac{\|m_2\|\left(1+C_1^*+C_2^*\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2+\beta_2}+C_3^*b_1^{\alpha_2+\beta_2}\right)}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

Further, we obtain

$$\|W_1(x_1,x_2)(t)+W_2(x_1,x_2)\| \leq Rr'+r_2' \leq r'.$$

Hence, $W_1(x_1,x_2)(t)+W_2(x_1,x_2)(t) \in B_{r'}$.

For $(x_1,y_1),(x_2,y_2) \in B_{r'}$ and $t \in [0,1]$, we have

$$
\begin{aligned}
|T_2(x_1,y_1)-T_2(x_2,y_2)| \leq & \frac{|\lambda_1|\left(1+B_1^*+\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}B_2^*+B_3^*a_1^{\alpha_1}\right)\|x_1-x_2\|}{\Gamma(\alpha_1+1)} \\
& + \frac{|\lambda_2|\left(B_2^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}B_1^*+B_4^*b_1^{\alpha_2}\right)\|y_1-y_2\|}{\Gamma(\alpha_2+1)}, \\
|R_2(x_1,y_1)-R_2(x_2,y_2)| \leq & \frac{|\lambda_1|\left(\sum_{j=1}^{m}\delta_j u_j^{\alpha_1}C_1^*+C_2^*+C_4^*a_1^{\alpha_1}\right)\|x_1-x_2\|}{\Gamma(\alpha_1+1)} \\
& + \frac{|\lambda_2|\left(1+C_1^*+\sum_{i=1}^{n}\gamma_i s_i^{\alpha_2}C_2^*+C_3^*b_1^{\alpha_2}\right)\|y_1-y_2\|}{\Gamma(\alpha_2+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|W_2(x_1,y_1)-W_2(x_2,y_2)\| \leq & R\|x_1-x_2\|+R\|y_1-y_2\| \\
\leq & R\|(x_1-x_2,y_1-y_2)\|.
\end{aligned}
$$

As $R<1$, then W_2 is a contraction.

Next, we prove that W_1 is compact and continuous. The continuity of f,g implies that the operator W_1 is continuous. Moreover, W_1 is uniformly bounded on $B_{r'}$.

Suppose that $0 \leq t_1 < t_2 \leq 1$. We have

$$
\begin{aligned}
|T_1(x,y)(t_2) - T_1(x,y)(t_1)| \leq & \Bigg| \frac{\int_0^{t_2} (t_2 - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)} \\
& - \int_0^{t_1} (t_1 - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds \Bigg| \\
& + |B_1(t_2) - B_1(t_1)| \\
& \times \Bigg| \frac{\sum_{i=1}^{n} \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)} \\
& - \frac{\int_0^{1} (1 - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)} \Bigg| \\
& + |B_2(t_2) - B_2(t_1)| \Bigg| \sum_{j=1}^{m} \delta_j \\
& \times \frac{\int_0^{u_j} (u_j - s)^{\alpha_1+\beta_1-1} f(s,x(s),y(s),\Phi y(s))ds}{\Gamma(\alpha_1+\beta_1)} \\
& - \frac{\int_0^{1} (1 - s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),\Psi x(s))ds}{\Gamma(\alpha_2+\beta_2)} \Bigg| \\
& + \frac{|B_3(t_2) - B_3(t_1)|}{\Gamma(\alpha_1+\beta_1)} \Bigg| \int_0^{a_1} (a_1 - s)^{\alpha_1+\beta_1-1} \\
& \times f(s,x(s),y(s),\Phi y(s))ds \Bigg| + \frac{|B_4(t_2) - B_4(t_1)|}{\Gamma(\alpha_2+\beta_2)} \\
& \times \Bigg| \int_0^{b_1} (b_1 - s)^{\alpha_2+\beta_2-1} g(s,x(s),y(s),\Psi x(s))ds \Bigg| \\
\leq & \frac{\|m_1\|(t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1})}{\Gamma(\alpha_1+\beta_1+1)} + |B_1(t_2) - B_1(t_1)| \\
& \times \left[\frac{\|m_2\| \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)} \right]
\end{aligned}
$$

$$+ |B_2(t_2) - B_2(t_1)| \left[\frac{\|m_1\| \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} \right] + \frac{|B_3(t_2) - B_3(t_1)| \cdot \|m_1\| a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}$$
$$+ \frac{|B_4(t_2) - B_4(t_1)| \cdot \|m_2\| b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}.$$

Similarly, we obtain that

$$\begin{aligned} |R_1(x,y)(t_2) - R_1(x,y)(t_1)| \leq & \frac{\|m_2\|(t_2^{\alpha_2+\beta_2} - t_1^{\alpha_2+\beta_2})}{\Gamma(\alpha_2+\beta_2+1)} + |C_1(t_2) - C_1(t_1)| \\ & \times \left[\frac{\|m_1\| \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{\|m_2\|}{\Gamma(\alpha_2+\beta_2+1)} \right] \\ & + |C_2(t_2) - C_2(t_1)| \left[\frac{\|m_2\| \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{\|m_1\|}{\Gamma(\alpha_1+\beta_1+1)} \right] + \frac{|C_3(t_2) - C_3(t_1)| \|m_2\| b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} \\ & + \frac{|C_4(t_2) - C_4(t_1)| \|m_1\| a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}. \end{aligned}$$

Therefore, the operator W_1 is equicontinuous. Thus, W_1 is relatively compact on $B_{r'}$. Then by Arzela Ascoli theorem, the operator W_1 is compact on $B_{r'}$. In conclusion, all terms of Krasnoselskii's theorem have been applied perfectly. Hence, (1.1) and (1.2) has at least one solution on $B_{r'}$. □

4. Ulam-Hyers Stability

Definition 4.1. For some $\varepsilon_1,\ \varepsilon_2 > 0$, we consider the system of inequalities

$$\begin{cases} \left| {}^c D^{\beta_1} ({}^c D^{\alpha_1} + \lambda_1(t)) x^*(t) - f(t, x^*(t), y^*(t), \Phi y^*(t)) \right| < \varepsilon_1, & t \in [0,1], \\ \left| {}^c D^{\beta_2} ({}^c D^{\alpha_2} + \lambda_2(t)) y^*(t) - g(t, x^*(t), y^*(t), \Psi x^*(t)) \right| < \varepsilon_2, & t \in [0,1]. \end{cases} \tag{4.1}$$

Then System (1.1)–(1.2) is Ulam-Hyers stable if there exist $C_1, C_2 > 0$, such that there is a unique solution (x, y) of Problem (1.1)–(1.2), with

$$\|(x^*, y^*) - (x, y)\| \leq C_1 \varepsilon_1 + C_2 \varepsilon_2.$$

Remark. (x^*, y^*) is a solution of system of inequalities (4.1) if we can find $\rho_1, \rho_2 \in \Big(C[0,1];\mathbb{R}\Big)$ such that $|\rho_1(t)| \le \varepsilon_1$, $|\rho_2(t)| \le \varepsilon_2$, $t \in [0,1]$ and

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x^*(t) = f(t, x^*(t), y^*(t), \Phi y^*(t)) + \rho_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y^*(t) = g(t, x^*(t), y^*(t), \Psi x^*(t)) + \rho_2(t), & t \in [0,1]. \end{cases} \tag{4.2}$$

Theorem 4.1. *If* (H_1), (H_2) *and* $r_{11} + r_{22} < 1$ *are satisfied, then Problem* (1.1)-(1.2) *is Ulam-Hyers stable.*

Proof. Let (x, y) be unique solution of System (1.1)-(1.2) and (x^*, y^*) be a solution of (4.1). Then we can find $\rho_1, \rho_2 \in \Big(C[0,1];\mathbb{R}\Big)$ such that

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + \lambda_1(t))x^*(t) = f(t, x^*(t), y^*(t), \Phi y^*(t)) + \rho_1(t), & t \in [0,1], \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + \lambda_2(t))y^*(t) = g(t, x^*(t), y^*(t), \Psi x^*(t)) + \rho_2(t), & t \in [0,1], \\ x(0) = 0, \quad x(a_1) = 0, \quad x(1) = \sum_{i=1}^{n} \gamma_i y(s_i), \\ y(0) = 0, \quad y(b_1) = 0, \quad y(1) = \sum_{j=1}^{m} \delta_i x(u_j), \\ 0 < a_1 < b_1 < s_1 < s_2 < \cdots < s_n < u_1 < u_2 < \cdots < u_m < 1. \end{cases} \tag{4.3}$$

By Lemma 2.3, we can obtain

$$\begin{aligned} x^*(t) =& \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1}\Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds \\ &- \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} + B_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} \\ &- \frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} + \frac{\sum_{i=1}^{n}\gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} \\ &\times \Big(g(s, x^*(s), y^*(s), \Psi x^*(s)) + \rho_2(s)\Big)ds - \frac{\int_0^1 (1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} \\ &\times \Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds\Bigg] + B_2(t)\Bigg[\frac{1}{\Gamma(\alpha_2)} \\ &\times \int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds - \frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j} (u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} \\ &+ \frac{\sum_{j=1}^{m}\delta_j \int_0^{u_j} (u_j-s)^{\alpha_1+\beta_1-1}\Big(f(s, x^*(s), y^*(s), \Phi y^*(s)) + \rho_1(s)\Big)ds}{\Gamma(\alpha_1+\beta_1)} \end{aligned}$$

$$- \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] + B_3(t)$$

$$\times \Bigg[\frac{\int_0^{a_1} (a_1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\Big(\rho_1(s)$$

$$+ f(s,x^*(s),y^*(s),\Phi y^*(s))\Big)ds\Bigg] + B_4(t)\Bigg[\frac{\int_0^{b_1} (b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}$$

$$- \frac{\int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg],$$

$$y^*(t) = \frac{\int_0^t (t-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)} - \frac{1}{\Gamma(\alpha_2)}$$

$$\times \int_0^t (t-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds + C_1(t)\Bigg[\frac{\int_0^1 (1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}$$

$$- \frac{\sum_{j=1}^m \delta_j \int_0^{s_i} (u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_1+\beta_1)}$$

$$\times \sum_{j=1}^m \delta_j \int_0^{u_j} (u_j-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s)) + \rho_1(s)\Big)$$

$$- \frac{\int_0^1 (1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}\Bigg] + C_2(t)$$

$$\times \Bigg[\frac{\int_0^1 (1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)}$$

$$+ \frac{\sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds}{\Gamma(\alpha_2+\beta_2)}$$

$$- \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s)) + \rho_1(s)\Big)ds\Bigg]$$

$$+ C_3(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}$$
$$\times \int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds\Bigg]$$
$$+ C_4(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{1}{\Gamma(\alpha_1+\beta_1)}$$
$$\times \int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}\Big(f(s,x^*(s),y^*(s),\Phi y^*(s)) + \rho_1(s)\Big)ds\Bigg].$$

Using, $|\rho_1(t)| \leq \varepsilon_1$ and $|\rho_2(t)| \leq \varepsilon_2$, $t \in [0,1]$, we have

$$\Big|x^*(t) - \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds$$
$$- \frac{\int_0^t (t-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds + B_1(t)\Big[\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)}$$
$$- \frac{1}{\Gamma(\alpha_2)}\sum_{i=1}^n \gamma_i \int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds + \frac{\sum_{i=1}^n\gamma_i}{\Gamma(\alpha_2+\beta_2)}$$
$$\times \int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1} g(s,x^*(s),y^*(s),\Psi x^*(s))ds - \frac{1}{\Gamma(\alpha_1+\beta_1)}$$
$$\times \int_0^1 (1-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] + B_2(t)\Bigg[\frac{1}{\Gamma(\alpha_2)}$$
$$\times \int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds - \frac{\sum_{j=1}^m \delta_j\int_0^{u_j}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)}$$
$$+ \frac{\sum_{j=1}^m \delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1} f(s,x^*(s),y^*(s),\Phi y^*(s))ds}{\Gamma(\alpha_1+\beta_1)} - \frac{1}{\Gamma(\alpha_2+\beta_2)}$$
$$\times \int_0^1(1-s)^{\alpha_2+\beta_2-1}\Big(g(s,x^*(s),y^*(s),\Psi x^*(s)) + \rho_2(s)\Big)ds\Bigg]$$
$$+ B_3(t)\Bigg[\frac{\int_0^{a_1}(a_1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{a_1}(a_1-s)^{\alpha_1+\beta_1-1}$$

$$
\begin{aligned}
&\times f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] + B_4(t)\Bigg[\frac{\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&- \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg]\Bigg| \\
\le&\frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)}\left(1+B_1^*+B_3^*a_1^{\alpha_1+\beta_1}+B_2^*\sum_{j=1}^m\delta_j u_j^{\alpha_1+\beta_1}\right) \\
&+ \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)}\left(B_2^*+B_1^*\sum_{i=1}^n\gamma_i s_i^{\alpha_2+\beta_2}+B_4^*b_1^{\alpha_2+\beta_2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg|y^*(t) - \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds - \frac{1}{\Gamma(\alpha_2)} \\
&\times\int_0^t(t-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds + C_1(t)\Bigg[\frac{\int_0^1(1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&- \frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^m\delta_j\int_0^{s_i}(u_j-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds + \frac{1}{\Gamma(\alpha_1+\beta_1)} \\
&\times\sum_{j=1}^m\delta_j\int_0^{u_j}(u_j-s)^{\alpha_1+\beta_1-1}f(s,x^*(s),y^*(s),\Phi y^*(s)) \\
&- \frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg] + C_2(t) \\
&\times\Bigg[\frac{\int_0^1(1-s)^{\alpha_1-1}\lambda_1(s)x^*(s)ds}{\Gamma(\alpha_1)} - \frac{\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds}{\Gamma(\alpha_2)} \\
&+ \frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^n\gamma_i\int_0^{s_i}(s_i-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds \\
&- \frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x^*(s),y^*(s),\Phi y^*(s))ds\Bigg] \\
&+ C_3(t)\Bigg[\frac{1}{\Gamma(\alpha_2)}\int_0^{b_1}(b_1-s)^{\alpha_2-1}\lambda_2(s)y^*(s)ds - \frac{1}{\Gamma(\alpha_2+\beta_2)} \\
&\times\int_0^{b_1}(b_1-s)^{\alpha_2+\beta_2-1}g(s,x^*(s),y^*(s),\Psi x^*(s))ds\Bigg] + C_4(t)\Bigg[\frac{1}{\Gamma(\alpha_1)} \\
&\times\int_0^{a_1}(a_1-s)^{\alpha-1}\lambda_1(s)x^*(s)ds - \frac{1}{\Gamma(\alpha_1+\beta_1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^{a_1} (a_1 - s)^{\alpha_1+\beta_1-1} f(s, x^*(s), y^*(s), \Phi y^*(s))ds\Bigg]\Bigg| \\
\leq & \frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)} \left(C_1^* \sum_{j=1}^m \delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1} \right) \\
&+ \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)} \left(1 + C_1^* + C_2^* \sum_{i=1}^n \gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2} \right).
\end{aligned}
$$

By (H_2), we get

$$
\begin{aligned}
&|x^*(t) - x(t)| \\
\leq & \frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)} \left(1 + B_1^* + B_3^* a_1^{\alpha_1+\beta_1} + B_2^* \sum_{j=1}^m \delta_j u_j^{\alpha_1+\beta_1} \right) \\
&+ \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)} \left(B_2^* + B_1^* \sum_{i=1}^n \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2} \right) \\
&+ \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^t (t-s)^{\alpha_1+\beta_1-1} \Big| f(s, x^*(s), y^*(s), \Phi y^*(s)) \\
&- f(s, x(s), y(s), \Phi y(s)) \Big| ds + \frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} x^*(s) ds \\
&+ |B_1(t)| \Bigg[\frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} |x^*(s) - x(s)| ds \\
&+ \frac{|\lambda_2| \sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2-1} |y^*(s) - y(s)| ds}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_2+\beta_2)} \\
&\times \sum_{i=1}^n \gamma_i \int_0^{s_i} (s_i - s)^{\alpha_2+\beta_2-1} \Big| g(s, x^*(s), y^*(s), \Psi x^*(s)) \\
&- g(s, x(s), y(s), \Psi x(s)) \Big| ds \\
&- \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^1 (1-s)^{\alpha_1+\beta_1-1} \Big| f(s, x^*(s), y^*(s), \Phi y^*(s)) \\
&- f(s, x(s), y(s), \Phi y(s)) \Big| ds \Bigg] + |B_2(t)| \Bigg[\frac{|\lambda_2| \int_0^1 (1-s)^{\alpha_2-1} |y^*(s) - y(s)| ds}{\Gamma(\alpha_2)} \\
&+ \frac{|\lambda_1|}{\Gamma(\alpha_1)} \sum_{j=1}^m \delta_j \int_0^{u_j} (u_j - s)^{\alpha_1-1} |x^*(s) - x(s)| ds + \sum_{j=1}^m \delta_j \\
&\times \frac{\int_0^{u_j} (u_j - s)^{\alpha_1+\beta_1-1} \Big| f(s, x^*(s), y^*(s), \Phi y^*(s)) - f(s, x(s), y(s), \Phi y(s)) \Big| ds}{\Gamma(\alpha_1+\beta_1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha_2+\beta_2)} \int_0^1 (1-s)^{\alpha_2+\beta_2-1} \Big| g(s, x^*(s), y^*(s), \Psi x^*(s)) \\
&- g(s, x(s), y(s), \Psi x(s)) \Big| ds \Bigg] + |B_3(t)| \Bigg[\frac{|\lambda_1| \int_0^{a_1} (a_1-s)^{\alpha_1-1} |x^*(s)-x(s)| ds}{\Gamma(\alpha_1)} \\
&- \frac{1}{\Gamma(\alpha_1+\beta_1)} \int_0^{a_1} (a_1-s)^{\alpha_1+\beta_1-1} \Big| f(s, x^*(s), y^*(s), \Phi y^*(s)) \\
&- f(s, x(s), y(s), \Phi y(s)) \Big| ds \Bigg] + |B_4(t)| \\
&\Bigg[\frac{|\lambda_2| \int_0^{b_1} (b_1-s)^{\alpha_2-1} |y^*(s)-y(s)| ds}{\Gamma(\alpha_2)} - \frac{1}{\Gamma(\alpha_2+\beta_2)} \int_0^{b_1} (b_1-s)^{\alpha_2+\beta_2-1} \\
&\times \Big| g(s, x^*(s), y^*(s), \Psi x^*(s)) - g(s, x(s), y(s), \Psi x(s)) \Big| ds \Bigg] \Big| \\
\leq& \frac{\varepsilon_1}{\Gamma(\alpha_1+\beta_1+1)} \left(1 + B_1^* + B_3^* a_1^{\alpha_1+\beta_1} + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} \right) \\
&+ \frac{\varepsilon_2}{\Gamma(\alpha_2+\beta_2+1)} \left(B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2} \right) \\
&+ r_{11} \Big(\|x^*-x\| + \|y^*-y\| \Big).
\end{aligned}
$$

So, $(1-r_{11})\|x^*-x\| \leq \Theta_1\varepsilon_1 + \Theta_2\varepsilon_2 + r_{11}\|y^*-y\|$, where

$$
\begin{aligned}
\Theta_1 =& \frac{1 + B_1^* + B_3^* a_1^{\alpha_1+\beta_1} + B_2^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}, \\
\Theta_2 =& \frac{B_2^* + B_1^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + B_4^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}
$$

In the same fashion, we have, $(1-r_{12})\|y^*-y\| \leq \Theta_3\varepsilon_1 + \Theta_4\varepsilon_2 + r_{12}\|x^*-x\|$, where

$$
\begin{aligned}
\Theta_3 =& \frac{C_1^* \sum_{j=1}^{m} \delta_j u_j^{\alpha_1+\beta_1} + C_2^* + C_4^* a_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}, \\
\Theta_4 =& \frac{1 + C_1^* + C_2^* \sum_{i=1}^{n} \gamma_i s_i^{\alpha_2+\beta_2} + C_3^* b_1^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)},
\end{aligned}
$$

then, we get

$$\|x^* - x\| \leq \frac{\Theta_1(1-r_{12}) + r_{11}\Theta_3}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}\varepsilon_1 + \frac{\Theta_2(1-r_{12}) + r_{11}\Theta_4}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}\varepsilon_2$$

and

$$\|y^* - y\| \leq \frac{\Theta_3(1-r_{11}) + r_{12}\Theta_1}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}\varepsilon_1 + \frac{\Theta_4(1-r_{11}) + r_{12}\Theta_2}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}\varepsilon_2,$$

which implies that

$$\begin{aligned}\|x^* - x\| + \|y^* - y\| \leq & \frac{\Theta_1(1-r_{12}) + r_{11}\Theta_3}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}\varepsilon_1 + \frac{\Big(\Theta_2(1-r_{12}) + r_{11}\Theta_4\Big)\varepsilon_2}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}} \\ & + \frac{\Big(\Theta_3(1-r_{11}) + r_{12}\Theta_1\Big)\varepsilon_1}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}} + \frac{\Big(\Theta_4(1-r_{11}) + r_{12}\Theta_2\Big)\varepsilon_2}{(1-r_{11})(1-r_{12}) - r_{11}r_{12}}.\end{aligned}$$

Hence, System (1.1)–(1.2) is Ulam-Hyers stable. □

5. Examples

Example 5.1. Consider the following system of fractional integro-differential Langevin equations:

(5.1)

$$\begin{cases} {}^cD^{\frac{12}{7}}\left({}^cD^{\frac{6}{7}} + \frac{t}{10^4}\right)x(t) = \frac{t^2}{3\times 10^4}\left(\frac{x(t)+y(t)}{4} + \frac{\int_0^t t^4 s^3 y(s)ds}{10^3}\right), & t \in [0,1], \\ {}^cD^{\frac{13}{8}}\left({}^cD^{\frac{7}{8}} + \frac{t}{10^4}\right)y(t) = \frac{\left(\sin(x(t)) + \cos(y(t)) + \frac{\int_0^t t^5 s^4 x(s)ds}{10^3}\right)}{4\times 10^4 + t^2}, & t \in [0,1], \\ x(0) = 0, \quad x\left(\frac{1}{1000}\right) = 0, \quad x(1) = \frac{1}{3000}\Big(y(\frac{1}{50}) + y(\frac{1}{40}) + y(\frac{1}{30})\Big), \\ y(0) = 0, \quad y\left(\frac{1}{100}\right) = 0, \quad y(1) = \frac{1}{4000}\Big(x(\frac{1}{25}) + x(\frac{1}{12}) + x(\frac{1}{6})\Big), \end{cases}$$

where $\beta_1 = \frac{12}{7}$, $\alpha_1 = \frac{6}{7}$, $\beta_2 = \frac{13}{8}$, $\alpha_2 = \frac{7}{8}$, $\lambda_1 = \lambda_2 = \frac{1}{10000}$ and

$$\begin{aligned} f(t,x,y,z) =& \frac{t^2}{30000}\Big(\frac{x(t)+y(t)}{4} + z(t)\Big), \\ g(t,x,y,z) =& \frac{1}{40000+t^2}\left(\sin(x(t)) + \cos(y(t)) + \frac{z(t)}{2}\right), \\ \Phi y(t) =& \frac{1}{250}\int_0^t t^4 s^3 y(s)ds), \quad \Psi x(t) = \frac{1}{1000}\int_0^t t^5 s^4 x(s)ds, \end{aligned}$$

$a_1 = \frac{1}{1000}$, $b_1 = \frac{1}{100}$, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3000}$, $s_1 = \frac{1}{50}$, $s_2 = \frac{1}{40}$, $s_3 = \frac{1}{30}$, $\delta_1 = \delta_2 = \delta_3 = \dfrac{1}{4000}$, $u_1 = \frac{1}{25}$, $u_2 = \frac{1}{12}$, $u_3 = \frac{1}{6}$. Clearly, $\delta_0 = \dfrac{1}{5000}$, $\lambda_0 = \dfrac{1}{1000}$ and $\sigma_1^* = \dfrac{1}{120000}$,

$\sigma_2^* = \dfrac{1}{40000}$. Furthermore, we have

$$r_{11} + r_{12} \approx 0.175 < 1.$$

Thus, by Theorem 3.1, System (5.1) has a unique solution.

Example 5.2. Consider the following problem:

(5.2)

$$\begin{cases} {}^cD^{\frac{14}{8}}\Big({}^cD^{\frac{6}{8}} + \dfrac{t}{2\times 10^4}\Big)x(t) = \dfrac{t\Big(\dfrac{x(t)+y(t)}{2} + \dfrac{1}{10^3}\displaystyle\int_0^t t^4 s^3 y(s)ds\Big)}{6\times 10^4}, & t\in[0,1], \\ {}^cD^{\frac{13}{7}}\Big({}^cD^{\frac{6}{7}} + \dfrac{t}{2\times 10^4}\Big)y(t) = \dfrac{t^2\Big(x(t)+y(t) + \dfrac{1}{10^3}\displaystyle\int_0^t t^5 s^4 x(s)ds\Big)}{4\times 10^4} & t\in[0,1], \\ x(0)=0, \quad x\Big(\frac{1}{500}\Big)=0, \quad x(1) = \dfrac{1}{6000}\Big(y(\frac{1}{90}) + y(\frac{1}{70}) + y(\frac{1}{60})\Big), \\ y(0)=0, \quad y\Big(\frac{1}{300}\Big)=0, \quad y(1) = \dfrac{1}{5000}\Big(x(\frac{1}{50}) + x(\frac{1}{40}) + x(\frac{1}{10})\Big), \end{cases}$$

where $\beta_1 = \frac{14}{8}$, $\alpha_1 = \frac{6}{8}$, $\beta_2 = \frac{13}{7}$, $\alpha_2 = \frac{6}{7}$, $\lambda_1 = \lambda_2 = \frac{1}{20000}$ and

$$f(t,x,y,z) = \frac{t}{60000}\Big(\frac{x(t)+y(t)}{2} + z(t)\Big), \quad g(t,x,y,z) = \frac{t^2}{40000}\Big(x(t)+y(t)+\frac{z(t)}{2}\Big),$$

$$\Phi y(t) = \frac{1}{250}\int_0^t t^4 s^3 y(s)ds), \quad \Psi x(t) = \frac{1}{1000}\int_0^t t^5 s^4 x(s)ds,$$

$a_1 = \frac{1}{500}$, $b_1 = \frac{1}{300}$, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{6000}$, $s_1 = \frac{1}{90}$, $s_2 = \frac{1}{70}$, $s_3 = \frac{1}{60}$, $\delta_1 = \delta_2 = \delta_3 = \frac{1}{5000}$, $u_1 = \frac{1}{50}$, $u_2 = \frac{1}{40}$, $u_3 = \frac{1}{10}$.

Clearly, $\delta_0 = \frac{1}{5000}$, $\lambda_0 = \frac{1}{2000}$ and $\sigma_1^* = \frac{1}{120000}$, $\sigma_2^* = \frac{1}{40000}$.

After calculating, we obtain $R \approx 0.0526 < 1$. So, by Theorem 3.2, Problem (5.2) has a least one solution.

6. Conclusion

In this paper, we suggested a new coupled fractional Langevin equation. More precisely, we have improved the existence and uniqueness results for a coupled system of nonlinear fractional Langevin equations via variable coefficient supplemented with multipoint boundary conditions by the application of the Banach contraction principle and Krasnoselskii's fixed point theorem. Further, we have established Ulam stability to the solution of mentioned system. Finally, we have presented two examples to demonstrate our results.

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